Isoperimetry and the concentration of measure phenomenon

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The basic question

Among all closed planar curves of a given length, which one encloses the greatest area?

or equivalently

Among all closed planar curves enclosing a fixed area, which one minimizes the perimeter?

The “obvious” answer is

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Modern version

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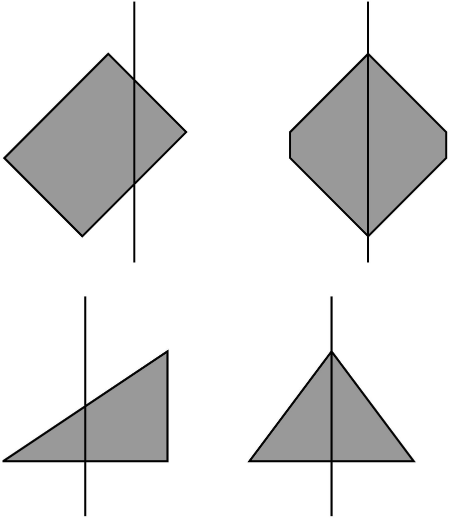
Steiner's symmetrization (1841)

Definition

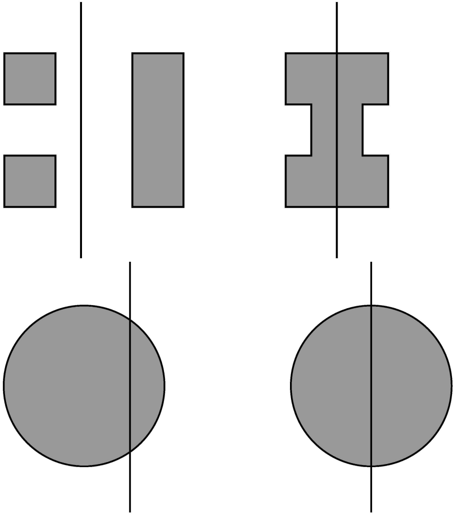
Fix a hyperplane H of codimension 1. To obtain Steiner's symetrization of set A with respect to H (call it $S_H A$), for each line L perpendicular to H replace $L \cap A$ by a segment $I \subseteq L$, symmetric wrt H and such that

$$\lambda_1(A) = \lambda_1(I).$$

Steiner's symmetrization



Steiner's symmetrization



Steiner's symmetrization

Properties of Steiner's symmetrization

- $\lambda_n(A) = \lambda_n(S_H A)$
- $\lambda_{n-1}(\partial A) \geq \lambda_{n-1}(\partial S_H A)$.

Steiner's symmetrization

Fact

For any compact set A we can find a sequence of hyperplanes H_n, such that $S_{H_n} S_{H_{n-1}} \ldots S_2 S_1 A$ "converges" to a Euclidean ball.
Hence to prove the isoperimetric inequality, it is enough to show that in this limit the Lebesgue measure is preserved and the boundary measure does not increase.

A simple observation

For "nice" sets A,

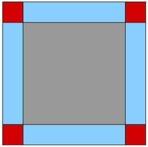
$$\lambda_{n-1}(\partial A) = \lim_{\varepsilon \to 0} \frac{\lambda_n(A_\varepsilon) - \lambda_n(A)}{\varepsilon},$$

where

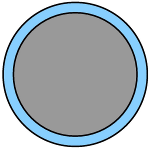
$$A_\varepsilon = \{x \in \mathbb{R}^n, \mathrm{dist}(x, A) \le \varepsilon\}$$

A simple observation

For example

- 

$$\frac{4a\varepsilon + 4\varepsilon^2}{\varepsilon} \to 4a$$

- 

$$\frac{\pi(r+\varepsilon)^2 - \pi r^2}{\varepsilon} = \frac{2\pi r\varepsilon + \varepsilon^2}{\varepsilon} \to 2\pi r$$

The isoperimetric inequality revisited

Theorem

Let A be a Borel measurable set in $\mathbb{R}^n$ and B – a Euclidean ball of the same measure. Then for any $\varepsilon > 0$ we have

$$\lambda_n(A_\varepsilon) \geq \lambda_n(B_\varepsilon) = \lambda_n(B(0, r + \varepsilon)),$$

where r is the radius of B.

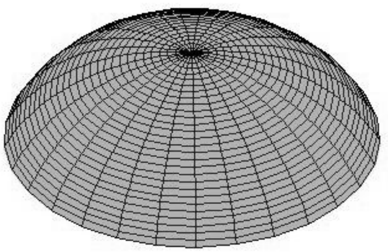
The isoperimetric inequality on a sphere

Theorem (Levy, Schmidt, Beckner)

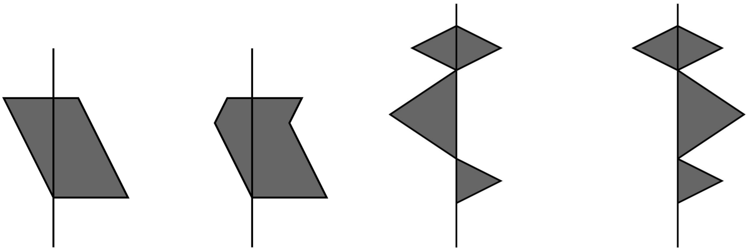
If A is a Borel measurable subset of S^{n-1} (equipped with the Euclidean or geodesic distance ρ and the surface measure μ) and B – a "cap" of the same measure, than

$$\mu(A_\varepsilon) \geq \mu(B_\varepsilon)$$

A cap



A digression on Beckner's proof – two point symmetrization



Some computations – concentration of measure

Fact

Let us now consider the normalized surface measure μ (i.e. $\mu(S^{n+1}) = 1$). Assume that $\mu(A) \geq 1/2$. Let ρ be the geodesic or Euclidean distance. Then

$$\mu(A_\varepsilon^\rho) \geq 1 - \sqrt{\frac{\pi}{8}} e^{-n\varepsilon^2/2}.$$

A little bit of Science-Fiction

Corollary (The “thick” equator)

If our earth was a high-dimensional sphere, we would all live in the tropics.

The median - a short revision of probability

Definition

For every random variable X there exists a number M_X (not necessarily unique), such that both

$$\mathbb{P}(X \geq M_X) \geq 1/2$$

and

$$\mathbb{P}(X \leq M_X) \geq 1/2$$

We call M_X a *median* of X.

Concentration of measure again - Lipschitz functions

Theorem

Let $f\colon S^{n+1} \to \mathbb{R}$ be a 1-Lipshitz function (wrt to the Euclidean or geodesic metric). Then, for all $t \geq 0$

$$\mu(\{x \in S^{n+1} \colon f(x) \geq M_f + t\}) \leq \sqrt{\frac{\pi}{8}} e^{-nt^2/2}.$$

In consequence

$$\mu(|f - M_f| \geq t) \leq \sqrt{\frac{\pi}{2}} e^{-nt^2/2}.$$

In other words:
In high dimensions, all 1-Lipschitz functions on the unit sphere are essentially constant.

How can it be useful? Proofs of existence

If we have sets $A_1, \ldots, A_n$ such that

$$\sum_{i=1}^{n} \mu(A_i^c) < 1,$$

then

$$\mu(\bigcap_{i=1}^{n} A_i) > 0,$$

so there exists

$$x \in \bigcap_{i=1}^{n} A_i.$$

The Dvoretzky theorem

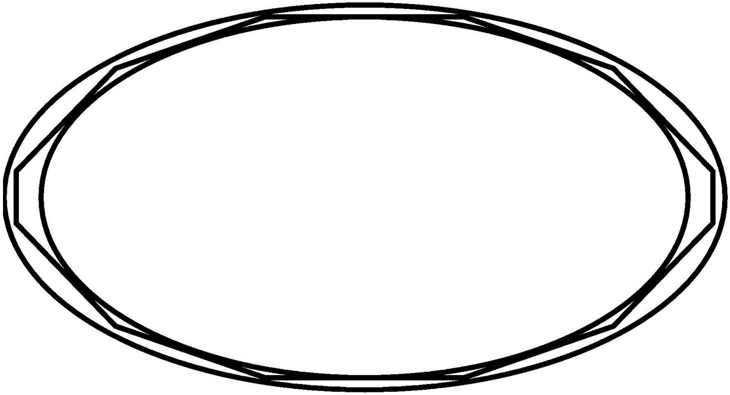
Theorem (Dvoretzky, Milman)

Let K be a convex, symmetric body in $\mathbb{R}^n$. Then, for every $\varepsilon > 0$, we can find a hyperplane H of dimension greater than $c(\varepsilon)\log n$ and an ellipsoid $\mathcal{E} \subseteq H$, such that

$$\mathcal{E} \subseteq H \cap K \subseteq (1+\varepsilon)\mathcal{E}.$$

In other words $H \cap K$ looks "almost" like an ellipsoid.

The Dvoretzky theorem



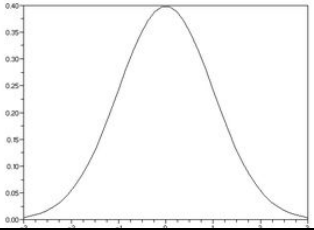
Another revision of probability - Gaussian measures

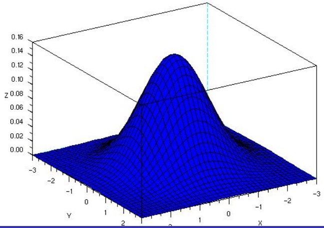
Definition

The standard Gaussian measure on $\mathbb{R}^n$ is the measure γ_n with density

$$g_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2},$$

where $|x|$ denotes the Euclidean norm.





The Poincaré observation

Theorem

Let $X = (X_1, \ldots, X_N)$ be a random vector distributed according to the normalized surface measure on S^{N-1}. Moreover, let ν_N be the distribution of $\sqrt{N}(X_1, \ldots, X_n)$. Then, for any n

$$\nu_N \xrightarrow{\alpha} \gamma_n, \quad \textit{as} \quad N \to \infty,$$

where $\xrightarrow{\alpha}$ denotes convergence in distribution. Even more is true, for every Borel set $A \subseteq \mathbb{R}^n$, we have

$$\nu_N(A) \to \gamma_n(A).$$

Gaussian concentration inequality

Theorem (Sudakov, Tsirelson, Borell)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a 1-Lipschitz function. Then for all $t \geq 0$,

$$\gamma_n(\{x\colon f(x) \geq \mathbb{E}_{\gamma_n} f + t\}) \leq e^{-t^2/2}.$$

In consequence

$$\gamma_n(\{x\colon |f(x) - \mathbb{E}_{\gamma_n} f| \geq t\}) \leq 2e^{-t^2/2}.$$

Random matrices

Consider an $n \times n$ symmetric matrix M_n , whose entries on and above the diagonal are i.i.d. $\mathcal{N}(0,1)$ random variables (Gaussian Orthogonal Ensemble). Let

$$\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \ldots \geq \lambda_n^{(n)}$$

be the eigenvalues of M_n.

Corollary

For each $k \leq n$ and $t \geq 0$

$$\mathbb{P}(|\lambda_k^{(n)} - \mathbb{E}\lambda_k^{(n)}| \geq t) \leq 2e^{-t^2/4}$$

It can be proven that

$$\frac{\mathbb{E}\lambda_1^{(n)}}{\sqrt{n}} \to 2.$$

Thus, concentration of measure for $\lambda_1^{(n)}$ and the Borel-Cantelli Lemma give

Theorem

With probability 1

$$\frac{\lambda_1^{(n)}}{\sqrt{n}} \to 2.$$

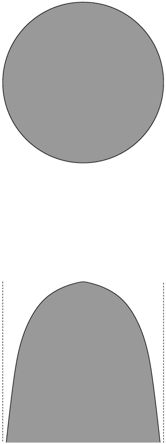
Is there a corresponding Gaussian isoperimetric inequality?

Theorem (Sudakov, Tsirelson, Borell, Ehrhard)

Let A be a Borel measurable subset of $\mathbb{R}^n$ and H a halfspace, such that $\gamma_n(H) = \gamma_n(A)$. Then, for all $\varepsilon > 0$,

$$\gamma_n(A_\varepsilon) \geq \gamma_n(H_\varepsilon).$$

Ehrhard's symmetrization



Bakry-Emery criterion

Theorem

Let ν be a Borel probability measure on $\mathbb{R}^n$, with density

$$\frac{\nu(dx)}{dx} = e^{-V(x)},$$

where $V\colon \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\mathrm{Hess}\, V \geq \lambda \mathrm{Id},$$

for some $\lambda > 0$. Then for all 1-Lipschitz functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and all $t \geq 0$,

$$\nu(\{x\colon f(x) \geq \mathbb{E}_\nu f + t\}) \leq e^{-\lambda t^2/2}$$

Convex functions

Theorem (Talagrand)

Let ν be arbitrary **product** *measure on $[0,1]^n$. Then for any* **convex** *1-Lipschitz function $f\colon [0,1]^n \to \mathbb{R}$ and any $t \geq 0$ we have*

$$\nu(\{x\colon |f(x) - \mathbb{E}_\nu f| \geq t\}) \leq 2e^{-t^2/2}.$$

- this immediately implies e.g. the Khintchine inequality (in Banach spaces)
- often enough for applications, e.g. to norms of random vectors
- we do not know if it is sufficient for eigenvalues of random matrices (some open problems)

Other developments

- more general class of measures (log-concave measures),
- concentration on abstract product spaces,
- isoperimetry on graphs (expander graphs, with applications in computer science),
- isoperimetry on other spaces (homogeneous spaces, Riemannian manifolds),
- connections to parabolic pde's, entropy methods, convergence to equilibrium,
- connections with fixed point properties for group actions,
- applications in statistics.