Information Sciences 709 (2025) 122078

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Toward rough set based insightful reasoning in intelligent systems

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ARTICLE INFO

Keywords:
Artificial intelligence
Granular computing
Rough sets
Granular approximation process
Reasoning over granular computations

ABSTRACT

This paper explores a rough set-based approach for supporting insightful reasoning in Intelligent Systems (ISs). The novelty lies in the introduction of a new concept for approximate reasoning processes based on granular computations. Although many rough set theory extensions developed over time focus on reasoning about (partial) set inclusion, these approximation spaces sometimes fall short when dealing with crucial aspects of approximate reasoning within ISs. Specifically, these systems aim to construct high-quality approximations of compound decision granules that represent solutions. Here, we present the basis for insightful reasoning realized through approximate reasoning processes grounded in granular computations. By doing so, we provide a sufficiently rich basis for designing IS problem solvers. This basis allows ISs to restructure or adapt their reasoning based on the generated granular computations, ultimately leading to high-quality granular solutions.

1. Introduction

One can observe a growing interest in real-world applications of methods that support the design of various types of Intelligent Systems (ISs) (see, *e.g.*, [1]). In particular, this includes areas such as human-computer interaction (see, *e.g.*, [2]), various types of complex systems including systems of systems (see, *e.g.*, [3]), agentic systems (see, *e.g.*, [4]), digital twins (see, *e.g.*, [5]).

We discuss an extension of the rough set approach from the approximation of concepts [6,7] to the approximation of complex granules (c-granules, in short) which are targets of ISs. In this way, we emphasize the fundamental role of rough sets in the design of ISs. Those c-granules can be related to different tasks such as

- inducing the high quality classifiers or clustering algorithms for given sets of training data sets;
- preserving the high quality of the controlled object safeness as a behavioral invariant;
- constructing from available physical parts a robot making it possible to perform efficiently given tasks;
- discovering a chemical component with specific properties;
- constructing *trustworthy* ISs.

While a deeper analysis is certainly needed to make these tasks truly understandable by ISs, this highlights the connection between these tasks and the approximate reasoning processes which should be employed by ISs to generate solutions. This paper will later discuss how dialogues with experts and/or chatbots can aid in decomposing tasks into a form that is truly understandable for ISs.

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https://doi.org/10.1016/j.ins.2025.122078
Received 30 May 2024; Received in revised form 14 February 2025; Accepted 8 March 2025

Available online 12 March 2025

ISs construct solutions to problems in the form of diverse granules, ranging from classifiers, robots, and drugs. However, these seemingly disparate solutions share a common origin: they are all built from more fundamental granules such as clusters, patterns, new features, or even physical components such as robot parts or drug compounds. This construction process is facilitated by the approximate reasoning capabilities of ISs, which allow them to create both abstract and physical granules. These granules serve as essential computational building blocks for cognition,[1] enabling ISs to understand the perceived situation on the way to solutions.

When inducing learning algorithms and generate using them classifiers, approximate reasoning techniques should aid in, *e.g.*, the development of parameterized learning algorithms and the optimization of their parameters.

It is important to note that in fields like robot design or drug discovery, positive cases (solutions) are often not readily available at the outset. In these scenarios, approximate reasoning processes should support the search for high-quality solutions through a combination of unsuccessful trials, dialogues with domain experts, interactions with domain knowledge bases, and interactions with the physical world for performing testing experiments.

We are aiming to develop the fundamental principles for designing ISs dealing with complex phenomena. To achieve this, we leverage the experience gained from successful and important ISs applications in the aforementioned domains.

One can see that in each of the above mentioned applications, ISs should construct objects, called here c-granules, satisfying to a satisfactory degree a given specification. This can be expressed by requirement that the generated granules should belong, *e.g.*, to the lower approximation of the concept consisting granules representing approximate solutions. Hence, ISs should be equipped with advanced reasoning methods for performing approximate reasoning processes along which such complex granules may be constructed during generation of granular computations. In developing models of such approximate reasoning processes, one should answer several important questions. Among them are the following ones:

- How are the target complex granules specified?
- How are the target complex granules processed by the ISs?
- What are the objects (c-granules) over which approximation reasoning processes run?
- How are approximations of c-granules defined and constructed along the reasoning performed by ISs? How are calculi of granules at different hierarchical levels defined and/or discovered? On what kind of reasoning are strategies based for the discovery of relevant calculi of granules? What are the basic steps of reasoning that support the construction of high-quality solutions of the given problem from a given c-granule g representing the specification?
- What are the methods of reasoning over granular computations generated by ISs aimed at steering them towards solutions satisfying given specifications to a satisfactory degree? Which methods can provide insightful reasoning for the control of ISs?
- What kind of quality measures should be used for estimating the quality of granules in the construction of solutions (*e.g.*, classifiers, clusters, plans, and other complex abstract and physical objects) evaluated along the approximate reasoning processes?
- What kind of computational model should ISs use to guarantee the feasibility of constructing high quality approximate solutions realized in the physical world?
- What kind of testing methods should ISs use to verify the quality of the provided solutions?

This paper focuses on ISs that handle complex phenomena. When solving problems with such ISs, two key issues arise:

1. the appropriate computing model,
2. the approximate reasoning processes that guide computations within this model.

These problems involve ISs generating computations that lead to solutions within a lower approximation of a concept encompassing all solutions to a given problem. This formally expresses that the generated solution is of high quality or sufficiently satisfactory. We aim to discuss these two critical issues. This exploration lays the groundwork for addressing these aforementioned issues and further establishes the foundation for this computing model.

We propose to base the approach on Interactive Granular Computing (IGrC) model (see, *e.g.*, [8–10] and the papers on https://dblp.org/pid/s/AndrzejSkowron.html) based on c-granules, which allow to link abstract and physical objects as opposed to information granules in Granular Computing (GrC) (see, *e.g.*, [11–13]). IGrC aims to synchronize issues of reasoning, language, and action and perception, as proposed in [14] for Turing test. It should be noted that information granules, closed in the abstract space, can be considered as a special case of (interactive) c-granules. One can restrict considerations to information granules assuming that formally correct, *e.g.*, information storage, retrieval, and reasoning algorithms function correctly (according to a pre-defined specification) when implemented in computer hardware (excluding environmental interaction, damage, etc.).

A critical aspect of c-granules lies in their interactions. Understanding complex systems of any kind hinges on this concept [15]. Within the IGrC framework, interactions are assumed to occur in the physical world. By generating relevant c-granule configurations, ISs can:

- Perceive certain object properties from these configurations and their interactions.
- Represent these properties within the informational layer of the IS control (c-granules).

[1] https://people.seas.harvard.edu/~/researchinterests.htm.

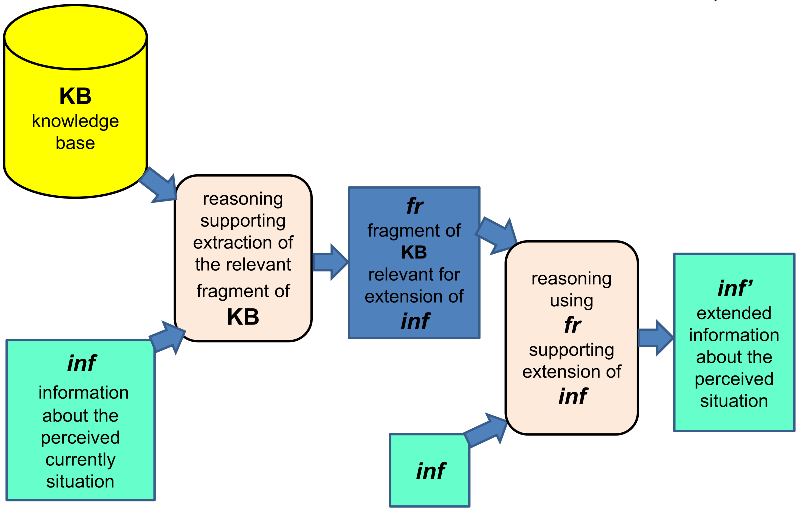

Fig. 1. Reasoning performed on information granules, including the knowledge base KB and information about the current situation (inf) perceived in the physical world.

Therefore, in IGrC, interactions between c-granules serve as the primary mechanism governing the dynamics of both individual c-granules and their more complex forms, such as networks of c-granules.

Reasoning processes realized by IS control are based on networks of c-granules, the fundamental objects of IGrC. Consider an illustrative example. Fig. 1 shows the reasoning performed on information granules, including the knowledge base (denoted in this figure by KB). In the first step, based on information about the current situation (inf) perceived in the physical world, the control aims to extract the relevant piece (fr) from KB that could be used to extend inf in a useful way. This requires a proper knowledge representation in KB to support the control's efficient extraction of fr. Next, using inf and fr, the control reasons to return an extended version of the information about the perceived situation, denoted by inf'. This extension is expected to be helpful, *e.g.*, in resolving conflicts between control rules that match inf'. Due to its generality, this type of reasoning may require decomposition through several levels before the control can realize it using accessible information granules.

The control of ISs should provide tools to generate the appropriate granular computations over such granules. We outline an approach for such control of ISs. Special attention is paid to reasoning methods performed by ISs over c-granules, which allow ISs to control granular computations towards the construction of high-quality approximate solutions. In the paper, we outline the discussed approach, emphasizing how the existing methods of concept approximation in the rough-set approach should be extended to deal with this much broader context in which ISs should operate. In particular, we emphasize the need to use the IGrC model with granules not only embedded in the abstract (mathematical) world but with granules connecting the abstract and physical worlds. The proposed approach allows us to use in the designed ISs reasoning methods that in a much deeper way than the existing rough set methods make it possible to analyze and control the construction of approximate solutions of high quality. This approach is a step towards the development of insightful reasoning for ISs based on rough sets. In particular, the discussed approach can be further developed towards reasoning methods that support changing solution search strategies by allowing a deeper view on some aspects of the problem to be solved. This is consistent with the [16] view of insightful reasoning:

> *To* [...] *this basic definition theorists often add the requirement that the problem solver has to restructure or change his or her thinking about some aspect of the problem or the solution in order to achieve insight.*

Fig. 2 illustrates a paradigm shift in approximation from the Pawlak model to the model presented in this paper. Approximation of concepts in the Pawlak model involves concepts that are represented as subsets of a given finite universe of objects. These concepts are approximated by so-called definable sets, which are unions of indiscernibility classes derived from an equivalence relation within a given approximation space. A fundamental goal of IS is to provide high-quality approximate solutions to the problems that IS addresses. These solutions are constructed along granular computations guided by IS control. This process is facilitated by advanced reasoning techniques and/or expert collaboration. The IS control behavior is based on adaptive models of so-called complex games (see Section 6), composed of rules that dictate necessary transformations within the current state of granular computation when relevant conditions are satisfied (to a satisfactory degree). The quality of approximate solutions is evaluated using appropriate quality measures. A primary objective is to generate, within the space of approximate solutions for a given problem, examples of approximate solutions definitively (with certainty) belonging to the concept of high-quality solutions, *i.e.*, the lower approximation of this concept. It is important to note that such examples are often not available in advance.

Examples of related projects.

Let's explore examples of related projects in more detail:

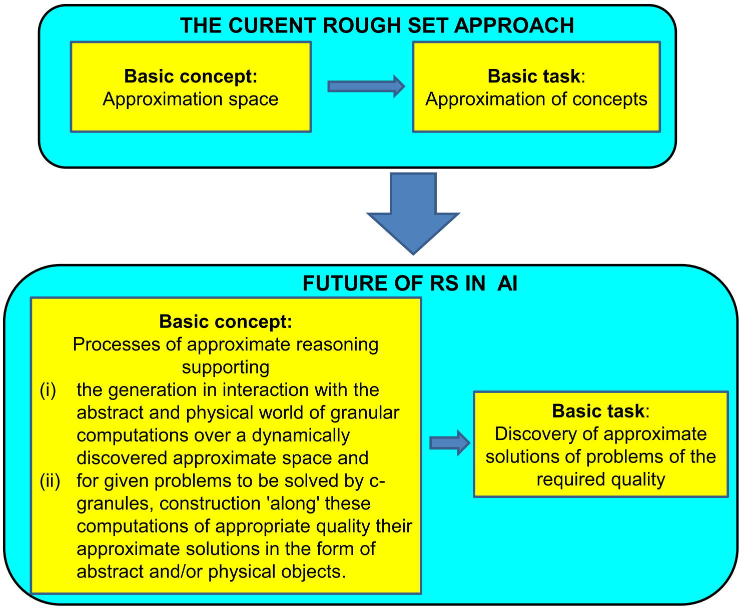

Fig. 2. From approximation of concepts to approximate solutions of problems.

- *Construction of learning algorithms and Classifiers* [17,18]: The ISs takes training data, quality requirements for the classifier, and databases containing information relevant to classifier construction (*e.g.*, search strategies) as input. The ISs then returns a classifier that meets the specified quality requirements. This can be seen as searching for a complex structure (granule) that satisfies a given specification to a satisfactory degree, as measured by the quality of the constructed classifier. In other words, the returned classifier should be an example of complex granule belonging (with the high degree) to the lower approximation of the concept *classifiers satisfying the given specification to a satisfactory degree*.
- *Automatic Design of Robots* [19]: The ISs takes databases of robot parts and a specification for the robot (*e.g.*, concerning desired tasks and performance metrics) as input. Then it returns a physical robot that meets the given specification to a satisfactory degree. Similarly to the first example, this corresponds to construction of a complex structure (c-granule) that satisfies the desired criteria. In this case, the returned robot is an example of c-granule from the lower approximation of the concept *robots satisfying the given specification to a satisfactory degree*.
- *Drug Discovery* [20]: The IS uses various relevant knowledge bases (e.g. medical and chemical data) and a specification for the desired drug (e.g. expected effects) as input. It then returns a drug that satisfactorily meets the specified requirements. Again, this means searching for a complex structure (granule) that meets the defined criteria. Here, the returned drug should belong (with a high degree) to the lower approximation of the concept *drug satisfying the given specification to a satisfactory degree*.

It is worthwhile to discuss more projects [21–23] closely related to the approach proposed in this work. These include two supervised PhD theses [21,23] and one commercial project [22] in which the first author was involved. These projects aim to clarify the core concept of rough set generalization presented in this paper. The goal of these projects was to develop reasoning methods that support the IS control in constructing (dynamic) c-granules that belong with high quality to c-granules (given in the form of vague specifications) during granular computations facilitated by the IS control.

The aim of [21] was to develop a high-quality learning algorithm for imbalanced data. The approach is based on a generalization of approximation spaces [24]. Compared to the original Pawlak model, which constructs concept approximations through simple reasoning about the (partial) inclusion of specified granules around test cases into decision classes corresponding to approximated concepts, the approach in [21] requires an advanced analysis of training cases within these granules.

Essentially, for a given test case, a granule of relevant training cases is identified as its neighborhood using the k-nearest neighbors (k-NN) method (with a specialized distance measure and optimized k). In particular, a specific rule is then constructed for each training case in the neighborhood. The granules defined by the left-hand sides of these rules are *narrowed down*, and it is checked whether each training case in the neighborhood forms a subgranule with cases labeled with the same decision. If confirmed, this strengthens the argument for assigning the same decision label to the test case. The learning algorithm, called RIONIDA, is obtained after optimizing meta-parameters on carefully selected benchmark datasets. RIONIDA outperforms state-of-the-art learning algorithms. This description shows that RIONIDA was initially constructed by a human expert using parameters that were subsequently optimized.

The Labeling-in-the-Loop (LITL) project [22] takes a more advanced approach involving expert dialogues. LITL aims to develop automatic and semi-automatic systems for data labeling in large datasets using machine learning. It provides several methods developed through expert interaction, including: (i) active learning sample selection, (ii) expert label assignment, (iii) expert consensus for ground truth estimation in conflicting votes, (iv) expert quality estimation, and (v) identification of new classes. Unlike [21], LITL optimizes learning through various methods that address data-related challenges, such as case selection for learning and feature selection and extraction. Additionally, LITL incorporates methods to enhance the reliability and efficiency of expert collaboration with LITL. Rough set methods have proven valuable in LITL development [25,26]. LITL offers tools for query selection based on similarity, such as identifying cases dissimilar to previous ones or cases where the model is uncertain.

Compared to RIONIDA, LITL provides advanced methods for expert interaction to efficiently construct high-quality learning algorithms.

In the case of [23], the objective was to develop advanced methods for optimizing traffic lights in urban environments such as Warsaw. An advanced traffic simulator based on cellular automata was developed to achieve this goal. The aim was to control traffic lights to ensure a *smooth flow of vehicles*. While *smooth flow of vehicles* is admittedly a complex and ambiguous term, advanced metaheuristics were developed to learn high-quality control strategies. It is crucial to note that the number of potential traffic light configurations within a city is immense. The simulator must also be able to adapt to real-world changes caused by weather conditions, accidents or unexpected traffic incidents, and human communication. Therefore, a complex dynamic model (c-granule) implemented through an adaptive cellular automata simulator is necessary to achieve high-quality traffic light control. This challenge is crucial for the solution of the discussed problem and is related to the adaptive complex games discussed in the paper. (see Section 6).

In the paper, we outline the process of evolution of approximation spaces in the rough set approach since the first model introduced by Professor Pawlak. In particular, we emphasize that classification problems in machine learning require the use of approximation spaces with large sets of indiscernibility relations (defining partitions of the object universes) and relevant optimization techniques (e.g., based on Boolean reasoning) to generate (semi-)optimal indiscernibility relations (partitions of the object universes) for approximation of concepts. Next, for ISs dealing with complex phenomena, we claim that it is necessary to base their control on IGrC, with complex granules as the basic objects, and on insightful reasoning supporting the adaptation of approximation spaces, from which granular networks are generated as states of granular computations. Along these granular networks, computational building blocks are constructed for approximating vague concepts used as triggers for transformations (specifying necessary sensory measurements/actions and interactions with domain bases or physical laws). Moreover, we emphasize the role of new, very important for many applications of ISs, new approximation spaces with states created by (parts of) granular computations generated by control of IS, along which approximate solutions of problems to be solved by ISs are constructed.

One can observe a growing interest in real-world applications of methods that support the design of various types of ISs (see, *e.g.*, [1]). Further progress in these areas depends heavily on the development of foundations. For example, an opinion is presented in [5]:

> *Our findings indicate that there is a lack of in-depth modeling approaches regarding the digital twin, while many articles focus on the implementation and testing of the artificial intelligence component. The majority of publications do not demonstrate a virtual-to-physical connection between the digital twin and the real-world system. Further, only a small portion of studies bases their digital twin on real-time data from a physical system, implementing a physical-to-virtual connection*

The paper is a step towards building foundations based on IGrC for the design and analysis of such AI systems. In particular,

- This paper proposes an extension of rough set theory based on Interactive Granular Computing. This extension aims to provide a basis for the design of Intelligent Systems (ISs) that can handle complex phenomena.
- The extension emphasizes approximate reasoning processes that make use of complex granules, rather than relying solely on approximation spaces used so far. Along these reasoning processes, solutions to problems tackled by ISs are in interactive granular computation.
- The paper shows how interactive granular computations can be used to a basis for controlling ISs in their search for high-quality solutions.
- Ultimately, the goal is to provide a foundation for ISs that deal with complex phenomena.

It should be noted that building these foundations will also improve communication between different application domains, which have often used specific modeling jargon [27]. One can also observe that testing the quality of the IS based on the proposed approach requires realization in the physical world, not only in the abstract world.

This paper is structured as follows. In Section 2 we recall the Pawlak standard rough set model emphasizing granular spaces and reasoning used to approximate concepts. In Section 2.2 we present examples related to Machine Learning. These examples justify usefulness of inductive reasoning in approximate reasoning processes. In Section 3 we emphasize the role of reasoning supporting searching for optimal multirelational approximation spaces. Section 4 is divided into two parts. In the first part we recall the discussion on uncertainty functions in approximation spaces introduced in [28]. In the second part, we discuss an extension of uncertainty functions to associations providing links between abstract and physical objects perceived by ISs. In Section 5 we investigate two kinds of granular spaces. In Section 6 we discuss other very important aspects related to the control of ISs and supporting IT reasoning. The characterization of the idea of c-granules with control is described in Section 7. Section 8 concludes the paper.

2. Approximation processes for the Pawlak rough set model and its generalizations

In this section, we recall the Pawlak model [6,7] and using this model we distinguish some important components of approximation process related to it. In the abstract setting the approximation space was defined as a pair (U, R), where U is a finite set of objects and $R \subseteq U \times U$ is an equivalence (indiscernibility) relation. This relation defines a partition of U into equivalence classes called

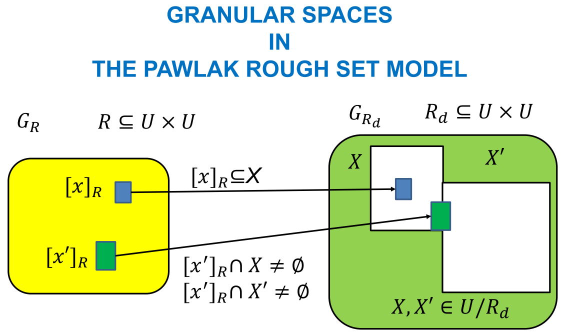

Fig. 3. Linking elementary granules from G_R to decision granules from G_{R_d}.

indiscernibility classes. These classes $[x]_R$ (for $x \in U$) are elementary granules. The aggregation operation on granules is defined producing new granules from families of elementary granules by taking the union of them. One should note that in the Pawlak model the universe U is treated as a given set and the issues how objects form this set are perceived in the physical world are outside of this model.

Together with the indiscernibility relation is considered a partition of U into decision classes (decision granules). Let us denote the corresponding equivalence relation to this partition by R_d. In this way we obtain a triple $AS = (U, R, R_d)$ called approximation space. For any decision granule $X \subseteq U$ is defined its lower approximation relative to R by $LOW(AS, X) = \{x \in U : [x]_R \subseteq X\}$[2] and the upper approximation of X by $UPP(AS, X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$. The set $Bd(AS, X) = UPP(AS, X) \setminus LOW(AS, X)$ is called the boundary region of X relative to R. If $Bd(AS, X) \neq \emptyset$ then X is called rough (relative to R), otherwise X is crisp (relative to R).

Hence, one can observe that in this model we deal with two granular spaces G_R and G_{R_d} corresponding to the partitions defined by relations R and R_d: $G_R = (\{[x]_R\}_{x \in U}, agg)$, where $agg : P(\{[x]_R\}_{x \in U}) \to P(U)$ and $agg(Z) = \bigcup Z$ for $Z \subseteq \{[x]_R\}_{x \in U}$ and $G_{R_d} = U/R_d$ where $U/R_d = \{[x]_{R_d}\}_{x \in U}$. The first component of G_R contains a family of elementary granules defined by R and the second component is an aggregation operation generating definable granules from elementary granules. In the case of G_{R_d}, we have only one component with the family of decision granules.

The granular spaces G_R, G_{R_d} are connected by an interface which contains some relations between granules from these two granular spaces as well as some reasoning methods which allow to find relevant relations between granules from these spaces. In the case of the discussed model, this interface has a simple form and is defined by

- the set theoretical inclusion operation between definable granules from G_R and granules from G_{R_d} and
- reasoning mechanism on which is based a searching heuristic for the maximal (relative to set inclusion) definable granules included into decision granules for the lower approximation of decision granules and minimal (relative to set inclusion) definable granules including decision granules for the upper approximation of decision granules.

On the basis of the reasoning mechanism mentioned above configuration (based on granular spaces and interface) is changing by adding new links between granules from G_R to G_{R_d} recording discovered by the used heuristic new inclusions and non-empty intersections of elementary granules from G_R into decision granules G_{R_d} (see Fig. 3).

The illustrative example in Fig. 3 shows a simple granular network that summarizes our previous considerations. The network is built from two granular spaces, G_R and G_{R_d}, which consist of elementary information granules (equivalence classes). These granules are defined by conditional attributes that determine the relation R in the universe of objects U. Additionally, a decision attribute determines the relation R_d in U. Both the description of these elementary information granules and their meaning (physical semantics) within the universe U are defined in an abstract (mathematical) space. The network itself is constructed from two sets of elementary granules, one from each space. Arrows connect these granules, indicating two types of links: inclusion or partial inclusion. In general, these links can represent more complex relationships between granules from different, more advanced granular spaces. For instance, information granules might contain details about the location and time of a measurement stored in their informational layers. They could also concern properties of various segments of multi-time series data or the aggregation of this data into clusters or more complex structural objects. Notably, these granules can also include specifications for so-called associations, which allow the control of ISs (c-granules) to generate their meaning in the physical world (see Section 4 for further details). Links can even represent relationships between different granules or their parts that are dependent on time.

In this way, granular computations have states representing granular networks and the discovered links. The IS control specifies the realized transitions between states. In the final state, all elementary granules are linked to decision granules and approximations of decision granules are computed. This simple example illustrates an important aspect of approximation processes running over structures defined by granular spaces and interfaces between them along which are constructed approximations of specified target

[2] We will write also $LOW(R, X)$ instead of $LOW(AS, X)$.

granules. In the discussed example decision granules play the role of target granules. In different applications different complex objects of the high quality required to be constructed, *e.g.*, learning algorithms, classifiers, clusters, complex physical or abstract objects, or the whole granular computations with the required properties.

The current state of approximation process is represented by granules from G_R and G_{R_d} which have been used in reasoning (and not yet forgotten) and already established links between granules from G_R to granules from G_{R_d} representing inclusion. For example, granules from G_R linked to at least two decision granules from G_{R_d} belong to the boundary region.

In applications, the relation R is defined by a finite set of attributes A, where any $a \in A$ is a function from U into a value set V_a. For any $x \in A$, $Inf_A(x)$ denotes the signature of x relative to A defined by $\{(a, a(x)) : a \in A\}$. For any $x, y \in U$ it is assumed that xRy if and only if $Inf_A(x) = Inf_A(y)$. Then R is denoted by R_A. In the considered case, the definition of granules is modified accordingly. Elementary granules are equivalence classes of R_A labeled by signatures of objects defining them, *i.e.*, for any $x \in U$ the equivalence class $[x]_{R_A}$ is labeled by $Inf_A(x)$. Instead of signature $Inf_A(x)$ one can consider the conjunction $\bigwedge_{a\in A} a = a(x)$. The semantics of $\bigwedge_{a\in A} a = a(x)$ is defined by $\| \bigwedge_{a\in A}(a = a(x)) \|_U = \{y \in U : a(y) = a(x) \text{ for } a \in A\}$. Obviously,

$$[x]_{R_A} = \| \bigwedge_{a\in A}(a = a(x)) \|_U .$$

Hence, the granule g_x corresponding to $x \in U$ can be treated as a pair $(\bigwedge_{a\in A}(a = a(x)), [x]_{R_A})$, with its syntax $\bigwedge_{a\in A}(a = a(x))$ and the semantics $[x]_{R_A}$. It is worthwhile mentioning that this approach is important when one would like to consider inductive extensions of approximation spaces. In this case the aggregation operation in G_{R_A} is restricted to unions of elementary granules defined by formulas obtained by dropping some conditions from Inf_A. Among such formulas, the computational building blocks for approximating decision concepts from G_{R_d} are selected. Any equivalence class $[x]_{R_A}$ is linked to a formula α_x obtained from $\bigwedge_{a\in A}(a = a(x))$ by dropping the maximal number of conditions under assumption that the semantics of α_x has non-empty intersection with exactly the same decision granules from G_d as the semantics of $\bigwedge_{a\in A} a = a(x)$. Then, *e.g.* for $X \in U/R_d$ its lower approximation is obtained by the union of semantics of all formulas α_x included in X. Hence, in this case we consider reasoning supporting linking the elementary granules $[x]_{R_A}$ from G_{R_A} with α_x and α_x with decision classes for $x \in U$.

The process of inducing approximations of decision granules can be realized on tolerance decision systems (for more information about tolerance rough sets see e.g. [28,29]). The three-way rough set based approach for approximation of decision granules in Intelligent Systems is discussed in [30].

An important aspect of the approximation is related to the quality of the resulting approximation of the decision granules from G_{R_d}. In the discussed model, the positive region of G_{R_d} relative to G_R, i.e.,

$$POS_{G_R}(G_{R_d}) = \bigcup_{X\in G_{R_d}} LOW(R, X)$$

is used to define the quality of approximation of G_{R_d} relative to G_R by

$$\gamma = \frac{|POS_{G_R}(G_{R_d})|}{|U|},$$

where $|U|$ denotes the cardinality of U.

However, in applications, the description length of the representation of the constructed approximations may also be important. Shorter descriptions may be preferred over longer from the point of view of readability or explainability of approximations. In this case, reasoning methods based on reduction of the length of representation may be applied. In particular, Boolean reasoning can be applied to support search for the minimal description of approximations of decision granules or classification defined by them. Efficient methods for generation of different kinds of reducts have been developed and used for this purpose [31–33]. In the discussed case, one can add a second component to the quality of approximation of decision granules, namely the component related to the description length.

The quality of approximation in the case of Pawlak's model is related to the given universe of objects U. This approach can be extended to the case when U is only a training sample and the derived approximations should be also extended on testing objects (cases) not seen so far, like in Machine Learning. This issue will be discussed shortly in one of the following sections. We would like to only stress here that in such a case one more component related to testing the derived models of approximation should be developed.

2.1. Generalization of the Pawlak model

Summarizing the discussed above example, we consider approximation issues in a much wider context than it was proposed in the Pawlak model. In the Pawlak model, an approximation space was defined as a pair of (U, R) and other issues discussed above were outside of formalization. In our discussion, we have distinguished some other important components on which the approximation process is based. Among them are the following ones:

- vague specifications of complex tasks to be solved represented by abstract (information) granules;
- associations – establishing links between abstract objects and physical objects, responsible for perception of situations (objects) in the physical world; this issue is elaborated more in papers concerning IGrC (see, *e.g.*, [8–10] and the papers on https://dblp.org/pid/s/AndrzejSkowron.html));

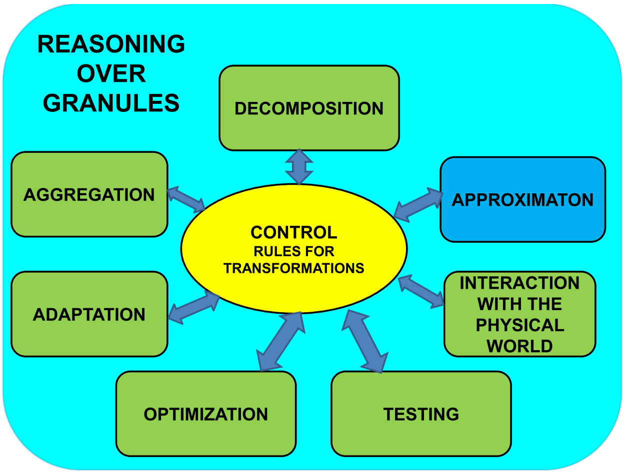

Fig. 4. Different components of reasoning in intelligent systems.

- networks of granular spaces linked by interfaces – making it possible to generate actual states representing information about the perceived situations together with other sources of information concerning *e.g.*, domain knowledge bases, sets of rules aiming to transform the current states, etc.;
- reasoning tools over generated granular computations or their states – aiming to control them toward obtaining the high quality solutions (represented by c-granules) such as classifiers, clusters, complex physical objects, plans etc.;
- methods for approximation of generated objects – based on reasoning over generated computations or their states;
- quality measures – making it possible to estimate the quality of the synthesized solutions;
- methods for testing the constructed solutions in the physical world.

In the sequel we present more examples to illustrate in more detail the idea concerning generalization of reasoning supporting approximation processes.

Fig. 4 presents key reasoning components of the IS control system. A central element is the rule set known as a complex game. Rules within a complex game consist of pairs: complex vague concept approximations (classifiers) and specifications for transformations to be applied to the current state of granular computation. Initial complex games of the IS control are developed in collaboration with experts or learned from available data. The implementational module (IM) of the IS control is responsible for executing these transformation specifications in the physical world. Special reasoning methods are used to select appropriate rules based on the currently perceived situation. If a transformation specification cannot be directly realized in the physical world by IM, it may need to be decomposed at several levels. This process is facilitated by dedicated decomposition reasoning methods. These methods are also crucial for decomposing vaguely defined problem specifications, a task often accomplished in collaboration with experts and chatbots. The complex game must adapt to perceived changes, and reasoning methods supporting this adaptation are essential. To comprehend the current physical situation, reasoning methods for aggregating and de-aggregating computational building blocks are utilized. Backtracking strategies are vital in case relevant blocks cannot be discovered. Reasoning methods are used to construct and optimize learning algorithms to approximate complex vague concepts. In addition, reasoning methods analyze the performance of the current complex game or computational building blocks to support adaptation. It is important to note that complex games, computational building blocks, the IS control system, and the IS itself can be viewed as c-granules of varying complexity. Reasoning methods employed for these diverse purposes can rely on deduction, induction, or abduction. Nevertheless, the input of experts is indispensable because it enables the use of their expertise in commonsense and experience-based reasoning, areas not yet fully addressed by formal methods. Furthermore, reasoning methods should be augmented by inference rules extracted from data (see, *e.g.*, [34]).

Fig. 5 shows a finite granular computation executed by the IS control (c-granule). Granular networks $N_1, N_2, \ldots, N_k$ represent different computational steps. These networks comprise families of c-granules constructed by the control system. Their informational layers contain various types of information, including: (i) spatio-temporal specifications of windows (addresses) w defining regions of physical space relevant to the physical realization of transformation specifications, (ii) $spec$ – specifications of transformations in parts of the physical space corresponding to w, (iii) $prop$ – perceived properties: information about physical objects within the scope of the realized transformation, acquired through measurements or inferred using knowledge from c-granules representing knowledge bases. This information may have compound structure composed of c-granule families from different granular spaces connected by relations expressing interdependencies between structural components (see Fig. 3 and more advanced example in Section 5.3). Transformation specifications $tr_1, tr_2, \ldots, tr_k$ are executed by IM in the physical world. These transformations may involve creating new c-granules, deleting existing ones, or suspending certain c-granules. These transformations are selected by IS control from the currently used set of rules.

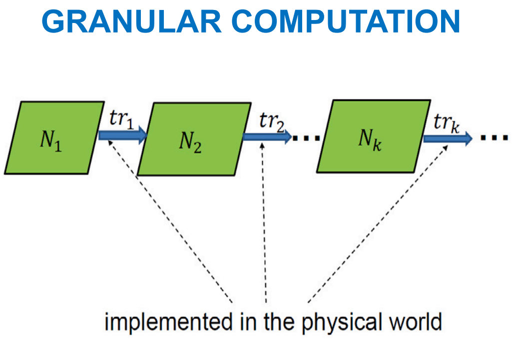

Fig. 5. Computation over granular networks $N_1, \ldots, N_k$ and transformations $tr_1, \ldots, tr_k$ realized in the physical world.

2.2. *Inductive reasoning in approximate processes*

In this subsection, we present examples from machine learning that justify the usefulness of inductive reasoning in classifier construction. We illustrate this with simple examples of rule-based and k-NN classifiers, but it can be observed that this applies to all other classifiers as well.

Let us consider an approximation space $AS^* = (U^*, R^*_A, R^*_d)$ and its restriction to a sample $U \subseteq U^*$, *i.e.* $AS = (U, R_A, R_d)$, where $R_A = R^*_A \cap (U \times U)$ and $R_d = R^*_d \cap (U \times U)$. We would like to inductively extend the approximations of decision granules from AS^*. This is especially true for objects x^* such that $[x^*]_A \cap U$ does not belong to U/R_A.

In the case of rule-based classifiers, we use formulas of the form α_x for $x \in U$ defined before. In fact we use decision rules of the form

$$\alpha_x \rightarrow d(x),$$

where $d(x)$ is the decision granule containing x. For a given $x^* \in U^* \setminus U$ we select all rules matching the description (relative to A) of this object x^* and we use a procedure for resolving possible conflicts between these rules.

In the case of the k-NN method, [18] the first granulation space over U^* is defined by neighborhoods for $x \in U^*$ consisting of the k-closest objects to x (relative to a given distance measure) from U. Next, in the simplest case, the majority decision for objects from this neighborhood is assigned to x. Many other advanced reasoning methods have been developed to predict decisions based on the distribution of decisions in neighborhoods. For example, in [21] decision-making for a given $x \in U^* \setminus U$ follows a certain process already characterized in discussion about projects in Section 1.

3. Optimization of approximate reasoning processes for multirelational granular spaces

In the literature different multirelational approximation spaces $(U, \{R_i\}_{i \in I})$ (where R_i is an equivalence relation over the finite universe U of objects and I is a set of indices) have been investigated. In this section, we want to emphasize the role of reasoning in supporting the search for optimal spaces. One can distinguish many different cases. Among them are the following ones where:

- I is a finite set;
- I is an infinite set;
- *reduction in case of finite set I*: searching for minimal subsets I' of a finite set I preserving discernibility of objects from U, *i.e.*, minimal subsets $I' \subseteq I$ such that

  $$\bigcap_{i \in I} R_i = \bigcap_{i \in I'} R_i;$$

  in this case Boolean reasoning methods were successfully used; this case is related to feature selection in Machine Learning;
- *optimal voting*: searching for the optimal aggregation operations of votes obtained on the basis of (U, R_i) ($i \in I$, where I is a finite set) relative to the quality of approximation of concepts over U; this case is related to Distributed Data Mining;
- *discovery strategies for sets of relations*: this case is related to feature extraction (or feature engineering) in Machine Learning and discovery of new sensors;
- *reduction in case of infinite case of I*: searching for minimal finite subsets of I preserving discernibility of objects from U, this case is related to feature selection from infinite sets of features in Machine Learning; reasoning is realized by Boolean reasoning, genetic or evolutionary algorithms;
- *discovery of new granular spaces with structured objects and attributes over such objects*: this case is related to hierarchical learning and federated learning; reasoning is supported, *e.g.*, by aggregation of information or decision systems.

3.1. *Optimization of parameterized approximation spaces and quality measures for approximate reasoning processes*

Approximation spaces can be treated as c-granules used for concept approximation. The approximation spaces discussed in this section (see, *e.g.*, [28], [29]) are some special parameterized relational structures. By tuning the parameters, it is possible to search for relevant approximation spaces relative to given concepts.

Definition 1. A parameterized approximation space is a system
$AS_{\#,\$} = \left(U, I_{\#}, \nu_{\$}\right)$, where

- U is a non-empty set of objects,
- $I_{\#} : U \rightarrow P(U)$ is an uncertainty function,
- $\nu_{\$} : P(U) \times P(U) \rightarrow [0,1]$ is a rough inclusion function,

where #, $ denote vectors of parameters (the indexes #, $ will be omitted if it does not lead to misunderstanding) and for every non-empty set U, let $P(U)$ denote the set of all subsets of U (the power set of U).

The lower and upper approximations of subsets of U are defined as follows.

Definition 2. For any approximation space $AS_{\#,\$} = \left(U, I_{\#}, \nu_{\$}\right)$ and any subset $X \subseteq U$, the lower and upper approximations are defined by

$$LOW\left(AS_{\#,\$}, X\right) = \left\{x \in U : \nu_{\$}\left(I_{\#}(x), X\right) = 1\right\},$$
$$UPP\left(AS_{\#,\$}, X\right) = \left\{x \in U : \nu_{\$}\left(I_{\#}(x), X\right) > 0\right\}.$$

Typical examples of uncertainty function are defined by $I_{\#}(x) = [x]_R$, where R is an equivalence or tolerance relation. Then parameters denoted by # are, *e.g.*, subsets of attributes from a given set of attributes and/or distance measures from a given family of such measures, used in the definition of R.

In case of uncertainty measure $\nu_{\$}$, let us consider the following example, where $ is a parameter defined by a threshold value $t \in (0, 0.5)$:

$$\nu_t(X,Y) = \begin{cases} 1 & \text{if} \quad \nu_{SRI}(X,Y) \geq 1-t \\ \frac{\nu_{SRI}(X,Y)-t}{1-2t} & \text{if} \quad t \leq \nu_{SRI}(X,Y) < 1-t \\ 0 & \text{if} \quad \nu_{SRI}(X,Y) \leq t \end{cases} \tag{1}$$

The parameter to be optimized is t and ν_{SRI} is the standard rough inclusion defined for $X, Y \subseteq U$ by:

$$\nu_{SRI}(X,Y) = \begin{cases} \frac{card(X \cap Y)}{card(X)} & \text{if} \quad X \neq \emptyset \\ 1 & \text{if} \quad X = \emptyset. \end{cases} \tag{2}$$

The rough inclusion function ν_t is used in the variable precision rough set approach [35].

Usually the optimization of the parameters #, $ should be done relative to the quality of the used approximation measure. One can observe that in a more general setting, this is related to the Minimum Description Length (MDL) principle, where also the description length of the approximation is considered, *i.e.*, it is required the proper balance between the quality of the approximation and the description length of the approximation, *i.e.*, this balance should be tuned. To illustrate, consider a family $ASF_{\#,\$}$ of approximation spaces $AS_{\#,\$}$ over the universe of objects U and an approximation measure quality $Quality_1$:

$$Quality_1 : ASF_{\#,\$} \times P(U) \rightarrow [0,1].$$

In the simplest case $Quality_1$ can be defined by:

Example 1. If $UPP(AS_{\#,\$}, X) \neq \emptyset$ for $AS_{\#,\$} \in ASF_{\#,\$}$ and $X \subseteq U$ then

$$Quality_1(AS_{\#,\$}, X) = \nu_{SRI}(UPP(AS_{\#,\$}, X), LOW(AS_{\#,\$}, X)).$$

In applications, we usually use another quality measure analogous to the MDL principle, where also the description length of approximation is included. Let us denote by $description(AS_{\#,\$}, X)$ the description length of approximation of $X \subseteq U$ in $AS\_\#, \$$.

Example 2. The description length may be measured, *e.g.*, by the sum of description lengths of algorithms testing membership for neighborhoods used in construction of the lower approximation, the upper approximation, and the boundary region of the set X. Then the quality $Quality_2(AS_{\#,\$}, X)$ can be defined by

$$Quality_2(AS_{\#,\$}, X) = g(Quality_1(AS_{\#,\$}, X), description(AS_{\#,\$}, X)),$$

where g is a relevant function used for fusion of values $Quality_1(AS_{\#,\$}, X)$ and $description(AS_{\#,\$}, X)$. This function g can represent weights given by experts relative to both criteria and can be defined, *e.g.*, by $g(r, r') = \lambda r + (1-\lambda) r'$, for $r, r' \in [0,1]$ and a weight λ to be tuned.

Table 1
Sample decision system, indiscernibility classes $[x]_{IND(A)} = I_A(x)$ for attribute set $A = \{a, b, c\}$ and generalized decision $\delta_A(x)$, respectively.

	a	b	c	d	$[x]_{IND(A)}$	$\delta_A(x)$
x_1	0	0	2	2	$\{x_1\}$	$\{2\}$
x_2	0	1	2	2	$\{x_2\}$	$\{2\}$
x_3	0	2	1	2	$\{x_3, x_4\}$	$\{1,2\}$
x_4	0	2	1	1	$\{x_3, x_4\}$	$\{1,2\}$
x_5	1	0	2	2	$\{x_5\}$	$\{2\}$
x_6	1	1	0	2	$\{x_6, x_7\}$	$\{1,2\}$
x_7	1	1	0	1	$\{x_6, x_7\}$	$\{1,2\}$
x_8	1	2	1	1	$\{x_8\}$	$\{1\}$
x_9	2	0	1	2	$\{x_9\}$	$\{2\}$
x_{10}	2	1	0	2	$\{x_{10}, x_{11}\}$	$\{1,2\}$
x_{11}	2	1	0	1	$\{x_{10}, x_{11}\}$	$\{1,2\}$
x_{12}	2	2	2	1	$\{x_{12}\}$	$\{1\}$
x_{13}	-1	0	1	3	$\{x_{13}\}$	$\{3\}$
x_{14}	-1	1	2	3	$\{x_{14}\}$	$\{3\}$

Table 2
Discernibility matrix for sample decision system.

		x_4		x_8		x_{11}		x_{14}
x_1		b, c		a, b, c		a, b, c		a, b
x_2	…	b, c	…	a, b, c	…	a, c	…	a
x_3		-		a		a, b, c		a, b, c
x_4		-		-		-		a, b, c
x_5				b, c		a, b, c		a, b
x_6				b, c		a		a, c
x_7				-		-		a, c
x_8	…		…	-	…	-	…	a, b, c
x_9						b, c		a, b, c
x_{10}						-		a, c
x_{11}						-		a, c
x_{12}								a, b
x_{13}								-
x_{14}								-

3.2. Illustrative example

Let us consider an illustrative example related to the concepts discussed so far.

Example 3. In this example, we illustrate the definitions and formulas used in the paper. Let $DS = (U, A, d)$ be a decision system, where the set of objects $U = \{x_1, \dots, x_{14}\}$, $d : U \to V_d$ is the decision attribute, and the set of conditional attributes $A = \{a, b, c\}$ (see Table 1). The set V_d of decision values is equal to $\{1, 2, 3\}$ and $\delta_A : U \to P(\{1, 2, 3\})$ is a generalized decision function defined by $\delta_A(x) = d([x]_{IND(A)})$ (see the column δ_A in Table 1).

DS is the decision system with decision classes $\{X_1, X_2, X_3\}$ creating a partition of U, where $X_1 = \{x_4, x_7, x_8, x_{11}, x_{12}\}$, $X_2 = \{x_1, x_2, x_3, x_5, x_6, x_9, x_{10}\}$ and $X_3 = \{x_{13}, x_{14}\}$. Based on the column $[x]_{IND(A)}$ from Table 1 we obtain that the lower approximation $LOW_A(X_1)$ of X_1 with respect to A is equal to $\{x_8, x_{12}\}$ and the lower approximation $LOW_A(X_2)$ of X_2 is equal to $\{x_1, x_2, x_5, x_9\}$. The upper approximation $UPP_A(X_1)$ of X_1 with respect to the attribute set A is equal to $\{x_3, x_4, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}\}$ and the upper approximation $UPP_A(X_2)$ of X_2 is equal to $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}\}$. The generalized decision $\delta_A : U \to P(\{1, 2, 3\})$ partitions the universe U of objects as follows:
$PART_{DS} = \{\delta_A^{-1}(\{1\}), \delta_A^{-1}(\{2\}), \delta_A^{-1}(\{3\}), \delta_A^{-1}(\{1, 2\})\} =$
$\{\{x_8, x_{12}\}, \{x_1, x_2, x_5, x_9\}, \{x_{13}, x_{14}\}, \{x_3, x_4, x_6, x_7, x_{10}, x_{11}\}\}$. All elements of $PART_{DS}$ are definable sets (relative to A).

For sample decision system presented in Table 1, we construct discernibility matrix [32] $(DM_{i,j})_{14\times 14}$, where

$$DM_{i,j} = \{at \in A : at(x_i) \neq at(x_j) \ \& \ d(x_i) \neq d(x_j)\}$$

(see Table 2).

The discernibility function $dis\_fun(a, b, c)$ is defined by

$$\bigwedge_{1 \le i < j \le 14, DM_{i,j} \neq \emptyset} (\bigvee DM_{i,j})$$

Thus, after simplification we obtain $dis\_fun(a, b, c) = (b \vee c) \wedge a$. Hence, we obtain two decision reducts $\{a, b\}$ and $\{a, c\}$.

The lower approximation of X_2 is equal to $\{x_1, x_2, x_5, x_9\}$ and we obtain, for example, the following certain decision rules:

$$b = 0 \ \& \ c = 2 \rightarrow \delta_A = \{2\}, \quad \{x_1, x_5\},$$

$$a = 0 \ \& \ c = 2 \rightarrow \delta_A = \{2\}, \quad \{x_1, x_2\},$$

$$a = 2 \ \& \ b = 0 \rightarrow \delta_A = \{2\}, \quad \{x_9\}.$$

The boundary region of X_2 is equal to

$$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}\} \setminus \{x_1, x_2, x_5, x_9\} = \{x_3, x_4, x_6, x_7, x_{10}, x_{11}\}$$

and we obtain, for example, the following rough decision rules:

$$c = 0 \rightarrow \delta_A = \{1, 2\}, \quad \{x_6, x_7, x_{10}, x_{11}\},$$

$$a = 0 \ \& \ b = 2 \rightarrow \delta_A = \{1, 2\}, \quad \{x_3, x_4\}.$$

4. Interaction with the physical world: from uncertainty functions to associations

This section is divided into two parts. In the first part, we recall the discussion about uncertainty functions in approximation spaces introduced in [28]. In the second part, we discuss an extension of uncertainty functions to associations providing links between abstract and physical objects perceived by ISs. For details of the discussion on the issues concerning the second part the reader is referred to papers on IGrC (see, *e.g.*, [8–10] and the papers on https://dblp.org/pid/s/AndrzejSkowron.html). Here, we limit our discussion to issues related to the physical realization of associations.

In the first part, we consider an approximation space $AS_{\#,\$}$. The uncertainty function $I_\#$ in this space defines for every object $x \in U$, a set of objects described similarly to x. The set $I_\#(x)$ is called the neighborhood of x (see, *e.g.*, [28]).

We assume that the values of the uncertainty function are defined using a *sensory environment*, i.e., a pair $(L, \|\cdot\|_U)$, where L is a set of formulas, called the *sensory formulas*, and $\|\cdot\|_U : L \longrightarrow P(U)$ is the *sensory semantics*. We assume that for any sensory formula α and any object $x \in U$ is available the information if $x \in \|\alpha\|_U$ holds. The set $\{\alpha : x \in \|\alpha\|_U\}$ is the *signature of* x over L. In $AS_{I_{\#,\$}}$ we restrict discussion to a finite set of attributes $A \subseteq L$ and the signature of x relative to A is denoted by $Inf_A(x)$.[3] For any $x \in U$ the *set* $\mathcal{N}_{AS_A(x)}$ of *neighborhoods of* x in $AS_{\#,\$}$ is defined by $\{\|\alpha\|_U : x \in \|\alpha\|_U \ \&\alpha \in A\}$ and from this set the neighborhood $I_\#(x)$ is constructed. For example, $I_\#(x)$ is defined by selecting an element from the set $\{\|\alpha\|_U : x \in \|\alpha\|_U \&\alpha \in A\}$ or the neighborhood is defined by $I_\#(x) = \bigcap \mathcal{N}_{AS_{\#,\$}}(x)$.

Note that each sensory environment $(L, \|\cdot\|_U)$ defines an information system with the universe U of objects. Any row of such an information system for an object x consists of binary information if $x \in \|\alpha\|_U$ holds, for any sensory formula α. We already presented examples of uncertainty functions in information systems with parameters to be optimized.

Note that any sensory environment $(L, \|\cdot\|_U)$ can be treated as a parameter of the search for the relevant finite subset A of L of the approximation space. This question is related to the discovery of the language of attributes (features) from which the relevant sets of attributes are selected.

In the second part of this section, we outline the problems associated with interpreting objects from the universe of objects U (or its extension U^*) of information system and physical realization of associations defining attributes. Contrary to the previous case, we would like to consider associations as non-purely mathematical objects linking abstract objects with the physical objects to be perceived by ISs. Hence, as objects in the considered information systems one can consider triplets (or their sequences)

$$(w, task_{spec}, inf),$$

where w is a specification of spatio-temporal window (address) describing a part of the physical space on which the task specified by $task_{spec}$ (together with the expected results) should be realized and inf consists of information perceived (so far) on realization of the association by ISs. It is assumed that ISs have the ability to create the relevant configuration of physical objects based on the specification w, to initiate interactions between them, and to perceive properties of these objects and their interactions that lead to information that can be treated as the value of the association for $(w, task_{spec}, inf)$. For example, $task_{spec}$ may specify requirement related to performing some measurements in the region of the physical world specified by w by sensor like camera or thermometer. For more details, the reader is referred to the papers related to IGrC. Here, we would like to emphasize problems related to reasoning for ISs related to associations and their physical realization. Among them there are problems, *e.g.*, emphasized in Data Science related to data governance. The reasoning tools of ISs should provide information

what, where, how, when

to search for new data toward satisfying the specification of the realized tasks. It should be noted that the discussed utilities of ISs are very important especially, for IS's dealing with complex phenomena where it is necessary to continuously search for new data sources of information and data making it possible to adapt the existing models to perceived changes in the environment.

[3] For simplicity of reasoning, we restrict our discussion to binary attributes. This approach can be easily extended for non-binary attributes.

Our discussion about information systems raises an important point. The rough set approach typically treats information systems as predetermined, without delving into the details of *how*, *why*, or *when* the information is collected.

While some research explores dynamic information systems, the concept of "dynamic" is often predefined or based on the assumption of random updates (see, *e.g.*, [36,37]). However, our analysis suggests that the dynamics of information systems are determined by the control mechanisms of the interacting ISs with the physical world. This implies that dynamic information systems should be considered within a broader framework that accounts for control-driven, granular computations (see, *e.g.*, [38]).

In addition, the literature on approximating vague concepts emphasizes the need to create adaptive rough sets [39,40]. Philosophers contend that crisp sets, as formulated in the Pawlak rough set model, cannot definitively define boundary regions consisting of borderline cases. These regions are defined within the model relative to a specific sets of objects and attributes. However, when these sets change, the boundary regions may also shift. This reinforces the importance of understanding the dynamics of these changes.

5. Granular spaces and granular networks

We discuss two types of granular spaces: simple and compound. In the latter case, the granules in such spaces can be of different types.

5.1. Simple and compound granular spaces

A simple granular space $\mathcal{G}$ is a tuple $(U, L, \|\cdot\|, G, Rel)$, where

- U is a space (universe) of objects (*e.g.*, a training set);
- L is a language of names of granules;
- $\|\cdot\| : L \to P(U)$ is the semantics of L;
- G is a set of granules consisting of pairs $(\alpha, \|(\alpha)\|)$, where $\alpha \in L$ (note that G should be constructively defined);
- Rel is a set of relations of different arity over G.

For example, G is defined by specifying a set G_0 of generators and a set of aggregation operations making it possible to generate new granules from G_0 and already generated granules. We have already discussed an example of simple granular space in Section 2. For example, one can assume that $r(g, g')$ (where $r \in Rel$ and $g = (\alpha, \|(\alpha)\|), g' = (\beta, \|(\beta)\|)$) holds iff $\|(\alpha)\| \subseteq \|(\beta)\|$. We assume that the language L consists of descriptors (*i.e.*, expressions of the form $a = v$ for $a \in A$ and $v \in V_a$) and their conjunctions.

Note that in the above definition of simple granular space, the semantics $\|\cdot\|$ is a pure mathematical function. However, one can assume that this semantics is not abstract, but physical, as it has been described in Section 4.

From already defined granular spaces new ones can be defined by applying to them an aggregation operation.

Another relation may concern the proximity of granules (in physical space, *e.g.*, in memory, or outside of ISs). Suppose the attribute a is used to identify the localization of objects and the distance of objects is measured by a distance metric ρ. Moreover, let $\epsilon > 0$ be a threshold expressing closeness in the following sense. Objects $x, y \in U$ are ϵ close iff $\rho(a(x), a(y)) < \epsilon$. Then we can define the ϵ-closeness of granules with signatures $Inf_A(x), Inf_A(y)$ by $r_\epsilon(Inf_A(x), Inf_A(y))$ iff $\rho(a(x), a(y)) < \epsilon$.

For simplicity, let us assume two granular spaces $\mathcal{G}_i$, where $i = 1, 2$ are defined. A compound granular space $Comp(\mathcal{G}_1, \mathcal{G}_2)$ can be obtained from these spaces as follows. $Comp(\mathcal{G}_1, \mathcal{G}_2)$ consists granules from $G_1 \cup G_2$ and some new relations

$$r_{k,l} \subseteq ((G_1)^k \times (G_2)^l)$$

for some $k, l > 0$, where $(G_1)^k, (G_2)^l$ are k-ary and l-ary Cartesian products of G_1, G_2, respectively. These new relations belong to the interface between $\mathcal{G}_1, \mathcal{G}_2$.

5.2. Granular networks

A granular network over granular space $\mathcal{G}$ is a finite set of granules from $\mathcal{G}$ and a finite set of facts over these granules from $\mathcal{G}$ expressed by relations from Rel (*e.g.*, a simple fact $r(g_1, g_2)$ expressing that the relation r holds between granules g_1, g_2 of $\mathcal{G}$). For example, such a relation r can be used to construct new c-granules from given subsets $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{G}$ by filtering the Cartesian product $\mathcal{G}_1 \times \mathcal{G}_2$ to retain only pairs of c-granules that satisfy the relation. Specifically, r could represent pairs of c-granules that are spatially proximate for a certain duration. In another application related to searching for relevant information granules by aggregation of information granules represented by information systems into new information systems, an n-ary relation $r(g_1, \ldots, g_n)$ can be used to keep in the constructed information system only those tuples $g_1, \ldots, g_n$ of c-granules as objects in the aggregated information systems that satisfy the relation r.

In Section 2, we presented a simple example of compound granular space created by linking (U, R) with granular space consisting decision granules with one linking them relation:

$$r_t([x]_A, X) \text{ holds iff } \nu_{SRI}([x]_A, X) \geq t.$$

Certainly, already constructed compound granular spaces and given simple spaces can be linked into a new compound granular space. It should be noted that the relevant relations $r_{k,l}$ should be learned from data, *e.g.*, to allow the IS control reason about approximation of granules from a higher hierarchical level by aggregation of some granules (patterns) from the lower hierarchical level.

It is well known that the lowest level of sensory (elementary) granules is often not sufficient to directly obtain the high quality of approximation of decision granules. On the way from sensory granules to decision granules, one should embed a chain of granular spaces, where each next granular space in this chain is obtained from the previous one by the corresponding generalization. Let us illustrate this by an example in which on the lowest level we have sensory measurements of values of attributes from A for the successive moments i of time, where $i = 1, \ldots, n$ for some n, *i.e.* elementary granules g_i are of the form (i, v), where v is the vector of values of attributes from A measured at the moment i. On the next levels one can consider different generalizations of this granular space. For example, one can create new granular space by forgetting the exact time of measurement, or one can consider clusters of similar descriptions of objects. Then in the interface between a given granular space and its generalization is based on the relation of inclusion of semantics of granules.

However, despite the great progress that has been made, reasoning that supports such generalization is still a challenging problem. [18].

5.3. Illustrative example of hierarchical network of granules

In the following illustrative example, we roughly explain the idea of c-granules, their different kinds, especially networks of c-granules in the context of hierarchical learning.

Let us start with an explanation of the idea behind sensory granules. In our illustrative example, we assume that the IS control has decided to obtain a picture of a particular object using a sensor, in this case a camera, from the cellular phone of another person who has access to that object. This task is specified in a given language of the IS control by an expression and the IS control is storing this expression in its buffer along with the phone number of the receiver. The phone number plays the role of specifying the address or window for the more general description of the sensory granule. We assume that the ISs is connected to a cellular network (a network of physical objects), enabling it to send messages from the buffer to receivers specified by phone numbers after pressing the relevant button. Based on knowledge of cellular network behavior, the IS control expects that the message stored in the buffer will be encoded into the relevant signal and transmitted by the cellular network to the receiver. Moreover, the ISs expect that (after some period of time) the receiver, based on the encoded message from the received signal, will take the required picture and send it back to the ISs using the cellular network again. Finally, the ISs expect to obtain the picture in a suitable format, such as a bitmap, in another of its buffers.

This bitmap may be stored in an information or decision system as a new object. In this case, the cellular network plays the role of configuring the physical objects organized or created by the IS control to realize the physical specification of the transformation related to the sensory measurement. In the illustrative example described, this includes obtaining a picture of a given object. It is assumed that the IS control has an implementation module (IM) responsible for carrying out such tasks.

It is important to note that we are discussing the expected behavior of the IS control, as the cellular network may unexpectedly interact with the environment, potentially leading to unexpected damage to the network.

One can easily imagine more complicated examples related to sensory measurements, such as returning video, pictures annotated with text, or video obtained after making changes realized by transformations specified by the control in a particular part of the physical space specified by spatio-temporal window. Sensory measurements are special cases of such transformations, and it is usually assumed that sensory measurements do not make (or make negligible) changes to the physical objects being measured.

It is also important to note that sensory measurement requires communication between the abstract and physical worlds.

Now let us assume that the results of sensory measurements were stored in a decision system. The objects in this system are in the form of pairs, consisting of the time when the measurement was received and the result of the measurement. Conditional attributes are now defined on such objects, and they may be related to the localization of the measurement and some features of the objects like size, color, speed, etc. For this decision system, we obtain two granular spaces defined by the conditional and decision attributes, as shown in Fig. 3.

In hierarchical learning, several more levels may be relevant. For example, on the next level to the decision system discussed earlier, one can consider a decision system with objects being information granules created by time windows of a given length. These time windows are labeled by value vectors of the conditional attributes of the previously defined decision system. Conditional attributes are now defined over these new labeled time windows, new granules. They may concern, *e.g.*, changes in the localization of objects or changes in speed, etc. One may also consider decision systems with objects being collections of objects from the initial decision system, such as those that are ‘close’ to each other.

The next level may consist of decision systems with objects being information granules formed by similarity classes of objects (relative to some similarity relation) from the previous hierarchical level. This process can be continued, constructing on the next level a decision system with objects being information granules formed by tuples of objects from the previous level, filtered by some conditions (constraints).

On each level, the relevant features of the new objects should be defined, enabling the IS control to derive the relevant computational building blocks for cognition, *i.e.*, the approximation of decision classes (concepts) on a given level and, finally, on the highest level of the hierarchy. It should be noted that when the IS control recognizes that the derived granules (structures) are not relevant, *e.g.*, in the approximation of complex vague concepts at the highest level of the hierarchy, it should then utilize reasoning techniques that support the search for the relevant granules through backtracking.

It is important to note that instead of links on each level, as shown in Fig. 3, in hierarchical modeling, we have some links between levels, showing relationships between objects from successive levels. For example, these links can specify procedures to check if a

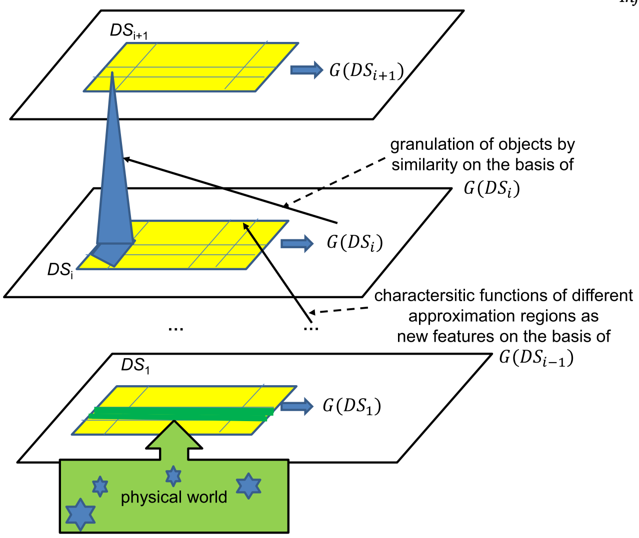

Fig. 6. Illustrative hierarchical network of granules.

given object from a lower level is 'a part' of a given object on the current level. In this way, we obtain networks of c-granules related to many different granular spaces, with links showing the relationships of granules from different levels of the hierarchy.

It should also be noted that on higher hierarchical levels, one can consider sensory attributes defined in relation to objects from this level. However, the considered aggregations of objects in our example were defined in the abstract (mathematical) space only. In this case, one can use information granules, which are special cases of c-granules, as discussed in the presentation on the differences between Granular Computing (GrC) and Interactive Granular Computing (IGrC).

In Fig. 6, on different levels of the hierarchy are decision systems (DS_i) and their corresponding granular networks $G(DS_i)$, as previously discussed (see Fig. 3). Granules from these networks can be employed, *e.g.*, to connect different hierarchical levels by proposing novel features at higher levels based on approximated regions at lower levels or by granularizing objects according to their discovered similarities at lower levels. Links between granules at different levels are replaced by procedures that identify generalizations at the higher level. These procedures determine whether an object from the lower level belongs to (or is a part of) the identified generalization. On the lower level of hierarchy, the decision system is updated through sensory measurement by the IM based on a given specification of the relevant association.

The c-granules discussed in this paper can be with or without control. For simplicity, let us assume that the only controlled c-granule represents the IS control. This granule consists of two layers: informational and physical.

The informational layer stores perceived information about the physical world, essential for subsequent operations. At each moment of the control's local time, this information takes the form of a family of spatio-temporal window specifications (addresses) labeled with relevant information. This information consists of, *e.g.*, previously perceived information from interactions with physical objects or knowledge bases, specifications of the currently implemented transformation, and control-relevant properties of the ongoing computation.

The physical layer consists of physical objects that interact with the IM module responsible for associations. At each computational step, the IS control determines whether a new relevant transformation specification should be selected for execution.

The following section provides more details about the IS control. In particular, we will discuss the key modules of the c-granule that are responsible for modeling this control. It is assumed that the c-granule information layer accurately represents the modules, their functions, and their overall behavior. In addition, we assume a correct implementation of the physical layer, consistent with the information layer specifications.

6. Further comments on behavior of the IS control and reasoning supporting it

Among the reasoning methods in the reasoning module of the control are methods based on deduction, induction, or abduction [17,41]. However, there are some *white spots* that require further research. Among them are reasoning (judgment) methods based on experience (*e.g.*, reasoning by analogy) [42,43] or mechanisms supporting discovery [44]. This means that in some applications, especially those dealing with complex phenomena (e.g. medicine), it is not possible to avoid dialogues with domain experts [45] and/or chatbots [46].

We have already discussed some issues related to reasoning by control of ISs. Fig. 4 illustrates different components of reasoning module of ISs. In particular, we have already emphasized the role of reasoning in supporting the optimization of different tasks performed by ISs in searching for high quality solutions to given problems to be solved, *e.g.*, optimization of parameters of approximation spaces, quality measures, learning algorithms. In this section, we discuss some others of these components. To summarize, let us remind that reasoning by control is performed on complex granules (c-granules) among which are granules linking abstract and

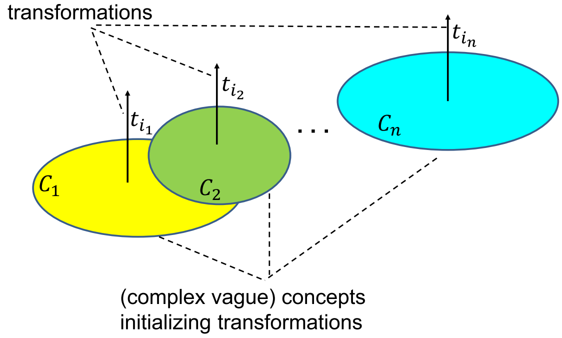

Fig. 7. Complex granule consisting of set of rules interpreted as a complex game with granules $C_1, \dots C_n$ representing vague concepts and their classifiers labeled by transformations $tr_{i_1}, \dots, tr_{i_n}$.

physical objects. Control is performing reasoning on granular networks generated by control of ISs and along performed reasoning are generated granular computations aiming to lead to the high quality of approximate solutions of considered problems.

The states of ISs are networks of granules from different granular spaces linked by relations. Granules from different granular spaces are linked by more compound relations than granules linked from the same granular space. Granular networks store properties of individual granules and properties of computations over granular networks. The links represent various relationships between granules. Some granules represent domain knowledge, others are used to represent components of IS control. There is also a distinct granule that represents the state of perception of the currently perceived situation.

Control of ISs is deciding in each state what kind of transformation of the current granular network to perform. The transformation may concern suspending some measurements, elimination of some granules from the network, updating the network with new granules or updating the existing granules. The control decides which transformation to select by matching the rules available to it with the granule representing information about the currently perceived situation. The set of rules can be treated as a complex game (see. *e.g.*, [8–10]) consisting of concepts (often, complex vague concepts) labeled by transformations which should de realized if the concept is satisfied (to satisfactory degree) (see Fig. 7).

The complex game illustrated in Fig. 7 consists of pairs (C_j, tr_{j_i}) for $j = 1, \dots, n$, where each C_j is a classifier for a vague concept triggering the transformation specification t_{j_i}. If C_j is satisfied (to a satisfactory degree) in the current perceived situation, the IS control registers this as a match. Subsequently, the control attempts to resolve conflicts among all matched classifiers to select a transformation specification for the IM to execute in the physical world. This is supported by appropriate reasoning methods. If successful, the selected specification is transmitted to the IM; otherwise, the IS control seeks additional information about the perceived situation (*e.g.*, through sensory measurements or knowledge derived from c-granules representing domain knowledge bases). The IM executes the transformation specification in the physical world through the following steps: (i) creating (or organizing) a configuration of physical objects defining the scope of the transformation, (ii) initiating interactions within this configuration, (iii) allowing the IS control to perceive relevant object properties and interactions over the relevant period of time and recording results in the informational layers of appropriate c-granules, and (iv) treating the appropriate recorded values as the transformation's outcomes. It's important to note that these outcomes may deviate from expectations due to environmental interactions.

Both the concept and its labeling transformation are complex granules (c-granules) in IGrC. The satisfiability of the concept to a satisfactory degree (under the additional assumption that the conflict resolution strategy between rules satisfiable to a satisfactory degree points to the concept) initiates the interaction with the granule - transformation leading to its realization.

Based on the results of this matching, the control attempts to resolve conflicts between rules that match the granule with information about the currently perceived situation. If this matching is successful, the control starts the realization of the selected transformation, otherwise is trying to get more information about the perceived situation, *e.g.*, by measurement or interaction with granules representing domain knowledge bases or with human experts.

One should note that the rules set of the control of ISs may have a compound structure, *e.g.*, representing distributed control or priorities of rules. Moreover, ISs should be equipped with reasoning methods that support the development of strategies for adapting rules. Fig. 8 presents a simplified example of an adaptive complex game. Classifiers for complex vague concepts, denoted as $C_1, \dots, C_m$, trigger complex games $G_1, \dots G_m$. Using these classifiers, the IS control identifies concepts $C_1, \dots, C_m$ that match the current perceived situation and employs appropriate reasoning methods to resolve conflicts between them. If conflicts can be resolved, a complex game from $G_1, \dots, G_m$ is selected; otherwise, the control seeks additional information about the perceived situation to facilitate conflict resolution. The challenging problems are related to learning the high quality classifiers for $C_1, \dots, C_m$. One should also note that the IC control must determine whether the goals outlined in the problem specification can be achieved. If not, it should adjust the goals accordingly.

In particular, complex games and adaptive complex games can be viewed as more compound c-granules. Their behavior arises from interactions between granules representing vague concepts. These interacting granules are labeled by c-granules representing decisions (*e.g.*, plans or complex games relevant to the domains defined by these vague concepts). In addition, reasoning processes influence these interactions by resolving conflicts between votes on different decisions predicted by the concepts. This reasoning ultimately determines the interaction that decides which plan or complex game to launch.

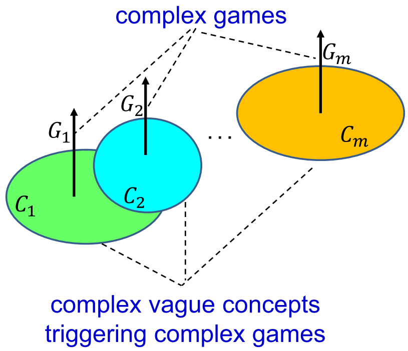

Fig. 8. Adaptive complex game consisting of vague concepts with classifiers $C_1, \ldots, C_m$ labeled by complex games $G_1, \ldots, G_m$.

More details on these important issues will be presented in our next paper. Here, we would put attention of the reader to the importance and richness of the required reasoning methods supporting the control of ISs.

Let us consider in more detail two more examples related to reasoning methods related to aggregation and decomposition of granular networks.

Granular aggregation is widely used in machine learning, *e.g.*, in feature engineering or hierarchical learning [18,47] and Granular Computing (GrC). In feature engineering, new features are defined over the existing ones. Hence, a partition defined by a new feature is defined by partitions of some known already features. This may be interpreted as a transformation of granules from one granular space into a new one.

In general, the decomposition problems are challenging. According to the opinion of Judea Pearl [48]:

Traditional statistics is strong in devising ways of describing data and inferring distributional parameters of sample. Causal inference requires two additional ingredients:

- *a science-friendly language for articulating causal knowledge, and*
- *a mathematical machinery for processing that knowledge, combining it with data and drawing new causal conclusions about a phenomenon.*

This transition from *a science-friendly language for articulating causal knowledge* to *a mathematical machinery for processing that knowledge* usually requires decomposition through several layers.

Decomposition (degranulation [12]) of granules is an important topic in GrC. Here we discuss some important problems related to the decomposition of granules, especially those described in natural language. The language of granules consists of complex vague concepts expressed in natural language. Their semantics are defined by classifiers induced from data. We emphasize reasoning methods that lead to the relevant decomposition. Let us mention two illustrative examples: decomposition of specification of tasks of ISs and decomposition of specifications of transformations in realization of associations.

In Section 1 we have presented examples of such tasks. In each of these examples, the description of task is presented using complex vague concepts what can be hardly understandable by ISs without additional support. It is worthwhile to cite here an opinion of Lotfi Zadeh [49]:

Information granulation plays a key role in implementation of the strategy of divide-and-conquer in human problem-solving.

One can ask how to acquire or learn such a strategy. One possible approach is to provide dialogue methods with domain experts. This is especially important for medical applications [45]. Another one is related to applications of chat boots [46]. It may be necessary to go through several levels of decomposition before the level with concepts which can be efficiently and with high quality approximated can be reached on the basis of available data. In the case of realization of specifications of associations for this last level a direct realization of this level in the physical world should be possible. This approach proved to be successful for several real-life projects where so called the rough set-based ontology approximation of concepts was applied: after decomposition to the lowest level the obtained ontology of concepts was approximated using bottom-up strategy (see, *e.g.*, [47]).

ISs also require reasoning mechanisms to handle situations where the current search strategy for relevant granules is unlikely to succeed. In such cases, the reasoning system of ISs should guide the backtracking process by appropriately decomposing (degranulating) some already generated granules up to a suitable level. Additionally, more advanced ISs should leverage information from unsuccessful trials to learn and adapt their search paths towards discovering the relevant granules.

7. Summary: characterization of c-granules with control

Finally, let us consider a concise characterization of c-granules with control, clarifying their nature. In particular, this concerns the IS control considered as a network of c-granules.

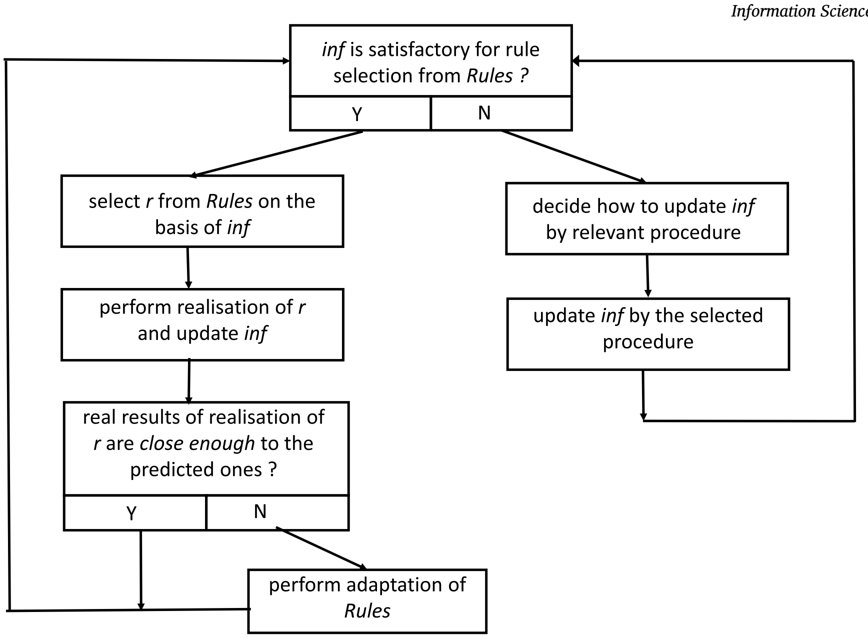

Fig. 9. Algorithmic cycle of c-granule control.

A c-granule with control, denoted by g, comprises two layers at any given local time t: an informational layer, $inf_l(t)$, and a physical layer, $phy_l(t)$.

$inf\_l(t)$ is a family of labeled information granules, *i.e.*, pairs

$$\{(w_i, g_i)\}_{i\in I},$$

where

- w_i specifies a spatio-temporal window (address) used by the control of g to localize a portion of physical space, $hard\_suit_i$, corresponding to w_i. The control of g designates transformations, termed associations, to be executed on $hard\_suit_i$.
- g_i is an information granule encapsulating various types of data/information depending on the application, including:
  (i) Information gathered thus far about $hard\_suit_i$ and/or its properties.
  (ii) A description of the currently active transformation on this part of the physical space.
  (iii) Specifications of spatio-temporal windows enabling g to identify portions of $phy_l(t)$ named $soft\_suit_i$ and $link\_suit_i$. These consist of directly accessible physical objects and objects facilitating interaction between (w_i, g_i) and $hard\_suit_i$, respectively.

The control of g can generate specialized (sub-)c-granules for direct communication between (w_i, g_i) and $soft\_suit_i$. These c-granules allow the control of g to encode information from g_i into physical objects within $soft\_suit_i$ and conversely, decode information from $soft\_suit_i$ to update g_i.

It is crucial to note that realizing current transformations of $phy\_l(t)$ and $inf\_l(t)$ might necessitate considering subsets of $\{(w_i, g_i)\}_{i\in I}$, where labeled information granules are interconnected by pertinent relations. This gives rise to granular networks, as previously discussed.

As mentioned, $\{(w_i, g_i)\}_{i\in I}$ includes also pairs describing the behavior of g. Specifically, this encompasses different modules of the control of g, notably the rule module (RM), the implementational module (IM), the adaptation module (AM) and the reasoning module (ReM).

In Fig. 9 is illustrated the basic cycle of c-granule control. In this figure, *inf* is information about the currently perceived situation and *Rules* is the set of control rules from RM. Note that the control of the c-granule should be supported by relevant reasoning techniques that support the realization of the basic cycle. For example, to decide whether *inf* is satisfactory for selecting a rule, the control should perform reasoning aimed at resolving conflicts between rules from $Rules$ that match inf.

The control of g selects an appropriate rule from the RM to transform the current pair $(inf_l(t), phy_l(t))$ into a new one based on the transformation specified by the rule's right-hand side. Transformations occur at two levels: abstract, defined by mathematical operations on information granules (*e.g.*, aggregation or de-aggregation), and physical, executed by the IM in the physical space to create what we call physical semantics. The pair $(inf_l(t), phy_l(t))$ is updated according to the transformations indicated by the selected rule. Importantly, the execution of associations that update $(inf_l(t), phy_l(t))$ is not limited to the informational layer. It depends on the realization of physical semantics in the real world. Consequently, our c-granule with control model cannot be considered purely mathematical.

8. Conclusions

We propose an extension of approximation spaces in rough sets to approximate solutions for problems solved by ISs. This approach leverages granular computations generated and controlled by ISs interaction with the physical world. Solutions, such as learning algorithms, classifiers, clustering algorithms, new materials, compound physical sensors, or robots, are represented by complex granules in IGrC. The approach allows to connect abstract concepts with physical objects.

The control of ISs aims to generate relevant granular computations that lead to high-quality solution approximations. These approximations are granules that satisfy given, often vague, specifications to a satisfactory or high degree that is interpreted that they belong to the lower approximation of the vague concepts. We emphasize the importance of reasoning methods applied to c-granules and the computations themselves. Using approximate reasoning over networks of c-granules, solution approximations are synthesized (constructed).

It's important to note that approximation spaces in the traditional rough set approach only create one component represented by corresponding granules for approximate reasoning. We propose that this new approach is fundamental for designing ISs, especially those dealing with complex phenomena. By providing a much wider context for approximating complex objects, we also open the door for insightful reasoning processes that support ISs in discovering high-quality approximate solutions.

CRediT authorship contribution statement

Andrzej Skowron: Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Jaroslaw Stepaniuk:** Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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