OPERATOR $\ell_p \to \ell_q$ NORMS OF RANDOM MATRICES WITH IID ENTRIES

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ABSTRACT. We prove that for every $p, q \in [1, \infty]$ and every random matrix $X = (X_{i,j})_{i \le m, j \le n}$ with iid centered entries satisfying the α-regularity assumption $\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_\rho$ for every $\rho \ge 1$, the expectation of the operator norm of X from ℓ_p^n to ℓ_q^m is comparable, up to a constant depending only on α, to

$$m^{1/q} \sup_{t \in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_{q \wedge \operatorname{Log} m} + n^{1/p^*} \sup_{s \in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_{i,1} \Big\|_{p^* \wedge \operatorname{Log} n}.$$

We give more explicit formulas, expressed as exact functions of p, q, m, and n, for the two-sided bounds of the operator norms in the case when the entries $X_{i,j}$ are: Gaussian, Weibullian, log-concave tailed, and log-convex tailed. In the range $1 \le q \le 2 \le p$ we provide two-sided bounds under the weaker regularity assumption $(\mathbb{E}X_{1,1}^4)^{1/4} \le \alpha (\mathbb{E}X_{1,1}^2)^{1/2}$.

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1. Introduction and main results

Let $X = (X_{i,j})_{i \le m, j \le n}$ be an $m \times n$ random matrix with iid entries. Seginer proved in [13] that if the entries $X_{i,j}$ are symmetric, then the expectation of the spectral norm of X is of the same order as the expectation of the maximum Euclidean norm of rows and columns of X. In this article we address a natural question: do there exist similar formulas for operator norms of X from ℓ_p^n to ℓ_q^m, where $p, q \in [1, \infty]$? Recall that if $A = (A_{i,j})_{i \le m, j \le n}$ is an $m \times n$ matrix, then

$$\|A\|_{\ell_p^n \to \ell_q^m} = \sup_{t \in B_p^n} \|At\|_q = \sup_{t \in B_p^n, s \in B_{q^*}^m} s^T A t = \sup_{t \in B_p^n, s \in B_{q^*}^m} \sum_{i \le m, j \le n} A_{i,j} s_i t_j$$

denotes its operator norm from ℓ_p^n to ℓ_q^m; by ρ^* we denote the Hölder conjugate of $\rho \in [1, \infty]$, i.e., the unique element of $[1, \infty]$ satisfying $\frac{1}{\rho} + \frac{1}{\rho^*} = 1$, and by $\|x\|_\rho = (\sum_i |x_i|^\rho)^{1/\rho}$ we denote the ℓ_ρ-norm of a vector x (a similar notation, $\|Z\|_\rho = (\mathbb{E}|Z|^\rho)^{1/\rho}$ is used for the L_ρ-norm of a random variable Z). Whenever we write $p \ge p_1$ or $p \le p_2$ we mean $p \in [p_1, \infty]$ or $p \in [1, p_2]$, respectively. If $p = 2 = q$, then $\|A\|_{\ell_p^n \to \ell_q^m}$ is the spectral norm of A, so the case $p = 2 = q$ corresponds to the aforementioned result by Seginer.

Let us note that bounds for $\mathbb{E}\|X\|_{\ell_p^n \to \ell_q^m}$ yield both tail bounds for $\|X\|_{\ell_p^n \to \ell_q^m}$ and bounds for $(\mathbb{E}\|X\|_{\ell_p^n \to \ell_q^m}^\rho)^{1/\rho}$ for every $\rho \ge 1$, provided that the entries of X satisfy a mild regularity assumption; see [1, Proposition 1.16] for more details. Thus, estimating the expectation of the operator norm automatically gives us more information about the behaviour of the operator norm.

Not much is known about the non-asymptotic behaviour of the operator norms of iid random matrices if $(p, q) \ne (2, 2)$; see the introduction to article [11] for an overview of the state of the art. In the case when $X_{i,j} = g_{i,j}$ are iid standard $\mathcal{N}(0, 1)$ random variables one may use the

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classical Chevet's inequality [4] to derive the following two-sided bounds (see [11] for a detailed calculation; compare also with [7, Remark 1.5]):

$$\mathbb{E}\big\|(g_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \sim \begin{cases} m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\le 2,\\ \sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*}m^{1/q-1/2}+m^{1/q}, & q\le 2\le p^*,\\ n^{1/p^*}+\sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}n^{1/p^*-1/2}, & p^*\le 2\le q,\\ \sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*}+\sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}, & 2\le q,p^* \end{cases}$$

$$\sim \sqrt{p^*\wedge \operatorname{Log} n}\, m^{(1/q-1/2)\vee 0}n^{1/p^*}+\sqrt{q\wedge \operatorname{Log} m}\, n^{(1/p^*-1/2)\vee 0}m^{1/q}, \tag{1}$$

where

$$\operatorname{Log} x=\max\{1,\ln x\},\quad \text{for } x>0,$$

and for two nonnegative functions f and g we write $f\gtrsim g$ (or $g\lesssim f$) if there exists an absolute constant C such that $Cf\ge g$; the notation $f\sim g$ means that $f\gtrsim g$ and $g\gtrsim f$. We write $\lesssim_\alpha$, $\sim_{K,\gamma}$, etc. if the underlying constant depends on the parameters given in the subscripts. Equation (1) yields that for $n=m$ we have

$$\mathbb{E}\big\|(g_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n} \sim \begin{cases} n^{1/q+1/p^*-1/2}, & p^*,q\le 2,\\ \sqrt{p^*\wedge q\wedge \operatorname{Log} n}\, n^{1/(p^*\wedge q)}, & p^*\vee q\ge 2. \end{cases}$$

However, even in the case of exponential entries it was initially not clear for us what the order of the expected operator norm is. This question led us to deriving in [11] two-sided Chevet type bounds for iid exponential and, more generally, Weibull random vectors with shape parameter $r\in[1,2]$. In consequence, we obtained the desired non-asymptotic behaviour of the operator norm in the Weibull case when $r\in[1,2]$ ($r=1$ is the exponential case). Note that this does not cover the case of a matrix $(\varepsilon_{i,j})_{i,j}$ with iid Rademacher entries, which corresponds to the case $r=\infty$. It is well known (by [2, 3], cf. [1, Remark 4.2]) that in this case

$$\mathbb{E}\big\|(\varepsilon_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \sim_{p,q} \begin{cases} m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\le 2,\\ m^{1/q-1/2}n^{1/p^*}+m^{1/q}, & q\le 2\le p^*,\\ n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*\le 2\le q,\\ n^{1/p^*}+m^{1/q}, & 2\le p^*,q. \end{cases} \tag{2}$$

Moreover, it is not hard to show that constants in lower bounds do not depend on p and q, whereas [12, Lemma 173] shows that in the case of square matrices the constants in (2) may be chosen independent of p and q, i.e.,

$$\mathbb{E}\big\|(\varepsilon_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n} \sim \begin{cases} n^{1/q+1/p^*-1/2}, & p^*,q\le 2,\\ n^{1/(p^*\wedge q)}, & p^*\vee q\ge 2. \end{cases}$$

It is natural to ask if the upper bound in (2) does not depend on p and q also in the rectangular case. Surprisingly, the answer to this question is negative — in Corollary 14 below we provide an exact two-sided bound (different than the one in (2)) up to a constant non-depending on p and q.

The two-sided bounds for operator norms in all the aforementioned special cases may be expressed in the following common form:

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \sim m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jX_{1,j}\Big\|_{q\wedge \operatorname{Log} m}+n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_iX_{i,1}\Big\|_{p^*\wedge \operatorname{Log} n}.$$

Therefore, it is natural to ask if this formula is valid for other distributions of entries. We are able to prove it for the class of random variables $X_{i,j}$ satisfying, for some $\alpha\in[1,\infty)$, the following mild regularity condition

$$\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_{\rho} \quad \text{for all } \rho \ge 1 \tag{3}$$

This condition was investigated in [10] and is sometimes called the α-regularity, and random variables satisfying it are called α-regular. This condition may be rephrased in terms of tails of random variables $X_{i,j}$ (see Proposition 9). The class of α-regular random variables contains, among others, Gaussian, Rademacher, log-concave, and Weibull random variables with any parameter $r \in (0,\infty)$. Although condition (3) is not very rigorous, it fails for some natural classes of random variables, such as lognormal and β-stable variables with $\beta \in (0,2)$.

The main result of this paper is the following two-sided bound.

Theorem 1. *Let $(X_{i,j})_{i\le m, j\le n}$ be iid centered random variables satisfying α-regularity condition* (3) *and let $p,q \in [1,\infty]$. Then*

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n \to \ell_q^m} \sim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \operatorname{Log} m} + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*\wedge \operatorname{Log} n}.$$

Remark 2. If $q \le 2 \le p$, then the assertion of Theorem 1 holds under a weaker condition that random variables $X_{i,j}$ are independent, centered, have equal variances, and satisfy $\|X_{i,j}\|_4 \le \alpha \|X_{i,j}\|_2$. We prove this in Subsection 6.1.

Remark 3. In the case when random variables $X_{i,j}$ are not necessarily centered, Theorem 1 and Jensen's inequality imply that (see Subsection 3.3 for a detailed proof)

$$\begin{aligned}\mathbb{E}\big\|(X_{i,j})_{i,j}\big\|_{\ell_p^n \to \ell_q^m} \sim_\alpha\ & m^{1/q} n^{1/p^*} |\mathbb{E}X_{1,1}| + m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j (X_{1,j} - \mathbb{E}X_{1,1})\Big\|_{q\wedge \operatorname{Log} m} \\ & + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i (X_{i,1} - \mathbb{E}X_{1,1})\Big\|_{p^*\wedge \operatorname{Log} n}\end{aligned} \tag{4}$$

provided that iid random variables $X_{i,j}$, $i \le m, j \le n$, satisfy

$$\|X_{i,j} - \mathbb{E}X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j} - \mathbb{E}X_{i,j}\|_{\rho} \quad \text{for all } \rho \ge 1. \tag{5}$$

The formula in Theorem 1 looks quite simple but, because of the suprema appearing in it, it is not always easy to see how the right-hand side depends on p and q. In Section 3 we give exact formulas for quantities comparable to the one from Theorem 1 in the case when the entries are Weibulls (this includes exponential and Rademacher random variables) or, more generally, when the entries have log-concave or log-convex tails.

The next proposition reveals how the two-sided bound from Theorem 1 depends on p and q in the case when $n = m$ and $p^* \vee q \ge 2$.

Proposition 4. *Let $p, q \in [1,\infty]$ and $p^* \vee q \ge 2$. Let $X_{i,j}$ be iid centered random variables satisfying* (3)*. Then*

$$n^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \operatorname{Log} n} + n^{1/p^*} \sup_{s\in B_{q^*}^n} \Big\|\sum_{i=1}^n s_i X_{i,1}\Big\|_{p^*\wedge \operatorname{Log} n} \sim_\alpha n^{1/(p^*\wedge q)} \|X_{1,1}\|_{p^*\wedge q\wedge \operatorname{Log} n}.$$

Moreover, if one of the parameters p^*, q is not larger than 2, then in the general rectangular case one of the terms from the formula in Theorem 1 can be simplified in the following way.

Proposition 5. *For $\tilde q \in [1,2]$, $p \in [1,\infty]$ and centered iid random variables X_i we have*

$$\frac{1}{2\sqrt{2}} n^{(1/p^*-1/2)_+} \|X_1\|_{\tilde q} \le \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_j\Big\|_{\tilde q} \le n^{(1/p^*-1/2)_+} \|X_1\|_2.$$

Similarly, for $\tilde{p} \in [1,2]$ and $q \in [1,\infty]$,

$$\frac{1}{2\sqrt{2}} m^{(1/q-1/2)_+} \|X_1\|_{\tilde{p}} \le \sup_{s\in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_i \Big\|_{\tilde{p}} \le m^{(1/q-1/2)_+} \|X_1\|_2.$$

In particular, if $1 \le p^*, q \le 2$, and $X_{i,j}$'s are iid random variables satisfying $\widetilde{\alpha}^{-1}\|X_{i,j}\|_1 \ge \|X_{i,j}\|_2 = 1$, then

$$m^{1/q} \sup_{t\in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_{i,1} \Big\|_{p^*} \sim_{\widetilde{\alpha}} m^{1/q} n^{1/p^*-1/2} + n^{1/p^*} m^{1/q-1/2}.$$

Theorem 1 and the last part of Proposition 5 imply that under the regularity assumption (3) the behaviour of $\mathbb{E}\|(X_{i,j})_{i,j=1}^n\|_{\ell_p^n\to\ell_q^n}$ in the range $1 \le p^*, q \le 2$ is the same as in the case of an iid Gaussian matrix (see (1)), whose entries have the same variance as $X_{1,1}$.

Propositions 4 and 5 yield that in the case of square matrices the bound from Theorem 1 may be expressed in a more explicit way in the whole range of p and q:

Corollary 6. *Let $(X_{i,j})_{i,j\le n}$ be iid centered random variables satisfying regularity condition (3) and let $1 \le p, q \le \infty$. Then*

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n} \sim_\alpha \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & p^*, q \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge \operatorname{Log} n}, & p^* \vee q \ge 2. \end{cases}$$

The rest of this article is organized as follows. In Section 2 we review properties of random variables satisfying α-regularity condition (3). In Section 3 we provide explicit functions of parameters p^*, q, n, m comparable to the bounds from Theorem 1 for some special classes of distributions, and prove Remark 3. In Section 4 we establish the lower bound of Theorem 1, and in Section 5 we give proofs of Propositions 4 and 5. Section 6 contains the proof of the upper bound of Theorem 1. It is divided into several subsections corresponding to particular ranges of (p, q), since the arguments we use in the proof vary depending on the range we deal with. In Subsections 6.3 and 6.4 we reveal the methods and tools, respectively, used in the most challenging parts of the proof.

2. Properties of α-regular random variables

In this section we discuss crucial properties of random variables satisfying α-regularity condition (3). We also show how to express this condition in terms of tails.

One of the important consequences of α-regularity condition (3) is the comparison of weak and strong moments of linear combinations of independent centered variables $X_{i,j}$, proven in [10], stating that for every $\rho \ge 1$ and every nonempty bounded $U \subset \mathbb{R}^{nm}$,

$$\Big(\mathbb{E}\sup_{u\in U}\Big|\sum_{i,j} X_{i,j}u_{i,j}\Big|^\rho\Big)^{1/\rho} \sim_\alpha \mathbb{E}\sup_{u\in U}\Big|\sum_{i,j} X_{i,j}u_{i,j}\Big| + \sup_{u\in U}\Big\|\sum_{i,j} X_{i,j}u_{i,j}\Big\|_\rho. \tag{6}$$

Another property of independent centered variables satisfying (3) is the following Khintchine–Kahane-type estimate, derived in [10, Lemma 4.1],

$$\Big\|\sum_{i,j} u_{i,j}X_{i,j}\Big\|_{\rho_1} \lesssim_\alpha \Big(\frac{\rho_1}{\rho_2}\Big)^\beta \Big\|\sum_{i,j} u_{i,j}X_{i,j}\Big\|_{\rho_2} \qquad \text{for every } \rho_1 \ge \rho_2 \ge 1, \tag{7}$$

where $\beta := \frac{1}{2} \vee \log_2 \alpha$ and u is an arbitrary $m \times n$ deterministic matrix.

For iid random variables $X_{i,j}$ we define their log-tail function $N\colon [0,\infty) \to [0,\infty]$ via the formula

$$\mathbb{P}(|X_{i,j}| \ge t) = e^{-N(t)}, \quad t \ge 0. \tag{8}$$

The function N is nondecreasing, but not necessary invertible. However, we may consider its generalized inverse $N^{-1}\colon [0,\infty) \to [0,\infty)$ defined by

$$N^{-1}(s) = \sup\{t \geq 0 \colon N(t) \leq s\}.$$

Lemma 7. *Suppose that condition* (3) *holds and N is defined by* (8)*. Then for every $\rho \geq 1$,*

$$\|X_{i,j}\|_\rho \sim_\alpha N^{-1}\big(\rho \vee (2\ln(2\alpha))\big).$$

Proof. To simplify the notation set $\gamma := 2\ln(2\alpha)$. Note that $\alpha \geq 1$ and $\gamma > 1$.

For $t < N^{-1}(\rho \vee \gamma)$ we have by Chebyshev's inequality

$$\|X_{i,j}\|_\rho \geq \mathbb{P}(|X_{i,j}| \geq t)^{1/\rho} t \geq e^{-(1\vee(\gamma/\rho))} t \geq e^{-\gamma} t.$$

Hence, $N^{-1}(\rho \vee \gamma) \leq 4\alpha^2 \|X_{i,j}\|_\rho$.

To derive the opposite bound, observe that the Paley-Zygmund inequality and α-regularity assumption (3) yield that for every $r \geq 1$,

$$\mathbb{P}\Big(|X_{i,j}| \geq \frac{1}{2}\|X_{i,j}\|_r\Big) = \mathbb{P}(|X_{i,j}|^r \geq 2^{-r}\mathbb{E}|X_{i,j}|^r) \geq (1-2^{-r})^2 \frac{(\mathbb{E}|X_{i,j}|^r)^2}{\mathbb{E}|X_{i,j}|^{2r}} \geq \frac{1}{4}\alpha^{-2r} \geq e^{-\gamma r}.$$

Therefore, $N^{-1}(\gamma r) \geq \frac{1}{2}\|X_{i,j}\|_r$ for every $r \geq 1$, so by taking $r = 1 \vee (\rho/\gamma)$ and applying (3) multiple times we get

$$N^{-1}(\rho \vee \gamma) \geq \frac{1}{2}\|X_{i,j}\|_{1\vee(\rho/\gamma)} \geq \frac{1}{2}\alpha^{-\lceil \log_2 \gamma \rceil}\|X_{i,j}\|_\rho \geq \frac{1}{2}(2\gamma)^{-\log_2 \alpha}\|X_{i,j}\|_\rho. \qquad \square$$

Remark 8. The proof above shows that

$$\frac{1}{e}N^{-1}(\rho) \leq \|X_{i,j}\|_\rho \leq 2(4\ln(2\alpha))^{\log_2 \alpha} N^{-1}(\rho) \quad \text{ for } \rho \geq 2\ln(2\alpha).$$

The next proposition shows how to rephrase condition (3) in terms of tails of $X_{i,j}$.

Proposition 9. *Let X be a random variable and $\mathbb{P}(|X| \geq t) = e^{-N(t)}$ for $N\colon [0,\infty) \to [0,\infty]$. Then the following conditions are equivalent*
i) there exists $\alpha_1 \in [1,\infty)$ such that $\|X\|_{2\rho} \leq \alpha_1 \|X\|_\rho$ for every $\rho \geq 1$;
ii) there exist $\alpha_2 \in [1,\infty)$, $\beta_2 \in [0,\infty)$ such that $N^{-1}(2s) \leq \alpha_2 N^{-1}(s)$ for every $s > \beta_2$;
iii) there exist $\alpha_2 \in [1,\infty)$, $\beta_2 \in [0,\infty)$ such that $N(\alpha_2 t) \geq 2N(t)$ for every $t > 0$ satisfying $N(t) > \beta_2$.

Proof. i)$\Rightarrow$ ii) By Lemma 7 we have for $s > 2\ln(2\alpha_1)$,

$$N^{-1}(2s) \sim_{\alpha_1} \|X\|_{2s} \leq \alpha_1 \|X\|_s \sim_{\alpha_1} N^{-1}(s).$$

Equivalence of ii) and iii) is standard.

iii)$\Rightarrow$ i) Let us fix $\rho \geq 1$. We have $\|X\|_\rho^\rho \geq t^\rho \mathbb{P}(|X| \geq t) = t^\rho e^{-N(t)}$. Thus, $N(t) > \beta_2$ for $t > t_0 := e^{\beta_2/\rho}\|X\|_\rho$, and so

$$\begin{aligned}
\|X\|_{2\rho}^{2\rho} &= \alpha_2^{2\rho}\int_0^\infty 2\rho t^{2\rho-1} e^{-N(\alpha_2 t)}\,dt \leq \alpha_2^{2\rho}\Big(t_0^{2\rho} + 2\rho\int_{t_0}^\infty t^{2\rho-1}e^{-N(\alpha_2 t)}\,dt\Big) \\
&\leq \alpha_2^{2\rho}\Big(t_0^{2\rho} + 2\rho\int_{t_0}^\infty t^\rho e^{-N(t)} t^{\rho-1} e^{-N(t)}\,dt\Big) \leq \alpha_2^{2\rho}\Big(t_0^{2\rho} + 2\|X\|_\rho^\rho \rho\int_{t_0}^\infty t^{\rho-1}e^{-N(t)}\,dt\Big) \\
&\leq \alpha_2^{2\rho}\Big(t_0^{2\rho} + 2\|X\|_\rho^\rho \rho\int_0^\infty t^{\rho-1}e^{-N(t)}\,dt\Big) = \alpha_2^{2\rho}\|X\|_\rho^{2\rho}(e^{2\beta_2}+2) \leq (\alpha_2(e^{\beta_2}+\sqrt{2}))^{2\rho}\|X\|_\rho^{2\rho}. \square
\end{aligned}$$

Remark 10. Remark 8 and the proof above show that i) implies ii) and iii) with constants $\alpha_2 = 2e\alpha_1(4\ln(2\alpha_1))^{\log_2 \alpha_1}$, $\beta_2 = 2\ln(2\alpha_1)$, and conditions ii), iii) imply i) with constants $\alpha_1 = \alpha_2(e^{\beta_2} + \sqrt{2})$.

3. Examples

In this section we focus on two particular classes of distributions: with log-concave and log-convex tails. They include Rademachers, subexponential Weibulls, and heavy-tailed Weibulls. Our aim is to provide an explicit function of parameters p^*, q, n, m comparable to the bounds from Theorem 1; such a function in the case of iid Gaussian matrices is given in (1).

Throughout this section, we assume that $X_{i,j}$ are iid symmetric random variables and their log-tail function $N\colon [0,\infty) \to [0,\infty]$ is given by (8).

3.1. **Variables with log-concave tails.** In this subsection we consider variables with log-concave tails, i.e., variables with convex log-tail function N. Since $N(0) = 0$ and N is convex, for every $s > t > 0$ we have

$$\frac{N(s)}{s} \geq \frac{N(t)}{t}. \tag{9}$$

In particular, Proposition 9 yields that a random variable with log-concave tails satisfy (3) with a universal constant α. Hence, in the square case Corollary 6 and Lemma 7 imply that

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n} \sim \begin{cases} n^{1/q+1/p^*-1/2}N^{-1}(1) & p^*, q \leq 2, \\ n^{1/(p^*\wedge q)}N^{-1}(p^*\wedge q\wedge \operatorname{Log} n) & p^*\vee q \geq 2 \end{cases}$$
$$\sim N^{-1}(p^*\wedge q\wedge \operatorname{Log} n)n^{1/(p^*\wedge q)}n^{(1/(p^*\vee q)-1/2)\vee 0}.$$

In the case of log-concave tails it is more convenient to normalize random variables in such a way that $N^{-1}(1) = 1$ rather than $\|X_{i,j}\|_2 = 1$. Observe that Lemma 7 and (9) yield that $\|X_{i,j}\|_2 \sim N^{-1}(1)$.

Lemma 11. *Let $X_1, \ldots, X_n$ be iid symmetric random variables with log-concave tails such that $N^{-1}(1) = 1$. Then for every $p, q \geq 1$,*

$$\sup_{t\in B_p^n}\Big\|\sum_{i=1}^n t_iX_i\Big\|_q \sim \max_{1\leq k\leq q\wedge n} k^{1/p^*}N^{-1}(q/k) + (q\wedge n)^{1/(p^*\vee 2)}n^{(1/p^*-1/2)\vee 0}.$$

Proof. The result of Gluskin and Kwapień [6] states that

$$\Big\|\sum_{i=1}^n t_iX_i\Big\|_q \sim \sup\Big\{\sum_{i\leq q\wedge n} t_i^*s_i\colon \sum_{i\leq q\wedge n} N(s_i)\leq q\Big\} + \sqrt{q}\Big(\sum_{i>q}|t_i^*|^2\Big)^{1/2},$$

where $t_1^*, \ldots, t_n^*$ is the nonincreasing rearrangement of $|t_1|, \ldots, |t_n|$.

Let us fix $t \in B_p^n$. Then for every $q > n$,

$$\sum_{i\leq q} t_i^* + \sqrt{q}\Big(\sum_{k>q}(t_k^*)^2\Big)^{1/2} = \sum_{i\leq n} t_i^* \leq n^{1-1/p} = n^{1/2-1/p}\sqrt{q\wedge n} = (q\wedge n)^{1/p^*}.$$

For $p \geq 2$ and $q < n$ we have

$$\sum_{i\leq q} t_i^* + \sqrt{q}\Big(\sum_{k>q}(t_k^*)^2\Big)^{1/2} \leq q^{1-1/p} + q^{1/2}(n-q)^{1/2-1/p} \sim q^{1/2}n^{1/2-1/p} = n^{1/2-1/p}\sqrt{q\wedge n}.$$

Note that $t_k^* \leq t_q^*$ whenever $k > q$. Therefore, for $p \in [1,2]$, $q < n$ we obtain

$$\sum_{i\leq q} t_i^* + \sqrt{q}\Big(\sum_{k>q}(t_k^*)^2\Big)^{1/2} \leq \sum_{i\leq q} t_i^* + \sqrt{q}(t_q^*)^{(2-p)/2}\Big(\sum_{k>q}(t_k^*)^p\Big)^{1/2} \leq q^{1-1/p} + q^{1/2}(t_q^*)^{1-p/2}$$
$$\leq 2q^{1-1/p} = 2(q\wedge n)^{1/p^*}.$$

The estimates above might be reversed up to universal constants if we take $t=\sum_{i=1}^n n^{-1/p}e_i$ for $p\ge 2$, and $t=\sum_{i=1}^{q\wedge n}(q\wedge n)^{-1/p}e_i$ for $p\in[1,2]$. Thus, in any case,

$$\sup_{t\in B_p^n}\Big(\sum_{i\le q\wedge n} t_i^* + \sqrt{q}\Big(\sum_{i>q}|t_i^*|^2\Big)^{1/2}\Big) \sim (q\wedge n)^{1/(p^*\vee 2)} n^{(1/p^*-1/2)\vee 0}.$$

Moreover, since $N^{-1}(1)=1$,

$$\begin{aligned}\sqrt{q}\Big(\sum_{i>q}|t_i^*|^2\Big)^{1/2} &\le \sum_{i\le q\wedge n} t_i^* + \sqrt{q}\Big(\sum_{i>q}|t_i^*|^2\Big)^{1/2}\\ &\le \sup\Big\{\sum_{i\le q\wedge n} t_i^* s_i\colon \sum_{i\le q\wedge n} N(s_i)\le q\Big\} + \sqrt{q}\Big(\sum_{i>q}|t_i^*|^2\Big)^{1/2}.\end{aligned}$$

Hence, it remains to prove that

$$\begin{aligned}\sup_{t\in B_p^n}\sup\Big\{\sum_{i\le q\wedge n} t_i^* s_i\colon \sum_{i\le q\wedge n} N(s_i)\le q\Big\} &= \sup\Big\{\Big(\sum_{i\le q\wedge n}|s_i|^{p^*}\Big)^{1/p^*}\colon \sum_{i\le q\wedge n} N(s_i)\le q\Big\}\\ &\sim \max_{1\le k\le q\wedge n} k^{1/p^*}N^{-1}(q/k).\end{aligned}$$

The lower bound is obvious since $N(N^{-1}(u))\le u$ for every $u\ge 0$. To show the upper estimate let

$$a := \max_{1\le k\le q\wedge n} k^{1/p^*}N^{-1}(q/k),$$

where the maximum runs through integers k satisfying $1\le k\le q\wedge n$. Then (9) implies that

$$\sup_{1\le t\le q\wedge n} t^{1/p^*}N^{-1}(q/t)\le 2a,$$

where the supremum runs through all $t\in\mathbb{R}$ satisfying $1\le t\le q\wedge n$. Hence,

$$N(s)\ge q\Big(\frac{s}{2a}\Big)^{p^*} \qquad \text{whenever } 2a\ge s\ge 2a(q\wedge n)^{-1/p^*}.$$

Therefore, condition $\sum_{i\le q\wedge n} N(s_i)\le q$ yields that $s_i\le a$ and so

$$\sum_{i\le q\wedge n} s_i^{p^*} \le (2a)^{p^*}\sum_{i\le q\wedge n}\Big(\frac{1}{q\wedge n}+\frac{1}{q}N(s_i)\Big)\le 2(2a)^{p^*}\le (4a)^{p*}. \qquad \square$$

Theorem 1 and Lemma 11 yield the following corollary.

Corollary 12. *Let $(X_{i,j})_{i\le m,j\le n}$ be iid symmetric random variables with log-concave tails such that $N^{-1}(1)=1$. Then for every $p,q\ge 1$,*

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}$$
$$\sim_\alpha \begin{cases} m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\le 2,\\ n^{1/p^*}\Big(\sqrt{p^*\wedge m\wedge \operatorname{Log} n}\, m^{1/q-1/2}+\sup\limits_{l\le p^*\wedge m\wedge\operatorname{Log} n} l^{1/q}N^{-1}\big(\frac{p^*\wedge \operatorname{Log} n}{l}\big)\Big)+m^{1/q}, & q\le 2\le p^*,\\ n^{1/p^*}+m^{1/q}\Big(\sqrt{q\wedge n\wedge\operatorname{Log} m}\, n^{1/p^*-1/2}+\sup\limits_{k\le q\wedge n\wedge \operatorname{Log} m} k^{1/p^*}N^{-1}\big(\frac{q\wedge\operatorname{Log} m}{k}\big)\Big), & p^*\le 2\le q,\\ n^{1/p^*}\Big((p^*\wedge m\wedge\operatorname{Log} n)^{1/q}+\sup\limits_{l\le p^*\wedge m\wedge\operatorname{Log} n} l^{1/q}N^{-1}\big(\frac{p^*\wedge\operatorname{Log} n}{l}\big)\Big) & \\ \quad +m^{1/q}\Big((q\wedge n\wedge\operatorname{Log} m)^{1/p^*}+\sup\limits_{k\le q\wedge n\wedge\operatorname{Log} m} k^{1/p^*}N^{-1}\big(\frac{q\wedge\operatorname{Log} m}{k}\big)\Big), & 2\le p^*,q.\end{cases}$$

3.1.1. *Subexponential Weibull matrices.* Let $X_{i,j}$ be symmetric Weibull random variables with parameter r, i.e., $N(t)=t^r$. If $X_{i,j}$ are subexponential, i.e. $r\geq 1$, then N is convex, and $\|X_{i,j}\|_\rho=(\Gamma(1+\rho/r)^{1/\rho})\sim\rho^{1/r}$. Thus, Corollary 6 implies that

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n}\sim\begin{cases}n^{1/q+1/p^*-1/2} & p^*,q\leq 2,\\ (p^*\wedge q\wedge \operatorname{Log} n)^{1/r}n^{1/(p^*\wedge q)}, & p^*\vee q\geq 2\end{cases}$$
$$\sim (p^*\wedge q\wedge \operatorname{Log} n)^{1/r}n^{1/(p^*\wedge q)}n^{(1/(p^*\vee q)-1/2)\vee 0}.$$

To obtain a formula in the rectangular case we first observe that $N^{-1}(1)=1$ and

$$\sup_{1\leq k\leq l}k^{1/p^*}N^{-1}(q/k)=q^{1/r}l^{(1/p^*-1/r)\vee 0}.$$

If $r\in[1,2]$, then $1/p^*-1/r\leq 0$ for $p^*\geq 2$ and Corollary 12 allows to recover the following bound from [11, Corollary 1.7].

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_p^n\to\ell_q^m}$$
$$\sim\begin{cases}m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\leq 2,\\ (p^*\wedge \operatorname{Log} n)^{1/r}n^{1/p^*}m^{(1/q-1/r)\vee 0}+\sqrt{p^*\wedge\operatorname{Log} n}\,n^{1/p^*}m^{1/q-1/2}+m^{1/q}, & q\leq 2\leq p^*,\\ n^{1/p^*}+(q\wedge\operatorname{Log} m)^{1/r}m^{1/q}n^{(1/p^*-1/r)\vee 0}+\sqrt{q\wedge\operatorname{Log} m}\,m^{1/q}n^{1/p^*-1/2}, & p^*\leq 2\leq q,\\ (p^*\wedge\operatorname{Log} n)^{1/r}n^{1/p^*}+(q\wedge\operatorname{Log} m)^{1/r}m^{1/q}, & 2\leq p^*,q\end{cases}$$
$$\sim (p^*\wedge\operatorname{Log} n)^{1/r}m^{(1/q-1/r)\vee 0}n^{1/p^*}+\sqrt{p^*\wedge\operatorname{Log} n}\,m^{(1/q-1/2)\vee 0}n^{1/p^*}$$
$$+(q\wedge\operatorname{Log} m)^{1/r}n^{(1/p^*-1/r)\vee 0}m^{1/q}+\sqrt{q\wedge\operatorname{Log} m}\,n^{(1/p^*-1/2)\vee 0}m^{1/q}.$$

In the case $r>2$ Corollary 12 yields the following.

Corollary 13. *Let $(X_{i,j})_{i\leq m,j\leq n}$ be iid Weibull random variables with parameter $r\geq 2$. Then for every $p,q\geq 1$,*

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_p^n\to\ell_q^m}$$
$$\sim\begin{cases}m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\leq 2,\\ m^{1/q-1/2}(p^*\wedge\operatorname{Log} n)^{1/r}(p^*\wedge m\wedge\operatorname{Log} n)^{1/2-1/r}n^{1/p^*}+m^{1/q}, & q\leq 2\leq p^*,\\ n^{1/p^*}+n^{1/p^*-1/2}(q\wedge\operatorname{Log} m)^{1/r}(q\wedge n\wedge\operatorname{Log} m)^{1/2-1/r}m^{1/q}, & p^*\leq 2\leq q,\\ (p^*\wedge\operatorname{Log} n)^{1/r}(p^*\wedge m\wedge\operatorname{Log} n)^{(1/q-1/r)\vee 0}n^{1/p^*} & \\ \quad+(q\wedge\operatorname{Log} m)^{1/r}(q\wedge n\wedge\operatorname{Log} m)^{(1/p^*-1/r)\vee 0}m^{1/q}, & 2\leq p^*,q\end{cases}$$

$$\sim m^{(1/q-1/2)\vee 0}(p^*\wedge\operatorname{Log} n)^{1/r}(p^*\wedge m\wedge\operatorname{Log} n)^{(1/(q\vee 2)-1/r)\vee 0}n^{1/p^*}$$
$$+n^{(1/p^*-1/2)\vee 0}(q\wedge\operatorname{Log} m)^{1/r}(q\wedge n\wedge\operatorname{Log} m)^{(1/(p^*\vee 2)-1/r)\vee 0}m^{1/q}.$$

In particular, when $r=\infty$ we get the following two-sided bound for matrices with iid Rademacher entries $\varepsilon_{i,j}$.

Corollary 14. *If $1\leq p,q\leq\infty$, then*

$$\mathbb{E}\big\|(\varepsilon_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_p^n\to\ell_q^m}\sim\begin{cases}m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\leq 2,\\ \sqrt{p^*\wedge m}\,m^{1/q-1/2}n^{1/p^*}+m^{1/q}, & q\leq 2\leq p^*,\\ n^{1/p^*}+\sqrt{q\wedge n}\,n^{1/p^*-1/2}m^{1/q}, & p^*\leq 2\leq q,\\ (p^*\wedge m)^{1/q}n^{1/p^*}+(q\wedge n)^{1/p^*}m^{1/q}, & 2\leq p^*,q.\end{cases}$$
$$\sim (p^*\wedge m)^{1/(q\vee 2)}m^{(1/q-1/2)\vee 0}n^{1/p^*}+(q\wedge n)^{1/(p^*\vee 2)}n^{(1/p^*-1/2)\vee 0}m^{1/q}.$$

Remark 15. In [11, Theorem 3.3] we provide two-sided bounds for $\mathbb{E}\|(a_i b_j X_{i,j})_{i\le m, j\le n}\|_{\ell_p^n\to\ell_q^m}$, where the vectors $a\in\mathbb{R}^m$ and $b\in\mathbb{R}^n$ are arbitrary, and $X_{i,j}$'s are Weibull random variables with parameter $r\in[1,2]$. We do not know similar formulas for $r>2$.

3.2. **Variables with log-convex tails.** In this subsection we assume that $X_{i,j}$ have log-convex tails, i.e., the function N given by (8) is concave.

Lemma 16. *Let $(X_{i,j})$ be iid symmetric random variables with log-convex tails and assume that* (3) *holds. Then for every $p,q\ge 1$,*

$$\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q \sim_\alpha \|X_{i,j}\|_q + \sqrt{q}\|X_{i,j}\|_2 n^{(1/p^*-1/2)\vee 0}.$$

Proof. If $q\le 2$, then (7) yields

$$\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q \sim_\alpha \sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_2 = \sup_{t\in B_p^n}\|t\|_2\|X_{i,j}\|_2 = n^{(1/p^*-1/2)\vee 0}\|X_{i,j}\|_2$$
$$\sim \|X_{i,j}\|_q + \sqrt{q}\|X_{i,j}\|_2 n^{(1/p^*-1/2)\vee 0}.$$

Now assume that $q>2$. By [8, Theorem 1.1] we have

$$\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q \sim \Big(\sum_{j=1}^n |t_j|^q\mathbb{E}|X_{1,j}|^q\Big)^{1/q} + \sqrt{q}\Big(\sum_{j=1}^n |t_j|^2\mathbb{E}|X_{1,j}|^2\Big)^{1/2}$$
$$= \|t\|_q\|X_{i,j}\|_q + \sqrt{q}\|t\|_2\|X_{i,j}\|_2 \gtrsim \|t\|_\infty\|X_{i,j}\|_q + \sqrt{q}\|t\|_2\|X_{i,j}\|_2.$$

We shall show that the last estimate may be reversed up to a constant depending only on α. To this aim put $a := \|t\|_\infty\|X_{i,j}\|_q + \sqrt{q}\|t\|_2\|X_{i,j}\|_2$. Then

$$\|t\|_q\|X_{i,j}\|_q \le (\|t\|_\infty\|X_{i,j}\|_q)^{(q-2)/q}(\|t\|_2\|X_{i,j}\|_q)^{2/q} \le a(\|X_{i,j}\|_q/\|X_{i,j}\|_2)^{2/q} \lesssim_\alpha a,$$

where the last estimate follows by (7). Thus, for $q>2$,

$$\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q \sim_\alpha \sup_{t\in B_p^n}(\|t\|_\infty\|X_{i,j}\|_q + \sqrt{q}\|t\|_2\|X_{i,j}\|_2) \sim \|X_{i,j}\|_q + \sqrt{q}\|X_{i,j}\|_2 n^{(1/p^*-1/2)\vee 0}.$$

□

Remark 17. Since N is concave, N^{-1} is convex and $N^{-1}(0)=0$, hence $N^{-1}(q)\ge \frac{q}{2}N^{-1}(2)$ whenever $q\ge 2$. So (3) and Lemma 7 imply that $\|X_{i,j}\|_q \sim_\alpha N^{-1}(q) \gtrsim_\alpha q\|X_{i,j}\|_2$. Thus, we get by Lemma 16,

$$\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q \sim_\alpha \|X_{i,j}\|_q \quad \text{for } p^*, q\ge 2.$$

Theorem 1, Lemma 16, and Remark 17 yield the following corollary.

Corollary 18. *Let $(X_{i,j})_{i\le m,j\le n}$ be iid symmetric random variables with log-convex tails such that* (3) *holds. Then*

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}$$
$$\sim_\alpha \begin{cases} (m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q})\|X_{i,j}\|_2, & p^*,q\le 2,\\ n^{1/p^*}(m^{1/q-1/2}\sqrt{p^*\wedge \operatorname{Log} n}\|X_{i,j}\|_2 + \|X_{i,j}\|_{p^*\wedge\operatorname{Log} n}) + m^{1/q}\|X_{i,j}\|_2, & q\le 2\le p^*,\\ n^{1/p^*}\|X_{i,j}\|_2 + m^{1/q}(n^{1/p^*-1/2}\sqrt{q\wedge\operatorname{Log} m}\|X_{i,j}\|_2 + \|X_{i,j}\|_{q\wedge\operatorname{Log} m}), & p^*\le 2\le q,\\ n^{1/p^*}\|X_{i,j}\|_{p^*\wedge\operatorname{Log} n} + m^{1/q}\|X_{i,j}\|_{q\wedge \operatorname{Log} m}, & 2\le p^*,q \end{cases}$$

3.2.1. *Heavy-tailed Weibull random variables.* Weibull random variables with parameter $r \in (0,1]$ have log-convex tails. Moreover, in this case $\|X_{i,j}\|_\rho = (\Gamma(1+\rho/r)^{1/\rho}) \sim_r \rho^{1/r}$, so the $X_{i,j}$'s satisfy (3) with $\alpha \sim_r 2^{1/r}$ and thus Corollary 6 implies that

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n \to \ell_q^n} \sim_r \begin{cases} n^{1/q+1/p^*-1/2} & p^*, q \le 2, \\ (p^* \wedge q \wedge \operatorname{Log} n)^{1/r} n^{1/(p^* \wedge q)}, & p^* \vee q \le 2 \end{cases}$$
$$\sim (p^* \wedge q \wedge \operatorname{Log} n)^{1/r} n^{1/(p^* \wedge q)} n^{(1/(p^* \vee q)-1/2)\vee 0}.$$

In the rectangular case Corollary 18 yields the following.

Corollary 19. *Let $(X_{i,j})_{i \le m, j \le n}$ be iid Weibull random variables with parameter $r \in (0,1]$. Then for every $1 \le p, q \le \infty$ we have*

$$\mathbb{E}\big\|(X_{i,j})_{i \le m, j \le n}\big\|_{\ell_p^n \to \ell_q^m} \sim_r (q \wedge \operatorname{Log} m)^{1/2} n^{(1/p^*-1/2)\vee 0} m^{1/q} + (q \wedge \operatorname{Log} m)^{1/r} m^{1/q}$$
$$+ (p^* \wedge \operatorname{Log} n)^{1/2} m^{(1/q-1/2)\vee 0} n^{1/p^*} + (p^* \wedge \operatorname{Log} n)^{1/r} n^{1/p^*}.$$

3.3. **Non-centered random variables.** In this subsection we prove (4) under centered regularity assumption (5). Note that

$$\|(\mathbb{E}X_{i,j})\|_{\ell_p^n \to \ell_q^m} = |\mathbb{E}X_{1,1}| \cdot \|(1)_{i,j}\|_{\ell_p^n \to \ell_q^m} = |\mathbb{E}X_{1,1}| \cdot \sup_{t \in B_p^n} \Big(\sum_{i=1}^m \Big| \sum_{j=1}^n t_j \Big|^q \Big)^{1/q}$$
$$= |\mathbb{E}X_{1,1}| \cdot m^{1/q} \sup_{t \in B_p^n} \Big| \sum_{j=1}^n t_j \Big| = m^{1/q} n^{1/p^*} |\mathbb{E}X_{1,1}|.$$

By the triangle inequality we have

$$\mathbb{E}\|(X_{i,j})\|_{\ell_p^n \to \ell_q^m} \le \mathbb{E}\big\|(X_{i,j} - \mathbb{E}X_{i,j})\big\|_{\ell_p^n \to \ell_q^m} + \|(\mathbb{E}X_{i,j})\|_{\ell_p^n \to \ell_q^m}$$
$$= \mathbb{E}\big\|(X_{i,j} - \mathbb{E}X_{i,j})\big\|_{\ell_p^n \to \ell_q^m} + m^{1/q} n^{1/p^*} |\mathbb{E}X_{1,1}|,$$

so Theorem 1 implies the upper bound in (4). Moreover, Jensen's inequality yields $\mathbb{E}\|(X_{i,j})\|_{\ell_p^n \to \ell_q^m} \ge \|(\mathbb{E}X_{i,j})\|_{\ell_p^n \to \ell_q^m}$, so applying the triangle inequality we get

$$\mathbb{E}\|(X_{i,j})\|_{\ell_p^n \to \ell_q^m} \ge \frac{1}{2}\mathbb{E}\|(X_{i,j})\|_{\ell_p^n \to \ell_q^m} + \frac{1}{2}\Big(\mathbb{E}\big\|(X_{i,j} - \mathbb{E}X_{i,j})\big\|_{\ell_p^n \to \ell_q^m} - \|(\mathbb{E}X_{i,j})\|_{\ell_p^n \to \ell_q^m}\Big)$$
$$\ge \frac{1}{2}\mathbb{E}\big\|(X_{i,j} - \mathbb{E}X_{i,j})\big\|_{\ell_p^n \to \ell_q^m}.$$

Hence, Theorem 1 and another application of the inequality $\mathbb{E}\|(X_{i,j})\|_{\ell_p^n \to \ell_q^m} \ge \|(\mathbb{E}X_{i,j})\|_{\ell_p^n \to \ell_q^m} = m^{1/q} n^{1/p^*} |\mathbb{E}X_{1,1}|$ yield the lower bound in (4).

4. Lower bounds

In this section we shall prove the lower bound in Theorem 1. The crucial technical result we use is the following lower bound for ℓ_r-norms of iid sequences.

Lemma 20. *Let $r \ge 1$ and $Y_1, Y_2, \ldots, Y_k$ be iid nonnegative random variables satisfying the condition $\|Y_i\|_{2r} \le \alpha \|Y_i\|_r$ for some $\alpha \in [1, \infty)$. Assume that $k \ge 4\alpha^{2r}$. Then*

$$\mathbb{E}\Big(\sum_{i=1}^k Y_i^r\Big)^{1/r} \ge \frac{1}{128\alpha^2} k^{1/r} \|Y_1\|_r.$$

Proof. Define

$$Z := \sum_{i=1}^{k} 1_{A_i}, \quad A_i := \Big\{ Y_i^r \geq \frac{1}{2}\mathbb{E}Y_i^r \Big\} = \Big\{ Y_i^r \geq \frac{1}{2}\mathbb{E}Y_1^r \Big\}.$$

The Paley-Zygmund inequality yields

$$\mathbb{P}(A_i) \geq \frac{1}{4}\frac{(\mathbb{E}Y_i^r)^2}{\mathbb{E}Y_i^{2r}} \geq \frac{1}{4}\alpha^{-2r}.$$

Since $k \geq 4\alpha^{2r}$, this gives

$$\mathbb{E}Z = \sum_{i=1}^{k}\mathbb{P}(A_i) \geq \frac{k}{4}\alpha^{-2r} \geq 1$$

and

$$\mathbb{E}Z^2 = 2\sum_{1\leq i<j\leq k}\mathbb{P}(A_i)\mathbb{P}(A_j) + \sum_{i=1}^{k}\mathbb{P}(A_i) \leq (\mathbb{E}Z)^2 + \mathbb{E}Z \leq 2(\mathbb{E}Z)^2.$$

Applying again the Paley-Zygmund inequality we obtain

$$\mathbb{P}\Big(Z \geq \frac{1}{2}\mathbb{E}Z\Big) \geq \frac{1}{4}\frac{(\mathbb{E}Z)^2}{\mathbb{E}Z^2} \geq \frac{1}{8}.$$

Hence,

$$\mathbb{E}\Big(\sum_{i=1}^{k} Y_i^r\Big)^{1/r} \geq \mathbb{P}\Big(Z \geq \frac{1}{2}\mathbb{E}Z\Big)\Big(\frac{1}{2}\mathbb{E}Z\frac{1}{2}\mathbb{E}Y_1^r\Big)^{1/r} \geq \frac{1}{8}\Big(\frac{k}{16}\alpha^{-2r}\mathbb{E}Y_1^r\Big)^{1/r} \geq \frac{1}{128\alpha^2}k^{1/r}\|Y_1\|_r. \quad \square$$

Proof of the lower bound in Theorem 1. Let us fix $t \in B_p^n$ and put $Y_i := |\sum_{j=1}^n t_j X_{i,j}|$. Then $Y_1, \ldots, Y_m$ are iid random variables. Moreover, by (7), $\|Y_i\|_{2r} \leq \tilde{\alpha}\|Y_i\|_r$ for $r \geq 1$, where a constant $\tilde{\alpha} \geq 1$ depends only on α.

If $m \geq 4\tilde{\alpha}^{2q}$, then by Lemma 20 we get

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m, j\leq n}\big\|_{\ell_p^n\to\ell_q^m} \geq \mathbb{E}\Big(\sum_{i=1}^{m} Y_i^q\Big)^{1/q} \geq \frac{1}{128\tilde{\alpha}^2}m^{1/q}\|Y_i\|_q.$$

If $m \leq 4\tilde{\alpha}^2$, then by (7) we have

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m, j\leq n}\big\|_{\ell_p^n\to\ell_q^m} \geq \|Y_i\|_1 \gtrsim_\alpha \|Y_i\|_{\operatorname{Log} m} \sim_\alpha m^{1/q}\|Y_i\|_{q\wedge \operatorname{Log} m}.$$

If $4\tilde{\alpha}^2 \leq m \leq 4\tilde{\alpha}^{2q}$, then $m = 4\tilde{\alpha}^{2\tilde{q}}$ for some $1 \leq \tilde{q} \leq q$. Moreover, in this case $m^{1/q} \sim_\alpha 1 \sim_\alpha m^{1/\tilde{q}}$ and $\tilde{q} \sim_\alpha q \wedge \operatorname{Log} m$. Hence, Lemma 20 and (7) yield

$$\begin{aligned}\mathbb{E}\big\|(X_{i,j})_{i\leq m, j\leq n}\big\|_{\ell_p^n\to\ell_q^m} &\geq \mathbb{E}\Big(\sum_{i=1}^{m} Y_i^q\Big)^{1/q} \sim_\alpha \mathbb{E}\Big(\sum_{i=1}^{m} Y_i^{\tilde{q}}\Big)^{1/\tilde{q}} \\ &\geq \frac{1}{128\tilde{\alpha}^2}m^{1/\tilde{q}}\|Y_i\|_{\tilde{q}} \sim_\alpha m^{1/q}\|Y_i\|_{q\wedge\operatorname{Log} m}.\end{aligned}$$

The argument above shows that

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m, j\leq n}\big\|_{\ell_p^n\to\ell_q^m} \gtrsim_\alpha m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^{n} t_j X_{1,j}\Big\|_{q\wedge\operatorname{Log} m}.$$

The bound by the other term follows by the following duality

$$\big\|(X_{i,j})_{i\leq m, j\leq n}\big\|_{\ell_p^n\to\ell_q^m} = \big\|(X_{j,i})_{j\leq n, i\leq m}\big\|_{\ell_{q^*}^m\to\ell_{p^*}^n}. \tag{10}$$

$\square$

5. Formula in the square case

This section contains proofs of Propositions 4 and 5, which immediately yield the equivalence of formulas from Theorem 1 and Corollary 6 in the square case.

Proof of Proposition 4. By duality it suffices to show that for $p^* \geq q \vee 2$,

$$n^{1/q} \sup_{t\in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_{q\wedge \operatorname{Log} n} + n^{1/p^*} \sup_{s\in B_{q^*}^n} \Big\| \sum_{i=1}^n s_i X_{i,1} \Big\|_{p^*\wedge \operatorname{Log} n} \sim_\alpha n^{1/q} \|X_{1,1}\|_{q\wedge \operatorname{Log} n}. \tag{11}$$

The lower bound is obvious (with constant 1). To derive the upper bound we observe first that if we substituted q and p^* by $q \wedge \operatorname{Log} n$ and $p^* \wedge \operatorname{Log} n$, respectively, then both sides of (11) would change only be a constant factor. So it is enough to consider the case $\operatorname{Log} n \geq p^* \geq q \vee 2$.

Now we shall show that

$$\Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q \lesssim_\alpha \|X_{1,1}\|_q \quad \text{for every } t \in B_p^n. \tag{12}$$

To this end fix $t \in B_p^n$ and assume without loss of generality that $t_1 \geq t_2 \geq \cdots \geq t_n \geq 0$. If $1 \leq q \leq 4$, then by (7) we have

$$\Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q \lesssim_\alpha \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_2 = \|t\|_2 \|X_{1,1}\|_2 \leq \alpha \|X_{1,1}\|_1 \leq \alpha \|X_{1,1,}\|_q.$$

If $q \geq 4$, then

$$\Big\| \sum_{j\leq e^{4q}} t_j X_{1,j} \Big\|_q \leq \sum_{j\leq e^{4q}} |t_j| \|X_{1,1}\|_q \leq e^{4q/p^*} \|t\|_p \|X_{1,1}\|_q \leq e^4 \|X_{1,1}\|_q.$$

Moreover, by Rosenthal's inequality [5, Theorem 1.5.11],

$$\Big\| \sum_{j>e^{4q}} t_j X_{1,j} \Big\|_q \leq C \frac{q}{\operatorname{Log} q} (\|(t_j)_{j>e^{4q}}\|_2 \|X_{1,1}\|_2 + \|(t_j)_{j>e^{4q}}\|_q \|X_{1,1}\|_q).$$

If $j > e^{4q}$, then $t_j \leq j^{-1/p} \leq e^{-4q/p}$, so for $p^* \geq q \geq 4$ we have

$$\|(t_j)_{j>e^{4q}}\|_q \leq \|(t_j)_{j>e^{4q}}\|_2 \leq \|t\|_p^{p/2} \max_{j>e^{4q}} t_j^{(2-p)/2} \leq \|t\|_p^{p/2} (e^{-4q/p})^{1-p/2} \leq e^{-q}$$

and (12) follows.

To conclude the proof it is enough to show that for $\operatorname{Log} n \geq p^* \geq q \vee 2$,

$$n^{1/p^*} \sup_{s\in B_{q^*}^n} \Big\| \sum_{i=1}^n s_i X_{i,1} \Big\|_{p^*} \lesssim_\alpha n^{1/q} \sup_{t\in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q + n^{1/q} \|X_{1,1}\|_q. \tag{13}$$

For $k = 0, 1, \ldots$ define $\rho_k := 32\beta^2 \operatorname{Log}^{(k)}(p^*)$, where $\operatorname{Log}^{(k+1)} x := \operatorname{Log}(\operatorname{Log}^{(k)} x)$, $\operatorname{Log}^{(0)} x := x$, and $\beta = \frac{1}{2} \vee \log_2 \alpha$. Observe that $(\rho_k)_k$ is nonincreasing and for large k we have $\rho_k = 32\beta^2$.

If $p^*/q \leq 32\beta^2$, i.e., $p^* \leq 32\beta^2 q$, then (7) implies that

$$\Big\| \sum_{i=1}^n s_i X_{i,1} \Big\|_{p^*} \lesssim_\alpha \Big(\frac{p^*}{q}\Big)^\beta \Big\| \sum_{i=1}^n s_i X_{i,1} \Big\|_q \lesssim_\alpha \Big\| \sum_{i=1}^n s_i X_{i,1} \Big\|_q.$$

Moreover, $B_{q^*}^n \subset n^{1/q-1/p^*} B_p^n$, so in this case

$$n^{1/p^*} \sup_{s\in B_{q^*}^n} \Big\| \sum_{i=1}^n s_i X_{i,1} \Big\|_{p^*} \lesssim_\alpha n^{1/q} \sup_{t\in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q$$

and (13) follows.

Now suppose that $\rho_k < p^*/q \le \rho_{k-1}$ for some $k \ge 1$. Define $q_k := 2p^*/\rho_k \ge q \vee 2$. Estimates (7) and (12), applied with $p^* := q_k$ and $q := q_k \ge 2$, yield

$$\sup_{s\in B_{q_k^*}} \Big\|\sum_{i=1}^n s_i X_{i,1}\Big\|_{p^*} \lesssim_\alpha \rho_k^\beta \sup_{s\in B_{q_k^*}} \Big\|\sum_{i=1}^n s_i X_{i,1}\Big\|_{q_k} \lesssim_\alpha \rho_k^\beta \|X_{1,1}\|_{q_k} \lesssim_\alpha \Big(\frac{\rho_k q_k}{q}\Big)^\beta \|X_{1,1}\|_q.$$

Since $q_k \ge q$ we have $B_{q^*}^n \subset n^{1/q - 1/q_k} B^n_{q_k^*}$. Therefore,

$$n^{1/p^*} \sup_{s\in B^n_{q^*}} \Big\|\sum_{i=1}^n s_i X_{i,1}\Big\|_{p^*} \lesssim_\alpha \Big(\frac{p^*}{q}\Big)^\beta n^{1/p^* - 1/q_k} n^{1/q} \|X_{1,1}\|_q = \Big(\frac{p^*}{q}\Big)^\beta n^{\frac{2-\rho_k}{2p^*}} n^{1/q} \|X_{1,1}\|_q.$$

Hence, it is enough to show that

$$\Big(\frac{p^*}{q}\Big)^\beta n^{\frac{2-\rho_k}{2p^*}} \le 1. \tag{14}$$

Observe that $p^*/q \ge 32\beta^2 \ge 8$, so $\operatorname{Log} n \ge p^* \ge 8q \ge 8$, $\operatorname{Log}(p^*/q) = \ln(p^*/q)$, and $\operatorname{Log} n = \ln n$. Thus, (14) is equivalent to

$$\frac{\rho_k - 2}{2\beta \operatorname{Log}(\frac{p^*}{q})} \ge \frac{p^*}{\operatorname{Log} n}. \tag{15}$$

We have $p^*/\operatorname{Log} n \le 1$ and

$$\frac{\rho_k - 2}{2\beta\operatorname{Log}(\frac{p^*}{q})} \ge \frac{24\beta^2 \operatorname{Log}^{(k)}(p^*)}{2\beta \operatorname{Log} \rho_{k-1}} \ge \frac{24\beta^2 - 2\beta + 2\beta \operatorname{Log}^{(k)}(p^*)}{2\beta \ln(32\beta^2) + 2\beta \operatorname{Log}^{(k)}(p^*)} \ge 1,$$

where in the first inequality we used $\operatorname{Log}^{(k)} x \ge 1$ and $8\beta^2 \ge 2$, in the second one $\operatorname{Log}(ab) \le \ln a + \operatorname{Log} b$ for $a \ge 1$, and in the last one $\ln(32e\beta^2) \le 12\beta$ for $\beta \ge 1/2$. □

Now we move to the proof of Proposition 5. Observe that m, n are arbitrary (not necessarily $m = n$).

Proof of Proposition 5. It is enough to establish the first part of the assertion. We have

$$\sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_j\Big\|_{\tilde q} \le \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_j\Big\|_2 = \sup_{t\in B_p^n} \|t\|_2 \|X_1\|_2$$

and the upper bound immediately follows.

If $p \le 2$, then $(1/p^* - 1/2)_+ = 0$ and the lower bound is obvious (with constant 1 instead of $1/2\sqrt{2}$). Assume that $p > 2$. Let $(X_j')_j$ be an independent copy of $(X_j)_j$, and let ε_i's be iid Rademachers independent of all other random variables. Then

$$\begin{aligned}\sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_j\Big\|_{\tilde q} &\ge n^{-1/p}\Big\|\sum_{j=1}^n X_j\Big\|_{\tilde q} \ge \frac12 n^{-1/p}\Big\|\sum_{j=1}^n (X_j - X_j')\Big\|_{\tilde q} = \frac12 n^{-1/p}\Big\|\sum_{j=1}^n \varepsilon_j (X_j - X_j')\Big\|_{\tilde q} \\ &\ge \frac12 n^{-1/p}\Big\|\sum_{j=1}^n \varepsilon_j (X_j - \mathbb{E}X_j')\Big\|_{\tilde q} = \frac12 n^{-1/p}\Big\|\sum_{j=1}^n \varepsilon_j X_j\Big\|_{\tilde q}.\end{aligned}$$

Moreover, Khintchine's and Hölder's inequalities yield (recall that $\tilde q \in [1,2]$)

$$\mathbb{E}\Big|\sum_{j=1}^n \varepsilon_j X_j\Big|^{\tilde q} \ge 2^{-\tilde q/2}\mathbb{E}\Big(\sum_{j=1}^n X_j^2\Big)^{\tilde q/2} \ge 2^{-\tilde q/2} n^{\tilde q/2 - 1}\mathbb{E}\sum_{j=1}^n |X_j|^{\tilde q} = 2^{-\tilde q/2} n^{\tilde q/2}\mathbb{E}|X_1|^{\tilde q}. \qquad \square$$

6. Upper bounds

To prove the upper bound in Theorem 1 we split the range $p^*, q \geq 1$ into several parts. In each of them we use different arguments to derive the asserted estimate.

6.1. Case $p^*, q \leq 2$.

In this subsection we shall show that the two-sided bound from Theorem 1 holds in the range $p^*, q \leq 2$ under the following mild 4th moment assumption

$$(\mathbb{E}X_{1,1}^4)^{1/4} \leq \alpha(\mathbb{E}X_{1,1}^2)^{1/2}. \tag{16}$$

Observe that then Hölder's inequality yields

$$\mathbb{E}X_{1,1}^2 \leq (\mathbb{E}X_{1,1}^4)^{1/3}(\mathbb{E}|X_{1,1}|)^{2/3} \leq \alpha^{4/3}(\mathbb{E}X_{1,1}^2)^{2/3}(\mathbb{E}|X_{1,1}|)^{2/3},$$

so

$$\mathbb{E}|X_{1,1}| \geq \alpha^{-2}(\mathbb{E}X_{1,1}^2)^{1/2}. \tag{17}$$

Let us first consider the case $p = q = 2$. Then we shall see that it may be easily extrapolated into the whole range of $p^*, q \leq 2$.

Proposition 21. *Let $(X_{i,j})_{i\leq m, j\leq n}$ be iid centered random variables satisfying* (16). *Then*

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_2^n\to\ell_2^m} \sim_\alpha (\mathbb{E}X_{1,1}^2)^{1/2}(\sqrt{n}+\sqrt{m}).$$

Proof. By [9, Theorem 2] we have

$$\begin{aligned}\mathbb{E}\big\|(X_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_2^n\to\ell_2^m} &\lesssim \max_j \sqrt{\sum_i \mathbb{E}X_{i,j}^2} + \max_i \sqrt{\sum_j \mathbb{E}X_{i,j}^2} + \sqrt[4]{\sum_{i,j}\mathbb{E}X_{i,j}^4} \\ &\leq (\mathbb{E}X_{1,1}^2)^{1/2}(\sqrt{n}+\sqrt{m}+\alpha\sqrt[4]{nm}) \lesssim_\alpha (\mathbb{E}X_{1,1}^2)^{1/2}(\sqrt{n}+\sqrt{m}).\end{aligned}$$

To get the lower bound we use Jensen's inequality and (17):

$$\begin{aligned}\mathbb{E}\big\|(X_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_2^n\to\ell_2^m} &\geq \max\Big\{\mathbb{E}\big\|(|X_{i,1}|)_{i\leq m}\big\|_2, \mathbb{E}\big\|(|X_{1,j}|)_{j\leq n}\big\|_2\Big\} \\ &\geq \max\Big\{\big\|(\mathbb{E}|X_{i,1}|)_{i\leq m}\big\|_2, \big\|(\mathbb{E}|X_{1,j}|)_{j\leq n}\big\|_2\Big\} \geq \alpha^{-2}(\mathbb{E}X_{1,1}^2)^{1/2}\sqrt{n\vee m}. \qquad \square\end{aligned}$$

Corollary 22. *Let $(X_{i,j})_{i\leq m,j\leq n}$ be iid centered random variables satisfying* (16). *Then for $p^*, q \leq 2$ we have*

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_p^n\to\ell_q^m} \sim_\alpha (\mathbb{E}X_{1,1}^2)^{1/2}(m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}).$$

Proof. Let $\varepsilon_{i,j}$'s be iid Rademacher random variables independent of $(X_{i,j})$. Symmetrization (as in the proof of Proposition 5) and (17) yields

$$\begin{aligned}\mathbb{E}\big\|(X_{i,j})_{i\leq m,j\leq n}^n\big\|_{\ell_p^n\to\ell_q^m} &\geq \frac{1}{2}\mathbb{E}\big\|(\varepsilon_{i,j}|X_{i,j}|)_{i\leq m,j\leq n}^n\big\|_{\ell_p^n\to\ell_q^m} \geq \frac{1}{2}\mathbb{E}\big\|(\varepsilon_{i,j}\mathbb{E}|X_{i,j}|)_{i\leq m,j\leq n}\big\|_{\ell_p^n\to\ell_q^m} \\ &\gtrsim_\alpha (\mathbb{E}X_{1,1}^2)^{1/2}\mathbb{E}\big\|(\varepsilon_{i,j})_{i\leq m,j\leq n}^n\big\|_{\ell_p^n\to\ell_q^m}.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\big\|(\varepsilon_{i,j})_{i\leq m,j\leq n}^n\big\|_{\ell_p^n\to\ell_q^m} &\geq n^{-1/p}\mathbb{E}\Big\|\Big(\sum_{j=1}^n\varepsilon_{i,j}\Big)_{i\leq m}\Big\|_q \sim n^{1/p^*-1}\Big(\mathbb{E}\Big\|\Big(\sum_{j=1}^n\varepsilon_{i,j}\Big)_{i\leq m}\Big\|_q^q\Big)^{1/q} \\ &= n^{1/p^*-1}m^{1/q}\Big\|\sum_{j=1}^n\varepsilon_{1,j}\Big\|_q \sim n^{1/p^*-1/2}m^{1/q},\end{aligned}$$

where in the first line we used the Kahane-Khintchine and in the second one the Khintchine inequalities. By duality (10) we get

$$\mathbb{E}\big\|(\varepsilon_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} = \mathbb{E}\big\|(\varepsilon_{i,j})_{i\le n,j\le m}\big\|_{\ell_{q^*}^m\to\ell_{p^*}^n} \gtrsim_\alpha m^{1/q-1/2}n^{1/p^*},$$

so the lower bound follows.

To get the upper bound we use Proposition 21 together with the following simple bound

$$\begin{aligned}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} &\le \|\mathrm{Id}\|_{\ell_p^n\to\ell_2^n}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_2^n\to\ell_2^m}\|\mathrm{Id}\|_{\ell_2^m\to\ell_q^m}\\ &= n^{1/2-1/p}m^{1/q-1/2}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_2^n\to\ell_2^m}.\end{aligned}$$ □

Corollary 22, Proposition 5 and (17) yield that under condition (16) Theorem 1 holds whenever $p^*, q \le 2$. Moreover, one may prove by repeating the same arguments that the two-sided estimate

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \sim_\alpha m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}$$

holds for every $p^*, q \le 2$ and independent random variables $X_{i,j}$ satisfying (16) and $\mathbb{E}X_{i,j}^2 = 1$ (we do not need to assume that $X_{i,j}$'s are identically distributed).

6.2. **Case $p^* \ge \operatorname{Log} n$ or $q \ge \operatorname{Log} m$.** In this subsection we shall show that Theorem 1 holds under the regularity assumption (3) if $p^* \ge \operatorname{Log} n$ or $q \ge \operatorname{Log} m$.

Remark 23. For $p^* \ge \operatorname{Log} n$, $\tilde{q} \in [1,\infty)$ and iid random variables X_i we have

$$\|X_1\|_{\tilde{q}} \le \sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jX_j\Big\|_{\tilde{q}} \le e\|X_1\|_{\tilde{q}}.$$

Similarly, for $q \ge \operatorname{Log} m$ and $\tilde{p} \in [1,\infty)$,

$$\|X_1\|_{\tilde{p}} \le \sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_iX_i\Big\|_{\tilde{p}} \le e\|X_1\|_{\tilde{p}}.$$

Proof. The lower bounds are obvious. To see the first upper bound it is enough to use the triangle inequality in $L_{\tilde{q}}$ and observe that $\|t\|_1 \le n^{1/p^*}\|t\|_p \le e$ for $p^* \ge \operatorname{Log} n$ and $t \in B_p^n$. □

By Remark 23, Theorem 1 in the case $p^* \ge \operatorname{Log} n$ or $q \ge \operatorname{Log} m$ reduces to the following statement.

Proposition 24. *Let $(X_{i,j})_{i\le n,j\le n}$ be iid centered random variables such that* (3) *holds. Then for $q \ge \operatorname{Log} m$,*

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \sim_\alpha \sup_{t\in B_p^n}\Big\|\sum_{j\le n} t_jX_{1,j}\Big\|_{\operatorname{Log} m} + n^{1/p^*}\|X_{1,1}\|_{p^*\wedge\operatorname{Log} n}.$$

Analogously, for $p^* \ge \operatorname{Log} n$,

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \sim_\alpha \sup_{s\in B_{q^*}^m}\Big\|\sum_{i\le m} s_iX_{i,1}\Big\|_{\operatorname{Log} n} + m^{1/q}\|X_{1,1}\|_{q\wedge\operatorname{Log} m}.$$

Proof. The lower bounds follow by Section 4 and Remark 23. Hence, we should establish only the upper bounds.

By duality (10) it is enough to consider the case $q \ge \operatorname{Log} m$. We have $\|(x_i)_{i\le m}\|_\infty \le \|(x_i)_{i\le m}\|_q \le e\|(x_i)_{i\le m}\|_\infty$, so

$$\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \sim \max_{i\le m}\big\|(X_{i,j})_{j\le n}\big\|_{p^*}.$$

Note that for arbitrary random variables $Y_1, \dots, Y_k$ we have

$$\mathbb{E}\max_{i\le k}|Y_i| \le \big\|\max_{i\le k}|Y_i|\big\|_{\operatorname{Log} k} \le \Big(\sum_{i\le k}\mathbb{E}|Y_i|^{\operatorname{Log} k}\Big)^{1/\operatorname{Log} k} \le e\max_{i\le k}\|Y_i\|_{\operatorname{Log} k}, \tag{18}$$

Hence,

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \lesssim \big\|\big\|(X_{1,j})_{j\le n}\big\|_{p^*}\big\|_{\operatorname{Log} m}.$$

Inequality (6) (applied with $m=1$, $U=\{1\}\otimes B_p^n$, and $\rho=\operatorname{Log} m$) implies

$$\Big\|\big\|(X_{1,j})_{j\le n}\big\|_{p^*}\Big\|_{\operatorname{Log} m} \sim_\alpha \mathbb{E}\big\|(X_{1,j})_{j\le n}\big\|_{p^*} + \sup_{t\in B_p^n}\Big\|\sum_{j\le n}t_jX_{1,j}\Big\|_{\operatorname{Log} m}.$$

If $p^*\ge \operatorname{Log} n$, then

$$\mathbb{E}\big\|(X_{1,j})_{j\le n}\big\|_{p^*} \sim \mathbb{E}\max_{j\le n}|X_{1,j}| \lesssim \|X_{1,1}\|_{\operatorname{Log} n},$$

where the last bound follows by (18). In the case $p^*\le\operatorname{Log} n$ we have

$$\mathbb{E}\big\|(X_{1,j})_{j\le n}\big\|_{p^*} \le \Big(\mathbb{E}\big\|(X_{1,j})_{j\le n}\big\|_{p^*}^{p^*}\Big)^{1/p^*} = n^{1/p^*}\|X_{1,1}\|_{p^*}. \qquad \square$$

6.3. **Outline of proofs of upper bounds in remaining ranges.** Let us first note that we may assume that random variables $X_{i,j}$ are symmetric, due to the following remark.

Remark 25. It suffices to prove the upper bound from Theorem 1 under additional assumption that random variables X_{ij} are symmetric.

Proof. Let $(X'_{i,j})_{i\le m,j\le n}$ be an independent copy of a random matrix $(X_{i,j})_{i\le m,j\le n}$, and let $Y_{i,j}=X_{i,j}-X'_{i,j}$. Then (3) implies for every $\rho\ge 1$,

$$\begin{aligned}\|Y_{i,j}\|_{2\rho} &\le \|X_{i,j}\|_{2\rho}+\|X'_{i,j}\|_{2\rho} = 2\|X_{i,j}\|_{2\rho} \le 2\alpha\|X_{i,j}\|_\rho = 2\alpha\|X_{i,j}-\mathbb{E}X'_{i,j}\|_\rho\\ &\le 2\alpha\|X_{i,j}-X'_{i,j}\|_\rho = 2\alpha\|Y_{i,j}\|_\rho.\end{aligned}$$

Therefore, $(Y_{i,j})_{i\le m,j\le n}$ are iid symmetric random variables satisfying (3) with $\alpha \coloneqq 2\alpha$. Moreover,

$$\begin{aligned}\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}X_{i,j}s_it_j &= \mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}(X_{i,j}-\mathbb{E}X'_{i,j})s_it_j\\ &\le \mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}(X_{i,j}-X'_{i,j})s_it_j = \mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}Y_{i,j}s_it_j,\end{aligned}$$

so it suffices to upper bound $\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}Y_{i,j}s_it_j$ by

$$\begin{aligned}&m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jY_{1,j}\Big\|_{q\wedge\operatorname{Log} m} + n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_iY_{i,1}\Big\|_{p^*\wedge\operatorname{Log} n}\\ &\qquad\le 2m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jX_{1,j}\Big\|_{q\wedge\operatorname{Log} m} + 2n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_iX_{i,1}\Big\|_{p^*\wedge\operatorname{Log} n}. \qquad\square\end{aligned}$$

We shall also assume without loss of generality that $\alpha\ge\sqrt{2}$. Then (7) holds with $\beta=\log_2\alpha$.

One of the ideas used in the sequel is to decompose certain subsets S of $B_{q^*}^m$ and T of B_p^n in the following way. Let T be a monotone subset of B_p^n (we need the monotonicity only to guarantee that if $t\in T$ and $I\subset[n]$, then $(tI_{\{i\in I\}})\in T$). Fix $a\in(0,1]$ and write $t\in T$ as $t=(t_iI_{\{|t_i|\le a\}})+(t_iI_{\{|t_i|>a\}})$. Since $a^p\,|\{i\colon |t_i|>a\}|\le\|t\|_p^p\le 1$, we get $T\subset T_1+T_2$, where

$$T_1 = T\cap aB_\infty^n, \qquad T_2=\{t\in T\colon |\operatorname{supp} t|\le a^{-p}\}.$$

Choosing $a = k^{-1/p}$ we see that for every $1 \le k \le n$ we have $T \subset T_1 + T_2$, where

$$T_1 = T \cap k^{-1/p} B_\infty^n, \qquad T_2 = \{t \in T \colon |\operatorname{supp} t| \le k\}.$$

Similarly, we may also decompose monotone subsets S of $B_{q^*}^m$ into two parts: one containing vectors with bounded ℓ_∞-norm and the other containing vectors with bounded support.

Once we decompose B_p^n and $B_{q^*}^m$ as above, we need to control the quantities of the form $\mathbb{E}\sup_{s\in S, t\in T} \sum X_{i,j} s_i t_j$ provided we have additional information about the ℓ_∞-norm or the size of the support (or both of them) for vectors from S and T. In the next subsection we present a couple of lemmas allowing to upper bound this type of quantities in various situations.

6.4. **Tools used in proofs of upper bounds in remaining ranges.**

Lemma 26. *Assume that $k, l \in \mathbb{Z}_+$, $p^*, q \ge 1$, $a, b > 0$ and $(X_{i,j})_{i\le m, j\le n}$ are iid symmetric random variables satisfying* (3) *with $\alpha \ge \sqrt{2}$, and $\mathbb{E}X_{i,j}^2 = 1$. Denote $\beta = \log_2 \alpha$.*

If $q \ge 2$, $S \subset B_{q^*}^m \cap aB_\infty^m$ and $T \subset \{t \in B_p^n \colon |\operatorname{supp}(t)| \le k\}$, then

(19)
$$\mathbb{E}\sup_{s\in S, t\in T} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \lesssim_\alpha m^{1/q} \sup_{t\in T} \Big\| \sum_{j=1}^n X_{1,j} t_j \Big\|_q + \big(n \wedge (k \operatorname{Log} n)\big)^\beta k^{(1/p^*-1/2)\vee 0} a^{(2-q^*)/2}.$$

If $p^* \ge 2$, $S \subset \{s \in B_{q^*}^m \colon |\operatorname{supp}(s)| \le l\}$ and $T \subset B_p^n \cap bB_\infty^n$, then

(20)
$$\mathbb{E}\sup_{s\in S, t\in T} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \lesssim_\alpha n^{1/p^*} \sup_{s\in S} \Big\| \sum_{i=1}^m X_{i,1} s_i \Big\|_{p^*} + \big(m \wedge (l \operatorname{Log} m)\big)^\beta l^{(1/q-1/2)\vee 0} b^{(2-p)/2}.$$

Proof. It suffices to prove (19), since (20) follows by duality.

Without loss of generality we may assume that $k \le n$. Let T_0 be a $\frac{1}{2}$-net (with respect to ℓ_p^n-metric) in T of cardinality at most $5^n \wedge \big(\binom{n}{k} 5^k\big) \le 5^n \wedge (5n)^k = e^d$, where $d = (n \ln 5) \wedge (k \ln(5n))$.

Then by (18) we get

$$\mathbb{E}\sup_{s\in S, t\in T} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \le 2\mathbb{E}\sup_{t\in T_0}\sup_{s\in S} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \le 2e \sup_{t\in T_0} \Big(\mathbb{E}\sup_{s\in S} \Big| \sum_{i\le m, j\le n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d}$$

$$\le 2e \sup_{t\in T} \Big(\mathbb{E}\sup_{s\in S} \Big| \sum_{i\le m, j\le n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d}. \tag{21}$$

Fix $t \in T$. By (6) applied with $U = \{(s_i t_j)_{i,j} \colon s \in S\}$ and $\rho = d$ we have

$$\Big(\mathbb{E}\sup_{s\in S} \Big| \sum_{i\le m, j\le n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d} \lesssim_\alpha \mathbb{E}\sup_{s\in S} \Big| \sum_{i\le m, j\le n} X_{i,j} s_i t_j \Big| + \sup_{s\in S} \Big\| \sum_{i\le m, j\le n} X_{i,j} s_i t_j \Big\|_d. \tag{22}$$

Since $S \subset B_{q^*}^m$,

$$\mathbb{E}\sup_{s\in S} \Big| \sum_{i\le m, j\le n} X_{i,j} s_i t_j \Big| \le \Big(\mathbb{E} \Big\| \Big(\sum_{j=1}^n X_{i,j} t_j \Big)_{i\le m} \Big\|_q^q \Big)^{1/q} = m^{1/q} \Big\| \sum_{j=1}^n X_{1,j} t_j \Big\|_q. \tag{23}$$

Since $\alpha \ge \sqrt{2}$, $\beta = \frac{1}{2} \vee \log_2 \alpha$, so by inequality (7)

$$\sup_{s\in S} \Big\| \sum_{i\le m, j\le n} X_{i,j} s_i t_j \Big\|_d \lesssim_\alpha d^\beta \sup_{s\in S, t\in T} \|s\|_2 \|t\|_2 \le d^\beta \sup_{s\in S} \|s\|_\infty^{(2-q^*)/2} \|s\|_{q^*}^{q^*/2} \sup_{t\in T} k^{(1/2-1/p)\vee 0} \|t\|_p$$

$$\le d^\beta k^{(1/p^*-1/2)\vee 0} a^{(2-q^*)/2}. \tag{24}$$

Inequalities (21)-(24) yield (19). □

In the sequel $(g_{i,j})_{i\le m, j\le n}$ are iid standard Gaussian random variables.

Lemma 27. *Let $(X_{i,j})_{i\le m,j\le n}$ be iid symmetric random variables satisfying* (3) *and* $\mathbb{E}X_{i,j}^2=1$. *Let $\beta=\log_2\alpha$. Then for any nonempty bounded sets $S\subset\mathbb{R}^m$ and $T\subset\mathbb{R}^n$ we have*

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}X_{i,j}s_it_j\lesssim \mathrm{Log}^{\beta}(mn)\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}g_{i,j}s_it_j.$$

Proof. Since $X_{i,j}$'s are independent and symmetric, $(X_{i,j})_{i\le m,j\le n}$ has the same distribution as $(\varepsilon_{i,j}|X_{i,j}|)_{i\le m,j\le n}$, where $(\varepsilon_{i,j})_{i\le m,j\le n}$ are iid Rademachers independent of $X_{i,j}$'s. By the contraction principle

$$\begin{aligned}\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}X_{i,j}s_it_j&=\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}\varepsilon_{i,j}|X_{i,j}|s_it_j\\ &\le \mathbb{E}\max_{i\le m,j\le n}|X_{i,j}|\cdot\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}\varepsilon_{i,j}s_it_j.\end{aligned}\tag{25}$$

Moreover, by (18) and regularity assumption (3) we have

$$\mathbb{E}\max_{i\le m,j\le n}|X_{i,j}|\le e\|X_{1,1}\|_{\mathrm{Log}(mn)}\lesssim\mathrm{Log}^{\beta}(mn)\|X_{1,1}\|_2=\mathrm{Log}^{\beta}(mn).\tag{26}$$

Jensen's inequality yields

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}\varepsilon_{i,j}s_it_j\sim\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}\varepsilon_{i,j}\mathbb{E}|g_{i,j}|s_it_j\lesssim\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}g_{i,j}s_it_j.\tag{27}$$

Inequalities (25)-(27) yield the assertion. □

The next result is an immediate consequence of the contraction principle (see also (25) together with (27)), but turns out to be helpful.

Lemma 28. *Let $(X_{i,j})_{i\le m,j\le n}$ be centered random variables. Then*

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}X_{i,j}s_it_j\lesssim\max_{i,j}\|X_{i,j}\|_\infty\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}g_{i,j}s_it_j.$$

Let us recall Chevet's inequality from [4]:

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}g_{i,j}s_it_j\lesssim\sup_{s\in S}\|s\|_2\mathbb{E}\sup_{t\in T}\sum_{j\le n}g_jt_j+\sup_{t\in T}\|t\|_2\mathbb{E}\sup_{s\in S}\sum_{i\le m}g_is_i.\tag{28}$$

We use it to derive the following two lemmas.

Lemma 29. *Let $q\ge 2$, $p\ge 1$, $l\le m$, $S\subset\{s\in B_{q^*}^m\colon\ |\mathrm{supp}(s)|\le l\}\cap aB_\infty^m$, and $T\subset B_p^n$. Then*

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}g_{i,j}s_it_j\lesssim\sqrt{p^*}a^{(2-q^*)/2}n^{1/p^*}+n^{(1/p^*-1/2)\vee 0}\sqrt{\mathrm{Log}\,m}\,l^{1/q}.$$

If we assume additionally that $l=m$, $p^*\ge 2$, and $T\subset bB_\infty^n$, then

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}g_{i,j}s_it_j\lesssim\sqrt{p^*}a^{(2-q^*)/2}n^{1/p^*}+\sqrt{q}b^{(2-p)/2}m^{1/q}.\tag{29}$$

Proof. We have

$$\sup_{t\in T}\|t\|_2\le\sup_{t\in B_p^n}\|t\|_2=n^{(1/p^*-1/2)\vee 0},$$

$$\sup_{s\in S}\|s\|_2\le\sup_{s\in S}\|s\|_{q^*}^{q^*/2}\|s\|_\infty^{(2-q^*)/2}\le a^{(2-q^*)/2},$$

$$\mathbb{E}\sup_{t\in T}\sum_{j=1}^n g_jt_j\le\mathbb{E}\sup_{t\in B_p^n}\sum_{j=1}^n g_jt_j=\mathbb{E}\|(g_j)_{j=1}^n\|_{p^*}\le(\mathbb{E}\|(g_j)_{j=1}^n\|_{p^*}^{p^*})^{1/p^*}=\|g_1\|_{p^*}n^{1/p^*}\le\sqrt{p^*}n^{1/p^*},$$

and

$$\mathbb{E}\sup_{s\in S}\sum_{i=1}^m g_i s_i \le \mathbb{E}\sup_{I\subset[m],|I|\le l}\Big(\sum_{i\in I}|g_i|^q\Big)^{1/q} \le l^{1/q}\mathbb{E}\max_{i\le m}|g_i| \lesssim l^{1/q}\sqrt{\operatorname{Log} m}.$$

The first assertion follows by Chevet's inequality (28) and the four bounds above.

In the case when $l = m$, $p^* \ge 2$, and $T \subset bB_\infty^n$ we use a different bound for $\sup_{t\in T}\|t\|_2$, namely

$$\sup_{t\in T}\|t\|_2 \le \sup_{t\in T}\|t\|_p^{p/2}\|t\|_\infty^{(2-p)/2} \le b^{(2-p)/2},$$

and for $\mathbb{E}\sup_{s\in S}\sum_{i=1}^m g_i s_i$, namely

$$\mathbb{E}\sup_{s\in S}\sum_{i=1}^m g_i s_i \le \mathbb{E}\sup_{s\in B_{q^*}^m}\sum_{i=1}^m g_i s_i \le \sqrt{q}m^{1/q}. \qquad \square$$

The next lemma is a slight modification of the previous one.

Lemma 30. *Let $2 \le p^*, q \le \gamma$, $l \le m$, $S \subset \{s \in B_{q^*}^m \colon\ |\operatorname{supp}(s)| \le l\} \cap aB_\infty^m$ and $T \subset B_p^n$. Then*

$$\mathbb{E}\sup_{s\in S, t\in T}\sum_{i\le m, j\le n} g_{i,j}s_i t_j \lesssim \sqrt{\gamma}\Big(a^{(2-q^*)/2}n^{1/p^*} + \sqrt{\operatorname{Log}(m/l)}\, l^{1/q}\Big).$$

Proof. We proceed as in the previous proof, observing that $\sqrt{p^*} \le \sqrt{\gamma}$ and, by [11, Lemmas 3.12 and 4.2],

$$\mathbb{E}\sup_{I\subset[m],|I|\le l}\Big(\sum_{i\in I}|g_i|^q\Big)^{1/q} \lesssim \sqrt{\gamma \vee \operatorname{Log}(m/l)}\, l^{1/q}. \qquad \square$$

The next proposition is a consequence of the $\ell_2^n \to \ell_2^m$ bound from [9].

Lemma 31. *Let $(X_{i,j})_{i\le m, j\le n}$, be be iid symmetric random variables satisfying (3) with $\alpha \ge \sqrt{2}$ and $\mathbb{E}X_{i,j}^2 = 1$. Then for $M > 0$,*

$$\mathbb{E}\big\|\big(X_{i,j}I_{\{|X_{i,j}|\ge M\}}\big)_{i\le m,j\le n}\big\|_{\ell_2^n\to\ell_2^m} \lesssim_\alpha (\sqrt{n}+\sqrt{m})\exp\Big(-\frac{\ln\alpha}{10}M^{1/\log_2\alpha}\Big).$$

Proof. By [9, Theorem 2] we have

$$\mathbb{E}\big\|\big(X_{i,j}I_{\{|X_{i,j}|\ge M\}}\big)_{i\le m,j\le n}\big\|_{\ell_2^n\to\ell_2^m} \le \max_{i\le m}\Big(\sum_{j\le n}\mathbb{E}X_{i,j}^2 I_{\{|X_{i,j}|\ge M\}}\Big)^{1/2} + \max_{j\le n}\Big(\sum_{i\le m}\mathbb{E}X_{i,j}^2 I_{\{|X_{i,j}|\ge M\}}\Big)^{1/2} + \Big(\sum_{i\le m,j\le n}\mathbb{E}X_{i,j}^4 I_{\{|X_{i,j}|\ge M\}}\Big)^{1/4}.$$

Regularity condition (3) and the normalization $\|X_{i,j}\|_2 = 1$ yields $\|X_{i,j}\|_\rho \le \alpha^{\log_2\rho}$ for all $\rho \ge 1$. Thus, for all $\rho \ge 4$,

$$\big(\mathbb{E}X_{i,j}^2 I_{\{|X_{i,j}|\ge M\}}\big)^{1/2} \le \big(\mathbb{E}X_{i,j}^4 I_{\{|X_{i,j}|\ge M\}}\big)^{1/4} \le (M^{4-\rho}\mathbb{E}|X_{i,j}|^\rho)^{1/4} \le M\Big(\frac{\alpha^{\log_2\rho}}{M}\Big)^{\rho/4}.$$

Let us choose $\rho := \frac{1}{2}M^{1/\log_2\alpha}$. If $M \ge \alpha^3$, then $\rho \ge 4$, so

$$M\Big(\frac{\alpha^{\log_2\rho}}{M}\Big)^{\rho/4} = M\alpha^{-\rho/4} = M\exp\Big(-\frac{\ln\alpha}{8}M^{1/\log_2\alpha}\Big) \lesssim_\alpha \exp\Big(-\frac{\ln\alpha}{10}M^{1/\log_2\alpha}\Big).$$

If $M \le \alpha^3$, then

$$\big(\mathbb{E}X_{i,j}^2 I_{\{|X_{i,j}|\ge M\}}\big)^{1/2} \le \big(\mathbb{E}X_{i,j}^4 I_{\{|X_{i,j}|\ge M\}}\big)^{1/4} \le (\mathbb{E}X_{ij}^4)^{1/4} \le \alpha \lesssim_\alpha \exp\Big(-\frac{\ln\alpha}{10}M^{1/\log_2\alpha}\Big). \qquad \square$$

6.5. Case $p^* \gtrsim_\alpha \operatorname{Log} m$ or $q \gtrsim_\alpha \operatorname{Log} n$.

Proposition 32. *Theorem 1 holds in the case $p^* \gtrsim_\alpha \operatorname{Log} m$ or $q \gtrsim_\alpha \operatorname{Log} n$.*

Proof. Without loss of generality we may assume that $\|X_{i,j}\|_2 = 1$. By Remark 25 it suffices to assume that $X_{i,j}$'s are symmetric and $\alpha \geq \sqrt{2}$, and by duality (10) it suffices to consider the case $q \geq C_0(\alpha) \operatorname{Log} n$, where

$$C_0(\alpha) = 8\beta = 8 \log_2 \alpha.$$

In particular $q \geq 4$, so $q^* \leq 4/3$. By Subsection 6.2 it suffices to consider the case $p^* \leq \operatorname{Log} n$.

Define

$$S_1 = B_{q^*}^m \cap e^{-q} B_\infty^m, \quad S_2 = \{s \in B_{q^*}^m : |\operatorname{supp}(s)| \leq e^{qq^*}\}.$$

Then $B_{q^*}^m \subset S_1 + S_2$.

If $s \in S_2$, then

$$\|s\|_1 \leq \|s\|_{q^*} |\operatorname{supp}(s)|^{-1/q^*+1} \leq e^{q^*} \leq e^{4/3},$$

so $S_2 \subset e^{4/3} B_{\infty^*}^m$. Thus, Proposition 24 and (7) imply

$$\begin{aligned}\mathbb{E} \sup_{s\in S_2, t\in B_p^n} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j &\lesssim \mathbb{E}\big\|(X_{i,j})_{i\leq m, j\leq n}\big\|_{\ell_p^n\to\ell_\infty^m} \\ &\sim_\alpha \sup_{t\in B_p^n} \Big\|\sum_{j\leq n} t_j X_{1,j}\Big\|_{\operatorname{Log} m} + n^{1/p^*}\|X_{1,1}\|_{p^*} \\ &\lesssim_\alpha \Big(1 \vee \frac{\operatorname{Log} m}{q}\Big)^\beta \sup_{t\in B_p^n} \Big\|\sum_{j\leq n} t_j X_{1,j}\Big\|_q + n^{1/p^*}\|X_{1,1}\|_{p^*}.\end{aligned}$$

Since the function $0 < q \mapsto \frac{1}{q}\ln m + \beta \ln q$ attains its minimum at $q = \ln m/\beta$, where the function's value is equal to $-\beta\ln(\beta/e) + \beta \ln\ln m$, we have $(\operatorname{Log} m/q)^\beta \lesssim_\alpha m^{1/q}$. Hence, the previous upper bound yields

$$\mathbb{E} \sup_{s\in S_2, t\in B_p^n} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j \lesssim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j\leq n} t_j X_{1,j}\Big\|_q + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*}. \tag{30}$$

Moreover, (19) from Lemma 26 applied with $S = S_1$, $T = B_p^n$, $a = e^{-q}$, and $k = n$, together with the inequality $q^* \leq 4/3$, implies that

$$\mathbb{E} \sup_{s\in S_1, t\in B_p^n} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j \lesssim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j\leq n} t_j X_{1,j}\Big\|_q + n^{\beta + ((1/p^*-1/2)\vee 0)} e^{-q/3}. \tag{31}$$

Since $q \geq C_0(\alpha) \operatorname{Log} n \geq 3\beta \ln n$ and $\|X_{1,1}\|_{p^*} \gtrsim_\alpha \|X_{1,1}\|_2 = 1$, inequalities (30) and (31) yield the assertion. □

6.6. Case $p^*, q \geq 3$.

By Subsection 6.2 we may assume that $p^* \leq \operatorname{Log} n$ and $q \leq \operatorname{Log} m$. In this subsection we restrict ourselves to to the case $p^*, q \geq 3$. However, similar proofs work also in the range $p^*, q \geq 2 + \varepsilon$, where $\varepsilon > 0$ is arbitrary — in this case the constants in upper bounds depend also on ε and blow up when ε approaches 0. If p^* or q lies above and close to 2, then we need different arguments, which we show in next subsections.

Lemma 33. *Assume that $3 \leq p^*, q \leq \operatorname{Log}(mn)$, $(X_{i,j})_{i\leq m, j\leq n}$ are iid symmetric random variables satisfying (3) with $\alpha \geq \sqrt{2}$, $\mathbb{E}X_{i,j}^2 = 1$, $S \subset B_{q^*}^m \cap \operatorname{Log}^{-8\beta}(mn) B_\infty^m$, and $T \subset B_p^n \cap \operatorname{Log}^{-8\beta}(mn) B_\infty^n$, where $\beta = \log_2 \alpha$. Then*

$$\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j \lesssim m^{1/q} + n^{1/p^*}.$$

Proof. Lemma 27 and inequality (29) yield

(32)
$$\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \lesssim \mathrm{Log}^{1/2+\beta}(mn)\big(m^{1/q}\,\mathrm{Log}^{-4\beta(2-p)}(mn) + n^{1/p^*}\,\mathrm{Log}^{-4\beta(2-q^*)}(mn)\big).$$

Since $p^* \ge 3$, $(2-p) \ge 1/2$, so

$$\mathrm{Log}^{-4\beta(2-p)}(mn) \le \mathrm{Log}^{-2\beta}(mn) \le \mathrm{Log}^{-\beta-1/2}(mn),$$

and similarly

$$\mathrm{Log}^{-4\beta(2-q^*)}(mn) \le \mathrm{Log}^{-\beta-1/2}(mn),$$

This together with bound (32) implies the assertion. □

Now we are ready to prove the upper bound in Theorem 1 in the case when p^*, q are separated from 2.

Proposition 34. *Let $(X_{i,j})_{i\le m, j\le n}$ be iid symmetric random variables such that (3) holds with $\alpha \ge \sqrt{2}$. Then the upper bound in Theorem 1 holds whenever $3 \le q \le \mathrm{Log}\, m$ and $3 \le p^* \le \mathrm{Log}\, n$.*

Proof. Without loss of generality we assume that $\mathbb{E}X_{i,j}^2 = 1$ and that $q \ge p^*$ (the opposite case follows by duality (10)).

Recall that $\beta = \log_2 \alpha \ge 1/2$ and let us consider the following subsets of balls $B_{q^*}^m$ and B_p^n:

$$S_1 = B_{q^*}^m \cap e^{-q} B_\infty^m, \quad S_2 = \{s \in B_{q^*}^m \colon |\mathrm{supp}(s)| \le e^{qq^*}\},$$
$$S_3 = B_{q^*}^m \cap \mathrm{Log}^{-8\beta}(mn) B_\infty^m, \quad S_4 = \{s \in B_{q^*}^m \colon |\mathrm{supp}(s)| \le \mathrm{Log}^{8\beta q^*}(mn)\},$$
$$T_1 = B_p^n \cap e^{-p^*} B_\infty^n, \quad T_2 = \{t \in B_p^n \colon |\mathrm{supp}(t)| \le e^{pp^*}\},$$

and

$$T_3 = B_p^n \cap \mathrm{Log}^{-8\beta}(mn) B_\infty^n, \quad T_4 = \{t \in B_p^n \colon |\mathrm{supp}(t)| \le \mathrm{Log}^{8\beta p}(mn)\}.$$

Note that $B_{q^*}^m \subset S_1 + S_2$, $B_{q^*}^m \subset S_3 + S_4$, $B_p^n \subset T_1 + T_2$, and $B_p^n \subset T_3 + T_4$. In particular

(33)
$$\big\|(X_{i,j})_{i\le m, j\le n}\big\|_{\ell_p^n \to \ell_q^m} = \sup_{s\in B_{q^*}^m, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} s_i t_j$$
$$\le \sup_{s\in S_1, t\in T_1} \sum_{i\le m, j\le n} X_{i,j} s_i t_j + \sup_{s\in S_2, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} s_i t_j + \sup_{s\in B_{q^*}^m, t\in T_2} \sum_{i\le m, j\le n} X_{i,j} s_i t_j.$$

If $s \in S_2$, then

$$\|s\|_1 \le \|s\|_{q^*} |\mathrm{supp}(s)|^{-1/q^*+1} \le e^{q^*} \le e^{3/2} < 5,$$

so $S_2 \subset 5B_1^m = 5B_{\infty^*}^m$ and we may proceed as in the proof of (30) to get

(34)
$$\mathbb{E} \sup_{s\in S_2, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \lesssim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j\le n} t_j X_{1,j}\Big\|_q + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*}$$

and, by duality,

(35)
$$\mathbb{E} \sup_{s\in B_{q^*}^m, t\in T_2} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \lesssim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j\le n} t_j X_{1,j}\Big\|_q + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*}.$$

Bounds (33)-(35) imply that it suffices to prove that

(36)
$$\sup_{s\in S_1, t\in T_1} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \lesssim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*}.$$

Recall that $q \ge p^* \ge 3$. Let us consider three cases.

Case 1, when $q, p^* \geq 60\beta^2 \operatorname{Log}\operatorname{Log}(mn)$. Then $e^{-q}, e^{-p^*} \leq \operatorname{Log}^{-8\beta}(mn)$, so $S_1 \subset S_3$ and $T_1 \subset T_3$. Thus, (36) follows by Lemma 33.

Case 2, when $q \geq 60\beta^2 \operatorname{Log}\operatorname{Log}(mn) \geq p^*$. Then $S_1 \subset S_3$ and $T_1 \subset B_p^n \subset T_3 + T_4$, so

$$\mathbb{E} \sup_{s\in S_1, t\in T_1} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j \leq \mathbb{E} \sup_{s\in S_3, t\in T_3} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j + \mathbb{E} \sup_{s\in S_1, t\in T_4} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j.$$

The first term on the right-hand side may be bounded properly by Lemma 33. In order to estimate the second term we apply (19) from Lemma 26 with $a = e^{-q}$ and $k = \lfloor \operatorname{Log}^{12\beta}(mn)\rfloor \geq \lfloor \operatorname{Log}^{8\beta p}(mn)\rfloor$ (the inequality follows by $p \leq 3^* = \frac{3}{2}$) to get

$$\mathbb{E} \sup_{s\in S_1, t\in T_4} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j \lesssim_\alpha m^{1/q} \sup_{t\in T_4} \Big\| \sum_{j=1}^n X_{1,j} t_j \Big\|_q + (\operatorname{Log}^{14\beta}(mn))^\beta e^{-q(2-q^*)/2}.$$

Since $q^* \leq 3^* = 3/2$, we have

$$(\operatorname{Log}^{14\beta}(mn))^\beta e^{-q(2-q^*)/2} \leq \operatorname{Log}^{14\beta^2}(mn) e^{-q/4} \leq 1,$$

so (36) holds.

Case 3, when $60\beta^2 \operatorname{Log}\operatorname{Log}(mn) \geq q, p^*$. Since $T_1 \subset T_3 + T_4$ and $S_1 \subset S_3 + S_4$, we have

$$\begin{aligned}\mathbb{E} \sup_{s\in S_1, t\in T_1} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j \leq{}& \mathbb{E} \sup_{s\in S_3, t\in T_3} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j + \mathbb{E} \sup_{s\in B_{q^*}^m, t\in T_4} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j \\ &+ \mathbb{E} \sup_{s\in S_4, t\in B_p^n} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j.\end{aligned}$$

The first term on the right-hand side may be bounded by Lemma 33. Now we estimate the second term — the third one may be bounded similarly (by using (20) from Lemma 26 instead of (19)). By (19) applied with $a = 1$ and $k = \lfloor \operatorname{Log}^{12\beta}(mn)\rfloor \geq \lfloor \operatorname{Log}^{8\beta p}(mn)\rfloor$ we have

$$\mathbb{E} \sup_{s\in B_{q^*}^m, t\in T_4} \sum_{i\leq m, j\leq n} X_{i,j} s_i t_j \lesssim_\alpha m^{1/q} \sup_{t\in T_4} \Big\| \sum_{j=1}^n X_{1,j} t_j \Big\|_q + \operatorname{Log}^{14\beta^2}(mn).$$

For a fixed $\beta = \log_2 \alpha \geq 1/2$ there exists $C(\beta) \geq 3$ such that for every $x \geq C(\beta) =: C_0(\alpha)$ we have $28\beta^2 \ln x \leq x/(60\beta^2 \ln x)$. Hence, if $mn \geq e^{C_0(\alpha)}$ and $p^* \leq q \leq 60\beta^2 \operatorname{Log}\operatorname{Log}(mn)$, then

$$14\beta^2 \ln \operatorname{Log}(mn) \leq \frac{1}{2} \ln(mn)/q \leq \frac{1}{2}(\ln m/q + \ln n/p^*) \leq \max\{\ln m/q, \ln n/p^*\},$$

so for every $m, n \in \mathbb{N}$,

$$\operatorname{Log}^{14\beta^2}(mn) \lesssim_\alpha \max\{m^{1/q}, n^{1/p^*}\},$$

and (36) follows. □

6.7. **Case** $q \geq 24\beta \geq 3 \geq p^*$ **or** $p^* \geq 24\beta \geq 3 \geq q$**.** In this subsection we assume (without loss of generality – see Remark 25) that $X_{i,j}$ are iid symmetric random variables satisfying (3) with $\alpha \geq \sqrt{2}$. We also use the notation $\beta = \log_2 \alpha \geq 1/2$, so $24\beta \geq 3$. By duality (10) it suffices to consider the case $q \geq 24\beta \geq 3 \geq p^*$. In particular, $q^* \leq 3/2$ whenever $q \geq 24\beta$. By Subsections 6.2 and 6.5 it suffices to consider the case $\operatorname{Log} m \wedge (C(\alpha) \operatorname{Log} n) \geq q$. In this case Theorem 1 follows by the following two lemmas.

Lemma 35. *If* $\operatorname{Log} m \geq q \geq 3 \geq p^*$, $n^{1/3} \geq m^{1/q} q^\beta$, *and* $\|X_{1,1}\|_2 = 1$, *then*

$$\mathbb{E}\|(X_{i,j})_{i\leq m, j\leq n}\|_{\ell_p^n \to \ell_q^m} \lesssim_\alpha n^{1/p^*}.$$

Proof. By (7) we get

$$\sup_{t\in B^n_{3/2}} \Big\|\sum_{i=1}^m t_i X_{i,1}\Big\|_q \lesssim_\alpha q^\beta \sup_{t\in B^n_{3/2}} \Big\|\sum_{i=1}^m t_i X_{i,1}\Big\|_2 = q^\beta \sup_{t\in B^n_{3/2}} \|t\|_2 = q^\beta.$$

This together with the assumption $n^{1/3} \ge m^{1/q} q^\beta$ and the estimate in the case $p^* = 3 \le q$ (already obtained in Subsection 6.6) gives $\mathbb{E}\|(X_{i,j})\|_{\ell^n_{3/2}\to\ell^m_q} \lesssim_\alpha n^{1/3}$. Therefore, for every $p^* \le 3$,

$$\mathbb{E}\|(X_{i,j})\|_{\ell^n_p\to\ell^m_q} \le \|\operatorname{Id}\|_{\ell^n_p\to\ell^n_{3/2}} \mathbb{E}\|(X_{i,j})\|_{\ell^n_{3/2}\to\ell^m_q} \lesssim_\alpha n^{2/3-1/p} n^{1/3} = n^{1/p^*}. \qquad \square$$

Lemma 36. *Assume that* $\operatorname{Log} m \wedge C(\alpha)\operatorname{Log} n \ge q \ge 24\beta \ge 3 \ge p^*$ *and* $q^\beta m^{1/q} \ge n^{1/3}$*. Then the upper bound in Theorem 1 holds.*

Proof. Without loss of generality we may assume that $\mathbb{E}X_{i,j}^2 = 1$ and $C(\alpha) \ge 2$. Let

$$\widetilde{S}_1 = \big\{s \in B^m_{q^*}\colon\ |\operatorname{supp}(s)| \le \operatorname{Log}^{4\beta q^*}(mn)\big\}, \quad S_1 = B^m_{q^*} \cap \operatorname{Log}^{-4\beta}(mn) B^m_\infty.$$

Then $B^m_{q^*} \subset S_1 + \widetilde{S}_1$.

If $\operatorname{Log} m \le C^2(\alpha)\operatorname{Log}^2 n$, then inequality (20) from Lemma 26 (applied with $b = 1$, $p\wedge 2$ instead of p and $l = \operatorname{Log}(mn)^{4\beta q^*} \le \operatorname{Log}(mn)^{6\beta}$) yields

$$\begin{aligned}\mathbb{E} \sup_{s\in\widetilde{S}_1, t\in B^n_{p\wedge 2}} \sum_{i\le m, j\le n} X_{i,j} s_i t_j &\lesssim_\alpha n^{1/(p^*\vee 2)} \sup_{s\in\widetilde{S}_1} \Big\|\sum_{i=1}^m X_{i,1} s_i\Big\|_{p^*\vee 2} + (\operatorname{Log} n)^{C_1(\alpha)} \\ &\lesssim_\alpha n^{1/(p^*\vee 2)} \sup_{s\in B^m_{q^*}} \Big\|\sum_{i=1}^m X_{i,1} s_i\Big\|_{p^*} + n^{1/3} \\ &\lesssim_\alpha n^{1/(p^*\vee 2)} \sup_{s\in B^m_{q^*}} \Big\|\sum_{i=1}^m X_{i,1} s_i\Big\|_{p^*}.\end{aligned}$$

In the case $\operatorname{Log} m \ge C^2(\alpha)\operatorname{Log}^2 n$ we have $m^{1/q} \ge e^{\operatorname{Log} m/(C(\alpha)\operatorname{Log} n)} \ge e^{(\operatorname{Log} m)^{1/2}}$, so now inequality (20) yields

$$\begin{aligned}\mathbb{E} \sup_{s\in\widetilde{S}_1, t\in B^n_{p\wedge 2}} \sum_{i\le m, j\le n} X_{i,j} s_i t_j &\lesssim_\alpha n^{1/(p^*\vee 2)} \sup_{s\in\widetilde{S}_1} \Big\|\sum_{i=1}^m X_{i,1} s_i\Big\|_{p^*\vee 2} + (\operatorname{Log} m)^{C_2(\alpha)} \\ &\lesssim_\alpha n^{1/(p^*\vee 2)} \sup_{s\in B^m_{q^*}} \Big\|\sum_{i=1}^m X_{i,1} s_i\Big\|_{p^*} + m^{1/q}.\end{aligned}$$

Thus, in any case

$$\begin{aligned}\mathbb{E} \sup_{s\in\widetilde{S}_1, t\in B^n_{p}} \sum_{i\le m, j\le n} X_{i,j} s_i t_j &\le n^{(1/p^*-1/2)\vee 0}\, \mathbb{E} \sup_{s\in\widetilde{S}_1, t\in B^n_{p\wedge 2}} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \\ &\lesssim_\alpha n^{1/p^*} \sup_{s\in B^m_{q^*}} \Big\|\sum_{i=1}^m X_{i,1} s_i\Big\|_{p^*} + m^{1/q} n^{(1/p^*-1/2)\vee 0}. \qquad (37)\end{aligned}$$

Let

$$\begin{aligned} S_2 &= \big\{s \in B^m_{q^*}\colon\ |\operatorname{supp}(s)| \le m \operatorname{Log}^{-q(\beta+1)}(mn)\big\} \cap \operatorname{Log}^{-4\beta}(mn) B^m_\infty, \\ S_3 &= B^m_{q^*} \cap m^{-1/q^*} \operatorname{Log}^{(\beta+1)q/q^*}(mn) B^m_\infty. \end{aligned}$$

Then $S_1 \subset S_2 + S_3$.

Lemmas 27 and 29 (applied with $l = m\operatorname{Log}^{-q(\beta+1)}(mn)$ and $a = \operatorname{Log}^{-4\beta}(mn)$), and inequality $q^* \le \frac{3}{2}$ yield

$$\begin{aligned}\mathbb{E}\sup_{s\in S_2, t\in B_p^n}&\sum_{i\le m, j\le n} X_{i,j}s_it_j\\ &\le \operatorname{Log}^{\beta}(mn)\Big(\operatorname{Log}^{-2\beta(2-q^*)}(mn)n^{1/p^*} + n^{(1/p^*-1/2)\vee 0}m^{1/q}\operatorname{Log}^{-\beta-1/2}(mn)\Big)\\ &\le n^{1/p^*} + n^{(1/p^*-1/2)\vee 0}m^{1/q}. \qquad (38)\end{aligned}$$

Moreover, if $\operatorname{Log} m \le C^2(\alpha)\operatorname{Log}^2 n$, then inequalities $n^{1/3} \le m^{1/q}q^\beta \le m^{1/q}\operatorname{Log}^\beta m$ and $q/(3q^*) \ge 4\beta + q/(12q^*)$ imply

$$m^{1/q^*}\operatorname{Log}^{-(\beta+1)q/q^*}(mn) \ge n^{q/(3q^*)}\operatorname{Log}^{-\beta q/q^*} m\operatorname{Log}^{-(\beta+1)q/q^*}(mn) \gtrsim_\alpha n^{4\beta},$$

and if $\operatorname{Log} m \ge C^2(\alpha)\operatorname{Log}^2 n \ge \operatorname{Log}^2 n$, then

$$\begin{aligned}m^{1/q^*}\operatorname{Log}^{-(\beta+1)q/q^*}(mn) &\ge e^{\operatorname{Log} m/q^*}\operatorname{Log}^{-C_3(\alpha)q} m \ge \exp\big((\operatorname{Log} m)/2 - C_4(\alpha)\operatorname{Log} n\cdot\ln(\operatorname{Log} m)\big)\\ &\gtrsim_\alpha e^{(\operatorname{Log}^2 n)/4} \gtrsim_\alpha n^{4\beta}.\end{aligned}$$

Since $q^* \le \frac{3}{2}$, in both cases we have

$$(m^{1/q^*}\operatorname{Log}^{-(\beta+1)q/q^*}(mn))^{(2-q^*)/2} \gtrsim_\alpha n^\beta.$$

Therefore, inequality (19) from Lemma 26 (applied with $a = m^{-1/q^*}\operatorname{Log}^{(\beta+1)q/q^*}(mn)$ and $k = n$) yields

$$\mathbb{E}\sup_{s\in S_3, t\in B_p^n}\sum_{i\le m, j\le n} X_{i,j}s_it_j \lesssim_\alpha m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n X_{1,j}t_j\Big\|_q + n^{(1/p^*-1/2)\vee 0}. \qquad (39)$$

Since

$$n^{(1/p^*-1/2)\vee 0} = \sup_{t\in B_p^n}\|t\|_2 \le \sup_{t\in B_p^n}\Big\|\sum_{j=1}^n X_{1,j}t_j\Big\|_q,$$

estimates (37)-(39) yield the assertion. □

6.8. **Case** $24\beta \ge q \ge p^*$ **or** $24\beta \ge p^* \ge q$**.** Once we prove the upper bound in the case $24\beta \ge q \ge p^*$, the upper bound in the case $24\beta \ge p^* \ge q$ follows by duality (10). We first deal with the case $p^* \ge 2$ and then move to the case $2 \ge p^*$ at the end of this subsection.

Let us begin with the proof in the case $p^* = q \ge 2$, where an interpolation argument works.

Lemma 37. *If $p^* = q \ge 2$, then the upper bound in Theorem 1 holds.*

Proof. By Subsections 6.1 and 6.6 we know that the assertion holds when $p^* = q \in \{2\}\cup[3,\infty]$.

Assume without loss of generality that $\mathbb{E}X_{i,j}^2 = 1$. Fix $p^* = q \in (2,3)$ and let $\theta \in (0,1)$ be such that $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{3}$, i.e., $\frac{1}{p} = 1 - \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{3^*}$. Then (7) implies that

$$\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jX_{1,j}\Big\|_{q\wedge\operatorname{Log} m} \sim_\alpha \sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jX_{1,j}\Big\|_2 = 1, \qquad (40)$$

and similarly

$$\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_iX_{i,1}\Big\|_{p^*\wedge\operatorname{Log} n} \sim_\alpha 1, \qquad (41)$$

By the Riesz-Thorin interpolation theorem, Hölder's inequality, (40) and (41) we get

$$\begin{aligned}\mathbb{E}\big\|(X_{i,j})_{i,j}\big\|_{\ell_p^n\to\ell_q^m} &\le \mathbb{E}\big(\big\|(X_{i,j})_{i,j}\big\|^{\theta}_{\ell_2^n\to\ell_2^m}\big\|(X_{i,j})_{i,j}\big\|^{1-\theta}_{\ell_{3^*}^n\to\ell_3^m}\big)\\ &\le \Big(\mathbb{E}\big\|(X_{i,j})_{i,j}\big\|_{\ell_2^n\to\ell_2^m}\Big)^{\theta}\Big(\mathbb{E}\big\|(X_{i,j})_{i,j}\big\|_{\ell_{3^*}^n\to\ell_3^m}\Big)^{1-\theta}\\ &\lesssim_\alpha (n\vee m)^{\theta/2}(n\vee m)^{(1-\theta)/3} = (n\vee m)^{1/q} \sim n^{1/p^*}+m^{1/q}. \qquad \square\end{aligned}$$

Proof of the upper bound in Theorem 1 in the case $24\beta \ge q \ge p^* \ge 2$. By Remark 25 it suffices to assume that $X_{i,j}$'s are symmetric and $\alpha \ge \sqrt{2}$. Then $\beta = \log_2\alpha \ge 1/2$. Inequality (7) implies that in the case $24\beta \ge q \ge p^* \ge 2$ the upper bound in Theorem 1 is equivalent to

$$\mathbb{E}\|(X_{i,j})_{i,j}\|_{\ell_p^n\to\ell_q^n} \lesssim_\alpha n^{1/p^*}+m^{1/q}. \tag{42}$$

If $m \le n$, then Lemma 37 yields

$$\mathbb{E}\|(X_{i,j})_{i,j}\|_{\ell_p^n\to\ell_q^m} \le \|\mathrm{Id}\|_{\ell_p^n\to\ell_{q^*}^n}\mathbb{E}\|(X_{i,j})_{i,j}\|_{\ell_{q^*}^n\to\ell_q^m} = n^{1/q^*-1/p}\mathbb{E}\|(X_{i,j})_{i,j}\|_{\ell_{q^*}^n\to\ell_q^m} \lesssim_\alpha n^{1/p^*}.$$

Thus, in the sequel we assume that $2 \le p^* \le q \le 24\beta$ and $m \ge n$. Define

$$k_0 := \inf\Big\{k\in\{0,1,\ldots\}\colon\ 2^k \ge \frac{5}{\ln\alpha}\frac{2-q^*}{q^*}\operatorname{Log} m\Big\}.$$

Observe that

$$k_0 = 0 \quad\text{or}\quad 2^{k_0} \le \frac{10}{\ln\alpha}\frac{2-q^*}{q^*}\operatorname{Log} m. \tag{43}$$

By Lemma 31 and the definition of k_0 we have

$$\begin{aligned}\mathbb{E}\big\|\big(X_{i,j}I_{\{|X_{i,j}|\ge\alpha^{k_0}\}}\big)_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} &\le \mathbb{E}\big\|\big(X_{i,j}I_{\{|X_{i,j}|\ge\alpha^{k_0}\}}\big)_{i\le m,j\le n}\big\|_{\ell_2^n\to\ell_2^m}\\ &\lesssim_\alpha \sqrt{m}\exp\big(-\frac{\ln\alpha}{10}2^{k_0}\big)\\ &\le m^{\frac12-\frac{2-q^*}{2q^*}} = m^{1/q}.\end{aligned}$$

By Lemma 28 and two-sided bound (1) we have

$$\mathbb{E}\big\|\big(X_{i,j}I_{\{|X_{i,j}|\le1\}}\big)_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m} \lesssim \mathbb{E}\|(g_{i,j})_{i\le m,j\le n}\|_{\ell_p^n\to\ell_q^m} \lesssim_\alpha n^{1/p^*}+m^{1/q}.$$

We have $B_{q^*}^m \subset S_1+S_2$, where

$$S_1 = \{s\in B_{q^*}^m\colon\ |\operatorname{supp}(s)| \le m^{1/(2\beta q)}\}, \quad S_2 = B_{q^*}^m \cap m^{-1/(2\beta qq^*)}B_\infty^m.$$

Inequality (20) from Lemma 26 applied with $b=1$, $l=m^{1/(2\beta q)}$ shows that

$$\mathbb{E}\sup_{s\in S_1,t\in B_p^n}\sum_{i\le m,j\le n}X_{i,j}s_it_j \lesssim_\alpha n^{1/p^*}+m^{1/q}. \tag{44}$$

Since $2\beta qq^* \le 100\beta^2$ we have

$$S_2 \subset S_3 := B_{q^*}^m \cap m^{-1/(100\beta^2)}B_\infty^m.$$

Thus, to finish the proof it is enough to upper bound the following quantity

$$\mathbb{E}\sup_{s\in S_3,t\in B_p^n}\sum_{i\le m,j\le n}X_{i,j}I_{\{1\le|X_{i,j}|<\alpha^{k_0}\}}s_it_j \le \sum_{k=1}^{k_0}\mathbb{E}\sup_{s\in S_3,t\in B_p^n}\sum_{i\le m,j\le n}X_{i,j}I_{\{\alpha^{k-1}\le|X_{i,j}|<\alpha^k\}}s_it_j.$$

Let $u_1,\ldots,u_{k_0}$ be positive numbers to be chosen later. We decompose the set S_3 in the following way, depending on k:

$$S_3 \subset S_{4,k}+S_{5,k},$$

where

$$S_{4,k} := \{s \in B_{q^*}^m \colon \ |\operatorname{supp} s| \le m/u_k\} \cap m^{-1/(100\beta^2)} B_\infty^m, \quad S_{5,k} := B_{q^*}^m \cap \Big(\frac{u_k}{m}\Big)^{1/q^*} B_\infty^m.$$

Thus,

$$\begin{aligned}\mathbb{E} &\sup_{s\in S_3, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} I_{\{\alpha^{k-1}\le |X_{i,j}|<\alpha^k\}} s_i t_j \\ &\le \mathbb{E} \sup_{s\in S_{4,k}, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} I_{\{|X_{i,j}|<\alpha^k\}} s_i t_j + \mathbb{E} \sup_{s\in S_{5,k}, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} I_{\{\alpha^{k-1}\le |X_{i,j}|\}} s_i t_j.\end{aligned}$$

Observe that $B_p^n \subset B_2^n$ and

$$\sup_{s\in S_{5,k}} \|s\|_2 \le \sup_{s\in S_{5,k}} \|s\|_{q^*}^{q^*/2} \|s\|_\infty^{(2-q^*)/2} \le \Big(\frac{u_k}{m}\Big)^{\frac{2-q^*}{2q^*}}.$$

Hence, Lemma 31 yields

$$\begin{aligned}\mathbb{E} \sup_{s\in S_{5,k}, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} I_{\{\alpha^{k-1}\le |X_{i,j}|\}} s_i t_j &\le \Big(\frac{u_k}{m}\Big)^{\frac{2-q^*}{2q^*}} \mathbb{E} \|(X_{i,j} I_{\{\alpha^{k-1}\le |X_{i,j}|\}})\|_{\ell_2^m \to \ell_2^n} \\ &\lesssim_\alpha m^{1/q} u_k^{\frac{2-q^*}{2q^*}} \exp\Big(-\frac{\ln\alpha}{10} 2^{k-1}\Big).\end{aligned}$$

Thus, if we choose

$$u_k := \exp\Big(\frac{q^* \ln\alpha}{20(2-q^*)} 2^k\Big),$$

we get

$$\sum_{k=1}^{k_0} \mathbb{E} \sup_{s\in S_{5,k}, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} I_{\{\alpha^{k-1}\le |X_{i,j}|\}} s_i t_j \lesssim_\alpha \sum_{k=1}^{\infty} m^{1/q} \exp\Big(-\frac{\ln\alpha}{40} 2^k\Big) \lesssim_\alpha m^{1/q}.$$

Lemmas 28 and 30 applied with $l = \frac{m}{u_k}$, $a = m^{-1/(100\beta^2)}$, and $\gamma = 24\beta$ yield

$$\mathbb{E} \sup_{s\in S_{4,k}, t\in B_p^n} \sum_{i\le m, j\le n} X_{i,j} I_{\{|X_{i,j}|<\alpha^k\}} s_i t_j \lesssim_\alpha \alpha^k \Big(m^{-\frac{(2-q^*)}{200\beta^2}} n^{1/p^*} + \sqrt{\operatorname{Log} u_k} (m/u_k)^{1/q}\Big).$$

Property (43) yields

$$\begin{aligned}\sum_{k=1}^{k_0} \alpha^k m^{-\frac{(2-q^*)}{200\beta^2}} n^{1/p^*} &\lesssim \alpha^{k_0} I_{\{k_0\ne 0\}} m^{-\frac{(2-q^*)}{200\beta^2}} n^{1/p^*} \lesssim_\alpha \Big(\frac{10}{\ln\alpha} \frac{2-q^*}{q^*} \operatorname{Log} m\Big)^{\log_2\alpha} e^{-\frac{(2-q^*)}{200\beta^2} \ln m} n^{1/p^*} \\ &\lesssim_\alpha n^{1/p^*} \sup_{x>0} x^{\log_2 \alpha} e^{-x} \lesssim_\alpha n^{1/p^*}.\end{aligned}$$

Finally, since $q \le 24\beta$ and $u_k \ge 1$ we get $\sqrt{\operatorname{Log} u_k}(m/u_k)^{1/q} \lesssim_\alpha m^{1/q} u_k^{-1/(2q)}$, so

$$\sum_{k=1}^{k_0} \alpha^k \sqrt{\operatorname{Log} u_k} (m/u_k)^{1/q} \lesssim_\alpha m^{1/q} \sum_{k\ge 1} \alpha^k \exp\Big(-\frac{q^*\ln\alpha}{40q(2-q^*)} 2^k\Big) \lesssim_\alpha m^{1/q}. \qquad \square$$

The case $2 \ge p^*, q$ was considered in Subsection 6.1. The proof in the case $24\beta \ge q \ge 2 \ge p^*$ is easy and bases on the already proven case when $q \ge 2 = p^*$ (see the proof above).

Proof of the upper bound in Theorem 1 in the case $24\beta \geq q \geq 2 \geq p^*$. Inequality (7) implies that in the case $24\beta \geq q \geq 2 \geq p^*$ the upper bound in Theorem 1 is equivalent to

$$\mathbb{E}\|(X_{i,j})_{i,j}\|_{\ell_p^n \to \ell_q^m} \lesssim_\alpha n^{1/p^*} + m^{1/q} n^{1/p^*-1/2}. \tag{45}$$

In particular, an already obtained upper bound in the case $24\beta \geq q \geq 2 = p^*$ yields

$$\mathbb{E}\|(X_{i,j})_{i,j}\|_{\ell_2^n \to \ell_q^m} \lesssim_\alpha n^{1/2} + m^{1/q},$$

so

$$\begin{aligned}\mathbb{E}\|(X_{i,j})_{i,j}\|_{\ell_p^n \to \ell_q^m} &\leq \|\operatorname{Id}\|_{\ell_p^n \to \ell_2^n} \mathbb{E}\|(X_{i,j})_{i,j}\|_{\ell_2^n \to \ell_q^m} \lesssim_\alpha n^{1/p^*-1/2}(n^{1/2} + m^{1/q}) \\ &= n^{1/p^*} + m^{1/q} n^{1/p^*-1/2},\end{aligned}$$

and thus, (45) holds. □

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