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Black Box Analysis and Rough Top Equality

by

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Summary. A black box is considered to be an ordered triple consisting of a finite set I of inputs, a finite set O of outputs, and of a mapping f of $\mathbf{B}(I)$ into O where $\mathbf{B}(I)$ is the family of all subsets of I. Under some natural hypotheses on f the relation $f^{-1} \circ f$ is a rough top equality on $\mathbf{B}(I)$, i.e., a relation investigated in connection with information systems. For this case the possibility of reducing the black box is investigated.

1. Introduction. When studying rough top equality (cf. [4], [5]), we looked for applications of the characterization theorem. One of these applications is the so called good black box that will be studied in the present paper. We prove that good black boxes lead to rough top equilities in a natural way. Thus, recognizing rough top equalities is the same as recognizing good black boxes. To any good box there exists an irreducible subbox and a homomorphism of the box onto the subbox. This means that the subbox acts, roughly speaking, as the given box and that it cannot be reduced without offending against the way of its activity.

2. Admissible black box. Let I, O be nonempty finite sets; the elements in I are called *inputs*, the elements in O *outputs*. Let f be a mapping of $\mathbf{B}(I)$ into O where $\mathbf{B}(I)$ denotes the family of all subsets of the set I. Then the ordered triple $S = \langle I, O, f \rangle$ is called a *black box* or, shortly, a *box*.

This is interpreted as follows. For any set $X \in \mathbf{B}(I)$ of inputs activated, the box S provides the output $f(X) \in O$. We can imagine that any input is represented by a push-button and can be activated by pushing it. The output depends only on the set of inputs activated, not on a string of inputs.

1. Example. Let a room be given, let I be the set of its windows. We suppose that the temperature in the room is 20°C when all windows are shut and the outside temperature is 0°C. For any set $X \subseteq I$, we define $f(X)$ to be the temperature in the room after all windows of the set X have been open during one minute. Put $O = \{f(X); X \in \mathbf{B}(I)\}$. Then $\langle I, O, f\rangle$ is a box.

2. Example. Let I be a finite set and $(I, \mathbf{B}(I), p)$ a probability space. We set $O = \{p(X); X \in \mathbf{B}(I)\}$. Then $\langle I, O, p\rangle$ is a box.

3. Example. Let $(\mathbf{B}(I), \cup, \cap, Co, \emptyset, I)$ $(\mathbf{B}(O), \cup, \cap, Co, \emptyset, O)$ be Boolean algebras of sets where I, O are finite nonempty sets, $\cup, \cap$ denote the operations of union and intersection, respectively, and Co is the operation of complementation. Suppose that h is a homomorphism of the first algebra into the second. Then $\langle I, \mathbf{B}(O), h\rangle$ is a box.

We shall deal with boxes of particular type.

Let $S = \langle I, O, f\rangle$ be a box. It is said to be *admissible* if the following condition is satisfied.

(i) For any X, $Y \in \mathbf{B}(I)$ and any $x \in I$ the condition $f(X) = f(Y)$ implies $f(X \cup \{x\}) = f(Y \cup \{x\})$.

By (i), an output of an admissible box changes by activating one further input in a unique way without regard to the set of inputs that has produced the given output.

Admissible boxes have some particular properties.

4. Lemma. *Let $S = \langle I, O, f\rangle$ be an admissible box. Then the following assertions hold.*

(a) *If X, $YZ \in \mathbf{B}(I)$ and $f(X) = f(Y)$, then $f(X \cup Z) = f(Y \cup Z)$.*

(b) *If $x, y \in I$, $Z \in \mathbf{B}(I)$, and $f(\{x\}) = f(\{y\})$, then $f(Z \cup \{x\}) = f(Z \cup \{y\})$.*

(c) *If X, $Y \in \mathbf{B}(I)$, x, $y \in I$, and $f(X) = f(Y)$, $f(\{x\}) = f(\{y\})$, then $f(X \cup \cup \{x\}) = f(Y \cup \{y\})$.*

(d) *If X, Y, Z, $T \in \mathbf{B}(I)$ and $f(X) = f(Y)$, $f(Z) = f(T)$, then $f(X \cup Z) = f(Y \cup T)$.*

(e) *If $n \geqslant 1$, $x_1, \ldots, x_n \in I$, $y_1, \ldots, y_n \in I$, and $f(\{x_i\}) = f(\{y_i\})$ for any $i \in \{1, \ldots, n\}$, then $f(\{x_1, \ldots, x_n\}) = f(\{y_1, \ldots, y_n\})$.*

Proof. (a) follows from (i) by easy induction, (b) is a particular case of (a). Furthermore, (c) is a consequence of (i) and (b), (d) is obtained from (c) by induction. For $n = 2$, (e) is a particular case of (d); the general case can be obtained by induction. □

5. Characterization Theorem for Admissible Boxes. *A box $S = \langle I, O, f\rangle$ is admissible if and only if $f^{-1} \circ f$ is a congruence on the semillattice $(\mathbf{B}(I), \cup)$.*

Proof. If S is admissible and X, Y, $Z \in \mathbf{B}(I)$, $(X, Y) \in f^{-1} \circ f$, then $f(X) = f(Y)$ which implies $f(X \cup Z) = f(Y \cup Z)$ by 4(a) and, thus, $(X \cup Z, Y \cup Z) \in f^{-1} \circ f$. Hence, $f^{-1} \circ f$ is a congruence on $(\mathbf{B}(I), \cup)$.

If $f^{-1} \circ f$ is a congruence on $(\mathbf{B}(I), \cup)$ and X, $Y \in \mathbf{B}(I)$, $x \in I$, are such that $f(X) = f(Y)$, then $(X, Y) \in f^{-1} \circ f$ which implies that $(X \cup \{x\}, Y \cup \{x\}) \in f^{-1} \circ f$ and, therefore, $f(X \cup \{x\}) = f(Y \cup \{x\})$. Hence, S is admissible. □

Let I, J be sets, h a mapping of I into J. For any $X \in \mathbf{B}(I)$, we put $f[X] = \{f(x); x \in X\}$. Clearly, $f[X \cup Y] = f[X] \cup f[Y]$ for any X, $Y \in \mathbf{B}(I)$.

Let $S = \langle I, O, f \rangle$, $T = \langle J, O, g \rangle$ be boxes, h a mapping of I into J. Then h is said to be a *homomorphism* of S into T if $f(X) = g(h[X])$ for any $X \in \mathbf{B}(I)$. Clearly, the activity of S can be simulated in T: an output in S produced by the set X can be obtained in T by means of the set $h[X]$.

6. LEMMA. *The composite of two homomorphism is a homomorphism.*

Proof. If h is a homomorphism of $\langle I, O, f \rangle$ into $\langle J, O, g \rangle$ and k a homomorphism of $\langle J, O, g \rangle$ into $\langle K, O, l \rangle$, then $f(X) = g(h[X]) = l(k[h[X]]) = l((k \circ h)[X])$ for any $X \in \mathbf{B}(I)$. □

Surjective homomorphisms preserve admissibility:

7. THEOREM. *Let $S = \langle I, O, f \rangle$, $T = \langle J, O, g \rangle$ be boxes, h a surjective homomorphism of S onto T. Then S is admissible if and only if so is T.*

Proof. (1) If S is admissible and U, $V \in \mathbf{B}(J)$, $u \in J$, are such that $g(U) = g(V)$, then there exist X, $Y \in \mathbf{B}(I)$, $x \in I$ such that $h[X] = U$, $h[Y] = V$, $h(x) = u$ regarding the surjectivity of h. Hence $f(X) = g(h[X]) = g(U) = g(V) = g(h[Y]) = f(Y)$ which implies $g(U \cup \{x\}) = g(h[X] \cup \{h(x)\}) = g(h[X] \cup h[\{x\}]) = g(h[X \cup \{x\}]) = f(X \cup \{x\}) = f(Y \cup \{x\}) = g(h[Y \cup \{x\}]) = g(h[Y] \cup h[\{x\}]) = g(h[Y] \cup \{h(x)\}) = g(V \cup \{x\})$ regarding the admissibility of S. Thus, T is admissible.

(2) If T is admissible and X, $Y \in \mathbf{B}(I)$, $x \in I$ are such that $f(X) = f(Y)$, then $g(h[X]) = g(h[Y])$ which implies that $f(X \cup \{x\}) = g(h[X \cup \{x\}]) = g(h[X] \cup h[\{x\}]) = g(h[X] \cup \{h(x)\}) = g(h[Y] \cup \{h(x)\}) = g(h[Y] \cup h[\{x\}]) = g(h[Y \cup \{x\}]) = f(Y \cup \{x\})$ and S is admissible. □

We now assign a "simpler" box to any admissible box, its skeleton. Before introducing it, we need the following.

8. LEMMA. *Let $S = \langle I, O, f \rangle$ be an admissible box. We put $\varphi(x) = f(\{x\})$ for any $x \in I$. Then for any X, $Y \in \mathbf{B}(I)$, the condition $\varphi[X] = \varphi[Y]$ implies that $f(X) = f(Y)$.*

Proof. For any $u \in \varphi[X]$, we put $X_u = \{x \in X; \varphi(x) = u\}$, $Y_u = \{y \in Y; \varphi(y) = u\}$, $k(u) = \max\{\text{card } X_u, \text{card } Y_u\}$, let $(x_1^u, \ldots, x_{k(u)}^u)$, $(y_1^u, \ldots, y_{k(u)}^u)$ be sequences of all elements in X_u, Y_u, respectively. Clearly, $f(\{x_i^u\}) = f(\{y_i^u\})$ for any $u \in \varphi[X]$ and any $i \in \{1, \ldots, k(u)\}$. By 2.4(e), we obtain $f(X) = f(\{x_i^u;$

$u \in \varphi[X], 1 \leqslant i \leqslant k(u)\}) = f(\{y_i^u; u \in \varphi[Y], 1 \leqslant i \leqslant k(u)\}) = f(Y)$. □

Let $S = \langle I, O, f \rangle$ be an admissible box. We put $\varphi(x) = f(\{x\})$ for any $x \in I$ and $F(U) = f(X)$ for any $U \in \mathbf{B}(\varphi[I])$ where $X \in \mathbf{B}(I)$ is an arbitrary set such that $U = \varphi[X]$. Then $T = \langle \varphi[I], O, F \rangle$ is a box that is called the *skeleton* of S and φ is said to be the *normal mapping* of S onto T.

By 8, $F(U)$ is uniquely determined by U and the definition of F is correct. Since $f(X) = F(\varphi[X])$ for any $X \in \mathbf{B}(I)$, we obtain.

9. Lemma. *The normal mapping of an admissible box onto its skeleton is a surjective homomorphism.* □

10. Corollary. *The skeleton of an admissible box is admissible.*

This is a consequence of 9 and 7. □

11. Example. If $S = \langle I, O, f \rangle$ is a box such that the mapping f is injective, then S is admissible. Its skeleton $\langle \varphi[I], O, F \rangle$ is defined as follows: $\varphi(x) = f(\{x\})$ for any $x \in I$, $F(U) = f(\varphi^{-1}[U])$ where $\varphi^{-1}[U] = \{x \in I; \varphi(x) \in U\}$ for any $U \in \mathbf{B}(\varphi[I])$.

12. Examples. (a) Consider the box $S = \langle I, O, f \rangle$ of 1. If there are two windows of the same size, say x_1, x_2, then $f(\{x_1\}) = f(\{x_2\})$, i.e., opening either the first or the second window (but not both!) has the same consequence. It is logical to suppose that opening both windows causes a greater decrease of temperature than opening only one of them. Thus, $f(\{x_1\}) \neq f(\{x_1, x_2\}) \neq f(\{x_2\})$ and condition (i) is not satisfied; S is not admissible.

(b) If $S = \langle I, O, p \rangle$ is the box of 2 and if there exist $x_1, x_2 \in I$ such that $x_1 \neq x_2$ and $p(\{x_1\}) = p(\{x_2\}) > 0$, then $p(\{x_1, x_2\}) = 2p(\{x_1\})$ and S is not admissible.

(c) If $\langle I, B(O), h \rangle$ is the box of 3, then h is a homomorphism and, consequently, $h^{-1} \circ h$ is a congruence on the algebra $(\mathbf{B}(I), \cup, \cap, Co, \emptyset, I)$ and, a fortiori, a congruence on the semilattice $(\mathbf{B}(I), \cup)$. By 5, the box is admissible.

3. Good boxes. An admissible box $S = \langle I, O, f \rangle$ is said to be *good* if the following condition is satisfied.

(ii) For any $X \in \mathbf{B}(I)$ and any $x \in I$ with $f(X \cup \{x\}) = f(X)$, there exists $y \in X$ such that $f(\{y\}) = f(\{x\})$.

1. Example. If $S = \langle I, O, f \rangle$ is a box such that f is an injective mapping, then S is good.

This leads us to the following definition. A box $S = \langle I, O, f \rangle$ is said to be *very good* if f is an injective mapping.

By 1, a very good box is good.

2. Characterization Theorem for Good Boxes. *Let $S = \langle I, O, f \rangle$ be an admissible box. Then the following conditions are equivalent.*

(a) *S is good.*
(b) *The skeleton of S is very good.*

Proof. (1) If S is good, if $T = \langle \varphi[I], O, F \rangle$ is its skeleton, and if $U, V \in \mathbf{B}(\varphi[I])$ are such that $F(U) = F(V)$, there exist $X, Y \in \mathbf{B}(I)$ such that $U = \varphi[X]$, $V = \varphi[Y]$. Let $u \in U$ be arbitrary. There exists $x \in X$ such that $u = \varphi(x)$. We have $f(X) = F(\varphi[X]) = F(U) = F(V) = F(\varphi[Y]) = f(Y)$ and, thus, $f(Y \cup \{x\}) = F(\varphi[Y \cup \{x\}]) = F(\varphi[Y] \cup \varphi[\{x\}]) = F(V \cup \{u\}) = F(U \cup \cup \{u\}) = F(U) = f(X) = f(Y)$ by 2.10. By (ii), there exists $y \in Y$ such that $f(\{y\}) = f(\{x\})$. Hence, $u = \varphi(y) \in \varphi[Y] = V$. We have proved that $U \subseteq V$; similarly, $V \subseteq U$, and, thus, $U = V$. Hence, F is injective and (a) implies (b).

(2) If the skeleton $T = \langle \varphi[I], O, F \rangle$ of S is very good and $X \in \mathbf{B}(I)$, $x \in I$ are such that $f(X \cup \{x\}) = f(X)$, then $F(\varphi[X \cup \{x\}]) = f(X \cup \{x\}) = f(X) = F(\varphi[X])$ and the injectivity of F implies that $\varphi[X \cup \{x\}] = \varphi[X]$. Thus, there exists $y \in X$ such that $f(\{x\}) = \varphi(x) = \varphi(y) = f(\{y\})$. Hence, S is good and (b) implies (a). □

We now describe a construction of all good boxes.

3. Construction Theorem for Good Boxes.

(a) *Let I, O be finite nonempty sets, φ a mapping of I into O, F an injection of $\mathbf{B}(\varphi[I])$ into O such that $F(\{y\}) = y$ for any $y \in \varphi[I]$. If putting $f(X) = F(\varphi[X])$ for any $X \in \mathbf{B}(I)$ and $S = \langle I, O, f \rangle$, S is a good box.*

(b) *Any good box can be obtained by a construction described in* (a).

Proof. (1) If the hypotheses of (a) are satisfied, then S is a box and $T = \langle \varphi[I], O, F \rangle$ is a very good box. For any $x \in I$ and any $X \in \mathbf{B}(I)$, we have $f(\{x\}) = F(\varphi[\{x\}]) = F(\{\varphi(x)\}) = \varphi(x)$ and $f(X) = F(\varphi[X])$. By definition, T is the skeleton of S and S is good by 2.

(2) If $S = \langle I, O, f \rangle$ is a good box and $T = \langle \varphi[I], O, F \rangle$ its skeleton, then $f(X) = F(\varphi[X])$ for any $X \in \mathbf{B}(I)$ and $F(\{\varphi(x)\}) = F(\varphi[\{x\}]) = f(\{x\}) = \varphi(x)$ for any $x \in I$ and, thus, $F(\{y\}) = y$ for any $y \in \varphi[I]$. Thus, there exist some objects I, O, φ, F, f with the properties described in (a). □

4. Example. Let us have $I = \{a, b, c\}$, $O = \{m, n, p, q\}$, suppose that φ, F are given by the following tables.

φ	a	b	c
	m	n	n

F	$\emptyset$	$\{m\}$	$\{n\}$	$\{m, n\}$
	p	m	n	q

Then f is given by the following table.

f	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	I
	p	m	n	n	q	q	n	q

Thus, $\langle I, O, f \rangle$ is a good box.

4. Rough top equality. As it was said in the Introduction, good boxes are motivated by looking for examples of rough top equalities. The role of rough top equalities follows from the investigation of information systems. We sketch briefly the relationship between these notions.

An information system (cf. [3]) is an ordered quadruple $\langle X, A, V, \varrho\rangle$ where X, A, V are finite nonempty sets and ϱ is a mapping of $X \times A$ into V. The elements of X are interpreted as objects, the elements of A as attributes, the elements of V as values of attributes. The function ϱ assigns to any object x in X and to any attribute a in A the value $\varrho(x, a)$ in V that the attribute a assumes for the object x. For any a in A, we put $\mathbf{e}(a) = \{(x, y) \in X \times X; \varrho(x, a) = \varrho(y, a)\}$. Clearly, $\mathbf{e}(a)$ is an equivalence on X. Furthermore, we put $R = \bigcap \{\mathbf{e}(a); a \in A\}$. Then R is an equivalence on X; its blocks are called elementary sets. A union of a family of elementary sets is said to be a definable set. The information system can construct only definable sets; other sets can be only approximated by means of definable sets. For any $Z \subseteq X$, we set $(\mathbf{u}R)(Z) = \bigcup \{Q; Q \in X/R, Q \cap Z \neq \emptyset\}$. Clearly, $(\mathbf{u}R)(Z)$ is the least definable set including Z; it is said to be the upper approximation of Z in the given information system.

1. Example. Let X be a set of patients in a hospital, A a set of medical criteria, as body temperature, blood-pressure, blood coagulation, etc., whose values are stated for any patient at a given moment. We suppose that any of these criteria has only a finite number of values. E.g., the body temperature is given by one of the following numbers: 35,0; 35,1; ...; 41,9; 42,0; similarly for the other criteria. Let V be the set of values of all criteria. For any $x \in X$, $a \in A$ let $\varrho(x, a)$ be the value of the criterion a for the patient x. Then $S = \langle X, A, V, \varrho\rangle$ is an information system.

An elementary set consists of all patients that present the same value for any criterion.

Let $Z \subseteq X$ be a set of patients that are suffering from a certain disease s. This set has been found by means of experts and we suppose that it is representative enough to say that any patient whose values of criteria do not agree with the values of at least one patient of the set Z does not suffer from the disease s. It follows that only patients of the set $(\mathbf{u}R)(Z)$ are suspected to have the disease s.

Let s_1 be another disease with a representative set Z_1 of sick patients found by means of experts. Then only patients in the set $(\mathbf{u}R)(Z_1)$ can be suspected to suffer from s_1. It can happen that $(\mathbf{u}R)(Z) = (\mathbf{u}R)(Z_1)$. Then the information system S is not able to distinguish between the diseases s and s_1 if any disease is measured by means of the set of patients suspected to suffer from the disease. From this point of view, the sets Z, Z_1 play the same role; they are said to be roughly top equal.

Now, it is not important that the equivalence R has been constructed by means of an information system. We introduce the concept of roughly top equal sets and of rough top equality in a more abstract form (cf. [1], [2], [3], [4]).

Let U be a finite nonempty set, R an equivalence on U. For any $X \subseteq U$, we put $(\mathbf{u}R)(X) = \cup \{Q; Q \in U/R, Q \cap X \neq \emptyset\}$. Clearly, $(\mathbf{u}R)(X) = \{y \in U$; there exists $x_y \in X$ with $(y, x_y) \in R\}$. Furthermore, we set $\mathbf{E}(R) = \{(X, Y) \in \mathbf{B}(U) \times \mathbf{B}(U); (\mathbf{u}R)(X) = (\mathbf{u}R)(Y)\}$. The equivalence $\mathbf{E}(R)$ on $\mathbf{B}(U)$ is said to be the *rough top equality corresponding to the equivalence* R. An equivalence E on $\mathbf{B}(U)$ is called a *rough top equality if* there exists an equivalence R on U such that $E = \mathbf{E}(R)$.

There is a natural relationship between rough top equalities and good boxes.

2. THEOREM. *A box $S = \langle I, O, f \rangle$ is good if and only if $f^{-1} \circ f$ is a rough top equality.*

Proof. (1) Let S be good. We put $R = \{(x, y) \in I \times I; f(\{x\}) = f(\{y\})\}$.

If $X \in \mathbf{B}(I)$ and $(\mathbf{u}R)(X) = \{y_1, \ldots, y_p\}$, then, by the definition of $(\mathbf{u}R)(X)$, there exist $x_1, \ldots, x_p$ in X such that $f(\{y_i\}) = f(\{x_i\})$ for any $i \in \{1, \ldots, p\}$; for any $y_i \in X$, we may choose $x_i = y_i$. By 2.4(e), we obtain $f((\mathbf{u}R)(X)) = f(\{y_1, \ldots, y_p\}) = f(\{x_1, \ldots, x_p\}) = f(X)$, i.e., $(X, (\mathbf{u}R)(X)) \in f^{-1} \circ f$.

If $Y \in \mathbf{B}(I)$, $Y \supseteq X$, $x \in Y$, and $(X, Y) \in f^{-1} \circ f$, then $f(X) = f(Y)$ and, therefore, $f(X \cup \{x\}) = f(Y \cup \{x\}) = f(Y) = f(X)$ regarding (i). By (ii), there exists $y \in X$ such that $(x, y) \in R$ and, thus, $x \in (\mathbf{u}R)(X)$. Hence, $Y \supseteq X$, $(X, Y) \in f^{-1} \circ f$ imply $Y \subseteq (\mathbf{u}R)(X)$ and, therefore, $(\mathbf{u}R)(X)$ is the greatest $Y \in \mathbf{B}(I)$ such that $(X, Y) \in f^{-1} \circ f$. Thus, $(X, Y) \in f^{-1} \circ f$ means $(\mathbf{u}R)(X) = (\mathbf{u}R)(Y)$ which is $(X, Y) \in \mathbf{E}(R)$. We have proved that $f^{-1} \circ f = \mathbf{E}(R)$.

(2) If $f^{-1} \circ f$ is a rough top equality, there exists an equivalence R on I such that $f^{-1} \circ f = \mathbf{E}(R)$. If X, $Y \in \mathbf{B}(I)$, $x \in I$, and $f(X) = f(Y)$, then $(X, Y) \in \mathbf{E}(R)$ which implies that $(\mathbf{u}R)(X) = (\mathbf{u}R)(Y)$. Thus $(\mathbf{u}R)(X \cup \{x\}) = (\mathbf{u}R)(X) \cup (\mathbf{u}R)(\{x\}) = (\mathbf{u}R)(Y) \cup (\mathbf{u}R)(\{x\}) = (\mathbf{u}R)(Y \cup \{x\})$ by (A 6) in 1.3 of [2], and, therefore, $f(X \cup \{x\}) = f(Y \cup \{x\})$ which is (i). If $X \in \mathbf{B}(I)$, $x \in I$ are such that $f(X \cup \{x\}) = f(X)$, then $(\mathbf{u}R)(X) \cup (\mathbf{u}R)(\{x\}) = (\mathbf{u}R)(X \cup \{x\}) = (\mathbf{u}R)(X)$ and, thus, $x \in (\mathbf{u}R)(\{x\}) \subseteq (\mathbf{u}R)(X)$ which implies the existence of $y \in X$ such that $(x, y) \in R$, i.e., $f(\{x\}) = f(\{y\})$ which is (ii). □

Thus, recognizing rough top equalities among all congruences is the same as recognizing good boxes among all admissible boxes.

5. Reducibility of boxes. If $S = \langle I, O, f \rangle$ is a box and $K \subseteq I$, then $T = \langle K, O, f \restriction \mathbf{B}(K) \rangle$ is a box that is referred to as a *subbox* of S; $f \restriction \mathbf{B}(K)$ denotes the restriction of f to the set $\mathbf{B}(K)$. A subbox T of S is said to be *proper* if $T \neq S$.

A subbox T of S is said to be a *reduct* of S if there exists a surjective homomorphism of S onto T. Clearly, the activity of a box can be simulated in any of its reducts.

By 2.6, we obtain

1. LEMMA. *If T is a reduct of S and U a reduct of T, then U is a reduct of S.* □

A box is said to be *reducible* if it has at least one proper reduct. A box that is not reducible is said to be *irreducible*.

2. THEOREM. *Any box has an irreducible reduct.*

Proof. The family of all reducts of a box S is nonempty because S is a reduct of S; this family is ordered by the relation "to be a subbox" on the basis of 1. A minimal element of this family is irreducible as it is easy to see. □

3. LEMMA. *A very good box is irreducible.*

Proof. Let $S = \langle I, O, f \rangle$ be very good and $T = \langle J, O, f \upharpoonright \mathbf{B}(J) \rangle$ its reduct, h a homomorphism of S onto T. Then h is a mapping of I onto J and $f(X) = (f \upharpoonright \mathbf{B}(J))(h[X]) = f(h[X])$ for any $X \in \mathbf{B}(I)$. The injectivity of f implies that $X = h[X]$ for any $X \in \mathbf{B}(I)$. Particularly, $\{x\} = h[\{x\}] = \{h(x)\}$ and, therefore, $h(x) = x$ for any $x \in I$ and, thus, $J = h[I] = I$, $h = \mathrm{id}_I$. Thus, the only reduct of S is S; hence, S is irreducible. □

Let $S = \langle I, O, f \rangle$ be a good box. By 4.2, there exists an equivalence R on I such that $f^{-1} \circ f = \mathbf{E}(R)$. Let J be a subset of I such that $J \cap C$ has exactly one element for any block $C \in I/R$. Then $T = \langle J, O, f \upharpoonright \mathbf{B}(J) \rangle$ is a subbox of S; it is called a *kernel* of S. For any $x \in I$, let $h(x)$ be the only element $y \in J$ such that $(x, y) \in R$. Then h is a mapping of I onto J that will be called the *canonical mapping* of S onto T.

4. LEMMA. *If S is a good box, T a kernel of S, and h the canonical mapping of S onto T, then h is a surjective homomorphism of S onto T and T is very good.*

Proof. If $S = \langle I, O, f \rangle$, $T = \langle J, O, f \upharpoonright \mathbf{B}(J) \rangle$, then $(\mathbf{u}R)(X) = (\mathbf{u}R)(h[X])$ by the definition of J and h and, hence, $(X, h[X]) \in \mathbf{E}(R) = f^{-1} \circ f$ which implies $f(X) = f(h[X]) = (f \upharpoonright \mathbf{B}(J))(h[X])$ for any $X \in \mathbf{B}(I)$. Hence, h is a surjective homomorphism of S onto T. If $X, Y \in \mathbf{B}(J)$, $X \neq Y$, then $(\mathbf{u}R)(X) \neq (\mathbf{u}R)(Y)$ by definition of J. Thus, $(X, Y) \notin \mathbf{E}(R) = f^{-1} \circ f$ and, hence, $f(X) \neq f(Y)$. Therefore, $f \upharpoonright \mathbf{B}(J)$ is injective and T is very good. □

5. COROLLARY. *A kernel of a good box is its very good reduct.* □

6. THEOREM. *A good box is irreducible if and only if it is very good.*

Proof. By 3, a very good box is irreducible. An irreducible good box coincides with any of its kernels and is very good by 5. □

7. EXAMPLE. Let $S = \langle I, O, f \rangle$ be the good box of 3.4. Then $f^{-1} \circ f$ has the following blocks: $\{\emptyset\}$, $\{\{a\}\}$, $\{\{b\}, \{c\}, \{b, c\}\}$, $\{\{a, b\}, \{a, c\}, I\}$. It follows that R has the blocks $\{a\}$, $\{b, c\}$. We take $J = \{a, b\}$, $h(a) = a$, $h(b) = b = h(c)$, $g(\emptyset) = p$, $g(\{a\}) = m$, $g(\{b\}) = n$, $g(J) = q$. Then $T = \langle J, O, g \rangle$ is a kernel of S and h is the corresponding canonical mapping.

6. Concluding remarks. We have seen that a good box may be replaced by a very good box that can be simpler than the given box and that the activity of the given box can be simulated by the activity of the simpler very good box. This property of good boxes and the possibility of constructing kernels seems to be important enough to motivate mathematical means for recognizing good boxes. We have seen that rough top equalities belong to such means and that recognizing rough top equalities is the same as recognizing good boxes.

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М. Новотны, З. Павляк, **Анализ черных и проближенные равенства**

В настоящей работе выделяется некоторый класс черных ящиков, характеризуемых посредством приближенных равенств. Для данного класса дается метод приведения черных ящиков к более простым ящикам, сохраняя их действие.