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Characterization of Rough Top Equalities and Rough Bottom Equalities

by

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Summary. In the articles [1] and [2], the rough top equality between subsets of a finite set was introduced. In [4], the rough top equality was proved to be a congruence on a certain semilattice. In this article, we characterize rough top equalities among all congruences of the mentioned semilattice. A similar result is obtained for rough bottom equalities.

1. Introduction. Rough top equalities appeared in connection with information systems. We describe briefly the relationship between these notions.

An information system (cf. [3] p. 16) is an ordered quadruple $\langle X, A, V, \varrho\rangle$ where X, A, V are finite nonempty sets and ϱ is a mapping of $X \times A$ into V. The elements in X are called objects, the elements in A attributes, the elements in V values of attributes. The mapping ϱ assigns to any element x in X and to any attribute a in A the value $\varrho(x, a) \in V$ that the attribute a assumes for the object x. For any $a \in A$, we put $\mathbf{e}(a) = \{(x, y) \in X \times X;\ \varrho(x, a) = \varrho(y, a)\}$, furthermore, we set $R = \bigcap \{\mathbf{e}(a);\ a \in A\}$. Then R is an equivalence on X. For any $Z \subseteq X$, we set $(\mathbf{u}R)(Z) = \bigcup \{Q;\ Q \in X/R, Q \cap Z \neq \emptyset\}$ and the set $(\mathbf{u}R)(Z)$ is said to be the upper approximation of Z in the information system. If $Z \subseteq X$, $T \subseteq X$, and $(\mathbf{u}R)(Z) = (\mathbf{u}R)(T)$, then the sets Z, T are said to be roughly top equal.

1. Example. Let X be the set of all pupils of a school, let $A = \{class\}$ where the values of the attribute *class* are: 1*a*, 1*b*, 1*c*, ..., 5*a*, 5*b*, 5*c*. For any pupil x in X, the function $\varrho(x, class)$ denotes the class attended by the pupil x. Then $\langle X, A, V, \varrho\rangle$ is an information system where V denotes the set of all values of the attribute *class*. Clearly, $R = \mathbf{e}(class)$ is the equivalence on X whose blocks are exactly the classes of the school.

Suppose that an infectious disease has appeared in the school and that $Z \subseteq X$ is the set of all pupils suffering from this disease at a certain moment. Generally, the sick pupils and their class mates are supposed to spread the infectious disease. For this reason, they are ordered to be in quarantine. Thus, the set of all pupils in quarantine is $(\mathbf{u}R)(Z)$.

If one knows only the set of all pupils in quarantine, one cannot deduce the set of pupils suffering from the disease. All sets roughly top equal to the set of sick pupils would produce the same set of pupils in quarantine.

In what follows, we shall investigate rough top equalities in a more abstract way.

2. Congruences on finite semilattices. In this paragraph, $(S, \vee)$ is a finite semilattice.

1. LEMMA. *If K is a congruence on $(S, \vee)$, then any block of K has a greatest element.*

Proof. If $X = \{s_1, \ldots, s_p\}$ is a block of K and if $(s_1, s_1 \vee \ldots \vee s_{i-1}) \in K$ for some $i \in \{2, \ldots, p\}$, then $(s_1, s_i) \in K$ and $s_1 \vee s_1 = s_1$ imply that $(s_1, s_1 \vee \ldots \vee s_i) \in K$. By induction, we obtain $(s_1, s_1 \vee \ldots \vee s_p) \in K$ where $s_1 \vee \ldots \vee s_p$ is the greatest element in X. □

Let K be a congruence on $(S, \vee)$. For any $x \in S$, we denote by $(\mathbf{g}K)(x)$ the greatest element $y \in S$ such that $(x, y) \in K$.

2. LEMMA. *$(\mathbf{g}K)$ is a closure operator on $(S, \vee)$, i.e., $(\mathbf{g}K)$ is extensive, monotone, and idempotent.*

Proof. By definition of $(\mathbf{g}K)$, we have $x \leqslant (\mathbf{g}K)(x)$, $(\mathbf{g}K)\big((\mathbf{g}K)(x)\big) = (\mathbf{g}K)(x)$ for any $x \in S$. If $x \leqslant y$, then $\big(x, (\mathbf{g}K)(x)\big) \in K$, $\big(y, (\mathbf{g}K)(y)\big) \in K$ imply $\big(x \vee y, (\mathbf{g}K)(x) \vee (\mathbf{g}K)(y)\big) \in K$, i.e., $\big(y, (\mathbf{g}K)(x) \vee (\mathbf{g}K)(y)\big) \in K$ whence $(\mathbf{g}K)(x) \leqslant (\mathbf{g}K)(x) \vee (\mathbf{g}K)(y) \leqslant (\mathbf{g}K)(y)$. □

3. LEMMA. *$(x, y) \in K$ if and only if $(\mathbf{g}K)(x) = (\mathbf{g}K)(y)$, for any $x, y \in S$.*

Proof. By definition of $(\mathbf{g}K)$, $(x, y) \in K$ implies $(\mathbf{g}K)(x) = (\mathbf{g}K)(y)$. On the other hand, we have $\big(x, (\mathbf{g}K)(x)\big) \in K$, $\big(y, (\mathbf{g}K)(y)\big) \in K$ and, therefore, $(\mathbf{g}K)(x) = (\mathbf{g}K)(y)$ implies $(x, y) \in K$. □

For any congruence K on $(S, \vee)$, we set $\mathbf{C}(K) = \{(\mathbf{g}K)(x); x \in S\}$.

By 2, we obtain easily

4. LEMMA. *For any $x \in S$, $(\mathbf{g}K)(x)$ is the least element $t \in \mathbf{C}(K)$ such that $x \leqslant t$.* □

5. LEMMA. *Let K_1, K_2 be congruences on $(S, \vee)$. If $\mathbf{C}(K_1) = \mathbf{C}(K_2)$, then $K_1 = K_2$.*

Proof. By 4, we have $(\mathbf{g}K_1)(x) = (\mathbf{g}K_2)(x)$ for any $x \in S$. The assertion follows by 3. □

3. Rough top equality. For any set U, we denote by $\mathbf{B}(U)$ the family of all subsets of U, by $Co\, X$ the set $U - X$, for any $X \in \mathbf{B}(U)$.

If U is a finite nonempty set, then $(\mathbf{B}(U), \cup)$ is a finite semilattice. Let R be an equivalence on U. For an arbitrary $X \in \mathbf{B}(U)$, we set

$$(\mathbf{u}R)(X) = \bigcup \{Q;\, Q \in U/R,\, Q \cap U \neq \emptyset\},\ (\mathbf{l}R)(X) = \bigcup \{Q;\, Q \in U/R,\, Q \subseteq X\}.$$

Then the set $(\mathbf{u}R)(X)$ is said to be the *upper approximation of* X, the set $(\mathbf{l}R)(X)$ the *lower approximation of* X *with respect to* R.

We shall need some properties of the operators $(\mathbf{u}R)$, $(\mathbf{l}R)$.

1. LEMMA. $(\mathbf{u}R)$ *is a closure operator.*

For the proof see (A1), (A6), (A4) of 1.3 in [2] □

2. LEMMA. $(\mathbf{l}R)((\mathbf{u}R)(X)) = (\mathbf{u}R)(X)$, $(\mathbf{u}R)((\mathbf{l}R)(X)) = (\mathbf{l}R)(X)$ *for any* $X \in \mathbf{B}(U)$.

See (A4), (A5) of 1.3 in [2]. □

We set $\mathbf{F}(R) = \{(\mathbf{u}R)(X);\, X \in \mathbf{B}(U)\}$.

3. LEMMA. $(\mathbf{F}(R), \cup, \cap, Co, \emptyset, U)$ *is a Boolean algebra.*

Proof. Clearly, $\emptyset = (\mathbf{u}R)(\emptyset) \in \mathbf{F}(R)$, $U = (\mathbf{u}R)(U) \in \mathbf{F}(R)$, $(\mathbf{u}R)(X) \cup (\mathbf{u}R)(Y) = (\mathbf{u}R)(X \cup Y) \in \mathbf{F}(R)$ for any X, $Y \in \mathbf{B}(U)$ by (A6) of 1.3 in [2], and $Co\,(\mathbf{u}R)(X) = (\mathbf{l}R)(Co\, X) = (\mathbf{u}R)((\mathbf{l}R)(Co\, X)) \in \mathbf{F}(R)$ for any $X \in \mathbf{B}(U)$ by (A8) of 1.3 in [2] and by 2. □

A subset C of $\mathbf{B}(U)$ is said to be *closed* in the Boolean algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$ if $(C, \cup, \cap, Co, \emptyset, U)$ is a subalgebra of $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$.

4. LEMMA. *If* C *is a set closed in the Boolean algebra* $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$, *then there exists an equivalence* R *on* U *such that* $\mathbf{F}(R) = C$.

Proof. $(C, \cup, \cap, Co, \emptyset, U)$ is a finite Boolean algebra. Thus, its atoms are pairwise disjoint and their union is U. Hence, they are blocks of an equivalence R on U. It is easy to see that $C = \mathbf{F}(R)$. □

is an equivalence on $\mathbf{B}(U)$; it is called the *rough top equality corresponding to* R. An equivalence K on $\mathbf{B}(U)$ is said to be a *rough top equality* if there exists an equivalence R on U such that $K = \mathbf{K}(R)$.

By (A3) of 2.2 in [2], we have

5. LEMMA. $\mathbf{K}(R)$ *is a congruence on the semilattice* $(\mathbf{B}(U), \cup)$.

Cf. also 1.1 of [4] □

Thus, the operator $(\mathbf{gK}(R))$ on $(\mathbf{B}(U), \cup)$ may be defined.

6. LEMMA. $(\mathbf{gK}(R)) = (\mathbf{u}R)$.

Proof. For any $X \in \mathbf{B}(U)$, $(\mathbf{gK}(R))(X)$ is the greatest element $Y \in \mathbf{B}(U)$ with the property $(X, Y) \in \mathbf{K}(R)$, i.e., the greatest element $Y \in \mathbf{B}(U)$ such that $(\mathbf{u}R)(X) = (\mathbf{u}R)(Y)$. Put $Y_0 = (\mathbf{u}R)(X)$. Then $(\mathbf{u}R)(Y_0) = (\mathbf{u}R)(X)$ and for any $Y \in \mathbf{B}(U)$ with $(\mathbf{u}R)(X) = (\mathbf{u}R)(Y)$, we obtain $Y \subseteq (\mathbf{u}R)(Y) = (\mathbf{u}R)(X) = Y_0$. Thus, $Y_0 = (\mathbf{u}R)(X)$ is the greatest element $Y \in \mathbf{B}(U)$ with $(\mathbf{u}R)(Y) = (\mathbf{u}R)(X)$. It follows that $(\mathbf{gK}(R))(X) = (\mathbf{u}R)(X)$ for any $X \in \mathbf{B}(U)$. □

By 6, we obtain

7. LEMMA. $\mathbf{C}(\mathbf{K}(R)) = \mathbf{F}(R)$. □

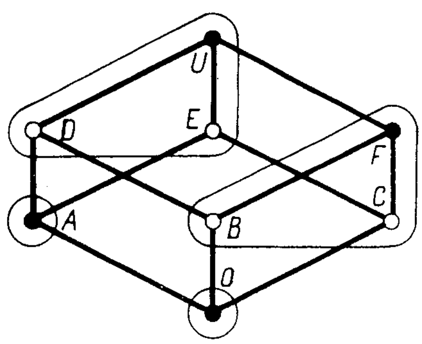
8. CHARACTERIZATION THEOREM FOR ROUGH TOP EQUALITIES. *Let K be a congruence on a semilattice $(\mathbf{B}(U), \cup)$ where U is a finite nonempty set. Then K is a rough top equality if and only if the set $\mathbf{C}(K)$ is closed in the Boolean algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$.*

Proof. If K is a rough top equality, there exists an equivalence R on U such that $K = \mathbf{K}(R)$. By 7, we obtain $\mathbf{C}(K) = \mathbf{F}(R)$ and this set is closed in $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$ by 3.

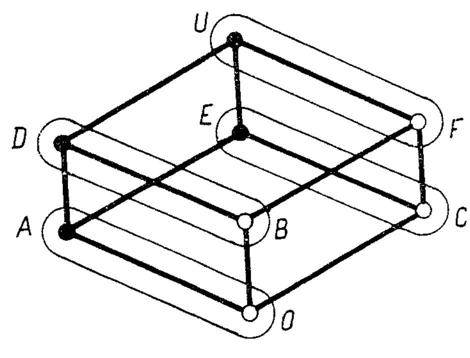
If $\mathbf{C}(K)$ is closed in $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$, there exists an equivalence R on U such that $\mathbf{F}(R) = \mathbf{C}(K)$ by 4. By 7, we obtain $\mathbf{C}(K) = \mathbf{C}(\mathbf{K}(R))$ which implies $K = \mathbf{K}(R)$ by 2.5. □

4. Examples. In both examples, we suppose $U = \{a, b, c\}$, $O = \emptyset$, $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{a, b\}$, $E = \{a, c\}$, $F = \{b, c\}$.

1. EXAMPLE. Let K be a congruence whose blocks are $\{O\}$, $\{A\}$, $\{B, C, F\}$, $\{D, E, U\}$. Then $\mathbf{C}(K) = \{O, A, F, U\}$. Since the set $\mathbf{C}(K)$ is closed in the algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$, K is a rough top equality.



2. EXAMPLE. Let K be a congruence whose blocks are $\{O, A\}$, (B, D), $\{C, E\}$, $\{F, U\}$. Then $\mathbf{C}(K) = \{A, D, E, U\}$ which is not closed in the algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$. Thus, K is not a rough top equality.



5. Algorithm for recognizing rough top equalities.

Data: Let a finite nonempty set U be given. Suppose that the Boolean algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$ is given by tables for binary operations $\cup$, $\cap$, and for the unary operation Co. Let a congruence K on $(\mathbf{B}(U), \cup)$ be given in such a way that the elements of any block of K are enumerated.

Preprocessing of data: For any $X \in \mathbf{B}(U)/K$, $X = \{M_1^X, \ldots, M_{k(X)}^X\}$, we construct $\bar{X} = (\ldots(M_1^X \cup M_2^X) \cup \ldots \cup M_{k(X)-1}^X) \cup M_{k(X)}^X$ by means of the table of the operation $\cup$. We put $\mathbf{C}(K) = \{\bar{X}; X \in \mathbf{B}(U)/K\}$.

Algorithm:

(1) If $\emptyset \notin \mathbf{C}(K)$, reject K; otherwise go to (2).

(2) If $Co\, X \notin \mathbf{C}(K)$ for some $X \in \mathbf{C}(K)$, reject K; otherwise go to (3).

(3) If either $X \cup Y \notin \mathbf{C}(K)$ or $X \cap Y \notin \mathbf{C}(K)$ for some X, $Y \in \mathbf{C}(K)$, reject K; otherwise K is a rough top equality.

6. Applications. Let $S = \langle I, O, f\rangle$ be an ordered triple where I, O are finite nonempty sets and f is a mapping of $\mathbf{B}(I)$ into O. Then S is said to be a *black box*. Black boxes can be realized in various ways. A black box is said to be *admissible* if the following condition is satisfied.

(i) For any X, $Y \in \mathbf{B}(I)$ and any $x \in I$ the condition $f(X) = f(Y)$ implies $f(X \cup \{x\}) = f(Y \cup \{x\})$.

In [5], we have proved, that a black box $\langle I, O, f\rangle$ is admissible if and only if $f^{-1} \circ f$ is a congruence on the semilattice $(\mathbf{B}(I), \cup)$.

An admissible black box $\langle I, O, f\rangle$ is said to be *good* if the following condition is satisfied.

(ii) For any $X \in \mathbf{B}(I)$ and any $x \in I$ with $f(X \cup \{x\}) = f(X)$, there exists $y \in X$ such that $f(\{y\}) = f(\{x\})$.

Good boxes have some simple properties: e.g., they can be replaced by their so called kernels that are simpler. A kernel of a good box acts in the same way as the box. Moreover, we have found a construction that provides all good boxes. For this reason, it seems to be important to recognize good boxes among admissible boxes. We have proved that an admissible box $\langle I, O, f\rangle$ is good if and only if $f^{-1} \circ f$ is a rough top equality.

For the details on boxes see [5].

7. Rough bottom equality. Let U be a finite nonempty set and R an equivalence on U. We now investigate lower approximations with respect to R. They can be reduced to upper approximations in the following sense.

1. LEMMA. *If U is a finite nonempty set, R an equivalence on U, and $X \subseteq U$, then* $(\mathbf{l}R)(X) = Co\,(\mathbf{u}R)(Co\,X)$.

For the proof see (A9) of 1.3 in [2]. □

We set $\mathbf{L}(R) = \{(X, Y) \in \mathbf{B}(U) \times \mathbf{B}(U); (\mathbf{l}R)(X) = (\mathbf{l}R)(Y)\}$. Then $\mathbf{L}(R)$ is said to be the *rough bottom equality corresponding to* R. An equivalence L on $\mathbf{B}(U)$ is said to be a *rough bottom equality* if there exists an equivalence R on U such that $L = \mathbf{L}(R)$.

By (A4) of 2.2 in [2], we have

2. LEMMA. $\mathbf{L}(R)$ *is a congruence on the semilattice* $(\mathbf{B}(U), \cap)$. □

We now give a complete characterization of rough bottom equalities among all congruences on $(\mathbf{B}(U), \cap)$. For the formulation and the proof of the result, we need some definitions.

Let $(S, \wedge)$ be a finite semilattice and L a congruence on $(S, \wedge)$. For any $x \in S$, we denote by $(\mathbf{h}L)(x)$ the least element y in S such that $(x, y) \in L$. Furthermore, we set $\mathbf{D}(L) = \{(\mathbf{h}L)(x); x \in S\}$.

We use some results proved for rough top equalities. Transition from rough bottom equalities to rough top equalities and vice versa is enabled by the following definition.

Let U be a finite nonempty set and ϱ a binary relation on $\mathbf{B}(U)$. We put

$$\tilde{\varrho} = \{(X, Y) \in \mathbf{B}(U) \times \mathbf{B}(U); (Co\,X, Co\,Y) \in \varrho\}.$$

If A, B, C are sets, C a subset of A, and f a mapping of A into B, we put $f[C] = \{f(x); x \in C\}$.

3. LEMMA. *If U is a finite nonempty set and L a congruence on $(\mathbf{B}(U), \cap)$, then $\tilde{L}$ is a congruence on $(\mathbf{B}(U), \cup)$ and* $Co\,[\mathbf{D}(L)] = \mathbf{C}(\tilde{L})$.

Proof. If $(X, Y) \in \tilde{L}$ and $Z \in \mathbf{B}(U)$, then $(Co\,X, Co\,Y) \in L$ and, hence, $(Co\,(X \cup Z), Co\,(Y \cup Z)) = (Co\,X \cap Co\,Z, Co\,Y \cap Co\,Z) \in L$ which implies that $(X \cup Z, Y \cup Z) \in \tilde{L}$. Thus, $\tilde{L}$ is a congruence on $(\mathbf{B}(U), \cup)$.

If $X \in \mathbf{B}(U)$, then any two consecutive conditions in the following sequence are equivalent.

$X \in \mathbf{D}(L)$;

$(Y, X) \in L$ implies $Y \supseteq X$;

$(Co\,Y, Co\,X) \in \tilde{L}$ implies $Co\,Y \subseteq Co\,X$;

$Co\,X \in \mathbf{C}(\tilde{L})$.

This implies that $Co\,[\mathbf{D}(L)] \subseteq \mathbf{C}(\tilde{L})$.

On the other hand, any two consecutive conditions in the following sequence are equivalent.

$X \in \mathbf{C}(\tilde{L})$;

$(Y, X) \in \tilde{L}$ implies $Y \subseteq X$;

$(Co\, Y, Co\, X) \in L$ implies $Co\, X \subseteq Co\, Y$;

$Co\, X \in \mathbf{D}(L)$.

This implies that $Co[\mathbf{C}(\tilde{L})] \subseteq \mathbf{D}(L)$ and, thus, $\mathbf{C}(\tilde{L}) \subseteq Co[\mathbf{D}(L)]$.

We have proved that $\mathbf{C}(\tilde{L}) = Co[\mathbf{D}(L)]$. □

4. CHARACTERIZATION THEOREM FOR ROUGH BOTTOM EQUALITIES. *If U is a finite nonempty set and L a congruence on the semilattice $(\mathbf{B}(U), \cap)$, then the following two assertions are equivalent.*

(i) *L is a rough bottom equality.*

(ii) *The set $\mathbf{D}(L)$ is closed in the Boolean algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$.*

Proof. By 1, (i) holds if and only if $\tilde{L}$ is a rough top equality which is equivalent with the following condition

(iii) $\mathbf{C}(\tilde{L})$ is closed in $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$

by 3.8 By 3, (iii) is equivalent with

(iv) $Co[\mathbf{D}(L)]$ is closed in $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$.

If (iv) holds, then $Co[\mathbf{D}(L)] = \mathbf{D}(L)$ and (ii) holds. If (ii) holds, then $Co[\mathbf{D}(L) = \mathbf{D}(L)$ and (iv) holds, too. Thus, (ii) and (iv) are equivalent which implies that (i) and (ii) are equivalent. □

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М. Новотны, З. Павляк, **Характеризация верхних и нижнних приближенных равенств**

В настоящей работе приводится характеризация приближенных равенств, в классе конгруэнций на верхней полурешетке всех подмножеств конечного множества.