Finite measure implies rectifiability for graphs of continuous functions

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1 Preliminaries

We shall use the notation of Federer; see [Fed69, pp. 669 – 671]. In the sequel n and m shall denote integers satisfying $1 \le m < n$.

1.1 Definition. Let $f : \mathbf{R}^m \to \mathbf{R}$ be $\mathscr{L}^m$ measurable. We say that f is of *locally bounded variation*, and write $f \in BV_{\mathrm{loc}}(\mathbf{R}^m)$, if $\mathbf{E}^m \llcorner f \in \mathbf{N}_m^{\mathrm{loc}}(\mathbf{R}^m)$.

1.2 Definition (cf. [Fed69, 4.5.10])**.** Let (Y, d) be a metric space, $f : \mathbf{R} \to Y$ be $\mathscr{L}^1$ measurable, $-\infty < a < b < \infty$, $I = \{t : a \le t \le b\}$. We define the *essential variation of f on I*, denoted $\operatorname{ess} \mathbf{V}_a^b f$, as the supremum of the set of numbers

$$\sum_{j=1}^{\nu} d(f(t_j), f(t_{j+1}))$$

corresponding to all finite sequences of points $t_1, t_2, \ldots, t_{\nu+1}$ of $\mathscr{L}^1$ approximate continuity of f with $a < t_1 \le t_2 \le \cdots \le t_{\nu+1} < b$.

1.3 Definition (cf. [Fed69, 4.5.9(27)])**.** For $i = 1, 2, \ldots, m$ and $z \in \mathbf{R}^{m-1}$ we define

$$\chi_{i,z} : \mathbf{R} \to \mathbf{R}^m \,, \qquad \chi_{i,z}(t) = (z_1, \ldots, z_{i-1}, t, z_i, \ldots, z_{m-1}) \,.$$

1.4 Lemma (cf. [Fed69, 4.5.10])**.** *Assume $f : \mathbf{R}^m \to \mathbf{R}$ is $\mathscr{L}^m$ measurable and $m \ge 2$. Then f is of locally bounded variation if and only if*

$$\int_K |f| \,\mathrm{d}\mathscr{L}^m < \infty \quad \text{whenever } K \subseteq \mathbf{R}^m \text{ is compact} \tag{1}$$

$$\text{and} \quad \int_{*Z} \operatorname{ess} \mathbf{V}_a^b (f \circ \chi_{i,z}) \,\mathrm{d}\mathscr{L}^{m-1}(z) < \infty \tag{2}$$

whenever $Z \subseteq \mathbf{R}^{m-1}$ is compact, $-\infty < a < b < \infty$, and $i \in \{1, 2, \ldots, m\}$.

1.5 Lemma (cf. [Fed69, 2.10.13])**.** *Let Y be a metric space, $g : \mathbf{R} \to Y$ be continuous. Then*

$$\operatorname{ess} \mathbf{V}_a^b g = \int N(g|\{t : a \le t \le b\}, y) \,\mathrm{d}\mathscr{H}^1(y) \quad \text{whenever } -\infty < a < b < \infty \,.$$

1.6 Lemma (cf. [Fed69, 2.10.25])**.** *Let X and Y be metric spaces, $f : X \to Y$ be Lipschitz, $A \subseteq X$, $0 \le k < \infty$, $0 \le l < \infty$. Then*

$$\int_Y^* \mathscr{H}^k(A \cap f^{-1}\{y\})\, \mathrm{d}\mathscr{H}^l(y) \le (\operatorname{Lip} f)^l \frac{\boldsymbol{\alpha}(k)\boldsymbol{\alpha}(l)}{\boldsymbol{\alpha}(k+l)} \mathscr{H}^{k+l}(A)$$

provided either $\{y : \mathscr{H}^k(A \cap f^{-1}\{y\}) > 0\}$ is a union of countably many sets of finite $\mathscr{H}^l$ measure or Y is boundedly compact.

2 The proof

2.1 Remark. Assume $m \ge 2$. Let $f : \mathbf{R}^m \to \mathbf{R}$ be $\mathscr{L}^1$ measurable, $F : \mathbf{R}^m \to \mathbf{R}^{m+1}$ be given by $F(x) = (x, f(x))$, and $i \in \{1, 2, \dots, m\}$, and $z \in \mathbf{R}^{m-1}$. Whenever $-\infty < s < t < \infty$ we have

$$\begin{aligned} |f \circ \chi_{i,z}(t) - f \circ \chi_{i,z}(s)| &\le |F \circ \chi_{i,z}(t) - F \circ \chi_{i,z}(s)| \\ &= \big(|f \circ \chi_{i,z}(t) - f \circ \chi_{i,z}(s)|^2 + |t-s|^2\big)^{1/2} . \end{aligned}$$

In particular, $\operatorname{ess} \mathbf{V}_a^b(f \circ \chi_{i,z}) \le \operatorname{ess} \mathbf{V}_a^b(F \circ \chi_{i,z})$.

2.2 Theorem. *Assume $m \ge 2$. Let $f : \mathbf{R}^m \to \mathbf{R}^{n-m}$ be continuous. Define $F : \mathbf{R}^m \to \mathbf{R}^n$ by $F(x) = (x, f(x))$ for $x \in \mathbf{R}^m$ and let $\Sigma = \operatorname{im} F$ be the graph of f. Suppose*

$$\mathscr{H}^m(\Sigma \cap K) < \infty \quad \text{whenever } K \subseteq \mathbf{R}^n \text{ is compact.}$$

Then $f \in BV_{\mathrm{loc}}(\mathbf{R}^m)^{n-m}$.

Proof. Since we want to show that each component function of f is in $BV_{\mathrm{loc}}(\mathbf{R}^m)$ we may, and shall, assume that $n - m = 1$. According to 1.4 it suffices to check conditions (1) and (2). Condition (1) is trivially satisfied since f is continuous. We shall check (2).

Let $Z \subseteq \mathbf{R}^{m-1}$ be compact, $-\infty < a < b < \infty$, and $i \in \{1, 2, \dots, m\}$. Set $J = \{t : a < t < b\}$. According to 2.1 and 1.5 for $z \in \mathbf{R}^{m-1}$ we have

$$\operatorname{ess} \mathbf{V}_a^b(f \circ \chi_{i,z}) \le \operatorname{ess} \mathbf{V}_a^b(F \circ \chi_{i,z}) = \int_{\mathbf{R}^n} N(F \circ \chi_{i,z}|J, y)\, \mathrm{d}\mathscr{H}^1(y) = \mathscr{H}^1(F \circ \chi_{i,z}[J]) ;$$

hence,

$$\int_{*Z} \operatorname{ess} \mathbf{V}_a^b(f \circ \chi_{i,z})\, \mathrm{d}\mathscr{L}^{m-1}(z) \le \int_Z^* \mathscr{H}^1(F \circ \chi_{i,z}[J])\, \mathrm{d}\mathscr{L}^{m-1}(z) .$$

Define

$$\begin{gathered} p_i : \mathbf{R}^n \to \mathbf{R}^{m-1} , \quad p_i(z) = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m) , \\ A_{i,J,Z} = \mathbf{R}^n \cap \{w : w_i \in J ,\ p_i(w) \in Z\} . \end{gathered}$$

Then $\operatorname{Lip} p_i = 1$ and

$$\operatorname{im} F \cap A_{i,J,Z} \cap p_i^{-1}[z] = F \circ \chi_{i,z}[J] \quad \text{for } z \in Z .$$

Therefore, according to 1.6 since $\Sigma \cap A_{i,J,Z}$ is compact

$$\begin{aligned}\int_{*Z} \operatorname{ess} \mathbf{V}_a^b(f \circ \chi_{i,z}) \, \mathrm{d}\mathscr{L}^{m-1}(z) &\le \int_Z^* \mathscr{H}^1(\operatorname{im} F \cap A_{i,J,Z} \cap p_i^{-1}[z]) \, \mathrm{d}\mathscr{L}^{m-1}(z) \\ &= \int_{\mathbf{R}^{m-1}}^* \mathscr{H}^1(\Sigma \cap A_{i,J,Z} \cap p_i^{-1}[z]) \, \mathrm{d}\mathscr{L}^{m-1}(z) \\ &\le \frac{\boldsymbol{\alpha}(1)\boldsymbol{\alpha}(m-1)}{\boldsymbol{\alpha}(m)} \mathscr{H}^m(\Sigma \cap A_{i,J,Z}) < \infty \,. \quad \square\end{aligned}$$

2.3 Corollary. *The graph of f is countably $(\mathscr{H}^m, m)$ rectifiable.*

Proof. Apply [Fed69, 4.5.9(4)(5)]. □

2.4 Corollary. *If F has the Lusin N property, i.e.,*

$$\mathscr{H}^m(Z) = 0 \quad \Rightarrow \quad \mathscr{H}^m(F[Z]) = 0 \,,$$

then $f \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^m, \mathbf{R}^{n-m})$.

Proof. Apply [Fed69, 4.5.9(30)]. □

References

[Fed69] Herbert Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

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