On the contributions of Helena Rasiowa to mathematical logic

Article *in* Multiple-Valued Logic · January 1999

1 author:

Josep Maria Font
University of Barcelona

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Josep Maria Font*

Faculty of Mathematics, University of Barcelona
Gran Via 585, E-08007 Barcelona, Spain
jmfont@ub.edu

Published in
Multiple-Valued Logic, An International Journal 4 (1999) 159–179

Abstract

After an overview of the main fields of mathematical logic where Rasiowa made significant contributions, the author analyzes with some detail the philosophical basis of Rasiowa's approach to the study of mathematical logic and the main technical points that characterize her contributions. In particular, it is shown that theses contributions are grounded on three key ideas of major Polish logicians, namely: (1) Lindenbaum's idea of treating the set of formulas as an abstract algebra; (2) Mostowski's idea of interpreting quantifiers as infinite conjunctions or disjunctions in the (ordered) set taken as model; and (3) Tarski's idea of defining a logic in general as a finitary closure operator on the power set of the set of formulas, completed by Łoś and Suszko's notion of 'structurality' (invariance under substitutions). These ideas and Rasiowa's own constructions are described in their historical context.

Keywords: Non-classical logics, algebraic logic, Lindenbaum-Tarski algebra, history of logic, metamathematics, completeness theorems, representation theorems, Rasiowa-Sikorski Lemma, S-algebras, closure operators.

*Research partly supported by Spanish dgicyt grant PB94-0920 and Generalitat de Catalunya grant 1996SGR-00052.

Introduction

Let me begin with some personal reminiscences. For me, writing a paper on the contributions of HELENA RASIOWA (1917–1994) to Mathematical Logic is the result of several unexpected coincidences[1].

My relationship with RASIOWA is academic, rather than personal. Her book An algebraic approach to non-classical logics [45] is one of the works most to blame for my professional dedication to research in Algebraic Logic, if some day I am taken to court for this extravagant behaviour. I first met RASIOWA in person as late as 1988, at the 18$^{\text{th}}$ ISMVL, held in Palma de Mallorca (Spain). There I contributed a paper ([19]; but see also [14]) on an abstract characterization of BELNAP's four-valued logic which used a mathematical tool inspired by some work done by BIAŁINICKI-BIRULA and RASIOWA in the fifties [4]. More precisely, I used a mathematical tool abstracted from the works of MONTEIRO on DE MORGAN algebras [30, 31], and these works, in turn, were based on that early work of BIAŁINICKI-BIRULA and RASIOWA. Two years later, that is, in 1990, I attended the 20$^{\text{th}}$ ISMVL, held in Charlotte (North-Carolina, U.S.A.). It so happened that this meeting included a session in memory of MONTEIRO.

ANTONIO MONTEIRO (1907–1980) was a Portuguese mathematician who settled in Bahía Blanca (Argentina) after spending some years in exile in Rio de Janeiro (Brazil) and in San Juan (also in Argentina). In all these places he promoted or initiated teaching and research in several areas of mathematics. His strong personality attracted the interest of a group of young mathematicians, and led to the creation of what one could describe as a *school*. In Bahía Blanca, MONTEIRO started a Mathematics Department in the newly founded Universidad Nacional del Sur, and in 1958 he invited ROMAN SIKORSKI and HELENA RASIOWA to lecture there, in order to give a new impulse to his group. With the influence of RASIOWA and SIKORSKI, together with MONTEIRO's own interests, there emerged a group of researchers in Algebraic Logic, which has since made a set of highly significant contributions to the field, and has developed a characteristic *style*. The works of this school, together with RASIOWA's book, deeply influenced the research of several people in Barcelona in the late seventies, and were partly responsible for the creation of a group of algebraic logicians in this city, to which I belong.

At the Charlotte symposium I contributed a paper ([17]; but see also [18]) that combined some of the latest of MONTEIRO's ideas with RASIOWA's general methods, along with other influences. When I arrived, I found that the person speaking in memory of MONTEIRO was RASIOWA herself. It is thus not surprising that I find myself writing an essay about her work in pure Mathematical Logic.

Giving a short account of all her research work in this area is no easy task; and it is even more difficult to make a selection of her most important achievements. For, unlike other well-known logicians, her contribution to Logic was not the discovery of a single outstanding theorem, but rather the detailed study of several non-classical logics, along with classical logic, by using certain typical mathematical tools, algebraic in nature, which she herself developed and whose

[1]This paper is an edited version of the talk delivered by the author to the Plenary Session in the Memory of HELENA RASIOWA at the 26$^{\text{th}}$ ISMVL held in Santiago de Compostela (Spain) on May 29–31, 1996. Some comments in this Introduction might be better understood by keeping this origin in mind.

strength and limits in applications she explored in more than 30 papers and 2 books published over more than 40 years (between 1947 and, say, 1989, for the topic I am concerned with[2]).

I can only try to convey to the reader a general picture of the significance of RASIOWA's work; so I will start by giving a *brief overview* of the main topics she dealt with, and will then look in some detail at *a few points* of her work that seem the most significant to me. Readers interested in more details can consult issue 3/4 of volume 25 (1995) of the *Bulletin of the Section of Logic*; a fairly complete bibliography appears in [1].

1 Overview

HELENA RASIOWA's research work in the area of pure Mathematical Logic can be classified, save for a few marginal papers, in the subfield of Algebraic Logic; indeed, her work has been one of the mainstreams in this small area, to such an extent that one way of defining Algebraic Logic for a quarter of a century was to say *"Algebraic Logic is what* RASIOWA *does"*.

The core of her work consists in the development of a rather general method to construct an algebraic semantics for certain logical systems. This means: Given a propositional or first-order logic $\mathcal{S}$, to find a class of algebras $\mathsf{Alg}^*\mathcal{S}$, all having an algebraic constant $\mathbf{1}$, such that every formula α can be interpreted in every algebra $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{S}$ as a mapping $\alpha^{\boldsymbol{A}}$ that associates with every interpretation $\overline{a}$ of the language into $\boldsymbol{A}$ an element $\alpha^{\boldsymbol{A}}(\overline{a}) \in A$; this element represents the *truth-value* of the formula α under the interpretation $\overline{a}$ in $\boldsymbol{A}$; so wee see that many-valuedness is indeed at the heart of RASIOWA's work. That the method fully succeeds means that the following ***Strong Completeness Theorem*** holds:

$$\Sigma \ \vdash_{\mathcal{S}} \alpha \quad \Longleftrightarrow \quad \Sigma^{\boldsymbol{A}}(\overline{a}) = \{\mathbf{1}\} \ \text{ implies } \ \alpha^{\boldsymbol{A}}(\overline{a}) = \mathbf{1}\ , \ \text{ for every } \boldsymbol{A} \in \mathsf{Alg}^*\mathcal{S} \text{ and every interpretation } \overline{a} \text{ in } \boldsymbol{A}\ , \tag{1}$$

where α is any formula, Σ is any set of formulas, and $\Sigma \vdash_{\mathcal{S}} \alpha$ means that α follows in the logic $\mathcal{S}$ from the assumptions in Σ. The proof is done by extending and generalizing the ***construction of Lindenbaum-Tarski quotients***, a method that in classical sentential logic was already known to produce the class of Boolean algebras. *Its success depends on a carefully chosen definition* of the class $\mathsf{Alg}^*\mathcal{S}$; with the help of suitable representation-like theorems, in some cases the class of algebras $\mathsf{Alg}^*\mathcal{S}$ can be substituted by a smaller class of algebras of sets or of some special kind of subsets of topological spaces or, in some particularly well-behaved cases, by a single algebra. A great deal of RASIOWA's early work consisted in applying these constructions to prove some metalogical results either for classical logic or for some non-classical logics.

During what we might define as the first period, say from 1950 to 1964, these logics were among the best-known and best-behaved ones, such as intuitionistic logic, LEWIS' modal system $\mathcal{S}4$, and positive and minimal logics. These are the logics treated in RASIOWA's famous book The mathematics of the metamathematics [55], published in 1963, written jointly with ROMAN SIKORSKI, with

[2]At the memorial session mentioned in footnote ([1]) as the origin of this paper, an analysis of RASIOWA's contributions to Applied Mathematical Logic was presented by Professor TON SALES of the Polytechnic University of Catalonia (Spain).

whom she also published another ten papers or so. In these cases, especially in that of classical logic, she obtains as applications of this algebraic semantics purely semantical proofs of a number of well-known metalogical properties of the logics, beginning with the Completeness Theorem [35, 48], Compactness [36], SKOLEM-LÖWENHEIM [49], HILBERT's ε-theorems about SKOLEM expansions [39], HERBRAND's theorem [28], and even GENTZEN's Cut-Elimination Theorem for Sequent calculus [54]. In the intuitionistic case, she obtains the Existential property and the Disjunction Property, and, for modal logic, their analogues [37, 38, 51, 52].

In 1957 she began to publish papers on several classes of algebras that correspond to lesser-known logics, namely DE MORGAN algebras [4] and NELSON algebras [5, 41], related to several logics with different kinds of negation, and, more importantly, POST algebras [42, 43, 46]. As the reader may know, POST algebras are the algebraic models of several many-valued logics whose set of truth-values is a finite linearly ordered set, and each POST algebra mirrors this structure by containing a set of constants inside it corresponding to the truth values. These logics fell within the scope of RASIOWA's methods after the work done in the sixties by ROUSSEAU [57, 58], who succeeded in axiomatizing them with an intuitionistic reduct. After finding the appropriate technical, algebraic results needed to apply her methodology to the many-valued predicate calculi associated with these logics, RASIOWA soon realized how to extend it to deal with more general many-valued logics, namely with logics whose truth-values form a denumerable chain of values with order-type $\omega + 1$ [44, 47], or, finally, an arbitrary partially ordered set with top [8, 9, 10]. These generalizations were directly responsible for her involvement in Algorithmic Logic, Fuzzy Logic, and other applications of Logic to Computer Science.

Most of these non-classical cases were incorporated in the even more famous book An algebraic approach to non-classical logics [45], published in 1974 by North-Holland in its series Studies in Logic and the Foundations of Mathematics. There is no doubt that the publication of the book in such a well-known and prestigious series, stocked in virtually every mathematical library, was one of the reasons for its enormous influence. But another reason is the maturity of its exposition in RASIOWA's book. Her development of a general method began early in the fifties, in collaboration with SIKORSKI, as several papers [40, 50, 53] and some chapters of their joint book [55] show, but its presentation in this book is superior.

In [45] RASIOWA singled out a wide class of sentential logics to which her methods and constructions apply; these logics she called ***standard systems of implicative extensional propositional calculi***, and are defined by some very natural conditions on an implication connective "$\rightarrow$" (I will give more details in Section 4.4). The scope of this book is wider than the previous one, but as a contrast (maybe reflecting a slight difference of interests between SIKORSKI and herself, or changes in such interests over the years) she gives less space to first-order logics. The differences between the logics she studies lie mainly in their propositional part, and in a Supplement she shows, in outline, a common procedure to associate a first-order logic with each of the propositional logics she studies; nonetheless, this treatment has had an important influence on recent work [33, 34] on a common algebraic treatment of several formalisms that incorporate some form of variable-binding, like first-order logic and lambda calculus.

Rasiowa's general method was characterized by a very tight design, in order to account for the algebraization of a distinct group of logics she had already dealt with, and reflects some features these logics have in common. Although its application might seem rather narrow, the number of new logics that have been studied by other people with this methodology is very large, because the requirements are extremely natural.

Actually, the class of logics she defined is, roughly speaking, the class of *all* logics to which her methods apply word for word (see Section 2.3). Therefore, it is not surprising that several *generalizations* have appeared that widen the scope of applications of the method by weakening some of the conditions in her definitions. Let me quote here the theory of *equivalential logics* developed by Janusz Czelakowski [11], and the theory of *algebraizable logics*, developed by Wim Blok and Don Pigozzi [6], which is more restricted than Czelakowski's, but still more general than Rasiowa's. These new developments have connected Rasiowa's approach with the more general *theory of logical matrices* developed also in Poland by other logicians like Łoś, Wójcicki, Wroński, Zygmunt, etc.; good accounts of this theory and of its diverse ramifications are [7, 12, 16, 63].

2 A philosophy of Mathematical Logic

Rasiowa's work presents us with a particular *perspective*, or *philosophy*, of Mathematical Logic. Let me say that, in order to appreciate the impact and influence of Rasiowa's work and perspective one has to adopt a *historical approach*: some aspects of this perspective are now smoothly incorporated into our logical heritage, even if we do not work in this line; however, as we shall see, this was certainly not the case in the late forties and the fifties.

One of the main elements of this philosophy is *the absence of any Philosophy in her work on Mathematical Logic*. This is beautifully explained and strongly defended in the preface of The Mathematics of the Metamathematics. The title of the book is itself a declaration of principles: No Philosophy, only Mathematics. Rasiowa and Sikorski propose the use of *infinitistic methods* in metamathematical investigations; by this they mean the actual use of any mathematical tool required, especially all the tools from abstract algebra, lattice theory, set theory, and topology, for instance the operations of forming the supremum or infimum of an infinite subset. They explicitly advocate deviating from the proof-theoretical approach of the formalist trend in metamathematics, which they judge to be an unnecessary limitation and complication that obscures the understanding of the deep, true nature of metamathematical notions. It is best to recall their own words:

> *The title of this book is not meant as a pun, although it may, at first sight, appear to be so.* [...]
>
> *The title of the book is inexact since not all mathematical methods used in metamatemathics are exposed in it.* [...] *The exact title of the book should be: Algebraic, lattice-theoretical, set-theoretical and topological methods in metamathematics.*
>
> [...]
>
> *The finitistic approach of Hilbert's school is completely abandoned*

> *in this book. On the contrary, the infinitistic methods, making use of the more profound ideas of mathematics, are distinctly favoured. This brings out clearly the mathematical structure of metamathematics. It also permits a greater simplicity and clarity in the proofs of the basic metamathematical theorems and emphasizes the mathematical contents of these theorems.*
> [...]
> *The theorem on the completeness of the propositional calculus is seen to be exactly the same as Stone's theorem on the representation of Boolean algebras.* [...] *It is surprising that the Gdel completeness theorem* [*of the predicate calculus*] *can be obtained, for example, as a result of the Baire theorem on sets of the first category in topological spaces, etc.* [55, pp. 5–6]

We can imagine that this might be a controversial issue in some academic contexts; and in fact this approach was criticized by some reviewers like KREISEL [27] or BETH [3]; in contrast, FEFERMAN [13] and ROBINSON [56] highlight the simplicity of the proofs obtained using these methods. Moreover, notice that RASIOWA and SIKORSKI's 1950 proof of the Completeness Theorem for first-order logics over languages of arbitrary cardinality [48] is almost contemporary with LEON HENKIN's famous 1949 proof [22], where infinitistic methods could not be avoided either, as FEFERMAN points out in his review:

> *In the opinion of the reviewer, this paper represents a distinct advance over all preceding proofs; for on the one hand, much less formal development from the axioms is required than in the proofs similar to Gdel's, and on the other hand, the doubly infinite passage to S_ω appearing in Henkin's proof is completely avoided here. Moreover, the present derivation* [...] *has the special advantage of bringing out the essentially algebraic character to the method first used by Henkin.* [13]

3 The mathematical context

RASIOWA's attitude towards logic is also shaped by three technical points which she takes from her masters and which automatically place her and her work inside a specific mathematical framework. Moreover, in these points I see certain *messages* for today's researchers in Logic, either pure or applied.

First, she adopts LINDENBAUM's idea of ***treating the set of formulas of a formal language as an abstract algebra***, namely the absolutely free algebra ***Fm*** generated by some set *Var* of *variables* or atomic formulas and where the operations correspond to the so-called *logical connectives* (either sentential or quantifiers); see [45, § VIII.2]. Historically, this was an important achievement on the way towards the mathematization of formal logic; by this idea, formal languages can be treated by the usual tools of algebra, substitutions and interpretations become just homomorphisms, and the subject is liberated from the slight degree of obscurantism or imprecision that pervaded its early history.

It is interesting to compare this with PAUL HALMOS' explanations of how he became involved in Algebraic Logic; his difficulties in starting show us that the modern "Polish approach" to logic was not widespread in western academic circles, even in the fifties:

> *An exposition of what logicians call the propositional calculus can annoy and mystify mathematicians. It looks like a lot of fuss about the use of parentheses, it puts much emphasis on the alphabet, and it gives detailed consideration to "variables" (which do not in any sense vary).* [...] *it is hard to see what genuine content the subject has.* [...] *Does it really have any mathematical value?*
>
> *Yes, it does.* [...] *rooting around in the library* [...] *bit by bit the light dawned. Question: what is the propositional calculus? Answer: the theory of free Boolean algebras with a countably infinite set of generators.* [...]
>
> *"Truth-tables", for instance, are nothing but the clumsiest imaginable way of describing homomorphisms into the two-element Boolean algebra.* [...] *The algebraic analogue of the logical concept of "semantic completeness" is semisimplicity.*
>
> [21, pp. 206–207]

Second, she follows her supervisor Andrzej Mostowski [32] in ***the way quantifiers are interpreted in the models***, namely as the generalized lattice-theoretic operations of join (for the existential quantifier) and meet (for the universal one) relative to the ordering relation existing in the models, which is defined by the implication (see Section 4.2). This choice, also taken independently by Henkin in [23], constitutes the main distinctive character of her approach to the algebraization of first-order logic. The two other best-known approaches, that of Tarski's school [24, 25, 26] with cylindric algebras, and that of Halmos [20] with polyadic algebras, both choose to represent quantifiers as independent primitive operations in the models (roughly speaking, one for each free variable in the first case, and one for each subset of the set of free variables in the second).

And third, she takes from Tarski [60] the idea of ***defining a logic as a finitary closure operator over the algebra of formulas***; see [45, §§VIII.4,5]. Although she assumes this operator is defined through the standard notion of proof in a formal system given by some axioms and inference rules, she hardly ever makes assumptions about the formal system itself, but only about the resulting closure operator. In this way, she emphasizes the *deductive character* of logic; this means that a logic is not just a collection of axioms and rules, or a collection of the associated theorems, as it is often understood, but a *relation* of consequence; in other words, she reminds us that *logic is about inference rather than about truth*.

If $\boldsymbol{Fm}$ is the formula algebra, with underlying set of formulas Fm, and $\vdash_{\mathcal{S}}$ represents the notion of proof from assumptions in the logic $\mathcal{S}$, then what she considers and studies is the closure operator $C_{\mathcal{S}} : P(Fm) \to P(Fm)$ defined by

$$\alpha \in C_{\mathcal{S}}(\Sigma) \iff \Sigma \vdash_{\mathcal{S}} \alpha \tag{2}$$

The properties postulated for this operator are the following, for all $\Sigma, \Delta \subseteq Fm$:

(C1) $\Sigma \subseteq C_{\mathcal{S}}(\Sigma)$.

(C2) If $\Sigma \subseteq \Delta$ then $C_{\mathcal{S}}(\Sigma) \subseteq C_{\mathcal{S}}(\Delta)$.

(C3) $C_{\mathcal{S}}(C_{\mathcal{S}}(\Sigma)) = C_{\mathcal{S}}(\Sigma)$.

(C4) If $\alpha \in C_{\mathcal{S}}(\Sigma)$ then there exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\alpha \in C_{\mathcal{S}}(\Sigma_0)$.

This is very close to the definition of sentential logic used in today's studies in Algebraic Logic, cf. [63], save that she does not mention as a general assumption the property traditionally called *structurality*[3]:

(C5) If $\alpha \in C_{\mathcal{S}}(\Sigma)$ then $\sigma(\alpha) \in C_{\mathcal{S}}(\sigma[\Sigma])$ for any substitution (i.e., homomorphism) σ of $\boldsymbol{Fm}$ into itself.

that is, that the relation of consequence should be invariant under substitutions; however, the only requirement she puts on the formal system is precisely that axioms and rules of inference have to be invariant under substitutions; thus she actually obtains (C5) for the operator defined by (2).

We can observe here an important element of Rasiowa's view of Mathematical Logic, and which is also present, and indeed prominent, in her contributions to applications of Logic. For Rasiowa, a logic is a mathematical object that is essentially algebraic in nature, as I have just pointed out; however, she establishes ***a sharp distinction between logics and algebras***. For her, a logic is a closure operator on the algebra of formulas, not just a particular algebra. Algebras can certainly be used as models of logics,; they can be used even to define logics (as in many-valued logic) but *algebras are not logics themselves*. By studying the relationships between logics and algebras while retaining the conceptual status of each she was able to uncover the usually implicit assumptions about these relationships, which eventually led her to succeed in her endeavour.

4 The main technical tools

In this section I will analyse the technical details of what I think are the more central points in Rasiowa's work in the area I am concerned with. Except for sections 4.2 and 4.6, I will refer only to the sentential case; this is because, on the one hand, the details are much longer and clumsier when a first-order language enters into the picture, and, on the other hand, because Rasiowa's treatment of first-order logics is just an extension of her treatment of propositional logics, as is clear from the "Supplement" to [45] (pp. 347–379): While she selects the class of propositional logics to be studied, and for each logic $\mathcal{S}$ in the class the corresponding class $\mathsf{Alg}^*\mathcal{S}$ of algebras is defined, what she does in the first order case is to associate a class of first-order logics (one for each first-order language) to each of these propositional logics, and to algebraize them through the study of the same class $\mathsf{Alg}^*\mathcal{S}$.

4.1 Interpretation of formulas as mappings

The first technical tool I want to highlight is that of interpreting formulas α as mappings $\alpha^{\boldsymbol{A}}$ on every algebra $\boldsymbol{A}$ of the same similarity type (signature) as the formal language; see [45, §VIII.3]. This idea is a generalization of Łukasiewicz

[3]Note that it has nothing to do with the *structural rules* of Proof Theory and Gentzen systems.

and Post's method of truth-tables (where one usually deals with a single, concrete algebra), and was extended to intuitionistic predicate logic (where one deals with the whole class of so-called Heyting algebras) by Mostowski.

In the sentential case, an interpretation $\overline{a}$ is obtained by just an assignment of values in $\boldsymbol{A}$ to the variables Var, that is, it is any mapping $\overline{a} : Var \to A$; since $\boldsymbol{Fm}$ is the absolutely free algebra generated by Var, one can define $\alpha^{\boldsymbol{A}}(\overline{a}) = \overline{a}(\alpha)$, the value of α under the homomorphism (denoted also as $\overline{a}$) from $\boldsymbol{Fm}$ to $\boldsymbol{A}$ that uniquely extends $\overline{a}$ in the usual way; for instance, if the language has just negation $\neg$ and implication $\to$ then the recursive clauses would be:

$$\text{If } \ p \in Var \ \text{ then } \ \overline{a}(p) \ \text{ is determined by the original mapping } \ \overline{a}. \tag{3}$$

$$\text{If } \ \alpha = \neg\beta \ \text{ then } \ \overline{a}(\alpha) = \neg^{\boldsymbol{A}}(\overline{a}(\beta)). \tag{4}$$

$$\text{If } \ \alpha = \beta \to \gamma \ \text{ then } \ \overline{a}(\alpha) = \overline{a}(\beta) \to^{\boldsymbol{A}} \overline{a}(\gamma). \tag{5}$$

We have denoted by $\neg^{\boldsymbol{A}}$ and $\to^{\boldsymbol{A}}$ the interpretations of the logical connectives as operations in the algebra $\boldsymbol{A}$, in order to emphasize the interplay between language and algebras; usually one denotes the operations in arbitrary algebras by the same symbols as those of the language.

4.2 Algebraic interpretation of quantifiers

In the first-order case, an interpretation $\overline{a}$ requires the specification of:

- A domain of individuals $D(\overline{a})$.
- Values in $D(\overline{a})$ for the constant symbols and the free variables of the language.
- A function with arguments and values in $D(\overline{a})$ for each functional symbol of the language.
- An $\boldsymbol{A}$-valued function with arguments in $D(\overline{a})$ for each of the predicate or relational symbols of the language.

Then the value $\alpha^{\boldsymbol{A}}(\overline{a})$ is obtained from the atomic cases (where it is given directly by the latter $\boldsymbol{A}$-valued functions) by using the algebraic structure of $\boldsymbol{A}$ for the propositional connectives as in (3)–(5) above, and by interpreting the quantifiers as the infinite lattice-theoretic operations as follows: Let $\alpha(x)$ be a formula with the free variable x, and let ξ be a bound variable not occurring in $\alpha(x)$ (Rasiowa takes two disjoint sets for the free and the bound variables, a little used trick which simplifies several technical points); denote by $\alpha(\xi)$ the substitution instance of $\alpha(x)$ with x replaced by ξ. Then:

$$\big(\exists\xi\,\alpha(\xi)\big)^{\boldsymbol{A}}(\overline{a}) \ = \bigvee_{i\in D(\overline{a})} \big(\alpha(x)\big)^{\boldsymbol{A}}\big(\overline{a}[x/i]\big) \tag{6}$$

$$\big(\forall\xi\,\alpha(\xi)\big)^{\boldsymbol{A}}(\overline{a}) \ = \bigwedge_{i\in D(\overline{a})} \big(\alpha(x)\big)^{\boldsymbol{A}}\big(\overline{a}[x/i]\big) \tag{7}$$

where $\overline{a}[x/i]$ is the interpretation that is exactly like $\overline{a}$ in every respect except that it gives the variable x the value i. Actually, definitions (6) and (7) are used only when the involved join and/or meet exists; otherwise one leaves the truth-value of $\exists\xi\,\alpha(\xi)$ or $\forall\xi\,\alpha(\xi)$ undefined. Because of this, simpler expositions of the semantics, like that in [55], use only *complete* lattices; but in other papers by RASIOWA several technical results about existence of bounds of certain families of elements, about completion of algebras of the relevant class, and about mappings preserving some infinite meets and joins, are proved; indeed, this purely algebraic, technical work is among the most difficult tasks that RASIOWA undertook.

We see here that the seed of *many-valuedness* is already present even in her treatment of classical logic: In classical model theory of first-order logic predicate symbols are represented by ordinary relations (of suitable arity) over the domain of individuals, that is, by **2**-valued functions. Here, when we speak of "interpretation in an algebra $\boldsymbol{A}$", what we mean is that this algebra is playing the rle of the set of truth-values; the relational symbols are interpretation as A-valued functions, therefore the interpretation of every formula is a value in A.

4.3 Lindenbaum-Tarski quotients

The specific technical construction that establishes a link between the logic $\mathcal{S}$ and the class of algebras $\mathbf{Alg}^*\mathcal{S}$ (see its definition in Section 4.5) and that makes it possible to prove the hard half ($\Leftarrow$) of the Strong Completeness Theorem (1) is the factorization of the formula algebra by an equivalence relation associated with every theory of the logic. This construction has to be credited to TARSKI, although in the first years after World War II it was initially credited to LINDENBAUM, particularly by Polish logicians, which explains the now usual denomination of *Lindenbaum-Tarski algebras.* The contents of footnote 1 on pages 245–246 of [55] make one think that this attribution to LINDENBAUM had a political or nationalistic component; but it was given particular impetus, in my opinion, by many people's misinterpretation of a remark in MCKINSEY's 1941 paper [29]. On page 122, lines 12–15 of [29] we read:

> *Proof: I first show, by means of an unpublished method of Lindenbaum,[7] that there is a matrix $\mathfrak{M}_1 = (K_1, D_1, -_1, *_1, \times_1)$ which is S2-characteristic, though not normal. Later I shall show how a normal S2-characteristic matrix can be constructed from $\mathfrak{M}_1$.*

Footnote 7 on the same page reads:

> [7] *This method is very general, and applies to any sentential calculus which has a rule of substitution for sentential variables. The method was explained to me by Professor Tarski, to whom I am also indebted for many other suggestions in connection with the present paper.*

The subsequent proof begins by constructing a matrix whose underlying algebra is the formula algebra, in accordance with LINDENBAUM's idea explained in Section 3, and after that a normal matrix is constructed by factorizing the first one; as the word *"Later"* on line 14 suggests, MCKINSEY himself was probably aware that this second step had not been invented by LINDENBAUM. In their completeness paper [48] RASIOWA and SIKORSKI name the factorized algebra after LINDENBAUM, but FEFERMAN, in his review [13] of this paper, points out that the usage of this construction seems to appear for the first time in Tarski's 1935 paper [61]; later on, Tarski himself claimed it as his own, see for instance footnote 4 on page 85 of [26].

The construction is as follows: For every theory Σ of the logic $\mathcal{S}$, the relation $\equiv_\Sigma$ is defined as:

$$\varphi \equiv_\Sigma \psi \iff \Sigma \vdash_\mathcal{S} \varphi \to \psi \text{ and } \Sigma \vdash_\mathcal{S} \psi \to \varphi \tag{8}$$

Then one has to check the following key facts:

(L1) The relation (8) is a *congruence* of the formula algebra $\boldsymbol{Fm}$.

(L2) The quotient algebra $\boldsymbol{Fm}/{\equiv_\Sigma} \in \mathsf{Alg}^*\mathcal{S}$.

(L3) The projection $\pi_\Sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}/{\equiv_\Sigma}$ given by $\pi_\Sigma(p) = p/{\equiv_\Sigma}$ is an interpretation into an algebra of the class $\mathsf{Alg}^*\mathcal{S}$, such that for every formula $\varphi \in Fm$, $\varphi^{\boldsymbol{Fm}/\equiv_\Sigma}(\pi_\Sigma) = \varphi/{\equiv_\Sigma}$.

(L4) In this quotient the theory Σ collapses exactly to the unit, that is, for every formula $\varphi \in Fm$, $\varphi/{\equiv_\Sigma} = \mathbf{1}$ if and only if $\Sigma \vdash_\mathcal{S} \varphi$.

Given these facts, the proof of part ($\Leftarrow$) of (1) is easy, working by contraposition: If $\Sigma \nvdash_\mathcal{S} \varphi$ then there is an algebra $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{S}$, namely $\boldsymbol{Fm}/{\equiv_\Sigma}$, and an interpretation $\overline{a}$ into it, namely π_Σ, such that $\Sigma^{\boldsymbol{A}}(\overline{a}) = \mathbf{1}$ while $\varphi^{\boldsymbol{A}}(\overline{a}) \neq \mathbf{1}$. As witnessed by Feferman's quotation from [13] reproduced in Section 2, this completeness proof is not far removed in spirit from Henkin's [22], although the method of construction of the model is very different: actually in both cases the models are obtained from the linguistic objects, the formulas. It is interesting to notice that Henkin himself, independently of Rasiowa and Sikorski, found essentially the same proof by following Mostowski's suggestions directly, and was quickly aware of the possibilities of generalizing such method; apparently, he was the first to notice that only implication was required for the whole process to work, and his paper [23] appeared in the same volume of *Fundamenta Mathematicae* as Rasiowa and Sikorski's [48].

4.4 Selection of the class of logics

The success of the proof in the preceding section determines the class of logics that can be treated with this method. One is tempted to think that the class identified in [45, §VIII.5] as ***standard systems of implicative extensional propositional calculi***, is the class of logics $\mathcal{S}$ such that for every theory Σ of $\mathcal{S}$ properties (L1) to (L4) hold. Actually, this is not strictly true: in the preceding section I stated the steps just needed for the proof to work, but in order to obtain exactly the same class of logics explicitly considered by Rasiowa one should consider the binary relation

$$\varphi \leq_\Sigma \psi \iff \Sigma \vdash_\mathcal{S} \varphi \to \psi \ , \tag{9}$$

and assume the following slightly stronger conditions:

(L1') The relation $\leq_\Sigma$ is a quasi-ordering (i.e., it is reflexive and transitive).

(L1") The relation $\equiv_\Sigma$ (which is now the symmetrization of $\leq_\Sigma$) is compatible with all the operations of the formula algebra $\boldsymbol{Fm}$ corresponding to the sentential connectives.

(L2) The quotient algebra $\boldsymbol{Fm}/{\equiv_\Sigma} \in \mathsf{Alg}^*\mathcal{S}$.

(L3) The projection $\pi_\Sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}/{\equiv_\Sigma}$ given by $\pi_\Sigma(p) = p/{\equiv_\Sigma}$ is an interpretation into an algebra of the class $\mathsf{Alg}^*\mathcal{S}$, such that for every formula $\varphi \in Fm$, $\varphi^{\boldsymbol{Fm}/\equiv_\Sigma}(\pi_\Sigma) = \varphi/{\equiv_\Sigma}$.

(L4') If $\Sigma \vdash_{\mathcal{S}} \varphi$ then $\psi \leq_\Sigma \varphi$ for every ψ.

(L4") If $\Sigma \vdash_{\mathcal{S}} \varphi$ and $\varphi \leq_\Sigma \psi$ then $\Sigma \vdash_{\mathcal{S}} \psi$.

It is straightforward that these conditions imply (L1) to (L4). As the reader can easily check, (L1) to (L4) as I have put them are enough for the argument to work; undoubtedly this was clear to anyone working in the field at that time. That Rasiowa preferred to take the more restrictive version (L1') to (L4") was probably because it is more natural, since then we have conditions more typical of the implication connective $\to$, while (L1) to (L4) are, in fact, conditions on the equivalence connective $\leftrightarrow$; if there is no conjunction in the language, then the set of two formulas $\{\varphi \to \psi, \psi \to \varphi\}$ can be taken collectively to act as an equivalence connective. Actually, Rasiowa only leaves out few examples, the best-known being the equivalential fragments of classical or intuitionistic logic. The first generalizations of her treatment were undertaken by Czelakowski in [11] precisely by adopting this approach, and gave rise to what he named *equivalential logics with an algebraic semantics* (which later on turned out to be a special case of the *algebraizable logics* of Blok and Pigozzi) and to the much more general class of *equivalential logics*, where (L4) or similar conditions are dropped.

4.5 The algebraic counterpart of a logic

The class of algebras $\mathsf{Alg}^*\mathcal{S}$ is determined by the requirement that the easy half ($\Rightarrow$) of the Completeness Theorem (1), that is, the part sometimes called the ***Soundness Theorem***, holds. According to [45, §VIII.6], an algebra $\boldsymbol{A}$ belongs to $\mathsf{Alg}^*\mathcal{S}$, and is called an ***$\mathcal{S}$-algebra***, if and only if it has a constant $\mathbf{1} \in A$ such that:

(A1) For every axiom α of $\mathcal{S}$ and every interpretation $\overline{a}$ into $\boldsymbol{A}$ $\alpha^{\boldsymbol{A}}(\overline{a}) = \mathbf{1}$.

(A2) For any inference rule $\alpha_1, \ldots, \alpha_n \vdash \beta$ of $\mathcal{S}$ and any interpretation $\overline{a}$ into $\boldsymbol{A}$, if $\alpha_i^{\boldsymbol{A}}(\overline{a}) = \mathbf{1}$ for $i = 1, \ldots, n$, then also $\beta^{\boldsymbol{A}}(\overline{a}) = \mathbf{1}$.

(A3) For any $a, b \in A$, if $a \to b = \mathbf{1}$ and $b \to a = \mathbf{1}$ then $a = b$.

Actually (A1) and (A2) together are equivalent to part ($\Rightarrow$) of (1), and amount to saying that the pair $\langle \boldsymbol{A}, \{\mathbf{1}\}\rangle$ is what has come to be called an ***$\mathcal{S}$-matrix***. The additional condition (A3) tells us that the algebras in the class are *reduced*, in some precise, technical sense, which roughly speaking means that, from among the models for $\mathcal{S}$, we want to select those algebras which turn logical equivalences into identities.

This definition of the class $\mathsf{Alg}^*\mathcal{S}$ *is historically the first general definition of what the algebraic counterpart of a logic* $\mathcal{S}$ *should be.* From any presentation of the logic $\mathcal{S}$ by axioms and rules the above conditions give a presentation of the class $\mathsf{Alg}^*\mathcal{S}$ by means of equations (A1) and quasi-equations (A2) and (A3); thus this class is always a *quasi-variety*. In [11], Czelakowski proved that for the logics treated by RASIOWA the class $\mathsf{Alg}^*\mathcal{S}$ defined by her coincides with the class of the algebraic reducts of the reduced $\mathcal{S}$-matrices, which is the class of algebras canonically associated with each logic $\mathcal{S}$ in the general theory of matrices.

The organization of material in [45] is striking: While conceptually its central topic is the study of logics, its first half is devoted to a systematic study of the algebraic and order-theoretic properties of several classes of algebras, and only in its second half is a general theory of sentential logics and its algebraization presented; after the general theory, the treatment of several particular logics uses the properties of the corresponding class of algebras contained in the first part of the book. The widest class of algebras she studies is that of *implicative algebras*; but the weakest logic she considers is HILBERT and BERNAYS' *logic of positive implication*, whose algebraic counterpart is a smaller class; only in Exercise VIII.1 (p. 208) does she ask the reader to construct a calculus whose associated class of algebras is exactly the class of implicative algebras[4].

4.6 Representation Theorems and the "Rasiowa-Sikorski Lemma"

A widespread criticism of the use of algebraic semantics such as $\mathsf{Alg}^*\mathcal{S}$ and of the significance of Completeness Proofs using LINDENBAUM-TARSKI quotients is that this semantics is not very different from syntax. Thus, it is particularly important to obtain Completeness Theorems for classes K of algebras more restricted than $\mathsf{Alg}^*\mathcal{S}$. The class K is usually the class of algebras whose universe is contained in a power set or in the family of open or closed sets of some topological space; in the most extreme case, K is constituted by a single algebra, for instance by the two-element Boolean algebra in the case of classical first-order logic.

If the class K is contained in $\mathsf{Alg}^*\mathcal{S}$, then part ($\Rightarrow$) of (1) holds also for K. To prove the converse by contraposition as in Section 4.3, one first obtains $\boldsymbol{Fm}/{\equiv_\Sigma}$ and π_Σ, and then applies some kind of *representation-like theorem* which maps the algebra $\boldsymbol{Fm}/{\equiv_\Sigma}$ to an algebra $\boldsymbol{A} \in \mathsf{K}$ in such a way that the "separation" of Σ and φ through π_Σ is preserved. The composition of π_Σ with the representation mapping becomes an interpretation into $\boldsymbol{A}$ which validates Σ but not φ, as desired. This kind of restricted completeness rests on the algebraic properties of the class of algebras $\mathsf{Alg}^*\mathcal{S}$. In the propositional case, this is all that is needed, and this explains why some of the purely algebraic works of RASIOWA such as [4, 41] are devoted to representability issues.

The last, but certainly not the least, of the points in RASIOWA's work that I want to highlight is a purely algebraic result known in the literature as *the* RASIOWA-SIKORSKI *Lemma*. It is a result in the representation theory of Boolean algebras as fields of sets, and becomes relevant to the topic of algebraization of classical *first-order* logic precisely through the application of the above procedure. In the case of first-order logics we need something more: representation theorems establish just algebraic homomorphisms, which in general may not be complete in the lattice-theoretical sense, that is, they may not preserve the join or meet of an infinite family (while preserving the finite ones). In

[4]This may be the appropriate place to say that Exercises II.1 and VIII.2 of [45], on the relation between the theories of this logic, the kernels of epimorphisms between implicative algebras, and special implicative filters, are wrong. See [15] for details.

order for the above mentioned composition to be an interpretation, these representations should at least preserve the infinite joins (6) and meets (7) needed to interpret the quantifiers.

In the case of classical first-order logic, we have Boolean algebras, and what is strictly called *the Rasiowa-Sikorski Lemma*; it has also been called "TARSKI's Lemma" (see for instance [2], pp. 21, 31) because RASIOWA and SIKORSKI's original proof was rather indirect and complicated, using STONE's representation of Boolean algebras and some topological properties, and TARSKI, as stated in [13], suggested a more natural proof, which has been much reproduced. The precise statement is:

> LEMMA. Let $\boldsymbol{A}$ be a Boolean algebra and $a \in A$, and assume that we have two countable collections of subsets $X_n, Y_n \subseteq A$ $(n \in \omega)$ such that for each $n \in \omega$ the elements $a_n = \bigvee X_n$ and $b_n = \bigwedge Y_n$ exist. Then there is a Boolean homomorphism h from $\boldsymbol{A}$ onto the two-element Boolean algebra $\mathbf{2}$ such that $h(a) = \mathbf{1}$ and for every $n \in \omega$, $h(a_n) = \bigvee h[X_n]$ and $h(b_n) = \bigwedge h[Y_n]$.

Since the epimorphisms from an arbitrary Boolean algebra onto $\mathbf{2}$ are determined by its ultrafilters, the Lemma is often formulated as stating the existence of an ultrafilter containing the given element a and preserving the two given denumerable families of joins and meets; but even in this case, the condition of preservation is formulated by using the homomorphism.

According to [59, p. 102], the result was originally found by SIKORSKI, but it was first published in the joint paper [48], and RASIOWA's name has also remained tied to it, together with SIKORSKI's; I think this is right, since in addition she generalized the Lemma to other classes of algebras whose representation theory she studied, in order to obtain strengthened completeness theorems for the corresponding predicate logics, [42, 46, 47]; moreover, if only a few mathematicians deserve the honour of having their name permanently attached to a mathematical result, RASIOWA is undoubtedly one of them.

References

[1] BARTOL, W., ORŁOWSKA, E., AND SKOWRON, A. Helena Rasiowa, 1917–1994. *Modern logic 5* (1995), 231–247.

[2] BELL, J. L., AND SLOMSON, A. B. *Models and ultraproducts: An introduction.* North-Holland, Amsterdam, 1971. Second revised printing.

[3] BETH, E. W. Review of [54]. *Mathematical Reviews*, MR22:1510.

[4] BIAŁINICKI-BIRULA, A., AND RASIOWA, H. On the representation of quasi-Boolean algebras. *Bulletin de l'Académie Polonaise des Sciences, Classe III 5* (1957), 259–261.

[5] BIAŁINICKI-BIRULA, A., AND RASIOWA, H. On constructible falsity in the constructive logic with strong negation. *Colloq. Math. 6* (1958), 287–310.

[6] BLOK, W., AND PIGOZZI, D. *Algebraizable logics*, vol. 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989. Out of print. Scanned copy available from `http://orion.math.iastate.edu:80/dpigozzi/`.

[7] Brown, D. J., and Suszko, R. Abstract logics. *Dissertationes Math. (Rozprawy Mat.) 102* (1973), 9–42.

[8] Cat-Ho, N., and Rasiowa, H. Semi-Post algebras. *Studia Logica 46* (1987), 149–160.

[9] Cat-Ho, N., and Rasiowa, H. Subalgebras and homomorphisms of semi-Post algebras. *Studia Logica 46* (1987), 161–175.

[10] Cat-Ho, N., and Rasiowa, H. Plain semi-Post algebras as a poset-based generalization of Post algebras and their representability. *Studia Logica 48* (1989), 509–530.

[11] Czelakowski, J. Equivalential logics, I, II. *Studia Logica 40* (1981), 227–236 and 355–372.

[12] Czelakowski, J. *Protoalgebraic logics*, vol. 10 of *Trends in Logic - Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[13] Feferman, S. Review of [48]. *The Journal of Symbolic Logic* (1952), XVII:72a.

[14] Font, J. M. Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the IGPL 5* (1997), 413–440.

[15] Font, J. M. On special implicative filters. *Mathematical Logic Quarterly 45* (1999), 117–126.

[16] Font, J. M., and Jansana, R. *A general algebraic semantics for sentential logics. Second revised edition*, vol. 7 of *Lecture Notes in Logic*. Association for Symbolic Logic, 2009. First edition published in 1996. Electronic version freely available through Project Euclid at `http://projecteuclid.org/euclid.lnl/1235416965`.

[17] Font, J. M., and Rius, M. A four-valued modal logic arising from Monteiro's last algebras. In *Proc. 20th Int. Symp. Multiple-Valued Logic* (Charlotte, 1990), The IEEE Computer Society Press, pp. 85–92.

[18] Font, J. M., and Rius, M. An abstract algebraic logic approach to tetravalent modal logics. *The Journal of Symbolic Logic 65* (2000), 481–518.

[19] Font, J. M., and Verdú, V. Abstract characterization of a four-valued logic. In *Proc. 18th Int. Symp. Multiple-Valued Logic* (Palma de Mallorca, 1988), The IEEE Computer Society Press, pp. 389–396.

[20] Halmos, P. R. *Algebraic logic.* Chelsea Publ. Co., New York, 1962.

[21] Halmos, P. R. *I want to be a mathematician. An automathography.* Springer-Verlag, Heidelberg and New York, 1985.

[22] Henkin, L. The completeness of the first order functional calculus. *The Journal of Symbolic Logic 14* (1949), 159–166.

[23] Henkin, L. An algebraic characterization of quantifiers. *Fundamenta Mathematicae 37* (1950), 63–74.

[24] Henkin, L., Monk, J. D., and Tarski, A. *Cylindric algebras, part I.* North-Holland, Amsterdam, 1971.

[25] Henkin, L., Monk, J. D., and Tarski, A. *Cylindric algebras, part II.* North-Holland, Amsterdam, 1985.

[26] Henkin, L., and Tarski, A. Cylindric algebras. In *Lattice Theory*, R. P. Dilworth, Ed., vol. 2 of *Proceedings of Symposia in Pure Mathematics.* American Mathematical Society, 1961, pp. 83–113.

[27] Kreisel, G. Review of [55]. *Mathematical Reviews*, MR29:1149.

[28] Łoś, J., Mostowski, A., and Rasiowa, H. A proof of Herbrand's theorem. *Journal de Mathématiques Pures et Appliquées 35* (1956), 19–24.

[29] McKinsey, J. C. C. A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. *The Journal of Symbolic Logic 6* (1941), 117–134.

[30] Monteiro, A. Matrices de Morgan caractéristiques pour le calcul propositionnel classique. *Anais da Academia Brasileira de Ciencias 33* (1960), 1–7.

[31] Monteiro, A. Algebras de Morgan (curso de algebra de la lógica III). 74 pp. Unpublished manuscript, Bahía Blanca (Argentina), 1966.

[32] Mostowski, A. Proofs of non-deducibility in intuitionistic functional calculus. *The Journal of Symbolic Logic 13* (1948), 204–207.

[33] Pigozzi, D., and Salibra, A. An introduction to lambda abstraction algebras. In *Proceedings of the IX Simposio Latino-Americano de Lógica Matemática* (Bahía Blanca, Argentina, 1992), M. Abad, Ed., vol. 38 of *Notas de Lógica Matemática*, Instituto de Matemática, Universidad Nacional del Sur, pp. 93–112.

[34] Pigozzi, D., and Salibra, A. The abstract variable-binding calculus. *Studia Logica 55* (1995), 129–179.

[35] Rasiowa, H. Algebraic treatment of the functional calculi of Heyting and Lewis. *Fundamenta Mathematicae 38* (1951), 99–126.

[36] Rasiowa, H. A proof of the compactness theorem for arithmetical classes. *Fundamenta Mathematicae 39* (1952), 8–14.

[37] Rasiowa, H. Constructive theories. *Bulletin de l'Académie Polonaise des Sciences, Classe III 2* (1954), 121–124.

[38] Rasiowa, H. Algebraic models of axiomatic theories. *Fundamenta Mathematicae 41* (1955), 291–310.

[39] Rasiowa, H. On the ε-theorems. *Fundamenta Mathematicae 43* (1956), 156–165.

[40] Rasiowa, H. Sur la méthode algébrique dans la méthodologie des systèmes déductifs élementaires. *Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 1 (49)* (1957), 223–231.

[41] Rasiowa, H. $\mathcal{N}$-lattices and constructive logic with strong negation. *Fundamenta Mathematicae 46* (1958), 61–80.

[42] Rasiowa, H. A theorem on the existence of prime filters in Post algebras and the completeness theorem for some many-valued predicate calculi. *Bulletin de l'Académie Polonaise des Sciences, Classe III 17* (1969), 347–354.

[43] Rasiowa, H. Ultraproducts of m-valued models and a generalization of the Löwenheim-Skolem-Gödel-Mal'cev theorem for theories based on m-valued logics. *Bulletin de l'Académie Polonaise des Sciences, Classe III 18* (1970), 415–420.

[44] Rasiowa, H. On generalized Post algebras of order ω^+ and ω^+-valued predicate calculi. *Bulletin de l'Académie Polonaise des Sciences, Classe III 21* (1973), 209–219.

[45] Rasiowa, H. *An algebraic approach to non-classical logics*, vol. 78 of *Studies in Logic and the Foundations of Mathematics.* North-Holland, Amsterdam, 1974.

[46] Rasiowa, H. Post algebras as a semantic foundation of m-valued logics. In *Studies in algebraic logic*, A. Daigneault, Ed., vol. 9 of *Studies in Mathematics*. The Mathematical Association of America, 1974, pp. 92–142.

[47] Rasiowa, H. Topological representations of Post algebras of order ω^+ and open theories based on ω^+-valued Post logic. *Studia Logica 44* (1985), 353–368.

[48] Rasiowa, H., and Sikorski, R. A proof of the completeness theorem of Gödel. *Fundamenta Mathematicae 37* (1950), 193–200.

[49] Rasiowa, H., and Sikorski, R. A proof of the Skolem-Löwenheim theorem. *Fundamenta Mathematicae 38* (1951), 230–232.

[50] Rasiowa, H., and Sikorski, R. Algebraic treatment of the notion of satisfability. *Fundamenta Mathematicae 40* (1953), 62–95.

[51] Rasiowa, H., and Sikorski, R. On satisfiability and decidability in non-classical functional calculi. *Bulletin de l'Académie Polonaise des Sciences, Classe III 1* (1953), 229–231.

[52] Rasiowa, H., and Sikorski, R. On existential theorems in non-classical functional calculi. *Fundamenta Mathematicae 41* (1954), 21–28.

[53] Rasiowa, H., and Sikorski, R. An application of lattices to logic. *Fundamenta Mathematicae 42* (1955), 83–100.

[54] Rasiowa, H., and Sikorski, R. On the Gentzen theorem. *Fundamenta Mathematicae 48* (1959), 57–69.

[55] Rasiowa, H., and Sikorski, R. *The mathematics of metamathematics.* P.W.N., Warszawa, 1963. (3rd edition, 1970).

[56] Robinson, A. Review of [39]. *Mathematical Reviews*, MR18:711g.

[57] Rousseau, G. Logical systems with finitely many truth-values. *Bulletin de l'Académie Polonaise des Sciences, Classe III 17* (1969), 189–194.

[58] Rousseau, G. Post algebras and pseudo-Post algebras. *Fundamenta Mathematicae 67* (1970), 133–145.

[59] Sikorski, R. *Boolean algebras*, 3rd. ed. Springer-Verlag, Berlin, 1969.

[60] Tarski, A. Über einige fundamentale Begriffe der Metamathematik. *C. R. Soc. Sci. Lettr. Varsovie, Cl. III 23* (1930), 22–29. English translation, revised and annotated by the author, in [62], pp. 30–37.

[61] Tarski, A. Grundzüge der Systemenkalkül. *Fundamenta Mathematicae 25;26* (1935;1936), 503–526;283–301. English translation, revised and annotated by the author, in [62], pp. 342–383.

[62] Tarski, A. *Logic, Semantics, Metamathematics. Papers from 1923 to 1938.* Clarendon Press, Oxford, 1956. 2nd edition by Hackett Pub. Co., Indianapolis, 1983.

[63] Wójcicki, R. *Theory of logical calculi. Basic theory of consequence operations*, vol. 199 of *Synthèse Library.* Reidel, Dordrecht, 1988.