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Abstract . Содержание . Streszczenie

In this note we compare notions of rough set and fuzzy set, and we show that these two notions are different.

Приближенные множества и нечеткие множества

В работе сравниваются понятия приближенного множества и размытого множества. Показано, что эти понятия разные.

Zbiory przybliżone i zbiory rozmyte

W nocie tej porównujemy pojęcie zbioru przybliżonego oraz zbioru rozmytego i pokazujemy, że są to pojęcia różne.

1. INTRODUCTION

The concept of a rough set has been introduced in Pawlak (1982), and some properties and application of this concept has been studies in many works (see for example Orłowska and Pawlak (1984)).

In this paper we compare this concept with that of fuzzy set, and we show that these two concepts are different.

2. ROUGH SET

In this section we recall, after Pawlak (1982), the concept of a rough set.

Let U be a set called <u>universe</u>, and let R be an equivalence relation on U, called an <u>indiscernibility</u> relation. Equivalence classes of the relation R are called <u>elementary sets</u> in A (an empty set is also elementary). Any union of elementary set is called a <u>composed</u> set in A. The family of all composed sets in A is denoted $Com(A)$. The pair $A = (U,R)$ will be called an <u>approximation space</u>.

Let $X \subseteq U$ be a subset of U. We define <u>lower</u> and <u>upper approximation</u> of X in A, denoted $\underline{A}(X)$ and $\overline{A}(X)$ respectively, as follows:

$$\underline{A}(X) = \{x \in U : [x]_R \subset X\}$$

$$\overline{A}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$$

where $[x]_R$ denotes an equivalence class of the relation R containing element x.

By $Fr_A(X) = \overline{A}(X) - \underline{A}(X)$ we denote the <u>boundary</u> of X in

Thus we may define two membership functions $\underline{\in}_A$, $\bar{\in}_A$, called strong and weak membership, respectively - as follows:

$$x \,\underline{\in}_A\, X \quad \text{iff} \quad x \in \underline{A}(X)$$
$$x \,\bar{\in}_A\, X \quad \text{iff} \quad x \in \bar{A}(X).$$

If $x \,\underline{\in}_A\, X$ we say that "x surely belongs to X in A" and if $x \,\bar{\in}_A\, X$ means "x possibly belongs to X in A".

One can easily check that the approximation space $A = (U,R)$ determines uniquely the topological space $T_A = (U, Com(A))$ and $Com(A)$ is the family of all open sets in T_A, and the family of all elementary sets in T_A is a base for T_A. From the definition of lower and upper approximation in A follows that $Com(A)$ is both the set of all open and closed sets in T_A and that $\underline{A}(X)$ and $\bar{A}(X)$ are interior and closure of set X in the topological space T_A. Thus $\underline{A}(X)$ and $\bar{A}(X)$ have the following properties:

1) $\underline{A}(X) \subseteq X \subseteq \bar{A}(X)$
2) $A(U) = A(U) = U$
3) $A(\emptyset) = A(\emptyset) = \emptyset$
4) $\bar{A}(X \cup Y) = \bar{A}(X) \cup A(Y)$
5) $\underline{A}(X \cup Y) \supseteq A(X) \cup A(Y)$
6) $\bar{A}(X \cap Y) \subseteq AX \cap AY$
7) $\underline{A}(X \cap Y) = \underline{AX} \cap \underline{AY}$
8) $\bar{A}(-X) = -\underline{AX}$
9) $\underline{A}(-X) = -\bar{A}X$

Moreover we have

10) $\underline{AAX} = \bar{A}\underline{AX} = \underline{AX}$
11) $\bar{A}\bar{A}X = \underline{A}\bar{A}X = \bar{A}X$

3. FUZZY SETS

We give now the definition of a fuzzy set introduced by Zadeh (see Zadeh (1965)).

Let U be a set called universe. A fuzzy set X in U is a membership function $\mu_X(x)$, which to every element $x \in U$ associate a real number from the interval $\langle 0,1 \rangle$, and $\mu_X(x)$ is the grade of membership of x in X.

The union and intersection of fuzzy sets X and Y are defined as follows:

$$\mu_{X \cup Y}(x) = \mathrm{Max}(\mu_X(x), \mu_Y(x))$$
$$\mu_{X \cap Y}(x) = \mathrm{Min}(\mu_X(x), \mu_Y(x))$$

for every $x \in U$.

The complement $-X$ of a fuzzy set X is defined by the membership function

$$\mu_X(x) = 1 - \mu_X(x)$$

for every $x \in X$.

4. ROUGH MEMBERSHIP FUNCTION

The question arises whether we may replace the concept of approximation by membership function similar to that inroduced by Zadeh.

Let $X \subseteq U$. We define membership function as follows:

$$\mu_X(x) = \begin{cases} 1 & \text{iff } x \in \underline{A}(X) \\ 1/2 & \text{iff } x \in Fr_A(X) \\ 0 & \text{iff } x \in -\bar{A}(X) \end{cases}$$

where $-X$ denotes $U - X$.

We shall show that such membership function cannot be extended to union and intersection of sets as in the previous section, i.e.

$$\mu_{X\cup Y}(x) \neq Max(\mu_X(x), \mu_Y(x)) \qquad a)$$

and

$$\mu_{X\cap Y}(x) \neq Min(\mu_X(x), \mu_Y(x)) \qquad b)$$

ad a) i) $\mu_{X\cup Y}(x) = 1 \equiv Max(\mu_X(x), \mu_Y(x)) = 1 \equiv \mu_X(x) = 1$

or $\mu_Y(x) = 1 \equiv x \in \underline{A}X$ or $x \in \underline{A}Y \equiv x \in \underline{A}X \cup \underline{A}Y$;

From the definition of the membership function for union of sets we have

ii) $\mu_{X\cup Y}(x) = 1 \equiv x \in \underline{A}(X\cup Y)$

From properties of interior operation we have

iii) $\underline{A}(X\cup Y) \supseteq \underline{A}(X) \cup \underline{A}(Y)$

Thus if $x \in Z = \underline{A}(X\cup Y) - (\underline{A}(X) \cup \underline{A}(Y))$, $\mu_{X\cup Y}(x) \neq 1$ with respect to i) and $\mu_{X\cup Y}(x) = 1$ according to ii) (Contradiction).

ad b) iv) $\mu_{X\cap Y}(x) = 0 \equiv Min(\mu_X(x), \mu_Y(x)) = 0 \equiv \mu_X(x) = 0$

or $\mu_Y(x) = 0 \equiv x \in -\bar{A}(X)$ or $x \in -\bar{A}(Y) \equiv$

$x \in -\bar{A}(X) \cup -\bar{A}(Y) \equiv x \in -(\bar{A}(X) \cap \bar{A}(Y))$

From the definition of membership function for intersection of sets we have

v) $\mu_{X\cap Y}(x) = 0 \equiv x \in -\bar{A}(X\cap Y)$

From properties of closure operation we have

vi) $\bar{A}(X\cap Y) \subseteq \bar{A}(X) \cap \bar{A}(Y)$

and consequently

vii) $-(\bar{A}(X) \cap \bar{A}(Y)) \subseteq -\bar{A}(X\cap Y)$

Thus if $x \in W = -\bar{A}(X\cap Y) - (-(\bar{A}(X) \cap \bar{A}(Y)))$

$\mu_{X\cap Y}(x) \neq 0$ according to iv)

and

$\mu_{X\cap Y}(x) = 0$ according to v).

(Contradiction).

This is to mean that membership function introduced in this section can not be extended to union and intersection of sets.

5. COMPLEMENT OF SETS

Membership function for complement of sets is the same for both fuzzy sets and rough sets, as shown below:

a) $\mu_{-X}(x) = 1 \equiv x \in \underline{A}(-X) \equiv x \in -\bar{A}(X) \equiv \mu_X(x) = 0 \equiv 1 - \mu_X(x) = 1$

b) $\mu_{-X}(x) = 0 \equiv x \in -\bar{A}(-X) \equiv x \in \underline{A}(X) \equiv \mu_X(x) = 1 \equiv 1 - \mu_X(x) = 0$

c) $\mu_{-X}(x) = 1/2 \equiv x \in \bar{A}(-X) - \underline{A}(-X) \equiv x \in \bar{A}(-X) \cap (-\underline{A}(-X)) \equiv$

$x \in \bar{A}(-X) \cap \bar{A}(X) \equiv x \in \bar{A}(X) \cap (-\underline{A}(X)) \equiv x \in \bar{A}(X) - \underline{A}(X) \equiv$

$\mu_X(x) = 1/2 \equiv 1 - \mu_X(x) = 1/2$

6. FINAL REMARKS

It follows from the above considerations that the idea of rough set cannot be reduced to the idea of fuzzy set by introducing membership function expressing the grade of membership. Moreover the concept of rough set is wider than the concept of fuzzy set; it reduces to fuzzy set if instead

$$\underline{A}(X\cup Y) \supseteq \underline{A}(X) \cup \underline{A}(Y)$$

and

$$\bar{A}(X\cap Y) \subseteq \bar{A}(X) \cap \bar{A}(Y)$$

the following is valid:

$$\underline{A}(X\cup Y) = \underline{A}(X) \cup \underline{A}(Y)$$

and

$$\bar{A}(X \cap Y) = \bar{A}(X) \cap \bar{A}(Y),$$

which of course in general case is not true.

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