Second order rectifiability of varifolds with bounded anisotropic mean curvature.

(joint work with Mario Santilli)

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1. Varifolds

Def. $U \subseteq \mathbb{R}^n$ open, $0 < k \leq n$

$\mathbb{V}_k(U)$ – Radon measures over $U \times G(n,k)$

$V \in \mathbb{V}_k(U)$

$\|V\|$ weight measure – Radon measure over U

$$\|V\|(B) = V(B \times G(n,k)) \quad \text{for } B \subseteq U$$

Most interesting type of varifolds.

$M \subseteq U$ a properly embedded k-dim. manifold of class $\mathcal{C}^1$.

$B \subseteq M$ a Borel set

$\mathbb{V}_k(B) \in \mathbb{V}_k(U)$

$$\mathbb{V}_k(B)(A) = \mathcal{H}^k(B \cap \{x : (x, \operatorname{Tan}(M,x)) \in A\})$$

As a functional

$$\mathbb{V}_k(B)(f) = \int_B f(x, \operatorname{Tan}(M,x))\, d\mathcal{H}^k(x)$$

$$\text{for } f \in \underbrace{\mathcal{K}(U \times G(n,k))}_{\text{continuous compactly supported functions}}$$

Rectifiable varifolds

$V \in \mathbb{RV}_k(\mathcal{U})$ iff.

$V \in \mathbb{V}_k(\mathcal{U})$ and there exist

* $M_1, M_2, \dots \subseteq \mathcal{U}$ k-dim manifolds of class $\mathcal{C}^1$
* $B_i \subseteq M_i$ Borel sets
* $r_1, r_2, \dots$ positive real numbers

$$V = \sum_{i=1}^{\infty} r_i \, \mathbb{V}_k(B_i)$$

In case $r_i \in \mathbb{Z}$ we say that V is integral.
and write $V \in \mathbb{IV}_k(\mathcal{U})$

Def. μ a Radon measure over $\mathcal{U}$, $x \in \mathcal{U}$

$$\Theta^k(\mu, x) = \lim_{r \downarrow 0} \frac{\mu \, \mathbb{B}(x, r)}{\alpha(k)\, r^k}.$$

(Hausdorff density)

2. Variational problems and the first variation

A Plateau-type problem

$B \subseteq \mathcal{U}$ a $(k-1)$-dimensional compact manifold

$\mathcal{R} = \{ M : M \subseteq \mathcal{U}$ a properly embedded $\mathcal{C}^1$ manifold with boundary $\partial M = B \}$

$$\Psi(M) = \mathcal{H}^k(M) = \| \mathbb{V}_k(M) \| (\mathcal{U})$$

Problem: Find M for which

$$\Psi(M) = \inf \Psi[\mathcal{R}]$$

(minimal surface)

Solutions exist in $\mathbf{IV}_k(U)$ but not necessarily in $\mathcal{R}$. $\rightsquigarrow$ Regularity theory

Generalisation

Anisotropy: rotations and reflections are not isometries

ϕ a non-Euclidean norm on $\mathbb{R}^n$

$\rightsquigarrow$ $\mathcal{H}^k_\phi$ k-dim Hausdorff measure over the metric space $(\mathbb{R}^n, \phi)$.

$$\mathcal{H}^k_\phi(B) = \int_B F_{BH}(\mathrm{Tan}(M, x))\, d\mathcal{H}^k(x)$$

for $B \subseteq M$ Borel, M a k-dim. manifold

where $F_{BH}(T) = \dfrac{\alpha(k)}{\mathcal{H}^k(T \cap \mathbb{B}^\phi(0,1))}$ for $T \in G(n,k)$

(Busemann-Hausdorff integrand)

This motivates considering

$$\Psi_F(M) = \int_M F(\mathrm{Tan}(M, x))\, d\mathcal{H}^k(x)$$

where F is a given continuous function of type $F : G(n,k) \longrightarrow \mathbb{R}_+$

The first variation

$g \in \mathcal{X}(U)$ – smooth compactly supported vectorfield on U.

$h_t : U \to U$ – the flow of g

$$\frac{d}{dt}\Big|_{t=0} h_t(x) = g(x) \quad \text{for } x \in U.$$

$$\frac{d}{dt}\Big|_{t=0} \Psi_F(h_t[M]) = \delta_F \mathbb{V}_k(M)(g)$$

In general, for $V \in \mathbb{V}_k(U)$ we have

$$\delta_F V(g) = \int B_F(T) \bullet Dg(x) \, dV(x,T)$$

where $B_F : G(n,k) \longrightarrow \mathrm{End}(\mathbb{R}^n)$

and
$$X \bullet Y = \mathrm{trace}(X^* \circ Y)$$
for $X, Y \in \mathrm{End}(\mathbb{R}^n)$.

Mean curvature

Def. The total F-variation measure.

$$\|\delta_F V\|(G) = \sup\{\delta_F V(g) : g \in \mathcal{X}(U), \ \mathrm{spt}\, g \subseteq G, \ |g| \le 1\}$$

for $G \subseteq U$ open

$$\|\delta_F V\|(B) = \inf\{\|\delta_F V\|(G) : B \subseteq G, \ G \subseteq U \text{ open}\}$$

Def. Assume $\|\delta_F V\|$ is Radon. Then

$$\delta_F V(g) = -\int \mathbb{h}_F(V,x) \bullet g(x) \, F(T) \, dV(x,T) + \int \eta_F(V,x) \bullet g(x) \, d\|\delta_F V\|_{sing}(x),$$

where $\|\delta_F V\|_{sing}$ is the part of $\|\delta_F V\|$ singular w.r.t. $\|V\|$.

$\mathbb{h}_F(V,\cdot) \in \mathbb{R}^n$ is the mean F-curvature vector of V.

Note.
In case $V = \mathbb{v}_k(M)$ for some k-dim. manifold of class $\mathcal{C}^2$ and $F \equiv 1$, then $\mathbb{h}_F(V,\cdot)$ coincides with the classical mean curvature vector of M.

Note. (minimal surfaces)
If V minimizes Ψ_F in a class closed under diffeomorphisms of $\mathcal{U}$ leaving $\partial \mathcal{U}$ fixed, then $\delta_F V = 0$; in particular,

$$\mathbb{h}_F(V,x) = 0 \text{ for } \|V\| \text{ a.a. } x$$

and $\|\delta_F V\|_{sing} = 0$.

Note.
In case $V = \mathbb{v}_k(M)$ and M is a graph of some Lipschitzian function $u : \mathbb{R}^k \to \mathbb{R}^{n-k}$, then conditions:

$$\begin{cases} \mathbb{h}_F(V,\cdot) \in \mathbb{L}_p(\|V\|, \mathbb{R}^n) \\ \|\delta_F V\|_{sing} = 0 \end{cases}$$

may be translated to certain PDE on u (elliptic only if F satisfies additional conditions) and the theory of PDE may be applied.
Regularity theory for varifolds starts a few steps earlier. The main thrust is on showing that V is locally a Lipschitz

graph around almost all points. Even mere rectifiability is an issue.

3. Rectifiability for varifolds.

Standing assumptions

$$\begin{cases} V \in \mathbb{V}_k(U), \quad 1 \le p \le \infty, \quad F : G(n,k) \to \mathbb{R}_+ \in \mathscr{C}^2, \\ \mathbb{h}_F(V,\cdot) \in \mathbb{L}_p(\|V\|, \mathbb{R}^n), \qquad \|\delta_F V\|_{\text{sing}} = 0, \\ \Theta^k(\|V\|, x) \ge 1 \qquad \text{for } \|V\| \text{ almost all } x. \end{cases}$$

Def. μ a Radon measure over U, $0 < k \le n$,

Always closed in U $\longrightarrow$ $\operatorname{spt} \mu = U \sim \bigcup \{ G : G \subseteq U \text{ open}, \mu(G) = 0 \}$

Borel set $\longrightarrow$ $\operatorname{carr}_k \mu = U \cap \{ x : \Theta^k(\mu, x) > 0 \}$

Note. Even if $\operatorname{carr}_k \|V\|$ is $(\mathcal{H}^k, k)$-rectifiable (i.e. can be covered by countably many k-dimensional manifolds of class $\mathscr{C}^1$ up to a set of $\mathcal{H}^k$- measure zero) the varifold V may not be rectifiable.

E.g. $S, T \in G(n,k)$, $S \ne T$

$$V(f) = \int_T f(x, S) \, d\mathcal{H}^k(x)$$

$$\text{for } f \in \mathscr{K}(\mathbb{R}^n \times G(n,k))$$

In other words, rectifiability of V means

coherent first order structure

* $S = \operatorname{Tan}^k(\|V\|, x)$ for V almost all (x, S)
* $(\mathcal{H}^k, k)$-rectifiability of $\operatorname{carr}_k \|V\|$.

Def. We say that $V \in \mathbf{IV}_k(U)$ is second order rectifiable, and write $V \in \mathbf{R}_2\mathbf{V}_k(U)$, if

* there exists a countable family $\mathcal{F}$ of k-dimensional manifolds of class $\mathcal{C}^2$ in U such that
$$\|V\|(U \sim \bigcup \mathcal{F}) = 0$$

(compatibility)
coherent second order structure

* $\mathbf{h}_F(V, x) = \mathbf{h}_F(M, x)$ for $\|V\|$-a.a. $x \in M$ whenever $M \in \mathcal{F}$. (locality of the mean curvature)

4. Summary of known results (non-comprehensive)

* Almgren / Allard 1972 (Ann. of Math)
$$\|\delta V\| \text{ Radon } \Rightarrow V \in \mathbf{RV}_k(U)$$

* Menne 2013 (J. Geom. Anal.)
$$V \in \mathbf{IV}_k(U) \text{ and } \|\delta V\| \text{ Radon } \Rightarrow V \in \mathbf{R}_2\mathbf{V}_k(U)$$
(only for integral varifolds)

* De Philippis, De Rosa, Ghiraldin 2018 (CPAM)
$$\|\delta_F V\| \text{ Radon } \Rightarrow V \in \mathbf{RV}_k(U)$$
given F satisfies the Atomic Condition
(certain weak-ellipticity condition)

* Santilli 2021 (Calc. Var. PDE)
$$\|\delta V\| \leq H \|V\| \Rightarrow \operatorname{spt}\|V\| \text{ is } (\mathcal{H}^k, k)\text{-rectifiable of class } 2.$$
for some $H \in \mathbb{R}$
(the varifold need not be integral)

Note. In case $F \equiv 1$ the monotonicity formula
(Allard 1972, 5.1, 8.3, 8.4)

yields that if $p \geq k$ $(\mathbb{h}(V,\cdot) \in \mathbb{L}_p)$, then

(a) $\operatorname{spt} \|V\| = \operatorname{conv}_k \|V\|$

(b) $\mathcal{H}^k \llcorner \operatorname{spt} \|V\| \ll \|V\|$.

(c) $\operatorname{Tan}(\operatorname{spt}\|V\|, x) = \operatorname{Tan}^k(\|V\|, x)$

for $\|V\|$ almost all x.

(classical tangent plane = approximate tangent)

In case F is not constant (a), (b), (c) remain unproven!

5. Main result

Def. (Kuratowski lower limit)

Let $S_1, S_2, \ldots \subseteq \mathcal{U}$.

$$\operatorname*{Li}_{j\to\infty} S_j = \mathcal{U} \cap \{ x : \lim_{i\to\infty} \operatorname{dist}(x, S_i) = 0 \}$$

Def. Let $\Sigma \subseteq \mathbb{R}^n$, $a \in \Sigma$, $0 < r < \infty$.

$$\Sigma_{a,r} = \left\{ \frac{b-a}{r} : b \in \Sigma \right\}$$

Def. Let $\Sigma \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$.

$$\operatorname{Set}\operatorname{Tan}(\Sigma, a) = \left\{ \operatorname*{Li}_{j\to\infty} \Sigma_{a,r_j} : \begin{array}{l} r_1, r_2 \ldots \in \mathbb{R} \\ r_j \downarrow 0 \text{ as } j \to \infty \end{array} \right\}$$

Def. Let $\Sigma \subseteq \mathcal{U}$ be rel. closed with $\mathcal{H}^{n-1}(\Sigma) < \infty$.

We say that Σ is good if for $\mathcal{H}^{n-1}$ almost all $a \in \Sigma$ there exist $\nu \in \mathbb{S}^{n-1}$ s.t.

$\nu \notin K$ and $-\nu \notin H$

for some $K, H \in \mathrm{SetTan}(\Sigma, a)$.

Theorem (K. and Santilli – in progress)

Suppose

$k = n-1$, $H \in \mathbb{R}_+$, $V \in \mathbb{V}_k(U)$,

ϕ is a uniformly convex smooth norm,

$F(\mathrm{span}\{\nu\}^\perp) = \phi(\nu)$ for $\nu \in S^{n-1}$,

$\|\delta_F V\| \leq H \|V\|$, $\mathrm{spt}\|V\|$ is good.

Then $V \in R_2\mathbb{V}_k(U)$.

Main ingredients of the proof.

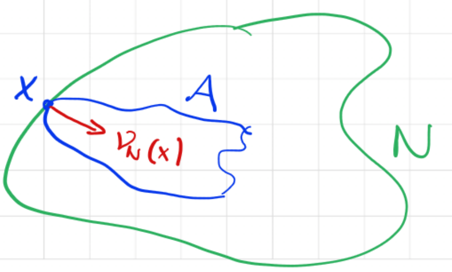
(1) Def. Assume $A \subseteq U$ is rel. closed, $0 < h < \infty$.

A is an $(n-1, h)$-set wrt. ϕ if:

for any open set $N \subseteq U$ with smooth ∂N s.t. $A \subseteq \mathrm{Clos}\, N$ there holds

$$\phi(\nu_N(x))\, \mathbb{h}_F(\partial N, x) \bullet \nu_N(x) \leq h \quad \text{for } x \in A \cap \partial N,$$

where $\nu_N(x)$ is the inward pointing unit normal vector to N at $x \in \partial N$ and $F(\mathrm{span}\{u\}^\perp) = \phi(u)$ for $u \in S^{n-1}$.



(2) De Philippis, De Rosa, Hirsch 2019 (DCDS)
De Rosa, K., Santilli 2020 (ARMA)

If $V \in \mathbb{V}_{n-1}(U)$ and $\phi(\mathbb{h}_F(V,\cdot)) < H$,
then

$$\operatorname{spt}\|V\| \text{ is an } (n-1, H)\text{-set wrt } \phi.$$

Note. This does not say that $\operatorname{spt}\|V\|$ can be touched by a smooth surface at any point. It only says that if one can touch $\operatorname{spt}\|V\|$ at some x with a smooth hypersurface M, then one gets a bound on the mean curvature of M at x.

(3) Theorem

$\Sigma \subseteq U$ is an $(n-1, h)$-set wrt. ϕ

$$S = \Sigma \cap \{x : \exists r > 0\ \exists \nu \in S^{n-1}\ \operatorname{dist}(x + r\nu, \Sigma) = r\}$$

(those $x \in \Sigma$ at which Σ may be touched by a ball)

$$\Sigma^* = \Sigma \cap \{x : \exists K \in \operatorname{SetTan}(\Sigma, x)\ K \neq \mathbb{R}^n\}$$

$\mathcal{H}^{n-1}(S \cap K) < \infty$ whenever $K \subseteq U$ compact.

Then $\mathcal{H}^{n-1}(\Sigma^* \sim S) = 0$

Remark

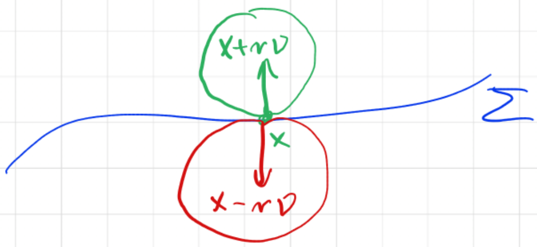
Since S is always $(\mathcal{H}^{n-1}, n-1)$-rectifiable (by a result of Menne & Santilli from 2019) of class $\mathcal{C}^2$ we get that Σ^* is as well.

Corollary

In case Σ is good we get that at $\mathcal{H}^{n-1}$ almost all $x \in \Sigma$ there are $\nu \in S^{n-1}$ and $0 < r < \infty$ s.t.

$$\operatorname{dist}(x + r\nu, \Sigma) = r = \operatorname{dist}(x - r\nu, \Sigma)$$

which implies quadratic height decay.



(4) Set $\Sigma = \operatorname{spt}\|V\|$ and assume it is good.

A Caccioppoli-type inequality provides quadratic L_2-tilt decay (oscillation of tangent planes)

(5) A classical argument of Brakke yields that $h_F(V, x) \in \operatorname{Tan}^k(\|V\|, x)^\perp$

(6) Using the technique of Ambrosio-Masnou we get locality of the mean curvature.

$\square$