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The stable homotopy of the classical groups.

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This is a detailed account of the proof of the important results of the author on the stable homotopy groups of the classical Lie groups as announced in an earlier note [Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 933–935; MR 21#1588]. The main steps in the treatment of the various groups have been unified in a single general proposition (Theorem I) which contains the periodicity theorems as special cases. This theorem and its proof are a striking illustration of the fact that geometrical methods in homotopy theory still have some interest.

Theorem I is as follows. Let M be a closed compact connected Riemannian manifold of class C'^{∞}. Let $\nu = (P, Q, h)$ consist of a pair of points P, Q on M and a homotopy class h of paths from P to Q. Let $\Omega_\nu M$ be the set of piecewise differentiable curves from P to Q on M, in the homotopy class h, parametrized from 0 to 1 proportionally to arc length. $\Omega_\nu M$ is given the topology induced by the metric

$$\rho(c, c') = \max \rho_0(c(t), c'(t)) + |J(c) - J(c')|,$$

where ρ_0 is the distance on M and $J(c)$ is the length of c. If $s \in \Omega_\nu M$ is a geodesic segment and $t \in I$, denote by $\lambda(s(t))$ the dimension of the vector space of Jacobi fields along s vanishing at $P = s(0)$ and $s(t)$. (The definition of a Jacobi field is recalled on page 319.) Define the index of a geodesic segment $s \in \Omega_\nu M$ to be the finite number $\sum_{0<t<1} \lambda(s(t))$. Let $|\nu|$ denote the minimum of the positive integers which occur as the index of a geodesic segment in $\Omega_\nu M$, and let M^ν be the subspace of $\Omega_\nu M$ consisting of the geodesic segments of minimal length. Theorem I: If M is a symmetric space (in the sense of E. Cartan), then M^ν is also a symmetric space (in particular a Riemannian manifold), and the inclusion $i: M^\nu \to \Omega_\nu M$ induces isomorphisms $i_*: \pi_k(M^\nu) \to \pi_k(\Omega_\nu M) \approx \pi_{k+1}(M)$ for $0 < k < |\nu| - 1$. In dimension $k = |\nu| - 1$, i_* is an epimorphism.

A procedure to compute $|\nu|$ in practice is the subject of § 6.

The proof of Theorem I is essentially in two steps. Step I: In § 3, a homotopy formulation of Morse theory is given following an idea inaugurated by R. Thom [C. R. Acad. Sci. Paris 228 (1949), 973–975; MR 10, 558]. Thom's argument is made more specific using J. H. C. Whitehead's notion of a CW-complex and is extended so as to include a more general type of critical sets, i.e., non-degenerate critical manifolds, a notion due to the author [Ann. of Math. (2) 60 (1954), 248–261; MR 16, 276]. Step II: Previous work of the author,

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and joint work with H. Samelson is then used to show that the Morse theory in its new formulation applies to the symmetric spaces and yields the result.

The paper ends with explicit computations of $|\nu|$ for the symmetric spaces arising from the classical Lie groups (§ 7 and § 8).

Although results from Morse theory and previous work of the author are used, the proofs can be followed if these results are taken for granted. Moreover, precise references are given whenever the proofs are not self-contained and the reconstitution of the full details is a routine matter. *M. A. Kervaire*