ALLARD'S WEAK MAXIMUM PRINCIPLE

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ABSTRACT. We reprove Allard's weak maximum principle [All86, 3.4(6)] using his own technique. We make no claim to originality or novelty.

1. WEAK MAXIMUM PRINCIPLE

1.1. As in [Fed69, 1.7] we define the standard *polarity* $\beta : \mathbf{R}^{n+1} \to \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R})$ by

$$\beta(u)v = u \bullet v \quad \text{for } u, v \in \mathbf{R}^{n+1}\,.$$

We assume ϕ is a uniformly convex $\mathscr{C}^2$-norm on $\mathbf{R}^{n+1}$. Uniform convexity implies that there exists an *ellipticity constant* $\gamma(\phi) > 0$ such that

$$\mathrm{D}^2\phi(u)(v,v) \ge \gamma(\phi)|v|^2 \quad \text{for } u \in \mathbb{S}^n \text{ and } v \in \ker \beta(u)\,.$$

We also define

$$c(\phi) = \sup\left\{\|\mathrm{D}^k\phi(\nu)\| : \nu \in \mathbb{S}^n,\ k \in \{0,1,2\}\right\} + \left(\inf\left\{\|\mathrm{D}^k\phi(\nu)\| : \nu \in \mathbb{S}^n,\ k \in \{0,1\}\right\}\right)^{-1}.$$

1.2. Suppose $N \subseteq \Omega$ is open and such that $\Omega \cap \partial N$ is a smooth hypersurface. We let $\nu_N : \Omega \cap \partial N \to \mathbb{S}^n$ to be the *inward pointing* unit-normal and V to be the Radon measure over $\Omega \times \mathbb{S}^n$ uniquely characterised by

$$\int \psi(x,\eta)\,\mathrm{d}V(x,\eta) = \frac{1}{2}\int_{\Omega\cap\partial N} \psi(x,\nu_N(x))\,\mathrm{d}\mathscr{H}^n(x) + \frac{1}{2}\int_{\Omega\cap\partial N} \psi(x,-\nu_N(x))\,\mathrm{d}\mathscr{H}^n(x) \quad \text{whenever } \psi \in \mathscr{K}(\Omega\times\mathbb{S}^n)\,.$$

Clearly, V is associated to the varifold $\mathbf{v}_n(\partial N \cap \Omega)$ by means of [DPDRG18, §5]. We shall tacitly identify co-dimension one varifolds with measures V as above. As in [All86, 3.1] we associate an integrand F with the norm ϕ by requiring that

$$F(x, \mathbf{1}_{\mathbf{R}^{n+1}} - \beta(\nu)^* \circ \beta(\nu)) = \phi(\nu) \quad \text{whenever } \nu \in \mathbb{S}^n \text{ and } x \in \Omega$$

and we write $\delta_\phi V$ for $\delta_F V$. Referring to [DPDRH19, Proposition 1] or [DRKS20, Remark 2.21] we get a formula for the first variation of V with respect to ϕ

$$\delta_\phi V(g) = -\int_{\Omega\cap\partial N} \phi(\nu_N(x))\mathbf{h}_\phi(V,x) \bullet g(x)\,\mathrm{d}\mathscr{H}^n(x) \quad \text{whenever } g \in \mathscr{X}(\Omega)\,,$$

where

$$\mathbf{h}_\phi(V,x) = \mathbf{h}_\phi(\partial N,x) = -\phi(\nu_N(x))^{-1}\,\mathrm{trace}\,\big(\mathrm{D}(\nabla\phi\circ\nu_N)(x)\big)\nu_N(x) \qquad \text{for } x\in\Omega\cap\partial N$$

denotes the *generalised mean ϕ-curvature vector* of V (or of ∂N).

1.3 *Remark*. Let $W \subseteq \mathbf{R}^{n+1}$ be open, $f : W \to \mathbf{R}$ be of class $\mathscr{C}^2$, and assume $\mathrm{D}f(z) \neq 0$ for $z \in W$. Define

$$\xi(z) = |\nabla f(z)|\,, \quad \nu(z) = \nabla f(z)\xi(z)^{-1}\,, \quad \eta(z) = \nabla\phi(\nu(z)) \quad \text{for } z\in W\,, \quad \text{and} \quad \Sigma_t = W\cap\{z : f(z) = t\} \quad \text{for } t\in\mathbf{R}\,.$$

Let $t \in \mathbf{R}$ and $z \in \Sigma_t$. Note that $\nu(z)$ is the unit normal vector of Σ_t at z pointing outside $W \cap \{z : f(z) < t\}$. Recalling 1.2 we get

$$\nu(z)\,\mathrm{trace}\,\mathrm{D}\eta(z) = -\phi(\nu(z))\mathbf{h}_\phi(\Sigma_t,z)\,.$$

1.4. **Theorem** (cf. [All86, 3.4(4)(5)(6)]). *Suppose*

$$\phi \text{ is a uniformly convex norm}\,, \quad 0 < t_0 < \infty\,, \quad W\subseteq\mathbf{R}^{n+1} \text{ is open}\,,$$
$$f : W\to\mathbf{R} \quad \text{is smooth}\,, \quad \nabla f(z)\neq 0 \quad \text{for } z\in W \text{ with } f(z) > t_0\,,$$
$$V\in\mathbf{V}_n(W)\,, \quad \mathrm{spt}\,\|V\|\cap\{z : f(z)\geq t_0\} \quad \text{is compact}\,,$$
$$0 < H < \infty\,, \quad \|\delta_\phi V\|(A) \leq H\|V_\phi\|(A) \quad \text{whenever } A\subseteq W\cap\{z : f(z) > t_0\}\,,$$
$$\eta = \nabla\phi\circ\nabla f\,, \quad \mathrm{trace}\,\mathrm{D}\eta(z)\geq H|\eta(z)| \quad \text{whenever } f(z) > t_0\,.$$

Then

$$\mathrm{spt}\,\|V\|\subseteq W\cap\big\{z : f(z)\leq t_0\big\}\,.$$

Proof. We reproduce the proof of [All86, 3.4(6)]. Define

$$\xi : W\to\mathbf{R}\,, \quad \nu : W\to\mathbf{R}^{n+1}\,,$$
$$\xi(z) = |\nabla f(z)| \quad \text{and} \quad \nu(z) = \nabla f(z)\xi(z)^{-1} \quad \text{for } z\in W\,,$$

Let $0 < \varepsilon < 1$ and $\zeta : \mathbf{R} \to \mathbf{R}$ be a smooth map such that

$$\zeta(t) = t \quad \text{for } \varepsilon < t < \infty\,, \quad \zeta(t) = 0 \quad \text{for } t\leq 0\,, \quad \text{and} \quad 0\leq\zeta'(t)\leq 1+2\varepsilon \quad \text{for } t\in\mathbf{R}\,.$$

Whenever $t_0 < t < \infty$ define

$$f_t(z) = \zeta(f(z)-t) \quad \text{and} \quad g_t(z) = f_t(z)\eta(z) \quad \text{for } z\in W\,.$$

Observe that g_t is a valid test function for $\delta_\phi V$ because $\mathrm{spt}\,\|V\| \cap \{z : f(z) \geq t\}$ is compact. Recall [DRKS20, Definition 2.16] to see that

$$B_\phi(u)\bullet L = \phi(u)\,\mathrm{trace}\,L - L(\nabla\phi(u))\bullet u \quad \text{for } u\in\mathbb{S}^n\,.$$

Since ϕ is positively 1-homogeneous we know that $\nabla\phi$ is 0-homogeneous; thus,

$$\eta = \nabla\phi\circ\nu\,, \quad \mathrm{D}\nabla\phi(u)u = 0\,, \quad \nabla\phi(u)\bullet u = \phi(u)\,,$$
$$\text{and} \quad \mathrm{D}\nabla\phi(u)v\bullet w = \mathrm{D}\nabla\phi(u)w\bullet v \quad \text{for } z\in W \text{ and } u,v,w\in\mathbb{S}^n\,.$$

For $t_0 < t < \infty$, $z \in W$, and $u \in \mathbb{S}^n$ there holds

$$\begin{aligned} B_\phi(u) \bullet \mathrm{D}g_t(z) &= \zeta'(f(z)-t)\xi(z)\phi(u)\eta(z) \bullet \nu(z) + \phi(u) f_t(z) \operatorname{trace} \mathrm{D}\eta(z) \\ &\quad - \zeta'(f(z)-t)\xi(z)\big(\nu(z) \bullet \nabla\phi(u)\big)\big(\nabla\phi(\nu(z)) \bullet u\big) - f_t(z)\mathrm{D}\eta(z)(\nabla\phi(u)) \bullet u \\ &= \zeta'(f(z)-t)\xi(z)\big(\phi(u)\phi(\nu(z)) - (\nu(z) \bullet \nabla\phi(u))(\nabla\phi(\nu(z)) \bullet u)\big) \\ &\qquad + \phi(u) f_t(z) \operatorname{trace} \mathrm{D}\eta(z) - f_t(z)\mathrm{D}\nabla\phi(\nu(z))u \bullet \mathrm{D}\nu(z)(\nabla\phi(u)) \,. \end{aligned}$$

Since $\mathrm{D}\nabla\phi(\nu(z))\nu(z) = 0$ we get for $z \in W$ and $u \in \mathbb{S}^n$

$$\mathrm{D}\nabla\phi(\nu(z))u = \mathrm{D}\nabla\phi(\nu(z))(u - \operatorname{sgn}(u \bullet \nu(z))\nu(z))$$

Define

$$d(u,v) = |u - \operatorname{sgn}(u \bullet v)v| = \sqrt{2}\big(1 - |u \bullet v|\big)^{1/2} \quad \text{for } u, v \in \mathbb{S}^n \,.$$

Assume the theorem is not true and set

$$\begin{gathered} \iota = \inf\big\{\operatorname{trace} \mathrm{D}\eta(z) - H|\eta(z)| : z \in \operatorname{spt}\|V\|,\ f(z) \ge t_0\big\} > 0 \,, \\ \kappa = \inf\big\{\xi(z) : z \in \operatorname{spt}\|V\|,\ f(z) \ge t_0\big\} \,. \end{gathered}$$

As in [All86, 3.2(6)] uniform convexity of ϕ yields

$$\phi(u)\phi(v) - (v \bullet \nabla\phi(u))(\nabla\phi(v) \bullet u) \ge \tfrac{1}{2}\gamma(\phi) d(u,v)^2 \quad \text{for } u, v \in \mathbb{S}^n \,;$$

thus,

$$\begin{aligned} B_\phi(u) \bullet \mathrm{D}g_t(z) &\ge \tfrac{1}{2}\zeta'(f(z)-t)\xi(z)\gamma(\phi)d(u,\nu(z))^2 \\ &\qquad + \phi(u)f_t(z)(H+\iota)|\eta(z)| - f_t(z)d(u,\nu(z))c(\phi)^2\|\mathrm{D}\nu(z)\| \\ &\qquad\qquad \text{for } t_0 < t < \infty,\ u \in \mathbb{S}^n, \text{ and } z \in W \,. \end{aligned}$$

Let

$$M = \sup\{\|\mathrm{D}^2\nu(z)\| : z \in \operatorname{spt}\|V\|\} \quad \text{and} \quad t_0 < t_1 = \sup f[\operatorname{spt}\|V\|] < \infty \,.$$

For any $t_0 < t < t_1$ there holds

$$\begin{aligned} H\int f_t(z)|\eta(z)|\,\mathrm{d}\|V_\phi\|(z) \ge |\delta_\phi V(g_t)| &\ge (H+\iota)\int f_t(z)|\eta(z)|\,\mathrm{d}\|V_\phi\|(z) \\ &\quad + \int \tfrac{1}{2}\zeta'(f(z)-t)\xi(z)\gamma(\phi)d(u,\nu(z))^2\,\mathrm{d}V(z,u) \\ &\quad - Mc(\phi)^2\int f_t(z)d(u,\nu(z))\,\mathrm{d}V(z,u) \,. \end{aligned}$$

Recall that that $\zeta'(t) = 1$ for $t > \varepsilon$, $|\eta(z)| \ge c(\phi)^{-1}$ and $\xi(z) \ge \kappa$ for $z \in \operatorname{spt}\|V\|$ with $f(z) \ge t_0$. Letting $\varepsilon \downarrow 0$ we get

$$\begin{aligned} 0 \ge \int_{\{z : f(z) \ge t\}} &(f(z)-t)c(\phi)^{-1}\iota + \tfrac{1}{2}\kappa\gamma(\phi)d(u,\nu(z))^2 \\ &\quad - (f(z)-t)c(\phi)^{-1}Mc(\phi)^3 d(u,\nu(z))\,\mathrm{d}V_\phi(z,u) \,. \end{aligned}$$

Now, we employ a technique borrowed from [DPDRH19, Theorem 3.4]. Define the function

$$p(\alpha, s) = \alpha\iota + \tfrac{1}{2}\kappa\gamma(\phi)s^2 - \alpha Mc(\phi)^4 s \quad \text{for } \alpha, s \in \mathbf{R} \,.$$

For $0 < \alpha < \infty$ the quadratic polynomial $p(\alpha, \cdot)$ attains its minimum at the point $s_\alpha = \alpha M c(\phi)^4 (\kappa \gamma(\phi))^{-1}$ with value

$$p(\alpha, s_\alpha) = \alpha\iota - \frac{\alpha^2 M^2 c(\phi)^8}{2\kappa\gamma(\phi)} .$$

Consequently, if $0 < \alpha < \alpha_0 = 2\iota\kappa\gamma(\phi) M^{-2} c(\phi)^{-8}$, then $p(\alpha, s) > 0$ for all $s \in \mathbf{R}$. Therefore, if $t_1 > t > t_1 - \alpha_0$ we get a contradiction. □

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