INDISCERNIBILITY, PARTITIONS AND ROUGH SETS

Zdzisław Pawlak

Polish Academy of Sciences

ABSTRACT. The concept of a rough set is a new mathematical tool intended to deal with imprecise knowledge. Some issues concerning basic properties of this concept will be discussed in this paper.

1. INTRODUCTION

We would like to discuss in this paper a basic assumption underlying the concept of a rough set (cf.[1]), which was introduced in order to deal with imprecise knowledge. Our primary assumption is that if we do not have precise knowledge about some objects - we are unable to discern them, i.e. our conjecture is that imprecision is manifested throughout indiscernibility. Moreover we assume that indiscernibility relation is an equivalence relation, and each equivalence (indiscernibility) class of the indiscernibility relation contains elements which can not be discerned at the present level of knowledge. Thus knowledge about certain set of objects is identified in our approach with the ability to classify this object into blocks of the partition induced by the indiscernibility relation. Thus the more knowledge we have about some objects the more exactly we can then classify. In order to formulate these ideas more precisely the concept of the rough set will be employed. Let us mention the idea of the rough set is basically different to that of fuzzy set introduced by Zadeh (cf. [4]).

2. APPROXIMATION SPACE

In this section we define the notion of an approximation space, which will be used as a mathematical model of imprecise knowledge. Informally the approximation space is a partition of a certain universe of discourse, such that the elements of the universe contained in the same block of the partition appear to be indistinguishable due to the lack of knowledge.

We begin our considerations with precise definition of the concept of the approximation space. The *approximation space* is a pair $A = (U,R)$, where U is a non-empty, finite set called the *universe* and R is a binary relation over U, called the *indiscernibility relation*. Elements of U will also be called *objects*. Objects can be anything we can think of, for example states, processes, moments of time, physical or abstract entities etc. In other words the universe is going to represent our domain of interest.

We assume that R is an equivalence relation. This assumption is motivated by many practical applications of the introduced concept.

If R is an indiscernibility relation then R^* is to denote the family of all equivalence classes of R (partition generated by R), i.e. $R^*=\{X_1,X_2,\dots,X_n\}$, where X_i are equivalence (indiscernibility) classes of R (or *blocks* of R^*). The equivalence class containing element $x \in U$ will be denoted by $[x]_R$ or $[x]$ when R is understood. If $R(x,y)$ we say that x and y are *R-indiscernible*.

If $A=(U,R)$ and $A'=(U',R')$ are two approximation spaces and $U' \subseteq U$, $R' \subseteq R$ we say that A' is an *approximation subspace* of A.

If $B = (U,Q)$ is the approximation subspace of $A=(U,R)$ we say that B is *finer* than A and A is *coarser* than B.

Intuitively, finer approximation space means that our knowledge about the universe is larger than in the case of coarser approximation space. The more knowledge we have the finer is the corresponding approximation space, the better is our ability to discern more precisely elements of the universe. Representing of knowledge by an indiscernibility relation seems to be natural and justified at least in some application areas.

It is worthwhile to mention that the inability to discern exactly objects might be not necessarily a drawback of our knowledge. Sometimes it may be a desired advantage. Suppose for example that we consider a concept of a green colour. Because human beings are able to discern many shades of green thus in order to have one concept of green colour we have to ignore small differences between them. In our terms it means a coarser approximation space, which is in fact a generalization of the concept of green colour, i.e. reduction of our ability to discern shades of green. Conversely, finer approximation space means specification of our knowledge. For example, in order to specify the concept of water we have to have more knowledge so that we are able to distinguish between oceans, lakes, rivers, puddles etc. Thus the concept of indiscernibility may have many applications. We shall concentrate in this paper on the investigation of indiscernibility as a model for imprecise knowledge independently of its possible meanings.

3. APPROXIMATION OF SETS

As we have already demonstrated in the previous section the incompleteness of our knowledge leads to practical difficulties in deciding membership of an object in a set. Consequently, in the rough set approach the basic role is played by the concept of the approximation of a set by another set.

To describe the idea, suppose we are given an approximation space $A = (U,R)$. With each subset $X \subseteq U$ we can associate two related subsets

$$\underline{R}X = \bigcup\{Y \in R^{*}: Y \subseteq X\}$$

$$\bar{R}X = \bigcup\{Y \in R^{*}: Y \cap X \neq \emptyset\}$$

called the *R-lower* and *R-upper approximation* of X respectively.

The set

$$BN_R(X) = \bar{R}X - \underline{R}X$$

will be called the *R-boundary* of X, or a *R-boundary region* of X.

We shall also employ the following denotations:

$POS_R(X) = \underline{R}X$, *R-positive region* of X

$NEG_R(X) = U - \bar{R}X$, *R-negative region of* X.

The positive region $POS_R(X)$ or the lower approximation of X is the collection of those objects which can be classified with full certainty as members of the set X. Similarly, the negative region $NEG_R(X)$ is the collection of objects about which it can be determined without any ambiguity that they do not belong to the set X, that is, there are contained in the complement -X.

The boundary region is, in a sense, undecidable area of the universe, i.e. none of the objects belonging to the boundary can be classified with certainty into X or complement $-X$.

Now we are able to define our basic concept of a rough set. Set $X \subseteq U$ is *rough with respect to R* if $\bar{R}X \neq \underline{R}X$; otherwise the set X is *exact*. Thus rough sets are sets with unsharp defined boundary, i.e. sets which can not be uniquely defined.

4. PROPERTIES OF APPROXIMATION

Directly from the definition of approxiamtions we can get the following properties of the *R*-lower and the *R*-upper approximations:

Proposition 1.

1) $\underline{R}X \subseteq X \subseteq \overline{R}X$
2) $\underline{R}\emptyset = \overline{R}\emptyset = \emptyset$; $\underline{R}U = \overline{R}U = U$
3) $\overline{R}(X \cup Y) = \overline{R}X \cup \overline{R}Y$
4) $\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y$
5) $X \subseteq Y$ implies $\underline{R}X \subseteq \underline{R}Y$ and $\overline{R}X \subseteq \overline{R}Y$
6) $\underline{R}(X \cup Y) \supseteq \underline{R}X \cup \underline{R}Y$
7) $\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y$
8) $\underline{R}(-X) = -\overline{R}X$
9) $\overline{R}(-X) = -\underline{R}X$
10) $\underline{R}\underline{R}X = \overline{R}\underline{R}X = \underline{R}X$
11) $\overline{R}\overline{R}X = \underline{R}\overline{R}X = \overline{R}X$

Proof.

1a) If $x \in \underline{R}X$, then $[x] \subseteq X$, but $x \in [x]$ hence $x \in X$ and $\underline{R}X \subseteq X$.

1b) If $x \in X$, then $[x] \cap X \neq \emptyset$ (because $x \in [x] \cap X$) hence $x \in \underline{R}X$, and $X \subseteq \underline{R}X$.

2a) From 1) $\underline{R}\emptyset \subsetneq \emptyset$ and $\emptyset \subseteq \underline{R}\emptyset$ (because the empty set is included in every set) thus $\underline{R}\emptyset = \emptyset$.

2b) Suppose $\overline{R}\emptyset \neq \emptyset$. Then there exists x such that $x \in \overline{R}\emptyset$. Hence $[x] \cap \emptyset \neq \emptyset$, but $[x] \cap \emptyset = \emptyset$, what contradicts the assumption, thus $\overline{R}\emptyset = \emptyset$.

2c) From 1) $\underline{R}U \subseteq U$. In order to show that $U \subseteq \underline{R}U$ let us observe that if $x \in U$, then $[x] \subseteq U$, hence $x \in \underline{R}U$, thus $\underline{R}U = U$.

2d) From 1) $\overline{R}U \supseteq U$, and obviously $\overline{R}U \subseteq U$, thus $\overline{R}U = U$.

3) $x \in \overline{R}(X \cup Y)$ iff $[x] \cap (X \cup Y) \neq \emptyset$

iff $[x] \cap X \cup [x] \cap Y \neq \emptyset$

iff $[x] \cap X \neq \emptyset \vee [x] \cap Y \neq \emptyset$ iff $x \in \overline{R}X \vee x \in \overline{R}Y$ iff $x \in \overline{R}X \cup \overline{R}Y$. Thus $\overline{R}(X \cup Y) = \overline{R}X \cup \overline{R}Y$.

4) $x \in \underline{R}(X \cap Y)$ iff $[x] \subseteq X \cap Y$ iff $[x] \subseteq X \wedge [x] \subseteq Y$ iff $x \in \underline{R}X \cap \underline{R}Y$.

5) Because $X \subseteq Y$ iff $X \cap Y = X$ by virtue of 4) we have $\underline{R}(X \cap Y) = \underline{R}X$ iff $\underline{R}X \cap \underline{R}Y = \underline{R}X$ which yields $\underline{R}X \subseteq \underline{R}Y$.

Because $X \subseteq Y$ iff $X \cup Y = Y$, hence $\overline{R}(X \cup Y) = \overline{R}Y$ and by virtue of 3) we have $\overline{R}X \cup \overline{R}Y = \overline{R}Y$ and hence $\overline{R}X \subseteq \overline{R}Y$.

6) Because $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, we have $\underline{R}X \subseteq \underline{R}(X \cup Y)$ and $\underline{R}Y \subseteq \underline{R}(X \cup Y)$ which yields $\underline{R}X \cup \underline{R}Y \subseteq \underline{R}(X \cup Y)$.

7) Because $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we have $\overline{R}(X \cap Y) \subseteq \overline{R}X$ and $\overline{R}(X \cap Y) \subseteq \overline{R}Y$ hence $\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y$.

8) $x \in \underline{R}(x)$ iff $[x] \subseteq X$ iff $[x] \cap -X = \emptyset$ iff $x \notin \overline{R}(-X)$ iff $x \in -\overline{R}(-X)$, hence $\underline{R}(X) = -\overline{R}(-X)$.

9) By substitution $-X$ for X in 9) we get $\overline{R}(X) = -\underline{R}(-X)$.

10a) From 1) $\underline{R}\underline{R}X \subseteq \underline{R}X$, thus we have to show that $\underline{R}X \subseteq \underline{R}\underline{R}X$. If $x \in \underline{R}X$ then $[x] \subseteq X$, hence $\underline{R}[x] \subseteq \underline{R}X$ but $\underline{R}[x] = [x]$, thus $[x] \subseteq \underline{R}X$ and $x \in \underline{R}\underline{R}X$, that is $\underline{R}X \subseteq \underline{R}\underline{R}X$.

10b) From 1) $\underline{R}X \subseteq \overline{R}\underline{R}X$, thus it is enough to show that $\underline{R}X \geq \overline{R}\underline{R}X$.

If $x \in \overline{R}\underline{R}X$, then $[x] \cap \underline{R}X \neq \emptyset$, i.e. there exists $y \in [x]$ such that $y \in \underline{R}X$, hence $[y] \subseteq X$ but $[x] = [y]$, thus $[x] \subseteq X$ and $x \in \underline{R}X$ which is to mean that $\underline{R}X \geq \overline{R}\underline{R}X$.

11a) From 1) $\overline{R}X \subseteq \overline{R}\overline{R}X$. We have to show, that $\overline{R}X \geq \overline{R}\overline{R}X$. If $x \in \overline{R}\overline{R}X$, then $[x] \cap \overline{R}X \neq \emptyset$ and for some $y \in [x]$ $y \in \overline{R}X$, hence $[y] \cap X \neq \emptyset$ but $[x] = [y]$, thus $[x] \cap X \neq \emptyset$, i.e. $x \in \overline{R}X$, which yields $\overline{R}X \geq \overline{R}\overline{R}X$.

11b) From 1) $\underline{R}\overline{R}X \subseteq \overline{R}X$. We have to show, that $\underline{R}\overline{R}X \geq \overline{R}X$. If $x \in \overline{R}X$ then $[x] \cap X \neq \emptyset$. Hence $[x] \subseteq \overline{R}X$ (because if $y \in [x]$, then $[y] \cap X = [x] \cap X \neq \emptyset$, i.e. $y \in \overline{R}X$) and $x \in \underline{R}\overline{R}X$, which gives $\underline{R}\overline{R}X \geq X$.

5. LOWER AND UPPER MEMBERSHIP

The concept of approximations of sets leads to a new conception of membership relation. Because definition of set in our approach is associated with knowledge (i.e.indiscernibility relation) hence also a membership relation must be related to knowledge. Formally this can be defined as follows:

$$x \underline{\in}_R X \text{ if and only if } x \in \underline{R}X$$

$$x \overline{\in}_R X \text{ if and only if } x \in \overline{R}X$$

where $\underline{\in}_R$ reads "x *surely belongs* to X with respect to R " and $\overline{\in}_R$ - "x possibly *belongs* to X with respect to R ".

Both membership relations again are referring to our knowledge, i.e. indiscernibility relation R.

From **Proposition 1** we obtain the following properties of membership relations (for simplicity we drop the subscript R):

1) $x \underline{\in} X$ implies $x \underline{\in} X$ implies $x \in X$ implies $x \overline{\in} X$
2) $x \underline{\in} \emptyset$ if and only if $x \overline{\in} \emptyset$ if and only if $x \in \emptyset$;
   $x \underline{\in} U$ if and only if $x \overline{\in} U$ if and only if $x \in U$;
3) $X \subseteq Y$ implies ($x \underline{\in} X$ implies $x \underline{\in} Y$ and $x \overline{\in} X$ implies $x \overline{\in} Y$)
4) $x \overline{\in} (X \cup Y)$ if and only if $x \overline{\in} X$ or $x \overline{\in} Y$
5) $x \underline{\in} (X \cap Y)$ if and only if $x \underline{\in} X$ and $x \underline{\in} Y$
6) $x \underline{\in} X$ or $x \underline{\in} Y$ implies $x \underline{\in} (X \cup Y)$
7) $x \overline{\in} (X \cap Y)$ implies $x \overline{\in} X$ and $x \overline{\in} Y$
8) $x \underline{\in} (-X)$ if and only if non $x \overline{\in} X$
9) $x \overline{\in} (-X)$ if and only if non $x \underline{\in} X$

6. ROUGH EQUALITY OF SETS

As we have already indicated the concept of rough set differs essentially from the ordinary concept of the set, because for the rough sets we are unable to define uniquely the membership relation and we have two of them instead (cf. section 5).

There is another important difference between those concepts namely that of equality of sets. In set theory two sets are equal if they have exactly the same elements. In our approach we need another concept of equality of sets, namely approximate (rough) equality.

In fact we need not one but three kinds of approximate equality of sets, as defined below:

Let $A = (U, R)$ and $X, Y \subseteq U$. We say that

a) Sets X and Y are *bottom R-equal* ($X \underset{\sim}{-}_R Y$) if $\underline{R}X = \underline{R}Y$.

b) Sets X and Y are *top R-equal* ($X \simeq_R Y$) if $\bar{R}X = \bar{R}Y$.

c) Sets X and Y are *R-equal* ($X \approx_R Y$) if $X \underset{\sim}{-}_R Y$ and $X \overset{\sim}{\simeq}_R Y$.

It is easy to see that $\underset{\sim}{-}_R$, $\simeq_R$ and $\approx_R$ are equivalence relations for any indiscernibility relation R.

We can associate the following interpretations, with the above notions of rough equality (For simplicity of notation we shall omit the subscripts.):

a) If $X \underset{\sim}{-} Y$, this means that positive examples of the sets X and Y are the same.

b) If $X \simeq Y$, then the negative examples of sets X and Y are the same.

c) If $X \approx Y$, then both positive and negative examples of sets X and Y are the same.

(By positive or negative examples of a set we mean the elements of the universe belonging to either positive or negative region of the set, respectively.)

The following exemplary properties of relations $\underset{\sim}{-}_R$, $\simeq_R$ and $\approx_R$ are immediate consequences of the definitions:

Proposition 2. For any indiscernibility relation we have the following properties:

1) $X \underset{\sim}{-} Y$ if $X \cap Y \underset{\sim}{-} X$ and $X \cap Y \underset{\sim}{-} Y$

2) $X \simeq Y$ if $X \cup Y \simeq X$ and $X \cup Y \simeq Y$

3) If $X \simeq X'$ and $Y \simeq Y'$, then $X \cup Y \simeq X' \cup Y'$

4) If $X \underset{\sim}{=} X'$ and $Y \underset{\sim}{=} Y'$, then $X \cap Y \underset{\sim}{=} X' \cap Y'$

5) If $X \simeq Y$, then $X \cup -Y \simeq U$

6) If $X \underset{\sim}{=} Y$, then $X \cap -Y \underset{\sim}{=} \emptyset$

7) If $X \subseteq Y$ and $Y \simeq \emptyset$, then $X \simeq \emptyset$

8) If $X \subseteq Y$ and $X \simeq U$, then $Y \simeq U$

9) $X \simeq Y$ if and only if $-X \simeq -Y$

10) $X \cap Y \underset{\sim}{=} \emptyset$ if and only if $X \underset{\sim}{=} \emptyset$ or $Y \underset{\sim}{=} \emptyset$

11) $X \cup Y \simeq U$ if and only if $X \simeq U$ or $Y \simeq U$

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Let us note that if we replace $\underset{\sim}{=}$ by $\simeq$ (or conversely) the above properties are not valid.

7. ROUGH INCLUSION OF SETS

One of the fundamental notions of the set theory is inclusion relation. Analogous notion can be introduced in rough sets framework. The rough inclusion of sets is defined here in much the same way as the rough equality of sets.

The formal definition of rough inclusion is as follows.

Let $A = (U,R)$ be an approximation space and $X,Y \subseteq U$.

We shall say that:

a) Set X is *bottom R-included* in Y ($X \underset{\sim}{\subset}_R Y$) if $\underline{R}X \subseteq \underline{R}Y$, ($\underline{R}X \neq \emptyset$)

b) Set X is *top R-included* in Y ($X \overset{\sim}{\subset}_R Y$) if $\bar{R}Y \subseteq \bar{R}X$.

c) Set X is *R-included* in Y ($X \underset{\sim}{\overset{\sim}{\subset}}_R Y$), if $X \underset{\sim}{\subset}_R Y$ and $X \overset{\sim}{\subset}_R Y$.

One can easily see that $\underset{\sim}{\subset}_R$, $\overset{\sim}{\subset}_R$ and $\underset{\sim}{\overset{\sim}{\subset}}_R$ are ordering relations.

In fact we have, as in the case of rough equality of sets, three rough inclusion relations of sets. The intuitive meaning of these inclusions is the following:

a) $X \underset{\sim}{\subset} Y$ means that the positive examples of the set X

are also positive examples of the set Y.

b) $X \tilde{\subset} Y$ means that negative examples of set Y are also the negative examples of the set X.

c) If $X \underset{\sim}{\tilde{\subset}} Y$, then both a) and b) hold.

It should be quite clear by now that rough inclusion of sets does not imply the inclusion of sets.

Immediately from the definitions we can derive the following simple properties:

Proposition 3.

1) If $X \subseteq Y$, then $X \underset{\sim}{\subset} Y$, $X \tilde{\subset} Y$ and $X \underset{\sim}{\tilde{\subset}} Y$.
2) If $X \underset{\sim}{\subset} Y$ and $Y \underset{\sim}{\subset} X$, then $X \underset{\sim}{=} Y$.
3) If $X \tilde{\subset} Y$ and $Y \tilde{\subset} X$, then $X \simeq Y$.
4) If $X \underset{\sim}{\tilde{\subset}} Y$ and $Y \underset{\sim}{\tilde{\subset}} X$, then $X \approx Y$.
5) $X \tilde{\subset} Y$ if and only if $X \cup Y \simeq Y$.
6) $X \underset{\sim}{\subset} Y$ if and only if $X \cap Y \underset{\sim}{=} Y$.
7) If $X \subseteq Y$, $X \underset{\sim}{=} X'$ and $Y \underset{\sim}{=} Y'$, then $X' \underset{\sim}{\subset} Y'$.
8) If $X \subseteq Y$, $X \simeq X'$ and $Y \simeq Y'$, then $X' \tilde{\subset} Y'$
9) If $X \subseteq Y$, $X \approx X'$ and $Y \approx Y'$ then $X' \underset{\sim}{\tilde{\subset}} Y'$.
10) If $X' \tilde{\subset} X$ and $Y' \tilde{\subset} Y$, then $X' \cup Y' \tilde{\subset} X \cup Y$.
11) If $X' \underset{\sim}{\subset} X$ and $Y' \tilde{\subset} Y$, then $X' \cap Y' \tilde{\subset} X \cap Y$.
12) $X \cap Y \underset{\sim}{\subset} X \tilde{\subset} X \cup Y$.
13) If $X \underset{\sim}{\subset} Y$ and $X \underset{\sim}{=} Z$, then $Z \underset{\sim}{\subset} Y$.
14) If $X \tilde{\subset} Y$ and $X \simeq Z$, then $Z \tilde{\subset} Y$.
15) If $X \underset{\sim}{\tilde{\subset}} Y$ and $X \approx Z$, then $Z \underset{\sim}{\tilde{\subset}} Y$.

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The above properties are not valid if we replace $\underset{\sim}{=}$ by $\simeq$ (or conversely).

8. CONCLUSION

It is interesting to compare the the concept of the rough sets with that of conventional set. Basic properties of roughs sets, like membership of elements, equality and inclusion of sets are related to our knowledge about the universe of discourse, expressed by the indiscernibility relation. Consequently whether an element belongs to a set or not is not an objective property of the element but depends upon our knowledge about it. Similarly equality and inclusion of sets are not decidable in absolute sense but depend on what we know about the sets in question. In general all properties of rough sets are not absolute but are related to what we know about them. In this sense the rough set approach could be viewed as a subjective counterpart of the "classical" set theory.

9. REFERENCES

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