Large Time Behavior via the Method of $\ell$-Trajectories

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The method of $\ell$-trajectories is presented in a general setting as an alternative approach to the study of the large-time behavior of nonlinear evolutionary systems. It can be successfully applied to the problems where solutions suffer from lack of regularity or when the leading elliptic operator is nonlinear. Here we concentrate on systems of a parabolic type and apply the method to an abstract nonlinear dissipative equation of the first order and to a class of equations pertinent to nonlinear fluid mechanics. In both cases we prove the existence of a finite-dimensional global attractor and the existence of an exponential attractor.

Key Words: large time behavior; global attractor; exponential attractor; finite fractal dimension; $\ell$-trajectory; fluids with shear-dependent viscosity; power-law fluids.

INTRODUCTION

The aim of this paper is to present a promising and powerful tool for dealing with the large-time behavior of nonlinear dissipative systems. This new approach, called the method of $\ell$-trajectories, is based on an observation that the limit behavior of solutions to a dynamical system in an original phase space can be equivalently captured by the limit behavior of $\ell$-trajectories; these are (continuous) parts of solution trajectories that are parametrized by time from an interval of the length $\ell$, $\ell > 0$.

In this paper we focus on systems of partial differential equations of a parabolic type and apply the method of $\ell$-trajectories to

(1) an abstract nonlinear dissipative equation of the first order,

(2) the system of equations describing the motion of a class of non-Newtonian incompressible fluids (fluids with shear-dependent viscosity).

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In both cases we prove the existence of a global attractor with finite fractal dimension and the existence of an exponential attractor. Most of the results are new.

The alternative way of describing the dynamics, which we are going to explain, allows us on one hand to weaken the requirements on the regularity of the solution and on the other hand enables us to formulate the results for a broader class of nonlinearities and, even more, to treat the problems where the leading elliptic operator is nonlinear. The reader can compare our results with the theories presented in [1, 4, 7, 11, 24], for example.

Before explaining the main features of the method, we want to emphasize that the method of $\ell$-trajectories is applicable not only to dissipative equations of parabolic type: The same approach can be successfully applied to hyperbolic type problems. As an example, the question of the finite-dimensionality of the global attractor for the wave equation with nonlinear damping is addressed in the forthcoming paper [20], where the author proves that the fractal dimension of a global attractor for the damped wave equation is finite provided that the damping function is strictly increasing of the polynomial growth – a result which to our knowledge does not have an analogy using other methods, such as that of the Lyapunov exponents, for example. And last, but not least, it turns out that the dynamical system of $\ell$-trajectories is also the natural description for studying the dynamics of nonlinear dissipative systems with bounded delay; see [19].

Now, let us explain the essence of the method of $\ell$-trajectories. Consider a nonlinear system of differential equations written as an abstract evolutionary problem,

$$
\begin{align*}
  u'(t) &= F(u(t)) \quad \text{in } X \quad (t > 0), \\
  u(0) &= u_0,
\end{align*}
$$

where $X$ is an (infinite-dimensional) Banach space, $F: X \to X$ is a nonlinear operator, and $u_0 \in X$. To give a brief characterization of the method of $\ell$-trajectories, let us assume for a while that solution operators $\{S_t\}_{t \geq 0}$ to (0.1) defined by $S_t u_0 = u(t)$ form a semigroup and in addition that $\{S_t\}_{t \geq 0}$ possesses a global attractor $\mathcal{A} \subset X$.

We describe such an arrangement of the dynamics in an equivalent way by introducing two mappings. The first mapping $b$ adds to any $u_0 \in X$ the $\ell$-trajectory that begins at $u_0$ (see Fig. 1); i.e., we consider $b$ as a mapping from $X$ into a subset of $X_\ell = L^2(0, \ell; X)$ defined by

$$
\{b(u_0)\}(\tau) = S_\tau u_0, \quad \tau \in [0, \ell].
$$
The second mapping $e$ assigns to any $\ell$-trajectory $\chi$ its end point\(^2\) (see Fig. 2); i.e.,

$$e(\chi) := \chi(\ell).$$

Now, we use $b$ to introduce a new semigroup $\{L_t\}_{t \geq 0}$ acting on the set of $\ell$-trajectories defined as (see Fig. 3)

$$L_t(b(u_0)) := b(S_t u_0), \quad u_0 \in X.$$

Then, on setting

$$\mathcal{A}_t := b(\mathcal{A}) = \{b(u_0); \ u_0 \in \mathcal{A}\},$$

\(^2\)Trajectories are supposed to be continuous at least in the weak topology of $X$; the value $\chi(\ell)$ has then a clear meaning.
it usually turns out that $\mathcal{A}_{\ell}$ is a global attractor related to $\{L_t\}_{t \geq 0}$. The complete structure is drawn in Fig. 4.

One might ask: What is the advantage of this alternative viewpoint on $\mathcal{A}$? Clearly, instead of estimating the fractal dimension of $\mathcal{A}$ directly, we are given the possibility of estimating the fractal dimension of $\mathcal{A}_{\ell}$ in the topology of $X_{\ell}$, which is revealed to be an easier task.\(^3\) After proving the finiteness of the fractal dimension of $\mathcal{A}_{\ell}$ one observes (see Lemma 1.2 below) that if $e$ is Lipschitz (or at least $\alpha$-Hölder) continuous then the fractal dimension of $\mathcal{A}$ cannot increase (or increases at most $1/\alpha$ times).

Note that the roles of $b$ and $e$ are different. While $b$ transfers the dynamics from $X$ into $X_{\ell}$, the mapping $e$ is responsible for delivering the properties of $\mathcal{A}_{\ell}$ to $\mathcal{A}$. The roles of $e$ and its regularity are more important than the role

\(^3\) For example, as shown in this paper, the criterion stated in Lemma 1.3 below is applicable to $L_t$ with ease, while it can be applied to $S_t$ only in special (quite regular) cases.
of $b$. It can happen that $A$ or $\{S_t\}_{t \geq 0}$ is not defined in $X$. Such a case can occur when the solution is not unique or when one does not have enough regularity to construct $A$. Still, it might be possible to introduce $\{L_t\}_{t \geq 0}$. In this case the attractor $A_t$ is constructed first, and after evaluating its fractal dimension we set $A$ to be $e(A_t)$. Not only does the defined set $A$ have properties of the global attractor in $X$, but also its fractal dimension is finite (provided $e$ is Lipschitz or Hölder continuous).4

The paper is organized in the following way: Section 1 recalls the definitions of basic notions and provides general helpful assertions. Section 2 presents the method of $\ell$-trajectories in a general framework. In Section 3 we provide a class of evolutionary problems with a nonlinear (monotone) elliptic operator, for which the assumptions of the general scheme are verified directly. Models from nonlinear fluid mechanics that include the Smagorinski model of isotropic turbulence and other shear-thickening fluid models in a three-dimensional setting and shear-thinning fluids in two dimensions are studied in Section 4; several new results are obtained both for the space periodic and the Dirichlet problems. Conclusions and perspectives are presented in the last section, which also includes bibliographical notes.

1. DEFINITIONS AND BASIC LEMMAS

In this section we recall several notions from the theory of dynamical systems.

Let $X$ be (a subset of) a normed space. One parameter family of (nonlinear) mappings $\Sigma_t: X \to X \ (t \geq 0)$ is called the semigroup provided that

$$\Sigma_{t+s} = \Sigma_t \Sigma_s \text{ for all } t, s \geq 0 \quad \text{and} \quad \Sigma_0 = I.$$ 

A typical example is a semigroup formed by the solution operators for a certain evolutionary problem, defined on some suitable space of initial conditions for which there exists a unique global solution.

The couple $(\Sigma_t, X)$ is usually referred to as a dynamical system.

A set $A \subset X$ is called a global attractor to $(\Sigma_t, X)$ if (i) $A$ is compact in $X$, (ii) $\Sigma_t A = A$ for all $t \geq 0$, and (iii) for any $B \subset X$ that is bounded,

$$\text{dist}(\Sigma_t B, A) \to 0 \text{ as } t \to \infty,$$

where $\text{dist}(B, A) = \sup_{b \in B} \inf_{a \in A} \|b - a\|_X$. Note that a dynamical system can have at most one global attractor.

A set $C \subset X$ is called positively invariant w.r.t $\Sigma_t$ if for all $t \geq 0$, $\Sigma_t C \subset C$, and it is called uniformly absorbing w.r.t $\Sigma_t$ if for any $B \subset X$ that is bounded there exists $t_0 = t_0(B)$ such that $\Sigma_t B \subset C$ for all $t \geq t_0$.

4 An example of such a situation is given in Section 4.1.
Lemma 1.1. Let $\Sigma_t, \mathcal{X}$ be a dynamical system. Assume that there exists a compact set $B^1 \subset \mathcal{X}$ which is uniformly absorbing and positively invariant w.r.t. $\Sigma_t$. Let moreover $\Sigma_t$ be continuous on $B^1$. Then $(\Sigma_t, \mathcal{X})$ has a global attractor.

Proof. We simply set $A$ to be the $\omega$-limit set of $B^1$; cf. [24].

Finally, the fractal dimension of a compact set $\mathcal{C} \subset \mathcal{X}$, denoted by $d^f_{\mathcal{X}}(\mathcal{C})$, is defined as

$$d^f_{\mathcal{X}}(\mathcal{C}) := \limsup_{\varepsilon \to 0} \frac{\log N^\varepsilon_{\mathcal{X}}(\mathcal{C})}{\log(1/\varepsilon)},$$

where $N^\varepsilon_{\mathcal{X}}(\mathcal{C})$ is the minimal number of $\varepsilon$-balls (with respect to the metric of $\mathcal{X}$) needed to cover $\mathcal{C}$.

Lemma 1.2. Let $\mathcal{X}, \mathcal{Y}$ be metric spaces and $F: \mathcal{X} \to \mathcal{Y}$ be $\alpha$-Hölder continuous on $\mathcal{C}$. Then

$$d^\mathcal{Y}_{\mathcal{X}}(F(\mathcal{C})) \leq \frac{1}{\alpha} d^\mathcal{X}_{\mathcal{Y}}(\mathcal{C}).$$

In particular, the fractal dimension does not increase under a Lipschitz continuous mapping.

Proof. Since $\|F(u) - F(v)\|_\mathcal{Y} \leq \alpha \|u - v\|_\mathcal{X}$, it holds that

$$F(B^\mathcal{X}(u,(\varepsilon/c)^{1/\alpha})) \subset B^\mathcal{Y}(F(u), \varepsilon).$$

Thus $N^\mathcal{Y}_\varepsilon(F(\mathcal{C})) \leq N^\mathcal{X}_\eta(\mathcal{C})$, where $\eta = (\varepsilon/c)^{1/\alpha}$. Therefore

$$\frac{\log N^\mathcal{Y}_\varepsilon(F(\mathcal{C}))}{\log(1/\varepsilon)} \leq \frac{\log N^\mathcal{X}_\eta(\mathcal{C})^{1/\alpha}}{\log(1/\varepsilon)} = \frac{\log N^\mathcal{X}_\eta(\mathcal{C})}{\alpha \log(1/\eta) - \log c},$$

which leads to the conclusion letting $\varepsilon \to 0$ ($\Rightarrow \eta \to 0$).

The importance of the notion of a finite fractal dimension is illustrated by a result of Foias and Olson [5]: if $C$ is a compact metric space such that $d^f_{\mathcal{X}}(C) < \frac{m}{2}$, $m \in \mathbb{N}$, then there exists an injective Lipschitz continuous mapping $P: C \to \mathbb{R}^m$ such that its inverse is Hölder continuous. In other words, if $d^f_{\mathcal{X}}(C) < \frac{m}{2}$, then $C$ is placed in the graph of a Hölder continuous mapping that maps the compact subset of $\mathbb{R}^m$ onto $C$. Moreover, if $C$ is a subset of a Hilbert space then $P$ can in addition be an orthogonal projector.

By this sentence we simply mean that $\alpha$-Hölder continuity is considered between metrics of $\mathcal{X}$ and $\mathcal{Y}$ respectively, though the mapping $F$ can only be defined on a proper subset $\mathcal{C}$ of $\mathcal{X}$.
**Lemma 1.3.** Let \( \mathcal{X}, \mathcal{Y} \) be normed spaces such that \( \mathcal{Y} \hookrightarrow \hookrightarrow \mathcal{X} \) and \( \mathcal{C} \subset \mathcal{X} \) be bounded. Assume that there exists a mapping \( \mathcal{L} \) such that \( \mathcal{L}^* \mathcal{C} \subset \mathcal{C} \) and \( \mathcal{L} : \mathcal{X} \rightarrow \mathcal{Y} \) is Lipschitz continuous on \( \mathcal{C} \). Then \( d^x_\mathcal{X}(\mathcal{C}) \) is finite.

**Proof.** Let \( \kappa \) be a Lipschitz constant of \( \mathcal{L} \) and \( N \) be the number of balls in \( \mathcal{X} \) of radii \( 1/4\kappa \) necessary to cover the unit ball in \( \mathcal{Y} \). Let us choose \( R > 0 \) and \( u \in \mathcal{C} \) such that \( \mathcal{C} \subset B^\mathcal{X}(u, R) \). Then we have
\[
\mathcal{C} \subset \mathcal{L}^* \mathcal{C} \subset B^\mathcal{Y}(\mathcal{L}u, \kappa R) \subset \bigcup_{1 \leq i \leq N} B^\mathcal{X}(\hat{u}_i, R/4),
\]
where \( \hat{u}_i \in \mathcal{X} \). We can assume that \( \mathcal{C} \cap B^\mathcal{X}(\hat{u}_i, R/4) \neq \emptyset \), which leads to the covering
\[
\mathcal{C} \subset \bigcup_{1 \leq i \leq N} B^\mathcal{X}(u_i, R/2),
\]
where \( u_i \in \mathcal{C} \).

Repeating the scheme inductively, we have \( N_{R/2^k}(\mathcal{C}) \leq N^k \). Now, for any positive \( \varepsilon \leq R \) there exists an integer \( k \geq 0 \) such that \( R/2^k \geq \varepsilon > R/2^{k+1} \). Then
\[
\log N_{R/2^k}(\mathcal{C}) \leq (k + 1) \log N \leq \frac{(k + 1) \log N}{\log(1/\varepsilon)} \leq \frac{k \log 2 - \log R}{\log 2},
\]
and consequently
\[
d^x_\mathcal{X}(\mathcal{C}) \leq \frac{\log N}{\log 2}.
\]

Furthermore, following [4], we introduce the concept of the exponential attractor. Under the assumptions of Lemma 1.1 we say that the set \( \mathcal{B} \subset \mathcal{B}^1 \) is an exponential attractor w.r.t. the dynamical system \((\Sigma_t, \mathcal{B}^1)\) if

(i) \( \mathcal{B} \) is compact,

(ii) \( \mathcal{B} \) is positively invariant w.r.t. \( \Sigma_t \),

(iii) \( d^x_\mathcal{X}(\mathcal{B}) \) is finite,

(iv) there exist \( c_1, c_2 > 0 \) such that \( \text{dist}^\mathcal{X}(\Sigma_t \mathcal{B}^1, \mathcal{B}) \leq c_1 e^{-c_2 t} \) for all \( t \geq 0 \).

Note that necessarily \( \mathcal{A} \subset \mathcal{B} \), so the basic idea behind the exponential attractor is to enlarge the global attractor so that the rate of convergence becomes exponential, yet keep the “good” properties (i)–(iii). The following lemma resumes a criterion on the existence of the exponential attractor obtained in [4].

**Lemma 1.4.** Let \( \mathcal{X} \) be a Hilbert space. Let \( \mathcal{B}^1 \subset \mathcal{X} \) satisfy the assumptions of Lemma 1.1. Assume that there exists \( \tau > 0 \) such that
(P1) $\Sigma_t: \mathcal{X} \mapsto \mathcal{X}$ is Lipschitz continuous on $\mathcal{B}^1$,

(P2) there exist $\delta \in (0, 1/4)$ and a finite-dimensional orthogonal projector $P: \mathcal{X} \mapsto \mathcal{X}$ such that for all $x_1, x_2 \in \mathcal{B}^1$ there holds either

$$\|\Sigma_t x_1 - \Sigma_t x_2\|_{\mathcal{X}} \leq \sqrt{2} \|P(\Sigma_t x_1 - \Sigma_t x_2)\|_{\mathcal{X}}$$

or

$$\|\Sigma_t x_1 - \Sigma_t x_2\|_{\mathcal{X}} \leq \delta \|x_1 - x_2\|_{\mathcal{X}},$$

and

(P3) the mapping $G: \mathcal{X} \times [0, \tau] \mapsto \mathcal{X}$ defined by $G(x, t) := \Sigma_t x$ is on $\mathcal{B}^1 \times [0, \tau]$ Hölder continuous.

Then the dynamical system $(\Sigma_t, \mathcal{B}^1)$ possesses an exponential attractor.

**Proof.** See [4, Chaps. 2 and 3].

We will also use this elementary lemma.

**Lemma 1.5.** Let $\mathcal{X}, \mathcal{Y}$ be normed spaces such that $\mathcal{Y} \hookrightarrow \hookrightarrow \mathcal{X}$, let moreover $\mathcal{X}$ be a Hilbert space. Then for a given $\varepsilon > 0$ there exists a finite-dimensional subspace $\mathcal{X}^n \subset \mathcal{X}$ such that, denoting by $P$ the ortho-projector to $\mathcal{X}^n$,

$$\|(I - P)u\|_{\mathcal{X}} \leq \varepsilon \|u\|_{\mathcal{Y}}$$

for any $u \in \mathcal{Y}$.

**Proof.** Without loss of generality we assume that $u \in S = \{v \in \mathcal{Y}; \|v\|_{\mathcal{Y}} = 1\}$. But the set $S \subset \mathcal{X}$ is compact, and denoting by $u_1, \ldots, u_n$ its $\varepsilon$-net, we see that the space $\mathcal{X}^n$ spanned by $u_1, \ldots, u_n$ has the desired property.

Finally, for the reader’s convenience we formulate the celebrated so-called Aubin–Lions lemma as it plays an important role in our paper.

**Lemma 1.6.** Let $p_1 \in (1, \infty], \ p_2 \in [1, \infty)$. Let $X$ be a Banach space and $Y, Z$ be separable and reflexive Banach spaces such that $Y \hookrightarrow \hookrightarrow X \hookrightarrow Z$. Then for any $\tau \in (0, \infty)$,

$$\{u \in L^{p_1}(0, \tau; Y); \ u' \in L^{p_2}(0, \tau; Z)\} \hookrightarrow \hookrightarrow L^{p_1}(0, \tau; X).$$

**Proof.** See [23], for example.

2. GENERAL SCHEME

The method of $\ell$-trajectories can be used for various purposes in studying the large-time behavior of dynamical systems. Depending on the purpose,
one needs certain assumptions to be satisfied. In order to make this presentation transparent we divide the general scheme, and correspondingly also the assumptions, into several subsections that emphasize their specific role. The titles of these subsections are:

1. Dynamical system on the set of \( \ell \)-trajectories,
2. \( \mathcal{A}_\ell \) – Attractor in the set of \( \ell \)-trajectories,
3. Finite fractal dimension of \( \mathcal{A}_\ell \),
4. \( \mathcal{A} \) – Attractor in the original space,
5. Finite fractal dimension of \( \mathcal{A} \),
6. \( \mathcal{E}_\ell \) – Exponential attractor in the set of \( \ell \)-trajectories,
7. \( \mathcal{E} \) – Exponential attractor in the original space.

(1) **Dynamical System on the Set of \( \ell \)-Trajectories**

The first assumptions concern the existence and uniqueness of the solution to (0.1).

Let \( (X, \| \cdot \|_X) \), \( (Y, \| \cdot \|_Y) \), and \( (Z, \| \cdot \|_Z) \) be three Banach spaces, \( X \) being reflexive and separable, such that

\[
Y \hookrightarrow \hookrightarrow X \quad \text{and} \quad X \hookrightarrow Z. \tag{2.1}
\]

For \( p_1 \in [2, \infty) \), \( p_2 \in [1, \infty) \), and \( \tau > 0 \) fixed we denote

\[
X_\tau := L^2(0, \tau; X),
\]

\[
Y_\tau := \{ u \in L^{p_1}(0, \tau; Y); \ u' \in L^{p_2}(0, \tau; Z) \}.
\]

Then Lemma 1.6 implies

\[
Y_\tau \hookrightarrow \hookrightarrow X_\tau. \tag{2.2}
\]

The space \( C([0, \tau]; X_w) \) is defined as

\[
C([0, \tau]; X_w) := \{ u \in L^\infty(0, \tau; X); \ \langle u(\cdot), \varphi \rangle_{X, X^*} \in C([0, \tau]) \ \text{for all} \ \varphi \in X^* \}.
\]

From now on, by solution to (0.1) on the interval \([0, T]\) with initial condition \( u_0 \) – or by solution, for short – we mean a function \( u \in C([0, T]; X_w) \cap Y_T \) satisfying (0.1) in some weak sense we have chosen; we assume that this weak formulation makes sense for a considered class of functions.

We require that:

(A1) For any \( u_0 \in X \) and arbitrary \( T > 0 \) there exists (not necessarily unique) \( u \in C([0, T]; X_w) \cap Y_T \), a solution to (0.1) on \([0, T]\) with \( u(0) = u_0 \);
moreover, for any solution the estimates of $||u||_{Y_t}$ are uniform with respect to $||u(0)||_X$.

(A2) There exists a bounded set $B^0 \subset X$ with the following properties: if $u$ is an arbitrary solution to (0.1) with initial condition $u_0 \in X$ then (i) there exists $t_0 = t_0(||u_0||_X)$ such that $u(t) \in B^0$ for all $t \geq t_0$ and (ii) if $u_0 \in B^0$ then $u(t) \in B^0$ for all $t \geq 0$.

Now, let $\ell > 0$ be an arbitrary fixed number. By the $\ell$-trajectory we mean any solution on the time interval $[0, \ell]$. The set of all $\ell$-trajectories is denoted by $X_\ell$ and equipped with the topology of $X_\ell = L^2(0, \ell; X)$. Note that since $X_\ell \subset C([0, \ell]; X_w)$, it makes sense to talk about the point values of trajectories. On the other hand, it is not clear whether $X_\ell$ is closed in $X_\ell$ and hence $X_\ell$ in general is not a complete metric space.

Since we do not require uniqueness of the solution, it is possible that more than one trajectory will start from a point $u_0 \in X$. We will impose a weaker condition:

(A3) Each $\ell$-trajectory has among all solutions unique continuation.

In other words, from an end point of an $\ell$-trajectory there starts at most one solution. Combined with the assumption (A1) about the global existence of solutions this in particular implies that if $\xi \in X_\ell$ and $T > \ell$ then there exists a unique $u$ which is a solution to (0.1) on $[0, T]$ such that $\xi = u|_{[0, T]}$.

Using (A3), we can define the semigroup $L_{\ell}$ on $X_\ell$ by

$$\{L_{\ell, t}\}(\tau) := u(t + \tau), \quad \tau \in [0, \ell],$$

where $u$ is the unique solution on $[0, \ell + \tau]$ such that $u|_{[0, \ell]} = \xi$.

(2) $\mathcal{A}_\ell$ – Attractor in the Set of $\ell$-Trajectories

We define $\mathcal{B}_\ell^0$ as the set of all $\ell$-trajectories starting at any point of $B^0$ from (A2). In symbols,

$$\mathcal{B}_\ell^0 := \{\xi \in X_\ell; \xi(0) \in B^0\}.$$

Observe that owing to (A2), $\mathcal{B}_\ell^0$ is positively invariant w.r.t. $L_{\ell}$. We add two more assumptions:

(A4) For all $t > 0$, $L_{\ell}: X_\ell \mapsto X_\ell$ is continuous on $\mathcal{B}_\ell^0$,

(A5) For some $\tau > 0$, $L_{\ell}(\mathcal{B}_\ell^0)^X_{\ell} \subset \mathcal{B}_\ell^0$.

7This assumption is suited to cover the situation in fluid mechanics where inner points of trajectories belong to a better space than $X$, hence solutions starting from them are more regular and consequently unique even in the wider class of weak solutions. Similarly, (A3) would be satisfied if we work with equations containing terms delayed (in time) by $\ell$, at most; cf. [19].
The assumption (A5) represents the crucial step in overcoming the problem of incompleteness of \( \mathcal{X}_\ell \), since it asserts that the closure (= completion) of \( L_t(\mathcal{B}^0) \) remains in \( \mathcal{X}_\ell \). Yet the assumption (A5) is naturally fulfilled provided that \( \mathcal{B}^0 \) is (weakly) closed and we have the “compactness” of solutions; cf. (E2) in Section 3 or the proof of Theorem 4.1 in Section 4.

We define

\[
\mathcal{B}^1_\ell := L_t(\mathcal{B}^0)_{X_\ell}.
\] (2.3)

The observation that \( \mathcal{B}^1_\ell \) is in fact a compact subset of \( \mathcal{X}_\ell \) is a key step in the proof of the following theorem.

**Theorem 2.1.** Let (A1)–(A5) hold. Then the dynamical system \( (L_t, \mathcal{X}_\ell) \) possesses a global attractor \( \mathcal{A}_\ell \).

**Proof.** Consider a set \( \mathcal{B}^1_\ell \subset \mathcal{X}_\ell \) defined in (2.3). Clearly, \( \mathcal{B}^1_\ell \) is closed, and by (A1), (A2) it is bounded in \( Y_\ell \leftrightarrow X_\ell \), and hence compact. Moreover, by the continuity of \( L_t \) – cf. (A4) – and the positive invariance of \( \mathcal{B}^0_\ell \), we have

\[
L_t \mathcal{B}^1_\ell = L_t(L_t(\mathcal{B}^0)_{X_\ell}) \subset L_{t+\tau}(\mathcal{B}^0)_{X_\ell} \subset L_t(\mathcal{B}^0)_{X_\ell} = \mathcal{B}^1_\ell.
\]

In order to show that \( \mathcal{B}^1_\ell \) is uniformly absorbing w.r.t. \( L_t \), it is enough to verify that \( \mathcal{B}^0_\ell \) is uniformly absorbing. Let \( \mathcal{B} \subset \mathcal{X}_\ell \) be bounded by some constant \( C \). Then for \( \chi \in \mathcal{B} \), one has \( \int_0^\ell ||\chi(t)||_{X_\ell}^2 \, dt \leq C \) and hence there is \( \tau \in [0, \ell] \) such that \( ||\chi(\tau)||_{X_\ell} \leq C/\sqrt{\ell} \). But then by (A2) there exists \( t_0 > 0 \) such that \( L_t \chi \subset \mathcal{B}^0_\ell \) for \( t \geq t_0 \), \( t_0 \) depending on \( C \) only.

By these considerations, the assumptions of Lemma 1.1 with \( \Sigma_t = L_t \), \( \mathcal{X} = \mathcal{X}_\ell \), and \( \mathcal{B}^1 = \mathcal{B}^1_\ell \) are satisfied and the existence of the global attractor follows.  

(3) **Fractal Dimension of \( \mathcal{A}_\ell \)**

The assumption which leads to the finiteness of the fractal dimension and which is also a key step in constructing the exponential attractor reads

(A6) There exists a space \( W_\ell \) with \( W_\ell \leftrightarrow X_\ell \) and \( \tau > 0 \) such that \( L_\ell \colon X_\ell \mapsto W_\ell \) is Lipschitz continuous on \( \mathcal{B}^1_\ell \).

It is worth noting that, as noted in [MN], there are important applications where such a condition can be verified; typically, with

\[
W_\ell = \{ u \in L^2(0, \ell; W); \ u' \in L^1(0, \ell; U^*) \},
\]

where \( W \leftrightarrow X \) and \( U \) is some space of very regular functions.
Theorem 2.2. Let (A1)–(A6) hold. Then the fractal dimension of $\mathcal{A}_\ell$ in $X_\ell$ is finite.

Proof. We apply Lemma 1.3 with $\mathcal{X} = X_\ell$, $\mathcal{Y} = W_\ell$, $\mathcal{L} = L_\ell$ and $\mathcal{C} = \mathcal{A}_\ell$. Since $\mathcal{A}_\ell \subset \mathcal{X}_\ell$ is compact, $L_\ell \mathcal{A}_\ell = \mathcal{A}_\ell$, and (A6) holds, we see that all assumptions of the lemma are fulfilled. 

(4) $\mathcal{A}$ – Attractor in the Original Space

Now, we introduce a mapping $e: \mathcal{X}_\ell \mapsto X$ which to a given $\ell$-trajectory assigns its end point. In symbols,

$$e(\mathcal{X}) = \chi(\ell).$$

In this manner we construct a one-way bridge between the set $\mathcal{X}_\ell$ on one side and the space $X$ on the other side. Note that the definition of $e$ is meaningful since, due to (A1), trajectories are weakly continuous. We define

$$B^1 := e(B^1).$$

Observe that by (A3) to a given initial condition $u_0 \in B^1$ there corresponds a unique solution to (0.1), hence solution operators $S_t$ are defined on $B^1$. Moreover, $B^1$ is positively invariant w.r.t. $S_t$.

Next, supposing that (A7) $e: X_\ell \mapsto X$ is continuous on $B^1$

and defining

$$\mathcal{A} := e(\mathcal{A}_\ell),$$

we obtain the following theorem.

Theorem 2.3. Let (A1)–(A5) and (A7) hold. Then $\mathcal{A}$ defined in (2.4) is a global attractor to dynamical system $(S_t, B^1)$.

Proof. Since $\mathcal{A}$ is a continuous image of a compact set, it is compact. Also, since $L_\ell(\mathcal{A}_\ell) = \mathcal{A}_\ell$, we have

$$S_t(\mathcal{A}) = S_t(e(\mathcal{A}_\ell)) = e(L_\ell(e(\mathcal{A}_\ell))) = e(\mathcal{A}_\ell) = \mathcal{A}.$$

To verify the attracting property of $\mathcal{A}$, we proceed by contradiction: let there exist sequences $u_n \in B^1$, $t_n \to \infty$, and a $\delta > 0$ such that

$$\operatorname{dist}^X(S_{t_n}u_n, \mathcal{A}) \geq \delta.$$  (2.5)
By the definition of $B^1$ there exists a sequence $\{\chi_n\} \subset \mathcal{B}_1^1$ such that $e(\chi_n) = u_n$. Since $\{\chi_n\}$ is bounded and $\mathcal{A}_\ell$ is an attractor w.r.t. $L_t$, we can assume – coming to a subsequence if necessary – that $L_t\chi_n \to \chi \in \mathcal{A}_\ell$. But by the continuity of $e$, $S_t u_n = e(L_t \chi_n) \to e(\chi) \in \mathcal{A}$, which contradicts (2.5).

**Remark 2.1.** The set $\mathcal{A}$ is also a global attractor to the dynamics of (0.1) on the whole space $X$ in the following sense: if $B \subset X$ is bounded and $B_\tau$ denotes the set of all values of all solutions to (0.1), starting from $B$, at time $t$, then

$$\text{dist}^X(B_t, \mathcal{A}) \to 0 \quad \text{as} \quad t \to \infty.$$ 

Indeed, by (A2), $B_t \subset B^0$ for $t \geq t_0$, hence $B_t \subset B^1$ for $t \geq t_0 + \tau$ and $B_t \subset S_{t-(t_0+\tau)}B^1$ for $t$ sufficiently large.

(5) *Finite Fractal Dimension of $\mathcal{A}$*

If we strengthen (A7) and require that

$$\text{(A8) } e: X_\ell \to X \text{ is } \alpha\text{-Hölder continuous on } \mathcal{B}_1^1,$$

we come to the following assertion.

**Theorem 2.4.** Let (A1)–(A6) and (A8) hold. Then the fractal dimension of $\mathcal{A}$ in $X$ is finite and

$$d_f^X(\mathcal{A}) \leq \frac{1}{\alpha} d_f^X(\mathcal{A}_\ell).$$

**Proof.** The proof is a consequence of Theorem 2.2, (A8), and Lemma 1.2, where we take $\mathcal{X} = X_\ell$, $\mathcal{Y} = X$, $F = e$, and $C = \mathcal{A}_\ell$.

(6) $\mathcal{E}_\ell – \text{Exponential Attractor in the Set of } \ell\text{-Trajectories}$

To construct an exponential attractor we will require

$$\text{(A9) For all } \tau > 0 \text{ the operators } L_t: X_\ell \to X_\ell \text{ are uniformly (with respect to } t \in [0, \tau]) \text{ Lipschitz continuous on } \mathcal{B}_1^1,$$

$$\text{(A10) For all } \tau > 0 \text{ there exists } c > 0 \text{ and } \beta \in (0, 1] \text{ such that for all } \chi \in \mathcal{B}_1^1 \text{ and } t_1, t_2 \in [0, \tau] \text{ it holds that } ||L_{t_1}\chi - L_{t_2}\chi||_{X_\ell} \leq c|t_1 - t_2|^\beta.$$ 

**Theorem 2.5.** Let $X$ be a Hilbert space and let assumptions (A1)–(A6) and (A9)–(A10) hold. Then $(L_t, \mathcal{B}_1^1)$ possesses an exponential attractor $\mathcal{E}_\ell$. 
Proof. We will apply Lemma 1.4 with $X = X_\tau$, $\Sigma_t = L_t$ and $B^1 = B^1_\tau$. Note that since $X$ is the Hilbert space, $X_\tau$ is the Hilbert space as well. Let us verify the assumptions (P1)–(P3). We fix a $\tau > 0$ for which (A6) holds. Then (P1) follows from (A6) or from (A9).

Next, let $\kappa$ be the Lipschitz constant of the mapping $L_\tau: X_\tau \mapsto W_\tau$ on $B^1_\tau$. We apply Lemma 1.5 with $X = X_\tau$, $Y = W_\tau$, and $\varepsilon = \frac{1}{8\kappa}$; then there exists a finite-dimensional ortho-projector $P$ such that $\| (I - P) \|_{X_\tau} \leq 1/(8\kappa) \| \chi \|_{W_\tau}$ for all $\chi \in W_\tau$. Thus we have

$$\| L_\tau \chi_1 - L_\tau \chi_2 \|_{X_\tau}^2 = \| P[(L_\tau \chi_1 - L_\tau \chi_2)] \|_{X_\tau}^2 + \| (I - P)(L_\tau \chi_1 - L_\tau \chi_2) \|_{X_\tau}^2$$

$$\leq \| P[(L_\tau \chi_1 - L_\tau \chi_2)] \|_{X_\tau}^2 + \frac{1}{64 \kappa^2} \| L_\tau \chi_1 - L_\tau \chi_2 \|_{W_\tau}^2$$

$$\leq \| P[(L_\tau \chi_1 - L_\tau \chi_2)] \|_{X_\tau}^2 + \frac{1}{64} \| \chi_1 - \chi_2 \|_{X_\tau}^2,$$

which can be rewritten as

$$\frac{1}{2} \| L_\tau \chi_1 - L_\tau \chi_2 \|_{X_\tau}^2 + \frac{1}{2} \| L_\tau \chi_1 - L_\tau \chi_2 \|_{X_\tau}^2$$

$$\leq \| P[(L_\tau \chi_1 - L_\tau \chi_2)] \|_{X_\tau}^2 + \frac{1}{64} \| \chi_1 - \chi_2 \|_{X_\tau}^2.$$

But whenever it holds that $a + b \leq c + d$ then necessarily either $a \leq c$ or $b \leq d$, which is just (P2) with $\beta = (4\sqrt{2})^{-1}$.

Finally, (A9), (A10) imply (P3) since

$$\| G(\chi_1, t_1) - G(\chi_2, t_2) \|_{X_\tau} = \| L_{t_1} \chi_1 - L_{t_2} \chi_2 \|_{X_\tau}$$

$$\leq \| L_{t_1} \chi_1 - L_{t_1} \chi_2 \|_{X_\tau} + \| L_{t_2} \chi_2 - L_{t_2} \chi_2 \|_{X_\tau}$$

$$\leq \varepsilon |t_1 - t_2|^\beta + \| \chi_1 - \chi_2 \|_{X_\tau}.$$

The proof of Theorem 2.5 is complete. \qed

(7) $E$ – Exponential Attractor in the Original Space

Analogously to subsection (4) where we have obtained $A$ from $A_\tau$ via the mapping $e: X_\tau \mapsto X$ we obtain $E$ as an image of $E_\tau$. We put

$$E := e(E_\tau). \quad (2.6)$$

**Theorem 2.6.** Let (A1)–(A6), (A8)–(A10) hold. Then $E$ defined in (2.6) is an exponential attractor to the dynamical system $(S_t, B^1)$. \[ \]

**Proof.** It immediately follows from the facts that $E_\tau$ is an exponential attractor w.r.t. $(L_t, B^1_\tau)$ and the mapping $e$ is H"older continuous.
We conclude by two lemmas that are often useful in verifying the assumptions (A4), (A8), (A9), and (A10).

**Lemma 2.1.** Let \( \mathcal{C}_t \subset \mathcal{X}_t \) be a set of trajectories and let \( \mathcal{C} \subset X \) be defined by \( \mathcal{C} := \{ \chi(t); \chi \in \mathcal{C}_t, t \in [\ell/2, \ell] \} \). Assume that solutions operators \( S_t \) make sense in \( \mathcal{C} \) (i.e., we have uniqueness of solutions starting from points in \( \mathcal{C} \)) and let moreover \( S_t \colon X \mapsto X \) be uniformly (for \( t \in [0, \tau] \)) Lipschitz continuous on \( \mathcal{C} \). Then

(i) the operators \( L_t \colon X_t \mapsto X_t \) are uniformly (for \( t \in [0, \tau] \)) Lipschitz continuous on \( \mathcal{C}_t \),

(ii) the operator \( e \colon X_t \mapsto X \) is Lipschitz continuous on \( \mathcal{C}_t \).

**Proof.** Note that by our assumptions, \( L_t \) are defined on \( \mathcal{C}_t \). Let \( \chi_1, \chi_2 \in \mathcal{C}_t \) and \( u, v \) be solutions on \([0, \tau + \ell]\) such that their restrictions to \([0, \ell]\) are equal to \( \chi_1, \chi_2 \) respectively. To prove (i), it is clearly enough to show that

\[
\int_t^{t+\ell} \|u(s) - v(s)\|_{L^2}^2 \, ds \leq c\|\chi_1 - \chi_2\|_{X_t}^2. \tag{2.7}
\]

Let \( t \in [\ell/2, \tau + \ell/2] \). Then for any \( s \in (0, \ell/2) \) we have

\[
\|u(t + s) - v(t + s)\|_{L^2}^2 = \|S_{t-t/2}\{u(\ell/2 + s)\} - S_{t-t/2}\{v(\ell/2 + s)\}\|_{L^2}^2 \leq c\|u(\ell/2 + s) - v(\ell/2 + s)\|_{L^2}^2.
\]

Integrating over \( s \in (0, \ell/2) \) we conclude that

\[
\int_t^{t+\ell/2} \|u(s) - v(s)\|_{L^2}^2 \, ds \leq c\|\chi_1 - \chi_2\|_{L^2(\ell/2, \ell; X)}^2,
\]

and using this inequality repeatedly with suitably chosen \( t \)'s we conclude (2.7).

To prove (ii), note that \( e(\chi_1) = \chi_1(\ell) = S_{\ell-s}\{\chi_1(s)\} \) for any \( s \in [\ell/2, \ell] \), analogously for \( \chi_2 \). Therefore

\[
\|e(\chi_1) - e(\chi_2)\|_{L^2}^2 = \|S_{\ell-s}\{\chi_1(s)\} - S_{\ell-s}\{\chi_2(s)\}\|_{L^2}^2 \leq c\|\chi_1(s) - \chi_2(s)\|_{X_t}^2
\]

for any \( s \in [\ell/2, \ell] \) and by integrating this inequality over \( s \in [\ell/2, \ell] \) we obtain (ii). □

**Lemma 2.2.** Assume (A3) holds. Let \( \mathcal{C}_t \subset \mathcal{X}_t \) be a set of trajectories such that \( L_t \mathcal{C}_t \subset \mathcal{C}_t \) for all \( t \geq 0 \), and, let \( \{\chi'; \chi \in \mathcal{C}_t \} \) be bounded in space \( L^q(0, \ell; X) \) with some \( q \in (1, \infty] \). Then for all \( \chi \in \mathcal{C}_t \) and \( 0 \leq t_1 \leq t_2 \leq \tau \),
we have

$$\|L_{t_2} \chi - L_{t_1} \chi\|_{X_t} \leq c|t_2 - t_1|^{\beta},$$

where $\beta = 1 - 1/q$ if $q < \infty$ and $\beta = 1$ if $q = \infty$.

**Proof.** Let $\chi \in C_\ell$, $0 \leq t_1 \leq t_2 \leq \tau$, $s \in (0, \ell)$. By virtue of (A3), there exists one and only one $u$, a solution on $[0, \tau + \ell]$, such that $\chi = u_{|[0, \ell]}$. Then

$$\| (L_{t_2} \chi)(s) - (L_{t_1} \chi)(s) \|_X = \| u(t_2 + s) - u(t_1 + s) \|_X$$

$$= \left\| \int_{t_1 + s}^{t_2 + s} u'(\sigma) \, d\sigma \right\|_X$$

$$\leq |t_2 - t_1|^{1 - \frac{1}{q}} \left( \int_{t_1 + s}^{t_2 + s} \| u'(\sigma) \|_X^q \, d\sigma \right)^{\frac{1}{q}}.$$

Due to the positive invariance of $C_\ell$, the last integral is bounded by some constant depending on $\tau$ and $\ell$. Hence

$$\| (L_{t_2} \chi)(s) - (L_{t_1} \chi)(s) \|_X^2 \leq \tilde{c}^2 |t_2 - t_1|^{2(1 - \frac{1}{q})},$$

and integrating this inequality over $s \in [0, \ell]$ completes the proof. □

3. APPLICATION TO AN ABSTRACT PARABOLIC EQUATION

In this section we consider a class of evolutionary problems of the type

$$u'(t) + N(u(t)) + Q(u(t)) = f,$$

$$u(0) = u_0.$$  (3.1)

The choice of assumptions which we are going to impose on nonlinear operators $N$ and $Q$ is twofold. On one hand, they involve a class of *nonlinear* strongly monotone elliptic operators. As mostly the large time asymptotic is considered for the problems where the dissipative operator is linear, we see that (3.1) covers a larger class of problems than those usually studied. On the other hand, this class of problems is not too general so that we can verify the assumptions (A1)–(A10) of the general framework of the method of $\ell$-trajectories in a rather direct way.

Let $X$ be a Hilbert space and $Y$ be a Banach space such that $Y \hookrightarrow X$ and

$$\|u\|_X \leq c_0 \|u\|_Y.$$  (3.2)

We denote by $(\cdot, \cdot)$ the scalar product in $X$ and by $\langle \cdot, \cdot \rangle$ the duality between $Y^*$ and $Y$ so that for $\Phi \in Y^*$, $v \in Y$ we write $\Phi(v) = \langle \Phi, v \rangle$. The spaces $X$
and $Y$ are regarded as subspaces of $Y^*$ by the embedding $u \in X \mapsto u^* \in Y^*$, $\langle u^*, v \rangle := (u, v), \forall v \in Y$.

We assume that $N, Q$ are nonlinear operators from $Y$ to $Y^*$ such that

$$\langle N(u) - N(v), u - v \rangle \geq c_1 ||u - v||_Y^2,$$

$$\langle Q(u), u \rangle \geq -c_4 - \frac{c_1}{8} ||u||_Y^2,$$

(E1) $|\langle N(u) - N(v), \varphi \rangle| \leq c_2 ||u - v||_Y ||\varphi||_Y,$

$$|\langle Q(u) - Q(v), \varphi \rangle| \leq K[u, v] ||u - v||_{1-\beta}^\beta ||\varphi||_Y,$$

where

$$K[u, v] = c_3 (||u||_X + ||v||_X + 1) \gamma (||u||_Y + ||v||_Y + 1) \beta$$

for all $u, v$, and $\varphi \in Y$ with some $\beta \in (0, 1]$ and $\gamma \geq 0$.

Let $f \in Y^*$, $u_0 \in X$, and $T > 0$. A function $u$ of the class

$$u \in L^\infty(0, T; X) \cap L^2(0, T; Y),$$

$$u' \in L^2(0, T; Y^*),$$

(3.3)

will be called a solution to (3.1) on $[0, T]$ – or solution for short – provided that

$$\langle u'(t), \varphi \rangle + \langle N(u(t)), \varphi \rangle + \langle Q(u(t)), \varphi \rangle = \langle f, \varphi \rangle,$$

$$u(0) = u_0,$$

(3.4)

holds for any $\varphi \in Y$ for almost all $t \in (0, T)$.

Observe that (E1) implies, that

$$\langle N(u), u \rangle \geq \frac{3c_1}{4} ||u||_Y^2 - \tilde{c}_1,$$

$$|\langle N(u), \varphi \rangle| \leq \tilde{c}_2 (||u||_Y + 1) ||\varphi||_Y,$$

$$|\langle Q(u), \varphi \rangle| \leq \tilde{c}_3 (||u||_X + 1) \beta + \gamma (||u||_Y + 1) ||\varphi||_Y,$$

(3.5)

which in particular ensures that (3.4) makes sense for the functions (3.3); since such functions also belong to $C([0, T]; X)$ (see [6]), the initial condition is meaningful.

Further, we assume that the solutions to (3.1) are compact in the following sense:

to any sequence $u^n$ of solutions on $[0, T]$ such that $u^n(0)$ is bounded (E2) in $X$ there exists a subsequence converging ($\ast$) weakly in spaces (3.3) to a certain function $u$ such that $u$ is again a solution on $[0, T]$.
Finally, we assume that

\[ (E3) \quad \text{for arbitrary } u_0 \in X, \ f \in Y^*, \text{ and } T > 0 \text{ there exists at least one solution.} \]

The reader familiar with this type of parabolic equation will note that by using a priori estimates obtained below, (E2) could be derived solely on the basis of the assumptions in (E1). Namely, the limiting process in the equation can be done by compact embedding and monotonicity of \( N \). Along the same lines, the existence of a solution in (E3) can be obtained by the Galerkin method. However, to avoid such a lengthy and technical procedure we formulate (E2) and (E3) simply as assumptions.

In the following theorem, we verify that the general scheme of the previous section is applicable to Eq. (3.1).

**Theorem 3.1.** Let (E1)–(E3) hold. Then assumptions (A1)–(A9) are satisfied. Consequently, the dynamical system (3.1) has a global attractor with finite fractal dimension.

If moreover solutions to (3.1) satisfy

\[ u' \in L^0_{loc}(0, \infty; X) \]  

with estimates depending on \( \|u(0)\|_X \), then (A10) is also satisfied and, consequently, the dynamical system (3.1) possesses an exponential attractor.

**Proof.** The general scheme of Section 2 will be applied with \( \rho_1 = \rho_2 = 2 \). Particular assumptions will be verified in corresponding steps.

**Step 1.** The existential part of (A1) is just (E3). To prove the second part (the estimates in the spaces in (3.3)) note that by (3.5) we are justified to take \( \varphi = u(t) \) in (3.4). We use (E1) and the fact that \( 2(u(t), u(t)) = \frac{d}{dt}\|u(t)\|_X^2 \) (see [6]) to obtain that

\[ \frac{1}{2}\frac{d}{dt}\|u(t)\|_X^2 + \frac{c_1}{2}\|u(t)\|_Y^2 \leq \tilde{c}_1 + c_4 + \frac{2}{c_1}\|f\|_{Y^*}^2 \]  

holds for any solution \( u \) for almost all \( t \in (0, T) \). Integrating (3.7) over \( t \in (0, T) \) yields the first line of (3.3); the second line follows then from (3.4), (3.5) by the duality argument.

**Step 2.** Using the embedding (3.2) and denoting \( \tilde{c}_4 := 2\max(\tilde{c}_1 + c_4, \frac{\tilde{c}_1}{2}) \), we derive from (3.7) that

\[ \frac{d}{dt}\|u(t)\|_X^2 + \frac{c_1}{c_0}\|u(t)\|_X^2 \leq \tilde{c}_4(1 + \|f\|_{Y^*}^2). \]
Choosing $R > c_0 \sqrt{c_4(1 + \|f\|^2_{L^2})/c_1}$, we see that $B^0 := B^X(0, R)^X$ satisfies (A2).

**Step 3.** Let $u, v$ be two solutions corresponding to the initial conditions $u_0, v_0$, respectively; $w := u - v$. We take the difference of equations for $u$ and $v$ and choose $w(t)$ as a test function (we suppress the argument $t$ for simplicity). We have

$$\langle \langle w', w \rangle + \langle N(u) - N(v), w \rangle + \langle Q(u) - Q(v), w \rangle = 0. \tag{3.8}$$

From (E1)$_4$, the boundedness of $\|u(\cdot)\|_X$ and $\|v(\cdot)\|_X$, and the Young inequality we have

$$|\langle Q(u) - Q(v), w \rangle| \leq K[u, v]\|w\|_X^\beta \|w\|_Y^{2-\beta} \leq c_5(\|u\|_Y + \|v\|_Y + 1)\|w\|_X^\beta \|w\|_Y^{2-\beta} \leq \frac{c_1}{2}\|w\|_Y^2 + \frac{c_6}{2}(\|u\|_Y + \|v\|_Y + 1)^2 \|w\|_X^2.$$}

Hence (3.8) and (E1)$_1$ imply that

$$\frac{d}{dt}\|w\|_X^2 + c_1\|w\|_Y^2 \leq h\|w\|_X^2, \tag{3.9}$$

where $h = h(t) = c_6(\|u(t)\|_Y + \|v(t)\|_Y + 1)^2$ is clearly integrable due to (3.3). Neglecting the second term on the left, we conclude from the Gronwall inequality that

$$\|w(t)\|_X^2 \leq c_7\|w(s)\|_X^2 \tag{3.10}$$

for any $0 \leq s < t \leq T$ where

$$c_7 = \exp \left( \int_0^T h(\tau) \, d\tau \right). \tag{3.11}$$

In particular, we obtain the uniqueness of solutions, which clearly implies (A3). Moreover, solution operators $S_t$ are defined in $X$.

**Step 4.** By virtue of (A1), $c_7$ in (3.11) admits uniform estimates for $u_0, v_0 \in B^0$ and $0 \leq s \leq t \leq T$. Hence, operators $S_t$ are for $t \in [0, T]$ uniformly Lipschitz continuous on $B^0$ and (A4) follows from Lemma 2.1 with $\mathcal{E}_t = \mathcal{B}_t^0$.

**Step 5.** To verify (A5), it is enough to show that

$$\mathcal{B}_t^0 \subset \mathcal{B}_f^0.$$
Let \( \chi_n \in B^0_\ell \) be a sequence of trajectories such that \( \chi_n \to \chi_0 \) in \( X_\ell \). Then by (E2) \( \chi_0 \) is also a trajectory. It remains to show that \( \chi_0(0) \in B^0 \). By (A2), \( \chi_n(t) \in B^0 \) for all \( t \). Also, coming to a subsequence if necessary, \( \chi_n(t) \to \chi_0(t) \) in \( X \) for almost all \( t \), and since \( B^0 \) is closed, \( \chi_0(t) \in B^0 \) for almost all \( t \in (0, \ell) \). In particular, \( \chi_0(t) \in B^0 \) for points arbitrarily close to 0, which by the continuity of \( \chi_0 \); \( [0, \ell] \to X \) and the closedness of \( B^0 \) implies that \( \chi_0(0) \in B^0 \).

\textbf{Step 6.} We will verify (A6) with \( \tau = \ell \) and \( W_\ell := \{ u \in L^2(0, \ell; Y); \ u' \in L^1(0, \ell; Y^*) \} \). By Lemma 1.6, \( W_\ell \hookrightarrow \hookrightarrow X_\ell \). Let \( u, \ v \) be two solutions starting from \( B^0 \). Take \( s \in (0, \ell) \) and integrate (3.9) over \( \tau \in (s, 2\ell) \). Thus, we obtain

\[
||w(2\ell)||^2_X + c_1 \int_s^{2\ell} ||w(\tau)||^2_Y d\tau \leq \int_s^{2\ell} h(\tau)||w(\tau)||^2_X d\tau + ||w(s)||^2_X. \tag{3.12}
\]

By (3.10), \( ||w(\tau)||^2_Y \leq c_7 ||w(s)||^2_X \), and by uniform estimates of \( \int_s^{2\ell} h(\tau) d\tau \) one can reduce (3.12) to

\[
c_1 \int_s^{2\ell} ||w(\tau)||^2_Y d\tau \leq c_8 ||w(s)||^2_X. \tag{3.13}
\]

Integrating (3.13) over \( s \in (0, \ell) \) then yields the inequality that is equivalent to saying that

\[
||L_\ell \chi_1 - L_\ell \chi_2||_{L^2(0, \ell; Y)} \leq \sqrt{\frac{c_8}{c_1 \ell}} ||\chi_1 - \chi_2||_{X_\ell}. \tag{3.14}
\]

for any \( \chi_1, \chi_2 \in B^0_\ell \). This proves the first part of (A6). The proof will be completed if we show that

\[
||\chi_1 - \chi_2'||_{L^1(0, \ell; Y^*)} \leq c_9 ||\chi_1 - \chi_2||_{L^2(0, \ell; Y)}. \tag{3.15}
\]

This is done by the duality argument. Let \( \psi \) be from the unit ball in \( L^\infty(0, \ell; Y) \). We claim that

\[
\int_0^\ell |\langle w', \psi \rangle| \leq c_{10} ||w||_{L^2(0, \ell; Y)} \tag{3.16}
\]

with \( c_{10} \) independent of \( \psi \). (We suppress the time argument for the sake of simplicity.) From (3.4) it follows that

\[
\int_0^\ell |\langle w', \psi \rangle| \leq \int_0^\ell |\langle N(u) - N(v), \psi \rangle| + \int_0^\ell |\langle R(u) - R(v), \psi \rangle| = I_1 + I_2.
\]
By (E1) and by means of the H"older inequality we estimate

$$I_1 \leq c_2 \int_0^\ell \|w\|_Y \psi \|_Y \leq c_2 \sqrt{\ell} \|w\|_{L^2(0,\ell;Y)}.$$  \hfill (3.17)

Similarly, using (E1), the embedding (3.2), and the boundedness of $u$, $v$ in $L^\infty(0,\ell;X)$ and in $L^2(0,\ell;Y)$, we obtain

$$I_2 \leq \int_0^\ell K[u,v]\|w\|_X^\beta \|w\|_Y^{1-\beta} \|\psi\|_Y \leq c_{11} \int_0^\ell (\|u\|_Y + \|v\|_Y + 1)^\beta \|w\|_Y$$

$$\leq c_{11} \left(\int_0^\ell (\|u\|_Y + \|v\|_Y + 1)^{2\beta}\right)^{1/2} \|w\|_{L^2(0,\ell;Y)},$$  \hfill (3.18)

and the last integral is bounded since $\beta \leq 1$. Both (3.18) and (3.17) are estimates of the type (3.16) which implies – as $\psi$ is arbitrary – (3.15). Hence (A6) is verified.

**Step 7.** Since in Step 4 we have shown uniform continuity of $S_t$ on $B^0$, (A7), (A8), and (A9) follow from Lemma 2.1 with $\mathcal{C}_t = \mathcal{B}^0_t$. Assumption (A10) follows from Lemma 2.2 with $\mathcal{C}_t = \mathcal{B}^1_t$, since by (3.6) the set $\mathcal{B}^1_t$ is bounded in the space $L^q(0,\ell;X)$.

Since all assumptions of the general scheme are fulfilled we use this scheme to conclude the existence of a finite-dimensional global attractor and the existence of an exponential attractor both in $X$ and in $X'$; see Theorems 2.1–2.6.

The proof of Theorem 3.1 is complete. \hfill $\blacksquare$

4. APPLICATIONS IN NONLINEAR FLUID MECHANICS

In this section we deal with the system of equations in $(0,\infty) \times \mathbb{R}^d$ ($d = 2, 3$),

$$\frac{\partial \mathbf{v}}{\partial t} + v_k \frac{\partial \mathbf{v}}{\partial x_k} - \text{div} \mathcal{F}(\mathcal{D}(\mathbf{v})) = -\nabla P + \mathbf{f},$$
$$\text{div} \mathbf{v} = 0,$$
$$\mathbf{v}(0,\cdot) = \mathbf{v}_0(\cdot),$$  \hfill (4.1)

$v, P$ are periodic with period $L$ at each variable $x_i$, $i = 1, \ldots, d$,

where $\mathbf{v} = (v_1, \ldots, v_d)$ and $P$ are the velocity and the pressure; the initial velocity $\mathbf{v}_0$ and the external body force $\mathbf{f}$ are given: $\mathbf{f}$ is time-independent and $\mathbf{v}_0$ is at $x_i$ $L$-periodic and divergence-free.
We suppose that the constitutive relation between the stress tensor $\mathcal{T}$ and the symmetric velocity gradient $D(v)$, defined as $D(v) := \frac{1}{2}((\nabla v + (\nabla v)^T))$, is given through a scalar potential $\Phi : \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}^+_0$; i.e.,

$$\mathcal{T}(\eta) = \frac{\partial \Phi(\eta)}{\partial \eta} \quad \text{for all } \eta \in \mathbb{R}^{d \times d}_{\text{sym}}.$$ 

Furthermore, we assume that for a certain $p > 1$ there are constants $C > 0$ and $\varepsilon > 0$ such that for all $\eta, \xi \in \mathbb{R}^{d \times d}_{\text{sym}},$

$$C^{-1}(\varepsilon + |\xi|)^{p-2}|\eta|^2 \leq \eta^T \cdot \nabla^2 \Phi(\xi) \eta \leq C(\varepsilon + |\xi|)^{p-2}|\eta|^2,$$

$$\Phi(0) = 0, \quad \frac{\partial \Phi(0)}{\partial \eta} = 0. \quad (4.2)$$

Set $\Omega := (0, L)^d$ and define

$$L_0(\Omega) := \left\{ \varphi |_{\Omega}; \varphi \in C^\infty(\mathbb{R}^d), \ \text{div } \varphi = 0, \int_\Omega \varphi \ dx = 0, \ \varphi \ \text{is } L\text{-periodic at } x_i \right\}.$$

Then, we set

$$L^2_{\text{DIV}} := \text{closure of } L_0(\Omega) \text{ in } L^2(\Omega) \text{ norm},$$

$$W^{1,p}_{\text{DIV}} := \text{closure of } L_0(\Omega) \text{ in } W^{1,p}(\Omega) \text{ norm}.$$ 

Finally, we introduce the operators $N(\cdot)$ and $Q(\cdot)$ setting

$$\langle N(v), \varphi \rangle := \int_\Omega \mathcal{T}_{ij}(D(v))D_{ij}(\varphi) \ dx,$$

$$\langle Q(v), \varphi \rangle := \int_\Omega v_k \frac{\partial v_i}{\partial x_k} \varphi_i \ dx.$$ 

We are going to apply the method of $\ell$-trajectories to the following three situations:

1. the case where $d = 2, 3$ and $p \geq \frac{3d+2}{d+2} \ (\geq 2),$
2. the case where $d = 2$ and $p \in (1, 2),$
3. extensions to the Dirichlet problem.

Note that the first case involves the Navier–Stokes equations (NSEs) in two dimensions; however, the three-dimensional NSEs are not included.

The importance of the first case comes from the fact that three-dimensional NSEs are still not well understood, and the question of the global existence of a unique solution is an open matter. It is natural, as was suggested and confirmed by O.A. Ladyzhenskaya (see [8, 9]), to investigate
models with the constitutive law

\[ T(D(v)) = \mu D(v) + v[D(v)]^{p-2} D(v), \quad \mu, v > 0, \]  

(4.3)

which clearly satisfy (4.2) and which generalize the NSEs. It is interesting to ask for what range of parameters \( p \) is the model (4.1) with (4.3) well-posed. It turns out that it happens when the case (1) holds. For such \( p \)'s, (4.1) with (4.3) can be considered as a well-posed alternative (perturbation) of the NSEs. Interestingly, if \( p = 3 \) in (4.3) we obtain the Smagorinski model of turbulence (see [2] for further comments), which is thus involved in our analysis.

The importance of the second case comes from broad applications in various areas (see [18] for an illustrative list). Recall also that all power-law fluids fall into this category.

The beauty of these examples is also situated in the fact that properties of the solution change quite significantly. Even more, the choice of the phase space \( X \) differs from case to case.

Finally, we underline that the earlier methods of evaluating the fractal dimension of limit sets of two-dimensional NSEs seems inapplicable here; in the case of the method of Lyapunov exponents (see [3] and [24]) this is due to missing the regularity of the linearized problem to (4.1). The other scheme developed by Ladyzhenskaya in [10] requires that the leading elliptic operator commutes with orthogonal projectors.

Before starting the analysis of the first case we list several inequalities which are consequences of (4.2); see [15, pp. 198–199] for the proof.

For any \( p \in (1, \infty) \) it holds

\[ \mathcal{F}(\eta) \cdot \eta \geq K_1(\varepsilon + |\eta|)^{p-2}|\eta|^2, \]

|\( \mathcal{F}(\eta) \| \leq K_2(\varepsilon + |\eta|)^{p-2}|\eta|. \]  

(4.4)

If \( p \geq 2 \) then

\[ (\mathcal{F}(\eta) - \mathcal{F}(\tilde{\eta})) \cdot (\eta - \tilde{\eta}) \geq K_3|\eta - \tilde{\eta}|^2 \int_0^1 (\varepsilon + |\lambda\eta + (1 - \lambda)\tilde{\eta}|)^{p-2} d\lambda, \]

\[ |\mathcal{F}(\eta) - \mathcal{F}(\tilde{\eta})| \leq K_4|\eta - \tilde{\eta}| \int_0^1 (\varepsilon + |\lambda\eta + (1 - \lambda)\tilde{\eta}|)^{p-2} d\lambda. \]  

(4.5)

If \( p \in (1, 2) \) then

\[ (\mathcal{F}(\eta) - \mathcal{F}(\tilde{\eta})) \cdot (\eta - \tilde{\eta}) \geq K_5(\varepsilon + |\eta| + |\tilde{\eta}|)^{p-2}|\eta - \tilde{\eta}|^2, \]
\[ |\mathcal{T}(\eta) - \mathcal{T}(\tilde{\eta})| \leq K_6|\eta - \tilde{\eta}|. \]

1. **The Case** \( d = 2, 3 \) and \( p \geq (3d + 2)/(d + 2) \)

Let

\[ \varepsilon > 0, \; f \in L^2_{\text{DIV}}, \; \text{and} \; v_0 \in L^2_{\text{DIV}}. \]  

We say that

\[ v \in L^\infty(0, T; L^2_{\text{DIV}}) \cap L^p(0, T; W^{1,p}_{\text{DIV}}) \]  

with

\[ v' \in L^p(0, T; (W^{1,p}_{\text{DIV}})^*) \]

is a weak solution to (4.1) if \( v(0) = v_0 \) and

\[ \langle v'(t), \varphi \rangle + \langle N(v(t)), \varphi \rangle + \langle Q(v(t)), \varphi \rangle = \langle f, \varphi \rangle \]  

holds for all \( \varphi \in W^{1,p}_{\text{DIV}} \) almost everywhere in \((0, T)\). Note that the initial condition is meaningful since (4.8) and (4.9) imply

\[ v \in C([0, T]; L^2_{\text{DIV}}). \]  

The following results have been obtained before.

**Proposition 4.1.** Assume that (4.7) holds. Then there exists a weak solution to (4.1), (4.2) satisfying (4.8)–(4.11). Moreover, any weak solution satisfies

\[ \frac{d}{dt} \|v\|_2^2 + c_1 \|\nabla v\|_2^2 + c_2 \|\nabla v\|_p^p \leq c_3 \|f\|_2^2, \]  

and its norm in the spaces of (4.8), (4.9) can be estimated by some \( C = C(\|v_0\|_{L^2_{\text{DIV}}}) \).

**Proof.** The existence of a solution can be shown via monotone operator theory, and in fact it holds even for \( \varepsilon = 0 \). From the energy inequality (4.12) obtained from (4.10) by setting \( \varphi := v(t) \) there follow the estimates (4.8) and (4.9) by using the duality argument. See [9] or [13] for details.

**Proposition 4.2.** If \( v_0 \in W^{1,p}_{\text{DIV}} \), then there exists a weak solution \( v \) to (4.1) satisfying

\[ v \in L^\infty(0, T; W^{1,p}_{\text{DIV}}) \cap L^2(0, T; W^{2,2}_{\text{DIV}}) \cap L^p(0, T; W^{1,3,p}), \]
\[ v' \in L^2(0, T; L^2_{\text{DIV}}), \quad (4.14) \]

and \( v \), respectively \( v' \), in these norms are bounded by some \( C = C(\|\nabla v_0\|_p) \).

Moreover, if \( v, u \) are two solutions of (4.1) corresponding to the initial values \( v_0 \) and \( u_0 \) then the difference \( w := v - u \) satisfies

\[
\frac{d}{dt} \|w\|_2^2 + c_4 \int_\Omega |\mathcal{D}(w)|^2 \int_0^1 (\varepsilon + |\lambda \mathcal{D}(u) + (1 - \lambda) \mathcal{D}(v)|)^{p-2} \, d\lambda \, dx \\
+ c_5 \|\nabla w\|_2^2 \leq c_6 \|\nabla u\|_3^{2p-1} \|w\|_2^2. \tag{4.15} \]

**Proof.** See [15, Secs. 5.3 and 5.4].

Finally, we state the consequences of (4.15).

**Proposition 4.3.** Let \( u, v \) be two weak solutions to (3.1) where moreover \( v \in L^p(0, T; W^{1,3p}_{\text{DIV}}) \). Then for \( 0 \leq s \leq t \leq T \),

\[ \|u(t) - v(t)\|_2 \leq c_7 \|u(s) - v(s)\|_2, \tag{4.16} \]

where \( c_7 \) depends on the norm of \( v \) in \( L^p(0, T; W^{1,3p}_{\text{DIV}}) \). Further,

\[ \|u - v\|_{L^2(0, t; W^{1,2}_{\text{DIV}})} \leq c_8 \|u - v\|_{L^2(0, t; L^2_{\text{DIV}})}, \tag{4.17} \]

\[ \|u' - v'\|_{L^1(0, t; W^{3/2}_{\text{DIV}})} \leq c_9 \|u - v\|_{L^2(0, t; L^2_{\text{DIV}})}. \tag{4.18} \]

**Proof.** Since \( 2 \leq p \) implies \( 2p/(2p - 1) \leq p \), (4.16) follows from (4.15) – where we neglected the second and third terms on the left – by the Gronwall inequality.

Now, we choose \( s \in (0, \ell) \) and integrate (4.15) over \( t \in (s, 2\ell) \). Denoting

\[ I(u, v) := c_4 \int_\Omega |\mathcal{D}(w)|^2 \int_0^1 (\varepsilon + |\lambda \mathcal{D}(u) + (1 - \lambda) \mathcal{D}(v)|)^{p-2} \, d\lambda \, dx, \]

we obtain

\[ \|w(2\ell)\|_2^2 + \int_s^{2\ell} I(u, v) + c_5 \int_s^{2\ell} \|\nabla w\|_2^2 \leq c_6 \int_s^{2\ell} \|\nabla u\|_3^{2p-1} \|w\|_2^2 + \|w(s)\|_2^2. \]

Making use of (4.16) in the integral on the right we deduce after several obvious simplifications that

\[ \int_{\ell}^{2\ell} I(u, v) + c_5 \int_{\ell}^{2\ell} \|\nabla w\|_2^2 \leq c_{10} \|w(s)\|_2^2, \]
which after integration over \( s \in (0, \ell) \) yields
\[
\int_\ell^{2\ell} I(u, v) + c_5 \int_\ell^{2\ell} \|\nabla w\|_2^2 \leq \frac{c_{10}}{\ell} \int_0^{\ell} \|w\|_2^2.
\] (4.19)

This in particular implies (4.18). The proof of (4.19) uses the duality argument. Taking \( \psi \) from the unit ball in \( L^\infty(\ell, 2\ell; W_{\text{DIV}}^{3, 2}) = (L^1(\ell, 2\ell; (W_{\text{DIV}}^{3, 2})^*)^* \), we observe that
\[
\left| \int_\ell^{2\ell} \langle w', \psi \rangle \right| \leq \int_\ell^{2\ell} |\langle N(u) - N(v), \psi \rangle| + \int_\ell^{2\ell} |\langle Q(u) - Q(v), \psi \rangle|.
\]

Let us estimate the first term on the right; an estimate for the second term is straightforward. Using (4.5) and the fact that \( \nabla \psi \) is bounded by embedding we come to
\[
c_{11} \int_\ell^{2\ell} \int_\Omega |\mathcal{D}(w)| \left( \int_\Omega (\varepsilon + |\mathcal{D}(u) + (1 - \lambda)\mathcal{D}(v)|)^{p-2} d\lambda \right) dx
\]
\[
\leq c_{12} \left( \int_\ell^{2\ell} I(u, v) \right)^{1/2} \left( \int_\ell^{2\ell} \int_\Omega (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} dx \right)^{1/2}.
\]
The last integral is clearly bounded. Coming to the supremum over all \( \psi \) and using (4.19), we conclude (4.18). \( \blacksquare \)

As we will see below, the assertions of Propositions 4.1–4.3 are sufficient to apply the scheme developed in Section 2 to show the existence of an exponential attractor and the existence of a global attractor with finite fractal dimension.

The result concerning the exponential attractor is new; the second result has been proved earlier in [17]; however, the proof here is different: while in [17] (see also [14]) there is a restriction from above on the length of the trajectories \( \ell \), we do not need any bound here. This flexibility can be useful in obtaining the optimal estimates on the fractal dimension.

**Theorem 4.1.** Assume that (4.7) holds and
\[
p \geq \frac{3d + 2}{d + 2} \quad (d = 2, 3).
\]
Then system (4.1) possesses both a global attractor \( \mathcal{A} \) with a finite fractal dimension and an exponential attractor \( \mathcal{E} \).

**Proof.** We use the general scheme of Section 2 with \( X = L^2_{\text{DIV}}, Y = W^{1,p}_{\text{DIV}}, \) and \( Z = (W^{1,p}_{\text{DIV}})^* \); the parameter \( p_1 \) is set to be \( p \) and \( p_2 = p/(p - 1) \).
In such a way
\[ Y_\ell = \{ u \in L^p(0, \ell; W^{1,p}_{\text{DIV}}), \; u' \in L^{p'}(0, \ell; (W^{1,p}_{\text{DIV}})^*) \} \]
and
\[ X_\ell = L^2(0, \ell; L^2_{\text{DIV}}), \]
where \( \ell > 0 \) is arbitrary.

**Step 1.** Assumption (A1) follows from Proposition 4.1.

**Step 2.** Since (4.12) with the Poincaré inequality \( \lambda \|v\|_2^2 \leq \|\nabla v\|_2^2 \) leads to
\[
\frac{d}{dt} \|v(t)\|_2^2 + c_1 \lambda \|v(t)\|_2^2 \leq c_3 \|f\|_2^2,
\]
we see that \( B^0 := \{ v \in X; \|v\|_2 \leq \rho \} \) satisfies (A2) whenever \( \rho^2 > c_3 \|f\|_2^2 / (c_1 \lambda) \).

**Step 3.** Consider a trajectory \( \chi \in \mathcal{X}_\ell \). Since \( \chi \in L^p(0, \ell; W^{1,p}_{\text{DIV}}) \), there exists \( \tau \in (0, \ell) \) such that \( \chi(\tau) \in W^{1,p}_{\text{DIV}} \). By Proposition 4.2, there exists a solution starting from \( \chi(\tau) \) which belongs to the spaces in (4.13), which is by Proposition 4.3 unique in the class of weak solutions. This implies (A3).

**Step 4.** By the argument from Step 3, the set \( \{ \chi|_{[\ell/2, \ell]} \}; \chi \in B^0 \} \) is bounded in the space \( L^\infty(\ell/2, \ell; W^{1,p}_{\text{DIV}}) \). Hence (A4) follows from Lemma 2.1 with \( C_\ell = B^0 \).

**Step 5.** To verify (A5), it is enough to observe that the stronger assertion \( B^0_X \subseteq B^0 \) holds. Let \( \chi_n \in B^0 \) and \( \chi_n \to \chi_0 \) in \( X_\ell \). But \( \chi_n \) are bounded in spaces (4.8) and (4.9), and by the arguments (based on the compactness of \( Q \) and the monotonicity of \( N \) which are essentially the same as those that lead to the existence of the weak solution we see that \( \chi_0 \) is a weak solution. It remains to verify that \( \chi_0(0) \in B^0 \); we can assume that \( \chi_n(t) \to \chi_0(t) \) in \( X \) for almost every \( t \) and hence \( \chi_0(t) \in B^0 \) for almost every \( t \). But \( \chi_0 \) is continuous and \( B^0 \) is closed.

At this stage we can conclude from Theorem 2.1 that the dynamical system \( (L_\ell, \mathcal{X}_\ell) \) has global attractor \( \mathcal{A}_\ell \).

**Step 6.** Note also that the set \( B^1 \) has due to the smoothing arguments above better regularity; namely, it is bounded in the spaces in (4.13), and the time derivatives of its elements are bounded in \( L^2(0, \ell; L^2_{\text{DIV}}) \). This implies that (A6) is satisfied with
\[ W^1_\ell := \{ u \in L^2(0, \ell; W^{1,2}_{\text{DIV}}); u' \in L^1(0, \ell; (W^{2,2}_{\text{DIV}})^*) \} \]
and \(\tau = \ell\), as follows from (4.17) and (4.18) in Proposition 4.3. (Note that \(W_\ell \leftrightarrow X_\ell\) by Lemma 1.6.) By Theorem 2.2, \(W_\ell \subset X_\ell\) has a finite fractal dimension.

**Step 7.** Assumptions (A7)–(A9) follow from Lemma 2.1 with \(C_\ell = B_0\); cf. Steps 3 and 4 above. Finally, (A10) with \(b = 1/2\) follows from Lemma 2.2 with \(C_\ell = B_1\) since this set is bounded in \(L^2(0, \ell; L^2_{\text{DIV}})\).

Now, using Theorem 2.5 we conclude that the dynamical system \((L_\ell, X_\ell)\) has an exponential attractor. Finally, using Theorems 2.3, 2.4, and 2.6 we obtain a global attractor with finite fractal dimension and an exponential attractor also for the dynamics in the space \(X = L^2_{\text{DIV}}\).

(2) **Case** \(d = 2\) and \(p \in (1, 2)\)

In this part, we suppose that

\[
\varepsilon \geq 0, \ f \in L^p(\Omega), \ \text{and} \ v_0 \in W^{1,2}_{\text{DIV}}.
\]

We define

\[
I_p(z) := \int_\Omega (\varepsilon + |\nabla(v)|)^{p-2} |\nabla(z)|^2 \, dx
\]

and recall the assertions proved in [15].

**Proposition 4.4.** Let (4.20) be satisfied. Then there exists

\[
v \in L^\infty(0, T; W^{1,2}_{\text{DIV}}) \cap L^2(0, T; W^{2,p}(\Omega))
\]

with

\[
v' \in L^2(0, T; L^2_{\text{DIV}})
\]

such that the weak formulation

\[
\langle v'(t), \phi \rangle + \langle N(v(t)), \phi \rangle + \langle Q(v(t)), \phi \rangle = \langle f, \phi \rangle
\]

holds for every \(\phi \in W^{1,2}_{\text{DIV}}\) and for almost all \(t \in (0, T)\).

In addition, Galerkin approximations \(v^N\) that converge weakly to \(v\) in spaces (4.21) and (4.22) fulfill

\[
\frac{d}{dt} \|\nabla v^N(t)\|_2^2 + c_1 I_p(\nabla v^N(t)) \leq c_2 \|f\|_{L^p}^2,
\]

\[
\int_0^T \|v^N(t)\|_2^2 \, dt \leq C = C(||v^N(0)||_{1,2}, ||f||_{L^p}).
\]

**Proof.** See Theorem 5.4.21 in [15] and its proof.

Next, we formulate immediate consequences of the previous proposition.
Proposition 4.5. Let (4.20) hold. Then solution \( v \), introduced in Proposition 4.4, satisfies
\[
v \in L^2(0, T; W^{2, p}(\Omega)) \quad \text{with} \quad \alpha = \frac{4}{4 - p}.
\] (4.25)

In addition to (4.24), Galerkin approximations \( v^N \) satisfy
\[
\frac{d}{dt} \| \nabla v^N \|^2_2 + c_2 \| \Delta v^N \|^2_2 \| e + |\nabla v^N| \|^p - 2 \leq c_3 \| f \|^2_2 \| e \| + \| \nabla v^N \|^2_2 - p
\] (4.26)
and (for all \( \delta \in (0, T) \))
\[
\sup_{t \in [\delta, T]} \| (v^N)'(t) \|^2_2 + \int_\delta^T \| \nabla (v^N)'(t) \|^2_2 \, dt \leq c_4,
\] (4.27)
where \( c_4 = c_4(\delta, T, \| \nabla v_0 \|_2, \| f \|_{p'}) \).

Moreover, if \( u, v \in L^\infty(0, T; W^{1, 2}_{\text{DIV}}) \) are two solutions of (4.1) corresponding to the initial values \( u_0, v_0 \) respectively, then the difference \( w := u - v \) satisfies
\[
\frac{d}{dt} \| w \|^2_2 + c_5 \| \nabla w \|^2_2 \leq c_6 \| w \|^2_2,
\] (4.28)
where \( c_5 \) depends on \( \sup_{t \in (0, T)} (\| \nabla u(t) \|_2 + \| \nabla v(t) \|_2) \) and \( c_6 \) depends on \( \sup_{t \in (0, T)} \| \nabla u(t) \|_2 \).

Proof. It is based on two inequalities,
\[
I_p(z) \geq c_7 \| \nabla z \|^2_2 \| e + |\nabla v| \|^p - 2,
\] (4.29)
\[
\int_\Omega (e + |\mathcal{D}(u)| + |\mathcal{D}(v)|)^{p-2} |\mathcal{D}(w)|^2 \, dx \geq c_8 \| w \|^2_2 \| e \| + |\nabla u| + |\nabla v| \|^2_2 - 2,
\] (4.30)
which are proved by means of Hölder and Korn-like inequalities (see also Lemma 5.3.24 in [15]). Since \( v \) satisfies (4.21) we observe that (4.29) with \( z = \nabla v \) implies (4.25). Similarly, as \( v^N \) satisfies (4.24) we see that (4.29) with \( z = \nabla v^N \) leads to (4.26).

Next, we take the time derivative of the Galerkin system, multiply its \( r \)th equation (see [15, p. 207, Eq. (2.21)]) by \( \frac{d}{dt} c_r^N \) and sum it over \( r \) from 1 to \( N \). Thus we obtain
\[
\frac{1}{2} \frac{d}{dt} \| (v^N)' \|^2_2 + c_9 \int_\Omega (e + |\mathcal{D}(v^N)|)^{p-2} |\nabla \frac{d v^N}{dt}|^2 \, dx \leq \int_\Omega |\nabla v^N| \| (v^N)' \|^2 \, dx,
\]
which yields (with the help of (4.24)$_1$ and (4.29) with $z = (v^N)'$)

$$
\frac{1}{2} \frac{d}{dt} \| (v^N)' \|_2^2 + c_{10} \| \nabla (v^N)' \|_2^2 \leq \| \nabla v^N \|_2 \| (v^N)' \|_4^2
\leq c_{11} \| \nabla v^N \|_2 \| (v^N)' \|_2 \frac{2^{p-1}}{p} \| \nabla (v^N)' \|_2^\frac{2}{p}.
$$

Young’s inequality and (4.24)$_1$ then give

$$
\frac{d}{dt} \| (v^N)' \|_2^2 + c_{12} \| \nabla (v^N)' \|_2^2 \leq c_{13} \| (v^N)' \|_2^2.
$$

Multiplying the last inequality by a smooth cut-off function vanishing at $t = 0$, integrating it with respect to time between 0 and $t \in (0, T)$, and using (4.24)$_2$ lead then to (4.27).

Finally, using (4.23) and (4.6) we see that the difference $w$ satisfies

$$
\frac{1}{2} \frac{d}{dt} \| w \|_2^2 + K_5 \int_{\Omega} (\varepsilon + |\mathcal{D}(u)| + |\mathcal{D}(v)|)^{2/p} |\mathcal{D}(w)|^2 \, dx \leq \int_{\Omega} |w|^2 |\nabla u| \, dx.
$$

Since $u, v \in L^\infty(0, T; W^{1,2}_{\text{DIV}})$, and the right-hand side of (4.31) is bounded by

$$
\| \nabla u \|_2 \| w \|_4^2 \leq c_{14} \| \nabla u \|_2 \| w \|_2^{2/p} \| \nabla w \|_2^\frac{2}{p},
$$

we observe that the last two inequalities together with (4.30) lead to

$$
\frac{1}{2} \frac{d}{dt} \| w \|_2^2 + c_{15} \| \nabla w \|_2^2 \leq c_{16} \| w \|_2^{2/p} \| \nabla w \|_2^\frac{2}{p}.
$$

The Young inequality completes the proof of (4.28). \[\square\]

Now, we are ready to employ our general scheme to obtain the following theorem.

**Theorem 4.2.** Let (4.20) hold and $p \in (1, 2)$, $d = 2$. Then the dynamical system (4.1) possesses a global attractor with finite fractal dimension and an exponential attractor.

**Proof.** First, we set $X := W^{1,2}_{\text{DIV}}, Y := W^{2,2}(\Omega)$, where $\alpha = \frac{4}{4-p}$ and $p_1 = p_2 = 2$. Hence we have

$$
X_\ell = L^2(0, \ell; X),
Y_\ell = \{v \in L^2(0, \ell; Y); \; v' \in L^2(0, \ell; L^2_{\text{DIV}})\}.
$$
Also, we will work with the auxiliary spaces $X^0 := L^2_{\text{DIV}}$, $Y^0 := W^{1,\infty}(\Omega)$, and
\[ X_t^0 = L^2(0, \ell; X^0), \]  
\[ Y_t^0 = \{ v \in L^2(0, \ell; Y^0); \ v' \in L^2(0, \ell; (Y^0)^*) \}. \]

Note that $W^{2,\infty}_{\text{DIV}} \hookrightarrow W^{1,2/(2-p)}_{\text{DIV}}$. As $\frac{4}{2-p} > 2$ for $p \in (1, 2)$, one has $Y_t \hookrightarrow X_t$ and $Y_t^0 \hookrightarrow X_t^0$.

**Step 1.** By Propositions 4.4 and 4.5, for every $v_0 \in W^{1,2}_{\text{DIV}}$ and for all $T > 0$ one can construct a uniquely defined solution satisfying (4.21)–(4.23).

It follows directly from (4.24) and (4.29) that
\[ \int_0^T \|\nabla^2 v(t)\|^2_x \, dt + \int_0^T \|v'(t)\|^2_x \, dt \leq C = C(T, \|\nabla v_0\|_2, \|f\|_{p}). \quad (4.31) \]

Finally, using (4.21)–(4.23), it is not difficult to observe that
\[ v \in C([0, T]; X_{w}). \quad (4.32) \]

Thus, the assumption (A1) is verified.

**Step 2.** We want to construct an absorbing, invariant set $B^0$. We will work with Galerkin approximations first: the embedding inequality $\|\nabla v\|_2 \leq C\|\nabla^2 v\|_2$ and some elementary inequalities enable us to rewrite (4.26) into the form
\[ \frac{d}{dt} \|\nabla v^N\|_2^2 + \kappa_1 \|\nabla v^N\|_2^p - \kappa_2 (\|f\|_{2', \varepsilon}) \leq 0, \quad (4.33) \]

where $\kappa_1, \kappa_2 > 0$ are independent of $N$. Let $R$ be large enough so that
\[ \kappa_1 R^p - \kappa_2 (\|f\|_{2', \varepsilon}) > 0. \quad (4.34) \]

Now, it is elementary to see that a closed ball $B^0 := B^X(0, R)$ with $R$ satisfying (4.34) is due to (4.33) uniformly (also w.r.t. $N$) absorbing and positively invariant in the sense of (A2) for $v^N$.

We claim that such a set is also invariant and positively absorbing for the solutions themselves. Let us check the invariance (the absorbing property is verified analogously). If $v_0 \in B^0$ then the initial conditions $v^N_0 := P^N v_0$ for Galerkin approximations $v^N$ (where $P^N$ are orthogonal projectors) belong also to $B^0$. Hence $v^N$ remain pointwise in $B^0$. We can assume that $v^N(t) \to v(t)$ in $X$ for almost all $t$, where $v$ is the unique solution with initial condition $v_0$. Hence $v(t)$ belongs to $B^0$ for almost all $t$. But $v$ is weakly continuous and $B^0$ is closed and convex, hence weakly closed, which finishes the proof.
Step 3. Due to the uniqueness of solutions, (A3) is satisfied and, moreover, we have also the solution operators $S_t$ defined in $X$.

Step 4. By (4.28) and the Gronwall inequality, the operators $S_t: X^0 \rightarrow X^0$ are uniformly Lipschitz continuous for $t \in [0, T]$ on the set $B^0_t$. Hence, by Lemma 2.1 with $C = B^0_t$ where we replace $X$ with $X^0$, operators $L_t$ are uniformly Lipschitz continuous for $t \in [0, T]$ on $B^0_t$ with respect to $X^0$ topology. So if $\chi_n \in B^0_t$ and $\chi_n \rightarrow \chi$ in $X_t$, then $L_t \chi_n \rightarrow L_t \chi$ in $X^0$. But by (4.21) and (4.22) the sequence $L_t \chi_n$ is compact in $X_t$, hence $L_t \chi_n \rightarrow L_t \chi$ in $X_t$ and (A4) holds.

Step 5. Defining for some fixed $\tau > 0$,

$$B^1_t := \overline{L_t B^0_t}^X,$$  \hfill (4.35)

we see that the definition (4.35) differs from (2.3) in that the closure is taken with respect to $X^0$. Since $L_t B^0_t$ is bounded in the spaces (4.21), (4.22), and (4.25), so is $B^1_t$; and this regularity also enables us to limit passage in the equation, hence $B^1_t \subseteq X_t$. Moreover, if $\chi \in B^1_t$ then $\chi(t) \in B^0$ for almost all $t$, and since $B^0$ is weakly closed and $\chi$ is weakly continuous, we have $\chi(0) \in B^0$; i.e., $B^1_t \subseteq B^0_t$. Finally, $B^1_t$ is compact both in $X_t$ and $X^0_t$ since it is closed in the weaker of both topologies and totally bounded by regularity.

Hence we can use Theorem 2.1 to obtain the global attractor $\mathcal{A}_t$ to $(L_t, X_t)$.

Step 6. We will now show that $d^X_f(\mathcal{A}_t)$ is finite. First of all, we claim that for $\chi_1, \chi_2 \in B^1_t$,

$$||L_t \chi_1 - L_t \chi_2||_{L^2(0,t; W^{1,p}(\Omega))} \leq c_1 ||\chi_1 - \chi_2||_{X^0},$$  \hfill (4.36)

$$||L_t \chi_1'||_{L^1(0,t; (W^{3,2}_{\text{div}})^*)} - (L_t \chi_2)'||_{L^1(0,t; (W^{3,2}_{\text{div}})^*)} \leq c_2 ||\chi_1 - \chi_2||_{X^0}.$$  \hfill (4.37)

Now, (4.36) follows from (4.28) and (4.37) follows from (4.36) and (4.6) by the duality argument, the proof being similar to a proof of (4.17),(4.18) in Proposition 4.3 above.

It then follows from Lemma 1.3 with $X = X_t$, $\mathcal{L} = L_t$, $\mathcal{C} = \mathcal{A}_t$, and

$$\mathcal{V} := \{ \chi \in L^2(0, t; W^{1,p}(\Omega)); \chi' \in L^1(0, t; (W^{3,2}_{\text{div}})^*) \}$$  \hfill (4.38)

(note that $\mathcal{V} \preceq \mathcal{V} \preceq X_t$ by Lemma 1.6) that $d^X_f(\mathcal{A}_t) < \infty$. Due to the interpolation inequality,

$$||u||_{1,2} \leq c ||u||_{2}^{p/2} ||u||_{2,0}^{2-p/2} \quad \text{with} \quad \alpha = \frac{4}{4-p},$$
(4.25), and the Hölder inequality we have on $\mathcal{B}_1^0$ that
\[
\int_0^\ell \|X_1(s) - X_2(s)\|_{1,2}^2 ds \leq C \left( \int_0^\ell \|X_1(s) - X_2(s)\|^2_{2} ds \right)^{\frac{p}{p+2}}.
\] (4.39)

In other words, the identity mapping $I: X^0_\ell \mapsto X_\ell$ is $\alpha$-Hölder continuous on $\mathcal{B}_1^0$ with $\alpha = p/(p+2)$. Thus, by Lemma 1.2,
\[
d_{\mathcal{X}_\ell}^X (\mathcal{A}_\ell) \leq \left( 1 + \frac{2}{p} \right) d_{\mathcal{X}}^X (\mathcal{A}_\ell) < \infty.
\] (4.40)

**Step 7.** By Lemma 2.1 with $\mathcal{C}_\ell = \mathcal{B}_1^0$ and with $X$ replaced with $X^0_\ell$ one obtains that the mapping $e: X^0_\ell \mapsto X^0_\ell$ is Lipschitz continuous on $\mathcal{B}_1^0$. We need, however, the Hölder continuity from $X^0_\ell$ into $X^\ell$. We compute
\[
\|\nabla u(t)\|_{2}^2 = \int_{\Omega} |\nabla u(x, t)|^2 dx = \frac{1}{\ell} \int_0^\ell \int_{\Omega} \frac{d}{ds} |\nabla u(x, s)|^2 dx ds
\]
\[
\leq \frac{1}{\ell} \int_0^\ell \|\nabla u(s)\|_{X^\ell}^2 ds + \frac{c}{\ell} \int_0^\ell \int_{\Omega} |\nabla u(x, s)||\nabla u'(x, s)| dx ds
\]
\[
\leq \frac{1}{\ell} \|u\|_{X^\ell}^2 + \frac{c}{\ell} \int_0^\ell \|\nabla u'(s)\|^2_{X^\ell} ds
\]
\[
\leq \frac{1}{\ell} \|u\|^2_{X^\ell} + \frac{c}{\ell} \left( \int_0^\ell \|\nabla u(s)\|^4_{X^\ell} ds \right)^{\frac{1}{2}}
\]
\[
\times \left( \int_0^\ell \|\nabla^2 u(s)\|^2_{X^\ell} ds \right)^{\frac{2-p}{2p}} \left( \int_0^\ell \|\nabla u(s)\|^2_{X^\ell} ds \right)^{\frac{p-1}{p}}.
\]

With the help of (4.25) and (4.27) this implies that
\[
\|u(t)\|_{1,2} \leq C \|u\|^p_{X^\ell},
\]
and putting $u = X_1 - X_2$ we see that (A7) and (A8) hold. In particular, we obtain the existence of a global attractor with a finite fractal dimension in the space $X = W^{1,2}_{\text{DIV}}$.

**Step 8.** For the sake of the construction of an exponential attractor, we will start with the topology $X^0_\ell$ instead of $X^\ell$. Then (A9) holds by Lemma 2.1
with $C = B^1_\ell (X \text{ replaced with } X^0)$, and (4.27) gives (A10) with $\alpha = 1/2$ by Lemma 2.2. Hence by Theorem 2.5 we obtain $\mathcal{E}_\ell$, an exponential attractor for the dynamical system $(L_t, B^1_\ell)$ with all metric notions with respect to $X^0_t$. However, since the identity mapping $I: X^0_t \mapsto X^0_t$ is on $B^1_\ell$ Hölder continuous, it is straightforward to see that $\mathcal{E}_\ell$ is also an exponential attractor with respect to the metric of $X_t$. From Theorem 2.6 then follows the existence of an exponential attractor also in the space $X = W^{1,2}_\text{DIV}$.

(3) Extensions to the Dirichlet Problem

The final part of this section is aimed at extending the results presented above by considering instead of the space-periodicity of $v$ and $P$ (see (4.1)₄) the Dirichlet boundary condition

$$v = 0 \quad \text{at } (0, \infty) \times \partial \Omega,$$

where $\Omega$ is a bounded, open set in $\mathbb{R}^d$, $d = 2, 3$. For $p \geq 2$, we can formulate the following two theorems.

**Theorem 4.3.** Let $\Omega$ have the Lipschitz boundary $\partial \Omega$. Let (4.7) hold and let

$$p \geq \frac{d + 2}{2}.$$  

Then the dynamical system (4.1)–(4.2) with (4.41) possesses a global attractor with finite fractal dimension and an exponential attractor.

**Proof.** The proof uses three ingredients: First, the validity of Proposition 4.1 for the analyzed system (4.1) with (4.41); second, the uniqueness theorem for weak solutions if (4.42) holds; and third, the $L^\infty (0, T; W^{1,p}_\text{DIV}^1)$ regularity proved by “testing the equation by time derivative $v'$.” Both this regularity and the uniqueness require that $v \in L^p (0, T; W^{1,p}_\text{DIV}^1)$ only. See [12, 13, 14, 15] for details.

The rest of the proof is analogous to the proof of Theorem 4.1. In fact, because of the uniqueness for initial data in $X = L^2_\text{DIV}$, some steps of the proof are simpler. □

**Theorem 4.4.** Let $\Omega \subset \mathbb{R}^3$ have a $C^2$-boundary and let (4.7) hold. Assume that

$$p \geq \frac{9}{4} \quad \text{and} \quad d = 3.$$  

Then the assertion of Theorem 4.3 holds.
**Proof.** The steps of the proof are completely analogous to the steps of the proof of Theorem 4.1. The only difference comes from the fact that regularity results of type (4.13) are proved for the Dirichlet problem only if (4.43) holds; see [16].

Let us finally remark that nothing is known if \( p \in (1, 2), \ d = 2, \) and (4.1) is considered with Dirichlet boundary conditions (4.41). This is connected with the fact that no global (this means up to the boundary) regularity result of the type (4.13) has been proved for (4.1) with (4.41) for \( p < 2, \) to date.

5. CONCLUSION AND PERSPECTIVES

The origin of the method of \( \ell \)-trajectories dates back to 1992 when this approach, presented in [14], was developed during a study of the fluid mechanics system of the equations (4.1) with (4.3) in three dimensions: while the existence of a global attractor \( \mathcal{A} \) in the original space \( X = L^2_{\text{DIV}} \) for \( p \geq 5/2 \) has been proved by a standard method, the question of its finite-dimensionality and its existence for lower \( p \)'s was left open. Instead, for \( p \geq 11/5 \) the existence of an attractor \( \mathcal{A}_\ell \) in the set of short trajectories (the length \( \ell \) of the trajectories had to satisfy a certain smallness assumption at that time), together with its finite fractal dimension in \( X_\ell = L^2(0, \ell; X) \), was proved.

Later on, we observed in [17] that by considering the set of the end points of all short trajectories starting in \( \mathcal{A}_\ell \) one may obtain the finite-dimensional global attractor \( \mathcal{A} \) in the original phase space \( X \).

The present paper comes up with several novelties. First of all, we present a list of 10 assumptions (A1)–(A10) on the properties of the general dynamical system: when they are satisfied the existence of a global attractor with a finite fractal dimension and the existence of an exponential attractor to the dynamical system (0.1) are guaranteed. Second, we show how to verify these 10 assumptions for a class of nonlinear evolutionary problems if the dissipative term is a nonlinear monotone operator. Third, we return to the nonlinear model of fluid mechanics (4.1), (4.2) and show that

(i) the smallness condition on \( \ell \), which seems essential in [14] and [17], can be removed\(^8\);

\(^8\) In this paper \( \ell \) can be arbitrary, which gives a space for optimal quantitative analysis (explicit estimates of \( d_f(\mathcal{A}) \)) and for possible extensions to discrete dynamical systems. However, for different types of problems the method seems to work only when \( \ell \) is sufficiently large; cf. [19] and [20]. As the extreme case one can consider Sell’s study of \( \infty \)-trajectories of the three-dimensional NSEs; see [21]. Note that due to missing uniqueness, (A3) can only be satisfied if \( \ell = \infty \).
(ii) the crucial assumption (A6) not only leads to the finite fractal dimension of \( \mathcal{A} \), but also it is key to proving the existence of an exponential attractor for all \( p \geq \frac{3d+2}{d+2} \) for the space-periodic problem;

(iii) the results of (i) and (ii) hold also for the Dirichlet boundary conditions if \( p \geq 2 \) in two dimensions and \( p \geq \frac{9}{4} \) in three dimensions;

(iv) the approach is applicable to the case \( p \in (1, 2) \) in two dimensions. While the existence of an global attractor was proved earlier by Serégin in [22], we present here an alternative and perhaps simpler technique. Furthermore, the results concerning the finite fractal dimension and the exponential attractor are new.

We believe, and current studies (see [19] and [20]) support our optimism, that the method of \( \ell \)-trajectories allows us to resolve several large-time behavior problems concerning evolutionary equations in various areas, such as wave equations with nonlinear dissipation, equations with delay or memory terms, equations with nonlinear monotone leading operators, and Boussinesq approximations.

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