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Logic for Artificial Intelligence: A Rasiowa–Pawlak School Perspective

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"The problem of understanding of intelligence is said to be the greatest problem in science today and "the" problem for this century – as deciphering the genetic code was for the second half of the latest one.

Arguably, the problem of learning represents a gateway to understanding intelligence in brains and machines, to discovering how the human brain works and to making intelligent machines that learn from experience and improve their competence..."

THE MATHEMATICS OF LEARNING:
DEALING WITH DATA
T. Poggio, S. Smale, Notices AMS, Vol. 50, May 2003 [185]

Abstract. The Rasiowa–Pawlak school was established during the second half of the twentieth century. The school concentrates on studies in logics, foundations of computer science and artificial intelligence (AI). Its formation has been greatly influenced by the logician Andrzej Mostowski, a professor at Warsaw University [110,111], who, in particular, directed the doctoral dissertation of Helena Rasiowa. Nowadays, the disciples of the Rasiowa–Pawlak school are active in many research-development centres worldwide. The school founded its own journal, Fundamenta Informaticae. In this paper, we present selected trends in the studies of the school concerning applications of logic in AI. At the beginning, we briefly describe the genesis of the Rasiowa–Pawlak school. We then present the understanding, currently dominating within the school, on such basic concepts as AI and logic. Since the beginning of the 1950's, the focus of the research by Helena Rasiowa and her associates has been the application of algebraic and topological methods to the investigation of crucial problems of logic from an AI perspective. Amongst them are the completeness theorem, construction of deduction systems, construction of models, especially models for constructive mathematics [136,241,67] and related logics such as intuitionistic, intermediate, modal, and approximation logics. In the paper, we discuss the fundamental, in our opinion, ideas underlying these roots of the Rasiowa–Pawlak school. A great importance in the studies of the school is assigned to the search for optimal tools for reasoning about complex vague concepts, construction of knowledge representation systems, reasoning about knowledge as well as for the application of logics in learning, communication, perception, planning, action, cooperation, and competition.

It should be noted that as is the case with many other research centers, the Rasiowa– school studies pertaining to the application of logics in AI have also undergone an evolution which we present in this paper. We include extensive references to the literature on the approach presented in this paper.

Keywords. Logic, AI, algebraic logic, abstract logic, approximation, wisdom technology, adaptive rough-granular computing, rough sets, machine learning.

1. The Genesis of the Rasiowa–Pawlak School

Since Poland regained its independence during the XX century after about 150 years of annexation by Russia, Germany and Austria, the Polish intelligentsia (both political as well as academic) has placed great importance on the design and the deployment of many action plans aiming at establishing a firm position for Poland internationally. These plans were, to a great extent, results of the considerations and actions of intellectuals affiliated with positivism in the nineteenth century on Polish territories under foreign rules. One of many such action plans focused on designing a research stimulation program in mathematics, logic and philosophy in the free Poland. The most important components of this program were published in a work by Zygmunt Janiszewski [83]. The consistent realization of Janiszewski's program led to the birth of one of the most powerful research centres in mathematics and mathematical logic during that time. Leading figures of the pre-WWII Polish mathematical school were[1] Stefan Banach, Samuel Eilenberg, Kazimierz Kuratowski, Jan Łukasiewicz, Stanisław Mazur, Stanisław Saks, Juliusz Paweł Schauder, Wacław Sierpiński, Hugo Steinhaus, Alfred Tarski, Antoni Szczepan Zygmund, and many others. On the other hand, leaders of the pre-WWII logical school were Kazimierz Ajdukiewicz, Leon Chwistek, Stanisław Jaśkowski, Tadeusz Kotarbiński, Stanisław Leśniewski, Adolf Lindenbaum, Bolesław Sobociński, Alfred Tarski, Kazimierz Twardowski to name a few.[2]

The refulgent expansion of Polish pre-WWII mathematical and logical schools was tragically interrupted with the outbreak of the World II. Many Poles were striving to continue the education and research work within undercover structures organized by the Polish underground state during the war. In such a way, Helena Rasiowa studied logic under the supervision of Jan Łukasiewicz, Bolesław Sobociński, Andrzej Mostowski, Karol Borsuk. Her first Master's thesis supervised by Jan Łukasiewicz and Bolesław Sobociński, burnt up during the Warsaw Uprising. After the war, for a short time, she had worked as a teacher of mathematics and then, following the advice of Andrzej Mostowski, she returned to Warsaw University. In 1950, she defended under the supervision of Andrzej Mostowski her PhD thesis on algebraic methods in logics. During his lectures at Warsaw University, Andrzej Mostowski frequently recalled the previously known vision of building a *thinking machine*, i.e., a device capable of not only calculating arithmetic expressions but also of thought process computation. While at it, he used to say the following after Gottfried Wilhelm Leibniz:

[1] In alphabetical order.
[2] In alphabetical order.

> *If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, and say to each other: 'Let us calculate'* [115].

The idea to replace the intuitive process of reasoning with a process of formal evaluations of algebraic expressions was considered by Andrzej Mostowski as crucial and in this context he used to recall another quote from Leibniz:

> *No one else, I believe, has noticed this, because if they had ... they would have dropped everything in order to deal with it; because there is nothing greater that man could do* ([146, p. 57]).

While investigating the problem of undecidability of intuitionistic predicate calculus, Andrzej Mostowski proposed a novel approach to semantics by means of algebraic models with logical values in a pseudo-Boolean algebra [147]. This approach was further studied and extended to investigation of the properties of logics by Helena Rasiowa, Roman Sikorski [196] and many of their disciples (see, e.g., [193,194,19,20,37,60,4,142, 39,162,188]).

After the Second World War, Kazimierz Kuratowski also played a key role in the reconstruction of the Polish mathematical and logical school. He founded the State Institute of Mathematics (PIM)[3] and was its director from 1948 to 1968. At the very beginning of the PIM existence, Kazimierz Kuratowski came up with an initiative to build the first computer in Poland. To do that, he organized the 23rd December 1948 historical meeting attended by Andrzej Mostowski, Krystyn Bochenek, Henryk Greniewski, Leon Łukaszewicz and Romuald Marczyński and created within PIM the Group of Mathematical Apparatuses (GAM). During the meeting, Kazimierz Kuratowski announced that Polish mathematics should occupy itself with computing machines and that the meeting goal was to discuss the possibilities and plans for the construction of the first computer in Poland. As a result of the decisions made afterwards, a preliminary program started in 1952 aimed at building the first Polish computer within GAM [131]. In this project supervised by Romuald Marczyński, Zdzisław Pawlak participated. The first step in developing Polish electronic computers was the construction of mercury-based ultrasonographic memory. The choice of this kind of memory was influenced by the intention to build a computer with sufficiently high speed. Mercury memory influenced the construction of a sequence of Polish computers (EMAL, XYZ, EMAL-2, and BINEG) that continued until 1959. As a result of these experiments, works on first Polish computers were also started up at the Warsaw University of Technology, where a project and a prototype of a first generation vacuum-tube computer was developed in 1960. The prototype was later improved and initiated the UMC-1, the first serial production of computers in Poland. These computers were designed on the basis of an original arithmetic with base '-2' proposed by Zdzisław Pawlak, who engaged in research on models of computer architecture and the summary of his research results was published in 1963 in his habilitation thesis entitled *"Organization of Address-Less Machines"*. In this thesis, Zdzisław Pawlak proposed the investigation of parenthesis-free languages, a generalization of Polish notation introduced by Jan Łukasiewicz in 1924 (see, e.g., [23]). In further stages of his research on computational models, Zdzisław Pawlak paid more and more attention to logical as-

[3] Later incorporated into the system of Polish Academy of Sciences and currently called Mathematical Institute of Polish Academy of Sciences.

pects of computational models. His interests, at that time, were greatly influenced by a disciple of Andrzej Mostowski, namely, Andrzej Ehrenfeucht, who had particular interests in applications of games to problems of definability and decidability (especially in the theory of ordinals) between the 1950s and the 1960s. The results obtained by Andrzej Ehrenfeucht have many applications in, *e.g.*, studies in modern complexity theory (Ehrenfeucht–Fraïssé games).

Since the 1970s, Zdzisław Pawlak's interests were particularly focused on information retrieval, knowledge representation systems and, next, on the logical foundations of design and construction of algorithms devised to represent and process complex and vague concepts using computers operating on data in two-valued logics.

In the early 1970s Zdzisław Pawlak, in cooperation with Victor Marek and Witold Lipski, started investigations on mathematical foundations of information retrieval [166]. Intensive investigations led to deep results in the area (see., e.g., [132,121–123,81]). The close cooperation of Zdzisław Pawlak with Victor Marek continued for more than a decade. In particular, in the early 1980s, Victor Marek was a member of a research group at the Institute of Computer Science of the Polish Academy of Sciences, where Zdzisław Pawlak discovered rough sets and the idea of classifying objects by means of their attributes [167–169]. Zdzisław Pawlak also closely cooperated with many other researchers. We would like to mention here his close cooperation with Ewa Orłowska, Erhard Konrad, and Cecylia Rauszer on knowledge representation systems and rough sets (see, e.g., [157,158,102,103,160,159,171,134,87,217,201]). During the succeeding years, Zdzisław Pawlak refined and amplified the foundations of rough sets and their applications,[4] and nurtured worldwide research in rough sets that has led to over 4000 publications.[5] As a result of this research, rough set theory and its diverse applications, especially in representing and handling complex, vague concepts and perceptions, has emerged and flourished during the recent years (see, e.g., [170,172–174]).

The Rasiowa–Pawlak school has been created on the basis of nearly forty years of seminars along with lectures at Warsaw University given by Helena Rasiowa from the 1950s to the 1990s. The principal threads of the school consisted of the so-called Tuesday seminars on logic and Thursday seminars on application of logic to foundations of computer science. Many people participated actively in these seminars. The majority of the participants are currently scattered across the globe. The authors would like to emphasize the special and unique atmosphere of beneficent cooperation amongst participants in these events as well as the ample and fruitful discussions on the research problems current during that time. The school has managed its own international journal entitled Fundamenta Informaticae, initiated principally by Helena Rasiowa and Zdzisław Pawlak. Since 1977, Fundamenta Informaticae has been one of the main research presentation platforms of the Rasiowa–Pawlak school. Its topical scope is quite broad and includes the majority of research trends in artificial intelligence, logic, mathematics, and theoretical computer science. It is, hence, difficult to describe, even briefly, all these trends. We therefore do not pretend to provide all streams of the Rasiowa–Pawlak school research in this paper. Instead, we aim at surveying of certain aspects of this research related to the application of logic in AI, so characteristic to the Rasiowa–Pawlak school. As a result, many important trends of the school such as algorithmic logics, natural de-

[4] For example, Zdzisław Pawlak received the Best Paper Award for the paper on rough sets at the ACM 23rd Annual Conference on Computer Science in Nashville, TN, USA in 1995.

[5] See http://rsds.univ.rzeszow.pl/.

duction and non-classical-logic-resolution-based reasoning systems, logical aspects of concurrent processes, application of universal algebra to computational models, algebraic aspects of non-Fregean logics will be omitted. While selecting materials for this paper, we would like, first of all, to underscore the research directions on logics in the Rasiowa–Pawlak school, which we deem particularly important for the further advancement of AI. The interested readers are encouraged to consult many other detailed works, especially on the algebraic approach (see [193,194,196,19,20,4,162,37,198–200] and http://rsds.univ.rzeszow.pl/).

In the context of the studies on application of logic to computer science conducted in the 60s and 70s, in one of these trends, a special stress was placed on understanding algorithms and computer programs in purely logical terms. For this task, a special logic called algorithmic logic [4,142] has been developed within the school, and the programming language LOGLAN based on this logic was invented [105].

It is difficult to present a complete list of researchers whose investigations were substantially influenced in different periods of their scientific activity by Helena Rasiowa or Zdzisław Pawlak. Among them are Lech Banachowski, Mohua Banerjee, Wiktor Bartol, Jan Bazan, Malcolm Beynon, Leonard Bolc, Gianpiero Cattaneo, Mihir Kumar Chakraborty, Newton da Costa, Andrzej Czyżewski, Wiktor Dańko, Piotr Dembiński, Patrick Doherty, Jan Doroszewski, Albert Dragalin, Didier Dubois, Ivo Düntsch, George Epstein, Anna Gomolińska, Jerzy Grzymała-Busse, Petr Hajek, Tsutomu Hosoi, Masahiro Inuiguchi, Andrzej Jankowski, Jouni Järvinen, Jan Komorowski, Beata Konikowska, Bożena Kostek, Antoni Kreczmar, Churn J. Liau, Tsau Young Lin, Witold Lipski, Wing Liu, Witold Łukaszewicz, Larisa Maksimowa, Victor Marek, Antoni Mazurkiewicz, Ernestina Menasalvas, Grażyna Mirkowska-Salwicka, Mikhail Moshkov, Adam Mrózek, Maciej Mączyński, Daniele Mundici, Hung Son Nguyen, Cat Ho Nguyen, Sinh Hoa Nguyen, Tuan Trung Nguyen, Damian Niwiński, Hiroakira Ono, Ewa Orłowska, Sankar K. Pal, Eleonora Perkowska, James F. Peters, Jan Plaza, Lech Polkowski, Henri Prade, Andrzej Proskurowski, Halina Przymusińska, Slavian Radev, Sheela Ramanna, Zbigniew Raś, Cecylia Rauszer, Grzegorz Rozenberg, Leszek Rudak, Andrzej Salwicki, Giovanni Sambin, Dana Scott, Maria Semeniuk-Polkowska, Roman Sikorski, Dimiter Skordev, Andrzej Skowron, Roman Słowiński, Jerzy Stefanowski, Jarosław Stepaniuk, Zbigniew Suraj, Roman Suszko, Piotr Synak, Roman Swiniarski, Andrzej Szałas, Marcin Szczuka, Dominik Ślęzak, Helmut Thiele, Jerzy Tiuryn, Tadeusz Traczyk, Boris Trakhtenbrot, Shusaku Tsumoto, Paweł Urzyczyn, Dimiter Vakarelov, Alicja Wakulicz-Deja, Stanisław Waligórski, Quoyin Wang, Anita Wasilewska, Jakub Wróblewski, Wei-Zhi Wu, Urszula Wybraniec-Skardowska, JingTao Yao, YiYu Yao, Marek Zawadowski, Ning Zhong, Wojciech Ziarko, Tomasz Zieliński.

2. The Concept of Artificial Intelligence

Ever since its inception, the notion of *artificial intelligence* (AI) has been understood in a variety of ways. Along with advances in knowledge and studies in the world, the viewpoint on its understanding within the Rasiowa–Pawlak school has been changing as well. Currently the generally accepted understanding is in accordance with the classic introductory textbook to this field [207], viz.,

Premises and observations				Decisions and further actions		
Temperature	Visibility	Fuel	Clear road	Acceleration	Deceleration	Turn
…	…	…	…	…	…	…
…	…	…	…	…	…	…
…	…	…	…	…	…	…
…	…	…	…	…	…	…
…	…	…	…	…	…	…

Figure 1. Decision table.

> *[we] define AI as the study of agents that receive percepts from the environment and perform actions. … The AI enterprise is based around the idea of intelligent agents – systems that can decide what to do and then do it.*

There are many extensions and modifications of this classical definition of AI (see, e.g., [247]). The Rasiowa–Pawlak school concentrates especially on learning and improving of approximations of vague complex concepts (used on relevant levels of real-life problem solving) in dynamically changing environments (in which we have cooperating, communicating, and competing agents) by using uncertain and insufficient knowledge or resources.

The multitude in the number of different definitions of AI is a consequence of the divergence in the understanding of the concept of *intelligence* itself, both in regard to humans as well as machines. In this paper, *intelligence* is understood in accordance with the definition put forward by *Mainstream Science on Intelligence* and signed by 52 intelligence researchers in 1994 (*Wall Street Journal*):

> *[A] very general mental capability that, among other things, involves the ability to reason, plan, solve problems, think abstractly, comprehend complex ideas, learn quickly and learn from experience. It is not merely book learning, a narrow academic skill, or test-taking smarts. Rather, it reflects a broader and deeper capability for comprehending our surroundings – "catching on", "making sense" of things, or "figuring out" what to do* (reprinted in [64, p. 13]).

It should be noted that there is a formal view of intelligence that has its origins in natural language and philosophy.[6] It is worthwhile mentioning that, as a natural consequence of the understanding of an agent's intelligence by Stuart Russell and Peter Norvig [207], the essence of this intelligence could be described by a decision function, represented in the form of a very large table with first columns describing attributes related to an agent's observations (*percepts*) and assumptions, and then last columns related to the proposed agent actions. In other words, the table could be as in Figure 1.

This kind of table may represent a driver's behavior on a road, or a physician's actions while treating a patient. In this case, the decision of *intelligent agents – systems that can decide what to do and then do it* is represented by rough concepts which can be employed to implement algorithms solving specific problems by means of the advanced rough set techniques proposed by Zdzisław Pawlak [167,170,172–174]. For instance, in [39], the application of such techniques in a control algorithm for unmanned helicopters, e.g., monitoring road traffic, is well illustrated. From another perspective that

[6] Intelligence. 1. The faculty of understanding. 2. Understanding as a quality admitting of degree. 3. The action or fact of mentally apprehending something [161].

keys on a principal notion associated with intelligence (i.e., understanding), there is *an individual's perception or judgment of a situation* [161] to consider. The original work by Zdzisław Pawlak on classification of objects and perception that has inspired research concerning what might best be described as perceptual intelligence, adaptive learning and what is known as rough ethology (see, e.g., [175,176]).

The difficulty with approximation of vague concepts or perceptions lies in, among other things, the fact that we do not have precise mathematical definitions for those concepts, but only partial information based on limited knowledge of the features of objects and the concepts. Moreover, this information is usually imprecise, noisy, biased, insufficient, and is a subject of dynamic changes.

3. The Concept of Logic

The concept of *logic* has been intensively studied since ancient times. The dominating understanding of this concept within the Rasiowa–Pawlak school is due to Alfred Tarski [233], who in the first half of the 20th century initiated studies on abstract understanding of the consequence operation and of the satisfiability relation. The classic paper [234] contained the following statement:

> *The term "semantics" denotes certain relations between a language's expressions and objects.*

In [235] as well as in many other papers, Tarski wrote:

> *Semantics is a discipline which, speaking loosely, deals with certain relations between expressions of a language and the objects (or "states of affairs") "referred to" by those expressions.*

Elsewhere in the paper, he wrote:

> *... the words "designates", "satisfies", and "defines" express relations (between certain expressions and the objects "referred to" by these expressions)...*

Since the early works [233], Alfred Tarski had paid much attention to deduction theory in his research. He even invented theories in which primary notions were propositions and a consequence operation satisfying the so-called Tarski axioms for consequence.

Taking into consideration the traditions of the Polish school of logic already mentioned, by the term *logic* we intuitively understand a structure of the form

$$< M, L, \models, D >, \tag{1}$$

where $\models$ is any relation between the class M of models and the class L of language expressions, and D is a deduction system. Łukasiewicz's disciples stress here that the relation $\models$ is not necessarily a (partial) function from the product $M \times L$ into the two-element Boolean algebra. In addition, $\models$ can have values in some multi-valued structure. In this context, a little more general concept is investigated within the Rasiowa–Pawlak school. Namely, by an abstract *multi-valued logic structure* we understand a system consisting of the following components:

1. A class of *admissible worlds* including real or imaginary objects which could be used as models representing our knowledge. We assume that each admissible

world has an assigned set of *logical values*; in classical mathematics only two logical values *true* and *false* are considered.
2. A set of *expressions* used as a language for representation of our thoughts about properties and phenomena in the admissible worlds.
3. A *truth function* being a (partial) function which for every admissible world A assigns another function from the set of all relevant expressions into the set of logical values of A; this function enables us to estimate of the degree of credibility and verifiability of expressions in each admissible world. Put another way, Tarski writes that this means that *the truth or falsehood of any sentence obtained from that function by substituting whole sentences for variables depends exclusively on the truth or falsity of the sentences that have been substituted* [239]. In general, a truth function is a propositional function of truth values [34].
4. A *deductive system* that enables us to draw inferences about an admissible world, based on credibility and verifiability of knowledge about that world; in general, we can assume that it is a closure operation [144].

It is easy to imagine many examples of abstract multi-valued logical structures and related research problems such as completeness, compactness, theorem proving, concept approximation, as well as interpolation.

Example 1. Let us consider as expressions of a logical structure a set of sentences (i.e., formulas without free variables) of a theory T of the classical predicate calculus. The class of admissible worlds could be just the class of all relational structures for the theory T. Logical values are: *true* and *false*, i.e., the elements of two-element Boolean algebra.

Example 2. Let us consider as expressions the set of all formulas with free variables from an infinite set of variables V of a theory T of classical predicate calculus. Then, the admissible worlds can be once again the relational structures. However, the set of logical values and the truth function should be slightly more complex. Namely, for each relational structure $\mathcal{A}$, let S be the set of all valuations of free variables V into the universe of $\mathcal{A}$ (i.e., functions from V into the universe of $\mathcal{A}$). Then, the set of logical values of $\mathcal{A}$ is the set of elements of the Boolean algebra of all subsets of S. Let the truth function for $\mathcal{A}$ assign to each formula p, the set of all valuations v, making p true in the structure $\mathcal{A}$. Thus, for the classical predicate calculus, the class of admissible worlds may contain more than two logical values.

Although Alfred Tarski was probably the first to investigate the general concept of satisfiability as a binary relation, and Jan Łukasiewicz and Emil Post were the first to investigate multi-valued semantical structures, it may be worthwhile mentioning that these concepts are currently widely used in mathematics and AI under various names. Notice that ubiquitous mathematical concepts such as matrix and table may be considered as multi-valued semantical structures where

- **Worlds**: matrix row indices $\Rightarrow$ *Admissible worlds*,
- **Expressions**: matrix column indices $\Rightarrow$ *Set of expressions*,
- **Truth Values**: matrix values $\Rightarrow$ *truth values*.

From the viewpoint of the Rasiowa–Pawlak school, such tables represent the principal concept of rough sets called *Pawlak's information system* [170,172], in which *admis-*

sible worlds are the objects of the system, while *expressions* are functions *representing* attributes. The *truth function* can be used in the context of a truth table to evaluate the truth or falsity of a value associated with the function values associated with an attribute of a particular sample object. Notice that if we treat decision tables (such as presented in Figure 1) as an essence of AI then multi-valued logical structures could be also treated as an essence of AI.

However, putting aside the philosophical context, from a purely formal viewpoint, if a language possesses denotations for logical values and a logical equivalence relation, then abstract multi-valued logical structures may be considered as logical structures with two logical values, in which the fact that a formula has an intermediate value v can be expressed as it is true that this formula is equivalent to v. Obviously, this maneuver has a merely formal character and usage of many logical values directly is more convenient in many situations. For example, Boolean multi-valued models could be very convenient for an interpretation of the Heisenberg uncertainly principle (see [13, pp. 156–157]) and for proving of independence of axioms of set theory (see [13]).

In the case of abstract multi-valued logical structures having only two logical values, we simply refer to them as abstract logical structures [84,85].

The approach to the semantics of classical predicate calculus presented in the second example is a very special case of a more general semantics known as the Rasiowa–Sikorski Boolean models [193,194,196]. The concept of Rasiowa–Sikorski Boolean models for set theory was applied by Dana Scott and Robert Solovay in an elegant method in the proof of independence of the axiom of choice and the continuum hypothesis from the axioms of the ZF set theory [208,209,13]. However, the original proof of the independence of the axiom of choice done by Paul Cohen [35] used Kripke semantics. In 1973, Denis Higgs, and independently Dana Scott and his students [50], generalized the Rasiowa–Sikorski Boolean models to the case of category theory, and especially to topos theory [73,50]. In particular, Denis Higgs defined the notion of Ω-set and established for a complete Boolean algebra B, the equivalence of the topos of B-sets both with the category of sets and maps in the Boolean extension $V(B)$ of the universe of sets and with the category of canonical set-valued sheaves on B. Topos-based semantics is now used as a uniform generalization of the Rasiowa–Sikorski models and Kripke-style semantics [241,130]. For example, in [130, p. 277], one can find the concept of Cohen topos which is a powerful tool for analysis of the independence of axioms of set theory. There are many other applications of Rasiowa–Sikorski Boolean models. From the point of view of reasoning under uncertainty in AI, there is an interesting and yet not very popular application of the Rasiowa–Sikorski Boolean models to interpretation of uncertainty in terms of quantum theory (see [36] and [13, pp. 156–157]). This approach could be a starting point for a better understanding of imprecise and vague complex concepts by means of the Rasiowa–Sikorski Boolean models. For authors of this article, especially interesting is the next step in this direction, namely, exploring possible applications of the Rasiowa–Sikorski Boolean models in the discovery of the ontology of patterns in time series based on combination of wavelets, quantum mechanics, granularity, and fractal geometry [25, p. 25]. Next, these patterns are used for approximation of concepts or percepts that paves the way toward making predictions and economical or financial decisions.

One of the research directions in the Rasiowa–Pawlak school is a the characterization of relationship of classical logic to other logic. For better understanding of the role of classical logic, it is important to identify relationships between logics, in particular, between classical logic and non-classical logics. It is worthy to note that important stimuli to these studies in the Rasiowa–Pawlak school were papers by Jan Łukasiewicz [23,196] (interpretation of classical logic in intuitionistic logic by double negation), Andrzej Mostowski [148] (introduction of the concept of generalized quantifiers), and Rasiowa–Sikorski (relationships of classical predicate calculus with intuitionistic and modal predicate calculus (see, e.g., [196, pp. 408, 485]). From the point of view of applications of logic to AI, especially interesting are intuitionistic logic and its relation to classical logic [241]. For many years, intuitionistic logic has been explored as a framework for computer science foundations. Using societies of intelligent agents for modeling in AI is strongly related to a very interesting principle known as the Brouwer–Heyting–Kolmogorov interpretation (of intuitionistic logic). Under this principle, intuitionistic proofs of implicative formulas are functions and the existence of proofs requires witnesses – agents [241]. Another idea interesting for the AI foundations is Kolmogorov's interpretation of intuitionistic implication as a reduction problem [241, p. 31]. The history of relationships between intuitionistic logic and classical logic is very old. In particular, Glivenko's Theorem [61] (discovered independently by Jan Łukasiewicz [23]) says that: *An arbitrary propositional formula A is classically provable if and only if $\neg\neg A$ is intuitionistically provable*. For other translations of propositional calculus see, e.g., [241] and [152]. Glivenko's Theorem cannot directly be extended to predicate calculus, although there are some forms of this theorem which use special types of modification of the Glivenko negative translation (i.e., Gödel-Gentzen [241], Gödel [241], Kolmogorov [241], Kuroda [241], Kleene [99], Rasiowa–Sikorski [196]). Andrzej Mostowski [148] stimulated a different dimension of characterization of classical logic, viz., a characterization in terms of extension of this logic by some infinite logical connectives. Examples of results in this direction can be found in [120,6,7,16,28].

There is a characterization of classical logic obtained in the Rasiowa–Pawlak school in terms of relationships to other logics. Namely, *any logical structure $\mathcal{L}$ with countable set of formulas is embeddable into a classical logical structure if and only if $\mathcal{L}$ satisfies compactness and completeness theorems* [84,85]. This result, in some sense, could be treated as a reverse theorem to Glivenko's Theorem for propositional calculus and its modifications for the predicate calculus.

4. Roots of the Algebraic Approach to Logic by the Rasiowa–Pawlak School

The core techniques that constitute the roots of the algebraic approach to logic by the Rasiowa–Pawlak school were developed in the 1950s and 1960s, and have been further extended in later years. Since the very beginning, these techniques have been focused on the following topics:

1. Development of algebraic methods used in search of the most relevant semantical structures, i.e., structures that enable to search efficiently for constructive (algorithmic) problem solutions in specific types of domains (e.g., applications of modal logic to knowledge representation in multiagent systems, logical reasoning about rough concepts, logical models for quantum computation), in particular research on semantics and inference rules.

2. Algebraic construction of *canonical world* for a consistent set of expressions and application of this technique to the study of equivalence between deductive inference and inference based on the truth relation (the completeness theorem).
3. Construction and analysis of alternative deductive systems for a given logic, for example based on modification of approaches invented by Stanisław Jaśkowski, Gerhard Gentzen, David Hilbert, Jacques Herbrand, and Helena Rasiowa together with Roman Sikorski [196,128].
4. Analysis of the fundamental model-theoretic properties of the admissible worlds for a given logical structure (e.g., the Skolem–Löwenheim theorem, the compactness theorem, the omitting-types theorem).
5. Research on the *geometric* properties of the *space of models* [54,205,250]. In this framework, by a space of models we understand a space which has models (Q-filters in the Lindenbaum–Tarski algebra) as points and a topology generated by Stone's representation theorem. One can consider also a distance between points measuring similarity of models. This is a metaphor of Stone's representation theorem for Boolean algebras [211]. One of the most exciting intellectual experiences for the authors of this article is a proof of the Rasiowa–Sikorski Lemma using topological properties of the *space of models* (using the Baire property). Beside of this application of topological methods to proving the completeness theorem, there are many other very interesting logical properties of the *space of models* implied by the topological properties of this space. They concern results characterizing open theories and mechanisms for the construction of Herbrand alternatives [196].

Basically, tools employed by the Rasiowa–Pawlak school are, in a natural way, a continuation of the idea of concept calculus proposed by Gottfried Wilhelm Leibniz, and later developed by George Boole and his disciples, in particular by Alfred Tarski and Adolf Lindenbaum. The evolution of tools employed in the algebraic research of different aspects of logic can be summarized by the scheme presented in Figure 2.

To introduce with acuity the key elements characteristic to the algebraic approach to logic, typical for many trends in the Rasiowa–Pawlak school, let us assume that Q denotes a set of abstract logical operators representing some logical connectives. In the case of classical logic, the most important ones are conjunction, disjunction, implication, and negation. As it was noticed by Adolf Lindenbaum and Alfred Tarski, if we glue together all the sentences which represent the same thought in a deductively closed theory in classical logic, we get a Boolean algebra. Intuitively, this is an algebra of thoughts for the theory. Algebraic operators in the algebra of thoughts correspond to logical connectives. In particular, provable implication corresponds to a partial order of thoughts.

Greater value in the order intuitively means *more true*. Disjunction corresponds to supremum in the order generated by the implication, and conjunction corresponds to infimum in the same order. If we treat the existential quantifier of classical logic as infinite disjunction, then its algebraic interpretation is the supremum. Similarly, if we treat the universal quantifier as infinite conjunction, then it corresponds to the infimum.

The set of abstract logical operators represented by propositional connectives can be generalized to a set Q consisting of more logical operators. In the standard way, we define the concept of Q-algebra that includes all operators from the set Q [193,194], and concept of Q-homomorphism between two Q-algebras which preserves all Q-operators. In this paper, we assume that Q is enumerable and each Q-algebra has a special constant

Domain & Operators	*Natural Artthmetic*	*Algebra of subsets*	*Boolean Algebra*	*Logical concepts in Lindenbaum – Tarski algebra*	*Semantical models for constructive mathematics*	*Topoi*	*Wisdom Granular Computing for a given application domain*
X < Y	X is smaller than Y	X is a subset of Y	X is smaller than Y in Boolean algebra	Y could be deduced from X	Logical value of X is smaller than logical value of Y in a Heyting algebra	Morphism from X to Y	Wisdom granule Y is a consequence of wisdom granule X in the domain
0	Zero	Empty set	The smallest element	False	0 in Heyting algebra	Initial element	Smallest wisdom granule in the domain
1	One	Full set	The biggest element	True	1 in Heyting algebra	Terminal element	Biggest wisdom granule in the domain
+	Addition	Join of two sets	Maximum	Disjunction	Maximum	Coproduct	Relative coproduct of two wisdom granule
*	Multiplication	Intersection of two sets	Minimum	Conjunction	Minimum	Product	Relative product of two wisdom granule
X^Y	Exponentia-tion X to power Y	Join of (–Y) and X	Join of (–Y) and X	Implication (Y implies X)	Relative pseudo – complementation Y→X in Heyting algebra	Object corresponding to all morphisms from Y to X	Granule corresponding to all consequences from granule Y to granule X
Mod (X)	Modulo X calculus	Quotient algebra of the filter generated by set X	Quotient Boolean algebra of the filter generated by set X	Lindenbaum – Tarski algebra for a theory generated by a set of axioms X	Models for a theory generated by axioms X	Category of sheaves over X	All consequences from a given granule X
Logical values	True False	True False	True False	Algebra of logical values	Elements of Heyting algebra	Subobject classifier	Identification of subgranules of granules

ANCIENT	CONTEMPORARY	FUTURE

Figure 2. Evolution of computational models of logical concepts from the Rasiowa–Pawlak school perspective (the last column is hypothetical for further research).

1 for the logical value of truth. An interesting introduction to logics based on Q-algebras with implication (extensions of implicative algebras) can be found in [193,194].

We say that an abstract multi-valued logical structure is a Q-*logical structure*, if

– the *algebra of thoughts* for any theory is a Q-algebra,
– the logical values for each admissible world form a Q-algebra,
– the truth function is defined using a Q-homomorphism from the set of expressions into a Q-algebra of logical values for each admissible world.

The main idea of the algebraic approach to predicate calculus in the research papers of Helena Rasiowa and her students is based on two concepts:

– Q-*representability*,
– Q-*characterizability*.

These concepts have been used in several variants and have been applied as the main tool for solving of several important problems.

In order to introduce both concepts let us assume that we have two classes M and K of Q-algebras. We say that M is Q-*representable* by K, if for any algebra A from the class M there exists a Q-isomorphism from A to a Q-algebra in K. For example, if Q includes standard finite Boolean operators, then Stone's representation theorem for Boolean algebras means that the class of Boolean algebras is Q-representable by the class of all of Q-fields of sets. The Stone representation theorem has been generalized for infinite Boolean operators by Rasiowa and Sikorski. If Q is a set of standard finite Boolean operators and a countable set of operators corresponding to infinite infima and suprema in such Q-Boolean algebras, then the Rasiowa–Sikorski representation theorem [196,197] says that the class of all such Q-Boolean algebras is Q-representable by the class of all Q-fields of sets. By the definition it means that for any countable set Q of infima and suprema in a Boolean algebra, there exists a Q-isomorphism to a Q-field of sets.

We say that M is Q-*characterizable* by K, if for any algebra A from the class M and for any element x of A such that x is different from 1, there exists a Q-homomorphism from A to a Q-algebra from the class K such that the value of x after the transformation by the Q homomorphism is different than 1.

For example, if Q includes standard finite Boolean operators and K has only one two-element Boolean algebra, then each Q-homomorphism from a Boolean algebra A into the class K can be identified with a Q-prime filter of A, and any element a of A can be identified with a set of all Q-prime filters including a. Originally, the Rasiowa–Sikorski Lemma was expressed as follows: If Q is a set of standard finite Boolean operators and a countable set of operators corresponding to infinite infima and suprema in such Q-Boolean algebras, then the class of all such Q-Boolean algebras is Q-characterizable by the class which has only one two-element Boolean algebra.

The logical meaning of the concept of Q-representability of the class of Q-algebras of thoughts by the Q-algebras of logical values for admissible worlds of a logical structure L is that L satisfies the completeness theorem. The concept of Q-representability of the class of Q-algebras of thoughts by the Q-algebras of logical values for admissible worlds of L is used for construction of special admissible worlds called *canonical models*.

We say that a Q-logical structure L is a *Rasiowa–Sikorski logical structure* if the class of Q-algebras of thoughts is Q-characterizable by the Q-algebras of logical values for admissible worlds of L. There are several examples of Rasiowa–Sikorski logical structures based on logics such as: intuitionistic [196], Heyting–Brouwer [198], intermediate [199], modal [196,193,194], Post [193], semi-Post [194] and algorithmic [4,142], autoepistemic [200], knowledge for groups of agents [202,203,53,183].

It is possible to apply several schemes of analysis to the research of Rasiowa–Sikorski logical structures. An excellent compendium of such schemes can be found in [196,193,194]. The schemes can be used for typical model-theoretic applications such as the completeness theorem or construction of the canonical model, and also for not so typical ones like natural deduction systems in the Rasiowa–Sikorski style [196,128], or the proof of decidability of formulas in *prenex form* for constructive intuitionistic theories [196].

Along with the emergence of the concept of topos, generalizing the semantics of Boolean and pseudo-Boolean models, as well as the appearance of other approaches like Kripke semantics or Beth tableaux, interests to these methods have also grown within the Rasiowa–Pawlak school. On one hand, opportunity to generalize techniques characteristic for algebraic and Kripke models to topos has been investigated [60]. On the other hand, internal representations of some logics in others have also been researched [88].

5. Logic for Reasoning and Knowledge

In the early stages, reasoning was understood within the Rasiowa–Pawlak school as natural deduction systems by Stanisław Jaśkowski, Genzen's sequent calculus systems, Herbrand disjunctions, and their modifications [19,20,192,240]. In particular, in the book by Helena Rasiowa and Roman Sikorski [196], an elegant deduction system for predicate calculus, called in literature the Rasiowa–Sikorski deduction system [128], has been presented. Much effort has also been devoted to the deduction systems in non-classical logics. An exemplary survey containing many results in this field can be found in [19,20]. Various aspects of logic programming and deduction in quantum logics have also been studied [138,154].

Together with the advancement in understanding the specifications of complex vague concepts, a gradual shift of the center of studies has taken place, toward approximation logic and approximate reasoning (see, e.g., [194,133,195,200,202,19,20,188,162]).

In practical applications, there has been a still-increasing need for approximate reasoning about concepts and objects on the basis of incomplete, partial, biased and ever changing information about them. The concepts themselves are usually vague. Methods for approximate reasoning about such concepts based on rough sets have been developed. Problems considered in applications can usually be reduced to the construction of objects adhering to vague specifications to a satisfactory degree (e.g., fuzzy sets [252–254], rough sets [167,170,172–174], rough mereology [187,189]). As a result, constructed approximate solutions are significantly easier to obtain than exact solutions, which often prove impossible to attain due to the lack of exact model or the inhibiting costs associated with their computing. This way of thinking seems to be popular in humans while solving problems. The research on searching for approximate solutions adhering to vague specifications to a satisfactory degree is therefore considered one of the principal advanced directions in developing better intelligent systems (see, e.g., [177,178]). Rough set based methods have proven effective in relatively simple applications such as searching for reducts and relevant features for a given phenomenon, as well as in highly complex situations such as image recognition and processing (see, e.g., [21,22,180]), biologically-inspired adaptive learning controllers (see, e.g., [178,180]) or control and cooperation among autonomous robots monitoring and repairing power lines.[7] An interesting application is the design for a control system of an autonomous helicopter, where a joint approach between rough sets and nonmonotonic reasoning has been presented [39].

Reasoning considered as formal operations on features of linguistic features has high complexity and thus identification of formulas (or abstraction classes understood as granules), to which the reasoning can be limited to, is of great concern. However, this kind of applications requires specific knowledge and wisdom representation systems, as well as the resignation from hitherto existing formalizations of language in classical logic. In this context, an interesting trend in the seminars by Rasiowa consisted of attempts referring to pre-WWII ideas by Stefan Banach and his collaborators to use geometrical methods (relying on a switch from language to topological concept space) [196,54,205,250]. Another, also appealing direction, pertaining to Wittgenstein's proposal, was to look at language and satisfiability relation by means of the apparatus of game theory [251,210, 74]. These works laid the ground for a proposal of algebraic models for non-Fregean logics by Roman Suszko [232,18]. Another application of game theory, from a bit different perspective, to investigate features and reasoning in intuitionistic logic can be found in [60,74]. Game theory can also be applied to approximate reasoning about granular computing [91].

[7] For more details on applications the reader is referred to the bibliography in [172–174] and http://rsds.univ.rzeszow.pl/.

By the term *knowledge*, we understand a system consisting of information, its internal relations and inference rules making it possible to reason about phenomena occurring in a domain of the world. Ever since its inception, one of the main problems in AI has been the construction of effective knowledge representation systems. There have been many paradigms of knowledge representation in both structured systems (e.g., frames, rule-based systems, semantic networks, logic programming) and non-structured systems (e.g., neural networks, genetic algorithms, ant systems). With the unprecedented expansion of the Internet, there is a growing demand for effective systems to represent, process, search and provide information in texts, images, audio and video recordings. One of the many effective tools for knowledge representation, in particular for vague concepts description, are rough set based systems and their generalizations. Some of such tools were employed by the authors to build in 2000 an Internet search engine called Excavio capable of communication with users through simple dialogs using phrases. Recently, this kind of technology has became more and more popular in text mining [1,3, 32,48,69,72]. The main idea of these applications relies on using automatic document clustering algorithms in regard to sets of sequences of phrases potentially interesting to searchers. The precursors of an interesting direction of application of rough set methods to Internet search engines were the Japanese [76]. This paper was based on ideas presented in [218]. The algorithms were also developed and published by members of the Rasiowa–Pawlak school [150]. Vastly interesting systems applying paradigms of rough sets and algebraic aspects of logic have also appeared in Japan, China, India, Sweden, and Canada [172–174].

Within the Rasiowa–Pawlak school a system for representing knowledge from many fields and for reasoning about this knowledge primarily by means of rough set paradigms has been designed and implemented. The system can be accessed on the server of Warsaw University and is presented at the Internet address: http://logic.mimuw.edu.pl/~rses/.

Independently from knowledge representation systems using the rough set approach, another important trend in the Rasiowa–Pawlak school is a series of research on autoepistemic logics and application of logic to the negotiation in a society of cooperating or competing agents. These works are in a natural way related to the fundamental intuitions of modal operators in modal logic. Interesting results in this area can be found in [202,195,183].

6. Mathematical Approach to Vagueness

One of the fundamental motivations for Jan Łukasiewicz in inventing multi-valued logics was his belief that not all propositions can be considered as either true or false, due to the impossibility of humans to predict certain events. A standard example frequently used in Łukasiewicz's works is the sentence *Jan will be in Warsaw next year* [23]. Jan Łukasiewicz, referring to the indetermination principle in quantum mechanics, stated that assigning to this sentence either truth or falsehood is abusive. His works initiated, intensely developed until now, a research direction pertaining to *uncertainty* by means of multi-valued logics. Among others there is an interesting attempt to combine approximation logic based on rough set paradigm with Post's multi-valued logics [194].

Mathematics requires that all mathematical concepts (including set) must be exact, otherwise precise reasoning would be impossible. However, philosophers and, recently, computer scientists as well as other researchers have become interested in vague (imprecise) concepts.

We would like to remind that modern understanding of the notion of vague (imprecise) concept has a quite firmly established meaning context including the following issues [98]:

1. The presence of borderline cases.
2. Boundary regions of vague concepts are not crisp.
3. Vague concepts are susceptible to sorites paradoxes.

Moreover, it is usually assumed that the understanding (approximation) of vague concepts (their semantics is determined by the satisfiability relation) depends on the agent's knowledge, which is often changing. Hence the approximation by agent of vague concepts should also be considered changing with time (this is known as the concept drift). In the 20th century, it became obvious that new specialized logical tools need to be developed to investigate and implement practical problems involving vague concepts. One of such trends are rough sets.

Rough set theory, proposed by Zdzisław Pawlak in 1982 [167,170] can be seen as a new mathematical approach to vagueness. The rough set philosophy is founded on the assumption that with every object of the universe of discourse we associate some information (data, measurements, observations, patterns, knowledge). For example, if objects are patients suffering from a certain disease, symptoms of the disease from clinical observations and diagnoses about patients. Objects characterized by the same information are indiscernible (similar) in view of the available information about them. The indiscernibility relation generated in this way is the mathematical basis of rough set theory. This understanding of indiscernibility is related to the idea of Gottfried Wilhelm Leibniz that objects are indiscernible if and only if all available functionals take on them identical values (Leibniz's Law of Indiscernibility: The Identity of Indiscernibles) [117]. However, in the rough set approach indiscernibility is defined relative elementary sets of objects with matching descriptions based on values of functions representing selected attributes thought to be inherent in objects or feature-based measurements based on the appearances of objects [182]. Any set of all indiscernible (similar) objects is called an elementary set, and forms a basic granule (atom) of knowledge about the universe. Any union of some elementary sets that is a subset of a set of interest (e.g., set of objects X that are considered somehow acceptable) is referred to as crisp (precise) set; otherwise, for those elementary sets that have a non-empty intersection with a set X of interest and its complement, the attention turns to the boundary of X. Whenever the boundary is not empty, then the set X is considered rough. The size of the boundary provides a measure of the vagueness of our knowledge about X. Consequently, each rough set has boundary-line cases, i.e., objects which cannot with certainty be classified either as members of the set or of its complement. Obviously crisp sets have no boundary-line elements at all. This means that boundary-line cases cannot be properly classified by employing available knowledge. Thus, the assumption that objects can be *seen* only through the information available about them leads to the view that knowledge has granular structure. Due to the granularity of knowledge, some objects of interest cannot be discerned and appear as the same (or similar). As a consequence, vague concepts, in contrast to precise concepts, cannot be characterized in terms of information about their elements. Therefore, in the proposed approach, we assume that any vague concept is replaced by a lower and the upper approximation of the vague concept. The lower approximation consists of all objects which surely belong to the concept and the upper approximation contains all objects which possibly belong to the concept. The difference between the upper and the lower approximation constitutes the boundary region of the vague concept. Approximations are two basic operations in rough set theory. Hence, rough set theory expresses vagueness not by means of membership, but by employing a boundary region of a set. If the boundary region of a set is empty it means that the set is crisp, otherwise the set is rough (inexact). A non-empty boundary region of a set means that our knowledge about the set is insufficient to define the set precisely. Rough set theory is not an alternative to classical set theory but is embedded in it. Rough set theory can be viewed as a specific implementation of Frege's idea of vagueness, i.e., imprecision in this approach is expressed by a boundary region of a set. The boundary region is defined relative to a given set of attributes, i.e., it could change if the set of features changes. Moreover, the boundary region is changing according to data access. Hence, one can see that boundary region cannot be defined by only one crisp concept. In [204,179], sorites paradoxes in the framework of rough sets are discussed.

Rough set theory has attracted attention of many researchers and practitioners all over the world, who have contributed essentially to its development and applications. Rough set theory overlaps with many other theories. Still, rough set theory may be considered as an independent discipline in its own right. The rough set approach seems to be of fundamental importance in artificial intelligence and cognitive sciences, especially in research areas such as machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, knowledge discovery, decision analysis, and expert systems. The main advantage of rough set theory in data analysis is that it does not need any preliminary or additional information about data like probability distributions in statistics, basic probability assignments in Dempster-Shafer theory, a grade of membership or the value of possibility in fuzzy set theory. It should also be observed that functions representing either attributes or features of objects do reflect *a priori* reflection about the appropriate choice of functions that represent selected features of observed objects such as widgets and organisms such as beetles or ants or attributes of conceptual objects such as population demographics or numbers in a statistical study. In effect, this signals a strong tie between fuzzy set and rough set theory, since each feature (*e.g.*, *high, medium, low* relative to observed or anticipated distributions of values in an experiment) is represented by a membership function chosen beforehand in fuzzy set theory. One can observe the following aspects of the rough set approach:

- partition of sets of sample objects using some form of the indiscernibility relation,
- identifying neighborhoods of sample objects relative to elementary sets to facilitate perception and object recognition,
- approximation of each set of objects of interest with another set,
- introduction of efficient algorithms for finding hidden patterns in data,
- determination of optimal sets of data (data reduction),
- evaluation of the significance of data,
- generation of sets of decision rules from data,
- easy-to-understand formulation,
- straightforward interpretation of obtained results,
- suitability of many of its algorithms for parallel processing.

When recalling the sources motivating the creation of multi-valued logic, which inspired Jan Łukasiewicz during his analysis of uncertain concepts by means of intermediate logical values, we would like to stress that many works on rough sets were dedicated to the relations between rough sets and multi-valued logics by Jan Łukasiewicz and others (see, e.g., [37,135,162]).

Despite of the efforts to build computers operating on multi-valued logics (electronic as well as optical), currently the model based on two-valued Boolean logic is dominating. It is, hence, not the best architecture to implement concepts described in multi-valued non-classical logics. Although there are announcements on research on building quantum computers based on quantum logic which is a non-classical logic, they have not yet been realized for real-life applications [33,75,228]. Given this context, especially significant are numerous works attempting to find handy and effective computing models for the treatment of complex vague concepts. Solving this problem is the essence of the idea by Zdzisław Pawlak who, we remind, was the architect of one of the first of the world computers, based on a system different from the typical binary system, to enter serial production. Zdzisław Pawlak postulated that we should accept the fact that our conceptual apparatus is limited and we are able to describe the reality only by means of expressions comprehensible to us, whose truth value (as understood by two-valued logic) can be easily verified. Apart from such expressions, however, there exist expressions concerning complex vague concepts that cannot be easily verified (e.g., *a financial operation bears an unacceptable risk*, *Kowalski contracted disease X*, *the situation on a road is dangerous*). In such cases, we generally can express, in a language of easily verifiable concepts, which objects with certainty fall into the scope of the described concept (lower approximation) as well as which objects may fall into this scope with some certainty (upper approximation). This simple and obvious idea has become foundation for constructing highly effective tools supporting the representation and processing of vague concepts in practically all fields of applications [167,170,172–176].

Vague complex concepts are very often related to one another through a hierarchy induced by abstraction levels of these concepts. Such hierarchies occur, for instance, when some concepts are components of other concepts. In such a context, there is a great interest in investigating the relation *being a part-of*. It is a different approach from the ontology of modern mathematics (also known as *Cantor ontology*), which is based on the relation *being an element-of*. Such alternative ontology for mathematics was proposed by Stanisław Leśniewski [118,119] in 1929 and became the inspiration for an important research within the Rasiowa–Pawlak school called *rough mereology* [187,162]. Within this approach, the notion *rough inclusion relation* plays a central role. It describes to what degree some concepts are parts of other concepts. A rough mereological approach is based on the relation to be a part to a degree. It is interesting to note here that Jan Łukasiewicz in 1913 investigated the inclusion to a degree of concepts in his discussion on relationships between probability and logical calculi [23].

Certainly, the mereological approach is not the only one attempt for establishing links between vague concepts and rough sets. Among others are links based on treating vague concepts by means of logical values. In this case, one can build an algebra of such vague concepts as an algebra of logical values. Usually, this kind of algebra is a pseudo-Boolean algebra and the relationships between vague concepts can be expressed as relationships of logical values of an intermediate logic. Notice, that if we would like to prove relationships between some concepts in intuitionistic logic, then we should be able to use constructive evidence. For example, we cannot affirm that any two concepts x and y are either equal or unequal. In intuitionistic logic we have two kinds of *inequality*:

- *classical inequality* which means that x is equal to y leads to contradiction,
- *intuitionistic inequality* which is stronger and means that we can provide a constructive evidence which proves that x and y are not equal.

In particular, if x and y are real numbers, then in order to show that x and y are intuitionistically unequal it is necessary to provide a constructive evidence for that. In this case, we can construct an example of a rational number which separates x and y. In other words, our knowledge about relationships between concepts in intuitionistic logic requires constructive evidences. For example, if we show that the statement *John does not have pneumonia* leads to contradiction, then in intuitionistic logic it does not mean that *John has pneumonia* holds.

Having in mind the above remarks it is interesting to represent our knowledge in the framework of intuitionistic logic. In general, it is not easy because the free Heyting algebra is infinite, even in the case of one generator only. However, if we assume that:

- each concept can be represented by a finite set of objects and
- the universe of objects which can be used for representation of concepts is finite,

then some intermediate logics may be used instead of intuitionistic logic [87].

Naturally, the remarks above do not cover all aspects of techniques employed by the Rasiowa–Pawlak school in treating vague concepts. At the current stage, particularly intensive are studies on rough granular computing as well as in the direction of working out techniques for learning of complex vague concepts. We discuss these issues more broadly in the following section.

7. Logic for Learning, Communication, Perception, Planning, Action, Cooperation, and Competition

From the viewpoint of understanding of the definition of AI along the lines of Russell and Norvig [207], the essence of artificial intelligence is to equip an agent with a decision table using one complex concept that allows to implement the idea:

> … *intelligent agents – systems that can decide what to do and then do it.*

In Section 6, we outlined the idea of construction and description of vague concepts using rough sets. In other words, a solution to learning of complex vague concepts is also a solution to the fundamental goals in AI. In this context, developing techniques applicable to the AI field called machine learning is of primary concern. Unfortunately, we are still very far from fully satisfactory solutions, and the currently developed approaches to learning are suitable to deal with a *predictable* environment, where one can foresee which of the existing machine learning paradigms may perform the best. In practice, so far, it has not been possible to develop satisfactory methods for autonomous systems [124], neither to work out universal techniques that allow machine learning to be applied in environments with high unpredictability (we mean an unpredictability degree comparable to that of an ecosystem in which living organisms have, and successfully could, learn various concepts in order to adapt and survive) [57]. From the perspective of further advances in AI, it would be handy to have tools allowing to design, construct, and development of machines capable of:

1. Initialization of input knowledge provided in a simplified natural language based on an expert's domain knowledge.
2. Automatic planning of experiments and learning of new, complex vague concepts that allow better actions within the domain of operation of the machine.
3. The skilful and effective use of the possessed wisdom in order to perform best possible actions.

Basically speaking, these ambitious goals have come around in AI many times since its inception. They can be observed in various attempts from the discussions of Alan Turing's test [242] through programs that generate winning strategies in games [46,79, 223], discussion of key problems in knowledge discovery systems [114], to many other modern research trends in this direction [100]. Within the Rasiowa–Pawlak school there have also been many approaches to solving this crucial problem. The authors of this paper are supervising applied projects in medicine and economy, where they attempt to use a new, possibly universal approach to a better understanding of the path leading to solving the three problems mentioned above. The research concerns mainly construction of a hierarchical architecture for networks of multi-valued logical structures (see, e.g., [9–11,90,91,149,151,150,12,224]). These works intensively employ rough set paradigms, though it is worth mentioning that they bear meta-AI characteristics (analogously to meta-mathematical research within the Rasiowa–Pawlak school). We call this technology wisdom technology, in short. In a simplification, these techniques refer to one of the directions in KDD dubbed hierarchical multi-search discovery systems (see also, e.g., [92]). These issues are related to the fundamental problems of pattern recognition, machine learning and KDD, involving methods for the extraction of new features [100, 182,181].

In any case, the research trends mentioned previously were not equipped with two essentially new elements derived from wisdom technology, concerning simulation of judgment processes [68,96,184,211] and of processes that construct, adaptively refine and handle complex vague concepts by way of interaction with the environment [127]. These processes are present at each level of the hierarchical structure mentioned above. Such a hierarchical structure constitutes a metaphoric view of a well-known psychological concept – *Maslow Hierarchy of Human Needs* (see Figure 3).

The wisdom technology can be explained by the so-called the *wisdom equation* [90] which can be expressed metaphorically as follows:

$$wisdom = KSN + AJ + IP,$$

where KSN, AJ, IP denote knowledge sources network, adaptive judgment, and interactive processes, respectively. For the context of the wisdom equation see Figure 4.

Combination of the technologies represented in the wisdom equation offers an intuitive starting point for a variety of approaches to design and implementation of computational models for wisdom technology. In this sense, wisdom is a concept of a higher level than knowledge, information and data in the context of DIKW hierarchy suggested in a poem by T.S. Eliot (see Figure 5).

The basic concepts of wisdom technology outlined above do not constitute a complete solution of the three fundamental problems of AI as posed at the beginning of the

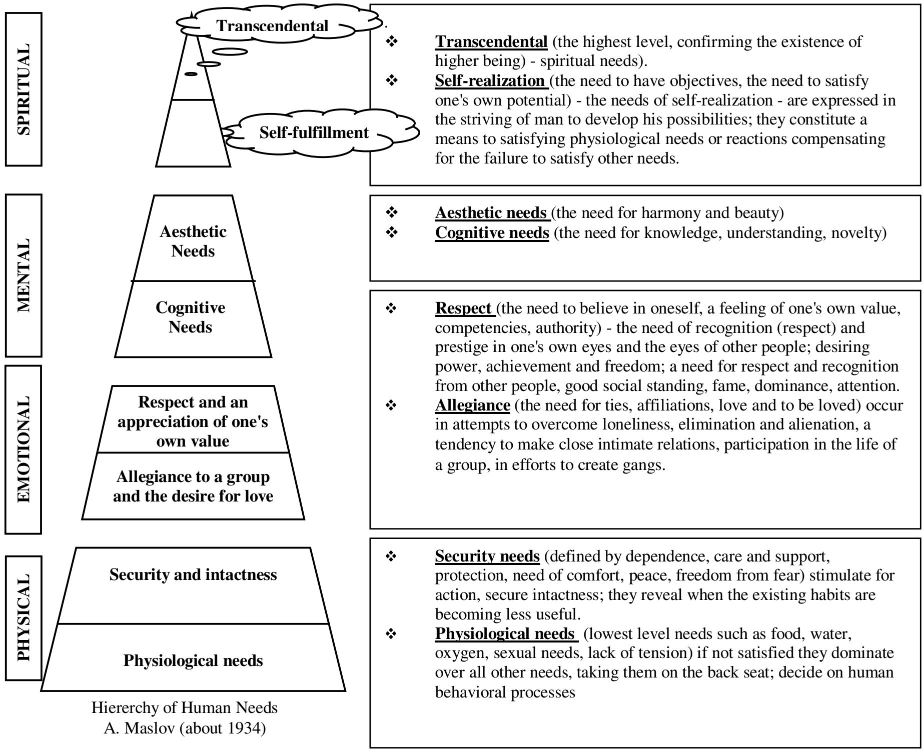
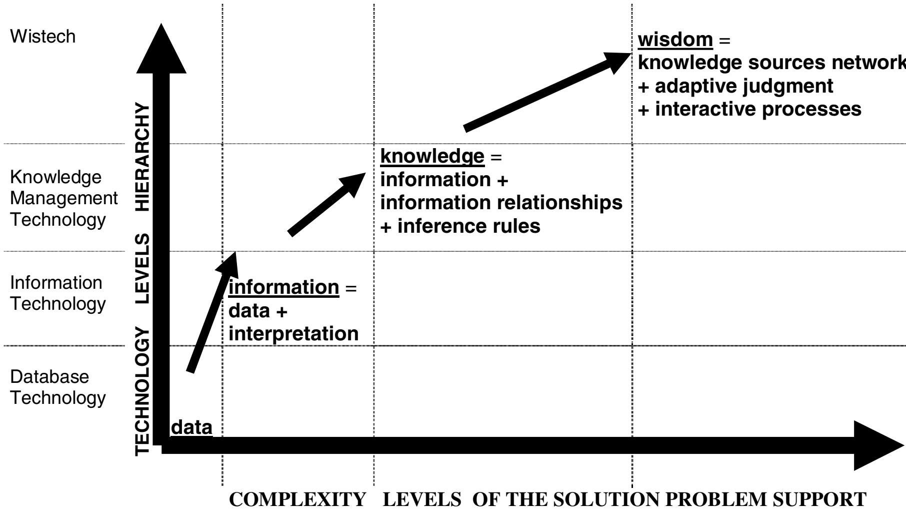


Figure 3. The Maslow Hierarchy of human needs (about 1934) as an example of judge hierarchy of habit controls.

section. Wisdom technology should rather be considered as a proposal indicating a potential research direction. Without doubt, in these studies, apart from mechanisms for the construction of metaphorically understood Maslow hierarchy for intelligent agents (in order to implement processes of judgment and of steering interactions with the environment), it will be critical to work out effective mechanisms for learning of new complex and vague concepts in the prospective domain of operation of an intelligent agent. Two directions may be distinguished in the learning mechanisms just mentioned:

1. Learning of concept hierarchies (including the metaphorically understood Maslow hierarchy for an intelligent agent), i.e., strategies for discovery of levels of the hierarchy, including relevant languages for each level and methods for selection of relevant concepts.
2. Learning concepts at particular nodes of the hierarchy.

No doubt, both kinds of processes are strongly interlaced with each other. However, a better understanding of the mechanism interlacing both groups would be more easily attained if a better understanding of specific features of each separated group was possible. From the viewpoint of studies conducted within the Rasiowa–Pawlak school, we consider the approach to understanding both process groups through a combination of rough set techniques and evolutionary programming [77–79,113] to the construction of abstract hierarchical multi-valued logical structures to be particularly essential. The simplest examples of such combination of rough sets and evolutionary programming can be



	Understanding	**Perception**	**Prediction**
Questions:	**Questions:** about data values and *additionally questions about data context, like:* ***Who? What? When? Where? How Much?***	**Questions:** information type questions and *additionally questions about explanation and prediction, like:* ***How? Why? What if?***	**Questions:** knowledge type *questions and additionally questions about correct judgments and decisions, action/ interaction planning / executing and justification, like:* ***What to do? Why to do it? When to do it? How to do it?***
Objects:	**Objects**: *Data and data explanation by a description, picture or other presentations.*	**Objects:** *Information and rules for information transformation (reasoning, ...), constrains, relationships between concepts, ideas and thought patterns.*	**Objects:** *Knowledge and correct judgments, decisions based on a hierarchy of being values or beliefs, action plans, incorporation of vision, design, plans and implementation standards based on being preferences.*
Time context:	**Time context:** *Usually information is a posteriori, known after the fact.*	**Time context:** *Usually knowledge is a priori, known before the fact and provides its meaning.*	**Time context:** *Usually deals with management interactions with environment to achieve future objectives.*
Measures:	**Measures:** *Logical values, uncertainty, completeness, amount.*	**Measures:** *Efficiency of problem solutions by applying theory to information, quality of problem solutions.*	**Measures:** *Priorities, culture values, profits, quality of action results and plan implementation.*

Figure 4. Wisdom equation context.

put straightforward. Namely, let us assume that our goal is to construct a very complex and vague concept for a decision table (as understood by rough sets described in the beginning of the paper) satisfying the idea:

> … *intelligent agents – systems that can decide what to do and then do it.*

At first one can assume that the table consisted of rules is provided by experts and that each rule is assigned a weight determined by means of rough set techniques (or of other paradigms). Next, the rules over a population of chromosomes constitute an input to the genetic algorithms under consideration. In the sequel, the approximation of our complex vague concept is refined using the implemented genetic algorithm and interactions with the environment and external experts. Although this demarche is described for a single

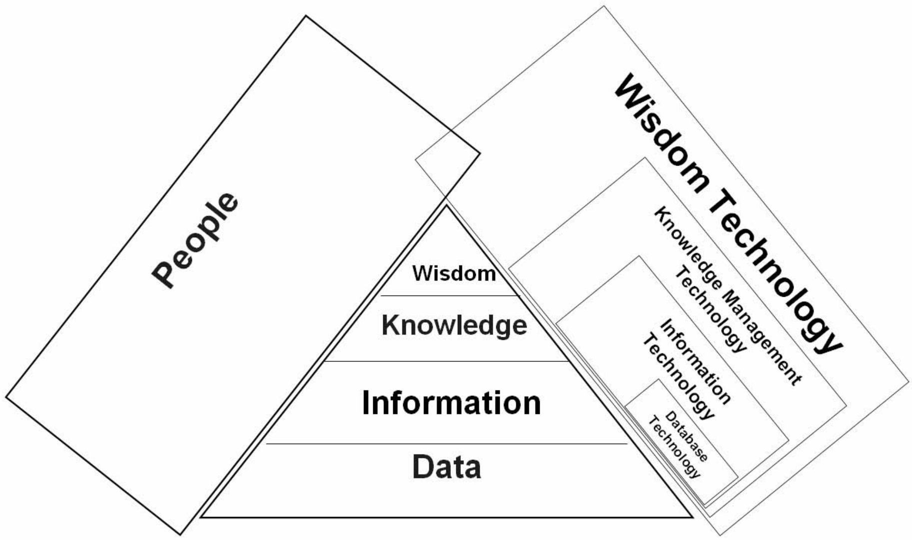
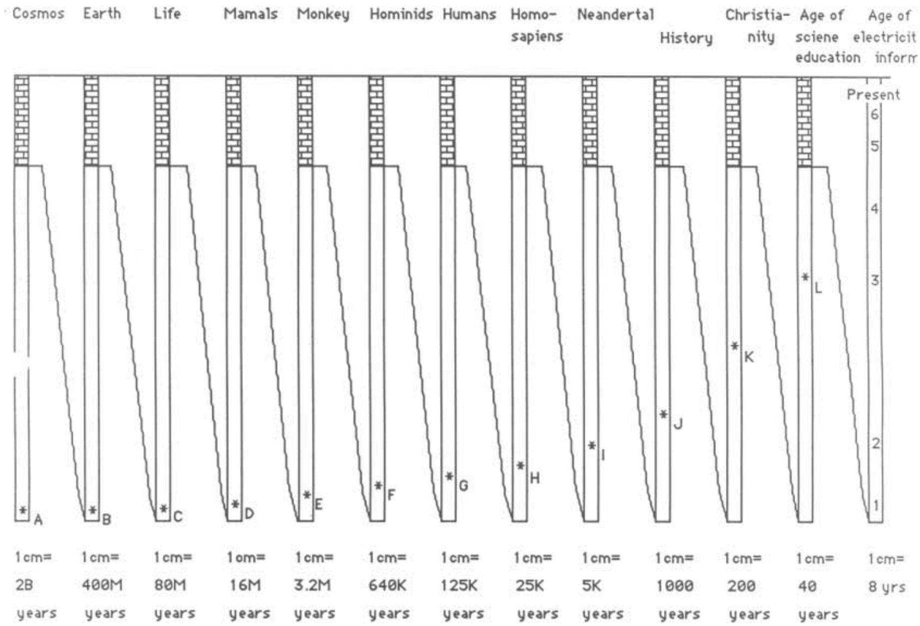


Figure 5. DIKW hierarchy.

node in the hierarchy, it is not difficult to imagine a similar construction (combining Pawlak decision tables with genetic algorithms) aiming at making best possible decisions for the architecture of the developed hierarchy of concepts. The combination of rough set techniques, genetic algorithms and simulations of interactions by agents with one another and with the environment using game theory provides a certain framework to granular computing (see [91]). The essence of these algorithms is searching for the descriptions of concept granules *optimal* for an agent, which can be later effectively employed by the agent to solve specific problems posed to it by the environment or by other agents (rules of the game). This search process is known to be highly complex and one should not expect that it will yield very precise granules. The goal here is merely to search for granular approximations sufficiently useful to the agent to construct solutions suitable to a satisfactory degree. The process leading to better and better approximation of granules can be based on genetic algorithms or on many other evolutionary models. However, due to high complexity of the concepts being approximated (e.g., because of the huge search space for relevant features [26,245]), this process is usually highly time-consuming in practical applications. Its acceleration is mainly possible by implementation of the broadest possible knowledge provided by domain experts [9–11,90,91,149,151,150,12,224], and in the case of genetic algorithms, by application of various techniques for speeding-up the evolution. This kind of evolution control techniques is sometimes referred to as *evolution of evolution* techniques [89,43]. This concept is a metaphor of observable phenomena for refining and accelerating evolution in the nature (see Figure 6, [86,89]).

To sum up, we would like to mention that the described proposals for learning mechanisms should be closely synchronized with specific agent's tasks in the environment and should effectively support its such fundamental functions as communication, perception, planning, action, cooperation, and competition [2,30,31,38,42,49,57,58,65,80, 94,100,104,124,125,164,207,216,226,227,230,238,240,243].



*A origin of the Universe and first chemical elements
*B origin of rocks on the Earth
*C origin of many-celled animals
*D origin of placetitals
*E origin of the propliopitehecus
*F origin of Australopithecus
*G origin of Homo erectus
*H origin of Homo sapiens
*I origin of Cro Magnan man
*J origin of the first civilizations
*K beginnings of the Renaissance
*L origin of modern science and technology of production

1 – first plane
2 – theory of quantum
3 – first atomic explosion; first computer
4 – first man in cosmos
5 – VLSI chips
6 – semiconductivity in high temperatures

Figure 6. The extra exponential growth in the speed of evolution.

8. Examples of Future Research Directions: From the Tarski Concept of Truth, Galois Connections, and Adjoint Functors to Adaptive Rough-Granular Computing by Agents

Alfred Tarski, in his research on satisfiability relation and concept of truth, investigated features of a class $Mod(A)$ of models satisfying some set A of expressions of the language as well as the set of features $Th(M)$ of a class of models M [235–237,70,238]. Clearly, by the definition, these functions Mod and Th are *adjoint*, i.e., they satisfy the condition:

$$M \subseteq Mod(A) \text{ if and only if } Th(M) \vdash A, \tag{2}$$

where

- M denotes models, worlds, memorized sequences of receptors receiving stimuli from the environment; sometimes subclasses of M are called *scenes*,
- A denotes a set of expressions of the language; the expressions are used to represent or denote concepts,

- $M_1 \subseteq M_2$ asserts inclusion of the model M_1 in M_2; for instance, a *scene reasoning* process may involve, for some reasons, certain models (e.g., when planning a trip by car, we consider all possible access paths at the beginning; then, depending on some other conditions, constraints, and other criteria, we gradually rule out irrelevant models and, as a consequence, we consider only a class of models M_1 with actual prospective access route models containing only the necessary information to make the trip),
- $Mod(A)$ denotes the class of all models satisfying expressions A,
- $Th(M)$ denotes the class of all language expressions that hold true in all models belonging to the class M,
- $A_1 \vdash A_2$ states that the all expressions belonging to A_2 can be derived in accordance with considered deduction rules from the set of formulas A_1.[8]

Condition (2) is usually expressed in the formalism of Galois connections or, more generally, adjoint functors as follows:

$$\frac{M \subseteq Mod(A)}{Th(M) \vdash A}, \tag{3}$$

and we say that Th is the left adjoint to functor Mod, whereas Mod is the right adjoint to functor Th. Using the language of category theory [130,129], we can also say that Th and Mod are adjoint to each other, and simply write

$$Th \dashv Mod. \tag{4}$$

Intuitively, $Th(M)$ can be regarded as a verbal description in the language of features of the models belonging to the class M. On the other hand, $Mod(A)$ can be intuitively considered as a projection of an agent's understanding of the expressions A about the class of models satisfying these expressions. In this way, we obtain the following metaphor of the adjoint functor above mentioned:

Linguistic description in a language of a class of models M	$\dashv$	*Imaginated models of a given set of language expressions A*

Similarly, a metaphoric evaluation of the aforementioned conjugation allows us to obtain, for a class of *perceived worlds* M and *expressions* A describing M, the following metaphors in natural language:

Description of M in a language	$\dashv$	*Imaginated interpretation of* A,
Symbolic reasoning about M	$\dashv$	*Scene reasoning described by expressions from* A,
Deductive reasoning in a language of M	$\dashv$	*Inductive generalization of properties specified by expressions* A,

[8] $\vdash$ (called a turnstile, first used by Gottlob Frege [51]) reads "provable from" or "is derivable from".

Judgments concerning M $\dashv$ *Action plans derived from expressions* A,

Analysis of features of models in M $\dashv$ *Synthesis of classes of models satisfying given features from* A,

Judgments concerning M $\dashv$ *Emotions concerning expressions from* A,

Logical functions of the left hemisphere (e.g., word computing) as perception effects by sensory organs of models in M, $\dashv$ *Imaginative functions of the right hemisphere (scene calculus) as effects of understanding of propositions from A by the brain.*

It can be seen from the metaphors above that adjoint functors can be regarded in a natural way as a generalization of the concept of semantics understood as a binary relation between model and language. It particularly concerns the Cartesian-closed categories, intensively investigated by Joachim Lambek and others [112]. In these categories, morphisms are considered as deductive reasoning operators. Hence, adjoint categories can be considered as pairs of categories corresponding to language and models. This kind of approach to semantics by means of adjoint functors could be a starting point to research about formula-less and model-less semantics by an analogy to point-less topology where *points* are represented by open sets including points (see, [93]). In other words, this kind of semantics deals with approximations of formulas and approximations of models instead of dealing directly with formulas and models only.

The above metaphors are only to draw attention to the adjoint relations between symbolic reasoning category and imaginative-scene reasoning category. It is worth stressing that it has become common in the cognitive studies and multi-agent interaction [227] to pay more and more attention to the role of the duality, illustrated above, in processes involving cognition and intelligence [226]. The intuitions illustrated here are basic for the implementation of granular computing [5,90,91,219,220,222,225], where an *intelligent* agent is equipped with two hemispheres (*of a brain*) – the left one is used for describing things in a symbolic language and for symbolic reasoning, while the right hemisphere deals with imagining by the agent the acceptable models satisfying certain features and with reasoning on possibilities of traversing from one model to the other (scene reasoning). Such an agent communicates with the world through sensors attached to the both brains hemispheres (see, e.g., [91]). Intuitively, the essence of granular computing is to *construct* the best possible ontological tools that are helpful in discovery of information granules which may help us wisely solve practical problems. In the context of rough sets and approximation spaces, granulation is rooted in the discovery of elementary sets that identify neighborhoods of objects of interest and give substance to our search for instances of classes of objects (i.e., concepts [161]) and which serve to establish a ground for perception [253]. A classical example illustrating these intuitions is the history of solving by our civilization, through nearly 2000 years since the ancient Greeks, problem of geometric constructions. This problem has become relatively easy due to a granulation method initiated by Évariste Galois. Using the Galois theory, certain problems in field theory may be reduced to group theory, which is in some sense simpler and better understood. In order to demonstrate the impossibility of such geometric constructions as squaring the circle, angle trisection or doubling the cube, we can express the problem in an algebraic language (instead of a geometric one) by means of the following observation: A given number can be constructed using a ruler and a compass if and only if its rank over the field of rationals is a natural power of 2. At the same time, using the Galois theory, one can express an algebraic problem by means of group theory, which can be solved relatively more easily [248,56].

In some sense, the Galois idea has been generalized by Garrett Birkhoff. He noticed in 1940 [17] that any binary relation yields two inverse dual isomorphism called *polarities*. He introduced this name because they also generalize the dual isomorphism between *polars* in analytic and projective geometry. This concept has been further generalized by Oystein Ore [153] and Cornelius J. Everett [44] to any partial order, and was called *Galois connections*. Everett showed that all Galois connections can be obtained by contraction from suitable polarities. Next, the concept of Galois connections has been generalized to category theory by Daniel M. Kan [97], who introduced the concept of *adjoint functors*. In 1946, Cornelius J. Everett and Stanisław Ulam[9] [45], using projective geometry, showed how to use this kind of connection to an algebraic interpretation of semantical meaning of quantifiers. In Ulam's opinion [244], it was the first algebraic interpretation of the universal and existential quantifiers semantic. This interpretation was based on an analogy with the projection operators in projective geometry. However, probably the first quantifier manipulation in an algebraic style could be find in early papers related to descriptive set theory (see [106–108]). This kind of algebraic quantifier manipulation is a basis for a certain type of measure of concept complexity, called the Kleene–Mostowski hierarchy (see [99,145]). For example, the well known Tarski–Kuratowski algorithm provides an easy way to get an upper bound on the classifications assigned to a formula and the set it defines. Notice that existentional and general quantifiers could be treated as adjoint functors. Having in mind this, one can generalize the Kleene–Mostowski hierarchy to arbitrary adjoint functors. This new hierarchy creates a measure of the granule complexity. The intuitions of algebraic interpretations of semantical issues of universal and existential quantifiers by an analogy with the projection operators in projective geometry, were further extended by Alfred Tarski and his research group [70]. Within the Rasiowa–Pawlak school an algebraic interpretation of quantifiers semantic is by means of *suprema* (existential quantifier) and *infima* (universal quantifier) (see [196]). William F. Lawvere and his successors unified all these ideas on the ground of category theory (see [130]) by showing how quantification can be constructed in suitable categories by using the idea of adjoint functors (i.e., by a construction which is a generalization of Galois connections).

In computer science, Galois connections are associated with several other names such as classification [8], contexts [55], or the Chu spaces [8,190]. It is interesting that applications of the Chu spaces to parallel and concurrent models were intensively studied [191]. These works are directly related to the idea of the use of algebraic approaches, developed by the Rasiowa–Pawlak school, to modeling of parallel and concurrent processes [63,137].

One of the key research directions within the Rasiowa–Pawlak school is the investigation of algebraic features of various kinds of logics. Especially important is the method

[9]See [206,244] for more details about Stanisław Ulam's life and work.

of proving completeness by means of the so-called *canonical models* (which are built of terms, language's expressions and a filter representing a given theory), described in the following section. The method shows how, having at disposal a description, to imagine models for it (to use the aforementioned metaphor, it is a cycle of back and forth shifts from the left hemisphere to the right one, and reverse). On the other hand, having models that can be transformed by suitable functors into models expressible in another language, one can build an approximation language for new models (a cycle of back and forth shifts from the right hemisphere to the left one, and reverse).

It is worth reminding that in modeling of concept approximations using rough sets, it is frequently necessary to discover relevant semantic granular structures (in right-hemisphere), syntactic granular structures (in left-hemisphere), and plans of interactions with environment and other agents. Next, based on the results of interactions it is necessary to upgrade the world perception, estimate the distance to the planned goals, and reconstruct plans. Intuitively speaking, this approach to adaptive granular computing is very similar to the quality improvement cycle known as PDCA cycle (Plan Do Check Act) or the Deming cycle. This is a fundamental challenge for the methods concerning approximate (inductive) defining of concepts from acquired information (e.g., sensor measurements represented by a given information system). This approach could be also applied to data mining (see, e.g., [100]), text mining (see, e.g., [100,15,47,66,249]), machine learning (see, e.g., [139,140,101,141,143,52,176–178,180]), and pattern recognition (see, e.g., [41,71,181]).

9. Conclusions

In this paper, some research trends within the Rasiowa–Pawlak school concerning the application of logic to AI have been selected and discussed. At the beginning, a concise genesis of the school was presented. In the next part, the understanding of AI and logic currently dominating within the school, along with the characteristic algebraic and topological tools employed by the school were described. We exposed rough set methods introduced by Zdzisław Pawlak and indicated by some trends in current studies concerning learning of complex vague concepts and their treatment. Particular attention was paid to wisdom technology and granular computing. It should be noted that this paper does not claim to be a complete and exhaustive presentation of research methods in application of logic to AI conducted within the Rasiowa–Pawlak school. These methods have significantly broader applications and have undergone a remarkable evolution during the past couple decades. This evolution, however, indicates certain directions for the future studies. For a better understanding of the evolutionary scope of the research directions conducted within the Rasiowa–Pawlak school, readers are encouraged to consult Figure 7.

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Evolution of AI models of computing in the Rasiowa-Pawlak School

Mathematical concepts and tools

	Many-valued and non-classical logic ⇒	**Approximate reasoning in distributed environments & natural computing: perception-based computing**
Logic	➢ Computability, uncertainty, natural deduction, algebraic semantics and language algebraic properties of different types of logic, especially in: • intuitionistic, • modal, • Post, intermediate, • with strong negation • implicative • algorithmic • program • non-Fregean • with infinite logical operators ➢ Abstract logics, relationship between them and characterization of classical and other logics ➢ Hierarchy of metalogics ➢ Logical aspects of programming paradigms ➢ Interpretation of logical operators in models of computation (generalized quantifiers, model operators, Post operators)	➢ Evolution of concepts: hierarchy of metalogics created by interactions with environment. ➢ Society of agents, represented by a set of modal operators. ➢ Consensus and emotional states as modal operators over non-classical truth values. ➢ Logic for distributed systems ➢ Reasoning under uncertainty in distributed systems. ➢ Vague concept approximation ➢ Boolean approximate reasoning: RSES ➢ Conflicts, negotiations, cooperation ➢ RS, FS, combination with nonmonotonic reasoning ➢ Approximate reasoning about knowledge ➢ Common sense reasoning ➢ Perception logic: evolving system of interacting local logics ➢ Computational models based on perception ➢ Computational models of behavior ➢ Learning and adaptation ➢ Autonomous computing
Algebra	➢ Algebraic models for non-classical and abstract predicate calculus (Q-algebras), generalization of Rasiowa – Sikorski Lemma ➢ Lattice theory, Boolean, Heyting, Brouwer, Post and other algebras ➢ Syntax and semantics as adjoint: concepts (Galois connections) ➢ Topos theory approach ➢ Internal representation of deduction by sheaves over closure spaces	➢ Algebraic structures for reasoning under uncertainty ➢ RS algebras, FS algebras ➢ Relational calculi ➢ Partial algebras ➢ Calculi of approximation spaces ➢ Mereological calculi of information granules
Geometry	➢ Topological properties of spaces of models and concepts ➢ "Distance" between theories which represent knowledge of agents ➢ Geometry of computations ➢ Cantor Space, as a geometric space of models for classical propositioinal calculus ➢ Topological interpretation of modal operators ➢ Closure spaces as generalized geometric spaces ➢ Heuristics based on geometry of computation space	➢ Measures of proximity (similarity): states and set of states of computations and concepts ➢ Similarity of cases and case-based reasoning ➢ Geometry of concepts ➢ Similarity of theories ➢ Granular space, information granulation and granular computing ➢ Discovery of granularity levels from data, e.g., relevant multi-valued logics.
	Philosophy, CS, Biology, Psychology, Sociology, … ⇒	**Approximate reasoning about complex vague concepts and objects in distributed & dynamically changing environments**

Inspirations outside of mathematics

Figure 7. Evolution of logical approaches to AI in the Rasiowa–Pawlak school.

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