those states. Another way to put it is that the following theory is implicit in the use of a state-space model.

Definition 9.27. The regular theory Th(S) associated with a state space S has the same types as Evt(S), that is, arbitrary sets $X, Y, \ldots$ of states of S, interpreted disjunctively. The theory is given by

$$\Gamma \vdash_{\mathrm{Th}(S)} \Delta \quad \text{iff} \quad \bigcap \Gamma \subseteq \bigcup \Delta.$$

Justification. We need to check that this theory is regular. It suffices to show that it satisfies Partition because it clearly satisfies Weakening. Suppose that $\Gamma \nvdash_{\mathrm{Th}(S)} \Delta$ and let $\sigma \in (\bigcap \Gamma - \bigcup \Delta)$. Let $\Gamma' = \{X \subseteq \mathrm{typ}(S) \mid \sigma \in X\}$ and $\Delta' = \{X \subseteq \mathrm{typ}(S) \sigma \mid \notin X\}$. Then $\langle \Gamma', \Delta' \rangle$ is a partition, $\langle \Gamma, \Delta \rangle \leq \langle \Gamma', \Delta' \rangle$, and $\Gamma' \nvdash_{\mathrm{Th}(S)} \Delta'$. □

9.4 Theory Interpretations

Just as classifications have infomorphisms and state spaces have projections, theories have their notion of map or morphism. We call it "theory interpretation."

Given a theory T, we write typ(T) for its set of types and $\vdash_T$ for its consequence relation.

Definition 9.28. A *(regular theory) interpretation* $f : T_1 \to T_2$ is a function from typ(T_1) to typ(T_2) such that for each $\Gamma, \Delta \subseteq \mathrm{typ}(T_1)$

if $\Gamma \vdash_{T_1} \Delta$, then $f[\Gamma] \vdash_{T_2} f[\Delta]$.

The following is sometimes useful for checking that a function is a theory interpretation.

Proposition 9.29. *Given regular theories T_1 and T_2, a function $f : \mathrm{typ}(T_1) \to \mathrm{typ}(T_2)$ is an interpretation if and only if for every consistent partition $\langle \Gamma', \Delta' \rangle$ of T_2, $\langle f^{-1}[\Gamma'], f^{-1}[\Delta'] \rangle$ is consistent in T_1.*

Proof. The routine proof is left as Exercise 9.11. □

Let us see how theory interpretations arise from infomorphisms and from state-space projections.

Definition 9.30. Given an infomorphism $f : \boldsymbol{A} \rightleftarrows \boldsymbol{B}$, we define

$$\mathrm{Th}(f) : \mathrm{Th}(\boldsymbol{A}) \to \mathrm{Th}(\boldsymbol{B})$$

to be the interpretation given by $\mathrm{Th}(f)(\alpha) = f^{\wedge}(\alpha)$.

Justification. In other words, Th(f) just forgets the action of f on tokens. If b is a counterexample to $\langle f[\Gamma], f[\Delta] \rangle$ in $\boldsymbol{B}$, then $f(b)$ is a counterexample to $\langle \Gamma, \Delta \rangle$ in $\boldsymbol{A}$. So, taking contrapositives, $\Gamma \vdash_{\boldsymbol{A}} \Delta$ entails $f^{\wedge}[\Gamma] \vdash_{\boldsymbol{B}} f^{\wedge}[\Delta]$. Hence Th($f$) is a interpretation. □

There is a similar operation that turns any state-space projection into a theory interpretation of the corresponding theories. Notice, however, the reversal of arrows.

Definition 9.31. Given a state-space projection $f : \boldsymbol{S}_1 \rightrightarrows \boldsymbol{S}_2$, let

$$\mathrm{Th}(f) : \mathrm{Th}(S_2) \to \mathrm{Th}(S_1)$$

be the interpretation defined by

$$\mathrm{Th}(f)(X) = f^{-1}[X]$$

for each set X of states of S_2.

Justification. We need to verify that if $f : S_1 \rightrightarrows S_2$ is a projection, then $\mathrm{Th}(f) : \mathrm{Th}(S_2) \to \mathrm{Th}(S_1)$ is an interpretation. Assume that $\Gamma_2 \vdash_{\mathrm{Th}(S_2)} \Delta_2$. Let $\Gamma_1 = \{f^{-1}(X) \mid X \in \Gamma_2\}$ and define Δ_1 similarly. We need to prove that $\Gamma_1 \vdash_{\mathrm{Th}(S_1)} \Delta_1$. Assume that this is not the case. Then there is a $\sigma_1 \in (\bigcap \Gamma_1 - \bigcup \Delta_1)$. But then $f(\sigma_1) \in (\bigcap \Gamma_2 - \bigcup \Delta_2)$, contradicting $\Gamma_2 \vdash_{\mathrm{Th}(S_2)} \Delta_2$. □

9.5 Representing Regular Theories

We have seen how any classification $\boldsymbol{A}$ gives rise to a regular theory Th($\boldsymbol{A}$). This theory is, of course, very dependent on just what tokens are present in the classification $\boldsymbol{A}$. The following result shows that any regular theory can be seen as the theory arising from some classification. (We also establish the analogous result for theory interpretations, that they all arise from infomorphisms of classifications.)

Definition 9.32.

1. Given a regular theory T, the classification Cla(T) *generated* by T is the classification whose

(a) tokens are the consistent partitions $\langle\Gamma, \Delta\rangle$ of typ(T),
(b) types are the types of T, such that
(c) $\langle\Gamma, \Delta\rangle \vDash_{Cla(T)} \alpha$ if and only if $\alpha \in \Gamma$ (equivalently, if and only if $\alpha \notin \Delta$).

2. Given an interpretation $f: T \to T'$, we define an infomorphism

$$\mathrm{Cla}(f): \mathrm{Cla}(T) \rightleftarrows \mathrm{Cla}(T')$$

by

(a) $\mathrm{Cla}(f)^\wedge(\alpha) = f(\alpha)$ for $\alpha \in \mathrm{typ}(T)$, and
(b) $\mathrm{Cla}(f)^\vee(\langle\Gamma, \Delta\rangle) = \langle f^{-1}[\Gamma], f^{-1}[\Delta]\rangle$ for any token $\langle\Gamma, \Delta\rangle$ of Cla(T').

Justification. We need to verify that $\mathrm{Cla}(f): \mathrm{Cla}(T) \rightleftarrows \mathrm{Cla}(T')$ and that it is an infomorphism. If $\langle\Gamma, \Delta\rangle$ is a token of Cla(T'), then it is a consistent partition of typ(T'). But then $\langle f^{-1}[\Gamma], f^{-1}[\Delta]\rangle$ is a partition of typ(T); it is consistent because f is an interpretation. Hence $\langle f^{-1}[\Gamma], f^{-1}[\Delta]\rangle$ is a token of Cla(T), as desired. To see that Cla(f) is an infomorphism, we need to verify that $\langle\Gamma, \Delta\rangle \vDash_{Cla(T')} f(\alpha)$ if and only if $\langle f^{-1}[\Gamma], f^{-1}[\Delta]\rangle \vDash_{Cla(T)} \alpha$. But this is clear because the former is equivalent to $f(\alpha) \in \Gamma$ and the latter is equivalent to $\alpha \in f^{-1}[\Gamma]$. □

Theorem 9.33 (Representation Theorem). *For any regular theory T, T =* Th(Cla(T)). *Similarly, for any interpretation f, f =* Th(Cla(f)).

Proof. Clearly both regular theories have the same set of types. Suppose $\Gamma \vdash_T \Delta$. We need to see that this is satisfied by every token in Cla(T). Such a token is a partition $\langle\Gamma', \Delta'\rangle$ of typ(T) such that $\Gamma' \nvdash_T \Delta'$. Suppose token $\langle\Gamma', \Delta'\rangle$ does not satisfy sequent $\langle\Gamma, \Delta\rangle$. Then $\Gamma \subseteq \Gamma'$ but $\Delta \cap \Gamma' = \emptyset$. But then $\langle\Gamma, \Delta\rangle \leq \langle\Gamma', \Delta'\rangle$ because the latter is a partition, but this contradicts Weakening. For the converse, suppose $\Gamma' \nvdash_T \Delta'$. Then by Partition there is a partition $\langle\Gamma', \Delta'\rangle$ of typ(T) extending $\langle\Gamma, \Delta\rangle$ such that $\Gamma' \nvdash_T \Delta'$. But then $\langle\Gamma', \Delta'\rangle$ is a counterexample to $\langle\Gamma, \Delta\rangle$ in Cla(T). The second statement is similar. □

The following is an immediate consequence of this representation theorem. The first sentence of it is equivalent to a theorem proved in Chapter 6 of Dunn and Hardegree's (1993) manuscript.

Corollary 9.34 (Abstract Completeness Theorem). *Every regular theory is* Th(A) *for some classification A. Every interpretation is* Th(f) *for some infomorphism f.*

We now investigate a related question: Which classifications are isomorphic to those arising from regular theories? It turns out that only very special classifications are of this form, namely, those that are separated, that is, have no indistinguishable tokens.

Recall the definition of the separated quotient Sep(A) of a classification A given in Example 5.12. It is obtained by keeping all types of A while identifying tokens that are indistinguishable from one another.

Proposition 9.35. *For any classification A,* Sep(A) *is isomorphic to* Cla(Th(A)).

Proof. The isomorphism is the type identical infomorphism that maps the indistinguishability class of each token to its state description. □

Corollary 9.36. *A classification A is isomorphic to* Cla(T) *for some regular theory T if and only if A is separated.*

Proof. The direction from left to right is immediate from Proposition 9.35. For the converse, suppose A is separated. Let T = Th(A). Then by Proposition 9.35, Cla(T) is isomorphic to Sep(A), that is, isomorphic to A if A is separated. □

Exercises

9.11. Prove Proposition 9.29.

9.12. Show that a regular theory T is algebraic if and only if Cla(T) is extensional.

9.13. (†) Show that the sum of regular theories is the coproduct in the category of regular theories and interpretations.

9.14. Prove that

$$\mathrm{Cla}(T + T') \cong \mathrm{Cla}(T) + \mathrm{Cla}(T').$$

(The sum of theories is defined in 10.1)

9.15. (†) Let $f: T \to \mathrm{Th}(A)$ be an interpretation. Show that there is a unique infomorphism $f^*: \mathrm{Cla}(T) \rightleftarrows A$ such that $f = \mathrm{Th}(f^*)$. This shows that the functor Cla is the left adjoint of the functor Th, with the identity as the unit of the adjunction. Because functors that are left adjoints are known to preserve colimits, this result can be viewed as a generalization of Exercise 9.14.

Lecture 10

Operations on Theories

All of the operations on classifications defined earlier have counterparts on local logics. With an eye toward defining these operations, we first explain how the operations work on theories.

10.1 Sums of Theories

Definition 10.1. The *sum* $T + T'$ of regular theories T and T' is the regular theory whose types are the disjoint union of typ(T) and typ(T') and whose consequence relation is such that for $\Gamma_1, \Delta_1 \subseteq$ typ(T) and $\Gamma_2, \Delta_2 \subseteq$ typ(T'),

$$\Gamma_1, \Gamma_2 \vdash_{T+T'} \Delta_1, \Delta_2 \quad \text{iff} \quad \Gamma_1 \vdash_T \Delta_1 \quad \text{or} \quad \Gamma_2 \vdash_{T'} \Delta_2.$$

Justification. We need to check that this theory is regular. To simplify notation, let's assume the types of T and T' are disjoint. Identity and Weakening are clear. We check Partition, or rather, its contrapositive. Suppose $\Gamma_1, \Delta_1 \subseteq$ typ(T) and $\Gamma_2, \Delta_2 \subseteq$ typ(T'), and that $\Gamma_1, \Gamma_2 \nvdash_{T+T'} \Delta_1, \Delta_2$. We want to show there is a partition $\langle \Gamma, \Delta \rangle$ of the disjoint union of typ(T) and typ(T') extending $\langle \Gamma_1 \cup \Gamma_2, \Delta_1 \cup \Delta_2 \rangle$ such that $\Gamma \nvdash_{T+T'} \Delta$. By the definition of $\vdash_{T+T'}$, $\Gamma_1 \nvdash_T \Delta_1$ and $\Gamma_2 \nvdash_{T'} \Delta_2$. Hence, by Partition for each of the theories T and T', there are partitions $\langle \Gamma_1', \Delta_1' \rangle$ of typ(T) and extending $\langle \Gamma_1, \Delta_1 \rangle$ and a partition $\langle \Gamma_2', \Delta_2' \rangle$ of typ(T') and extending $\langle \Gamma_2, \Delta_2 \rangle$ such that $\Gamma_1' \nvdash_T \Delta_1'$ and $\Gamma_2' \nvdash_{T'} \Delta_2'$. But then $\langle \Gamma_1' \cup \Gamma_2', \Delta_1' \cup \Delta_2' \rangle$ is a consistent partition of the sum that extends the original sequent. □

Proposition 10.2. *The functions $\sigma_T : T \to T + T'$ and $\sigma_{T'} : T' \to T + T'$ that take types to their copies in the disjoint union are both interpretations.*

Proof. The proof is obvious. □

The sum of theories is a rather trivial operation. For example, if T_1 is a theory of flashlight switches and T_2 is a theory of flashlight bulbs, then $T_1 + T_2$ is a theory of both but without any interactions because there would be no constraints in this theory relating bulbs and switches in any nontrivial manner.

Proposition 10.3. *Given classifications* $\mathbf{A}$ *and* $\mathbf{B}$,

$$\text{Th}(\mathbf{A} + \mathbf{B}) = \text{Th}(\mathbf{A}) + \text{Th}(\mathbf{B}).$$

Proof. Without loss of generality we assume the types of the classifications are disjoint. Let $T = \text{Th}(\mathbf{A} + \mathbf{B})$, and let $T' = \text{Th}(\mathbf{A}) + \text{Th}(\mathbf{B})$. It is immediate that every constraint of T' is a constraint of T. To go the other direction, we prove the contrapositive. Suppose that $\Gamma \nvdash_{T'} \Delta$. Let $\Gamma_A = \Gamma \cap \text{typ}(\mathbf{A})$ and define Δ_A, Γ_B, and Δ_B similarly. Then $\Gamma_A \nvdash_{\text{Th}(\mathbf{A})} \Delta_A$ and $\Gamma_B \nvdash_{\text{Th}(\mathbf{B})} \Delta_B$. Hence there are counterexamples to these sequents, say $a \in \text{tok}(\mathbf{A})$ and $b \in \text{tok}(\mathbf{B})$. But then $\langle a, b \rangle \in \text{tok}(\mathbf{A} + \mathbf{B})$ and is a counterexample to the sequent $\langle \Gamma, \Delta \rangle$ in $\mathbf{A} + \mathbf{B}$. Hence $\Gamma \nvdash_T \Delta$. □

10.2 A Partial Order on Theories

Definition 10.4. Let Σ be fixed. A natural partial order on regular theories on Σ is defined by $T_1 \sqsubseteq T_2$ if and only if each constraint of $\vdash_{T_1}$ is also a constraint of $\vdash_{T_2}$.

This can also be expressed by saying that the inclusion map typ(T_1) $\subseteq$ typ(T_2) is a theory interpretation.

Proposition 10.5. *Let T_1 and T_2 be regular theories on Σ. The least upper bound of T_1 and T_2 is the theory $\langle \Sigma, \vdash \rangle$, where $\vdash$ is the smallest regular consequence relation containing $\vdash_{T_1}$ and $\vdash_{T_2}$. The greatest lower bound of T_1 and T_2 in the $\sqsubseteq$-order is the theory $\langle \Sigma, \vdash \rangle$ such that*

$$\Gamma \vdash \Delta \quad \textit{iff} \quad \Gamma \vdash_{T_1} \Delta \quad \textit{and} \quad \Gamma \vdash_{T_2} \Delta.$$

Proof. The proof is straightforward. □

We write the least upper bound of T_1 and T_2 as $T_1 \sqcup T_2$, and also call this the *join* of T_1 and T_2.

The existence of least upper bounds and greatest lower bounds generalizes to show that the set of theories on a fixed set Σ of types is a complete lattice.

10.3 Quotients of Theories

The operation on theories corresponding to quotients of classifications is that of restriction.

Definition 10.6. Let T be a theory on a set Σ of types and let $\Sigma_0 \subseteq \Sigma$. Then $T\restriction\Sigma_0$ is the theory on Σ_0 whose consequence relation is that of T restricted to Σ_0-sequents.

We leave the following proposition to the reader to check. It is entirely routine.

Proposition 10.7. *Let T be a theory on a set Σ of types and let $\Sigma_0 \subseteq \Sigma$.*

1. *If T is regular, so is $T\restriction\Sigma_0$.*
2. *$T\restriction\Sigma_0$ is the largest theory on Σ_0 such that the identity map of Σ_0 into Σ is a theory interpretation.*
3. *If $T = \mathrm{Th}(A)$, then $T\restriction\Sigma_0 = \mathrm{Th}(A\restriction\Sigma_0)$.*

Corresponding to dual invariants on classifications, we have the following operation on theories.

Definition 10.8. Let $T = \langle\Sigma, \vdash\rangle$ be a theory and let R be a binary relation on Σ. The *(dual) quotient of T by R*, written T/R, is the theory defined as follows. Its set of types is the set Σ/R of equivalence classes $[\alpha]_R$ for $\alpha \in \Sigma$. Its consistent partitions are those partitions $\langle\Gamma', \Delta'\rangle$ of Σ/R such that the following is T-consistent:

$$\langle\{\alpha \in \Sigma \mid [\alpha]_R \in \Gamma'\}, \{\beta \in \Sigma \mid [\beta]_R \in \Delta'\}\rangle.$$

Justification. We are using here the fact that we can specify a regular theory by specifying its set of consistent partitions. □

Example 10.9. Let T be a theory with types $\alpha, \beta_1, \beta_2, \gamma$ and suppose $\alpha \vdash_T \beta_1$ and $\beta_2 \vdash_T \gamma$. If $\beta_1 R \beta_2$, then in the theory T/R we will have $[\alpha]_R \vdash [\gamma]_R$. (Prove this.)

Example 10.10. Given a regular theory T, let $\alpha R \beta$ if and only if $\alpha \vdash_T \beta$. Then $[\alpha]_R = [\beta]_R$ if and only if $\alpha \vdash_T \beta$ and $\beta \vdash_T \alpha$. The quotient T/R is called the *Lindenbaum theory* associated with T and is written $\mathrm{Lind}(T)$. This theory identifies types that are equivalent in T. It is an algebraic theory (in the sense of Exercise 9.2).

We leave the following result as an exercise for the reader, because its parts are very similar to (and simpler than) things we have done earlier.

Proposition 10.11. *Let $T = \langle\Sigma, \vdash\rangle$ be a theory on a set Σ of types and let R be a binary relation on Σ.*

1. *The theory T/R is the least regular theory on Σ/R such that the function $\alpha \mapsto [\alpha]_R$ is a theory interpretation.*
2. *If $J = \langle\mathrm{typ}(A), R\rangle$ is a dual invariant on A and $T = \mathrm{Th}(A)$, then $T/R = \mathrm{Th}(A/J)$.*
3. *Let $f : T \to T'$ be a theory interpretation that respects R in the sense that $f(\alpha) = f(\beta)$ whenever $\alpha R \beta$. There is a unique theory interpretation $f' : T/R \to T'$ such that $f(\alpha) = f'([\alpha]_R)$ for each $\alpha \in \Sigma$.*

Exercises

10.1. Let A and B be classifications. We say that A is an *informational subclassification* of B, written $A \sqsubseteq B$, if $\mathrm{typ}(A) \subseteq \mathrm{typ}(B)$, $\mathrm{tok}(B) \subseteq \mathrm{tok}(A)$, and the classification relations agree on the types and tokens in common to both.

1. Prove that if $A \sqsubseteq B$, then $\mathrm{Th}(A) \sqsubseteq \mathrm{Th}(B)$.
2. Does the converse of (1) hold?

10.4 Moving Theories

A major theme of this book is the idea of reasoning at a distance, that is, using a theory of one or more parts of a distributed system to reason about other parts. This was foreshadowed by our discussion of the rules of f-Intro and f-Elim in Lecture 2. We turn to these rules now, beginning with f-Elim.

Definition 10.12. Let $T' = \langle\Sigma', \vdash_{T'}\rangle$ be a regular theory and let $f : \Sigma \to \Sigma'$. The *inverse image of T'* under f, written $f^{-1}[T']$, is the theory with types Σ and consequence relation given by

$$\Gamma \vdash \Delta \quad \text{iff} \quad f[\Gamma] \vdash_{T'} f[\Delta].$$

Proposition 10.13. *Let $T' = \langle\Sigma', \vdash_{T'}\rangle$ be a regular theory and let $f : \Sigma \to \Sigma'$ be a function. Let T be the inverse image of T' under f. Then T is a regular theory. Indeed, it is the largest regular theory on Σ such that $f : T \to T'$ is an interpretation.*

Proof. Let $T = \langle\Sigma, \vdash\rangle$. We first show that $\vdash$ satisfies Weakening and Partition. The former is obvious. To prove the latter, suppose $\Gamma, \Delta \subseteq \mathrm{typ}(T)$ and $\Gamma \nvdash \Delta$.

We need to show that there is a partition $\langle \Gamma'', \Delta'' \rangle \geq \langle \Gamma, \Delta \rangle$ such that $\Gamma'' \nvdash \Delta''$. By the definition of $\vdash$, $f[\Gamma] \nvdash_{T'} f[\Delta]$. Hence, by Partition in T', there is a partition $\langle \Gamma', \Delta' \rangle \geq \langle f[\Gamma], f[\Delta] \rangle$ such that $\Gamma' \nvdash_{T'} \Delta'$. Let $\Gamma'' = f^{-1}[\Gamma']$ and $\Delta'' = f^{-1}[\Delta']$. It is clear that $\langle \Gamma'', \Delta'' \rangle$ is a partition, that $\langle \Gamma'', \Delta'' \rangle \geq \langle \Gamma, \Delta \rangle$, and that $\Gamma'' \nvdash \Delta''$. It is clear that this is the largest theory that makes $f : T \to T'$ an interpretation. □

Images of theories under maps are introduced similarly, but the definition is complicated by the absence of a result analogous to Proposition 10.13. Given a function $f : \mathrm{typ}(T) \to \Sigma'$, there is no guarantee that the consequence relation on Σ' defined by

$$\Gamma' \vdash \Delta' \quad \text{iff} \quad f^{-1}[\Gamma'] \vdash_T f^{-1}[\Delta']$$

is regular. For example, if $\alpha \in \Sigma'$ is a type outside the range of f, then this definition would have $\alpha \nvdash \alpha$. What we want is the smallest regular consequence relation containing the one just defined. Another way to get at the same thing is as follows.

Definition 10.14. Let $T' = \langle \Sigma, \vdash_T \rangle$ be a regular theory and let $f : \Sigma \to \Sigma'$; we define the *image of T under f*, $f[T]$ as follows. Its types are the elements of Σ'. The theory is given by specifying its consistent partitions as follows: a partition $\langle \Gamma, \Delta \rangle$ of Σ' is $f[T]$-consistent if and only if $\langle f^{-1}[\Gamma], f^{-1}[\Delta] \rangle$ is T-consistent.

Proposition 10.15. *Let T be a regular theory and let $f : \mathrm{typ}(T) \to \Sigma'$. $f[T]$ is the smallest regular theory T' on Σ' such that $f : \mathrm{typ}(T) \to T'$ is an interpretation.*

Proof. Let T' be any regular theory on Σ' such that $f : \mathrm{typ}(T) \to T'$ is an interpretation. Assume $\Gamma' \vdash_{f[T]} \Delta'$. We need to prove $\Gamma' \vdash_{T'} \Delta'$. By Partition, it suffices to prove $\Gamma'' \vdash_{T'} \Delta''$ for every partition $\langle \Gamma'', \Delta'' \rangle$ extending $\langle \Gamma', \Delta' \rangle$. Fix such a partition. By Weakening, we have $\Gamma'' \vdash_{f[T]} \Delta''$. Hence, by the definition of $f[T]$, $f^{-1}[\Gamma''] \vdash_T f^{-1}[\Delta'']$. Because $f : \mathrm{typ}(T) \to T'$ is an interpretation, $f[f^{-1}[\Gamma'']] \vdash_{T'} f[f^{-1}[\Delta'']]$. But $f[f^{-1}[\Gamma'']] \subseteq \Gamma''$ and $f[f^{-1}[\Delta'']] \subseteq \Delta''$, so we get the desired $\Gamma'' \vdash_{T'} \Delta''$ by Weakening. □

10.5 Families of Theories

Science develops partial theories about different phenomena, with the hope that these theories will one day be part of, or interpretable in, some grand,

unified theory of everything. In the introduction, we raised the question of how theories fit together. In this section we apply our earlier results to show that there is a sense in which it is always possible to put them together in an optimal manner, as long as the theories are regular. (Every theory has a regular closure, of course.)

Definition 10.16. A *family of theories* $\mathcal{T}$ consists of an indexed family $\mathrm{th}(\mathcal{T}) = \{T\}_{i \in I}$ of regular theories together with a set $\mathrm{inter}(\mathcal{T})$ of interpretations, all of which have both domain and codomain in the family $\mathrm{th}(\mathcal{T})$.

Using some of the results obtained earlier, we can prove the following.

Theorem 10.17. *Every family $\mathcal{T}$ of theories has a "limit," that is a weakest regular theory in which each T_i can be interpreted so as to respect whatever interpretations $f : T_i \to T_j$ are present in the family. This theory is unique up to theory isomorphism.*

It turns out to be easier to prove the theorem than to state the definitions needed to make it precise. Because we will not be using the result in what follows, we simply sketch the proof of the result, leaving it to the reader to fill in the details.

Given the family $\mathcal{T}$ of regular theories, we use the operation Cla to turn it into a distributed system. The classifications of the system consist of those classifications of the form $\mathrm{Cla}(T_i)$ for $T_i \in \mathrm{th}(\mathcal{T})$. The infomorphisms of the distributed system consist of the infomorphisms of the form $\mathrm{Cla}(f)$ for $f \in \mathrm{inter}(\mathcal{T})$. As we have seen, $f : T_i \to T_j$ is an interpretation if and only if $\mathrm{Cla}(f) : \mathrm{Cla}(T_i) \rightleftarrows \mathrm{Cla}(T_j)$ is an infomorphism, so this turns our family of theories into a distributed system. The limit of this system consists of a classification $\mathbf{C}$ and infomorphisms $g_i : \mathrm{Cla}(T_i) \rightleftarrows \mathbf{C}$. The limit of our family consists of the theory $\mathrm{Th}(\mathbf{C})$, together with the interpretations $\mathrm{Th}(g_i)$ for $i \in I$.

Lecture 11

Boolean Operations and Theories

In Lecture 7 we discussed the relationship between classifications and the Boolean operations. In this lecture, we study the corresponding relationship for theories. In particular, we discuss Boolean operations that take theories to theories, as well as what it would mean for operations to be Boolean operations in the context of a particular theory. In this way, we begin to see how the traditional rules of inference emerge from an informational perspective. The topic is a natural one but it is not central to the main development so this lecture could be skipped.

11.1 Boolean Operations on Theories

Given a regular theory $T = \langle \Sigma, \vdash \rangle$, one may define a consequence relation on the set $\text{pow}(\Sigma)$ of subsets of Σ in one of two natural ways, depending on whether one thinks of the sets of types disjunctively or conjunctively. This produces two new theories, $\vee T$ and $\wedge T$, respectively.

These operations should fit with the corresponding power operations $\vee A$ and $\wedge A$ on classifications A; we want $\vee\text{Th}(A)$ to be the same as the theory $\text{Th}(\vee A)$, for example. Thus, to motivate our definitions, we begin by investigating the relationship of the theory $\text{Th}(A)$ of a classification to the theories $\text{Th}(\vee A)$ and $\text{Th}(\wedge A)$ of its two power classifications.

Definition 11.1. Given a set Γ of subsets of Σ, a set Y is a *choice set* on Γ if $X \cap Y \neq \emptyset$ for each $X \in \Gamma$.

Proposition 11.2. *Let A be a classification, let $a \in \text{tok}(A)$, and let Γ, Δ be subsets of* $\text{pow}(\text{typ}(A))$*:*

1. *a satisfies the $\vee A$-sequent $\langle \Gamma, \Delta \rangle$ if and only if for every choice set Y on Γ, a satisfies the A-sequent $\langle Y, \bigcup \Delta \rangle$;*

2. *a satisfies the $\wedge A$-sequent $\langle \Gamma, \Delta \rangle$ if and only if for every choice set Y on Δ, a satisfies the A-sequent $\langle \bigcup \Gamma, Y \rangle$.*

Similarly, if Γ, Δ are subsets of $\text{typ}(A)$, *then a satisfies $\langle \Gamma, \Delta \rangle$ in $\neg A$ if and only if a satisfies $\langle \Delta, \Gamma \rangle$ in A.*

Proof. We prove the contrapositive version of (1). Note that a is a counterexample to the sequent $\langle \Gamma, \Delta \rangle$ in $\vee A$ if and only if

(a) for each $X \in \Gamma$, $a \models_{\vee A} X$, and
(b) for each $X \in \Delta$, $a \not\models_{\vee A} X$.

The clause (a) is equivalent to the statement that for each $X \in \Gamma$ there is a type $\alpha \in X$ such that $a \models_A \alpha$, which is just to say that there is a choice set Y for Γ such that $a \models_A \alpha$ for each $\alpha \in Y$. Clause (b) is equivalent to the statement that for each $X \in \Delta$ and each type $\alpha \in X$, $a \not\models_A \alpha$, which is just to say that $a \not\models_A \alpha$ for each $\alpha \in \bigcup \Delta$. Thus a is a counterexample to $\langle \Gamma, \Delta \rangle$ if and only if there is a choice set Y for Γ such that

(a′) $a \models_A \alpha$ for each $\alpha \in Y$, and
(b′) $a \not\models_A \alpha$ for each $\alpha \in \bigcup \Delta$.

This just says that a is a counterexample to the sequent $\langle Y, \bigcup \Delta \rangle$ in A. Part (2) is proved similarly, and part (3) is straightforward. □

As an immediate consequence, we obtain the following corollary.

Corollary 11.3. *Let A be a classification and let Γ, Δ be subsets of* $\text{pow}(\text{typ}(A))$.

1. *$\Gamma \vdash_{\vee A} \Delta$ if and only if for each choice set Y on Γ, $Y \vdash_A \bigcup \Delta$.*
2. *$\Gamma \vdash_{\wedge A} \Delta$ if and only if for each choice set Y on Δ, $\bigcup \Gamma \vdash_A Y$.*

Similarly, if Γ, Δ are subsets of $\text{typ}(A)$, *then $\Gamma \vdash_{\neg A} \Delta$ if and only if $\Delta \vdash_A \Gamma$.*

Observing that the right-hand side of each of these equivalences deals only with the theory of the classification A, the following is immediate.

Corollary 11.4. *Let A and B be classifications that have the same theory, that is,* $\text{Th}(A) = \text{Th}(B)$. *For any Boolean operation $\mathcal{B}$ ($\vee, \wedge, \neg$),* $\text{Th}(\mathcal{B}(A)) = \text{Th}(\mathcal{B}(B))$.

Recall that the Representation Theorem (Theorem 9.33) associated a canonical classification $\text{Cla}(T)$ with any theory T. Using this and the preceding corollary, we have a convenient way to get at canonical Boolean operations on theories.

Definition 11.5. For any regular theory T, define the theories $\vee T$, $\wedge T$, and $\neg T$ as follows:

1. $\vee T = \text{Th}(\vee\text{Cla}(T))$;
2. $\wedge T = \text{Th}(\wedge\text{Cla}(T))$;
3. $\neg T = \text{Th}(\neg\text{Cla}(T))$.

We unwind this definition as follows.

Corollary 11.6. *Let $T = \langle\Sigma, \vdash\rangle$ be a regular theory. The disjunctive power $\vee T = \langle\text{pow}(\Sigma), \vdash_\vee\rangle$ and conjunctive power $\wedge T = \langle\text{pow}(\Sigma), \vdash_\wedge\rangle$ of T are those regular theories on $\text{pow}(\Sigma)$ such that for each $\Gamma, \Delta \subseteq \text{pow}(\Sigma)$*

1. *$\Gamma \vdash_\vee \Delta$ if and only if for each choice set Y on Γ, $Y \vdash \bigcup\Delta$, and*
2. *$\Gamma \vdash_\wedge \Delta$ if and only if for each choice set Y on Δ, $\bigcup\Gamma \vdash Y$.*

The negation $\neg T = \langle\Sigma, \vdash_\neg\rangle$ is the regular theory on Σ such that for each $\Gamma, \Delta \subseteq \Sigma$, $\Gamma \vdash_\neg \Delta$ if and only if $\Delta \vdash \Gamma$.

The following shows that these definitions behave properly with respect to the Boolean operations on classifications.

Corollary 11.7. *For any classification A and any Boolean operation $\mathcal{B}$, $\mathcal{B}(\text{Th}(A)) = \text{Th}(\mathcal{B}(A))$.*

Proof. By the Representation Theorem,

$$\text{Th}(A) = \text{Th}(\text{Cla}(\text{Th}(A))).$$

Hence, by Corollary 11.4,

$$\text{Th}(\mathcal{B}(A)) = \text{Th}(\mathcal{B}(\text{Cla}(\text{Th}(A)))).$$

But the right-hand side of this equation is the definition of $\mathcal{B}(\text{Th}(A))$. □

Exercises

11.1. Let T be any regular theory. It follows from the definition of $\neg T$ and Proposition 9.35 that $\text{Cla}(\neg T) \cong \neg\text{Cla}(T)$. Find a natural isomorphism.

11.2. Investigate the relationship between the disjunctive classifications $\vee\text{Cla}(T)$ and $\text{Cla}(\vee T)$ and similarly for conjunction.

11.2 Boolean Operations in Theories

We now turn to the question of what it would mean for a theory T to have a conjunction, negation, or disjunction. Because it is in some ways the simplest, we begin with negation to get our bearings.

Negation

To determine what the correct notion of a Boolean connective in a theory should be, we again start the discussion with theories that arise from classifications. Our aim is to find out how Boolean operations behave on such theories and then generalize this to arbitrary regular theories.

Proposition 11.8. *Let A be a classification with a negation $\neg$. The consequence relation $\vdash_A$ generated by A has the following properties:*

$\neg$-Left: *If $\Gamma \vdash_A \Delta, \alpha$ then $\Gamma, \neg\alpha \vdash_A \Delta$.*
$\neg$-Right: *If $\Gamma, \alpha \vdash_A \Delta$ then $\Gamma \vdash_A \Delta, \neg\alpha$.*

Proof. These are both rather obvious. □

The rules of $\neg$-Left and $\neg$-Right are, of course, simply the standard rules for negation in a classical, Gentzen approach to classical validity.

Definition 11.9. Let $T = \langle\Sigma, \vdash\rangle$ be a regular theory. A function $\neg : \Sigma \to \Sigma$ is a negation on T if and only if T satisfies the following closure conditions:

$$\neg\text{-Left: } \frac{\Gamma \vdash_T \Delta, \alpha}{\Gamma, \neg\alpha \vdash_T \Delta}$$

$$\neg\text{-Right: } \frac{\Gamma, \alpha \vdash_T \Delta}{\Gamma \vdash_T \Delta, \neg\alpha.}$$

We have stated this in the rule format that is standard for Gentzen system; they should be read as "if . . . then . . .," as in the statement of Proposition 11.8.

We have stated this definition only for regular theories on purpose. Without the structural properties insured by regularity, the traditional rules simply do not guarantee that $\neg$ behaves anything like a negation. For example, in the theory defined in Example 9.23, the identity function satisfies $\neg$-Left and

$\neg$-Right. However, for regular theories $\neg$-Left and $\neg$-Right are all that are needed to ensure that a function behaves like a negation. This is shown in our Theorem 11.12, the abstract completeness theorem for negation. The following definition provides the key to this result.

Definition 11.10. Let T be a regular theory on Σ and $\neg : \Sigma \to \Sigma$. A partition $\langle \Gamma, \Delta \rangle$ *treats* $\neg$ *as a negation* if for all types α, $\neg\alpha \in \Gamma$ if and only if $\alpha \in \Delta$.

If A is a classification with a negation $\neg$, then any realized partition of the types of A must treat $\neg$ as a negation. Put the other way around, partitions that do not respect $\neg$ are spurious. Consequently, they should not be relevant in using the rule Partition. The following result shows that closure under the negation rules allows us to ignore these spurious partitions in applying Partition.

Proposition 11.11. *Let T be any regular theory on Σ and let $\neg : \Sigma \to \Sigma$. Then T satisfies $\neg$ -Left and $\neg$ -Right if and only if T satisfies the following condition:*

> **$\neg$ -Partition:** *If $\Gamma' \vdash_T \Delta'$ for every partition $\langle \Gamma', \Delta' \rangle \geq \langle \Gamma, \Delta \rangle$ that treats $\neg$ as a negation, then $\Gamma \vdash_T \Delta$.*

Proof. Assume that T is a regular theory on Σ satisfying $\neg$-Left and $\neg$-Right. Let us show that it satisfies $\neg$-Partition. Suppose that $\Gamma' \vdash_T \Delta'$ for every partition that both treats $\neg$ as a negation and extends $\langle \Gamma, \Delta \rangle$. We need to prove that $\Gamma \vdash_T \Delta$. To do this, we use Partition. Thus let $\langle \Gamma', \Delta' \rangle$ be *any* partition extending $\langle \Gamma, \Delta \rangle$. We need to prove that $\Gamma' \vdash_T \Delta'$. If $\langle \Gamma', \Delta' \rangle$ treats $\neg$ as a negation, we are done by our assumption. So we need only consider the case where it does not treat $\neg$ as a negation. There are two cases to consider.

Case 1. There is a $\neg\alpha \in \Gamma'$ such that $\alpha \notin \Delta'$. But then $\alpha \in \Gamma'$. Then we have our desired $\Gamma' \vdash_T \Delta'$ as follows:

$$\begin{array}{c} \alpha \vdash_T \alpha \\ \hline \Gamma' \vdash_T \Delta', \alpha \\ \hline \Gamma', \neg\alpha \vdash_T \Delta' \\ \hline \Gamma' \vdash_T \Delta' \end{array}.$$

The first step is an Identity, the second is by Weakening, the third is by $\neg$-Left. The final step is simply the third step rewritten, in view of the fact that $\neg\alpha \in \Gamma'$.

Case 2. There is a $\neg\alpha \notin \Gamma'$ such that $\alpha \in \Delta'$. But then $(\neg\alpha) \in \Delta'$. The conclusion follows as before, using $\neg$-Right.

Toward the converse, let us show $\neg$-Left, the other case being symmetric. Thus we want to show

$$\frac{\Gamma \vdash_T \Delta, \alpha}{\Gamma, \neg\alpha \vdash_T \Delta}.$$

Assume $\Gamma \vdash_T \Delta, \alpha$ but $\Gamma, \neg\alpha \nvdash_T \Delta$. Then there is a partition $\langle \Gamma', \Delta' \rangle$ extending $\langle \Gamma \cup \{\neg\alpha\}, \Delta \rangle$ such that $\Gamma' \nvdash_T \Delta'$ and such $\langle \Gamma', \Delta' \rangle$ treats $\neg$ as a negation. But then $\alpha \in \Delta'$ so we have $\Gamma' \vdash_T \Delta'$ from $\Gamma \vdash_T \Delta, \alpha$ by Weakening. □

This proposition allows us to prove the promised result, justifying the definition of a theory negation for regular theories.

Theorem 11.12 (Abstract Completeness for Negation). *Let $T = \langle \Sigma, \vdash \rangle$ be a regular theory and let $\neg : \Sigma \to \Sigma$. The following are equivalent:*

1. *$\neg$ is a negation on T;*
2. *there is a classification A and a negation infomorphism $n : \neg A \rightleftarrows A$ such that $T = \text{Th}(A)$ and $\neg = n^{\wedge}$;*
3. *let $A = \text{Cla}(T)$ and $n : \neg A \rightleftarrows A$ be the token identical contravariant pair that agrees with $\neg$ on types. Then $n : \neg A \rightleftarrows A$ is a negation infomorphism on A.*

Proof. The implication from (3) to (2) is trivial. That from (2) to (1) follows immediately from Proposition 11.8. To prove that (1) implies (3), assume (1). Recall that the types of A are those of T whereas the tokens of A are the consistent partitions of T. By Proposition 11.11, we know that all consistent partitions treat $\neg$ as a negation. To establish that n is a negation, we need to prove that for any such consistent partition $\langle \Gamma, \Delta \rangle$ and any type α, $\langle \Gamma, \Delta \rangle \models_{\neg A} \alpha$ if and only if $\langle \Gamma, \Delta \rangle \models_A \neg\alpha$. This is a consequence of the following chain of equivalences:

$$\begin{array}{rll} \langle \Gamma, \Delta \rangle \models_{\neg A} \alpha & \text{iff} & \langle \Gamma, \Delta \rangle \not\models_A \alpha \\ & \text{iff} & \alpha \notin \Gamma \\ & \text{iff} & \alpha \in \Delta \\ & \text{iff} & \neg\alpha \in \Gamma \\ & \text{iff} & \langle \Gamma, \Delta \rangle \models_A \neg\alpha. \end{array}$$ □

Corollary 11.13. *If $\neg$ is a negation on the regular theory T, then it is a regular theory interpretation from $\neg T$ to T.*

Proof. This is an immediate consequence of the previous result. □

One might be tempted to define a negation to be any interpretation from $\neg T$ to T. This, however, is far too weak. In other words, the converse of Corollary 11.13 does not hold.

Example 11.14. It does not follow from the fact that $\neg$ is an interpretation from $\neg T$ to T that a type and its negation are incompatible or that together they exhaust the possibilities. In other words, neither

$$\alpha, \neg\alpha \vdash_T \quad \text{nor} \quad \vdash_T \alpha, \neg\alpha$$

is a consequence of $\neg$ being a interpretation from $\neg T$ to T. To see this, let Σ be any set containing at least two types, say α and β, and let $T = \text{Triv}(\Sigma)$ be the least regular theory on Σ as characterized in Exercise 9.4. As we saw there, $\Gamma \vdash_T \Delta$ if and only if $\Gamma \cap \Delta \neq \emptyset$. Hence this consequence relation is symmetric so $\neg T = T$. Consequently, any permutation of Σ is a interpretation from $\neg T$ to T. For example, simply switching α and β is such an interpretation. However, clearly $\alpha, \beta \nvdash_T$ and $\nvdash_T \alpha, \beta$.

Corollary 11.15. *If T is a regular theory with a negation $\neg$ then*

Converse of $\neg$-Left: $\Gamma \vdash_T \Delta, \alpha$ *if* $\Gamma, \neg\alpha \vdash_T \Delta$.
Converse of $\neg$-Right: $\Gamma, \alpha \vdash_T \Delta$ *if* $\Gamma \vdash_T \Delta, \neg\alpha$.

Proof. Any negation on a classification with negation $\neg$ has these properties. □

Disjunction and Conjunction

We now proceed to the parallel considerations for disjunction and conjunction. We will not be as verbose in our discussion, because the main points have already been made with negation.

Proposition 11.16. *Let A be a classification. If A has a disjunction $\vee$, then its theory* Th(A) *satisfies the following:*

$\vee$-Left: *If $\Gamma, \alpha \vdash_A \Delta$ for each $\alpha \in \Theta$ then $\Gamma, \vee\Theta \vdash_A \Delta$.*
$\vee$-Right: *if $\Gamma \vdash_A \Delta, \Theta$ then $\Gamma \vdash_A \Delta, \vee\Theta$.*

If A has a conjunction $\wedge$, then its theory Th(A) *satisfies the following:*

$\wedge$-Left: *If $\Gamma, \Theta \vdash_A \Delta$, then $\Gamma, \wedge\Theta \vdash_A \Delta$.*
$\wedge$-Right: *If $\Gamma \vdash_A \Delta, \alpha$ for each $\alpha \in \Theta$, then $\Gamma \vdash_A \Delta, \wedge\Theta$.*

Proof. These are all easily proved. □

Definition 11.17. Let $T = \langle \Sigma, \vdash \rangle$ be a regular theory. A function $\vee : \text{pow}(\Sigma) \to \Sigma$ is a disjunction on T if and only if T satisfies

$$\vee\text{-Left:} \quad \frac{\Gamma, \alpha \vdash \Delta \quad \text{for each } \alpha \in \Theta}{\Gamma, \vee\Theta \vdash \Delta}$$

$$\vee\text{-Right:} \quad \frac{\Gamma \vdash \Delta, \Theta}{\Gamma \vdash \Delta, \vee\Theta}$$

Similarly, a function $\wedge : \text{pow}(\Sigma) \to \Sigma$ is a conjunction on T if and only if T satisfies

$$\wedge\text{-Right:} \quad \frac{\Gamma \vdash \Delta, \alpha \quad \text{for each } \alpha \in \Theta}{\Gamma \vdash \Delta, \wedge\Theta}$$

$$\wedge\text{-Left:} \quad \frac{\Gamma, \Theta \vdash \Delta}{\Gamma, \wedge\Theta \vdash \Delta}$$

Just as with negation, when we have an operation on types that is a disjunction (or conjunction), certain partitions of the types become spurious.

Definition 11.18. Let T be any regular theory on Σ. A partition $\langle \Gamma, \Delta \rangle$ is said to *treat $\vee$ as a disjunction* if for all sets Θ of types, $\vee\Theta \in \Delta$ if and only if $\Theta \subseteq \Delta$. The partition $\langle \Gamma, \Delta \rangle$ *treats $\wedge$ as a conjunction* if for all sets Θ of types, $\wedge\Theta \in \Gamma$ if and only if $\Theta \subseteq \Gamma$.

Proposition 11.19. *Let T be any regular theory on Σ.*

1. *If $\vee : \text{pow}(\Sigma) \to \Sigma$, then T satisfies $\vee$-Left and $\vee$-Right if and only if it satisfies the following condition:*

   $\vee$-Partition: *If $\Gamma' \vdash_T \Delta'$ for every partition $\langle \Gamma', \Delta' \rangle \geq \langle \Gamma, \Delta \rangle$ that treats $\vee$ as a disjunction, then $\Gamma \vdash_T \Delta$.*

2. *Similarly, if $\wedge : \text{pow}(\Sigma) \to \Sigma$, then T satisfies $\wedge$-Left and $\wedge$-Right if and only if it satisfies the following condition:*

   $\wedge$-Partition: *If $\Gamma' \vdash_T \Delta'$ for every partition $\langle \Gamma', \Delta' \rangle \geq \langle \Gamma, \Delta \rangle$ that treats $\wedge$ as a conjunction, then $\Gamma \vdash_T \Delta$.*

Proof. We prove that if T is a regular theory on Σ satisfying $\vee$-Left and $\vee$-Right, then it satisfies $\vee$-Partition, and leave the rest to the reader. Suppose that $\Gamma \vdash_T \Delta$ for every partition that treats $\vee$ as a disjunction and extends

$\langle \Gamma, \Delta \rangle$. We need to prove that $\Gamma \vdash_T \Delta$. To do this, we use partition. Thus let $\langle \Gamma', \Delta' \rangle$ be any partition extending $\langle \Gamma, \Delta \rangle$. We need to prove that $\Gamma' \vdash_T \Delta'$. If $\langle \Gamma', \Delta' \rangle$ treats $\vee$ as a disjunction, we are done. So we need only consider the case where it does not treat $\vee$ as a disjunction. There are two ways in which this might happen.

Case 1. There is a $\vee\Theta \in \Delta'$ such that $\Theta \not\subseteq \Delta'$. But then there is some $\alpha \in \Theta \cap \Gamma'$. Then we have our desired $\Gamma' \vdash_T \Delta'$ by Identity, Weakening, and $\vee$-Right.

Case 2. There is a $\vee\Theta \notin \Delta'$ such that $\Theta \subseteq \Delta'$. Because $\Theta \subseteq \Delta'$, we have $\alpha \vdash_T \Delta'$ for each $\alpha \in \Theta$. But then by $\vee$-Left, $\vee\Theta \vdash_T \Delta'$. But if $\vee\Theta \notin \Delta'$, then $\vee\Theta \in \Gamma'$, so $\Gamma' \vdash_T \Delta'$ by Weakening. □

We obtain abstract completeness results for disjunction and conjunction parallel to that for negation. We will only state the one for disjunction.

Theorem 11.20 (Abstract Completeness for Disjunction). *Let $T = \langle \Sigma, \vdash \rangle$ be a regular theory and let $\vee : \text{pow}(\Sigma) \to \Sigma$. The following are equivalent:*

1. *$\vee$ is a disjunction on T.*
2. *There is a classification $\mathbf{A}$ and a disjunction infomorphism $d : \vee\mathbf{A} \rightleftarrows \mathbf{A}$ such that $T = \text{Th}(\mathbf{A})$ and $\vee = d^{\wedge}$.*
3. *Let $\mathbf{A} = \text{Cla}(T)$ and $d : \vee\mathbf{A} \rightleftarrows \mathbf{A}$ be the token identical contravariant pair that agrees with $\vee$ on types. Then $d : \vee\mathbf{A} \rightleftarrows \mathbf{A}$ is a disjunction infomorphism on $\mathbf{A}$.*

Proof. The proof is similar to the result for negation. □

Corollary 11.21. *If T is a regular theory with a disjunction $\vee$ (or conjunction $\wedge$) then*

Converse of $\vee$-Left: $\Gamma, \alpha \vdash_T \Delta$ *for each* $\alpha \in \Theta$, *provided* $\Gamma, \vee\Theta \vdash_T \Delta$.
Converse of $\vee$-Right: $\Gamma \vdash_T \Delta, \Theta$ *if* $\Gamma \vdash_T \Delta, \vee\Theta$.

In the case of conjunction, T has the following properties:

Converse of $\wedge$-Left: $\Gamma, \Theta \vdash_T \Delta$ *if* $\Gamma, \wedge\Theta \vdash_T \Delta$.
Converse of $\wedge$-Right: $\Gamma \vdash_T \Delta, \alpha$ *for each* $\alpha \in \Theta$, *provided* $\Gamma \vdash_T \Delta, \wedge\Theta$.

Proof. Any consequence relation on a classification with disjunction (conjunction) has these properties. □

Definition 11.22. A regular theory T is *Boolean* if it has a disjunction, conjunction, and negation.

Corollary 11.23. *Let T be a regular theory. The following are equivalent:*

1. *T is a Boolean theory.*
2. *T is the theory of a Boolean classification.*
3. *$\text{Cla}(T)$ is a Boolean classification.*

Proof. This is an immediate consequence of Theorems 11.12, 11.20, our abstract completeness theorems for negation, disjunction, and conjunction. □

Exercises

11.3. Let T be a regular theory with a negation $\neg$. Show the following infinitary version of $\neg$-Left: for any sets Γ, Δ, Θ, if $\Gamma \vdash_T \Delta, \Theta$ then $\Gamma, \neg[\Theta] \vdash_T \Delta$ ($\neg[\Theta]$ is the image of Θ under $\neg$; note that if Θ is finite, this result follows from a finite number of applications of $\neg$-Left, but this strategy will not work if Θ is infinite). There is a corresponding infinitary version of $\neg$-Right.

11.4. Let T be an algebraic theory with a disjunction $\vee$. Show that for any set Γ of types, $\vee\Gamma$ is the least upper bound of Γ in the partial ordering $\leq_T$. Prove the analogous result for conjunction. What can you prove about $\neg$? Show that if T is a Boolean, algebraic theory, then it is in fact a Boolean algebra under the same operations.

11.5. Give a direct proof of Corollary 11.13, that is, one that does not go through the abstract completeness theorem, 11.12.

11.3 Boolean Inference in State Spaces

Let S be a state space with set of states Ω. Recall from Definition 9.27 the theory $\text{Th}(S)$ associated with S. This theory depends only on the set Ω of states of the state space and captures the idea that these states are exhaustive and mutually incompatible. By Proposition 8.18, $\text{Th}(S)$ is a Boolean theory with union, intersection, and complement acting as disjunction, conjunction, and negation, respectively. Furthermore, $\vdash \Omega$.

We now show that $\text{Th}(S)$ is the smallest such regular theory. Thus $\text{Th}(S)$ is the closure of the "axiom" $\vdash \Omega$ under the classical rules of inference associated with disjunction, conjunction, and negation along with Identity, Weakening, and Global Cut.

Theorem 11.24. *Let S be a state space with set of states Ω.* Th(S) *is the smallest regular theory on* pow(Ω) *satisfying the following conditions:*

1. $\vdash \Omega$,
2. *the operation* $\Theta \mapsto \bigcup \Theta$ *is a disjunction,*
3. *the operation* $\Theta \mapsto \bigcap \Theta$ *is a conjunction, and*
4. *the operation* $X \mapsto \Omega - X$ *is a negation.*

Proof. Let $\vdash$ be the smallest consequence relation on pow(Ω) such that (1)–(4) hold. Suppose $\Gamma, \Delta \subseteq$ pow(Ω) and $\Gamma \vdash_{Th(S)} \Delta$. We need to prove that $\Gamma \vdash \Delta$. By assumption, we have $\bigcap \Gamma \subseteq \bigcup \Delta$. Hence $\bigcup \Delta \cup -(\bigcap \Gamma) = \Omega$. By (1), then, $\vdash \bigcup \Delta \cup -(\bigcap \Gamma)$. By the converse of $\vee$-Right, we have $\vdash \bigcup \Delta, \Omega - (\bigcap \Gamma)$. By $\neg$ -Left, $\bigcap \Gamma \vdash \bigcup \Delta$. By the converse of $\vee$-Right, we obtain $\bigcap \Gamma \vdash \Delta$; and by the converse of $\wedge$-Left, we obtain the desired conclusion $\Gamma \vdash \Delta$. □

Lecture 12

Local Logics

With the groundwork laid in the preceding lectures, we come to the central material of the book, the idea of a local logic, which will take up the remainder of Part II. In this lecture we introduce local logics and proceed in the lectures that follow to show how local logics are related to channels and so to information flow.

If one is reasoning about a distributed system with components of very different kinds, the components will typically be classified in quite different ways, that is, with quite different types. Along with these different types, it is natural to think of each of the components as having its own logic, expressed in its own system of types. In this way, the distributed system gives rise to a distributed system of local logics. The interactions of the local logics reflect the behavior of the system as a whole.

In order to capture this idea, we introduce and study the notions of "local logic" and "local logic infomorphism" in this lecture. The main notions are introduced in the first two sections and studied throughout this lecture. The important idea of moving a logic along an infomorphism is studied in Lecture 13. In Lecture 14, we show that every local logic can be represented in terms of moving natural logics along binary channels. The idea of moving logics is put to another use in Lecture 15 to define the distributed logic of an information system. It is in this chapter that our picture of information flow is most fully articulated. Finally, in Lecture 16, we explore the relationship between local logics and state spaces in some detail.

12.1 Local Logics Defined

The notion of a local logic puts the idea of a classification together with that of a regular theory, but with an important added twist. In order to model reasonable but unsound inference, we introduce the notion of a "normal token" of a logic.

Definition 12.1. A *local logic* $\mathfrak{L} = \langle \text{tok}(\mathfrak{L}), \text{typ}(\mathfrak{L}), \models_{\mathfrak{L}}, \vdash_{\mathfrak{L}}, N_{\mathfrak{L}} \rangle$ consists of

1. a classification $\text{cla}(\mathfrak{L}) = \langle \text{tok}(\mathfrak{L}), \text{typ}(\mathfrak{L}), \models_{\mathfrak{L}} \rangle$,
2. a regular theory $\text{th}(\mathfrak{L}) = \langle \text{typ}(\mathfrak{L}), \vdash_{\mathfrak{L}} \rangle$, and
3. a subset $N_{\mathfrak{L}} \subseteq \text{tok}(\mathfrak{L})$, called the *normal tokens* of $\mathfrak{L}$, which satisfy all the constraints of $\text{th}(\mathfrak{L})$.

A token $a \in \text{tok}(\mathfrak{L})$ that fails to satisfy some constraint $\Gamma \vdash_{\mathfrak{L}} \Delta$ of $\mathfrak{L}$ is said to be an *exception* to this constraint. Part of the definition of a local logic insures that normal tokens are not exceptions to any constraints of the logic. Notice, however, that we do not assume that every nonnormal token is an exception to some constraint of the logic. It sometimes happens that a token satisfies all the constraints of a logic, but "by accident" as it were; the tokens in $N_{\mathfrak{L}}$ model the set of tokens that satisfy all the constraints for principled reasons.

Definition 12.2. A logic $\mathfrak{L}$ is *sound* if every token of $\text{tok}(\mathfrak{L})$ is normal. $\mathfrak{L}$ is *complete* if every sequent satisfied by every normal token is a constraint of the logic.

When we come to moving logics via infomorphisms, we will see that there are two natural ways to move logics. One preserves soundness, the other preserves completeness. Both, however, correspond to natural methods of reasoning about distributed systems.

Systematic Examples

We give three important examples of ways in which local logics arise. We will study these logics in detail in what follows.

Definition 12.3. Let A be a classification. The *local logic generated by* A, written $\text{Log}(A)$, has classification A, regular theory $\text{Th}(A)$, and all its tokens are normal. A logic is *natural* if it is generated by some classification.

Definition 12.4. Let S be a state space. The *local logic generated by* S, written $\text{Log}(S)$, has classification $\text{Evt}(S)$, regular theory $\text{Th}(S)$, and all its tokens are normal.

Justification. To see that $\text{Log}(S)$ is a local logic, we need to verify that every token of $\text{Evt}(S)$, that is, every token of S, satisfies every constraint of $\text{Th}(S)$. This is obvious. □

Definition 12.5. Let T be a regular theory. The *local logic generated by* T, written $\text{Log}(T)$, has classification $\text{Cla}(T)$, regular theory T, and all its tokens are normal. A logic is *formal* if it is generated by some regular theory.

Proposition 12.6. *Every formal logic is natural. Every natural logic* $\text{Log}(A)$ *on a separated classification* A *is isomorphic to a formal logic.*

Proof. The first follows from $\text{Log}(T) = \text{Log}(\text{Cla}(T))$. The second holds because if A is separated, then $A \cong \text{Cla}(\text{Th}(A))$, by Proposition 9.35. □

All of the examples of local logics given so far have been sound. Our main source of unsound local logics arises from moving logics via infomorphisms, a topic we take up in the next chapter. But it is easy enough to give other examples of unsound local logics. One example is given in Exercise 12.1. Others will be given in Lecture 19.

Exercises

12.1. Let $\mathfrak{L}$ be a local logic and let $\Theta \subseteq \text{typ}(\mathfrak{L})$ be a set of types. Suppose that in reasoning about the tokens in $\mathfrak{L}$ we have, in the course of a great deal of experience, never encountered any token that is not of all types in Θ. It might not be unreasonable to conclude that all tokens are of all the types in Θ and so ignore Θ in our reasoning. The *conditionalization of* $\mathfrak{L}$ *on* Θ is the logic $\mathfrak{L} \mid \Theta$ defined as follows. The classification of $\mathfrak{L} \mid \Theta$ is the same as that of $\mathfrak{L}$. The theory of $\mathfrak{L} \mid \Theta$ is given by

$$\Gamma \vdash_{\mathfrak{L}|\Theta} \Delta \quad \text{iff} \quad \Gamma, \Theta \vdash_{\mathfrak{L}} \Delta.$$

The normal tokens of $\mathfrak{L} \mid \Theta$ consist of those normal tokens of $\mathfrak{L}$ that are of all types in Θ. If $\Theta = \{\theta\}$ is a singleton, we write this conditionalized logic as $\mathfrak{L} \mid \theta$.

1. Show that $\mathfrak{L} \mid \Theta$ is indeed a local logic.
2. Give an everyday example of a sound local logic $\mathfrak{L}$ and a type θ such that $\mathfrak{L} \mid \theta$ is not sound.
3. Prove that for every local logic $\mathfrak{L}$ there is a sound local logic $\mathfrak{L}'$ and a type $\theta \in \text{typ}(\mathfrak{L}')$ such that $\mathfrak{L} = \mathfrak{L}' \mid \theta$.

12.2. Let S be a state space with set Ω of types. The trivial logic on S, written Triv(S), is the sound local logic with the classification S and with the consequence relation that is the smallest regular consequence relation such that $\vdash_{\text{Triv}(S)} \Omega$ and, for all distinct states $\sigma_1, \sigma_2 \in \Omega$, $\sigma_1, \sigma_2 \vdash_{\text{Triv}(S)}$.

1. Justify this definition.
2. Show that for any sets $\Gamma, \Delta \subseteq \text{typ}(S)$, $\Gamma \vdash_{\text{Triv}(S)} \Delta$ if and only if one of the following three conditions holds:
   (a) $\Gamma \cap \Delta \neq \emptyset$,
   (b) Γ has at least two elements, or
   (c) $\Delta = \Omega$.

12.2 Soundness and Completeness

In this section we make some simple remarks about sound and complete local logics. Note, though, that most interesting local logics are neither sound nor complete.

Proposition 12.7. *For any local logic $\mathfrak{L}$ on a classification A, the following are equivalent:*

1. *$\mathfrak{L}$ is natural.*
2. *$\mathfrak{L}$ is sound and complete.*
3. *$\mathfrak{L} = \text{Log}(A)$.*

Proof. For any classification A, Log(A) is the unique sound and complete local logic with classification A. □

Proposition 12.8. *Let S be a state space. The logic* Log(S) *is sound. It is complete if and only if S is complete.*

Proof. This is obvious from the definitions. □

We turn any logic into one that is sound, complete, or both, as follows.

Definition 12.9. For any local logic $\mathfrak{L}$, the *sound part of* $\mathfrak{L}$, written Snd($\mathfrak{L}$), is the local logic obtained by throwing away all the abnormal tokens of $\mathfrak{L}$ and restricting the classification relation accordingly. Everything else is left unchanged.

Example 12.10. Recall the theory of light bulbs from Example 9.17. The consistent partitions were given by the rows of the following truth table:

Lit	Unlit	Live
1	0	1
0	1	1
0	1	0

The theory was characterized in terms of sequents in Example 9.19. Let us turn this into a local logic $\mathfrak{L}$ by specifying the classification as follows:

	Lit	Unlit	Live
b_1	0	1	1
b_2	1	0	1
b_3	0	1	1
b_4	0	0	1

The set of normal tokens of our logic consists of all the tokens that satisfy the constraints, that is, b_1, b_2, and b_3. Token b_4 violates the constraint $\vdash$ LIT, UNLIT. (Perhaps b_4 is burning but extremely dimly.) The sound part Snd($\mathfrak{L}$) is the same logic, except on the classification corresponding to the first three rows of the above table, that is, the token b_4 is thrown out.

Definition 12.11. For any local logic $\mathfrak{L}$ the *completion of* $\mathfrak{L}$, written Cmp($\mathfrak{L}$), is the local logic obtained by adding a new normal token $n_{\langle\Gamma,\Delta\rangle}$ for each consistent partition $\langle\Gamma, \Delta\rangle$ that is not the state description of a normal token of $\mathfrak{L}$.[1] The new token $n_{\langle\Gamma,\Delta\rangle}$ is classified by type α of $\mathfrak{L}$ if and only if $\alpha \in \Gamma$. Everything else is left unchanged.

Example 12.12. Returning to the logic $\mathfrak{L}$ of Example 12.10, the completion Cmp($\mathfrak{L}$) is obtained by adding a normal token n to correspond to the one unrealized consistent partition, as follows:

	Lit	Unlit	Live
b_1	0	1	1
b_2	1	0	1
b_3	0	1	1
b_4	0	0	1
n	0	1	0

[1] Under normal circumstances, the most sensible way to do this would be to add the consistent partition $\langle\Gamma, \Delta\rangle$ itself as this new token. This could only be a problem if by some quirk this pair were already a token of the classification.

Someone using the logic $\mathfrak{L}$ is implicitly assuming that tokens like n are possible even if they do not happen to be realized.

Proposition 12.13. *Let $\mathfrak{L}$ be a local logic.*

1. Snd($\mathfrak{L}$) *is sound; it is complete if and only if $\mathfrak{L}$ is complete.*
2. *$\mathfrak{L}$ is sound if and only if* Snd($\mathfrak{L}$) = $\mathfrak{L}$.
3. Cmp($\mathfrak{L}$) *is complete; it is sound if and only if $\mathfrak{L}$ is sound.*
4. *$\mathfrak{L}$ is complete if and only if* Cmp($\mathfrak{L}$) = $\mathfrak{L}$.

Proof. See Exercise 12.3. □

Proposition 12.14. Snd(Cmp($\mathfrak{L}$)) = Cmp(Snd($\mathfrak{L}$))

Proof. See Exercise 12.3. □

Definition 12.15. For any local logic $\mathfrak{L}$, the *sound completion* of $\mathfrak{L}$, written SC($\mathfrak{L}$), is Snd(Cmp($\mathfrak{L}$)).

The sound completion of a local logic $\mathfrak{L}$ throws away the nonnormal tokens and adds in tokens to make the logic complete. It is thus a sound and complete logic, one with the same constraints as the original logic. Thus SC($\mathfrak{L}$) represents an idealization of the logic $\mathfrak{L}$, how the world would work if the logic were perfect. We will see that there is an important relationship between $\mathfrak{L}$ and SC($\mathfrak{L}$) in Lecture 14. For now we simply note that $\mathfrak{L}$ is sound and complete if and only if SC($\mathfrak{L}$) = $\mathfrak{L}$.

Exercise

12.3. Give proofs of Propositions 12.13 and 12.14.

12.3 Logic Infomorphisms

Suppose we have classifications A and C, representing the possible behaviors of some distributed system C and its components A. The part-whole relationship is modeled by means of an infomorphism $f : A \rightleftarrows C$. If we have local logics on A and C, respectively, we need to ask ourselves under what conditions f respects these logics.

The basic intuition is that if an instance c of the whole system is normal with respect to its logic, then the component $f(c)$ should be normal. Similarly, any constraint that holds of normal components must translate into something that holds of normal instances of the whole system. This is formalized in the following definition.

Definition 12.16. A *logic infomorphism* $f : \mathfrak{L}_1 \rightleftarrows \mathfrak{L}_2$ consists of a contravariant pair $f = \langle f^{\wedge}, f^{\vee} \rangle$ of functions such that

1. $f : \mathrm{cla}(\mathfrak{L}_1) \rightleftarrows \mathrm{cla}(\mathfrak{L}_2)$ is an infomorphism of classifications,
2. $f^{\wedge} : \mathrm{th}(\mathfrak{L}_1) \to \mathrm{th}(\mathfrak{L}_2)$ is a theory interpretation, and
3. $f^{\vee}[N_{\mathfrak{L}_2}] \subseteq N_{\mathfrak{L}_1}$.

We have seen how to associate a local logic with any classification, state space, or regular theory. The various morphisms between these structures (infomorphisms, projections, and interpretations, respectively) all give rise to logic infomorphisms, as we now show. First, though, we state a lemma that we will use several times.

Lemma 12.17. *Let $f : A \rightleftarrows B$ be an infomorphism, let Γ, Δ be sets of types of A, and let $b \in \mathrm{tok}(B)$. Then $f(b)$ satisfies $\langle \Gamma, \Delta \rangle$ in A if and only if b satisfies $\langle f[\Gamma], f[\Delta] \rangle$ in B.*

Proof. Assume that $f(b)$ satisfies $\langle \Gamma, \Delta \rangle$ in A. To show that b satisfies $\langle f[\Gamma], f[\Delta] \rangle$ in B, assume that b satisfies every $\beta \in f[\Gamma]$. Then, because f is an infomorphism, $f(b)$ satisfies every $\alpha \in \Gamma$. But then $f(b)$ satisfies some $\alpha' \in \Delta$, so b satisfies $f(\alpha')$, an element of $f[\Delta]$, as desired. The converse is similar. □

Definition 12.18. For any (classification) infomorphism $f : A \rightleftarrows B$, let Log(f) : Log(A) $\rightleftarrows$ Log(B) be the logic infomorphism that is the same as f as a pair of functions, but taken to have as domain and codomain the logics, rather than their underlying classification.

Justification. By Lemma 12.17, it is clear that if $\Gamma \vdash_{\mathrm{Log}(A)} \Delta$ then $f[\Gamma] \vdash_{\mathrm{Log}(B)} f[\Delta]$. Because these logics are sound, the condition on normal tokens is trivial. □

Definition 12.19. For any state space projection $f : S_1 \rightleftarrows S_2$ let Log(f) be the logic infomorphism Log(f) : Log(S_2) $\rightleftarrows$ Log(S_1) that, as a pair of functions, is the same as Evt(f); that is, Log(f) is the identity on tokens and takes inverse images of sets of states.

Justification. To see that Log(f) is a logic infomorphism, we need to check that if $\Gamma_2 \vdash_{\mathrm{Log}(S_2)} \Delta_2$, and if $\Gamma_1 = \{f^{-1}[X] \mid X \in \Gamma_2\}$, and $\Delta_1 = \{f^{-1}[X] \mid X \in \Delta_2\}$, then $\Gamma_1 \vdash_{\mathrm{Log}(S_1)} \Delta_1$. But $\Gamma_i \vdash_{\mathrm{Log}(S_i)} \Delta_i$ just means that every state of S_i that is in every $X \in \Gamma_i$ is in some $Y \in \Delta_i$, so the implication follows from general properties of inverse images of functions. □

Definition 12.20. For any regular theory interpretation $f : T_1 \to T_2$, let $\mathrm{Log}(f)$ be the logic infomorphism $\mathrm{Log}(f) : \mathrm{Log}(T_1) \rightleftarrows \mathrm{Log}(T_2)$ defined as follows. On types, $\mathrm{Log}(f)$ is just f. On tokens, $\mathrm{Log}(f)$ maps any consistent partition $\langle \Gamma, \Delta \rangle$ of T_2 to $\langle f^{-1}[\Gamma], f^{-1}[\Delta] \rangle$.

Justification. To see that $\mathrm{Log}(f)$ is a logic infomorphism, we need to check that if $\langle \Gamma, \Delta \rangle$ is a consistent partition of T_2, then $\langle f^{-1}[\Gamma], f^{-1}[\Delta] \rangle$ is a consistent partition of T_1. That it is a partition follows just from set-theoretic properties of functions. That it is consistent follows from the fact that f is a regular theory interpretation. □

Exercises

12.4. Let $\mathbf{o}$ be the unique sound local logic on the zero classification (cf. Exercise 3). Show that for every local logic $\mathfrak{L}$ there is a unique logic infomorphism $f : \mathbf{o} \rightleftarrows \mathfrak{L}$. Find a local logic $\mathbf{1}$ such that for every local logic $\mathfrak{L}$ there is a unique logic infomorphism $f : \mathfrak{L} \rightleftarrows \mathbf{1}$.

12.5. (†) Let cla be the forgetful functor from the category of local logics to the category of classifications, taking each logic $\mathfrak{L}$ to its classification $\mathrm{cla}(\mathfrak{L})$ and each logic infomorphism f to the infomorphism f constituted by the same pair of functions. Show that Log is left adjoint to the restriction of cla to complete logics, and right adjoint to the restriction of cla to sound logics.

12.4 Operations on Logics

We have already seen ways of combining classifications and regular theories. It is but a small step from that to combining local logics.

Sums of Logics

Definition 12.21. The sum $\mathfrak{L}_1 + \mathfrak{L}_2$ of local logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$ is the local logic with

1. classification $\mathrm{cla}(\mathfrak{L}_1) + \mathrm{cla}(\mathfrak{L}_2)$,
2. regular theory $\mathrm{th}(\mathfrak{L}_1) + \mathrm{th}(\mathfrak{L}_2)$, and
3. $N_{\mathfrak{L}_1+\mathfrak{L}_2} = N_{\mathfrak{L}_1} \times N_{\mathfrak{L}_2}$

The *canonical logic infomorphisms* $\sigma_{\mathfrak{L}_1} : \mathfrak{L}_1 \rightleftarrows \mathfrak{L}_1 + \mathfrak{L}_2$ and $\sigma_{\mathfrak{L}_2} : \mathfrak{L}_2 \rightleftarrows \mathfrak{L}_1 + \mathfrak{L}_2$ are defined as follows:

1. for each $\alpha \in \mathrm{typ}(\mathfrak{L}_i)$, $\sigma_{\mathfrak{L}_i}(\alpha) = \sigma_{\mathrm{cla}(\mathfrak{L}_i)}(\alpha)$;
2. for each pair $c \in \mathrm{tok}(\mathfrak{L}_1 + \mathfrak{L}_2)$, $\sigma_{\mathfrak{L}_i}(c) = \sigma_{\mathrm{cla}(\mathfrak{L}_i)}(c)$.

Justification. To see that $\mathfrak{L}_1 + \mathfrak{L}_2$ is a local logic, we need to see that every token $\langle a, b \rangle \in N_{\mathfrak{L}_1} \times N_{\mathfrak{L}_2}$ satisfies every constraint of $\mathrm{th}(\mathfrak{L}_1) + \mathrm{th}(\mathfrak{L}_2)$. But this is clear from the definitions. To see that $\sigma_{\mathfrak{L}_1} : \mathfrak{L}_1 \rightleftarrows \mathfrak{L}_1 + \mathfrak{L}_2$ and $\sigma_{\mathfrak{L}_2} : \mathfrak{L}_2 \rightleftarrows \mathfrak{L}_1 + \mathfrak{L}_2$ are indeed logic infomorphisms, there are three things to check, as follows:

1. $\sigma_{\mathfrak{L}_i}$ is an infomorphism because it is identical to $\sigma_{\mathrm{cla}(\mathfrak{L}_i)}$ on classifications;
2. $\sigma_{\mathfrak{L}_i}\hat{}$ is a theory interpretation because it is identical to $\sigma_{\mathrm{th}(\mathfrak{L}_i)}$ on regular theories; and
3. if $b = \langle a_1, a_2 \rangle \in N_{\mathfrak{L}_1+\mathfrak{L}_2}$, then $\sigma_{\mathfrak{L}_i}(b) = a_i$, which is a normal token of $\mathfrak{L}_i$. □

The above definition extends from the sum of two local logics to the sum of any indexed families of local logics without incident.

Proposition 12.22. *For any classifications A and B,*

$$\mathrm{Log}(A + B) = \mathrm{Log}(A) + \mathrm{Log}(B).$$

The same holds for arbitrary indexed families of classifications.

Proof. The only part that is not obvious is that the two logics have the same theories. But this is the content of Proposition 10.3. □

Proposition 12.23. *Let $\mathfrak{L}$ be the sum $\sum_{i \in I} \mathfrak{L}_i$ of an indexed family $\{\mathfrak{L}_i\}_{i \in I}$ of local logics. Given a family $\{f_i : \mathfrak{L}_i \rightleftarrows \mathfrak{L}'\}_{i \in I}$ of logic infomorphisms, the sum $\sum_{i \in I} f_i : \mathfrak{L} \rightleftarrows \mathfrak{L}'$ is a logic infomorphism.*

Proof. Suppose c is a normal token of $\mathfrak{L}'$. Then $f_i(c)$ is a normal token of A_i for each $i \in I$, and so $(\sum_{i \in I} f_i)(c)$ is a normal token of $\mathfrak{L}$. Suppose that $\Gamma \vdash_{\mathfrak{L}} \Delta$. We can write $\langle \Gamma, \Delta \rangle$ as

$$\langle \Gamma, \Delta \rangle = \left\langle \bigcup_{i \in I} \sigma_{\mathfrak{L}_i}[\Gamma_i], \bigcup_{i \in I} \sigma_{\mathfrak{L}_i}[\Delta_i] \right\rangle,$$

where $\langle \Gamma_i, \Delta_i \rangle$ is a sequent of $\mathfrak{L}_i$ for each $i \in I$. Because $\mathfrak{L} = \sum_{i \in I} \mathfrak{L}_i$, it follows that $\Gamma_i \vdash_{\mathfrak{L}_i} \Delta_i$ for some $i \in I$. But f_i is a logic infomorphism, so $f_i[\Gamma_i] \vdash_{\mathfrak{L}'} f_i[\Delta_i]$, and so by Weakening $(\sum_{i \in I} f_i)[\Gamma] \vdash_{\mathfrak{L}'} (\sum_{i \in I} f_i)[\Delta]$. □

Exercises

12.6. Show that $\mathfrak{L}_1 + \mathfrak{L}_2$ is sound if and only if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are both sound.

12.7. Show that $\mathfrak{L}_1 + \mathfrak{L}_2$ is complete if and only if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are both complete.

Joins of Logics

Definition 12.24. We define a partial order on local logics on a fixed classification A as follows:

$$\mathfrak{L}_1 \sqsubseteq \mathfrak{L}_2 \quad \text{iff} \quad \text{th}(\mathfrak{L}_1) \sqsubseteq \text{th}(\mathfrak{L}_2) \quad \text{and} \quad N_{\mathfrak{L}_2} \subseteq N_{\mathfrak{L}_1}.$$

Note the difference in the direction of the two inclusions. Stronger logics have more constraints but fewer normal tokens. This "contravariance" is something we have seen repeatedly. This is closely related to the problem of nonmonotonicity. A hint of this is given in Exercise 2.

Definition 12.25. The *join* $\mathfrak{L}_1 \sqcup \mathfrak{L}_2$ of logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$ on A is the local logic with regular theory $\text{th}(\mathfrak{L}_1) \sqcup \text{th}(\mathfrak{L}_2)$ and normal tokens $N_{\mathfrak{L}_1} \cap N_{\mathfrak{L}_2}$. The *meet* of local logics is defined dually.

Justification. We need to check that $\mathfrak{L}_1 \sqcup \mathfrak{L}_2$ is the least upper bound of logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$ in the $\sqsubseteq$-ordering on logics. This follows from the corresponding properties of classifications and regular theories. Greatest lower bounds are justified similarly. □

A straightforward generalization of the justification of joins and meets gives the following.

Proposition 12.26. *The ordering $\sqsubseteq$ on logics on A is a complete lattice.*

Example 12.27. Let A be a classification. Here are some simple applications of Proposition 12.26.

1. There is a smallest local logic $\mathfrak{L}$ on A. Its theory is given by $\Gamma \vdash_{\mathfrak{L}} \Delta$ if and only if $\Gamma \cap \Delta \neq \emptyset$. $\mathfrak{L}$ is sound. We call $\mathfrak{L}$ the *a priori logic* on A and denote it by AP(A). This logic looks trivial, but it is of some use.
2. There is a largest local logic on A. Its theory consists of all sequents, and so is inconsistent. It has an empty set of normal tokens. It is tempting to call this the postmodern logic on A.
3. Given any theory T on typ(A), there is a smallest logic $\mathfrak{L}$ for which all the constraints of T are constraints of $\mathfrak{L}$. Its theory is the regular closure of T and its normal tokens are the set of all tokens satisfying all the constraints of T.

Proposition 12.28. *Let $f : \mathfrak{L}_1 \rightleftarrows \mathfrak{L}_2$ be a logic infomorphism, and let $\mathfrak{L}_1'$ and $\mathfrak{L}_2'$ be logics with the same classifications as $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. If $\mathfrak{L}_1' \sqsubseteq \mathfrak{L}_1$ and $\mathfrak{L}_2 \sqsubseteq \mathfrak{L}_2'$, then f is also a logic infomorphism from $\mathfrak{L}_1'$ to $\mathfrak{L}_2'$.*

Proof. The proof is routine. □

Proposition 12.29. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be logics on the classification A. If $f : \mathfrak{L}_1 \rightleftarrows \mathfrak{L}$ and $f : \mathfrak{L}_2 \rightleftarrows \mathfrak{L}$ are logic infomorphisms, then f is also a logic infomorphism from $\mathfrak{L}_1 \sqcup \mathfrak{L}_2$ to $\mathfrak{L}$.*

Proof. Recall that the consequence relation of $\mathfrak{L}_1 \sqcup \mathfrak{L}_2$ is not just the union of the consequence relations of the individual logics but is the result of closing the union under Weakening. So suppose $\Gamma \vdash_{\mathfrak{L}_1 \sqcup \mathfrak{L}_2} \Delta$. Then for some $i = 1, 2$ and some $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash_{\mathfrak{L}_i} \Delta'$. Because $f : \mathfrak{L}_i \rightleftarrows \mathfrak{L}$ is a logic infomorphism, $f[\Gamma'] \vdash_{\mathfrak{L}} f[\Delta']$ and so $f[\Gamma] \vdash_{\mathfrak{L}} f[\Delta]$, by Weakening in $\mathfrak{L}$. The condition on normal tokens is clear. □

Quotients of Logics

Just as we had quotients and dual quotient of classifications and theories, so too do we have them for logics.

Definition 12.30. Let $\mathfrak{L}$ be a local logic on a classification A and let $I = \langle \Sigma, R \rangle$ be an invariant on A. The *quotient logic of $\mathfrak{L}$ by I*, written $\mathfrak{L}/I$, is the logic that has

1. classification A/I,
2. theory $\text{th}(\mathfrak{L}) \mid \Sigma$, and
3. the set of normal tokens $\{[a]_R \mid a \in N_{\mathfrak{L}}\}$.

Justification. We must verify that if $a \in N_{\mathfrak{L}}$, then $[a]_R$ satisfies all the constraints of $\text{th}(\mathfrak{L}) \mid \Sigma$. This follows from the fact that $\tau_I : A/I \rightleftarrows A$ is an infomorphism. □

This construction is not as important for our purposes as the dual, so we leave some of its properties to the exercises.

Definition 12.31. Let $\mathfrak{L}$ be a local logic on a classification A and let $J = \langle A, R\rangle$ be a dual invariant on A. The *quotient logic of* $\mathfrak{L}$ *by* J, written $\mathfrak{L}/J$, is the logic that has

1. classification A/J,
2. theory $\mathrm{th}(\mathfrak{L})/R$, and
3. the set of normal tokens $N_{\mathfrak{L}} \cap A$.

Justification. We must verify that if $a \in N_{\mathfrak{L}} \cap A$, then a satisfies all the constraints of $\mathrm{th}(\mathfrak{L})/R$. Equivalently, it suffices to show that if $\langle \Gamma, \Delta\rangle$ is the state description of a in A/J, and $\Gamma_0 = \tau_J{}^{-1}[\Gamma]$, $\Delta_0 = \tau_J{}^{-1}[\Delta]$, then a satisfies $\langle \Gamma_0, \Delta_0\rangle$ in A. This follows from the fact that $\tau_J : A \rightleftarrows A/J$ is an infomorphism. □

Example 12.32. Given a local logic $\mathfrak{L}$, let $\alpha R \beta$ if and only if $\alpha \vdash_T \beta$. Let $J = \langle N_{\mathfrak{L}}, R\rangle$. This is clearly a dual invariant, because all normal tokens satisfy all constraints and hence respect R. The quotient $\mathfrak{L}/J$ is called the *Lindenbaum logic* of $\mathfrak{L}$ and is written Lind ($\mathfrak{L}$). This logic identifies types that are equivalent in $\mathfrak{L}$. Notice that it is always a sound logic. It is complete if and only if $\mathfrak{L}$ is complete.

We say that a partition $\langle \Gamma, \Delta\rangle$ of Σ *respects the relation* R on Σ if for all $\alpha, \beta \in \Sigma$, if $\alpha R \beta$, then $\alpha \in \Gamma$ if and only if $\beta \in \Gamma$. This is equivalent to saying that if $\alpha \in \Gamma$, then $[\alpha]_R \subseteq \Gamma$.

Proposition 12.33. *Let $\mathfrak{L}$ be a local logic on A and let $J = \langle A, R\rangle$ be a dual invariant on A.*

1. *$\mathfrak{L}/J$ is sound if and only if $A \subseteq N_{\mathfrak{L}}$. Hence if $\mathfrak{L}$ is sound, then $\mathfrak{L}/J$ is sound.*
2. *$\mathfrak{L}/J$ is complete if and only if every $\mathfrak{L}$-consistent partition of* typ(*A*) *that respects R is the state description of some $a \in A \cap N_{\mathfrak{L}}$. Hence, if $\mathfrak{L}$ is complete and $N_{\mathfrak{L}} \subseteq A$, then $\mathfrak{L}/J$ is complete.*

Proof. Statement (1) is an immediate consequence of the definition. Statement (2) is almost immediate, given the rule Partition. □

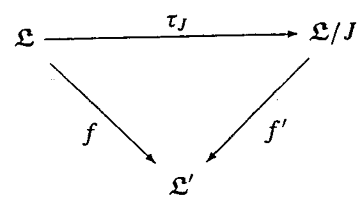
From this we obtain the following simple but useful characterization of a dual quotient as a natural logic.

Corollary 12.34. *A logic of the form $\mathfrak{L}/J$, where $J = \langle A, R\rangle$ is a dual invariant on* cla($\mathfrak{L}$), *is a natural logic if and only if $A \subseteq N_{\mathfrak{L}}$ and every $\mathfrak{L}$-consistent partition that respects R is the state description of some $a \in A$.*

Proof. Because a logic is natural if and only if it is sound and complete, this result follows immediately from Proposition 12.33. □

Proposition 12.35. *Let $\mathfrak{L}$ be a local logic on a classification A and let J be a dual invariant on A.*

1. *The logic $\mathfrak{L}/J$ is the least logic on A/J such that the function τ_J is a logic infomorphism.*
2. *Let $f : \mathfrak{L} \rightleftarrows \mathfrak{L}'$ be a logic infomorphism that respects J. There is a unique logic infomorphism $f' : \mathfrak{L}/J \rightleftarrows \mathfrak{L}'$ such that the following diagram commutes:*



Proof. Statement (1) follows easily from Proposition 10.11. Statement (2) follows directly from Propositions 5.21 and 10.11. □

Exercises

12.8. Recall the definition of the conditionalization $\mathfrak{L} \mid \Theta$ of a logic to a set of types given in Exercise 12.1. Let $\mathfrak{L}$ be a local logic and let $\Theta_0 \subseteq \Theta_1 \subseteq \mathrm{typ}(\mathfrak{L})$. Show that $\mathfrak{L} \mid \Theta_0 \sqsubseteq \mathfrak{L} \mid \Theta_1$.

12.9. Let $\mathfrak{L}$ be a local logic on a classification A. For any set $B \subseteq \mathrm{tok}(A)$, let $\mathrm{typ}(B) = \bigcap_{b\in B} \mathrm{typ}(b)$. That is, $\alpha \in \mathrm{typ}(B)$ if and only if for all $a \in B$, $a \models_A \alpha$. Suppose that B consists of all tokens in $\mathfrak{L}$ ever observed in the past, and suppose we are willing to reason on the assumption that anything that has always held true of past tokens will hold true of some future token. We can model this form of reasoning by considering the logic $\mathfrak{L} \mid \mathrm{typ}(B)$. The normal tokens of this logic consist of the set of all tokens that satisfy all the types that hold of tokens in B. Prove the following antimonotonicity result: if $B_0 \subseteq B_1$, then $\mathfrak{L} \mid \mathrm{typ}(B_1) \sqsubseteq \mathfrak{L} \mid \mathrm{typ}(B_0)$.

12.10. Dualize Proposition 12.35.

12.5 Boolean Operations and Logics

Our discussion of Boolean operations, classifications, and theories was prompted by the natural desire to relate Boolean operations and local logics. Readers who have skipped the chapters on Boolean operations will want to skip this section as well.

Again, we break the discussion into two parts. We first discuss Boolean operations on logics and then discuss what it means for an operation on the types of a single logic to be a disjunction, conjunction, or negation. We have done most of the work already, so the discussion will be rather brief.

Boolean Operations on Logics

We have laid the groundwork for the following definition in preceding chapters.

Definition 12.36. Let $\mathfrak{L}$ be a local logic on a classification A with theory T and set N of normal tokens. We define the disjunctive power logic $\vee\mathfrak{L}$, the conjunctive power logic $\wedge\mathfrak{L}$, and the negation $\neg\mathfrak{L}$ of $\mathfrak{L}$ as follows:

1. $\vee\mathfrak{L}$ is the local logic with classification $\vee A$, theory $\vee T$, and normal tokens N;
2. $\wedge\mathfrak{L}$ is the local logic with classification $\wedge A$, theory $\wedge T$, and normal tokens N; and
3. $\neg\mathfrak{L}$ is the local logic with classification $\neg A$, theory $\neg T$, and normal tokens N.

Justification. We need to see that every token in N satisfies the constraints of $\vee T$, $\wedge T$, and $\neg T$ in the classifications $\vee A$, $\wedge A$, and $\neg A$, respectively. This follows immediately from Proposition 11.2. □

Proposition 12.37. *For any local logic $\mathfrak{L}$, $\mathfrak{L}$ is sound if and only if $\vee\mathfrak{L}$ is sound, if and only if $\wedge\mathfrak{L}$ is sound, and if and only if $\neg\mathfrak{L}$ is sound. Similarly, $\mathfrak{L}$ is complete if and only if $\vee\mathfrak{L}$ is complete, if and only if $\wedge\mathfrak{L}$ is complete, and if and only if $\neg\mathfrak{L}$ is complete.*

Proof. Soundness is trivial. Let us show that $\mathfrak{L}$ is complete iff $\vee\mathfrak{L}$ is complete. First, suppose that $\mathfrak{L}$ is not complete. Thus there is a consistent sequent $\langle\Gamma, \Delta\rangle$ with no normal counterexample in A. Let $\Gamma' = \{\{\alpha\} \mid \alpha \in \Gamma\}$, and let $\Delta' = \{\Delta\}$. Then $\langle\Gamma', \Delta'\rangle$ has no normal counterexample in $\vee A$. We claim that $\langle\Gamma', \Delta'\rangle$ is consistent in $\vee A$. Suppose this is not the case. Then for every choice set Y for Γ', $Y \vdash_{\mathfrak{L}} \bigcup\Delta'$. But Γ is such a choice set and $\bigcup\Delta' = \Delta$, so $\Gamma \vdash_{\mathfrak{L}} \Delta$, contradicting the assumption that $\langle\Gamma, \Delta\rangle$ is consistent. Now let us prove the converse. Suppose $\langle\Gamma', \Delta'\rangle$ is consistent in $\vee A$. Then there is a choice set Y for Γ' such that $Y \nvdash_{\mathfrak{L}} \bigcup\Delta'$. If $\mathfrak{L}$ is complete, then this sequent has a normal counterexample a. But then, by Proposition 11.2, a is a normal counterexample to $\langle\Gamma', \Delta'\rangle$ in $\vee A$. □

Proposition 12.38. *For any Boolean operation $\mathcal{B}$ and any classification A, $\mathcal{B}(\text{Log}(A)) = \text{Log}(\mathcal{B}(A))$.*

Proof. The two logics clearly have the same classification and normal tokens. But they also have the same constraints by Corollary 11.4. □

Boolean Operations in Local Logics

We can define what it means for an operation on types to be a disjunction, conjunction, or negation on a local logic.

Definition 12.39. Let $\mathfrak{L}$ be a local logic on a classification A.

1. A function $\vee : \text{pow}(\text{typ}(A)) \to \text{typ}(A)$ is a *disjunction on* $\mathfrak{L}$ if $\vee$ is a disjunction on A and on $\text{th}(\mathfrak{L})$.
2. A function $\wedge : \text{pow}(\text{typ}(A)) \to \text{typ}(A)$ is a *conjunction on* $\mathfrak{L}$ if $\wedge$ is a conjunction on A and on $\text{th}(\mathfrak{L})$.
3. A function $\neg : \text{typ}(A) \to \text{typ}(A)$ is a *negation on* $\mathfrak{L}$ if $\neg$ is a negation on A and on $\text{th}(\mathfrak{L})$.

Exercises

12.11. Show that if $\neg$ is a negation on the regular theory T, then it is also a negation on the sound local logic $\text{Log}(T)$. Prove parallel statements for disjunction and conjunction.

12.12. Let S be an ideal state space, that is, one where the tokens and types are identical and the identity function is the state function. Prove that $\text{Log}(S) = \text{Log}(\text{Evt}(S))$.

12.13. Let S be a state space and its associated trivial logic $\text{Triv}(S)$ from Exercise 2.

1. Show that

$$\text{Log}(S) = \vee\,\text{Triv}(S).$$

2. Prove that the natural embedding $\eta^d_{\mathrm{Triv}(S)} : \mathrm{Triv}(S) \rightleftarrows \mathrm{Log}(S)$ is a logic infomorphism.

12.14. In order to state the following succinctly, let us use the following notation. If A and B are classifications with the same set of tokens and $f : \mathrm{typ}(A) \to \mathrm{typ}(B)$, let us write f_* for the unique token identical, contravariant pair $f_* : A \rightleftarrows B$ with $f = f_*^\wedge$. Let $\mathfrak{L}$ be a *complete* local logic. Prove the following results:

1. A function $\vee : \mathrm{pow}(\mathrm{typ}(A)) \to \mathrm{typ}(A)$ is a disjunction on $\mathfrak{L}$ if and only if $\vee_* : \vee\mathfrak{L} \rightleftarrows \mathfrak{L}$ is an infomorphism.
2. A function $\wedge : \mathrm{pow}(\mathrm{typ}(A)) \to \mathrm{typ}(A)$ is a conjunction on $\mathfrak{L}$ if and only if $\wedge_* : \wedge\mathfrak{L} \rightleftarrows \mathfrak{L}$ is an infomorphism.
3. A function $\neg : \mathrm{typ}(A) \to \mathrm{typ}(A)$ is a negation on $\mathfrak{L}$ if and only if $\neg_* : \neg\mathfrak{L} \rightleftarrows \mathfrak{L}$ is an infomorphism.

Give an example to show that the assumption of completeness is needed.

Lecture 13

Reasoning at a Distance

Suppose we are given an infomorphism $f : A \rightleftarrows B$. In Lecture 2, we discussed how we often implicitly use a logic on one of these classifications to reason about tokens in the other. There we expressed the idea in terms of inference rules we called f**-Intro** and f**-Elim**. In Section 10.4 we explored the idea in terms of regular theories. In this chapter, we amplify on this by showing how to move local logics. Most of the work has already been done in the discussion of theories.

The rule f**-Intro** corresponds to moving a logic $\mathfrak{L}$ from A to B via f; we call the new logic $f[\mathfrak{L}]$. The rule f**-Elim** corresponds to moving a logic $\mathfrak{L}$ from B to A via f^{-1}; we call the new logic $f^{-1}[\mathfrak{L}]$.

13.1 Moving Logics

We define both of the above logics here and then study them in turn in the subsections that follow. The rules mentioned above are used for motivation. The actual definition is phrased differently.

Definition 13.1. Given an infomorphism $f : A \rightleftarrows B$ and a local logic $\mathfrak{L}$ on A, the *image of* $\mathfrak{L}$ *under* f, denoted $f[\mathfrak{L}]$, is the local logic on the classification B with theory $f[\mathrm{th}(\mathfrak{L})]$ and with normal tokens

$$\{b \in \mathrm{tok}(B) \mid f(b) \in N_{\mathfrak{L}}\}.$$

Justification. We have seen in Proposition 10.15 that $f[\mathrm{th}(\mathfrak{L})]$ is a regular theory. We need to verify that every normal token satisfies every constraint of this theory. Recall that this theory was specified by giving its consistent partitions. Because closure under Partition is valid, it suffices to show that for each partition $\langle \Gamma, \Delta \rangle$ of typ($\boldsymbol{B}$) if $f(b)$ satisfies $f^{-1}[\Gamma] \vdash_{\mathfrak{L}} f^{-1}[\Delta]$, then b satisfies $\Gamma \vdash_{f[\mathfrak{L}]} \Delta$. This follows from Lemma 12.17. □

We define the inverse image of a logic in a parallel manner.

Definition 13.2. Given an infomorphism $f : \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ and a local logic $\mathfrak{L}$ on $\boldsymbol{B}$, the *inverse image of* $\mathfrak{L}$ *under* f, denoted $f^{-1}[\mathfrak{L}]$, is the local logic on $\boldsymbol{A}$ with theory $f^{-1}[\text{th}(\mathfrak{L})]$ and with normal tokens

$$\{a \in \text{tok}(\boldsymbol{A}) \mid a = f(b) \text{ for some } b \in N_{\mathfrak{L}}\}.$$

Justification. We need only check that every normal token satisfies every constraint, but this is clear. □

13.2 Images of Logics

We begin with the following result showing us that our definition gives us what we want, at least as far as the consequence relation is concerned. By "least" in the following, we mean with respect to the $\sqsubseteq$-partial ordering on local logics.

Theorem 13.3. *Let* $\mathfrak{L}$ *be a local logic on* $\boldsymbol{A}$ *and let* $f : \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ *be an infomorphism. The image of* $\mathfrak{L}$ *is the least logic* $\mathfrak{L}'$ *on* $\boldsymbol{B}$ *such that* f *is a logic infomorphism from* $\mathfrak{L}$ *to* $\mathfrak{L}'$.

Proof. To show that f is a logic infomorphism from $\mathfrak{L}$ to $f[\mathfrak{L}]$, suppose that $\Gamma \vdash_{\mathfrak{L}} \Delta$. If $\langle \Gamma', \Delta' \rangle$ is a partition of typ($\mathfrak{L}'$) extending $\langle f[\Gamma], f[\Delta] \rangle$, then $\langle f^{-1}[\Gamma'], f^{-1}[\Delta'] \rangle$ is a partition of typ($\boldsymbol{A}$) extending $\langle \Gamma, \Delta \rangle$. Then $f^{-1}[\Gamma'] \vdash_{\mathfrak{L}} f^{-1}[\Delta']$ by Weakening, and so $\Gamma' \vdash_{f[L]} \Delta'$ because $\langle f^{-1}[\Gamma'], f^{-1}[\Delta'] \rangle$ is a partition. Hence by Partition, $f[\Gamma] \vdash_{f[L]} f[\Delta]$. The condition for normal tokens is clearly satisfied.

Now assume that f is a logic infomorphism from $\mathfrak{L}$ to $\mathfrak{L}'$. To show that $f[\mathfrak{L}] \sqsubseteq \mathfrak{L}'$, we recall that f is already known to be a theory interpretation. So suppose that b is a normal token of $\mathfrak{L}'$. Then $f(b)$ is a normal token of $\mathfrak{L}$ because f is a logic infomorphism, and so b is a normal token of $f[\mathfrak{L}]$, by definition. Hence $N_{\mathfrak{L}'} \subseteq N_{f[\mathfrak{L}]}$. □

We now turn to the soundness and completeness of the f-Intro rule. Recall that a pair $f = \langle f^{\wedge}, f^{\vee} \rangle$ is *token surjective* if $f^{\vee}$ is surjective.

Proposition 13.4. *Let* $\mathfrak{L}$ *be a local logic on a classification* $\boldsymbol{A}$ *and let* $f : \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ *be an infomorphism.*

1. *If* $\mathfrak{L}$ *is sound, then* $f[\mathfrak{L}]$ *is sound.*
2. *If* f *is token surjective and* $\mathfrak{L}$ *is complete, then* $f[\mathfrak{L}]$ *is complete.*

Proof. For (1), suppose $\mathfrak{L}$ is sound. Then for each token b of $\boldsymbol{B}$, $f(b)$ is normal in $\mathfrak{L}$, and so b is normal in $f[\mathfrak{L}]$. For (2), suppose $\mathfrak{L}$ is complete and $f^{\vee}$ is surjective. Any consistent sequent $\langle \Gamma, \Delta \rangle$ of $f[\mathfrak{L}]$ is extendable to a consistent partition $\langle \Gamma', \Delta' \rangle$, by Partition, and so $\langle f^{-1}[\Gamma'], f^{-1}[\Delta'] \rangle$ is a consistent partition of $\mathfrak{L}$. By the completeness of $\mathfrak{L}$, there is a normal token a of $\mathfrak{L}$ with state description $\langle f^{-1}[\Gamma'], f^{-1}[\Delta'] \rangle$. By the surjectivity of $f^{\vee}$, there is a token b of $\boldsymbol{B}$ with $f(b) = a$, and so b has state description $\langle \Gamma', \Delta' \rangle$. Moreover, because a is a normal token of $\mathfrak{L}$, b is a normal token of the image $f[\mathfrak{L}]$. □

The restriction in the second part of Proposition 13.4 is crucial: If the infomorphism is not surjective on tokens, then the image of a complete logic is not necessarily complete. Indeed, we have the following.

Proposition 13.5. *Any logic* $\mathfrak{L}$ *is the image of the complete logic* Cmp($\mathfrak{L}$) *under the type-identical inclusion infomorphism* $\kappa_{\mathfrak{L}} : \text{cla}(\text{Cmp}(\mathfrak{L})) \rightleftarrows \text{cla}(\mathfrak{L})$.

Proof. That $\kappa_{\mathfrak{L}}$ is an infomorphism and Cmp($\mathfrak{L}$) is complete are immediate given their construction. That $\mathfrak{L}$ is $\kappa_{\mathfrak{L}}[\text{Cmp}(\mathfrak{L})]$ follows directly from our characterization of the image of a logic. □

Corollary 13.6. *A local logic is sound if and only if it is an image of a natural logic.*

Proof. By Proposition 12.7, every natural logic is sound and complete, so by Proposition 13.4, its image is sound. Conversely, if $\mathfrak{L}$ is sound then Cmp($\mathfrak{L}$) is both sound and complete, by Proposition 12.13, and so is natural by Proposition 12.7 again. Moreover, by Proposition 13.5, $\mathfrak{L}$ is an image of Cmp($\mathfrak{L}$), and so we are done. □

13.3 Inverse Images of Logics

We now explore the same set of issues with respect to inverse images of logics, or, if you like, the f-Elim rule. In some ways, this operation is better behaved, in other ways worse. First we have the following characterization of the inverse image of a logic.

Theorem 13.7. *Let* $f : \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ *be an infomorphism and let* $\mathfrak{L}'$ *be a logic on* $\boldsymbol{B}$. *The inverse image of* $\mathfrak{L}'$ *under* f *is the largest logic* $\mathfrak{L}$ *on* $\boldsymbol{A}$ *such that* f *is a logic infomorphism from* $\mathfrak{L}$ *to* $\mathfrak{L}'$.

Proof. First, f is a logic infomorphism from $f^{-1}[\mathfrak{L}']$ to $\mathfrak{L}'$ because $f^{\wedge}$ is a regular theory interpretation, by Proposition 10.13, and again the condition for normal tokens is clearly satisfied.

Assume that f is a logic infomorphism from $\mathfrak{L}$ to $\mathfrak{L}'$. To show that $\mathfrak{L} \sqsubseteq f^{-1}[\mathfrak{L}']$, suppose that a is a normal token of $f^{-1}[\mathfrak{L}']$. Then there is a normal token b of $\mathfrak{L}'$ such that $a = f(b)$, and so a is a normal token of $\mathfrak{L}$ because f is a logic infomorphism from $\mathfrak{L}$ to $\mathfrak{L}'$. Hence $N_{f^{-1}[\mathfrak{L}']} \subseteq N_{\mathfrak{L}}$. Moreover, if $\Gamma \vdash_{\mathfrak{L}} \Delta$, then $f[\Gamma] \vdash_{\mathfrak{L}'} f[\Delta]$ because f is a logic infomorphism from $\mathfrak{L}$ to $\mathfrak{L}'$, and then $\Gamma \vdash_{f^{-1}[\mathfrak{L}']} \Delta$, by Proposition 10.13. Hence $\mathfrak{L} \sqsubseteq f^{-1}[\mathfrak{L}']$, and we are done. □

Example 13.8. A simple application is the relationship between a logic $\mathfrak{L}$ on a sum $\boldsymbol{A} + \boldsymbol{B}$ of classifications and the logics on the summand $\boldsymbol{A}$ induced by $\mathfrak{L}$ and the canonical embedding $\sigma_A : \boldsymbol{A} \rightleftarrows \boldsymbol{A} + \boldsymbol{B}$ of $\boldsymbol{A}$ into $\boldsymbol{A} + \boldsymbol{B}$. The normal tokens of this logic are those $a \in \text{tok}(\boldsymbol{A})$ such that for some $b \in \text{tok}(\boldsymbol{B})$, $\langle a, b\rangle \in N_{\mathfrak{L}}$. The constraints of the logic are those sequents $\langle \Gamma, \Delta\rangle$ of $\boldsymbol{A}$ such that $\sigma_A[\Gamma] \vdash_{\mathfrak{L}} \sigma_A[\Delta]$. (The reason is simply that both of these logics are the largest logic on $\boldsymbol{A}$ making σ_A a logic infomorphism.) If the types of $\boldsymbol{A}$ and $\boldsymbol{B}$ are disjoint, this is simply the restriction of $\vdash_{\mathfrak{L}}$ to the sequents of $\boldsymbol{A}$.

Proposition 13.9. *Let $\mathfrak{L}$ be a local logic on a classification $\boldsymbol{B}$ and let $f : \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ be an infomorphism.*

1. *If $\mathfrak{L}$ is complete, then $f^{-1}[\mathfrak{L}]$ is complete.*
2. *If f is token surjective and $\mathfrak{L}$ is sound, then $f^{-1}[\mathfrak{L}]$ is sound.*

Proof. For (1), note that a counterexample to completeness in $f^{-1}[\mathfrak{L}]$ would give rise, via f, to a counterexample in $\mathfrak{L}$. The second statement follows directly from the definition of inverse images. □

The restriction in (2) is again crucial: if the infomorphism is not surjective on tokens, then the inverse image of a sound logic is not necessarily sound. Indeed, we have the following result.

Proposition 13.10. *Any local logic $\mathfrak{L}$ is the inverse image of the sound local logic* Snd($\mathfrak{L}$) *under the type identical inclusion $\iota_{\mathfrak{L}} : \text{cla}(\mathfrak{L}) \rightleftarrows \text{cla}(\text{Snd}(\mathfrak{L}))$.*

Proof. The proof is derived from the definition of inverse images. □

Proposition 13.11. *A local logic is complete if and only if it is an inverse image of a natural logic.*

Proof. By Proposition 12.7, every natural logic is sound and complete, so by Proposition 13.9, its inverse image is complete. Conversely, if $\mathfrak{L}$ is complete, then Snd($\mathfrak{L}$) is both sound and complete, by Proposition 12.13, and so is natural by Proposition 12.7 again. Moreover, by Proposition 13.10, $\mathfrak{L}$ is an inverse image of Snd($\mathfrak{L}$) and so we are done. □

Corollary 13.12. *If $f : \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ is a token surjective infomorphism, then, $f^{-1}[\text{Log}(\boldsymbol{B})] = \text{Log}(\boldsymbol{A})$ and $f[\text{Log}(\boldsymbol{A})] = \text{Log}(\boldsymbol{B})$.*

Proof. Log($\boldsymbol{B}$) is sound and complete. Completeness is preserved under inverse images and soundness is preserved for token-surjective infomorphisms, so its inverse image is also sound and complete. But by Proposition 12.7, Log($\boldsymbol{A}$) is the only sound and complete logic on $\boldsymbol{A}$. The second part follows similarly from Proposition 13.4. □

Analytic Truth

To illustrate these notions, let us work out a simple example having to do with the notion of analytic truth, that is, truth by virture of meaning.

We are going to set up two propositional languages as classifications $\boldsymbol{A}$ and $\boldsymbol{B}$. The language $\boldsymbol{A}$ has as types arbitrary sentences built up from the atomic sentences MOTHER, FATHER, and BACHELOR. For tokens, we take arbitrary truth assignments to these atomic sentences, with $a \models_A \alpha$ defined in the usual way. Thus the logic Log($\boldsymbol{A}$) is just the usual classical propositional logic on this set of atomic sentences. The constraints of this logic are those that are classically valid, which, by the completeness theorem for proposition logic, are those derivable in the classical Gentzen calculus. As a schematic example, we have

$$\alpha \to \neg(\beta \wedge \gamma), \alpha, \beta \vdash_{\text{Log}(A)} \neg\gamma,$$

where α, β and γ are arbitrary sentences.

The language $\boldsymbol{B}$ is similar, but its atomic sentences are FEMALE, MARRIED, PARENT, where the last is intended to connote the property of being a parent. Again, we take all truth assignments as tokens, and the constraints are those derivable in the classical calculus.

There is an important difference between these two classifications. Some of the truth assignments (tokens) of $\boldsymbol{A}$ represent spurious possibilities, things that could not really happen. For example, an assignment that assigned true (1) to both MOTHER and BACHELOR does not represent a real possibility (ignoring the possibility of sex-change operations). As a result, some seeming analytic truths are not constraints in the theory of A. For example, we note that

$$\text{MOTHER} \nvdash_A \neg\text{BACHELOR}$$

By contrast, every truth assignment of B represents a genuine possibility, because the atomic types of this classification are independent. Hence every analytic truth that can be expressed in this language is a constraint of B.

We can use B to see what is wrong with A. There is a natural infomorphism $f : A \rightleftarrows B$. Define f on atomic sentences as follows:

α	$f(\alpha)$
MOTHER	PARENT $\wedge$ FEMALE
FATHER	PARENT $\wedge$ $\neg$FEMALE
BACHELOR	$\neg$(FEMALE $\vee$ MARRIED)

Here f is defined on complex sentences so as to commute with the various logical operations.

On tokens, we define f in the natural way. Thus, given an assignment s for B, we define the assignment $s' = f(s)$ by means of the following:

$$\begin{array}{lll} s'(\text{MOTHER}) = 1 & \text{iff} & s \models \text{PARENT} \wedge \text{FEMALE} \\ s'(\text{FATHER}) = 1 & \text{iff} & s \models \text{PARENT} \wedge \neg\text{FEMALE} \\ s'(\text{BACHELOR}) = 1 & \text{iff} & s \models \neg(\text{FEMALE} \vee \text{MARRIED}) \end{array}$$

We can use the infomorphism f to move the natural logic Log(A) to B or to move the natural logic Log(B) to A. By our general results, we see that f[Log(A)] is sound on B and f^{-1}[Log(B)] is a complete logic on A.

The infomorphism f is *not* token surjective, however, because if a is one of the spurious truth assignments mentioned above, it is not in the range of f. Consequently, we do not expect f[Log(A)] to be complete. And, indeed, we see that

$$f(\text{MOTHER}) \nvdash_{f[\text{Log}(A)]} f(\neg\text{BACHELOR}),$$

whereas

$$f(\text{MOTHER}) \vdash_{\text{Log}(B)} f(\neg\text{BACHELOR}).$$

Similarly, because f is not token surjective, we know that f^{-1}[Log(B)] cannot be sound, because the normal tokens of this logic are just those in the range of f. Note, however, that these are exactly the nonspurious truth assignments. In other words, the normal tokens of f^{-1}[Log(B)] are just those tokens that represent genuine possibilities.

13.4 More About Moving Logics

In this section we collect together some simple but useful observations about the operations of taking images and inverse images of local logics.

Proposition 13.13. *Let $f : A \rightleftarrows B$ and $g : B \rightleftarrows C$ be infomorphisms.*

1. *For any logic $\mathfrak{L}$ on A, $gf[\mathfrak{L}] = g[f[\mathfrak{L}]]$.*
2. *For any logic $\mathfrak{L}$ on C, $(gf)^{-1}[\mathfrak{L}] = f^{-1}[g^{-1}[\mathfrak{L}]]$.*

Proof. The proof is immediate from the definitions. □

Proposition 13.14. *The operations of taking images and inverse images of logics are both order-preserving (with respect to $\sqsubseteq$).*

Proof. Let $f : A \rightleftarrows B$ be an infomorphism. We need to show the following:

1. For logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$ on A, if $\mathfrak{L}_1 \sqsubseteq \mathfrak{L}_2$ then $f[\mathfrak{L}_1] \sqsubseteq f[\mathfrak{L}_2]$.
2. For logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$ on B, if $\mathfrak{L}_1 \sqsubseteq \mathfrak{L}_2$ then $f^{-1}[\mathfrak{L}_1] \sqsubseteq f^{-1}[\mathfrak{L}_2]$.

For (1), note that by Theorem 13.7, f is a logic infomorphism from $\mathfrak{L}_2$ to $f[\mathfrak{L}_2]$ and so is also a logic infomorphism from $\mathfrak{L}_1$ to $f[\mathfrak{L}_2]$ by Proposition 12.28. By Theorem 13.7 again, $f[\mathfrak{L}_1]$ is the smallest logic making f a logic infomorphism from $\mathfrak{L}_1$, and so $f[\mathfrak{L}_1] \sqsubseteq f[\mathfrak{L}_2]$. Part (2) is proved similarly. □

Proposition 13.15. *The operations of taking images and inverse images of logics preserves joins in the $\sqsubseteq$ ordering.*

Proof. Let $f : A \rightleftarrows B$ be an infomorphism. We need to show the following:

1. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are both logics on A, then $f[\mathfrak{L}_1 \sqcup \mathfrak{L}_2] = f[\mathfrak{L}_1] \sqcup f[\mathfrak{L}_2]$.
2. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are both logics on B, then $f^{-1}[\mathfrak{L}_1 \sqcup \mathfrak{L}_2] = f^{-1}[\mathfrak{L}_1] \sqcup f^{-1}[\mathfrak{L}_2]$.

For (1), first note that by Corollary 13.14, $f[\mathfrak{L}_1] \sqsubseteq f[\mathfrak{L}_1 \sqcup \mathfrak{L}_2]$ and $f[\mathfrak{L}_2] \sqsubseteq f[\mathfrak{L}_1 \sqcup \mathfrak{L}_2]$. Therefore $f[\mathfrak{L}_1] \sqcup f[\mathfrak{L}_2] \sqsubseteq f[\mathfrak{L}_1 \sqcup \mathfrak{L}_2]$. For the other direction, note that by Theorem 13.7, f is a logic infomorphism from $\mathfrak{L}_i$ to $f[\mathfrak{L}_i]$, for $i = 1, 2$, and so also from $\mathfrak{L}_i$ to $f[\mathfrak{L}_1] \sqcup f[\mathfrak{L}_2]$, by Proposition 12.28. Thus by

Proposition 12.29, f is a logic infomorphism from $\mathfrak{L}_1 \sqcup \mathfrak{L}_2$ to $f[\mathfrak{L}_1] \sqcup f[\mathfrak{L}_2]$, and so by Theorem 13.7 again, $f[\mathfrak{L}_1 \sqcup \mathfrak{L}_2] \sqsubseteq f[\mathfrak{L}_1] \sqcup f[\mathfrak{L}_2]$. The proof of (2) is similar. □

Corollary 13.16. *Let $f : A \rightleftarrows B$ be an infomorphism.*

1. *For any logic $\mathfrak{L}$ on A, $\mathfrak{L} \sqsubseteq f^{-1}[f[\mathfrak{L}]]$.*
2. *For any logic $\mathfrak{L}$ on B, $f[f^{-1}[\mathfrak{L}]] \sqsubseteq \mathfrak{L}$.*

Proof. For (1), by Theorem 13.7, $f^{-1}[f[\mathfrak{L}]]$ is the largest logic on A such that f is a logic infomorphism from $f^{-1}[f[\mathfrak{L}]]$ to $f[\mathfrak{L}]$. But also by Theorem 13.7, f is a logic infomorphism from $\mathfrak{L}$ to $f[\mathfrak{L}]$, and so the result follows. The proof of (2) is similar. □

Corollary 13.17. $\mathfrak{L}_1 + \mathfrak{L}_2 = \sigma_{\mathfrak{L}_1}[\mathfrak{L}_1] \sqcup \sigma_{\mathfrak{L}_2}[\mathfrak{L}_2]$.

Proof. By Theorem 13.7, $\sigma_{\mathfrak{L}_1}[\mathfrak{L}_1] \sqsubseteq \mathfrak{L}_1 + \mathfrak{L}_2$ and $\sigma_{\mathfrak{L}_2}[\mathfrak{L}_2] \sqsubseteq \mathfrak{L}_1 + \mathfrak{L}_2$ because $\sigma_{\mathfrak{L}_1} : \mathfrak{L}_1 \rightleftarrows \mathfrak{L}_1 + \mathfrak{L}_2$ and $\sigma_{\mathfrak{L}_2} : \mathfrak{L}_2 \rightleftarrows \mathfrak{L}_1 + \mathfrak{L}_2$ are logic infomorphisms. Thus $\sigma_{\mathfrak{L}_1}[\mathfrak{L}_1] \sqcup \sigma_{\mathfrak{L}_2}[\mathfrak{L}_2] \sqsubseteq \mathfrak{L}_1 + \mathfrak{L}_2$.

For the other inequality, we must check that every normal token of $\sigma_{\mathfrak{L}_1}[\mathfrak{L}_1] \sqcup \sigma_{\mathfrak{L}_2}[\mathfrak{L}_2]$ is a normal token of $\mathfrak{L}_1 + \mathfrak{L}_2$ and that if $\Gamma \vdash_{\mathfrak{L}_1+\mathfrak{L}_2} \Delta$, then $\Gamma \vdash_{\sigma_{\mathfrak{L}_1}[\mathfrak{L}_1] \sqcup \sigma_{\mathfrak{L}_2}[\mathfrak{L}_2]} \Delta$. This is sufficient, because the classification of both sides is just $\mathrm{cla}(\mathfrak{L}_1) + \mathrm{cla}(\mathfrak{L}_2)$.

Suppose $\langle a, b\rangle$ is normal in $\sigma_{\mathfrak{L}_1}[\mathfrak{L}_1] \sqcup \sigma_{\mathfrak{L}_2}[\mathfrak{L}_2]$. Then it is normal in both $\sigma_{\mathfrak{L}_1}[\mathfrak{L}_1]$ and $\sigma_{\mathfrak{L}_2}[\mathfrak{L}_2]$, so a is normal in $\mathfrak{L}_1$ and b is normal in $\mathfrak{L}_2$. But $N_{\mathfrak{L}_1+\mathfrak{L}_2} = N_{\mathfrak{L}_1} \times N_{\mathfrak{L}_2}$, and so $\langle a, b\rangle$ is normal in $\mathfrak{L}_1 + \mathfrak{L}_2$.

Now suppose that $\Gamma \vdash_{\mathfrak{L}_1+\mathfrak{L}_2} \Delta$. We can write $\langle \Gamma, \Delta\rangle$ uniquely as

$$\langle \sigma_{\mathfrak{L}_1}\Gamma_1 \cup \sigma_{\mathfrak{L}_2}\Gamma_2, \sigma_{\mathfrak{L}_1}\Delta_1 \cup \sigma_{\mathfrak{L}_2}\Delta_2 \rangle,$$

where $\Gamma_1, \Delta_1 \subseteq \mathrm{typ}(\mathfrak{L}_1)$ and $\Gamma_2, \Delta_2 \subseteq \mathrm{typ}(\mathfrak{L}_2)$. So either $\Gamma_1 \vdash_{\mathfrak{L}_1} \Delta_1$ or $\Gamma_2 \vdash_{\mathfrak{L}_2} \Delta_2$, by the definition of $\mathfrak{L}_1 + \mathfrak{L}_2$. Thus $\sigma_{\mathfrak{L}_1}\Gamma_1 \vdash_{\sigma_{\mathfrak{L}_1}[\mathfrak{L}_1]} \sigma_{\mathfrak{L}_1}\Delta_1$ or $\sigma_{\mathfrak{L}_2}\Gamma_2 \vdash_{\sigma_{\mathfrak{L}_2}[\mathfrak{L}_2]} \sigma_{\mathfrak{L}_2}\Delta_2$. In either case, $\Gamma \vdash_{\sigma_{\mathfrak{L}_1}[\mathfrak{L}_1] \sqcup \sigma_{\mathfrak{L}_2}[\mathfrak{L}_2]} \Delta$ by weakening. □

Corollary 13.18. *Given an indexed family $\{f_i : \mathfrak{L}_i \rightleftarrows \mathfrak{L}\}_{i \in I}$ of logic infomorphisms,*

$$\sum_{i\in I} f_i \left[\sum_{i\in I} \mathfrak{L}_i\right] = \bigsqcup_{i\in I} f_i[\mathfrak{L}_i].$$

Proof. This is a straightforward computation given the above results:

$$\begin{aligned}
\left(\sum_{i\in I} f_i\right)\left[\sum_{i\in I}\mathfrak{L}_i\right] &= \left(\sum_{i\in I} f_i\right)\left[\bigsqcup_{i\in I}\sigma_{\mathfrak{L}_i}[\mathfrak{L}_i]\right] && \text{(Cor. 13.17)}\\
&= \bigsqcup_{i\in I}\left(\sum_{i\in I} f_i\right)[\sigma_{\mathfrak{L}_i}[\mathfrak{L}_i]] && \text{(Cor. 13.15)}\\
&= \bigsqcup_{i\in I}\left(\sum_{i\in I} f_i\right)\sigma_{\mathfrak{L}_i}[\mathfrak{L}_i] && \text{(Prop. 13.13)}\\
&= \bigsqcup_{i\in I} f_i[\mathfrak{L}_i]
\end{aligned}$$

The final identity follows from the definition of $\sum_{i\in I} f_i$. □

Proposition 13.19.

$$\left(\sum_{i\in I} f_i\right)^{-1}[\mathfrak{L}] = \sum_{i\in I}\left(f_i^{-1}[\mathfrak{L}]\right).$$

Proof. The proof is similar to that of 13.18. □

Exercises

13.1. Characterize the *a priori* logic AP(A) on A by moving the logic $\mathfrak{o}$ of Exercise 12.4.

13.2. Given an infomorphism $f : A \rightleftarrows B$ and a local logic $\mathfrak{L}$ on A, show that if f is surjective on types, then for all sequents $\langle \Gamma, \Delta\rangle$ of B, $\Gamma \vdash_{f[\mathfrak{L}]} \Delta$ if and only if $f^{-1}[\Gamma] \vdash_{\mathfrak{L}} f^{-1}\Delta$. Give an example showing that the surjectivity condition is necessary.