Application of Two-valued Lattices to the Realization of Many-valued Truth-tables

by

Z. PAWLAK

Presented by P. SZULKIN on June 29, 1961

The present paper contains a simple method of synthesis of switching circuits according to a given many-valued truth-table. It is shown that, in order to design circuits realizing any n-valued truth-table, only two basic logical elements, namely, "and" and "or" gates are necessary.
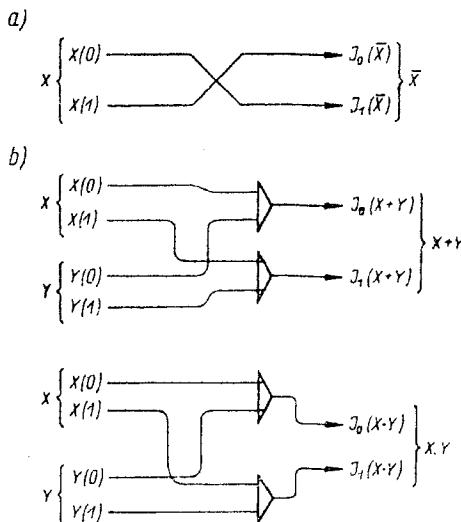

Fig. 1. Two-valued truth-table

Let A_n be a truth-table A with values $0, 1, ..., n-1$ and let $G(A_n)$ denote the corresponding truth-function. We shall introduce the two-valued function $J_k(G(A_n))$ $(0 \leqslant k < n)$ which will be defined as follows (see [1]):

   i. $J_k(G(A_n)) = T$ if $G(A_n) = k$,

   ii. $J_k(G(A_n)) = F$ if $G(A_n) \neq k$.

The function $J_k(G(A_n))$ may be read as "$G(A_n)$ has the truth-value k". Similarly, we define the function $J_k(X)$ which may be read "Variable X has the truth-value k". Instead of $J_k(X)$ we shall often write $X(k)$.

To each function $G(A_n)$ containing p independent n valued variables $X_1, X_2, ..., X_p$ we may associate the sequence $[J_0(G(A_n)), J_1(G(A_n)), ..., J_{n-1}(G(A_n))]$, where for all k $(0 \leqslant k \leqslant n-1)$

$$J_k(G(A_n)) = \bigcup_{k_1, k_2, ..., k_p} \bigcap_{i=1}^{p} \overline{X}_i(k_i)$$

and the sum $\bigcup$ is taken for all such values $k_1, k_2, ..., k_p$ of variables $X_1, X_2, ..., X_p$ for which $G(A_n) = k$. It is easy to show that functions $J_k(G(A_n))$ form a distributive lattice (see [2]). that is the following axioms hold (lower case italic letters denote two-valued variables):

1. $x \cap x = x$,
2. $x \cup x = x$,
3. $x \cap y = y \cap x$,
4. $x \cup y = y \cup x$,
5. $x \cap (y \cap z) = (x \cap y) \cap z$,
6. $x \cup (y \cup z) = (x \cup y) \cup z$,
7. $x \cap (x \cup y) = x$,
8. $x \cup (x \cap y) = x$,
9. $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$,
10. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$,
11. $(x \cap y) \cup (y \cap z) \cup (z \cap x) = (x \cup y) \cap (y \cup z) \cap (z \cup x)$,
12. $X(0) \cap X(1) \cap ... \cap X(n-1) = F$,
13. $X(0) \cup X(1) \cup ... \cup X(n-1) = T$.

Examples. Consider the following three, two-valued truth-tables

X	Y
0	1
1	0

X	Y	Z
0	0	0
0	1	1
0	0	1
1	1	1

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

and let the corresponding truth-functions be denoted by $\overline{X}$, $X+Y$ and $X \cdot Y$, respectively. Then we have:

$J_0(\overline{X}) = X(1)$,

$J_1(\overline{X}) = X(0)$,

or

$\overline{X} = [X(1), X(0)]$.

$J_0(X+Y) = X(0) \cap Y(0)$,

$J_1(X+Y) = (X(0) \cap Y(1)) \cup (X(1) \cap Y(0)) \cup (X(1) \cap Y(1)) = X(1) \cup Y(1)$,

or

$$X + Y = [X(0) \cap Y(0), X(1) \cup Y(1)],$$

$$J_0(X \cdot Y) = (X(0) \cap Y(0)) \cup (X(1) \cap Y(0)) \cup (X(0) \cap Y(1)) = X(0) \cup Y(0),$$

$$J_1(X \cdot Y) = X(1) \cap Y(1),$$

or

$$X \cdot Y = [X(0) \cup Y(0), X(1) \cap Y(1)].$$
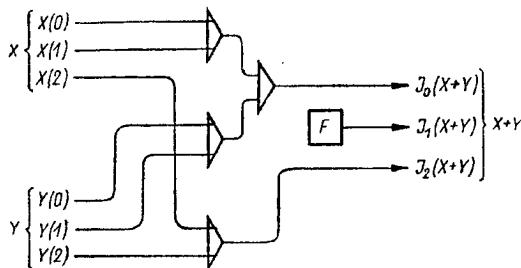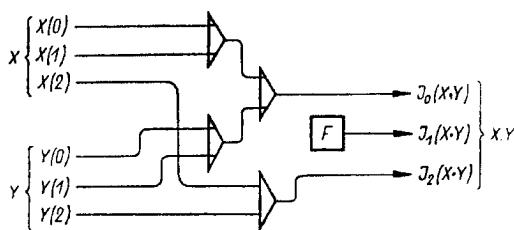
Fig. 2. Three-valued truth-table

Consider now two examples of three-valued functions $X + Y$ and $X \cdot Y$ defined by the tables:

$Y \backslash X$	0 1 2
0	0 0 2
1	0 0 2
2	2 2 2

$Y \backslash X$	0 1 2
0	0 0 0
1	0 0 0
2	0 0 2

$$J_0(X + Y) = (X(0) \cap Y(0)) \cup (X(1) \cap Y(0)) \cup (X(0) \cap Y(1)) \cup (X(1) \cap Y(1)) =$$
$$= (X(0) \cup X(1)) \cap (Y(0) \cup Y(1)),$$

$$J_1(X + Y) = F,$$

$$J_2(X + Y) = (X(0) \cap Y(2)) \cup (X(1) \cap Y(2)) \cup (X(2) \cap Y(2)) \cup (X(2) \cap$$
$$\cap Y(1)) \cup (X(2) \cap Y(0)) = X(2) \cup Y(2),$$

That is

$$X + Y = [(X(0) \cup X(1)) \cap (Y(1) \cup Y(0)), F, (X(2) \cup Y(2))].$$

Similarly for $X \cdot Y$ we have:

$$J_0(X \cdot Y) = (X(0) \cap Y(1)) \cup (X(0) \cap Y(1)) \cup (X(0) \cap Y(2)) \cup (X(1) \cap$$
$$\cap Y(0)) \cup (X(1) \cap Y(1)) \cup (X(1) \cap Y(2)) \cup (X(2) \cap Y(0)) \cup$$
$$\cup (X(2) \cap Y(1)) = (X(0) \cup X(1)) \cup (Y(0) \cup Y(1)),$$

$$J_1(X \cdot Y) = F,$$

$$J_2(X \cdot Y) = X(2) \cap Y(2),$$

or

$$X \cdot Y = [(X(0) \cup X(1)) \cup (Y(0) \cup Y(1)), F, X(2) \cap Y(2)].$$

Of course, the realization of a table which does not contain all the values, that is of an incomplete table, is also possible. Schemes of corresponding example are given in the figures, where the "and" gate and the "or" gate are represented in the usual manner. Values F and T are to be represented as the presence and the absence of a pulse, respectively, or else as a wide and narrow pulse.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] J. B. Rosser, A. R. Turqutte, *Axiom schemes for M-valued propositional calculi*, The Journ. of Symb. Logic, **10** (1945), 61.

[2] G. Birkhoff, *Lattice Theory*, 1948.