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Quadratic flatness and regularity for codimension-one varifolds with bounded anisotropic mean curvature.

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joint work with Mario Santilli

1. Anisotropic mean curvature

Setup. $\Omega \subseteq \mathbb{R}^{n+1}$, $\phi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ a smooth uniformly convex norm

$\Sigma \subseteq \Omega$ a smooth hypersurface

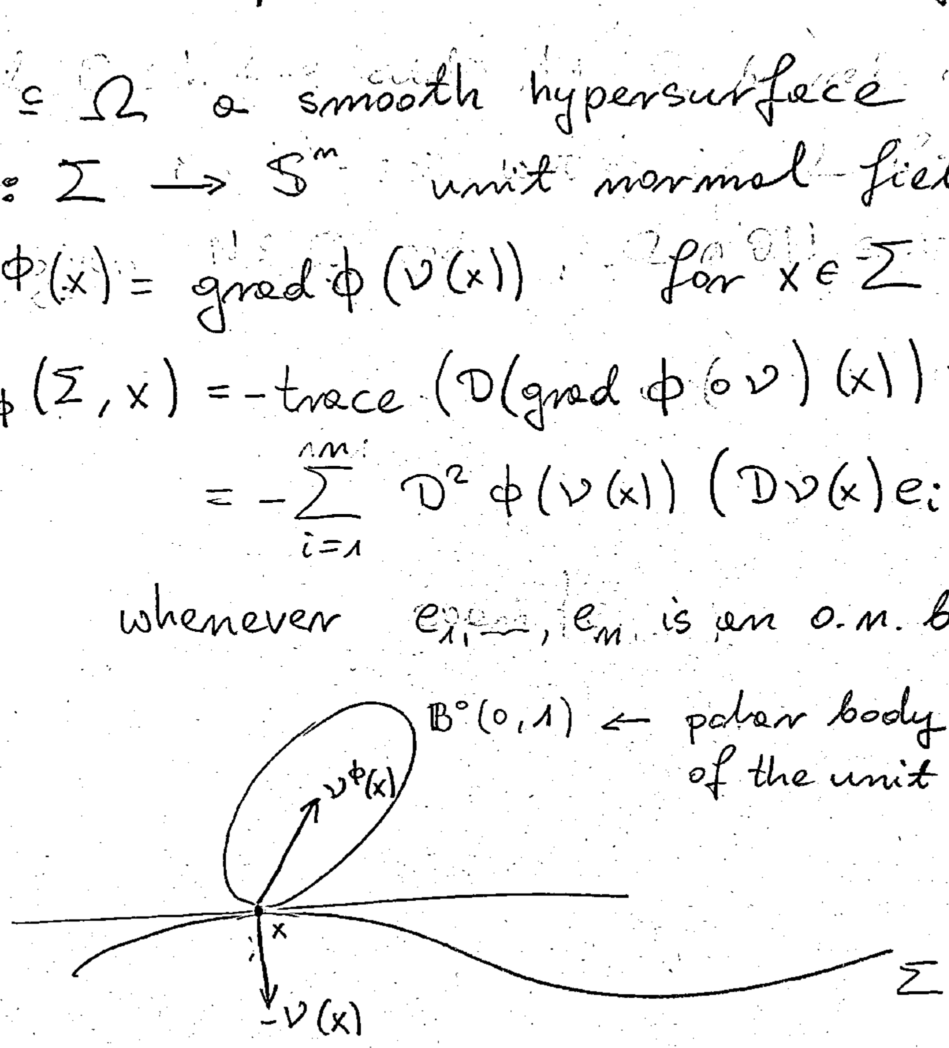
$\nu : \Sigma \to \mathbb{S}^n$ unit normal field

$\nu^\phi(x) = \operatorname{grad} \phi(\nu(x))$ for $x \in \Sigma$

$$\phi(\nu(x)) \cdot \mathbf{h}_\phi(\Sigma, x) = -\operatorname{trace}\left(D(\operatorname{grad}\phi \circ \nu)(x)\right) \nu(x) \in \mathbb{R}^{n+1}$$

$$= -\sum_{i=1}^{n} D^2\phi(\nu(x))\left(D\nu(x)e_i, e_i\right) \nu(x),$$

whenever $e_1, \dots, e_n$ is an o.n.b. of $\operatorname{Tan}(\Sigma, x)$.



Remark: If $\Sigma = \partial \mathbb{B}^\circ(0,1)$ then $\mathbf{h}_\phi(\Sigma, x)$ points inside $\mathbb{B}^\circ(0,1)$

Consider the functional

$$\Phi(\Sigma) = \int_\Sigma \phi(\nu(x)) \, d\mathcal{H}^n(x)$$

Given a vectorfield $g : \Omega \to \mathbb{R}^{n+1}$ smooth with compact support, and letting h_t be its flow, we get

$$\frac{d}{dt}\Big|_{t=0} \Phi(h_t[\Sigma]) = -\int_\Sigma \phi(\nu(x))\, \mathbb{h}_\phi(\Sigma,x)\bullet g(x)\, d\mathcal{H}^n(x)$$

2. <u>(n,h) - sets</u>

Let $0 \le h < \infty$. A rel. closed $A \subseteq \Omega$ is an <u>(n,h) - set wrt. ϕ</u> in Ω if:

[given $N \subseteq \Omega$ with $\partial N \cap \Omega$ smooth,
$\nu : \partial N \to S^n$ unit normal field, inward pointing,
such that $A \subseteq \operatorname{Clos} N$ there holds
$$\phi(\nu(x))\, \mathbb{h}_\phi(\partial N, x)\bullet \nu(x) \le h \quad \text{for } x \in A \cap \partial N.$$

3. <u>The normal bundle and its curvature</u>

$A \subseteq \mathbb{R}^{n+1}$ arbitrary

$$N(A) = (\operatorname{Clos} A \times S^n) \cap \{(a,\nu) : \operatorname{dist}(a + r\nu, A) = r \text{ for some } r > 0\}$$

$$\xi_A(x) = \operatorname{Clos} A \cap \{a : \operatorname{dist}(x,A) = |x-a|\} \quad \text{for } x \in \mathbb{R}^{n+1}$$

$$\nu_A(x) = \frac{x - \xi_A(x)}{\operatorname{dist}(x,A)} \quad \text{for } x \in \mathbb{R}^{n+1} \setminus \operatorname{Clos} A$$

$$\operatorname{Unp}(A) = \mathbb{R}^{n+1} \cap \{x : \mathcal{H}^0(\xi_A(x)) = 1\}$$

$$= \operatorname{dmn} D \operatorname{dist}(\cdot, A) \leftarrow \boxed{\text{points of differentiability of } \operatorname{dist}(\cdot, A)}$$

$\Rightarrow$ $\boxed{\mathcal{L}^{n+1}(\mathbb{R}^{n+1} \setminus \operatorname{Unp}(A)) = 0 \text{ by Rademacher}}$

$$\tilde{N}(A) = N(A) \cap \{(a,\nu) : a + r\nu \in \operatorname{dmn} D\nu_A \text{ for some } r > 0\}$$

<u>Proposition 0.</u> (1) $\mathcal{H}^n(N(A) \setminus \tilde{N}(A)) = 0$

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(2) For $(a,\nu) \in \tilde{N}(A)$ and $0 < r < \infty$ s.t. $a + r\nu = x \in \operatorname{dmn} D\nu_A$ we have $D\nu(x) \in \operatorname{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ is self-adjoint with $D\nu(x)\,\nu(x) = 0$ and all eigenvalues are $\leq \frac{1}{\operatorname{dist}(x,A)}$.

[Hug & Santilli 2022]

(3) Let $\chi_{A,1}(x) \leq \dots \leq \chi_{A,n}(x)$ be the eigenvalues of $D\nu_A(x) \,|\, \operatorname{span}\{\nu(x)\}^{\perp}$. Define <u>principal curvatures</u> $\kappa_{A,i} : \tilde{N}(A) \longrightarrow (-\infty, \infty]$ by
$$\kappa_{A,i}(a,\nu) = \frac{\chi_{A,i}(a+r\nu)}{1 - r\,\chi_{A,i}(a+r\nu)} \quad \text{for } i \in \{1 \ldots n\}.$$
These numbers do not depend on the choice of r for which $a + r\nu \in \operatorname{dmn} D\nu_A$.

<u>Remark</u>. If $r_A(a,\nu) = \sup\{r : \operatorname{dist}(a+r\nu, A) = r\}$,
[HS22] then $\kappa_{A,i}(a,\nu) \geq \frac{-1}{r_A(a,\nu)}$.

4. Quadratic flatness for (n,h) sets

<u>Lemma 1</u> $A \subseteq \Omega$ an (n,h) set wrt Φ, $(a,\nu) \in \tilde{N}(A)$. Then
$$\sum_{i=1}^{n} \kappa_{A,i}(a,\nu) \leq n\,\kappa_{A,n}(a,\nu) \leq n\,C\left(h + \frac{1}{r_A(a,\nu)}\right)$$
for some $C = C(\Phi, n)$

<u>Remarks on the proof.</u>
$\exists r > 0$. $x = a + r\nu \in \operatorname{dmn} D\nu_A$ and $\{z : \operatorname{dist}(z, A) = r\} = S(A,r)$ is locally, around x, a graph of some semiconcave function, $f : \operatorname{Tan}(S(A,r), x) \to \mathbb{R}$, which is twice pointwise differentiable at 0 and one touches A at x

with the shifted graph.

<u>Lemma 2</u> $A \subseteq \Omega$ rel. closed and $A \neq \Omega$ and assume

⊛ $\mathcal{R}_{A,m}(a,\nu) < \infty$ for $\mathcal{H}^m$ almost all $(a,\nu) \in N(A)$ with $a \in \Omega$.

Then

$$\mathcal{H}^m(\mathbb{p}[N(A)] \cap \mathbb{U}(a,r)) > 0 \quad \text{for } a \in \Omega \cap \partial A \text{ and } 0 < r < \infty.$$

<u>Remarks on the proof</u>

Use "coarea" formula from [HS22] to disintegrate the Lebesgue measure along fibres of ξ_A. Continuity of $\xi_A | \mathrm{Unp}(A)$ provides a non-empty open set $U \subseteq \Omega$ s.t. $\Omega \cap \xi_A^{-1}[\mathbb{U}(a,r)] \setminus A = U \cap \mathrm{Unp}(A)$ and we get

$$0 < \mathcal{L}^{m+1}(U) = \mathcal{L}^{m+1}(\Omega \cap \xi_A^{-1}[\mathbb{U}(a,r)] \setminus A)$$

$$\underset{(\text{coarea})}{\leq} \int_{\mathbb{p}[N(A)] \cap \mathbb{U}(a,r)} (\ldots)\, d\mathcal{H}^m$$

non-negative terms.

<u>Here</u> $\mathbb{p}: \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{m+1}$, $\mathbb{p}(x,\nu) = x$ for $x, \nu \in \mathbb{R}^{m+1}$.

Condition ⊛ says that ∂A is essentially m-dimensional.

<u>Lemma 3</u> $0 < h < \infty$, $A \subseteq \Omega$ is an (m,h) set wrt. ϕ, $\mathcal{L}^{m+1}(A) = 0$, $\mathcal{H}^m(K \cap \mathbb{p}[N(A)]) < \infty$ for $K \subseteq \Omega$ compact

Then for $\mathcal{H}^m$ almost all $a \in \mathbb{p}[N(A)] \cap \Omega$ there is $r > 0$ and $\nu \in \mathbb{S}^m$ s.t.

$$\operatorname{dist}(a + r\nu, A) = \operatorname{dist}(a - r\nu, A) = r.$$

[or $(a,\nu), (a,-\nu) \in N(A)$]

Remarks on the proof

* Previous resut of [HS22] gives that for $\mathcal{H}^n$ a.a. $a \in \mathbb{p}[N(A)]$ we have $N(A,a) \in \{1,2\}$.
  $\uparrow \{v : (a,v) \in N(A)\}$.
* Assume there is a compact set $K \subseteq \Omega \cap \mathbb{p}[N(A)]$ s.t. $0 < \mathcal{H}^n(K)$ and $\mathcal{H}^0(N(A,a)) = 1$ for $a \in K$.
* The set $\mathbb{p}[N(A)]$ is countably $(\mathcal{H}^n, n)$-rectifiable so K is $(\mathcal{H}^n, n)$ rectifiable ; hence,

  (lower Minkowski content) $\rightarrow$ $$\liminf_{r \to 0+} \frac{1}{2r} \mathcal{L}^{n+1}(K_r) \geq \mathcal{H}^n(K),$$
  where $K_r = \{x : \operatorname{dist}(x, K) < r\}$
* Disintegrate $\mathcal{L}^{n+1}$ along fibres of ξ_A to get
  $$\mathcal{L}^{n+1}(K_r) = \int_{N(A)|\Omega} J_A(a,v) \int_0^{\varrho(a,v,r)} \chi_{K_r}(a+tv) f(a,v,t)\, dt\, d\mathcal{H}^n(a,v),$$
  where $f(a,v,t) = \prod_{j=1}^{n} (1 + t\kappa_{A,j}(a,v))$

  and $\varrho(a,v,r) = \inf\{r, \tau_A(a,v)\}$

  $J_A(a,v)$ = approx. Jacobian of $\mathbb{p}|N(A)$
* Lemma 1 yields: $0 \leq f(a,v,t) \leq C(1+t)^n$ and we can use dominated convergence thm. to get
  $$\limsup_{r \to 0+} \frac{1}{2r} \mathcal{L}^{n+1}(K_r) \leq \frac{1}{2} \mathcal{H}^n(K)$$
  because $$\lim_{r \to 0+} \frac{1}{2r} \int_0^{\varrho(a,v,r)} \chi_{K_r}(a+tv) f(a,v,t)\, dt = \frac{1}{2}$$
  for $(a,v) \in \tilde{N}_n(A)|K$. $\lightning$

6/ **Def.** $A \subseteq \mathbb{R}^{n+1}$, $a \in A$

$$\mathrm{SetTan}(A,a) = \left\{ \underset{j\to\infty}{\mathrm{Li}}\, A_{a,r_j} : r_1, r_2, \ldots \in \mathbb{R},\ r_j \to 0^+ \text{ as } j\to\infty \right\}$$

where $A_{a,r} = \left\{ \frac{b-a}{r} : b \in A \right\}$

and $\underset{j\to\infty}{\mathrm{Li}}\, S_j = \left\{ x : \lim_{k\to\infty} \mathrm{dist}(x, S_k) = 0 \right\}$

(Painlevé - Kuratowski lower limit)

Def

$$A^* = A \cap \left\{ a : \mathrm{SetTan}(A,a) \neq \{\mathbb{R}^{n+1}\} \right\}$$

Lemma 4 $A \subseteq \Omega$ an (n,h)-set wrt. ϕ and $a \in A^*$

Then $\limsup_{r\to 0^+} \frac{1}{r^n} \mathcal{H}^n\left(\mathbb{B}(a,r) \cap \mathbf{p}[N(A)]\right) > 0$.

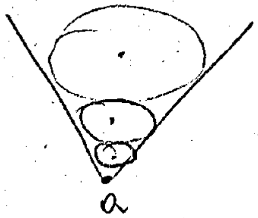
In particular, if $\mathcal{H}^n(K \cap \mathbf{p}[N(A)]) < \infty$ for $K \subseteq \Omega$ compact,

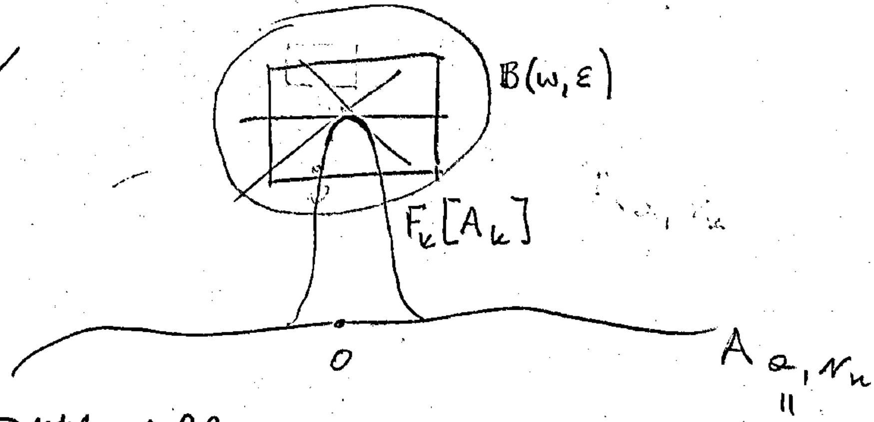
then $\mathcal{H}^n(A^* \setminus \mathbf{p}[N(A)]) = 0$

Remarks on the proof. Wlog $a = 0$.

There exists a sequence $r_k \to 0^+$, $w \in \mathbb{R}^{n+1}$, and $0 \le \varepsilon < \frac{1}{8}$ s.t.

$$\mathbb{B}(r_k w, 2\varepsilon r_k) \cap A = \emptyset \quad \forall k \in \mathbb{N}$$





$F_k : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ diffeo.

Theorem 5

Assume $A \subseteq \Omega$ is rel. closed an (n,h)-set, and such that $\mathcal{H}^n \llcorner p[N(A)]$ is a Radon measure, $\mathcal{L}^{n+1}(A) = 0$

Let $R = A \cap \{a : \exists r > 0 \; \exists \nu \in \mathbb{S}^n$

$$\mathbb{U}(a + r\nu, r) \cap A = \emptyset = \mathbb{U}(a - r\nu, r) \cap A\}$$

Then

$$\mathcal{H}^n(R \cap \mathbb{U}(a,r)) > 0 \quad \text{for } a \in A, \; r > 0$$

and $\mathcal{H}^n(A^* \setminus R) = 0$

5. Application to varifolds

Lemma 6 $V \in \mathbb{V}_n(\Omega)$, $0 < h < \infty$, $\|\delta_\phi V\| \le h \|V\|$.

[DRKS 20] Then $\operatorname{spt}\|V\|$ is an (n,h)-set wrt ϕ.

[DPDRH 19]

Theorem 7 If additionally $\mathcal{H}^n \llcorner \operatorname{spt}\|V\|$ is Radon, then Thm 5 applies with $A = \operatorname{spt}\|V\|$.

In case ϕ is Euclidean (isotropic case) the monotonicity formula gives a lower bound

$$\|V\|\,\mathbb{B}(a,r) \gtrsim r^n \quad \text{for } \|V\| \text{ almost all } a$$

and $r > 0$ small (controlled by h)

This implies that

$$\operatorname{Tan}^n(\|V\|, a) = \operatorname{Tan}(\operatorname{spt}\|V\|, a) \quad \text{for } \|V\| \text{ a.e. } a;$$

hence, at such points $\operatorname{spt}\|V\|$ becomes flat at small scales.

$$\mathbb{L}^\infty\text{-flatness} \xRightarrow{\text{Allard 1972}} \mathcal{C}^{1,\alpha}\text{-regularity}$$

In the anisotropic case (Φ non-Euclidean) monotonicity of density ratios is unproven

However

$$\left.\begin{array}{l} \mathbb{L}^\infty\text{-flatness} \\ + \ \mathcal{H}^m \llcorner \operatorname{spt}\|V\| \text{ Radon} \end{array}\right\} \xRightarrow{\text{Allard } 1986} \mathcal{C}^{1,\alpha}\text{-regularity.}$$

<u>Corollary 8</u>

$\mathcal{H}^m$ almost all points of $(\operatorname{spt}\|V\|)^*$ are $\mathcal{C}^{1,\alpha}$-regular points.