Quadratic flatness and regularity for codimension one varifolds with bounded anisotropic mean curvature

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joint work with Mario Santilli

Setup

$\Omega \subseteq \mathbb{R}^{n+1}$ open

$\phi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ smooth uniformly convex norm

ϕ° - dual norm

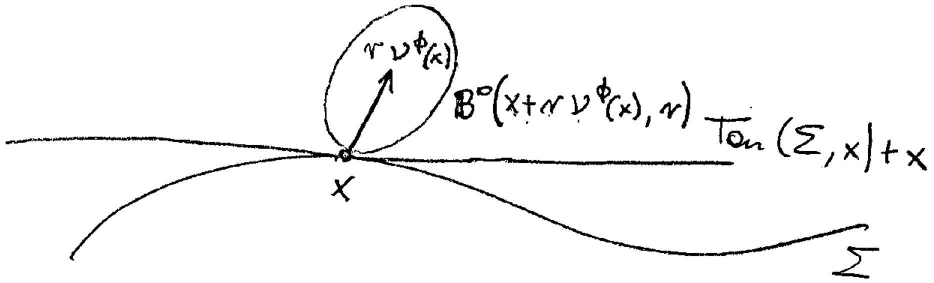
1. Anisotropic mean curvature

$\Sigma \subseteq \Omega$ smooth hypersurface

$\nu : \Sigma \longrightarrow \mathbb{S}^n$ smooth unit normal field

Def. $\nu^\phi(x) = \operatorname{grad} \phi(\nu(x))$ for $x \in \Sigma$

(anisotropic normal)



For small $0 < r < \infty$ and $x \in \Sigma$ we have

$$\mathbb{U}^\circ(x + r\nu^\phi(x), r) \cap \Sigma = \emptyset$$

Def $\phi(\nu(x))\, \mathbf{h}_\phi(\Sigma, x) = -\operatorname{trace}\left(D\nu^\phi(x)\right)\nu(x)$

(mean ϕ-curvature vector)

<u>Remark</u> If $\Sigma = \partial \mathbb{B}^\circ(0,1)$, then $\mathbb{h}_\phi(\Sigma, x)$ points inside $\mathbb{B}^\circ(0,1)$.

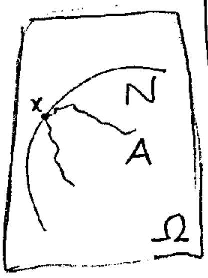
<u>Remark</u> Define
$$\Phi(\Sigma) = \int_\Sigma \phi(\nu(x))\, d\mathcal{H}^n(x)$$

Given a smooth vectorfield $g : \Omega \to \mathbb{R}^{n+1}$ with compact support, and its flow h_t we get

$$\underbrace{\frac{d}{dt}\Phi(h_t[\Sigma])}_{} = -\int_\Sigma \phi(\nu(x))\, \mathbb{h}_\phi(\Sigma,x) \bullet g(x)\, d\mathcal{H}^n(x)$$

First variation of Φ on Σ in direction g.

<u>Def</u>. Let $0 \le h < \infty$, $A \subseteq \Omega$ rel. closed.
A is (n,h)-set wrt. ϕ in Ω if:



given $N \subseteq \Omega$ open with $\partial N \cap \Omega$ smooth, $\nu : \partial N \to \mathbb{S}^n$ inward unit normal field, $A \subseteq \mathrm{Clos}(N)$, there holds
$$\phi(\nu(x))\, \mathbb{h}_\phi(\partial N, x) \bullet \nu(x) \le h \quad \text{for } x \in A \cap \partial N$$

see B. White „Area blow-up sets ..."

<u>Remark</u> If $V \in \mathbb{V}_n(\Omega)$, $\|\delta_\phi V\| \le h \|V_\phi\|$, $0 < h < \infty$, then $\operatorname{spt}\|V\|$ is an (n,h)-set wrt. ϕ.

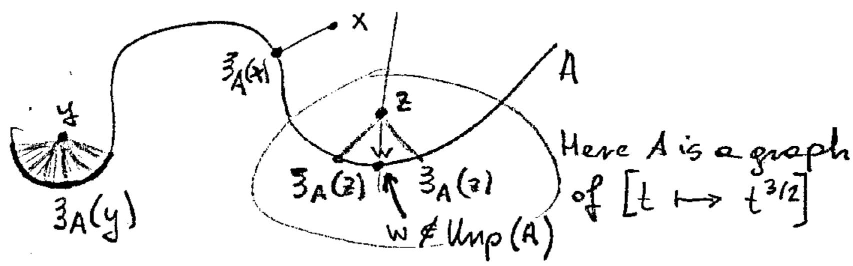
see De Rosa, K., Santilli, ARMA 2020

2. The normal bundle and its curvature

$A \subseteq \mathbb{R}^{n+1}$ arbitrary closed set

Def * $N(A) = (A \times \mathbb{S}^n) \cap \{(a,\nu) : \delta_A(a + r\nu) = r \text{ for some } r > 0\}$,

* where $\delta_A(x) = \inf\{|x-a| : a \in A\}$ ← Euclidean distance
* $\xi_A(x) = A \cap \{a : \delta_A(x) = |x-a|\}$ for $x \in \mathbb{R}^{n+1}$ ← nearest point projection
* $\nu_A(x) = \dfrac{x - \xi_A(x)}{\delta_A(x)}$ for $x \in \mathbb{R}^{n+1} \setminus A$
* $\mathrm{Unp}(A) = \mathbb{R}^{n+1} \cap \{x : \mathcal{H}^0(\xi_A(x)) = 1\}$ ← unique nearest point



Note * ν_A is defined outside of A

* if $p(x,\nu) = x$ for $x, \nu \in \mathbb{R}^{n+1}$,

then $p[N(A)]$ is the subset of A of points, where A can be touched by a ball.

Thm. $\mathrm{Unp}(A) = \mathrm{dmn}\, D\delta_A$ ← points of differentiability of δ_A

Cor $\mathcal{L}^{n+1}(\mathbb{R}^{n+1} \setminus \mathrm{Unp}(A)) = 0$

(by Rademacher since $\mathrm{Lip}\,\delta_A = 1$)

Def. $\tilde{N}(A) = N(A) \cap \{(a,\nu) : a + r\nu \in \mathrm{dmn}\, D\nu_A \text{ for some } r > 0\}$

Thm [Hug, Santilli]

(1) $\mathcal{H}^n(N(A) \setminus \tilde{N}(A)) = 0$

(2) If $(a,\nu) \in \tilde{N}(A)$, $0 < r < \infty$, and $x = a + r\nu \in \operatorname{dmn} D\nu_A$
then $D\nu_A(x) \in \operatorname{End}(\mathbb{R}^{n+1})$ is self-adjoint
with $D\nu_A(x)\nu = 0$ and all eigenvalues $\leq \frac{1}{\delta_A(x)}$.

(3) Let $\chi_{A,1}(x) \leq \dots \leq \chi_{A,n}(x)$ be the eigenvalues of $D\nu_A(x) \mid \operatorname{span}\{\nu_A(x)\}^{\perp}$.

Define <u>principal curvatures</u> $\kappa_{A,i} : \tilde{N}(A) \to (-\infty, \infty]$
by
$$\kappa_{A,i}(a,\nu) = \frac{\chi_{A,i}(a+r\nu)}{1 - r\chi_{A,i}(a+r\nu)} \quad \text{for } i \in \{1 \dots n\}$$

These numbers do not depend on the choice of $0 < r < \infty$ for which $a + r\nu \in \operatorname{dmn} D\nu_A$.

<u>Def</u> $r_A(a,\nu) = \sup\{r : \delta_A(a+r\nu) = r\}$ ← reach function

<u>Lemma</u> [Hug, Santilli]
$$\kappa_{A,i}(a,\nu) \geq \frac{-1}{r_A(a,\nu)}$$

3. <u>Main Theorem</u> [K., Santilli]

Assume $A \subseteq \Omega$ is an (n,h)-set wrt Φ s.t.
$\mathcal{H}^n \llcorner p[N(A)]$ is Radon, $\mathcal{L}^{n+1}(A) = 0$.

Let
$$R = A \cap \left\{ a : \exists r > 0 \ \exists \nu \in \mathbb{S}^n \ \begin{array}{l} \delta_A(a+r\nu) = r \\ \delta_A(a-r\nu) = r \end{array} \right\}$$

Then $\mathcal{H}^n(S \setminus R) = 0$ for any countably $(\mathcal{H}^n, n)$-rectifiable set $S \subseteq A$.

<u>Application to varifolds</u>

Assume $V \in \mathbf{IV}_n(\Omega)$, $0 \leq h < \infty$, $\|\delta_\Phi V\| \leq h \|V_\Phi\|$, $\mathcal{H}^n \llcorner \operatorname{spt}\|V\| \ll \|V\|$,
$\Theta^n(\|V\|, x) = 1$ for $\|V\|$ almost all x.

Then Main Theorem applies to $A = \operatorname{spt}\|V\|$.

In case ϕ is Euclidean (isotropic)
the monotonicity formula yields

$$\|V\| B(a,r) \gtrsim r^n \quad \text{for } \|V\| \text{ almost all } a \text{ and } 0 < r < r_0(h)$$

This gives $\mathrm{Tan}^n(\|V\|, a) = \mathrm{Tan}(\operatorname{spt}\|V\|, a)$ for $\|V\|$ a.a. a
and $\operatorname{spt}\|V\|$ is flat at small scales. Then

$$\mathbb{L}_\infty\text{-flatness} \underset{\text{Allard 1972}}{\Longrightarrow} \mathcal{C}^{1,\alpha}\text{-regularity}.$$

In the anisotropic case (ϕ non-Euclidean)
monotonicity of density ratios remains unproven.
However,

$$\left.\begin{cases} \mathbb{L}_\infty\text{-flatness} \\ + \mathcal{H}^n \llcorner \operatorname{spt}\|V\| \ll \|V\| \end{cases}\right. \xrightarrow{\text{Allard 1986}} \mathcal{C}^{1,\alpha}\text{-regularity}$$

Corollary $\mathcal{H}^n$ almost all points of $\operatorname{spt}\|V\|$
are $\mathcal{C}^{1,\alpha}$-regular.

4. Proof of the Main Theorem

Lemma 1 $A \subseteq \Omega$ an (n,h)-set wrt ϕ
$(a,\nu) \in \tilde{N}(A)$

Then

$$\sum_{i=1}^{n} \kappa_{A,i}(a,\nu) \le n\, \kappa_{A,n}(a,\nu) \le C\left(h + \frac{1}{r_A(a,\nu)}\right)$$

where $C = C(\phi, n)$.

Note $(a,\nu) \in \tilde{N}(A) \Rightarrow$ A can be touched by a part of $\{x : \delta_A(x) = r\}$ shifted towards A.

Lemma 2 $A \subseteq \Omega$ an (m,h)-set wrt. ϕ

$\mathcal{L}^{m+1}(A) = 0$, $K \subseteq A$ compact,

$\mathcal{H}^m(K) < \infty$.

Then

$$\lim_{r \to 0^+} \frac{1}{2r} \mathcal{L}^{m+1}(K + \mathbb{B}(0,r))$$

Minkowski content

$$= \mathcal{H}^m(K \cap \{a : \mathcal{H}^0(\mathbb{p}^{-1}\{a\} \cap N(A)) = 2\})$$

$$+ \frac{1}{2} \mathcal{H}^m(K \cap \{a : \mathcal{H}^0(\mathbb{p}^{-1}\{a\} \cap N(A)) = 1\})$$

Proof

Let $K_r = K + \mathbb{B}(0,r)$.

Disintegration of $\mathcal{L}^{m+1}$ along fibres of ξ_A.

$$\mathcal{L}^{m+1}(K_r) = \mathcal{L}^{m+1}(K_r \setminus A)$$

$$= \int_{N(A)} \operatorname{ap} J \mathbb{p}(a,v) \int_0^{\inf\{r, r_A(a,v)\}} \chi_{K_r}(a+tv) f(a,v,t)\, d\mathcal{L}^1(t)\, d\mathcal{H}^m(a,v),$$

where $f(a,v,t) = \prod_{j=1}^{m} (1 + t\kappa_{A,j}(a,v))$.

By Lemma 1

$$0 \le f(a,v,t) \le \Delta (1+t)^m \quad \text{for } (a,v) \in N(A)$$
$$0 < t < r_A(a,v);$$

$\Delta = \Delta(h, m, \phi)$

thus, one may use the dominated convergence theorem to pass to the limit under the integral.

Remark Actually, the above disintegration formula is valid as long $\kappa_{A,m}(a,v) < \infty$ for $\mathcal{H}^m$ almost all $(a,v) \in N(A)$ but this holds by Lemma 1.

Lemma 3 $A \subseteq \Omega$ an (n,h)-set wrt ϕ
$\mathcal{L}^{n+1}(A) = 0$, $R \subseteq A$ countably
$(\mathcal{H}^n, n)$-rectifiable

Then for $\mathcal{H}^n$ almost all $a \in R$ there are $\nu \in S^n$ and $0 < r < \infty$
s.t. $\delta_A(a + r\nu) = r = \delta_A(a - r\nu)$

Proof. Known estimate for the lower Minkowski content of a countably $(\mathcal{H}^n, n)$-rectifiable set

$$\liminf_{r \to 0+} \frac{1}{2r} \mathcal{L}^n(K_r) \geq \mathcal{H}^n(K) \quad \text{for } K \subseteq R \text{ compact.}$$

Combining with Lemma 2 gives

$$\mathcal{H}^n(K) \leq \mathcal{H}^n(K \cap A_2) + \tfrac{1}{2} \mathcal{H}^n(K \cap A_1)$$

where A_2 = points where one can touch with two balls,
A_1 = points where one can touch with only one ball.

In particular, if $K \subseteq A_1$, then

$$\mathcal{H}^n(K) \leq \tfrac{1}{2} \mathcal{H}^n(K) \Rightarrow \mathcal{H}^n(K) = 0$$

□

Remark [Hug, Santilli]

$$\mathcal{H}^0(\mathbb{p}^{-1}\{x\} \cap N(A)) \in \{1,2\} \quad \text{for } \mathcal{H}^n \text{ almost all } x \in \mathbb{p}[N(A)]$$

Remark For an arbitrary closed set A it might be that $\mathcal{H}^n(\mathbb{p}[N(A)]) = 0$ even if $\mathcal{H}^n(A) < \infty$
so Lemma 2 exhibits a property specific for (n,h)-sets.