Geometric curvature energies in calculus of variations

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FNP Fundacja na rzecz Nauki Polskiej

1. Main features

Let $\mathcal{A}$ be the set of all Σ, which can be represented as $\Sigma = \Psi(\Phi(M)) \subseteq \mathbb{R}^n$, where M is a smooth, m-dimensional manifold, $\Phi : M \to \mathbb{R}^n$ is a C^1-immersion and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is bilipschitz.

- A priori, elements of $\mathcal{A}$ may have self-intersections.

By a *geometric curvature energy* we mean an integral functional $\mathcal{E} : \mathcal{A} \to \mathbb{R}_+$ defined as the L^p norm of a certain function (called *discrete curvature*) which penalizes close approach of intrinsically distant points. One example is the inverse of the *tangent-point radius* $R_{tp}(x,y)^{-1}$ defined as the radius of the sphere passing through the point y and tangent to Σ at x. Other examples are known.
Main features of such energies are:

- Analogues of the classical Sobolev-Morrey embedding theorem hold. If the parameter p is larger than a certain constant p_0, depending only on the choice of the functional and the dimension m, and if $\Sigma \subseteq \mathbb{R}^n$ has finite energy, then it must be a *submanifold* of $\mathbb{R}^n$ of class $C^{1,1-p_0/p}$ [1, 4].
- The set of all submanifolds having uniformly bounded energy and measure and passing through a common point is compact in the topology of C^1-convergence and contains at most a definite number of isotopy types.
- In consequence, one can find minimizers of $\mathcal{E}$ as well as other functionals under topological constraints (e.g. given diffeomorphism type).

2. One dimensional example

Assume that $\Sigma = \gamma(S^1)$, where $\gamma : S^1 \to \mathbb{R}^n$ is an immersion such that $|\gamma'| \equiv 1$. The *Menger curvature* of three points $x, y, z \in \Sigma$ is given by

$$c(x,y,z) = R(x,y,z)^{-1},$$

where $R(x,y,z)$ is the radius of the circumcircle of x, y, z. For any $p > 0$ we define the *Menger curvature energy* by

$$\mathcal{M}_p(\Sigma) = \int_\Sigma \int_\Sigma \int_\Sigma c(x,y,z)^p \, d\mathcal{H}^1_x \, d\mathcal{H}^1_y \, d\mathcal{H}^1_z .$$

- If $\mathcal{M}_p(\Sigma) < \infty$ for some $p > 3$, then Σ is a *submanifold* of class $C^{1,1-3/p}$.
- If γ is a C^2 embedding, then $c(x,y,z)$ is bounded on $\Sigma \times \Sigma$. Hence $\mathcal{M}_p(\Sigma) < \infty$ for any $p > 0$.

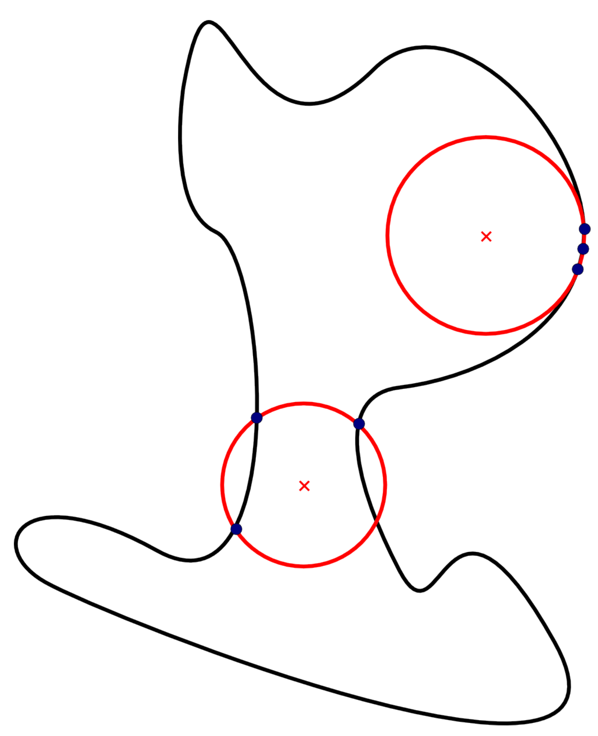

Figure 1: *Discrete curvatures capture both local and global behavior of sets.*

3. Obvious generalization

One could try to generalize the Menger curvature to m-dimensions by taking the inverse of the m-sphere passing through $m+2$ points of an m-surface. Unfortunately, this curvature would not be bounded on all smooth submanifolds of $\mathbb{R}^n$. Con- side xy.

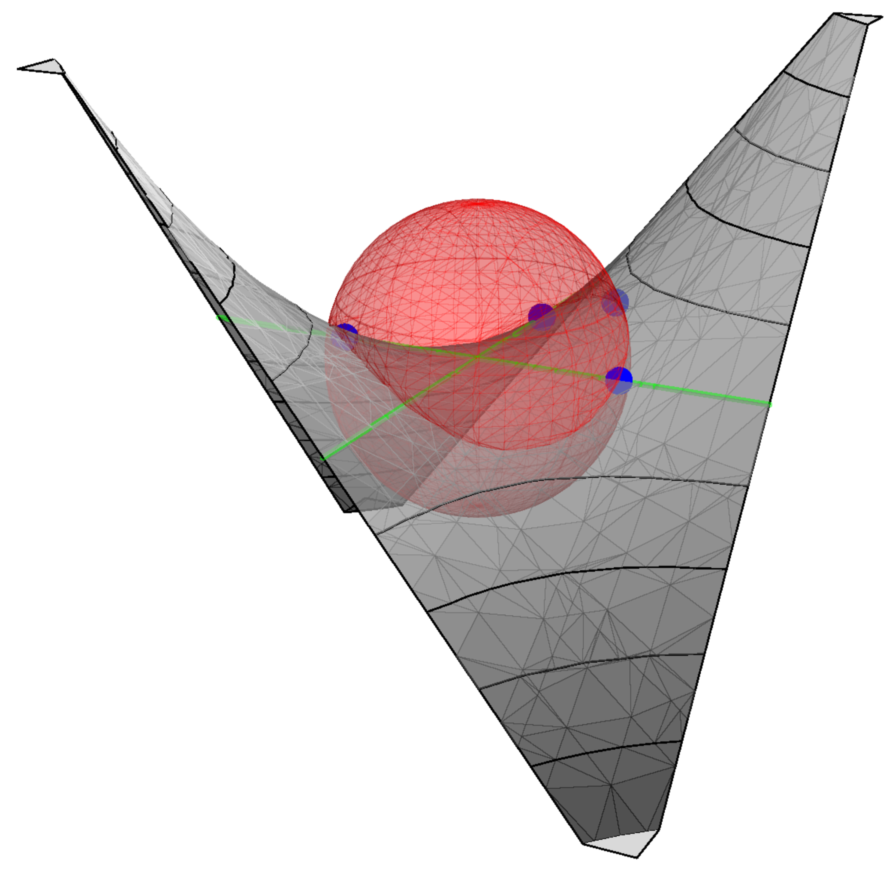

Figure 2: Σ *is a saddle surface. Green lines are the intersection of* Σ *with the plane* $\mathbb{R}^2 \times \{0\}$*. Four blue dots span the red sphere, which intersects* Σ *transversely. There exists a sequence of non-co-planar quadruples converging to the origin, such that the corresponding spheres also converge to a point and not to a tangent sphere.*

4. Tangent-point curvature

For $x, y \in \Sigma$ the *tangent-point curvature* is given by

$$K_{tp}(x,y) = R_{tp}(x,y)^{-1} = \frac{2|(T_x\Sigma^\perp)_\natural(y-x)|}{|y-x|^2}.$$

Here $R_{tp}(x,y)$ is the radius of an m-sphere tangent to Σ at x and passing through y.

- If $\Sigma \subseteq \mathbb{R}^n$ is embedded and of class C^2, then $\limsup_{y\to x} K_{tp}(x,y) = \|A(x)\|$.

For $p > 0$, we define the *tangent-point energy*

$$\mathcal{T}_p(\Sigma) = \int_\Sigma \int_\Sigma K_{tp}(x,y)^p \, d\mathcal{H}^m_x \, d\mathcal{H}^m_y .$$

- If $p > p_0$, then $\mathcal{T}_p(\Sigma)$ controls bending of Σ.

Regularity Theorem. *If $p > 2m$ and $\mathcal{T}_p(\Sigma) \le E$, then Σ is an embedded manifold of class $C^{1,\alpha}$, where $\alpha = 1 - \frac{2m}{p}$. Moreover, there exist $R > 0$ and $L > 0$ controlled by E, such that for each $x \in \Sigma$*

♣ $\big((\Sigma - x) \cap \mathbb{B}_R\big) = \operatorname{graph} f \cap \mathbb{B}_R$, *where* $f : T_x\Sigma \to T_x\Sigma^\perp$ *satisfies* $\|f\|_{C^{1,\alpha}} \le L$.

$$\mathcal{A}_p(E,A) = \big\{\Sigma \in \mathcal{A} : \mathcal{T}_p(\Sigma) \le E \,, \mathcal{H}^m(\Sigma) \le A \,, 0 \in \Sigma\big\}.$$

- If $\Sigma_1, \Sigma_2 \in \mathcal{A}_p(E,A)$ are close in the Hausdorff metric, then they are ambient C^1-isotopic.

Isotopy Theorem. *If $\Sigma_1, \Sigma_2 \in \mathcal{A}_p(E,A)$. Then there exists $R > 0$ controlled by E and A, such that if the Hausdorff distance $d_{\mathcal{H}}(\Sigma_1, \Sigma_2) = \rho \le R$, then Σ_1 and Σ_2 are ambient C^1-isotopic. Moreover, there exists a diffeomorphism $J : \mathbb{R}^n \to \mathbb{R}^n$ such that $J(\Sigma_1) = \Sigma_2$ and for $x, y \in \mathbb{R}^n$*

$$(1 - C\rho^{\frac{\alpha}{2}})|x-y| \le |J(x) - J(y)| \le (1 + C\rho^{\frac{\alpha}{2}})|x-y| .$$

5. $C^{1,\alpha}$-tubular neighborhoods

- For a $C^{1,\alpha}$ submanifold $\Sigma \subseteq \mathbb{R}^n$, one can construct a tubular neighborhood $U \supseteq \Sigma$ equipped with a C^1-projection $p : U \to \Sigma$ along almost normal spaces.

Proposition. *Assume $\Sigma \subseteq \mathbb{R}^n$ satisfies ♣ and $\operatorname{diam}\Sigma \le d$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ and a projection $p : \Sigma + \mathbb{B}_\delta \to \Sigma$ such that*

- *p is C^1-smooth*
- $|p(x) - x| \le 4\operatorname{dist}(x, \Sigma)$
- *for all $z \in \Sigma$ there exists $N \in G(n, n-m)$ such that $p^{-1}(z) \subseteq (z + N)$ and $\sphericalangle(N, T_z\Sigma^\perp) \le \varepsilon$*

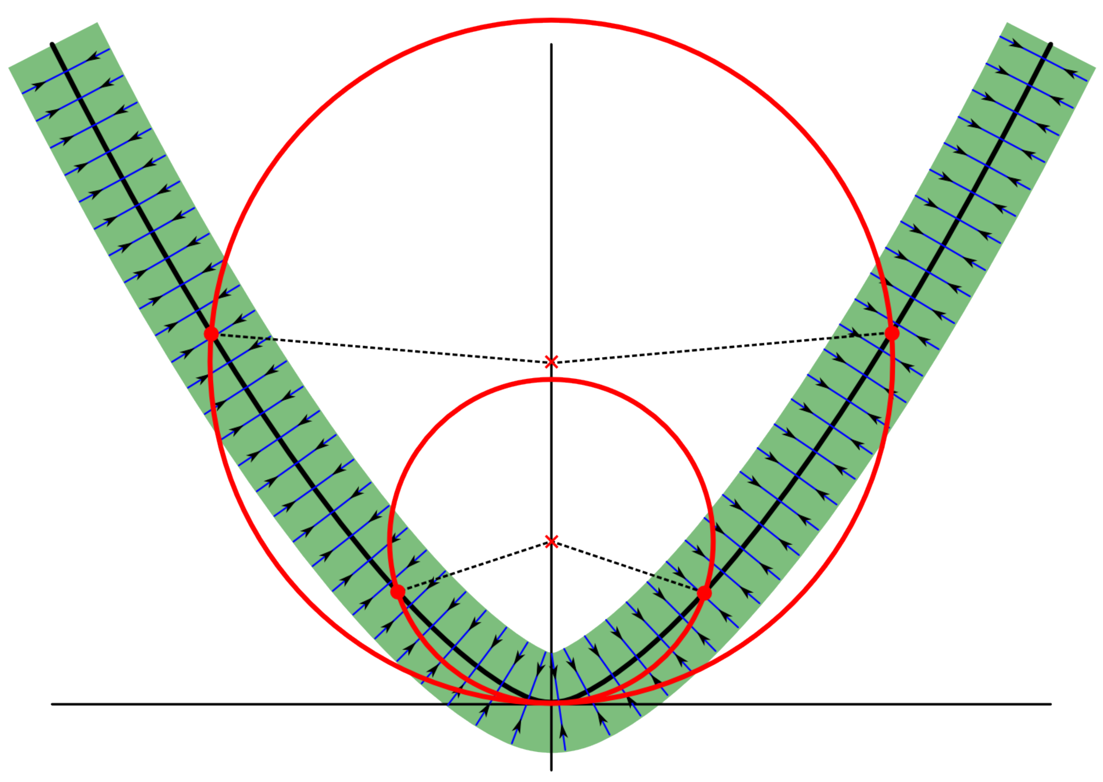

Figure 3: *For each point on the vertical line there are two points on Σ (the black line) which realize the distance to Σ. However we can still define an "almost nearest point projection".*

6. Variational problems

- Due to Blaschke's selection theorem, $\mathcal{A}_p(E,A)$ is compact in the Hausdorff metric.
- As a consequence of the Isotopy Theorem, we obtain that a sequence of manifolds $\Sigma_j \in \mathcal{A}_p(E,A)$ which converges in the Hausdorff metric, converges in a much stronger, C^1-sense. Moreover, almost all manifolds in the sequence are ambient isotopic to the limit manifold.

Finiteness Theorem. *The class $\mathcal{A}_p(E,A)$ contains only finitely many different isotopy classes of manifolds. Moreover the number of these classes can be bounded by a constant explicitly computable from the numbers E, A, m, n, p.*

- One can solve variational problems with topological constraints.

Existence of minimizers. *Let M be fixed reference manifold and let*

$$\mathcal{B}_M = \mathcal{A}_p(E,A) \cap \{\Sigma : \Sigma \text{ is diffeomorphic to } M\}.$$

Then there exists $\Sigma \in \mathcal{B}_M$ such that

$$\mathcal{T}_p(\Sigma) = \inf\nolimits_{K \in \mathcal{B}_M} \mathcal{T}_p(K).$$

- Of course, one can also find in $\mathcal{B}_M$ a minimizer of any functional which is l.s.c. with respect to C^1-convergence, e.g. there exists $\Sigma \in \mathcal{B}_M$ such that

$$\mathcal{H}^m(\Sigma) = \inf\nolimits_{K \in \mathcal{B}_M} \mathcal{H}^m(K).$$

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