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Rough sets

Basic notions

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Abstract . Содержание . Streszczenie

We investigate in this paper approximate operations on sets, approximate equality of sets and approximate inclusion of sets. The presented approach may be consider as an alternative to fuzzy sets theory and tolerance theory.

Приближенные сборы

В работе рассматривается приближенные основные операции над множествами, приближенные равности а также приближенное включение множеств. Предлагаемый подход обсуждается как альтернатива размытых сборов а также теории толерантности.

Zbiory przybliżone

W pracy rozważa się przybliżone działania na zbiorach, przybliżone równanie zbiorów oraz przybliżoną inkluzję zbiorów. Proponowane podejście może być traktowane jako alternatywa zbiorów rozmytych oraz teorii tolerancji.

INTRODUCTION

In many branches of computer applications approximate classification of objects is of primary concern. Mathematical tools offering methods to deal with such problems are fuzzy sets (see Zadeh [2]) and tolerance spaces (see Zeeman [3]). We propose here another approach based on ideas presented in Pawlak [1]. The kernel of our method lies in introducting two membership function $\underline{\in}$, $\overline{\in}$, which are to mean "surely belongs" and "possibly belongs" respectively, and the notion of an upper and lower approximation of a set in a relational structure $A = (U,R)$. The relation R is an equivalence relation, called here an approximation space. If x belongs to lower approximation of X we say that x surely belongs to X; if x belongs to upper approximation of X, we say that x possibly belongs to X. Axioms for approximations are given, which turns out to be counterparts of axioms of topological closure and interior operation.

1. PRELIMINARY

Let U be some set called here the <u>universum</u>, and let R be an equivalence relation on U. The pair $A = (U,R)$ will be called an <u>approximation space</u>. Subset of U will be denoted by X,Y,Z possibly with indices. The empty set will be denoted by 0, and set U will be also denoted by 1.

Equivalence classes of the relation R will be called elementary sets in A, or in short elementary sets, when A is known. Every union of some elementary sets in A and an empty set 0 will be called a composed set in A or in short – composed set.

Let X be some subset of U. The least composed set in A containing X will be called the best upper approximation of X in A, in short $\bar{A}X$; the greatest composed set in A contained in X will be called the best lower approximation of X in A, or in short $\underline{A}X$. Set $B_A(X) = \bar{A}X - \underline{A}X$ will be called the boundary of X in A. Set $E_A(X) = X-\underline{A}(X)$ is referred to as an edge of X in A.

The following are axioms for approximations valid in every approximation space:

(A1) $\bar{A}X \supset X \supset \underline{A}X$,

(A2) $\bar{A}1 = \underline{A}1 = 1$,

(A3) $\bar{A}0 = \underline{A}0 = 0$,

(A4) $\bar{A}\bar{A}X = \underline{A}\bar{A}X = \bar{A}X$,

(A5) $\underline{A}\underline{A}X = \bar{A}\underline{A}X = \underline{A}X$,

(A6) $\bar{A}(X \cup Y) = \bar{A}X \cup \bar{A}Y$,

(A7) $\underline{A}(X \cap Y) = \underline{A}X \cap \underline{A}Y$,

(A8) $\bar{A}X = -\underline{A}(-X)$,

(A9) $\underline{A}X = -\bar{A}(-X)$.

We use here the abbreviation $1-X = -X$. Let us remark that the above axioms are counterparts of axioms of topological closure and interior operations.

One can easily verify that the following properties of approximations are valid in every approximation space A.

1) $\bar{A}(X \cap Y) \subset \bar{A}X \cap \bar{A}Y$,

2) $\underline{A}(X \cup Y) \supset \underline{A}X \cup \underline{A}Y$,

3) $\bar{A}X - \bar{A}Y \subset \bar{A}(X-Y)$,

4) $\underline{A}X - \underline{A}Y \supset \underline{A}(X-Y)$.

The following are counterparts of the rule $X \cup -X = 1$:

5) $\bar{A}X \cup \underline{A}(-X) = 1$,

6) $\bar{A}X \cup \bar{A}(-X) = 1$,

7) $\underline{A}X \cup \bar{A}(-X) = -B_A(X)$,

8) $\underline{A}X \cup \underline{A}(-X) = -B_A(X)$.

The rule $X \cap -X = 0$ has the following analogues

9) $\bar{A}X \cap \underline{A}(-X) = 0$,

10) $\bar{A}X \cap \bar{A}(-X) = B_A(X)$,

11) $\underline{A}X \cap \underline{A}(-X) = 0$,

12) $\underline{A}X \cap \bar{A}(-X) = 0$.

De Morgan's laws have the following counterparts

13) $-(\underline{A}X \cup \underline{A}Y) = \bar{A}(-X) \cap \bar{A}(-Y)$,

14) $-(\underline{A}X \cup \bar{A}Y) = \bar{A}(-X) \cap \underline{A}(-Y)$,

15) $-(\bar{A}X \cup \underline{A}Y) = \underline{A}(-X) \cap \bar{A}(-Y)$,

16) $-(\bar{A}X \cup \bar{A}Y) = \underline{A}(-X) \cap \underline{A}(-Y)$,

17) $-(\underline{A}X \cap \underline{A}Y) = \bar{A}(-X) \cup \bar{A}(-Y)$,

18) $-(\underline{A}X \cap \bar{A}Y) = \bar{A}(-X) \cup \underline{A}(-Y)$,

19) $-(\bar{A}X \cap \underline{A}Y) = \underline{A}(-X) \cup \bar{A}(-Y)$,

20) $-(\bar{A}X \cap \bar{A}Y) = \underline{A}(-X) \cup \underline{A}(-Y)$.

Moreover we have

21) $-(\bar{A}X \cup \bar{A}Y) = \underline{A}(-(X \cup Y)) = \underline{A}(-X \cap -Y)$,

22) $-(\underline{A}X \cap \underline{A}Y) = \bar{A}(-(X \cap Y)) = \bar{A}(-X \cup -Y)$.

23) $-\bar{A}(X \cup Y) = -\bar{A}X \cap -\bar{A}Y$;

24) $-\underline{A}(X \cap Y) = -\underline{A}X \cup -\underline{A}Y$, etc.

Fig. 1 depicts the notion of an upper and lower approximation in a two dimension approximation space, consisting of a rectangle partitioned into elementary squares.

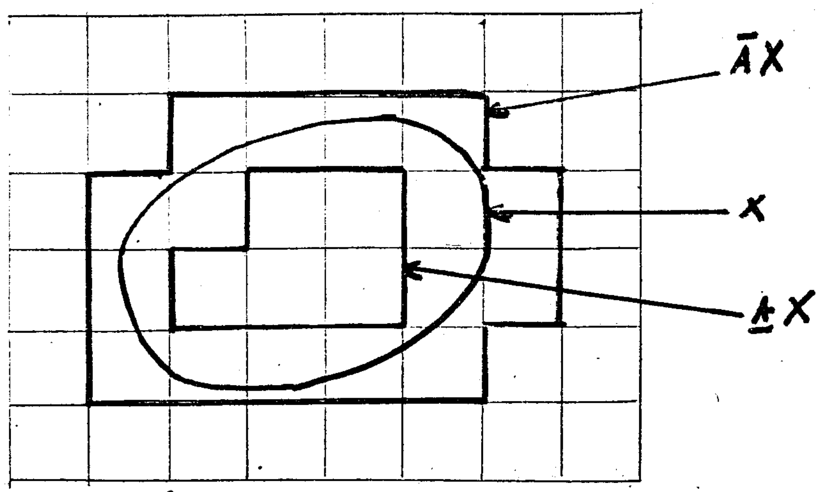

Fig. 1

Let us define two membership functions $\underline{\in}$, $\bar{\in}$ (called <u>strong</u> and <u>weak</u> membership respectively), as follows:

$x \underline{\in} X$ iff $x \in \underline{A}X$,

$x \bar{\in} X$ iff $x \in \bar{A}X$.

If $x \underline{\in} X$ we shall say "x surely belongs to X", and $x \bar{\in} X$ is to mean "x possibly belongs to X". Thus we may interprete approximations as counterparts of necessity and possibility in a model logic.

2. ROUGH EQUALITY

We define here three kinds of rough equality of sets:

1) $X \underset{A}{\eqsim} Y$ iff $\underline{A}X = \underline{A}Y$,

2) $X \underset{A}{\simeq} Y$ iff $\bar{A}X = \bar{A}Y$,

3) $X \underset{A}{\approx} Y$ iff $X \underset{A}{\eqsim} Y$ and $X \underset{A}{\simeq} Y$.

In what follows we shall omit index A if the approximation space A is understood.

It is easy to check that $\underset{A}{\eqsim}$, $\underset{A}{\simeq}$, $\underset{A}{\approx}$ are equivalence relations on $\mathcal{P}(U)$.

If X is a set then $\hat{X}$, $\underset{\vee}{X}$, $\check{\hat{X}}$ are equivalence classes containing X in the relations $\simeq$, $\eqsim$, $\approx$ respectively.

It is easy to check the following properties:

6) $X \underset{A}{\eqsim} 0$ iff $\underline{A}X = 0$,

7) $X \underset{A}{\simeq} 0$ iff $X = 0$,

8) $X \underset{A}{\eqsim} 1$ iff $X = 1$,

9) $X \underset{A}{\simeq} 1$ iff $\bar{A}X = 1$,

10) $E_A(X) \underset{A}{\eqsim} 0$.

Any set $X \in \check{0}$ (i.e. $\underline{A}X = 0$ or $X \underset{A}{\eqsim} 0$) will be called loose set in A. We shall identify all loose sets in A, and denoted by $\check{0}$. Any set $X \in \hat{1}$ (i.e. $\overline{A}X = 1$ or $X \underset{A}{\simeq} 1$), will be called representative set in A, and denoted by $\hat{1}$.

For any A and X we have:

11) $X \cap \check{0} \underset{A}{\eqsim} 0$,

12) $X \cup \hat{1} \underset{A}{\simeq} 1$,

13) $X - \check{0} \underset{A}{\eqsim} X$, if $\check{0} \subset E_A(X)$,

14) $X - \check{0} \underset{A}{\simeq} X$, if $\check{0} \subset \underline{A}X$,

15) $-\check{0} \underset{A}{\simeq} 1$, $-\hat{1} \underset{A}{\eqsim} 0$.

If X is both loose and representative ($X \in \check{0} \cap \hat{1}$), then X will be called dispersed. We shall denote any dispersed set by Δ.

For any dispersed set Δ and A we have

16) $X \cap \Delta \underset{A}{\eqsim} 0$, $X \cup \Delta \underset{A}{\simeq} 1$,

17) $-\Delta \underset{A}{\simeq} \Delta$, $-\Delta \underset{A}{\eqsim} \Delta$, $-\Delta \underset{A}{\approx} \Delta$.

Moreover we have

18) If $X \underset{A}{\simeq} Y$ then $X \cup Y \underset{A}{\simeq} X \underset{A}{\simeq} Y$,

19) If $X \underset{A}{\eqsim} Y$ then $X \cap Y \underset{A}{\eqsim} X \underset{A}{\eqsim} Y$,

20) If $X \underset{A}{\simeq} X'$ and $Y \underset{A}{\simeq} Y'$ then $X \cup Y \underset{A}{\simeq} X' \cup Y'$

21) If $X \underset{A}{\eqsim} X'$ and $Y \underset{A}{\eqsim} Y$ then $X \cap Y \underset{A}{\eqsim} X' \cap Y'$

22) If $X \underset{A}{\eqsim} Y$, then $-(-X) \underset{A}{\eqsim} Y$,

If $X \underset{A}{\simeq} Y$ then $-(-X) \underset{A}{\simeq} Y$,

and consequently

if $X \underset{A}{\approx} Y$ then $-(-X) \underset{A}{\approx} Y$.

3. ROUGH SUBSETS

We introduce three notions of subsets in a approximation space.

1) $X \underset{A}{\underset{\sim}{\subset}} Y$ iff $\underline{A}X \subset \underline{A}Y$,

2) $X \underset{A}{\overset{\sim}{\subset}} Y$ iff $\overline{A}X \ \overline{A}Y$,

3) $X \underset{A}{\underset{\sim}{\overset{\sim}{\subset}}} Y$ iff $X \underset{A}{\underset{\sim}{\subset}} Y$ and $X \underset{A}{\overset{\sim}{\subset}} Y$.

It is easy to check that all three relations are ordering relations, and

4) If $X \underset{A}{\underset{\sim}{\subset}} Y$ and $Y \underset{A}{\underset{\sim}{\subset}} X$ then $X \underset{A}{\eqsim} Y$,

5) If $X \underset{A}{\overset{\sim}{\subset}} Y$ and $Y \underset{A}{\overset{\sim}{\subset}} X$ then $X \underset{A}{\simeq} Y$,

6) If $X \underset{A}{\underset{\sim}{\overset{\sim}{\subset}}} Y$ and $Y \underset{A}{\underset{\sim}{\overset{\sim}{\subset}}} X$ then $X \underset{A}{\approx} Y$.

7) If $X \subset Y$ then $X \underset{A}{\underset{\sim}{\subset}} Y$, $X \underset{A}{\overset{\sim}{\subset}} Y$ and $X \underset{A}{\underset{\sim}{\overset{\sim}{\subset}}} Y$,

8) If $X \subset Y$ and $X \underset{A}{\underset{\sim}{=}} X'$, $Y \underset{A}{\underset{\sim}{=}} Y'$ then $X' \underset{A}{\underset{\sim}{\subseteq}} Y'$,

9) If $X \subset Y$ and $X \underset{A}{\overset{\sim}{=}} X'$, $Y \underset{A}{\overset{\sim}{=}} Y'$ then $X' \underset{A}{\overset{\sim}{\subset}} Y'$,

10) If $X \subset Y$ and $X \underset{A}{\approx} Y'$, $Y \underset{A}{\approx} Y'$ then $X' \overset{\sim}{\underset{\sim}{\subseteq}} Y'$,

11) If $X \underset{A}{\underset{\sim}{=}} X'$ and $Y \underset{A}{\underset{\sim}{=}} Y'$, then $X \cup Y \underset{A}{\underset{\sim}{\supseteq}} X' \cup Y'$,

12) If $X \underset{A}{\overset{\sim}{=}} X'$ and $Y \underset{A}{\overset{\sim}{=}} Y'$ then $X \cap Y \underset{A}{\overset{\sim}{\subset}} X' \cap Y'$,

13) $X \cup \underset{\vee}{0} \underset{A}{\underset{\sim}{\subseteq}} X \underset{A}{\overset{\sim}{\subset}} X \cup \underset{\vee}{0}$,

14) $X \cap \hat{1} \underset{A}{\underset{\sim}{\subseteq}} X \underset{A}{\overset{\sim}{\subset}} X \cap \hat{1}$.

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