# On the Complexity of Pattern Matching for Highly Compressed Two-Dimensional Texts

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**Abstract**

We consider the complexity of problems related to 2-dimensional texts (2d-texts) described succinctly. In a succinct description, larger rectangular sub-texts are defined in terms of smaller parts in a way similar to that of Lempel-Ziv compression for 1-dimensional texts, or in shortly described strings as in [9], or in hierarchical graphs described by context-free graph grammars. A given 2d-text $T$ with many internal repetitions can have a **hierarchical description** (denoted $Compress(T)$) which is up to exponentially smaller and which can be the only part of the input for a pattern-matching algorithm which gives information about $T$. Such a hierarchical description is given in terms of a straight-line program, see [9] or, equivalently, a 2-dimensional grammar.

We consider **compressed pattern-matching**, where the input consists of a 2d-pattern $P$ and of a hierarchical description of a 2d-text $T$, and **fully compressed pattern-matching**, where the input consists of hierarchical descriptions of both the pattern $P$ and the text $T$. For 1-dimensional strings there exist polynomial-time deterministic algorithms for these problems, for similar types of succinct text descriptions [2, 6, 8, 9]. We show that the complexity dramatically increases in a 2-dimensional setting.

For example, compressed 2d-matching is $\mathcal{NP}$-complete, fully compressed 2d-matching is $\Sigma_2^{\mathcal{P}}$-complete, and testing a given occurrence of a two dimensional compressed pattern is co-$\mathcal{NP}$-complete.

On the other hand, we give efficient algorithms for the related problems of randomized equality testing and testing for a given occurrence of an uncompressed pattern.

We also show the surprising fact that the compressed size of a subrectangle of a compressed two dimensional array can grow exponentially, unlike the one dimensional case.

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## 1 Introduction

We consider algorithms for problems dealing with *highly compressed* 2d-texts, *i.e.*, two dimensional arrays with entries from some finite alphabet. A 2d-text $T$ is represented hierarchically in a succinct way, denoted $Compress(T)$. Texts are as much as exponentially compressed. Our main problem is the **Fully Compressed Matching Problem**:

**Instance:** $Compress(P)$ and $Compress(T)$.

**Question:** does $P$ occur in $T$?

where $P$ and $T$ are rectangular 2d-texts. The **Compressed Matching Problem** is essentially the same, the only difference being that $P = Compress(P)$, in other words, the pattern is uncompressed. Our results show that an attempt to deal with exponentially compressed 2d-texts should fail algorithmically. The size of the problem is $n+m$, where $n = |Compress(T)|$ and $m = |Compress(P)|$. Let $N$ be the total uncompressed size of the problem. Note that in general $N$ can be exponential with respect to $n$, and any algorithm which decompresses $T$ takes exponential time in the worst case.

We also consider the problems of **Pattern Checking**, that is, testing an occurrence of a pattern at a given position. This problem has also its *compressed* and *fully compressed* versions. The hierarchical description of a 2d-text is in terms of *straight-line programs* (SLP's for short), or equivalently, two dimensional context-free grammars generating single objects with the following two operations:

$A \leftarrow BC$, which concatenates 2d-texts $B$ and $C$ (both of equal height)

$A \leftarrow B \ominus C$, which puts the 2d-text $B$ on top of $C$ (both of equal length)

An SLP of size $n$ consists of $n$ statements of the above form, where the result of the last statement is the compressed 2d-text. The only constants in our SLP's are symbols of an alphabet, interpreted as $1 \times 1$ images. We view SLP's as compressed (descriptions of) images.

The complexity of basic string problems for one dimensional texts is polynomial, see [4, 6, 8, 10]. Surprisingly, the complexity jumps if we pass to two dimensions. The compressed size of a subrectangle of a compressed two dimensional array $A$ can be exponential with respect to the compressed size of $A$, though such a situation cannot occur in the 1-dimensional case. This phenomenon appears to be responsible for the increase in the time complexity.

**Theorem 1.1** *For each $n$ there exists an* SLP *of size $n$ describing a text image $A_n$ and a subrectangle $B_n$ of $A_n$ such that the smallest* SLP *describing $B_n$ has exponential size.*

*Proof:* The proof is omitted in this version. □

**Example.** Hilbert's curve can be viewed as an image which exponentially compressible in terms of SLP's. An SLP which describes the $n^{\text{th}}$ Hilbert's curve, $H_n$, uses six (terminal) symbols , , , , , , and 12 variables $_i$ , $_i$ , $_i$ , $_i$ , $_i$ , $_i$ , $_i$ , $_i$ , $_i$ , $_i$ , $_i$ , $_i$ , for each $0 \leq i \leq n$. A variable with index $i$ represents a text square of size $2^i \times 2^i$ containing part of a curve. The dots in the boxes show the places where the curve enters and leaves the box.

The 2d-text $T$ = $_3$ describing the 3rd Hilbert's curve is shown in Figure 1. It is composed of four smaller square 2d-texts $_2$ , $_2$ , $_2$ , $_2$ according to the composition rule in Figure 1. $T$ consists of 64 (terminal) symbols.

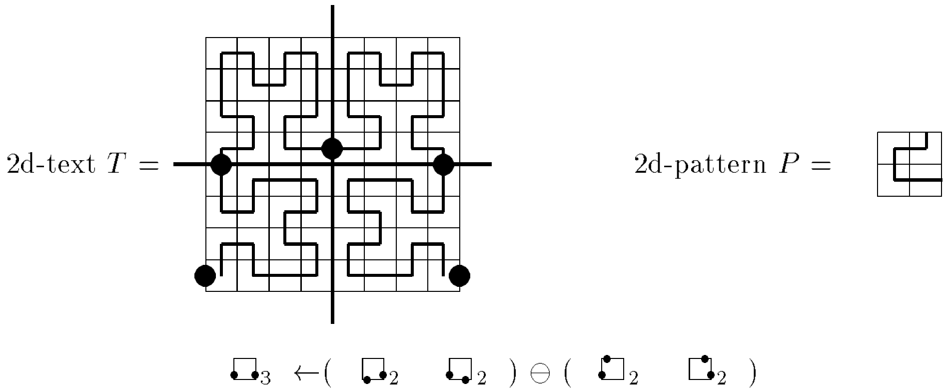
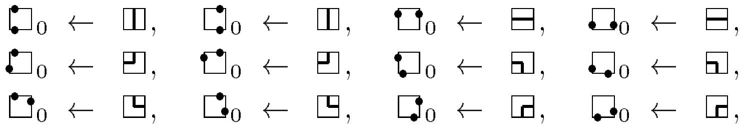
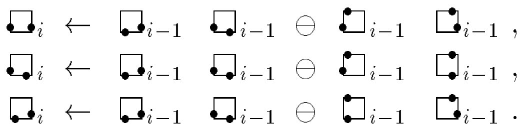



Figure 1: An example of a 2d-text $T$ and a pattern $P$. The pattern occurs twice in $T$. The black dots are not part of $T$.

The $1 \times 1$ text squares are described as follows.



The text squares for variables indexed by $i \geq 1$ are rotations of text squares for the variables $_i$ , $_i$ , $_i$ . These variables are composed according to the rules:



## 2 Equality testing in randomized polynomial time

We reduce equality of two 2d-texts $\mathcal{A}$ and $\mathcal{B}$ to equality of two polynomials $Poly_{\mathcal{A}}(x, y)$ and $Poly_{\mathcal{B}}(x, y)$. Assume that the symbols are integers in some small range. For an $n \times n$ 2d-text $\mathcal{Z}$ define its corresponding polynomial

$$Poly_{\mathcal{Z}}(x, y) = \sum_{i,j=1}^{n} \mathcal{Z}_{i,j} x^i y^j .$$

**Observation.**
Let $\mathcal{A}$ and $\mathcal{B}$ are two 2d-texts. Then $\mathcal{A} = \mathcal{B} \Leftrightarrow Poly_{\mathcal{A}} \equiv Poly_{\mathcal{B}}$.

**Fact 2.1** *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be 2d-texts corresponding to variables $A$, $B$, $C$ in some SLP.*

1. *If $A \leftarrow B \ominus C$ then $Poly_{\mathcal{A}}(x,y) = Poly_{\mathcal{C}}(x,y) \cdot x^{height(\mathcal{B})} + Poly_{\mathcal{B}}(x,y)$.*
2. *If $A \leftarrow BC$ then $Poly_{\mathcal{A}}(x,y) = Poly_{\mathcal{C}}(x,y) \cdot y^{width(\mathcal{B})} + Poly_{\mathcal{B}}(x,y)$.*
3. *for given values $(x_0, y_0)$ of arguments the value $Poly_{\mathcal{A}}(x_0, y_0) \bmod k$ can be computed in time polynomial w.r.t. the compressed size of $\mathcal{A}$ and the number of bits of $k$.*
4. $\mathrm{degree}(Poly_{\mathcal{A}}) \;=\; height(\mathcal{A}) \cdot width(\mathcal{A})$.

The following result is a version of theorems given by Schwartz and by Zippel [13].

**Lemma 2.2**
*Let $\mathcal{P}$ be a nonzero polynomial of degree at most $d$. Assume that we assign to each variable in $\mathcal{P}$ a random value from a set $\Omega$ of integers of cardinality $R$. Then*

$$Prob\{\mathcal{P}(\bar{x}) \neq 0\ \} \geq 1 - \tfrac{d}{R}.$$

**Theorem 2.3** There exists a polynomial time randomized algorithm for testing equality of two 2d-texts, given their hierarchical compressed representation.

*Proof:* Let $n$ be the total size of compressed description of $\mathcal{A}$, $\mathcal{B}$. Denote

$$deg = \max\{degree(\mathcal{A}),\ degree(\mathcal{B})\}.$$

The value of $deg$ corresponds to maximum size of 2d-texts and we have $deg \leq c^n$, for a constant $c$.

We can test equality of $\mathcal{A}$ and $\mathcal{B}$ in a randomized way, due to Lemma 2.2, by selecting random values $x_0$, $y_0$ of variables in the range $[1 \ldots 2{\cdot}deg]$ and testing $y_1 = y_2$, where $y_1 = Poly_{\mathcal{A}}(x_0, y_0)$ and $y_2 = Poly_{\mathcal{B}}(x_0, y_0)$.

However there is one technical difficulty, the numbers $y_1$, $y_2$ are exponential w.r.t. $deg$ and can need exponential number of bits, then, obviously, we are not able to compute them in polynomial time. Hence instead of computing the exact values of $y_1$, $y_2$ we choose a random prime number $p$ from a suitable interval and compute values $y_1$, $y_2$ *modulo* $p$. We refer the reader to Theorem 7.5 and the discussion in Example 7.1 in [13], for details about randomized testing of the equality of two number using prime numbers and modular arithmetic with exponentially smaller number of bits than the numbers to be tested.

If the computed values $y_1\ mod\ p$ and $y_2\ mod\ p$ are different, then the polynomials are different. Otherwise, by Lemma 2.2, the polynomials are identical with high probability. The computation of $y_1\ mod\ p$ and $y_2\ mod\ p$, where $p$ is a prime number with polynomially many bits, can be done in polynomial time w.r.t. $n$. This completes the proof. □

**Open Problem**: We designed a fast randomized algorithm for the equality of two compressed 2d-texts, and we conjecture that there is a polynomial time deterministic algorithm.

# 3 Compressed two dimensional pattern-matching

Recall that the compressed matching problem is to find, given an uncompressed pattern and compressed text, whether the pattern occurs within the text.

In our constructions we will use, as a building block, rectangles filled with one kind of letter only, say $a$. We will use $[a]^i_j$ to denote such an $i \times j$ 2d-text. It is easy to see that $[a]^i_j$ can be compressed to an SLP of size $O(\log(i) + \log(j))$. We will use $I, J, \ldots, P, Q, \ldots$ for uncompressed 2d-texts, and $\mathcal{I}, \mathcal{J}, \ldots, \mathcal{P}, \mathcal{Q}, \ldots$ for compressed ones. Given a compressed 2d-text $\mathcal{R}$ (uncompressed 2d-text $R$), we use $\mathcal{R}_{i,j}$ ($R_{i,j}$) to denote the symbol at position $(i, j)$; if the position $(i, j)$ is out of range, we will have $\mathcal{R}_{i,j} = \perp$. We will number the rows and columns starting from 0. We also use the convention that given a number $m$, $\widetilde{m}$ is a 0-1 vector $(a_0, ..., a_{k-1})$ such that $m = \sum_{i=0}^{k-1} 2^i a_i$. The length of $\widetilde{m}$ should be clear from context. Let $Positions(P) = \{(i, j) : P_{i,j} \neq \perp\}$ and $Positions(\mathcal{P}) = \{(i, j) : \mathcal{P}_{i,j} \neq \perp\}$.

First we consider the **Point test problem**: compute the symbol $\mathcal{I}_{i,j}$ for given $\mathcal{I}$, $i$ and $j$.

**Lemma 3.1** There exists an $O(n|P|)$ time algorithm for the point test problem, where $n$ is the size of $\mathcal{T}$.

**Theorem 3.2** Compressed matching for two dimensional 2d-texts is $\mathcal{NP}$-complete.

*Proof:* To see that compressed matching is in $\mathcal{NP}$, we express this problem as the following property of pattern $P$ and 2d-text $\mathcal{R}$:

$$\exists (i, j)\{\forall (k, l) \in Positions(P) \quad P_{k,l} = \mathcal{R}_{i+k,j+l}\}.$$

The equality inside the braces can be tested in polynomial time (Lemma 3.1), hence we have expressed the problem in the normal form for $\mathcal{NP}$.

To show $\mathcal{NP}$ hardness, we will use a reduction from 3SAT. Consider a set of clauses $C_0, \ldots, C_{k-1}$, where each clause is a Boolean function of some three variables from the set $\{x_0, \ldots, x_{n-1}\}$. The 3SAT question is whether there exists $m$ such that $0 \leq m \leq 2^n - 1$ and $C_i(\widetilde{m}) = 1$ for $i = 0, \ldots, k-1$.

Define a $k \times 2^n$ 2d-text $A$ by: $A_{i,m} = C_i(\widetilde{m})$. Then the 3SAT question is equivalent to the following: does $A$ contain a column consisting of $k$ 1's (i.e. the pattern $[1]^1_m$)? We will reduce 3SAT to the compressed matching problem by showing how to compress $A$ to an SLP with $O(kn)$ statements. It suffices to show that we can compress any row of $A$ to an SLP with $O(n)$ statements, because we combine the compressed rows using $k$ "$\ominus$" operations.

Consider a row $R$ of $A$ corresponding to a clause $C(x_h, x_i, x_j)$ where $h < i < j$. Define

$\imath(v_0, \ldots, v_{n-1}) = v_h + 2v_i + 4v_j$, then $R_m = C(\widetilde{\imath(\widetilde{m})})$. We will show how to compress the string $I$ over $\Sigma = \{0,1,2,3,4,5,6,7\}$ defined by $I_m = \imath(\widetilde{m})$, for $0 \leq m < 2^n$. Then obtain an SLP for $R$ from the SLP $\mathcal{I}$ for $I$ by replacing each constant $a \in \Sigma$ with $C(\widetilde{a})$.

We omit an easy proof of the following fact.

**Fact 3.3**

$$I \;=\; (((0^{2^h}1^{2^h})^{2^{i-h-1}}(2^{2^h}3^{2^h})^{2^{i-h-1}})^{2^{j-i-1}}((4^{2^h}5^{2^h})^{2^{i-h-1}}(6^{2^h}7^{2^h})^{2^{i-h-1}})^{2^{j-i-1}})^{2^{n-j-1}}$$

To compress $I$, write a constant length SLP that computes all subexpressions of $I$, then replace each statement of the form $K \leftarrow L^{2^i}$ with $i$ statements $L \leftarrow LL$ followed by $K \leftarrow L$. This results in an SLP with $O(n)$ statements. $\Box$

## 4 Fully compressed two dimensional pattern-matching

Recall that the fully compressed matching problem is to determine, given a pattern and a text that are both compressed, whether the pattern occurs within the text. We prove that this problem is $\Sigma_2^{\mathcal{P}}$-complete, see [14] for the definition of the class $\Sigma_2^{\mathcal{P}}$.

**Theorem 4.1 (main result)** Fully compressed matching for 2d-texts is $\Sigma_2^{\mathcal{P}}$-complete.

Given compressed pattern $\mathcal{P}$ and compressed 2d-text $\mathcal{I}$, the positive answer to the fully compressed two dimensional pattern matching question is equivalent to the following:

$$\exists(i,j)\forall(k,l) \in \mathit{Positions}(\mathcal{P}) \quad \{\mathcal{P}_{k,l} = \mathcal{I}_{i+k,j+l}\}$$

By Lemma 3.1, the equality in this formula can be checked in polynomial time, hence the problem can be formulated in the normal form of $\Sigma_2^{\mathcal{P}}$ problems.

This proof of $\Sigma_2^{\mathcal{P}}$-hardness requires two lemmas.

**Lemma 4.2** *There exists a log-*SPACE *function $f$ such that for any 3CNF formula $F$, $f(F) = (u, v, t)$, where $u$ and $v$ are vectors of non-negative integers, $t$ is an integer and*

$$\forall x \quad F(x) \equiv \exists y \;\; ux + vy = t.$$

*where the quantifiers range over 0-1 vectors of appropriate length.*

*Proof:* Assume that $F$ has $n$ variables, $a$ clauses with three literals, $b$ clauses with two literals and $c$ clauses with one literal. Vector $u$ will consists of $n$ numbers and vector $v$ of $7a + 3b$ numbers. We will describe each of these numbers, (and $t$ as well) using the identity $\tilde{d} = d^0 \ldots d^{(a+b+c-1)}$, where $d^{(k)}$ is *the fragment of $d$ corresponding to clause $C_k$*. The fragments corresponding to a clause with $l$ literals will have length $2l$. We describe in detail the case of a clause with three literals, the other cases being similar, only simpler.

Assume that clause $C_k$ contains three variables, $x_h$, $x_i$, $x_j$. The fragments of $u_h$, $u_i$, and $u_j$ corresponding to $C_k$ are 000100, 000010 and 000001 respectively, while for $l \notin \{h, i, j\}$ we have $u_l^{(k)} = 000000$.

There are 7 truth assignments for $(x_h, x_i, x_j)$ that satisfy $C(k)$, for each one we have an entry in vector $v$; if $v_l$ is the entry corresponding to a truth assignment $(b_0, b_1, b_2)$ for $C_k$, then $v_l^{(k)} = 100(1-b_0)(1-b_1)(1-b_2)$. Moreover, for $k' \neq k$ we have $v_l^{(k')} = 0\ldots0$. Finally, $t^{(k)} = 100111$.

Consider now $x$ such that $F(x)$ is true. Then the fragment of $\widehat{ux}$ corresponding to a clause $C_k$ is $000b_0b_1b_2$, where $(b_0, b_1, b_2)$ is a truth assignment satisfying $C_k$ (note that $x$ satisfies all the clauses of $F$). Let $v_l$ be the entry of $v$ corresponding to this truth assignment, and $v_{l_1}, \ldots, v_{l_6}$ be the entries corresponding to other truth assignments that may satisfy $C_k$. We set $y_l$ to 1 and $y_{l_1}, \ldots, y_{l_6}$ to 0; it is easy to see that the fragment of $\widehat{ux+vy}$ corresponding to $C_k$ is 100111, the same as the corresponding fragment of $t$. Since this is true for every fragment of $t$, we have $ux + vy = t$.

Now suppose that there exists $y$ such that $ux+vy = t$. If for every clause $C_k$ exactly one of the entries corresponding to the truth assignments that satisfy $C_k$ has coefficient 1 in the vector $y$, and if the addition is performed without carries, then each $C_k$ is satisfied. It can be proved by induction that this is indeed the case (note that in our string representations of numbers we write the least significant bit first). We leave the details are left to the reader.

Finally, the method of creating $(u, v, t)$ is so regular that it can be carried out by a deterministic log-SPACE Turing machine. $\Box$

Define the $\Sigma_2$(Subset Sum) problem as follows: given $(u, v, t)$ where $u$ and $v$ are vectors of positive integers and $t$ is an integer; the question is whether $\exists x \forall y \; ux + vy \neq t$, where the quantifiers range over 0-1 vectors of appropriate length.

**Lemma 4.3** *The $\Sigma_2$(Subset Sum) problem is $\Sigma_2^{\mathcal{P}}$-complete.*

*Proof:* Consider now an arbitrary property $L$ of binary strings that belongs to $\Sigma_2^{\mathcal{P}}$. In its normal form, $L$ is represented as

$$L(x) \quad \equiv \quad \exists y_1 \forall y_2 \; P(x, y_1, y_2)$$

where $P$ is a polynomial time predicate. Because $\mathcal{P} \subseteq \mathcal{NP} \cap \text{co-}\mathcal{NP}$, the predicate $P$ can be represented as

$$P(x, y_1, y_2) \quad \equiv \quad \neg(\exists y_3 F(x, y_1, y_2, y_3))$$

where $F$ is a 3CNF formula (computed using space which is logarithmic in the size of $x$ in unary). Let "$\cdot$" denote concatenation of vectors. By the previous lemma,

$$F(x, y_1, y_2, y_3) \equiv \exists y_4 \; u(x \cdot y_1 \cdot y_2 \cdot y_3) + v y_4 = t$$

where $(u, v, t)$ can be computed in logarithmic space from $F$. Define $\bar{u}, \bar{v}, \bar{w}$ and $\bar{t}$ so that $u(x \cdot y_1 \cdot y_2 \cdot y_3) + vy_3 = \bar{w}x + \bar{u}y_1\bar{v}(y_2 \cdot y_3 \cdot y_4)$ and $\bar{t} = t - \bar{w}x$. By substitution and the De Morgan laws, we have

$$\begin{aligned} L(x) &\equiv \exists y_1 \forall y_2 \neg(\exists y_3 \exists y_4 \; u(x \cdot y_1 \cdot y_2 \cdot y_3) + vy_4 = t) \\ &\equiv \exists y_1 \forall y_2 \forall y_3 \forall y_4 \; u(x \cdot y_1 \cdot y_2 \cdot y_3) + vy_4 \neq t \\ &\equiv \exists y_1 \forall y_2 \forall y_3 \forall y_4 \; \bar{w}x + \bar{u}y_1\bar{v}(y_2 \cdot y_3 \cdot y_4) \neq t \\ &\equiv \exists y_1 \forall (y_2 \cdot y_3 \cdot y_4) \; \bar{u}y_1\bar{v}(y_2 \cdot y_3 \cdot y_4) \neq \bar{t} \end{aligned}$$

Because the last of the above statements is an instance of $\Sigma_2$(Subset Sum), we have shown that $L$ can be reduced to $\Sigma_2$(Subset Sum). $\square$

To prove that fully compressed two dimensional pattern matching is $\Sigma_2^{\mathcal{P}}$ complete, it suffices to show how to translate an instance of $\Sigma_2$(Subset Sum). Consider an instance given by $(u, v, t)$. Recall the definition of $T^w$ from our proof of co-$\mathcal{NP}$ completeness.

Let $U$ be the 2d-text $T^u$ and let $V$ be the 2d-text $T^v$ with all rows reversed. Recall that dimensions of $U$ and $V$ are $2^n \times (1 + r)$ and $2^m \times (1 + s)$ respectively, where $m$ and $n$ are the lengths of $u$ and $v$, while $r$ and $s$ are their sums. We define the pattern and the test as follows:

$$\begin{array}{lll} P \leftarrow 1 \ominus [0]^1_{2^n+2^m} & S_0 \leftarrow [0]^{2^n}_{s-t} U & S_1 \leftarrow V[1]^{2^m}_{r-t} \\ S_2 \leftarrow [0]^{2^n}_{1+r+s-t} & T \leftarrow R_1 \ominus R_2 \ominus R_2 & \end{array}$$

The subrectangles $S_i$'s are *stripes* of the text $T$. Observe first that $T$ contains $P$ if and only if there exists a column of $T$, say $c$, that contains $P$. Because the length of $P$ equals the sum of heights of $S_1$ and $S_2$ plus 1, $P$ can start anywhere in the upper stripe $S_0$ but only there. Because $P$ starts with 1, it must start within $U$, so $c = s - t + a$ for some $a \geq 0$. Therefore column $c$ consists of column $a$ of $U$, column $s - t + a$ of $V$ and zeros at the bottom—we can easily exclude the case when this column crosses the middle stripe $S_1$ through the subrectangle consisting of 1's only.

Now, column $a$ of $U$ is column $a$ of $T^u$, so a 1 exists in this column if and only if for some $x < 2^n$ we have $u\tilde{x} = a$. Moreover, column $s - t + a$ of $V$ is column $s - (s - t - a) = t - a$ of $T^v$, and we have all 0's in this column if and only if $vy \neq t - a$ for every $y < 2^m$. Summarizing, $P$ occurs in $T$ if and only if there exists $x$ with the following property: for $a = ux$ the equality $vy = t - a \equiv a + vy = t \equiv ux + vy = t$ holds for no $y$. Therefore the positive answer to our pattern matching problem is equivalent to the positive answer to the original $\Sigma_2$(Subset Sum) problem. This concludes the proof of Theorem 4.1.

## 5 Fully compressed pattern checking

The problem of fully compressed pattern checking at a given location is to check, given pattern $\mathcal{P}$ and text $\mathcal{R}$ that are both compressed and a position within the text, whether $\mathcal{P}$ occurs within $\mathcal{R}$ at this particular place.

**Theorem 5.1** *Fully compressed pattern checking for d-texts is co-$\mathcal{NP}$-complete.*

*Proof:* We can use Lemma 3.1 to express this problem in the normal form of co-$\mathcal{NP}$:

$$\forall (k,l) \in \mathit{Positions}(\mathcal{P}) \;\; \mathcal{P}_{k,l} = \mathcal{R}_{k+i,l+j}.$$

To prove co-NP hardness, we will reduce co-(Subset Sum) to our problem. An instance of co-(Subset Sum) is a vector of integer weights $w = (w_0, \ldots, w_{n-1})$ and a target integer value $t$; the question is whether $\forall m < 2^n \; w\widetilde{m} \neq t$. (Here $w\widetilde{m}$ stands for the inner product; because $\widetilde{m}$ is a 0-1 vector, $w\widetilde{m}$ is a sum of a subset of the terms of $w$.) We can transform this question to a pattern checking question in a natural manner. Let $s = 1 + \sum_{i=0}^{n-1} w_i$, and let the 2d-text $T^w$ consists of 0's and 1's, with $T^w_{m,i} = 1$ if and only if $w\widetilde{m} = i$. Then our co-(Subset Sum) question is whether column $t$ of $T^w$ consists of 0's only. In terms of the pattern checking problem, we specify the text $T^w$, the pattern $[0]^1_{2^n}$ and the position $(t,0)$.

To finish the proof, we need to compress $T^w$. Observe that row $m$ of $T^w$ contains exactly one 1, at position $w\widetilde{m}$. Moreover, for $m < 2^{n-1}$ we have $w(\widetilde{m + 2^{n-1}}) = w(\widetilde{m}) + w_{n-1}$. Therefore when we split $T^w$ into upper and lower halves (each with $2^{n-1}$ rows), the pattern of 1's is very similar, the only difference being that in the lower half (with higher row numbers) the 1's are shifted by $w_{n-1}$ to the right. Moreover, if we remove the last $w_{n-1}$ zeros from each row in the upper half, we obtain $T^{w(n-1)}$, a 2d-text defined just as $T^w$, but where $w(n-1) = (w_0, \ldots, w_{n-2})$. Proceeding inductively, we compute $T^w$:

$T^{w(0)} \leftarrow 1$
**for** $i \leftarrow 0$ **to** $n-1$ **do**
  $U \leftarrow T^{w(i)}[0]^{2^i}_{w_i}$;  $L \leftarrow [0]^{2^i}_{w_i} T^{w(i)}$;  $T^{w(i+1)} \leftarrow U \ominus L$

To obtain an SLP for $T^w$, we combine $3n+1$ statements of the above program with SLP's that compute auxiliary 2d-texts $[0]^{2^i}_{w_i}$. The resulting SLP has $O(n^2 + b)$ statements, where $b$ is the total number of bits in the binary representations of the numbers in vector $w$. □

## 6 Compressed pattern checking

Recall that the compressed pattern checking problem is to check whether an uncompressed pattern $P$ occurs at a position $(x,y)$ of a 2d-text $T$ given by an SLP $\mathcal{T}$. Let $n$ be the size of $\mathcal{T}$ and $N$ be the size of $T$. The compressed pattern checking problem can be solved easily in polynomial time by using an algorithm for point test problem $m \cdot k$ times. By Lemma 3.1

there is an algorithm which solves the compressed pattern checking problem in $O(n|P|)$ time. We improve this by replacing $n$ by $\log N \log m$. This is similar to the approach of [6]. If the 2d-text is not very highly compressed then $\log(N)$ is close to $\log(n)$. The idea behind the algorithm is to consider point tests in groups, called a *queries*. Denote by $\mathcal{V}$ a text which is generated by a variable $V$. A *query* is a triple $(V, p, \mathcal{R})$ where $V$ is a variable in the SLP $\mathcal{T}$, $p$ is a position inside $\mathcal{V}$ and $\mathcal{R}$ is a subrectangle of the pattern $P$. Denote by $\mathcal{R}'$ the subrectangle of $\mathcal{V}$ which is placed at position $p$ in $\mathcal{V}$ and is of the same shape as the rectangle $\mathcal{R}$. We require that $\mathcal{R}'$ abut one of the sides of the rectangle $\mathcal{V}$. An *answer* to the query tells whether $\mathcal{R}' = \mathcal{R}$. Queries are answered by replacing them by equivalent "simpler" queries. We say that a query $(V, p, \mathcal{R})$ is *simpler* than a query $(V', p', \mathcal{R}')$ if $|\mathcal{V}| < |\mathcal{V}'|$. A query which contains a variable $V$ is called a *$V$-query*. An *atomic query* is a query $(V, p, \mathcal{R})$ such that $\mathcal{V}$ is a $1 \times 1$ square, which can be answered in $O(1)$ time.

The queries are divided into three classes: *strip queries*, *edge queries*, and *corner queries*. Let $(V, p, \mathcal{R})$ be a query. Let $\mathcal{R}'$ be the rectangle of the same shape as $\mathcal{R}$ which is positioned at $p$ in $\mathcal{V}$. $(V, p, \mathcal{R})$ is a *corner* query if $\mathcal{R}$ contains at least one side of the pattern or $\mathcal{R}$ is a corner subrectangle of the pattern and $\mathcal{R}'$ is a corner subrectangle of $\mathcal{V}$. $(V, p, \mathcal{R})$ is an *edge* query if $\mathcal{R}'$ contains one side of $\mathcal{V}$. There are four types of edge queries depending on which side of $\mathcal{V}$ is contained in $\mathcal{R}'$. We call these *down*, *left*, *right* and *up* queries. $(V, p, \mathcal{R})$ is a *strip* query if $\mathcal{R}$ is a strip of the pattern and $\mathcal{R}'$ is a strip of $\mathcal{V}$.

The algorithm CHECKING for the checking problem uses two procedures, *Split*$(V, Q)$ and *Remove_Edge_Queries*$(V, Q)$, where $V$ is a variable in $\mathcal{T}$ and $Q$ is a set of queries.

**Algorithm** CHECKING
{ input: an SLP $\mathcal{T}$, a pattern $P$ and a position $p$ }
{ output: true iff $P$ occurs at $p$ in a text described by $\mathcal{T}$ }
**begin**
  $V_1, V_2, \ldots, V_n$:= sort variables in $\mathcal{T}$ on the sizes of their texts, in descending order
  $Q$:=$\{(V_1, p, P)\}$
  **for** $i$:=1 **to** $n$ **do**
    $Q$:=$Remove\_Edge\_Queries(V_i, Q)$  $Q$:=$Split(V_i, Q)$
    {there are now only atomic queries in $Q$}  answer all atomic queries in $Q$
**end**

The procedure *Compress_Edge_Queries*$(V, Q)$ deals only with edge $V$-queries in $Q$. Its aim is to eliminate, for each type of edge query separately, all edge $V$-queries except the query which contains the largest subrectangle of the pattern. We describe how this procedure works for left-edge queries. Let $(V, (0,0), \mathcal{R})$ be a left-edge query and $\mathcal{R}$ be of maximal size among all left-edge $V$-queries in $Q$. Let $(V, (0,0), \mathcal{R}')$ be any other left-edge $V$-query. Then the rectangle of shape $\mathcal{R}'$ positioned at $(0,0)$ in $\mathcal{V}$ is a subrectangle of the rectangle of shape

$\mathcal{R}$ positioned at $(0,0)$ in $\mathcal{V}$. Hence, to answer both queries it is enough to answer the query $(V,(0,0),\mathcal{R})$ and to check whether the text $\mathcal{R}'$ occurs in $\mathcal{R}$ at $(0,0)$. Before removing each edge query equality of appropriate rectangles is checked and if the rectangles do not match then the procedure stops and the algorithm returns false.

Assume that $A{:=}BC$ or $A{:=}B \ominus C$ is an assignment for $A$. The *Split*$(A,Q)$ procedure replaces $A$-queries in $Q$ by equivalent $B$-queries and $C$-queries. Let $(A,p,\mathcal{R})$ be an $A$-query in $Q$. Consider a rectangle $R$ of shape $\mathcal{R}$ positioned at $p$ in $\mathcal{A}$. Then division of $A$ into $B$ and $C$ according to the assignment for $A$ causes that either $R$ to be wholly contained in $B$ or $C$, or to be divided into two smaller rectangles one of which is in $B$ and the other in $C$. In the latter case the split of a query is called a *division* of the query.

**Fact 6.1** *The total number of all divisions of queries during the work of the algorithm is exactly* $|P|-1$.

For each variable, edge and corner queries are stored in a list. The data structure for storing strip queries is more sophisticated. For each variable it is a 2-3-tree [1] in which keys are positions of strip rectangles in the variable. Recall that 2-3 trees provide operations *split* and *join* in $O(\log s)$ time where $s$ is the number of elements in the tree.

**Fact 6.2** *In each step of algorithm* CHECKING *the set* $Q$ *contains at most four corner queries and* $m$ *strip queries.*

Implementation of the *Split* operation, if it is not a division, requires merging 2-3 trees and this may result in a large number of splits of 2-3 trees. Fortunately, it is possible to prove the following lemma, using arguments similar to those of [6].

**Lemma 6.3** *The number of splits of 2-3 trees in algorithm* CHECKING *is* $O(m \log N)$.

**Theorem 6.4** *The algorithm* CHECKING *works in* $O(|P|+n+(m\log N)(\log m))$ *time.*

*Proof:* By Fact 6.1, the total cost of all divisions is $O(|P|)$. The total cost of all *Split*s which are not divisions is determined by the number of all corner queries and all edge queries which survive after the *Remove_Edge_Queries* operation during the execution of the algorithm and the number of splits of 2-3 trees. This gives, by Lemma 6.3, $O(n+(m\log N)(\log m))$. $\square$

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