Introduction to Stochastic Analysis 1

1. Find the law of $5W_1 - W_3 + W_7$.

2. For which parameters a and b random variables $aW_1 - W_2$ and $W_3 + bW_5$ are independent?

3. Find the law of the random vector $(W_{t_1}, W_{t_2}, \ldots, W_{t_n})$ for $0 < t_1 < t_2 < \ldots < t_n$.

4. Show that $\lim_{t\to\infty} \frac{W_t}{t} = 0$ a.s.

5. Prove that the following transformation of the Wiener process are also Wiener
a) $X_t = -W_t$ (reflection)
b) $Y_t = c^{-1/2}W_{ct}, c > 0$ (scaling of time)
c) $Z_t = tW_{1/t}$ for $t > 0$ and $Z_0 = 0$ (time inversion)
d) $U_t = W_{T+t} - W_T, T \geq 0$
e) $V_t = W_t$ for $t \leq T$, $V_t = 2W_T - W_t$ for $t > T$, where $T \geq 0$.

6. Show that with probability 1 trajectories of Wiener process are unbounded.

7. Show that with probability 1 trajectories of Wiener process are not uniformly continuous on $\mathbb{R}_+$.

8. Let $\pi_n = \{t_0^{(n)}, t_1^{(n)}, \ldots, t_{k_n}^{(n)}\}$, where $a = t_0^{(n)} < t_1^{(n)} < \ldots < t_{k_n}^{(n)} = b$ is a sequence of partition of the segment $[a, b]$ and let $\|\pi_n\| := \max_k |t_k^{(n)} - t_{k-1}^{(n)}|$ denote the diameter of π_n. Prove that

$$S_n := \sum_{k=1}^{k_n} |W_{t_k^{(n)}} - W_{t_{k-1}^{(n)}}|^2 \to b - a \quad \text{in } L^2(\Omega, \mathcal{F}, P), \text{ when } n \to \infty$$

if $\|\pi_n\| \to 0$ and $S_n \to b - a$ a.s. if $\sum_n \|\pi_n\| < \infty$.

9. Show that with probability one trajectories of the Wiener process have unbounded variation on each nondegenerate interval.

10. Prove that with probability 1 trajectories of the Wiener process are nowhere differentiable on $[0, \infty)$.
Hints:
a) If a function f is differentiable at some point in $[0, 1]$ then there exists $M < \infty$ and for sufficiently large n there exists $0 \leq j \leq n - 3$ such that $|f((j + 1)/n) - f(j/n)| \leq M/n$, $|f((j + 2)/n) - f((j + 1)/n)| \leq M/n$ and $|f((j + 3)/n) - f((j + 2)/n)| \leq M/n$.
b) Show that $\Pr(|W_{(j+1)/n} - W_{j/n}| \leq M/n, |W_{(j+2)/n} - W_{(j+1)/n}| \leq M/n, |W_{(j+3)/n} - W_{(j+2)/n}| \leq M/n) \leq CMn^{-3/2}$ and prove that this implies nowhere differentiability of Wiener process trajectories on $[0, 1]$.

Introduction to Stochastic Analysis 2

1. Prove that if a set $A \in \mathcal{B}(\mathbb{R}^T)$ then there exists a countable set $T_0 \subset T$ such that if $x, y \in \mathbb{R}^T$ and $x(t) = y(t)$ for $t \in T_0$ then $x \in A \Leftrightarrow y \in A$.

2. Let $T = [a, b]$ $a < t_0 < b$, prove, that the following sets do not belong to $\mathcal{B}(\mathbb{R}^T)$.
i) $A_1 = \{x \in \mathbb{R}^T : \sup_{t\in[a,b]} |x_t| \leq 1\}$;
ii) $A_2 = \{x \in \mathbb{R}^T : t \to x_t$ cige na [a,b] $\}$;
iii) $A_3 = \{x \in \mathbb{R}^T : \lim_{t\to t_0} x_t = 0\}$;
iv) $A_4 = \{x \in \mathbb{R}^T : t \to x_t$ cige w $t_0\}$.
Prove measurability of these sets under the assumption of continuity (right continuity) of trajectories, i.e. prove that all these sets after intersecting them with $C(T)$ ($RC(T)$ respectively) belong to $\mathcal{B}(\mathbb{R}^T) \cap C(T)$ ($\mathcal{B}(\mathbb{R}^T) \cap RC(T)$ resp.).

3. Let $T = [a, b]$. Prove that $\mathcal{F} = \{A \cap C(T) : A \in \mathcal{B}(\mathbb{R}^T)\}$ is the Borel σ-field (for the sup metric) on $C(T)$.

4. Prove that there exists a process $(X_t)_{t\geq 0}$ with independent increments, starting from 0 and such that $X_t - X_s$ has the Cauchy distribution with parameter $t - s$ (such a process is called a Cauchy process or a 1-stable process).

5. A process X is a modification of the Wiener process. Which of the following properties are satisfied by X:
a) independence of increments,
b) stationarity of increments,
c) continuity of trajectories,
d) $\lim_{t\to\infty} \frac{X_t}{t} = 0$ a.s.,
e) $\lim_{t\to\infty} \frac{X_t}{t} = 0$ in probability?

6. Consider the following properties of processes:
a) continuity of trajectories;
b) stochastic continuity (i.e. $X_t \xrightarrow{\mathbb{P}} X_s$ for $t \to s$);
c) continuity in L_p (i.e. $\mathbb{E}|X_t - X_s|^p \to 0$ for $t \to s$).
What implications between the following properties are satisfied?