Prime Normal Form and Equivalence of Simple Grammars

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Abstract. We give the first algorithm to decompose a *simple* context-free language into *primes*. The algorithm computes a canonical representation of a simple language, converting its simple context-free grammar into Prime Normal Form (PNF); a simple grammar is in PNF if its nonterminals define prime prefix codes, i.e., prefix-free languages which cannot be represented as concatenation of two non-trivial prefix-free languages.
We also improve on the complexity for testing the equivalence of simple grammars. The best previously known algorithm for the equivalence problem worked in $O(n^{13})$ time. We improve it to $O(n^7 \log^2 n)$ and $O(n^5 \operatorname{polylog} v)$ deterministic time, and $O(n^4 \operatorname{polylog} n)$ randomized time, where n is the total size of the grammars involved, and v is the length of a shortest string derivable from a nonterminal, maximized over all nonterminals.

1 Introduction

The fundamental question in language theory is, given a class of languages find a canonical representation of any language of the class. Such representation often permits to solve problems related to decidability or equivalence of laguages from the given class. Most often the canonical representation of the language is given by a special form of its grammar, called a normal form. In this paper we show that every simple language admits a unique representation by its grammar given in so-called Prime Normal Form (PNF). We give an algorithm to convert a simple grammar into PNF. The crucial question that our algorithm is confronted with, is whether a simple language is prime, i.e., not decomposable into a concatenation of two non-trivial prefix-free languages.

In general, the canonical representation of any type of language may be substantially larger than its original grammar. This is also the case for simple languages. Hence the equivalence of simple languages by means of canonical representations may be inefficient. Equivalence problem for simple context-free grammars is a classical question in formal language theory. It is a nontrivial problem, since the inclusion problem for simple languages is undecidable. A. Korenjak and J. Hopcroft, see [**?**,**?**], proved that the problem is decidable and they gave the first, doubly exponential time algorithm solving it. Their result was improved by D. Caucal to $O(n^3 v(G))$ time, see [**?**]. The parameter n is the size of the simple grammar and $v(G)$ is the length of a shortest string derived from a nonterminal, maximized over all nonterminals. Caucal's algorithm is of exponential complexity, since $v(G)$ can be exponential with respect to n. Y. Hirshfeld, M. Jerrum, F. Moller gave the first polynomial $O(n^{13})$ time algorithm for this problem in [**?**], we call it the HJM algorithm.

In the second part of this paper we revisit Caucal's algorithm and show that, by carefully implementing this algorithm, we can reduce the time complexity $O(n^{13})$ of [**?**] to $O(n^7 \log^2 n)$ and to $O(n^5 \operatorname{polylog}(v(G)))$. Similarly as the HJM algorithm, we apply the techniques used in the algorithmic theory of compressed strings, based on Lempel-Ziv string encoding. The idea of such encoding is that instead of representing a string explicitly we design a context-free grammar generating the string as a one-word language. As the combinatorial complexity of such grammar can be substantially smaller than the length of the word, it may be considered as a succinct representation of the word. Such encodings were recently considered by researchers, mainly in the context of efficient pattern matching. There is one problem of this domain which is of particular interest for us — the *compressed first mismatch problem* (*First-MP*). Given two strings encoded by a grammar, *First-MP* looks for the position of the first symbol at which the strings differ. Polynomial time algorithms for computing *First-MP* were given independently in [**?**] and [**?**], in completely disjoint settings (the problem has been called differently, the term "first mismatch problem" was introduced in [**?**]). More powerful algorithms were given in [**?**], where more complicated problem of fully compressed string-matching was solved.

Simple languages are applied by IDT Canada to perform packet classification at wire speed. The class of packets is described with the aid of a simple prefix code, and their recognition is made by so-called Concatenation State Machine, an efficient version of a stateless pushdown automaton, cf. [**?**]. The algorithms described in this paper are being used to represent classification policies, to manage them, and to automatically convert them to a program for *PAX.port*, a specialized processing unit for real-time network packet filtering and classification.

2 Simple languages

A context free grammar $G = (\Sigma, N, P)$ is composed of a finite set Σ of *terminals*, a finite set N of *nonterminals* disjoint from Σ, and a finite set $P \subset N \times (N \cup \Sigma)^*$ of *production rules*. For every $\beta, \gamma \in (N \cup \Sigma)^*$, if $(A, \alpha) \in P$, then $\beta A \gamma \to \beta \alpha \gamma$. A *derivation* $\beta \xrightarrow{*} \gamma$ is a finite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ such that $\beta = \alpha_0$, $\gamma = \alpha_n$, and $\alpha_{i-1} \to \alpha_i$ for $i \in [1, n]$.

For every sequence of nonterminals $\alpha \in N^*$ of a grammar $G = (\Sigma, N, P)$, we denote by $L_G(\alpha)$ the set of terminal strings derivable from α, i.e., $L_G(\alpha) \stackrel{\texttt{def}}{=} \{w \in \Sigma^* \mid \alpha \xrightarrow{*} w\}$. Often, if G is known from the context, we will write $L(\alpha)$ instead of $L_G(\alpha)$.

A grammar $G = (\Sigma, N, P)$ is in *Greibach normal form* if for every production rule $(A \to \alpha) \in P$, we have $\alpha \in \Sigma N^*$. A grammar $G = (\Sigma, N, P)$ is a *simple context-free grammar* (simple grammar) if G is a Greibach normal form grammar and such that whenever $A \to a\,\alpha_1$ and $A \to a\,\alpha_2$, for a same $a \in \Sigma$, then $\alpha_1 = \alpha_2$.

A language $L \subseteq \Sigma^*$ is a *simple language* (also called *s-language*) if there exists a simple grammar $G = (\Sigma, N, P)$ such that $L_G(A) = L$, for some $A \in N$. The definition implies that every nonterminal of a simple grammar defines a simple language. Since simple laguages are prefix codes and are closed by concatenation, the family of simple languages under concatenation constitutes a free monoid with $\{\varepsilon\}$ as unit (where ε denotes the empty word). Thus, every nonempty simple language L admits a unique decomposition into *prime* (i.e., undecomposable) simple languages, $L = P_1 P_2 \ldots P_n$.

3 Prime Normal Form for simple grammars

In this section we give an algorithm converting any simple grammar to its canonical representation called *Prime Normal Form*. A simple grammar is in Prime Normal Form (PNF) if each of its nonterminals represents a prime. We will use the following algebraic notation for left and right division in the free monoid of prefix codes. If $L = L_1 L_2$ for some prefix codes L, L_1, L_2, then by $L_1^{-1} L$ we denote L_2 and by $L L_2^{-1}$ we denote L_1. We call L_1 a left divider and L_2 a right divider of L.

Let L be a prefix code and $L = P_1 P_2 \ldots P_n$ be its decomposition into primes. Prime P_n will be called *final prime* of L, and it will be denoted by $f(L)$. In particular, if L is a prime, then $f(L) = L$.

Lemma 1. *Let $G = (\Sigma, N, P)$ be a simple grammar. For every $X \in N$, there exists $Y \in N$, such that $f(L(X)) = L(Y)$.*

Proof. Let $w \in L(X)f(L(X))^{-1}$, and $X \xrightarrow{*} w\alpha$ be the leftmost derivation in G, with $\alpha \in N^+$. Since $L(\alpha) = f(L(X))$ and $L(\alpha)$ is a prime, α consists of a single nonterminal, i.e., $\alpha \in N$. □

Let $w_0\,\alpha_0 \to \ldots \to w_i\,\alpha_i \to \ldots \to w_n\alpha_n$ be the leftmost derivation $X \xrightarrow{*} w$, with $w_0 = \varepsilon$, $\alpha_0 = X$, $w_n = w$, $\alpha_n = \varepsilon$, $w_i \in \Sigma^*$, and $\alpha_i \in N^*$, for $i \in [0, n]$. We are interested in the subsequence $\pi(X, w) = Y_0, Y_1, \ldots, Y_j$ of $\alpha_0, \alpha_1, \ldots, \alpha_n$, which consists of those elements of $\alpha_0, \ldots, \alpha_n$ that are single nonterminals. E.g., for the leftmost derivation of $abcdef \in L(X)$:

$$\underline{X} \to a\,YY \to ab\,\underline{Y} \to abc\,\underline{Y} \to abcd\,YZ \to abcde\,\underline{Z} \to abcdef$$

we have $\pi(X, abcdef) = X, Y, Y, Z$.

Definition 1. *Let $G = (\Sigma, N, P)$ be a simple grammar. We define relation $\mathcal{D}$ over $N \cup \{\varepsilon\}$ as follows. $(X, Y) \in \mathcal{D}$ if and only if:*

- *there exists a rule $(X \to a\alpha Y)$ in P for some $a \in \Sigma$ and $\alpha \in N^*$, or*
- *$Y = \varepsilon$ and there exists a rule $(X \to a)$ in P for some $a \in \Sigma$.*

Relation $\mathcal{D}$ can be seen as a digraph $(N \cup \{\varepsilon\}, \mathcal{D}, \varepsilon)$ with sink ε. In a digraph with a sink, vertex v is called a *d-articulation point* of vertex u if and only if v is present on every path from u to the sink. It was shown in [**?**] that the order of first (or last) occurrences of the d-articulation points of a vertex v is the same in all paths from v to the sink. Thus, it is natural to represent the set of all d-articulation points for a given vertex v as an ordered list of vertices, $(u_0, u_1, \ldots, u_n)$, where $u_0 = v$ and u_n is the sink.

In [**?**], it was shown that a prefix code L is prime if and only the initial state v_1 of the minimal deterministic automaton for L does not have any d-articulation point except sink and v_1 itself. Moreover, the list of d-articulation points $(v_1, v_2, \ldots, v_n)$ corresponds to the prime decomposition of L, the factors being the languages defined by automata having v_i as the initial state and v_{i+1} as the final state (with all outgoing transitions of the final state removed), for $i \in [1, n)$, respectively.

Lemma 2. *For every path π from X to ε in $\mathcal{D}$ there exists a word $w \in L(X)$, such that $\pi = \pi(X, w)$. Conversely, for every $w \in L(X)$, $\pi(X, w)$ defines a path from X to ε in $\mathcal{D}$.*

We say that a grammar $G = (\Sigma, N, P)$ is *reduced* if there is no two different nonterminals defining the same language, i.e., for all $X, Y \in N$, if $L(X) = L(Y)$ then $X = Y$. By Lemma 1, the set of nonterminals $F(X) \stackrel{\text{def}}{=} \{Y \in N \mid L(Y) = f(L(X))\}$ is nonempty. If the underlying grammar is reduced then $F(X)$ consists of a single nonterminal which, by convenient abuse of notation, will be denoted by $f(X)$.

Theorem 1. *Let $G = (\Sigma, N, P)$ be a reduced simple grammar. For every $X \in N$, $L(X)$ is prime if and only if X does not have d-articulation points in $\mathcal{D}$ except sink and X itself. Moreover, if $Y \in N$ is a d-articulation point of X then $L(Y)$ is a right divider of $L(X)$.*

Proof. By Lemma 1, since G is reduced, every derivation of w starting in X is of a form $X \xrightarrow{*} w'\,f(X) \xrightarrow{*} w$. Thus, for every $w \in L(X)$, $\pi(X, w)$ contains $f(X)$, i.e., $f(X)$ is a d-articulation point of X in $\mathcal{D}$.

Let Y be a d-articulation point of X in $\mathcal{D}$. By Lemma 2, every derivation starting in X passes by Y, thus Y is a d-articulation point for X in the (infinite) deterministic automaton for X, which implies that $L(Y)$ is a right divider of $L(X)$, cf. [**?**]. □

Theorem 2. *If a simple grammar $G = (\Sigma, N, P)$ is in a reduced form then we can find $f(X)$ for all $X \in N$ in linear time.*

Proof. By Theorem 1, the non-terminal $f(X)$ is exactly the second last d-articulation point for X in $\mathcal{D}$. Calculating $f(X)$ for all $X \in N$ can be done in linear time, by using an algorithm for finding dominators in flow graphs, cf. [**?**]. □

The algorithm for transforming a simple grammar $G = (\Sigma, N, P)$ into PNF, called $PNF(G, S)$, is presented in Figure 1.

Input: Simple grammar $G = (\Sigma, N, P)$ and $S \in N^+$.
Output: Simple grammar G' in PNF and $S' \in N^+$, such that $L_G(S) = L_{G'}(S')$.

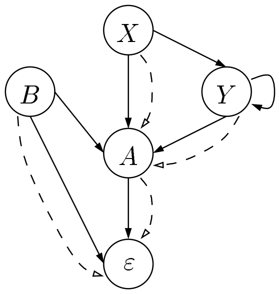
1. Reduce G.
   Find redundant nonterminals by checking if $L(X) = L(Y)$, for all $X, Y \in N$. Each redundant nonterminal is substituted in P and in S, and removed from N.
2. For every $X \in N$, find $f(X) \in N$.
   Construct the digraph $\mathcal{D}$ and find the second-last d-articulation point for X.
   If for every $X \in N$, $X = f(X)$, then **return** (G, S).
3. Construct a new grammar $G' = (\Sigma, N, P')$ and new S'.
   Define morphism $h : N \mapsto N^*$ as: $h(X) \overset{\text{def}}{=} \begin{cases} X & \text{if } X = f(X) \\ Xf(X) & \text{otherwise.} \end{cases}$
   Set S' to $h(S)$, and P' as follows, for $a \in \Sigma$, $X, Y \in N$, $\alpha \in N^*$:
   (a) If $(X \to a\alpha) \in P$ and $X = f(X)$, then $(X \to ah(\alpha))$ is in P'.
   (b) If $(X \to a\alpha f(X)) \in P$ and $X \neq f(X)$, then $(X \to ah(\alpha))$ is in P'.
   (c) If $(X \to a\alpha Y) \in P$, $X \neq f(X)$, and $Y \neq f(X)$, then $(X \to ah(\alpha)Y)$ is in P'.
4. Set G to G', S to S' and go to 1.

Fig. 1. Algorithm *PNF*(G, S)

We present an example of the execution of the algorithm. The input consists of a simple grammar $G = \{(X \to aAA), (X \to bYY), (Y \to aY), (Y \to bBA), (A \to a), (B \to aXA), (B \to b)\}$, and a simple language represented as a word $S = XA$ over nonterminals of G. We obtain the grammar in PNF while keeping track of the prime decomposition of S. For each iteration, we give the value of S, the grammar G, the digraph $\mathcal{D}$ (solid lines), the d-articulation tree (dotted lines), and the values $f(x)$ for $x \in \{X, Y, A, B\}$ and $h(x)$ for $x \in \{X, Y, A, B, S\}$.

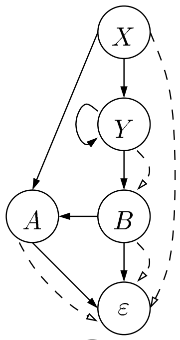
Iteration 1:

$$\begin{aligned} S &= XA \\ X &\to aAA + bYY \\ Y &\to aY + bBA \\ A &\to a \\ B &\to aXA + b \end{aligned}$$



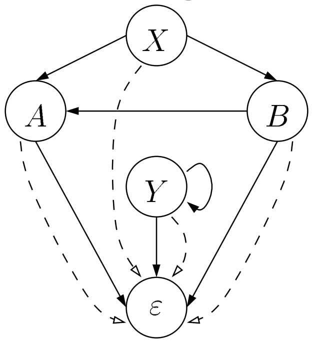
x	$f(x)$	$h(x)$
X	A	XA
Y	A	YA
A	A	A
B	B	B
S	–	XAA

$$\textbf{Iteration 2:}\quad \begin{aligned} S &= XAA \\ X &\to aA + bYAY \\ Y &\to aY + bB \\ A &\to a \\ B &\to aXAA + b \end{aligned}$$



x	$f(x)$	$h(x)$
X	X	X
Y	B	YB
A	A	A
B	B	B
S	–	XAA

$$\textbf{Iteration 3:}\quad \begin{aligned} S &= XAA \\ X &\to aA + bYBAYB \\ Y &\to aY + b \\ A &\to a \\ B &\to aXAA + b \end{aligned}$$



x	$f(x)$	$h(x)$
X	X	X
Y	Y	Y
A	A	A
B	B	B
S	–	XA

Theorem 3. *The algorithm PNF(G, S) correctly computes a PNF simple grammar G' and S' such that $L_G(S) = L_{G'}(S')$.*

Proof. Step 1 does not change the semantics of any nonterminal, assuming that $L(X) = L(Y)$ is correctly checked. Polynomial algorithms for deciding on $L(X) = L(Y)$ are investigated later in this paper.

Step 2 effectively finds final primes for all nonterminals, and it can be accomplished in linear time, which follows from Theorem 2. If all nonterminals are primes then the grammar is in PNF and the algorithm terminates.

Step 3 transforms the grammar G into G' by right-factorizing every non-prime nonterminal X by $f(X)$: If X is prime then $L_G(X) = L_{G'}(X)$, otherwise $L_G(X) = L_{G'}(Xf(X))$. Every production $(X \to \alpha) \in P$ is rewritten accordingly into a corresponding production $(X \to \beta) \in P'$. The step can be easily done in linear time.

Every iteration of the program cut the length of non-prime nonterminals, in terms of their prime decomposition, by one. Thus, the total number of iterations equals the maximum length of the prime decompositions of nonterminals of the initial grammar. □

Both steps, 2 and 3, of the algorithm may be computed in linear time, hence the complexity of each iteration of the main loop is dominated by grammar reduction from step 1.

The polynomial time algorithm from Section 6, repeated $\Omega(n^2)$ times may be used to perform the grammar reduction. However, for grammar $G = \{(A_1 \to aA_2A_2), (A_2 \to aA_3A_3), \ldots, (A_{n-1} \to aA_nA_n), (A_n \to a)\}$, language $L_G(A_1)$ has exponential number of primes with respect to the size of G. Hence the number of iterations of the main loop of *PNF*(G, S) may be exponential and so may be the size of the resulting PNF grammar. Since simple languages constitute a free monoid, the PNF form is unique.

Corollary 1. *Every simple language L can be represented by a PNF simple grammar $G = (\Sigma, N, P)$ and a starting word $S \in N^*$, such that $L_G(S) = L$. Such a representation is unique. The problem of constructing the PNF representation of L given by a simple grammar is decidable. The PNF representation may be of exponential size with respect to the size of the original grammar.*

4 Labeled transition system

A simple grammar $G = (\Sigma, N, P)$ can be seen as a compact description of a deterministic (infinite) automaton, where states are words over N and transitions are defined by production rules: $\alpha \xrightarrow{a} \beta$ iff $\alpha = A\alpha'$, $(A \to a\gamma) \in P$, and $\beta = \gamma\alpha'$.

A relation $\mathcal{R} \subseteq N^* \times N^*$ is a *bisimulation* iff, whenever $(\alpha, \beta) \in \mathcal{R}$, we have:

- if $\alpha \xrightarrow{a} \alpha'$ then $\beta \xrightarrow{a} \beta'$ for some β' such that $(\alpha', \beta') \in \mathcal{R}$; and
- if $\beta \xrightarrow{a} \beta'$ then $\alpha \xrightarrow{a} \alpha'$ for some α' such that $(\alpha', \beta') \in \mathcal{R}$.

We write $\alpha \sim \beta$ iff $(\alpha, \beta) \in \mathcal{R}$ for some bisimulation $\mathcal{R}$.

By $\alpha \xrightarrow{a}$ we denote that there exists an outgoing transition from α labeled by a, and by $\alpha \not\xrightarrow{a}$ that there does not. We will write $(\alpha, \beta) \xrightarrow{a} (\alpha', \beta')$ to say that $\alpha \xrightarrow{a} \alpha'$ and $\beta \xrightarrow{a} \beta'$.

Lemma 3. *Let $G = (\Sigma, N, P)$ be a simple grammar and $\alpha, \beta \in N^*$. $L_G(\alpha) = L_G(\beta)$ iff $\alpha \sim \beta$.*

Proof. Both implications are easily provable by contradictions. □

5 First mismatch-pair problem

Our approach to transform Caucal's algorithm for the equivalence problem of simple grammar, cf. [**?**], into a polynomial time one (with respect to the size of the input grammar) is to use compressed representations of sequences of nonterminals, instead of using explicit representations.

We will use the terminology of *acyclic morphisms* because it is more convenient in presenting our algorithms. It is basically equivalent to the representation of a single word by a context-free grammar generating exactly one word, or to a "straight line program".

A *morphism* over a monoid M is an application $H : M \mapsto M$ such that $H(1_M) = 1_M$ and $H(x \cdot y) = H(x) \cdot H(y)$, for all $x, y \in M$. A morphism $H : M \mapsto M$ is fully defined by providing the values for the generators of M. Thus, a morphism H over a finitely generated free monoid N^* is usually defined by providing $H : N \mapsto N^*$. A morphism $H : N \to N^+$ is said to be *acyclic* if we can order elements of N in such a way that for each $A \in N$, we have: $H(A) = A$ or $A > B$ for each symbol B occuring in the string $H(A)$. For an acyclic morphism H over N^* we denote $H^{|N|}$ by H^*, since $H^{|N|+1} = H^{|N|}$. If $H^*(\alpha) = w$ then we say that (H, α) is a compressed representation of w. The size of w can be exponential with respect to the size of its compressed representation.

Let $G = (\Sigma, N, P)$ be a simple grammar. We say that an acyclic morphism $H : N \mapsto N^+$ is *self-proving* in G iff for each $A \in N$ we have:

- If $A \xrightarrow{a} \alpha$ then $H(A) \xrightarrow{a} \beta$ and $H^*(\alpha) = H^*(\beta)$; and
- If $H(A) \xrightarrow{a} \beta$ then $A \xrightarrow{a} \alpha$ and $H^*(\alpha) = H^*(\beta)$.

The following lemma has been essentially shown in [**?**]. We use a different terminology so we provide a proof.

Lemma 4. [Correctness Lemma]
If H is an acyclic morphism self-proving in $G = (\Sigma, N, P)$, then $x \sim H(x)$, for every $x \in N^*$.

Proof. Firstly, we prove that if H is self-proving and $(x, H^i(x)) \xrightarrow{a} (\alpha, \beta)$, for $x, \alpha, \beta \in N^+$ and $a \in \Sigma$, then $H^*(\alpha) = H^*(\beta)$, for any $i \geq 0$.

For $i = 0$, $H^i(x) = x$, and thus the implication is obviously true. Let $A \in N$, $x, y, z \in N^*$, and $(Ax, H(Ax)) \xrightarrow{a} (yx, zH(x))$. Directly form the definition of *self-proving* morphism, we have

$H^*(yx) = H^*(y)H^*(x) = H^*(z)H^*(x) = H^*(z)H^*(H(x)) = H^*(zH(x))$. Let $(x, H^{k+n}(x)) \xrightarrow{a} (\alpha, \beta)$. Consider, $(x, H^k(x)) \xrightarrow{a} (\alpha, \gamma)$ and $(H^k(x), H^n(H^k(x)) \xrightarrow{a} (\gamma, \beta)$. By induction, such derivations exist and $H^*(\alpha) = H^*(\gamma) = H^*(\beta)$.

Secondly, we prove that for $x, y, x', y' \in N^*$, if $H^*(x) = H^*(y)$ and $(x, y) \xrightarrow{a} (x', y')$, then $H^*(x') = H^*(y')$. Since $H^*(x) = H^*(y)$, an existance of $(x, H^*(x)) \xrightarrow{a} (x', z)$ implies an existance of $(y, H^*(y)) \xrightarrow{a} (y', z)$. By the intermediate result proved above we have $H^*(x') = H^*(y')$.

Finally, we notice that relation $B \stackrel{\text{def}}{=} \{(x, y) \in N^* \times N^* \mid H^*(x) = H^*(y)\}$, contains all pairs $(x, H^*(x))$ and it is a bisimulation. □

The crucial tool in the polynomial-time algorithms is the compressed *first mismatch-pair* problem, *First-MP*:

Input: an acyclic morphism $H : N \mapsto N^+$ and two strings $x, y \in N^*$;
Output:
- *First-MP*$(x, y, H) =$ *nil*, if $H^*(x) = H^*(y)$;
- *First-MP*$(x, y, H) =$ *failure*, if one of $H^*(x)$, $H^*(y)$ is a proper prefix of the other;
- *First-MP*$(x, y, H) = (A, B) \in N \times N$, where (A, B) is the *first mismatch pair*, i.e., the first symbols occuring at the same position in $H^*(x)$ and in $H^*(y)$, repectively, which are different.

We say that a morphism H is binary if $|H(A)| \leq 2$ for each $A \in N$. The following fact can be shown using the algorithm from [**?**].

Lemma 5. *Assume that given acyclic morphism $H : N \mapsto N^+$ is binary, then we can solve the First-MP problem in time $O(k^2 \cdot h^2)$, where $k = |N|$ and $h \stackrel{\text{def}}{=} \min\{k \geq 0 \mid H^k = H^{k+1}\}$ is the depth of the morphism.*

6 The equivalence algorithm

Conceptualy it is much easier to deal with grammars in binary Greibach Normal Form (denoted GNF2). This means that each side of the production is of the form $(A \to a\alpha)$, where $a \in \Sigma$ and $\alpha \in \{\epsilon\} \cup N \cup N^2$.

Lemma 6. *For each simple grammar G of total size n (the total number of symbols describing G) there is an equivalent simple grammar G' in GNF2 with only $O(n)$ nonterminal symbols. G' can be constructed from G in $O(n)$ time.*

The total size of a grammar in GNF2 is of a same order as the size of N. Hence by the size of a grammar $G = (\Sigma, N, P)$, we mean $n = |N|$.

All known algorithms for the equivalence problem for simple grammars are based on the possibility of computing the quotient of one prefix language by another, assuming that the quotient exists and the languages are given by a simple grammar.

More precisely, let A and B be two nonterminals of a simple grammar $G = (\Sigma, N, P)$, such that $L(A) = L(B) \cdot L$, for some language $L \subseteq \Sigma^*$. The language L can be derived from A by a leftmost derivation following any word w from $L(B)$, i.e., $A \xrightarrow{*} w\gamma$, for $\gamma \in N^*$, and $L(\gamma) = L$.

Let $||A||$ denote the length of the shortest word derivable from A.

Lemma 7. *Let G be a simple grammar of size n. One can compute the lengths of shortest words derivable from all nonterminals of G in time $O(n \log n)$.*

Proof. Finding $||A||$ for all $A \in N$ corresponds to the single-source shortest paths problem in an *and/or* graph, which, using Dijkstra algorithm, can be solved in time $O(n \log n)$. □

Lemma 8. *Let A and B be two nonterminals of a simple grammar $G = (\Sigma, N, P)$ such that $L(A) = L(B) \cdot L$, for some $L \subseteq \Sigma^*$. We can compute $\gamma \in N^*$ such that $L(\gamma) = L$ in time $O(n)$. It is guaranteed that $|\gamma| \leq n$.*

Proof. Consider the parse tree for the derivation of the smallest word w from A. The idea is to go down the tree and cut the subtree which generates the prefix of length $||B||$. The depth of the tree is at most n, therefore $|\gamma| \leq n$ and computing the value takes $O(n)$ time. □

The result of algorithm for finding the quotient of A by B as described in the proof of Lemma 8 will be denoted by $quot(A, B)$. The algorithm will give a result for any pair of nonterminals A and B, as long as $||A|| \geq ||B||$. Notice that $L(A) = L(B\,quot(A, B))$ only if $L(B)$ is a left divider of $L(A)$.

Lemma 4 is the starting point for the design of the algorithm *EQUIVALENCE*. Assume that we have a fixed linear order $A_1 < A_2 \ldots < A_n$ of nonterminals, such that whenever $i < j$, we have $||A_i|| \leq ||A_j||$. The whole idea of the algorithm is to construct a self-proving morphism H or, in the process of its construction, to discover a failure which contradicts $L(A) = L(B)$. The main point of the algorithm is to keep pairs of long strings in compressed form. We keep only strings of linear length, their real value is determined by the morphism H. The algorithm is using only a single application of the relation $\stackrel{a}{\to}$. Each time a new rule is generated by setting $H(A) = B\gamma$, where $\gamma = quot(A, B)$, we create pairs generated from $(A, B\gamma)$ by application of $\stackrel{a}{\to}$, for every letter a of terminal alphabet. We keep the generated pairs in set Q. To each pair we apply operation *First-MP*, which "eliminates" next nonterminal, or finds that we have a pair of identical strings, such pairs are removed from Q. All the time the algorithm is checking locally for the proof of the nonequivalence of A and B. If the nonequivalence is not discovered and there is nothing to process, i.e., Q is empty, the algorithm returns the value TRUE, meaning $L(A) = L(B)$.

For technical reasons (to simplify the description of the algorithm) assume that *First-MP*(x, y, H) gives ordered pairs, if *First-MP*$(x, y, H) = (A, B)$ then $A > B$. See Figure 2.

Lemma 9. *The algorithm is correct. The algorithm makes $O(n)$ iterations.*

Proof. In each iteration, either a pair of strings is removed from Q, or a nonterminal is "eliminated" and no more than $|\Sigma|$ pairs of strings are inserted into Q. The crucial property is that whenever $H(A) = B\gamma$, then the nonterminals in $B\gamma$ are of smaller rank than A, ensuring that H is acyclic. Note also that *First-MP* returns a pair (A, B) such that $H(A) = A$, therefore a nonterminal can only be "eliminated" once. After at most $n - 1$ eliminations, *First-MP* will either find that $H^*(\beta_1) = H^*(\beta_2)$ and remove the pair from Q or return $failure$. The maximum number of iterations is thus $O(n)$.

Correctness follows from Lemma 4. □

Corollary 2. *The algorithm EQUIVALENCE(X, Y, G) works in time $O(n\,F(n))$, where n is the size of G, and $F(n)$ is the complexity of the First Mismatch-Pair Problem.*

Lemma 10. *Every instance of First Mismatch-Pair Problem, First-MP(α, β, H), in algorithm EQUIVALENCE(X, Y, G) can be solved in time:*

1. $O(n^6 \log^2 n)$ *and*
2. $O(n^4 \operatorname{polylog} v(G))$.

Input: Simple grammar $G = (\Sigma, N, P)$ and nonterminals $X, Y \in N$;
Output: TRUE if $L_G(X) = L_G(Y)$, FALSE otherwise.

```
Initialization:
    Q := {(X,Y)};
    for each A ∈ N do H[A] := A;
while Q is not empty do
    (β1, β2) := an element of Q;
    switch (First-MP(β1, β2, H)) do
        case nil : remove (β1, β2) from Q;
        case failure : return FALSE;
        case (A, B) :
            γ := quot(A, B);
            H[A] := Bγ;        /* The nonterminal A is "eliminated" */
            for each a ∈ Σ do
                if (A, Bγ) -a->(β1, β2) then insert (β1, β2) into Q;
                if (A -a-> and B -a/->) or (A -a/-> and B -a->) then return FALSE;
return TRUE;
```

Fig. 2. Algorithm *EQUIVALENCE*(X, Y, G).

where n is the size of G, and $v(G)$ is the length of a shortest string derivable from a nonterminal, maximized over all nonterminals.

Proof. In the proof we use twice Lemma 5.

1. Assume H is an acyclic morphism over N, where $n = |N|$ such that $|H(A)| \leq n$ for each A. Then we can construct a morphism H_b such that $H_b^* = H^*$, over the set of $k \leq n^2$ nonterminals and with depth $h = O(n \log n)$.
   The transformation of the morphism can be done similarly to a balanced transformation into a Chomsky normal form. If $H(A) = B_1 B_2 \ldots B_n$ then we introduce $n - 2$ new auxiliary nonterminals to change it into a balanced binary tree generating $B_1 B_2 \ldots B_n$ from A. We need $O(n)$ new nonterminals per each original one, altogether the number of nonterminals increases to $O(n^2)$, i.e., k is in $O(n)$. However the depth is changed only logarithmically. Observe that on each top down path in generation we have at most n original nonterminals, all of them should be different, and at most $O(n \log n)$ auxiliary nonterminals, i.e., h is in $O(n \log n)$. Now, point 1 follows from Lemma 5.
2. We can use the technique from [**?**] which transforms each grammar generating a single word u into a grammar of depth $O(\log |u|)$ by introducing $O(n \operatorname{polylog} n)$ new nonterminals. Then Lemma 5 can be applied. □

The series of lemmas gives directly the following theorem, due to the fact that after binarization of the morphism the number of variables grows quadratically and the depth only grows by a logarithmic factor.

Theorem 4. *The algorithm EQUIVALENCE(X, Y, G) deciding on the equivalence of two nonterminals X and Y in a simple grammar G, works in time $O(n^7 \log n)$ and $O(n^5 \operatorname{polylog} v(G))$, where n is the size of G, and $v(G)$ is the length of a shortest string derivable from a nonterminal, maximized over all nonterminals.*

7 Randomized algorithm for *First-MP*

We consider now randomized computations. We reduce equality of two compressed texts $H^*(A)$ and $H^*(B)$, to equality of two polynomials of degree at most $\max(|H^*(A)|, |H^*(B)|)$. It is essential that the uncompressed lengths of strings $H^*(A)$ and $H^*(B)$ is only singly exponential. It follows from the construction of the operation *quot* which involves only shortest strings derivable from nonterminals of the grammar G.

Lemma 11. *In the course of the algorithm* $|H^*(X)| \leq v(G)$ *for each* $X \in N$. *There is a constant* c *such that* $|H^*(X)| \leq c^n$.

Lemma 12 (Randomized Equality Testing). *We can check equality* $H^*(A) = H^*(B)$ *in* $O(n \operatorname{polylog} n)$ *randomized time.*

Proof. Let us identify nonterminals with integers $[1 \ldots n]$ according to our order $<$ of nonterminals. For $A \in N$ and morphism H denote by $Poly_A(x)$ a polynomial with coefficients given by the sequence $H^*(A)$, where symbols are treated as integer coefficients. Then $H^*(A) = H^*(B)$ if the polynomials $Poly_A$ and $Poly_B$ are identical, written $Poly_A \equiv Poly_B$. It is easy to see the following claim, since H^* can be interpreted as a straight-line program computing the value of $Poly_A(x)$. The degree of $Poly_A$ corresponds to the length of $H^*(A)$, which is singly exponential due to Lemma 11.

Claim We can compute $Poly_A(x)$ using $O(n)$ arithmetic operations, for a given value of the argument x. The degree of $Poly_A$ is at most singly exponential with respect to n.

The following results are versions of theorems given by Schwartz [**?**] and by Zippel [**?**] (see also Ibarra and Moran [**?**]).

Claim Let P be a nonzero polynomial of degree at most d. Assume that we assign to each variable in P a random value from a set Ω of integers of cardinality R. Then

$$Prob\{P(\bar{x}) \neq 0\ \} \geq 1 - \frac{d}{R}.$$

By the last claim we can test equality of A and B in a randomized way, by selecting random values of variables, say in the range $[1 \ldots 4 \cdot d]$ and testing equality of values. This guarantees that the probability of test correctness is at least $\frac{1}{2}$. The test can be repeated logarithmic number of times to guarantee that the probability of failure is at most $\frac{1}{n}$.

However there is one technical difficulty. The involved numbers are exponential with respect to degree and could need an exponential number of bits. In that case, we are not able to compute them in polynomial time. Hence instead of computing and comparing the exact values we choose random prime number p from the interval $[1 \ldots n^2]$ and compute the values *modulo* p. We refer the reader to Theorem 7.5 and the discussion in Example 7.1 of [**?**], for details about randomized testing of the equality of two number using modular arithmetic and prime numbers with an exponentially smaller number of bits than the numbers to be tested.

If the computed values modulo p are different, then the polynomials are different. Otherwise, by the claim, the polynomials are identical with high probability. The computation modulo p, where p is a prime number with polynomially many bits, can be done in polynomial time with respect to n. This completes the proof. □

Lemma 13. *The first mismatch-pair problem can be solved by a randomized algorithm in time* $O(n^2 \operatorname{polylog} n)$.

Proof. We can check equality of two prefixes of $H^*(A)$ and $H^*(B)$ in the same time as equality of $H^*(A)$ and $H^*(B)$. This can be done by changing H into H' which generates only corresponding segments. We omit the details. Then Lemma 12 can be applied.

If we can compute equality of prefixes then we can do binary search to compute the first mismatch. We have to add as a coefficient the number of iterations in the binary search. This number is logarithmic with respect to the lengths of uncompressed strings, hence it is $O(n)$ since the lengths are only singly exponential. This complete the proof. □

Theorem 5. *We can solve the equivalence problem for simple grammars by a randomized algorithm in* $O(n^4 \operatorname{polylog} n)$ *time.*

8 Conclusions

We have given an algorithm converting any simple grammar to its canonical representation called Prime Normal Form. We also improved the complexity of the algorithm verifying equivalence of simple languages. This result may be used to reduce simple grammars, which is the most expensive step of the PNF algorithm. Despite this improvement, this algorithm still works in exponential time in the worst case, since its output may be of exponential size. However, this theoretical limitation does not seem to occur in practice in the context of network packet filtering and classification.

One interesting open problem is to propose a canonical representation of a simple grammar, and an algorithm computing it, such that the size of this representation is polynomial with respect to the size of the original grammar.