PRACE IPI PAN • ICS PAS REPORTS

Zdzisław Pawlak

About conflicts

451

October 1981
WARSZAWA

INSTYTUT PODSTAW INFORMATYKI POLSKIEJ AKADEMII NAUK
INSTITUTE OF COMPUTER SCIENCE POLISH ACADEMY OF SCIENCES
00-901 WARSAW, P.O. Box 22, POLAND

Zdzisław Pawlak

ABOUT CONFLICTS

451

Warsaw, October 1981

Pracę zgłosił Andrzej Blikle

Mailing address: Zdzisław Pawlak
Institute of Computer Science
Polish Academy of Sciences
Palace of Culture and Science
00-901 Warsaw
B.O. Box 22

ISSN 0138-0648

P r i n t e d   a s   a   m a n u s c r i p t
N a   p r a w a c h   r ę k o p i s u

Nakład 700 egz. Ark wyd. 0,70; ark. dru. 1,25.
Papier offset.kl. III, 70 g, 70 x 100. Oddano do druku w październiku 1981 r. W.D.N. Zam. nr 559/81

Abstract . Содержание . Streszczenie

In this paper we introduce a mathematical model of conflict situations, based on three binary relations: alliance, conflict and neutrality. Axioms for alliance and conflict relations are given and some properties of these relations are investigated.

Further a strength of an object is introduced, Set of three above mentioned relations together with the strength of all objects is called situation. Some rules of transformation of situations are introduced and investigated.

Finally the notion of a capture is defined and the rules of sharing of the capture among objects in a given situation are formulated and some theorems concerning capture sharing are given.

О конфликтах

В работе вводится математическая модель конфликтных ситуаций, основанная на трех бинарных отношениях: совместного действия, конфликта и нейтралитета. Приводятся аксиомы для отношений конфликта и совместного действия, а также исследуются некоторые свойства этих отношений.

Вводится понятие силы объекта. Множество вышеуказанных отношений и сила создают ситуацию.

В заключительной части работы вводится понятие "добычи", определяется принцип ее дележа между объектами в данной ситуации, а также дается несколько теорем, касающихся дележа "добычи".

O konfliktach

W pracy wprowadzono matematyczny model konfliktowych sytuacji, oparty na trzech relacjach binarnych, współdziałania, konfliktu, neutralności . Podano aksjomaty dla relacji konfliktu i współdziałania oraz zbadano pewne własności tych relacji.

Dalej wprowadzono pojęcie siły obiektu. Zbiór powyższych relacji oraz siła tworzą sytuacje. Podano reguły zmiany sytuacji oraz zbadano pewne własności tych zmian.

Na końcu wprowadzono pojęcie "łupu", zdefiniowano zasadę jego podziału między obiekty w danej sytuacji oraz podano kilka twierdzeń dotyczących podziału łupu.

1. INTRODUCTION

We propose here a formal model of conflict situations, somewhat different from that considered in game theory.

Three binary relations: alliance, conflict and neutrality on some set of objects X are the departure point of our approach. Union of these relations is called configuration of X. Some axioms for alliance and conflict relations are given and configurations satisfying these axioms are investigated.

Further with each object from X a nonnegative real number is associated. It is called the strength of the object. Configuration of X together with the strength of all objects in X is called the situation of X, then some rules of transformation of situations are formulated and studied.

Finally the nonnegative real number called the capture is introduced and rules of sharing the capture among the objects in the situation are given and some theorems concerning capture sharing are formulated.

2. CONFIGURATIONS

Let X be some finite set. Elements of X will be called objects, which can be interpreted as human beings, trading organizations, political groups, governments, etc.

Let φ be a partial function which to each $(x,y) \in D_\varphi$ associates the number +1, or −1, that is $\varphi : X \times X \to \{+1,-1\}$.

We assume that φ is symmetric, i.e. $\varphi(x,y) = \varphi(y,x)$ for every $x,y \in X$.

If $\varphi(x,y) = +1$, we say that x,y are <u>allied</u> and if $\varphi(x,y) = -1$, we say that x,y are in <u>conflict</u>. If $\varphi(x,y)$ is undefined, we say that x,y are <u>neutral</u>.

The pair $C = (X, \varphi)$ will be called <u>configuration</u> of X, or if X is understood - configuration.

If $C = (X, \varphi)$ then we shall also write X_C, φ_C to denote that X, and φ form the configuration C.

Each configuration $C = (X, \varphi)$ defines three disjoint binary relations on X, denoted R_C^+, R_C^0, R_C^-, and defined as follows:

$$R_C^+(x,y) \Leftrightarrow \varphi(x,y) = +1,$$

$$R_C(x,y) \Leftrightarrow \varphi(x,y) = -1,$$

$$R_C^0(x,y) \Leftrightarrow \varphi(x,y) = \text{undefined}.$$

We shall call R_C^+, R_C^0, R_C^- the <u>alliance</u>, <u>conflict</u> and <u>neutrality</u> relations respectively.

If $R_C^- = \emptyset$ we shall say that C is conflictless configuration, otherwise C is conflict configuration.

If the relation R_C^+ satisfies the following axioms

A1. $R_C^+(x,x)$,

A2. $R_C^+(x,y) \Rightarrow R_C^+(y,x)$,

A3. $R_C^+(x,y) \& R_C^+(y,z) \Rightarrow R_C^+(x,z)$,

we shall say that R_C^+ is <u>regular</u>; otherwise R_C^+ is <u>non-regular</u>.

Thus the regular relation R_C^+ is an equivalence relation on X, and we shall call equivalence classes of R_C^+ <u>blocks</u> of the configuration C, and each relation $R_C^+ \cap B \times B$, where

B is some block in C, will be called coalition in C. In what follows we shall often identify blocks and corresponding coalitions.

The family of all blocks in C will be denoted by B_C and the block containing x - is denoted by B_x.

If the relation R_C^- satisfies the conditions:

B1. $\sim R_B^-(x,x)$,

B2. $R_C^-(x,y) \Rightarrow R_C^-(y,x)$,

B3. $R_C^-(x,y) \& R_C^-(y,z) \Rightarrow R_C^+(x,z)$,

B4. $R_C^-(x,y) \& R_C^+(y,z) \Rightarrow R_C^-(x,z)$,

B5. $R_C^+(x,y) \& R_C^-(y,z) \Rightarrow R_C^-(x,z)$,

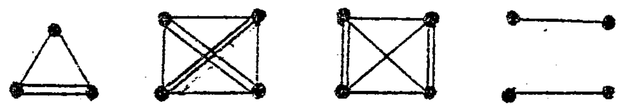
we shall say that R_C^- is <u>regular</u>, otherwise R_C^- is <u>nonregular</u>.

If R_C^+, R_C^- are both regular, then so is the configuration C, otherwise C is nonregular.

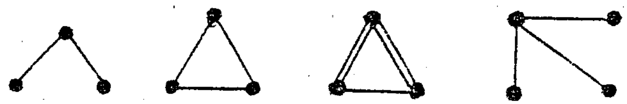
Each configuration $C = (X, \varphi)$ can be depicted by a graph; objects of C are interpreted as vertices of the graph. If $R_C^+(x,y)$, we shall connect vertices x and y by a double line, called a <u>positive</u> edge. If $R_C^-(x,y)$, then we shall connect vertices x,y by a single line, called a <u>negative</u> edge.

If $C = (X, \varphi)$ is a configuration then the associated graph will be denoted by G_C. In what follows we shall identify configurations and their graphs, and consequently we shall use graph theoretical terminology for configurations, like connected configuration, subconfiguration, loop in the configuration, etc.

Examples of regular and nonregular configurations are shown on Fig. 1.



Regular configurations



Nonregular configurations

Fig. 1

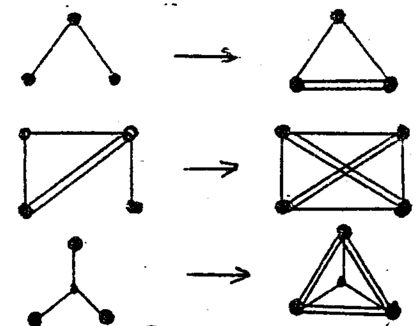
Let $C = (X, \varphi)$ and $C' = (X', \varphi')$ be the configurations. We say that C' is an extension of C (or C is a <u>subconfiguration</u> of C') if $\varphi' = \varphi/_{X' \times X'}$.

Suppose $C = (X, \varphi)$ is a nonregular configuration. If $C' = (X', \varphi')$ is the least regular extension of C, then C' will be called a <u>forced extension</u> of C.

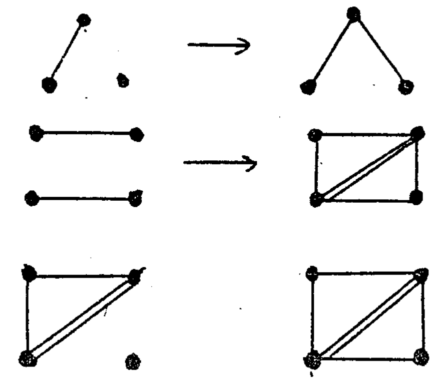
It is obvious that for every nonregular configuration there exist at most one forced extension of it.

If $C = (X, \varphi)$ is regular configuration, then every extension of C will be called <u>free extension of</u> C.

Examples of forced and free extensions are shown on Fig. 2.



Forced extensions



Free extensions

Fig. 2.

3. REDUCED GRAPHS

If the loop in the characteristic graph G_C contains an odd number of negative edges we shall call the loop <u>improper</u>; otherwise the loop is <u>proper</u>.

One can prove the following property of characteristic graphs:

If the characteristic graph G_C of the configuration $C = (X, \psi)$, contains improper loop, then $C = (X, \psi)$ is nonregular and there is no regular extension for C. If the graph G_C does not contain improper loop, then C has regular forced extension.

Let $B_x \neq B_y$ be two blocks of the configuration $C = (X, \varphi)$. If there are $x' \in B_x$ and $y' \in B_y$ such that $R_C^-(x', y')$, then we shall say, that blocks B_x, B_y are in a <u>weak conflict</u>, if for every $x' \in B_x$ and $y' \in B_y$, $R_C^-(x', y')$, we shall say that blocks B_x, B_y are in a <u>strong conflict</u>; If block B is not in a conflict with any other block in $C = (X, \varphi)$ we say that B is <u>neutral</u> in C.

The following theorem is true:

If $C = (X, \psi)$ is a regular configuration, then every maximal connected subconfiguration of C (i.e. maximal connected subgraph of the graph G_C) is a neutral block in C or a pair of blocks in strong conflict.

That is to say that with every regular configuration we can associate besides the characteristic graph G_C another graph $\bar{G}_C$, which vertices are blocks in C and two blocks B, B' are connected by an edge in $\bar{G}_C$ if and only if B, B' are in strong conflict in C. Graph $\bar{G}_C$ will be called <u>reduced graph</u> of the configuration C.

Exemples of characteristic graphs and reduced graphs of some configurations are shown on Fig. 3.

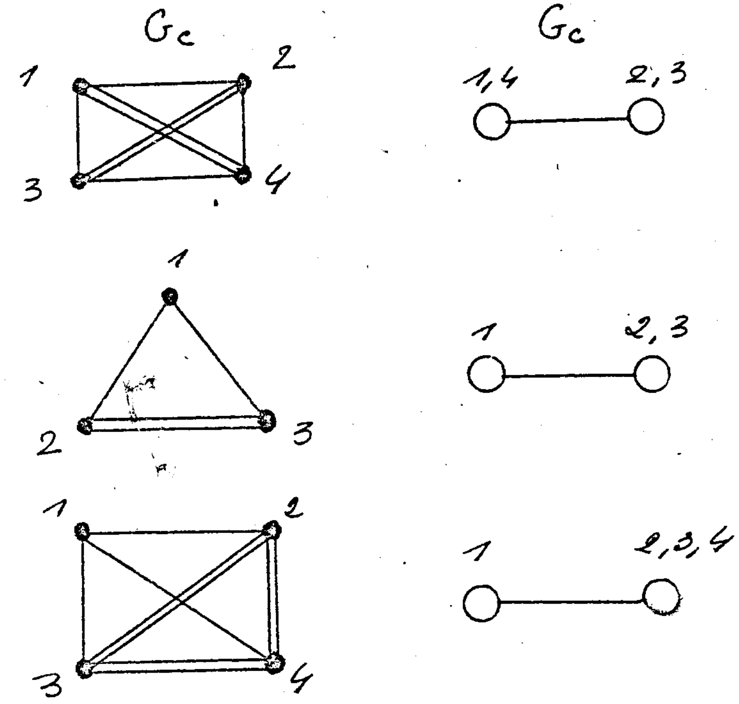


Fig. 3

4. STRENGTH OF OBJECTS

Let $C = (X, \psi)$ be a configuration and let $\mu: X \Rightarrow \mathcal{R}$ be a function which to each object $x \in X$ assigns nonnegative real number, called the <u>strength</u> of x.

The strength of $Y \subset X$ is defined as

$$\bar{\mu}(Y) = \sum_{x \in Y} \mu(x)$$

In particular $\bar{\mu}(X)$ denotes the strength of the whole configuration $C = (X, \varphi)$.

The ordered troplet $S = (X, \varphi, \mu)$ will be called the <u>situation</u> of X. If $S = (X, \varphi, \mu)$ is a situation we shall write X_S, φ_S, μ_S.

Let E_x denote the set of all objects being in conflict with x (enemies of x) in $C = (X, \varphi)$, i.e. $E_x = \{ y \subset X : R_C^-(x,y) \}$.

The function $\lambda : X \times X \to \langle 0,1 \rangle$ will be called the <u>strategy</u> in $S = (X, \varphi, \mu)$. The strategy λ says how the strength of each object in S is distributed against its enemies. We assume that the strategy λ satisfies the following conditions:

1°. $\lambda(x,x) = 0$,

2°. if $R_C^+(x,y)$, $R_C^0(x,y)$ then $\lambda(x,y) = 0$,

3°. if $R_C^-(x,y)$, then $\lambda(x,y) \neq 0$,

4° $\bar{\lambda}(x) \leq 1$, for every $x \in X$, where $\bar{\lambda}(x) = \sum_{y \in E_x} \lambda(x,y)$

If for some $x \in X$, $\bar{\lambda}(x) = 1$ we shall say that the strategy λ for x is <u>total</u> in S. If $\bar{\lambda}(x) = 1$ for every $x \in X_C^-$, where $X_C^- = \{ y \in X : \bigvee_z R_C^-(y,z) \}$, we say that λ is a <u>total strategy</u> in S.

The strategy λ in S will be called <u>balanced</u> if $\lambda(x,y)\mu(x) - \lambda(y,x)\mu(y) = 0$, for every $(x,y) \in R_C^-$.

We shall consider balanced strategies in this paper only.

Given a situation $S = (X, \varphi, \mu)$ and the strategy λ in S, we introduce a function ς called the <u>realization</u>

of the strategy λ in the situation S, which associates to each situation S and the strategy λ a new situation S', i.e.

$$\varsigma(S, \lambda) = S',$$

or

$$\varsigma_\lambda(S) = S'.$$

We introduce now some auxiliary nations needed to define the realization function ς.

Let $S = (X, \varphi, \mu)$ and $S' = (X', \varphi', \mu')$, be two situations such that $\varsigma_\lambda(S) = S'$ for some strategy λ.

The set $X_{S,\lambda}^0 \subset X$, defined as

$$X_{S,}^0 = \{ y \in X : \mu_{\varsigma_\lambda(S)}(y) = 0 \}$$

will be called the set of <u>looser</u> in the situation S and the strategy λ.

The set $X_{S,\lambda}^* \subset X$, defined as

$$X_{S,\lambda}^* = \{ x \in X_S : \mu_\lambda(x) > 0 \}, \text{ where } \mu_\lambda(x) = \mu(x)(1 - \bar{\lambda}(x)),$$

will be called the set of <u>winners</u> in the situation S and the strategy λ.

If $S = (X, \varphi, \mu)$ is a situation and λ a strategy in S, then

$$\varsigma_\lambda(S) = S' = (X', \varphi', \mu'),$$

is defined as follows:

1° $X' = \{ x \in X : \mu_\lambda(x) > 0 \}$,

2° $\varphi' = \varphi / X' \times X'$

3° $\mu' = \mu_\lambda / X'$,

of course if S is a conflictless situation, then for every strategy λ in S $S_\lambda(S) = S$.

Realization of the strategy λ in the situation S reduces the strength of each object being involved in conflict by the strength engaged against its enemies and eliminates all those objects which strength is reduced to zero.

Let $S = (X, \varphi, \mu)$ be a situation. We shall say that the situation $S = (X, \varphi, \mu)$ is <u>balanced</u> if there exists such a strategy λ in S, that $\mu_{S_\lambda(S)}(X_S) = 0$, i.e. all objects being involved in conflicts in the situation S are "destroyed", by the realization of the strategy λ .

One can show that $S = (X, \varphi, \mu)$ is balanced if the following set of linear equations has a solution.

$$\sum_{y \in E_{x_1}}^{-} \lambda(x_1, y) = 1,$$

$$\sum_{y \in E_{x_2}} \lambda(x_2, y) = 1,$$

$$\cdots\cdots\cdots\cdots$$

$$\sum_{y \in E_{x_n}} \lambda(x_n, y) = 1,$$

where

$$X_S = \{x_1, x_2, \ldots, x_n\},$$

or in short

$$\bigwedge_{X_i \in X_S^-} \bar{\lambda}(X_i) = 1,$$

$$\lambda(x_{i_1}, y_{i_1})\mu(x_{i_1}) - \lambda(y_{i_1}, x_{i_1})\mu(y_i) = 0$$

$$\lambda(x_{i_2}, y_{i_2})\mu(x_{i_2}) - \lambda(y_{i_2}, x_{i_2})\mu(y_i) = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda(x_{i_m}, y_{i_m})\mu(x_{i_m}) - \lambda(y_{i_m}, x_{i_n}) = (y_{i_m}) = 0$$

for all $(x_{i_j}, y_{i_j}) \in R_S^-$.

Thus the situation $S = (X, \varphi, \mu)$ is balanced if and only if there exist a total strategy in S.

For example the situation shown on Fig. 4 is balanced,

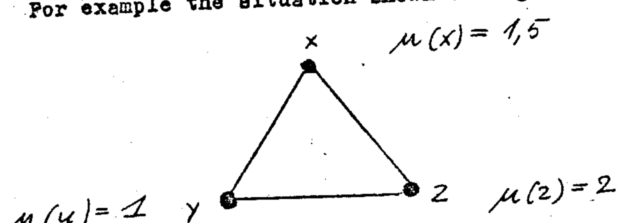


Fig. 4

because the set of equations

$$\lambda(x,y) + \lambda(x,z) = 1,$$
$$\lambda(y,x) + \lambda(y,z) = 1,$$
$$\lambda(z,y) + \lambda(z,x) = 1,$$
$$1{,}5\lambda(x,y) - \lambda(y,x) = 0,$$
$$1{,}5\lambda(x,y) - 2\lambda(z,x) = 0,$$
$$\lambda(y,z) - 2\lambda(z,y) = 0$$

has the solution

$$\lambda(x,y) = 1/6,$$
$$\lambda(y,x) = 2/8,$$
$$\lambda(x,z) = 5/6,$$
$$\lambda(z,x) = 5/8,$$

$\lambda(y,z) = 6/8,$

$\lambda(z,y) = 3/8.$

On the other hand the situation shown on Fig. 5

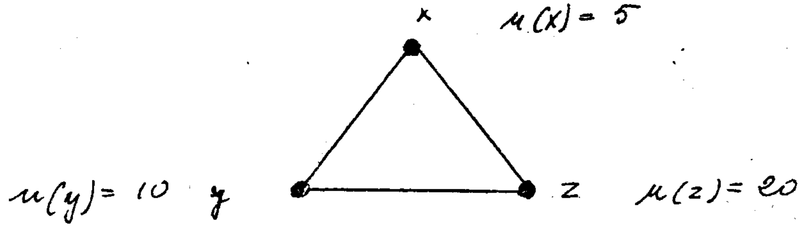


Fig. 5

is not balanced.

One can show that for every situation S there exists a strategy λ in S such that $\mathcal{S}_\lambda(S)$ is conflictless.

The strategy λ such that $\mathcal{S}_\lambda(S)$ is conflictless will be called <u>maximal</u>, and will be denoted by λ^*.

5. HOW DO CONFLICTS ARISE?

Let $S = (X, \varphi, \mu)$ be a situation and let q be some nonnegative real number, called the <u>capture</u>.

We assume that in each situation $S = (X, \varphi, \mu)$ the capture q is shared among objects of X, i.e. increases the strength of each object (possibly by zero), according to some rules expressed by the function σ.

If $S = (X, \varphi, \mu)$ is a conflictless situation, and λ^* some maximal strategy in S, then

$$\sigma(S, q, \lambda^*) = S = (X, \varphi, \mu'),$$

where

$$\mu'(x) = \mu(x) + \frac{\mu(x)}{\bar{\mu}(X)} q ;$$

If $S = (X, \varphi, \mu)$ is a conflict situation, then we assume that the capture is shared only among winners according to some prefixed rules, for example in proportion to the strength of each winner to the strength of all winners. (There are possible other principles but we shall not discuss them here).

Let $X = (X, \varphi, \mu)$ be a conflict situation, λ^* maximal strategy in S and q the capture. Then

$$\sigma(S, q, \lambda^*) = S' = (X', \varphi', \mu'),$$

where

1° $X' = \mathcal{S}_{\lambda^*}(X),$

2° $\varphi' = \varphi / X' \times X'$,

3° $$\mu'(x) = \begin{cases} \mu(x), & \text{if } x \in X^{\circ}_S \setminus X^*_{S,\lambda^*}, \\ \mu_\lambda(x) + \dfrac{\mu(x)}{\bar{\mu}(X^*_{S,\lambda^*})} q, & \text{if } x \in X^*_{S,\lambda^*}. \end{cases}$$

In the sequent in stead of μ' we use notation $\mu_{\sigma(S,q,\lambda^*)}$.

We shall say that the situation $S = (X, \varphi, \mu)$ is <u>better</u> for $x \in X$, than the situation $S' = (X, \varphi', \mu)$, if for every maximal strategies λ and λ' in S and S' respectively

$$\mu_{\sigma(S,q,\lambda)}(x) > \mu_{\sigma(S',q,\lambda')}(x),$$

for every q.

Let $S = (X, \varphi, \mu)$ be a situation. We shall say that the situation S is <u>stable</u> if for every situation $S' = (X, \varphi', \mu)$ and for every maximum strategies λ, λ' in S and S' respectively and every capture q, the situation S'

is not better than the situation S, for every $x \in X$; otherwise the situation S is <u>unstable.</u>

The following theorem is true:

A situation $S = (X, \varphi, \mu)$ is stable if and only if $card(X) = 2$ and $\mu(x_1) = \mu(x_2)$, where $X = \{x_1, x_2\}$.

One can show (by simple computation) the following (sad) theorem:

Let $S = (X, \varphi, \mu)$ be an unstable conflictless situation, and let q be a capture; then there exist the situation $S' = (X, \varphi', \mu)$ and a maximum strategy λ^*, such that S' is better than S for every $x \in X^*_{S', \lambda^*}$ if and only if $q > \bar{\mu}_S(X)$.

We shall say that the strategy λ is <u>better</u> than the strategy λ' for x in the situation $S = (X, \varphi, \mu)$, if

$$\mu_{\sigma(S,q,\lambda)}(x) > \mu_{\sigma(S,q,\lambda')}(x)$$

for every q.

The following theorem (which may be called "the minimum effort principle") is valid:

Let $S = (X, \varphi, \mu)$ be a situation; strategy λ is better than λ' for $x \in X^-_S$ in S if and only if

$$\bar{\lambda}(x) < \bar{\lambda'}(x).$$

One can also show easily the following theorem (which may be called "the principle of minimal ascendency"),

Let $S = (X, \varphi, \mu)$ and $S' = (X, \varphi', \mu)$ be two conflict situations and let λ^*, λ'^* be two maximal strategies

in S and S' respectively, such that $\bar{\lambda}^*(x) \leq \bar{\lambda}'^*(x)$ for some $x \in X$.

The situation S' is better for $x \in X^*_{S,\lambda^*} \cap X^*_{S',\lambda'^*}$ if and only if

$$\bar{\mu}(X^*_{S,\lambda^*}) > \bar{\mu}(X^*_{S',\lambda'^*}).$$

ACKNOWLEDGMENT

Thanks are due to dr T. Bromek, prof. A. Salwicki, prof. J. Winkowski for valuable comments.