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SHORT COMMUNICATION

ROUGH SETS AND FUZZY SETS

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In this note we compare notions of rough set and fuzzy set, and we show that these two notions are different.

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1. Introduction

The concept of a rough set has been introduced in Pawlak [2] and some properties and application of this concept have been studied in many works (see for example Orłowska and Pawlak [1]).

In this paper we compare this concept with that of fuzzy set, and we show that these two concepts are different.

2. Rough set

In this section we recall, after Pawlak [2], the concept of a rough set.

Let U be a set called *universe*, and let R be an equivalence relation on U, called an *indiscernibility* relation. Equivalence classes of the relation R are called *elementary sets* in A (an empty set is also elementary). Any union of elementary set is called a *composed* set in A. The family of all composed sets in A is denoted $\mathrm{Com}(A)$. The pair $A=(U, R)$ will be called an *approximation space.*

Let $X\subseteq U$ be a subset of U. We define *lower* and *upper approximation* of X in A, denoted $\underline{A}(X)$ and $\bar{A}(X)$ respectively, as follows:

$$\underline{A}(X)=\{x\in U:[x]_R\subset X\},$$
$$\bar{A}(X)=\{x\in U:[x]_R\cap X\neq\emptyset\},$$

where $[x]_R$ denotes the equivalence class of the relation R containing element x.

By $\mathrm{Fr}_A(X)=\bar{A}(X)-\underline{A}(X)$ we denote the *boundary* of X in A.

Thus we may define two membership functions $\underline{\in}_A$, $\bar{\in}_A$, called *strong* and *weak* membership, respectively, as follows:

$$x \underline{\in}_A X \quad \text{iff } x \in \underline{A}(X),$$
$$x \bar{\in}_A X \quad \text{iff } x \in \bar{A}(X).$$

If $x \underline{\in}_A X$ we say that "x *surely* belongs to X in A" and $x \bar{\in}_A X$ means "x *possibly* belongs to X in A".

One can easily check that the approximation space $A = (U, R)$ uniquely determines the topological space $T_A = (U, \mathrm{Com}(A))$ and $\mathrm{Com}(A)$ is the family of all open sets in T_A, and the family of all elementary sets in T_A is a base for T_A. From the definition of lower and upper approximation in A follows that $\mathrm{Com}(A)$ is both the set of all open and closed sets in T_A and that $\underline{A}(X)$ and $\bar{A}(X)$ are interior and closure of the set X in the topological space T_A. Thus $\underline{A}(X)$ and $\bar{A}(X)$ have the following properties:

(1) $\underline{A}(X) \subseteq X \subseteq \bar{A}(X)$,
(2) $\underline{A}(U) = \bar{A}(U) = U$,
(3) $\underline{A}(\emptyset) = \bar{A}(\emptyset) = \emptyset$,
(4) $\bar{A}(X \cup Y) = \bar{A}(X) \cup \bar{A}(Y)$,
(5) $\underline{A}(X \cup Y) \supseteq \underline{A}(X) \cup \underline{A}(Y)$,
(6) $\bar{A}(X \cap Y) \subseteq \bar{A}X \cap \bar{A}Y$,
(7) $\underline{A}(X \cap Y) = \underline{A}X \cap \underline{A}Y$,
(8) $\bar{A}(-X) = -\underline{A}X$,
(9) $\underline{A}(-X) = -\bar{A}X$.

Moreover we have

(10) $\underline{A}\underline{A}X = \bar{A}\underline{A}X = \underline{A}X$,
(11) $\bar{A}\bar{A}X = \underline{A}\bar{A}X = \bar{A}X$.

3. Fuzzy sets

We give now the definition of a fuzzy set introduced by Zadeh (see Zadeh [3]).

Let U be a set called *universe*. A *fuzzy set* X in U is a *membership* function $\mu_X(x)$, which to every element $x \in U$ associates a real number from the interval $\langle 0, 1 \rangle$, and $\mu_X(x)$ is the grade of membership of x in X.

The union and intersection of fuzzy sets X and Y are defined as follows:

$$\mu_{X \cup Y}(x) = \mathrm{Max}(\mu_X(x), \mu_Y(x)), \qquad \mu_{X \cap Y}(x) = \mathrm{Min}(\mu_X(x), \mu_Y(x))$$

for every $x \in U$. The complement $-X$ of a fuzzy set X is defined by the membership function

$$\mu_{-X}(x) = 1 - \mu_X(x)$$

for every $x \in X$.

4. Rough membership function

The question arises whether we may replace the concept of approximation by membership function similar to that introduced by Zadeh.

Let $X \subseteq U$. We define a membership function as follows:

$$\mu_X(x) = \begin{cases} 1 & \text{iff } x \in \underline{A}(X) \\ \frac{1}{2} & \text{iff } x \in \mathrm{Fr}_A(X), \\ 0 & \text{iff } x \in -\bar{A}(X), \end{cases}$$

where $-X$ denotes $U - X$.

We shall show that such a membership function cannot be extended to union and intersection of sets as in the previous section, i.e.

$$\mu_{X \cup Y}(x) \neq \mathrm{Max}(\mu_X(x)\mu_Y(x)) \tag{a}$$

and

$$\mu_{X \cap Y}(x) \neq \mathrm{Min}(\mu_X(x), \mu_Y(x)). \tag{b}$$

Ad (a):

$$\begin{aligned} \mu_{X \cup Y}(x) = 1 &\Leftrightarrow \mathrm{Max}(\mu_X(x), \mu_Y(x)) = 1 \\ &\Leftrightarrow \mu_X(x) = 1 \text{ or } \mu_Y(x) = 1 \\ &\Leftrightarrow x \in \underline{A}X \text{ or } x \in \underline{A}Y \Leftrightarrow x \in \underline{A}X \cup \underline{A}Y. \end{aligned} \tag{i}$$

From the definition of the membership function for union of sets we have

$$\mu_{X \cup Y}(x) = 1 \Leftrightarrow x \in \underline{A}(X \cup Y) \tag{ii}$$

From the properties of interior operation we have

$$\underline{A}(X \cup Y) \supseteq \underline{A}(X) \cup \underline{A}(Y). \tag{iii}$$

Thus if $x \in Z = \underline{A}(X \cup Y) - (\underline{A}(X) \cup \underline{A}(Y))$, then $\mu_{X \cup Y}(x) \neq 1$ by (i) and $\mu_{X \cup Y}(x) = 1$ according to (ii) (contradiction).

Ad (b):

$$\begin{aligned} \mu_{X \cap Y}(x) = 0 &\Leftrightarrow \mathrm{Min}(\mu_X(x), \mu_Y(x)) = 0 \Leftrightarrow \mu_X(x) = 0 \text{ or } \mu_Y(x) = 0 \\ &\Leftrightarrow x \in -\bar{A}(X) \text{ or } x \in -\bar{A}(Y) \Leftrightarrow x \in -\bar{A}(X) \cup -\bar{A}(Y) \\ &\Leftrightarrow x \in -(\bar{A}(X) \cap \bar{A}(Y)). \end{aligned} \tag{iv}$$

From the definition of the membership function for intersection of sets we have

$$\mu_{X \cap Y}(x) = 0 \Leftrightarrow x \in -\bar{A}(X \cap Y). \tag{v}$$

From the properties of closure operation we have

$$\bar{A}(X \cap Y) \subseteq \bar{A}(X) \cap \bar{A}(Y) \tag{vi}$$

and consequently

$$-(\bar{A}(X) \cap \bar{A}(Y)) \subseteq -\bar{A}(X \cap Y). \tag{vii}$$

Thus if $x \in W = -\bar{A}(X \cap Y) - (-(\bar{A}(X) \cap \bar{A}(Y)))$ then $\mu_{X \cap Y}(x) \neq 0$ according to (iv) and $\mu_{X \cap Y}(x) = 0$ by (v). (contradiction).

This is to mean that the membership function introduced in this section cannot be extended to union and intersection of sets.

5. Complement of sets

The membership function for the complement of sets is the same for both fuzzy sets and rough sets, as shown below:

$$\begin{aligned}
\mu_{-X}(x)=1 &\Leftrightarrow x\in \underline{A}(-X) \Leftrightarrow x\in -\bar{A}(X)\\
&\Leftrightarrow \mu_X(x)=0 \Leftrightarrow 1-\mu_X(x)=1, && \text{(a)}\\
\mu_{-X}(x)=0 &\Leftrightarrow x\in -\bar{A}(-X) \Leftrightarrow x\in \underline{A}(X)\\
&\Leftrightarrow \mu_X(x)=1 \Leftrightarrow 1-\mu_X(x)=0, && \text{(b)}\\
\mu_{-X}(x)=\tfrac{1}{2} &\Leftrightarrow x\in \bar{A}(-X)-\underline{A}(-X) \Leftrightarrow x\in \bar{A}(-X)\cap(-\underline{A}(-X))\\
&\Leftrightarrow x\in \bar{A}(-X)\cap \bar{A}(X) \Leftrightarrow x\in \bar{A}(X)\cap(-\underline{A}(X))\\
&\Leftrightarrow x\in \bar{A}(X)-\underline{A}(X) \Leftrightarrow \mu_X(x)=\tfrac{1}{2} \Leftrightarrow 1-\mu_X(x)=\tfrac{1}{2}. && \text{(c)}
\end{aligned}$$

6. Final remarks

It follows from the above considerations that the idea of rough set cannot be reduced to the idea of fuzzy set by introducing a membership function expressing the grade of membership. Moreover the concept of rough set is wider than the concept of fuzzy set; it reduces to fuzzy set if instead of

$$\underline{A}(X\cup Y)\supseteq \underline{A}(X)\cup \underline{A}(Y) \quad\text{and}\quad \bar{A}(X\cap Y)\subseteq \bar{A}(X)\cap \bar{A}(Y)$$

the following is valid:

$$\underline{A}(X\cup Y)= \underline{A}(X)\cup \underline{A}(Y) \quad\text{and}\quad \bar{A}(X\cap Y)= \bar{A}(X)\cap \bar{A}(Y),$$

which of course in the general case is not true.

References

[1] E. Orłowska and Z. Pawlak, Expressive power of knowledge representation, Internat. J. Man–Machine Stud. 20 (1984) 485–500.
[2] Z. Pawlak, Rough sets, Internat. J. Inform. Comput. Sci. 11 (5) (1982) 341–356.
[3] L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.

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BULLETIN

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1. Editorial

The Bulletin continues its series of short accounts research activities at different institutes around the world. In this issue, we present articles from Czechoslovakia and Turkey. Please do not wait to be invited before sending your institute's research brochure to the Bulletin.

We also carry a book announcement, and two book reviews provided by Henri Prade. The review of 'Aspects of Vagueness' has already appeared in Busefal. Busefal is the abbreviation for **BU**lletin pour les **S**ous **E**nsembles **F**lous et leurs **A**pp**L**ications. (There is an English version, but it sounds better in French.) Appearing approximately quarterly, Busefal is a 'communication medium' for publicising current ideas on research. It carries short articles, sometimes prior to formal publication elsewhere, as well as reviews, conference reports, etc. Further information may be obtained from Henri Prade, Laboratoire 'Langages et Systèmes Informatiques', Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France.

Two Calls for Papers also appear, for the 1986 NAFIPS meeting and the International Symposium on Multiple-valued Logic.

Many thanks to Ali Bulbul, Josef Drewniak and Vilem Novak for their contributions to the Bulletin, and to Henri Prade and Claudette Testemale for their book reviews.

2. King Sun Fu

It is with regret that we announce the death of King Sun Fu. King Sun was the first president of IFSA, and did not live to see the first IFSA Congress. He had been active in fuzzy sets for many years, but is also well known for his work on syntactic pattern recognition. He was Professor in the School of Electrical Engineering at Purdue University, West Lafayette, Indiana. King Sun has published many books and papers, and had reached high office in the IEEE. His sudden death has come as a great shock to us all.

3. Fuzzy Sets in Czechoslovakia

The development of the theory and applications of fuzzy sets in Czechoslovakia began in about 1975. First, the main work was concentrated on the theory. When