LECTURE NOTES

ON

NONCOMMUTATIVE GEOMETRY

AND

QUANTUM GROUPS

Edited by Piotr M. Hajac

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Notes taken and typeset

by

Paweł Witkowski

Preface

Piotr M. Hajac

Opening lecture

The origin of Noncommutative Geometry is twofold. On the one hand there is a wealth of examples of spaces whose coordinate algebra is no longer commutative but which have obvious geometric meaning. The first examples came from phase space in quantum mechanics but there are many others, such as the leaf spaces of foliations, duals of nonabelian discrete groups, the space of Penrose tilings, the noncommutative torus which plays a role in M-theory compactification, and finally the space of Q-lattices which is a natural geometric space carrying an action of the analogue of the Frobenius for global fields of zero characteristic.

On the other hand the stretching of geometric thinking imposed by passing to noncommutative spaces forces one to rethink about most of our familiar notions. The difficulty is not to add arbitrarily the adjective quantum behind our familiar geometric language but to develop far reaching extensions of classical concepts. This has been achieved a long time ago by operator algebraists as far as measure theory is concerned. The theory of nonabelian von-Neumann algebras is indeed a far reaching extension of measure theory, whose main surprise is that such an algebra inherits from its noncommutativity a god-given time evolution.

The development of the topological ideas was prompted by the Novikov conjecture on homotopy invariance of higher signatures of ordinary manifolds as well as by the Atiyah-Singer Index Theorem. It has led to the recognition that not only the Atiyah-Hirzebruch K-theory but more importantly the dual K-homology admit Noncommutative Geometry as their natural framework. The cycles in K-homology are given by Fredholm representations of the C*-algebra A of continuous functions. A basic example is the group ring of a discrete group and restricting oneself to commutative algebras is an obviously undesirable assumption.

The development of differential geometric ideas, including de Rham homology, connections and curvature of vector bundles, took place during the eighties thanks to cyclic homology which led for instance to the proof of the Novikov conjecture for hyperbolic groups but got many other applications. Basically, by extending the characteristic classes to the general framework it allows us for many concrete computations on noncommutative spaces.

The very notion of Noncommutative Geometry comes from the identification of the two basic concepts in Riemann's formulation of Geometry, namely those of manifold and of infinitesimal line element. It was recognized at an early stage that the formalism of quantum mechanics gives a natural place both to infinitesimals (the compact operators in Hilbert space) and to the integral (the logarithmic divergence in an operator trace). It was also recognized long ago by geometers that the main quality of the homotopy type of a manifold, (besides being defined by a cooking recipe) is to satisfy Poincare duality not only in ordinary homology but in K-homology.

In the general framework of Noncommutative Geometry the confluence of the two notions of metric and fundamental class for a manifold led very naturally to the equality ds=1/D which expresses the infinitesimal line element ds as the inverse of the Dirac operator D, hence under suitable boundary conditions as a propagator. The significance of D is two-fold. On the one hand it defines the metric by the above equation, on the other hand its homotopy class represents the K-homology fundamental class of the space under consideration.

We shall discuss three of the recent developments of Noncommutative Geometry. The first is the understanding of the noncommutative nature of spacetime from the symmetries of the Lagrangian of gravity coupled with matter. The starting point is that the natural symmetry group G of this Lagrangian is isomorphic to the group of diffeomorphisms of a space X, provided one stretches one's geometrical notions to allow slightly noncommutative spaces. The spectral action principle allows to recover the Lagrangian of gravity coupled with matter from the spectrum of the line element ds.

The second has to do with various appearances of Hopf algebras relevant to Quantum Field Theory which originated from my joint work with D.Kreimer and led recently in joint work with M.Marcolli to the discovery of the relation between renormalization and one of the most elaborate forms of Galois theory given in the Riemann-Hilbert correspondence and the theory of motives. A tantalizing unexplained bare fact is the appearance in the universal singular frame eliminating the divergence of QFT of the same numerical coefficients as in the local index formula. The latter is the corner stone of the definition of curvature in noncommutative geometry.

The third is the spectral interpretation of the zeros of the Riemann zeta function from the action of the idele class group on the space of Q-lattices and of the explicit formulas of number theory as a trace formula of Lefschetz type.

Alain Connes (Warszawa, 6 October 2004)

4 June 2008 Introduction
by Nigel Higson

Fall 2004/05 K-theory of operator algebras
by Rainer Matthes and Wojciech Szymański

Spring 04/05 Foliations, C*-algebras and index theory
by Paul F. Baum and Henri Moscovici

Fall 2005/06 Dirac operators and spectral geometry
by Joseph C. Varilly

Spring 05/06 From Poisson to quantum geometry
by Nicola Ciccoli

Fall 2006/07 Cyclic homology theory
by Jean-Louis Loday and Mariusz Wodzicki

Spring 06/07 Equivariant KK-theory and noncommutative index theory
by Paul F. Baum and Jacek Brodzki

Fall 2007/08 Galois structures
by Tomasz Brzeziński, George Janelidze, and Tomasz Maszczyk

Spring 07/08 The Baum-Connes conjecture, localisation of categories and quantum groups
by Paul F. Baum and Ralf Meyer

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Introduction

lecture by Nigel Higson

0.1 Spectral geometry

0.1.1 The Lorentz problem

The Lorentz problem (1910) was solved by Weyl.

$$\Delta u_n = \lambda_n u_n$$
$$u_n|_{\partial\Omega} = 0$$
$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{\mathrm{Area}(\Omega)} \tag{0.1}$$

The idea is to consider an inverse of Laplace operator $K = \Delta^{-1}$. For the eigenvalues of K one has

$$\lambda_n^{-1} = \min_{\dim(V)=n-1} \max_{v\perp V} \frac{||Kv||}{||v||}, \quad V \subseteq L^2(\Omega).$$

Denote

$$N(\lambda) = \sharp\{\lambda_n \leq \lambda\}, \quad \frac{N(\lambda)}{\lambda} \sim \frac{\mathrm{Area}(\Omega)}{4\pi}.$$

Domain dependence

$$\begin{matrix} K_1 \leq K_2 \\ \Omega_1 \leq \Omega_2 \end{matrix} \implies N_1(\lambda) \leq N_2(\lambda).$$

Divide Ω into squares. For each square I_n the Lorentz problem is easy.

0.1.2 Coefficient of logarithmic divergence

$$\mathrm{Tr}_\omega(\Delta^{-1}) = \lim_{\lambda\to\infty} \frac{1}{\log\lambda} \sum_{\lambda_n\leq\lambda} \lambda_n^{-1}. \tag{0.2}$$

This is a trace on operators with $\lambda_n(T) = O(\frac{1}{n})$. One has

$$\mathrm{Tr}_\omega(\Delta^{-1}) = \frac{1}{4\pi}\mathrm{Vol}(\Omega), \tag{0.3}$$

and for $f\colon \Omega \to \mathbb{C}$

$$\mathrm{Tr}_\omega(M_f\Delta^{-1}) = \frac{1}{4\pi}\int_\Omega f(x)dx, \tag{0.4}$$

where M_f is the operator of multiplication by f.

0.1.3 Zeta function of Δ

Let M be a manifold. Consider the Laplace operator with $\Delta u_n = \lambda_n u_n$. Define

$$\Delta^z u_n := \lambda_n^z u_n. \tag{0.5}$$

Theorem 0.1. *The trace $\mathrm{Tr}(\Delta^{\frac{z}{2}})$ is a meromorphic function on $\mathbb{C}$ with only simple poles.*

0.1.4 Noncommutative residue

$$\mathrm{NCRes}(P) := \mathrm{const} \cdot \mathrm{Res}_{z=0} \mathrm{Tr}(P\Delta^z) \tag{0.6}$$

The function $\mathrm{Tr}(P\Delta^z)$ is also meromorphic on $\mathbb{C}$ with only simple poles. We have

$$[\Delta^z, P] = z[\Delta, P]\Delta^z - 1 + \frac{z(z-1)}{2}[\Delta, [\Delta, P]]\Delta^{z-2} + \dots,$$

so NCRes is a trace:

$$\begin{aligned}\mathrm{NCRes}([P,Q]) &\sim \mathrm{Tr}([P,Q], \Delta^z) \\ &= \mathrm{Tr}(PQ\Delta^z - QP\Delta^z) \\ &= \mathrm{Tr}(Q[\Delta^z, P]) \\ &= 0.\end{aligned}$$

Tauberian theorem:

$$\mathrm{NCRes}(\Delta^{-\frac{\dim M}{2}}) = \mathrm{Tr}_\omega(\Delta^{-\frac{\dim M}{2}}). \tag{0.7}$$

$$\mathrm{NCRes}(\Delta^{-\frac{\dim M}{2}}) = \mathrm{Vol}(M). \tag{0.8}$$

From the equality $\mathrm{NCRes}([P,Q]) = 0$ one has

1. $\mathrm{Order(P)} = -\dim M$
2. NCRes depends only on th symbol of P. If $\sigma(P) = \{-,-\}$ (), then $\mathrm{NCRes} = 0$.

0.1.5 Residues an geometry (and physics?)

Connes' notation

$$\mathrm{NCRes} = ⨍ \tag{0.9}$$

Up to constants:

1. $⨍ \Delta^{-\frac{\dim M}{2}} = \mathrm{Vol}(M)$
2. $⨍ f\Delta^{-\frac{\dim M}{2}} = \int f d\mathrm{Vol}$
3. $⨍ \Delta^{-\frac{\dim M}{2}+1} = \int \kappa d\mathrm{Vol}$, where κ is a scalar curvature of M.

0.1.6 Square root of the Laplacian

Let Δ be a Laplace-type operator, D will denote a first order selfadjoint operator of "Dirac type"

$$D^2 = \Delta.$$

Introducing D adds

- Index (topology)
- Forms ("Fermions")

The spectrum of D is symmetric. D can be written as

$$\begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

Then the index

$$\mathrm{Index}(D) = \dim(\ker D_+) - \dim(\ker D_-) \tag{0.10}$$

is a topological invariant.

Furthermore

$$[D, f]^2 = -||df||^2 \mathrm{Id}, \tag{0.11}$$

and for complex fuunctions $f^0, \ldots, f^n$, on M, $n = \dim M$

$$⨍ f^0[D, f^1] \ldots [D, f^n]|D|^{-n} = \int_M f^0 df^1 \ldots df^n. \tag{0.12}$$

Spectral theory leads to integrals, index differential forms.

0.1.7 Spectral triples

Spectral triple (A, H, D) consists of

- algebra of bounded operators A
- Hilbert space H
- selfadjoint operator D such that D^{-1} is compact, and

$$||[D, a]|| < \infty \text{ for all } a \in A.$$

Example 0.2.

- Standard example
$$(C^\infty(M), L^2(M), D) \tag{0.13}$$
- Matrices
$$(M_n(\mathbb{C}), \mathbb{C}^n, F), \tag{0.14}$$
where F is any operator.
- Products
$$(C^\infty(M), L^2(M), D) \times (M_n(\mathbb{C}), \mathbb{C}^n, F) \tag{0.15}$$

Theorem 0.3 (Reconstruction theorem of A. Connes)**.** *Let A be a commutative algebra, and (A, H, D) a spectral triple, such that*

- $|\lambda_n(D)| = O(n^{-k})$,
- $a^0[D, a^1] \dots [D, a_n] = \mathrm{Id}$ *(orientation condition),*
- $[a^0, [D, a^1]] = 0$ *(D is local),*
- D^2 *is of Laplace type (regularity),*
- *H satisfies projectivity condition as an A-module, then $A = C^\infty(M)$ and D is of Dirac type.*

Suppose we have

- $a_1, \dots, a_n$ - functions in A,
- $X_1, \dots, X_n$ - elements of order 1 in the algebra generated by A and D ("vector fields").

$$\sum [Q, a_i] X_i = qQ + r, \quad q = \mathrm{Order(Q)},\ \mathrm{Order}(R) < q,$$

$$\sum [X_i, a_i] = n$$

Example 0.4. $a_i = x_i$ - coordinate functions on a manifold, $X_i = \frac{\partial}{\partial x\, i}$, Q - any differential operator. Since

$$(n+q)Q = \sum [Qa_i, x_i] - \sum [a_i Q, x_i] + R$$

there is no trace function on D.

The identities extend to $Q\Delta^{-z}$, "pseudodifferential".

$$\mathrm{Order}(Q\Delta^{-z}) = q - 2z$$

0.2 Singular spaces

0.2.1 Groupoids

Let $G \times X \to X$ be a group acting on a set. Denote

$$G \ltimes X := \{(x_1, g, x_2) \mid gx_2 = x_1\} \tag{0.16}$$

with

$$(x_1, g, x_2)(x_2, h, x_3) = (x_1, gh, x_3)$$

(collection of arrows).

Definition 0.5. *A* groupoid *is a (small) category in which every morphism is invertible.*

Denote by H the set of morphisms, and by K the set of objects in a given groupoid. There are source and range maps $s, r\colon H \to K$ and we denote

$$H \times_K H := \{(\gamma_1, \gamma_2) \mid s(\gamma_1) = r(\gamma_2)\}$$

Structure maps:

$$H \times_K H \xrightarrow{\circ} H$$
$$K \xrightarrow{\text{unit}} H$$
$$H \xrightarrow{\text{inverse}} H$$

Example 0.6.

- K - a set, $H \subseteq K \times K$ - equivalence relation
- the *action groupoid* (0.16)
- if $G \times M \to M$ is a principal action, then $H = M \times_G M$, $K = M/G$ is called a *fundamental groupoid*.

$$[m_1, m_2][m_2, m_3] = [m_1, m_3]$$

- Ehresmann's groupoid

$$H = (M \times X \times M)/G \tag{0.17}$$

$$[m_1, x, m_2][m_2, xm_3] = [m_1, x, m_3]$$

  For $M = G$
$$[g_1, x, g_2] \mapsto (g_1^{-1}x, g_1^{-1}g_2, g_2^{-1}x)$$
  gives an isomorphism with action groupoid.

  In general $K = (M \times X)/G$. For $M = S^1$, $G = \mathbb{Z}$ we can obtain a Kronecker foliation of a torus.

  If the action on X is free, then

$$(M \times X \times M)/G$$

  is an equivalence relation of foliation.

For any groupoid H denote

$$H_k := \{\gamma \in H \mid s(\gamma) = k\}$$

Definition 0.7. *For a groupoid H define a* groupoid algebra *of funtions $f\colon H \to \mathbb{C}$ with multiplication given by*

$$f_1 * f_2(\gamma) = \int_{H_{s(\gamma)}} f_1(\gamma\eta^{-1}) f_2(\eta) d\eta. \tag{0.18}$$

The formula (0.18) defines a representation of $C^\infty(H)$ on each $L^2(H_k)$.

Example 0.8. Let V be a vector space, and W an affine vector space over V. There is an action

$$\begin{aligned} V \times (W \times \mathbb{R}) &\to W \times \mathbb{R} \\ v \cdot (w,t) &\mapsto (w + tv, t) \end{aligned}$$

Then $H = V \ltimes (W \times \mathbb{R})$ is a family of groupoids over $\mathbb{R}$

$$H_t = \begin{cases} V \ltimes_t W & = \{(w_1, v, w_2)\} = \{(w_1, w_2)\} \text{ for } t \neq 0 \\ V \times_0 W & = V \ltimes W \text{ for } t = 0 \end{cases}$$

$$V \ltimes (W \times \mathbb{R}) \cong TW \times \{0\} \amalg W \times W \times \mathbb{R}^*.$$

The groupoid $V \rtimes (W \times \mathbb{R})$ depends only on W as a smooth manifold. It globalizes to Connes tangent groupoid

$$\mathbb{T}M := TM \times \{0\} \amalg M \times M \times \mathbb{R}^*. \tag{0.19}$$

If we form $C^*(\mathbb{T}M)$ we obtain a continuous field of C*-algebras. At $t = 0$, $C^*(TM) = C_0(T^*M)$, and at $t \neq 0$ $\mathcal{K}(L^2(M))$.

0.3 Index theorem

0.3.1 K-theory

For an algebra A there is a Grothendieck group of projective modules $\mathrm{K}(A)$.

$$\begin{aligned} \mathrm{K}(A) &= \pi_1(\mathrm{GL}_\infty(A)) \\ p,\ p^2 = p &\mapsto e^{2\pi i t p} \quad \text{(loop of invertible elements)} \end{aligned}$$

If A is a C*-algebra with unit, then typical projective module is pA, $p^2 = p$. If $||p - q|| < \varepsilon$ then $p = uqu^{-1}$ for some unitary u. If $p \in A_0$ for a continuous field of C*-algebras A_t, then there is a section p_t near $t = 0$. Hence if $\{A_t\}$ has $A_{t_1} = A_{t_2}$ for $t_1, t_2 \neq 0$, then we get $\mathrm{K}(A_0) \to \mathrm{K}(A_1)$. As a special case we have

$$\mathrm{K}(T^*M) \to \mathrm{K}(\mathrm{pt})$$

If A is an algebra with unit, and $\tau\colon A \to \mathbb{C}$ is a trace, $\tau(ab) = \tau(ba)$, then there is

$$\begin{aligned} \tau\colon\ \mathrm{K}(A) &\to \mathbb{C} \\ \tau([p_{ij}]) &= \sum \tau(p_{ii}) \end{aligned}$$

If (M, μ) is a manifold with measure μ, $A = C^\infty(M)$, then

$$\tau(p) := \int_M \operatorname{rank}(p(u)) d\mu(u).$$

Suppose we have $\varphi\colon A \times A \times A \to \mathbb{C}$. Define

$$\begin{aligned} b\varphi(a^0, a^1, a^2, a^3) := &\ \varphi(a^0 a^1, a^2, a^3) \\ &- \varphi(a^0, a^1 a^2, a^3) \\ &+ \varphi(a^0, a^1, a^2 a^3) \\ &- \varphi(a^3 a^0, a^1, a^2) \\ \Lambda\varphi(a^0, a^1, a^2) := &\ \varphi(a^2, a^0, a^1) \end{aligned}$$

Theorem 0.9. *If $b\varphi = 0$ and $\Lambda\varphi = \varphi$, then $p \mapsto \varphi(p,p,p)$ gives*

$$\varphi\colon \mathrm{K}(A) \to \mathbb{C}$$

0.3.2 Cyclic cocycles from Lie algebra actions

For a Lie algebra $\mathfrak{g}$ there is a chain complex

$$\cdots \xleftarrow{b} \Lambda^{n-1}\mathfrak{g} \xleftarrow{b} \Lambda^{n}\mathfrak{g} \xleftarrow{b} \Lambda^{n+1}\mathfrak{g} \xleftarrow{b} \cdots.$$

Let $\mathfrak{g} \times A \to A$ be an action by derivations. An invariant trace $\tau\colon A \to \mathbb{C}$ satisfies

$$\tau(X(a)) = 0, \text{ for } X \in \mathfrak{g}.$$

Let $c = X_1 \wedge \cdots \wedge X_n$. Denote

$$\varphi_c(a^0, \ldots, a^n) = \tau(a^0 X_1(a^1) \ldots X_n(a^n)) \quad \text{(anti-symmetrize)}$$

Then $b\varphi_c = 0$. For example, if δ_1, δ_2 are commuting derivations, then

$$\varphi(a^0, a^1, a^2) = \tau(a^0(\delta_1(a^1)\delta_2(a^2) - \delta_2(a^1)\delta_1(a^2))).$$

For the irrational rotation algebra A_θ we get

$$\mathcal{U}(\mathfrak{g}) \times A \to A.$$

Consider $G \subseteq \mathrm{Diff}(M)$, $A = C_c^\infty(G \ltimes M)$. Say $M = \mathbb{R}$. An algebra A has no trace in general. But form an "$ax+b$" group

$$P := \mathbb{R} \times \mathbb{R}_+, \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

with action $g(x,y) = (g(x), g'(x)^{-1}y)$. There is an invariant smooth measure.

Now consider

$$\mathrm{Diff}(\mathbb{R})^+ = P \cdot \{g \mid g(0) = 0,\ g'(0) = 1\}$$

$$\delta_n(g) = \frac{d^n}{dx^n} \log\left(\frac{dg^{-1}}{dx}\right)$$

Connes–Moscovici Hopf algebra H_1:

$$\begin{aligned}
\Delta Y &= Y \otimes 1 + 1 \otimes Y \\
\Delta X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \\
\Delta \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1
\end{aligned}$$

$$\begin{aligned}
[Y, X] &= X \\
[X, \delta_n] &= \delta_{n+1} \\
[Y, \delta_n] &= n\delta_n \\
[\delta_n, \delta_n] &= 0
\end{aligned}$$

0.3.3 Back to the tangent groupoid

Consider once more the tangent groupoid

$$\mathbb{T}M = \coprod_{t\in\mathbb{R}} \mathbb{T}_t M, \mathbb{T}_t M = \begin{cases} TM & \text{for } t = 0 \\ M \times M & \text{for } t \neq 0 \end{cases}$$

$$\begin{aligned} &V \ltimes (W \times \mathbb{R}) \cong \mathbb{T}M \\ &\quad (w_1, v, w_2) \mapsto (w_1, w_2) \text{ for } t \neq 0 \\ &\quad (w_1, v, w_2) \mapsto (v, w_2) \text{ for } t = 0,\ w_1 = w_2 \end{aligned}$$

We can think of

$$\mathbb{T}M = TM \times \{0\} \amalg M \times M \times \mathbb{R}^*$$

as of a family (equivariant)

$$\mathbb{T}M = \{\mathbb{T}M_{(m,t)}\}$$

If D is a PDO on M (of order 1), then

$$D_{(m,t)} = \begin{cases} D_m & \text{for } t = 0 \text{ on } \mathbb{T}M_{(m,0)} = T_m M \\ tD & \text{for } t \neq 0 \text{ on } \mathbb{T}M_{(m,t)} = M \end{cases}$$

is a smooth equivariant family.

Definition 0.10. *Family of PDO D is elliptic if the model operators D_p are elliptic for all $p \in M$*

Definition 0.11. *A constant coefficient operator D_p is elliptic if its Fourier transform (a function on T_p^*M) vanishes only at $\xi = 0$.*

0.3.4 Ellipticity and C*-algebras

Lemma 0.12. *D_p is elliptic if and only if $f(\widehat{D_p}) \in C_0(T_p^*M)$ for all $f \in C_0(\mathbb{R})$, that is if and only if $f(D_p) \in C^*(T_pM)$ for all $f \in C_0(\mathbb{R})$.*

Theorem 0.13. *If D is an elliptic family on a groupoid H with compact base, then $f(D) \in C^*(H)$.*

The following theorem represents most of the work in proving the Atiyah-Singer index theorem:

Theorem 0.14. *If D is elliptic on M, then the index map*

$$\mathrm{K}(T^*M) \to \mathrm{K}(\mathrm{pt})$$

takes the symbol of D to the analytic index of D.

0.3.5 Baum-Connes conjecture

Assume that G acts properly on M, and M is universal (e. g. M is a symmetric space of non-compact type). Then the index map

$$\mathrm{K}(C^*(G \ltimes T^*M)) \to \mathrm{K}(C^*G) \tag{0.20}$$

associated to $G \ltimes \mathbb{T}M$ is an isomorphism.

0.3.6 Contact manifolds

Definition 0.15. Contact manifold *is a pair* (M, S) *such that* M *is a manifold,* $S \subseteq T_M$ *is a subbundle such that locally there exists* $\alpha \in \Omega^1(M)$ *such that*

$$S = \ker \alpha$$
$$\alpha d\alpha \wedge \cdots \wedge d\alpha = \mathrm{vol}$$

Examples 0.16.

- $S^{2n-1} \subset \mathbb{C}^N$,
- SM, M Riemannian,
- Heisenberg groups

Theorem 0.17 (Darboux)**.** *Contact = locally Heisenberg*

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$H_5 = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & w \\ 0 & 0 & 1 & v \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

Start with Heisenberg group H_3 and define for $t \in \mathbb{R}$

$$t \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tx & t^2 z \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix}$$

Define
$$H \ltimes (H \times \mathbb{R}), \quad h \cdot (k, t) = ((t \cdot h)k, t).$$

This depends only on the contact structure. We get Heisenberg contact groupoid $\mathbb{H}M = HM$ and an index map

$$\mathrm{K}(C^*(HM)) \to \mathrm{K}(\mathrm{pt}).$$

Bibliography

Part I

K-theory of Operator Algebras

by

Rainer Matthes

Wojciech Szymański

Based on the lectures of:

- Rainer Matthes
  (*Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, ul. Hoża 74, Warszawa, 00-682 Poland*)
  – Chapters 1, 2, 3, 7, 8.

- Wojciech Szymański
  (*Dept. of Mathematics & Computer Science, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark*)
  – Chapters 4, 5, 6.

Chapter 1

Preliminaries on C*-algebras

These notes on K-theory owe a great deal to the book by Rørdam, Larsen and Laustsen [rll00], from which we borrowed both theoretical material and some exercises

1.1 Basic definitions

1.1.1 Definitions C*-algebra, $*$-algebra

Definition 1.1. *A C*-algebra A is an algebra over $\mathbb{C}$ with involution $a \mapsto a^*$ (*-algebra), equipped with a norm $a \mapsto \|a\|$, such that A is a Banach space, and the norm satisfies $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ (C^*-property).*

Immediate consequence: $\|a^*\| = \|a\|$. a^* is called adjoint of a.

A C*-algebra A is called *unital* if it has a multiplicative unit $1_A = 1$. Immediate consequence: $1^* = 1$, $\|1\| = 1$ ($\|1\| = \|1^2\| = \|1\|^2$). If A and B are C*-algebras, a $*$-*homomorphism* $\varphi : A \to B$ is a linear multiplicative map commuting with the involution. If A and B are unital, φ is called unital if $\varphi(1_A) = 1_B$. A surjective φ is always unital.

A C*-algebra A is called separable, if it contains a countable dense subset.

1.1.2 Sub-C^* and sub-$*$-algebras

A subset B of a C*-algebra A is called sub-$*$-algebra, if it closed under all algebraic operations (including the involution). It is called sub-C*-algebra, if it is also norm-closed. The norm closure of a sub-$*$-algebra is a sub-C*-algebra (from continuity of the algebraic operations).

If F is a subset of a C*-algebra A, the sub-C*-algebra generated by F, denoted by $C^*(F)$, is the smallest sub-C*-algebra containing F. It coincides with the norm closure of the linear span of all monomials in elements of F and their adjoints. A subset F is called self-adjoint, if $F^* := \{a^* \mid a \in F\} = F$.

1.1.3 Ideals and quotients

An ideal in a C*-algebra is a norm-closed two-sided ideal. Such an ideal is always self-adjoint, hence a sub-C*-algebra. ([d-j77, 1.8.2], [m-gj90, 3.1.3]) If I is an ideal in a C*-algebra A, the quotient $A/I = \{a + I \mid a \in A\}$ is a $*$-algebra, and also a C*-algebra with respect to the norm $\|a+I\| := \inf\{\|a+x\| \mid x \in I\}$. I is obviously the kernel of the quotient map $\pi : A \to A/I$. ([m-gj90, 3.1.4], [d-j77, 1.8.2])

A $*$-homomorphism $\varphi : A \to B$ of C*-algebras is always norm-decreasing, $\|\varphi(a)\| \leq \|a\|$. It is injective if and only if it is isometric. ([m-gj90, 3.1.5]). $\ker\varphi$ is an ideal in A, $\operatorname{im}\varphi$ a sub-C*-algebra of B. ([m-gj90, 3.1.6]). φ always factorizes as $\varphi = \varphi_0 \circ \pi$, with injective $\varphi_0 : A/\ker\varphi \to B$.

A C*-algebra is called *simple* if its only ideals are $\{0\}$ and A (trivial ideals).

1.1.4 The main examples

Example 1. Let X be a locally compact Hausdorff space, and let $C_0(X)$ be the vector space of complex-valued continuous functions that vanish at infinity, i.e., for all $\epsilon > 0$ exists a compact subset $K_\epsilon \subseteq X$ such that $|f(x)| < \epsilon$ for $x \notin K_\epsilon$. Equipped with the pointwise multiplication and the complex conjugation as involution, $C_0(X)$ is a $*$-algebra. With the norm $\|f\| := \sup_{x\in X}\{|f(x)|\}$, $C_0(X)$ is a (in general non-unital) commutative C*-algebra.

Theorem 1.2. *(Gelfand-Naimark) Every commutative C*-algebra is isometrically isomorphic to an algebra $C_0(X)$ for some locally compact Hausdorff space X.*

Idea of proof: X is the set of multiplicative linear functionals (characters (every character is automatically *-preserving, [d-j77, 1.4.1(i)], equivalently, the set of maximal ideals), with the weak-$*$-topology (i.e., the weakest topology such that all the functionals $\chi \mapsto \chi(a)$, $a \in A$, are continuous.

Additions:

(i) $C_0(X)$ is unital if and only if X is compact.

(ii) $C_0(X)$ is separable if and only if X is separable.

(iii) X and Y are homeomorphic if and only if $C_0(X)$ and $C_0(Y)$ are isomorphic.

(iv) Each proper continuous map $\eta : Y \to X$ induces a $*$-homomorphism $\eta^* : C_0(X) \to C_0(Y)$ ($\eta^*(f) = f \circ \eta$). Conversely, each $*$-homomorphism $\varphi : C_0(X) \to C_0(Y)$ induces a proper continuous map $\eta : Y \to X$ (map a character χ of $C_0(Y)$ to the character $\chi \circ \varphi$ of $C_0(X)$).

(v) There is a bijective correspondence between open subsets of X and ideals in $C_0(X)$ (the ideal to an open subset is the set of functions vanishing on the complement of the subset, to an ideal always corresponds the set of characters vanishing on the ideal, its complement in the set of all characters is the desired open set). If $U \subseteq X$ is open, then there is a short exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C_0(X) \longrightarrow C_0(X \setminus U) \longrightarrow 0, \tag{1.1}$$

where $C_0(U) \to C_0(X)$ is given by extending a function on U as 0 to all of X, and $C_0(X) \to C_0(X \setminus U)$ is the restriction, being surjective due to Stone-Weierstraß.

Example 2. Let $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ denote the set of all continuous linear operators on $\mathcal{H}$. Then $B(\mathcal{H})$ is an algebra with respect to addition, multiplication with scalars, and composition of operators, it is a $*$-algebra with the usual operator adjoint, and it is a C*-algebra with respect to the operator norm.

Theorem 1.3. *(Gelfand-Naimark) Every C*-algebra A is isometrically isomorphic to a closed C^*-subalgebra of some $B(\mathcal{H})$.*

Idea of proof: Consider the set of positive linear functionals ($\varphi(a^*a) \geq 0$) on A. Every such functional allows to turn the algebra into a Hilbert space on which the algebra is represented by its left action. Take as Hilbert space the direct sum of all these Hilbert spaces. Then the direct sum of these representations gives the desired injection.

1.1.5 Short exact sequences

A sequence of C*-algebras and $*$-homomorphisms

$$\ldots \longrightarrow A_k \xrightarrow{\varphi_k} A_{k+1} \xrightarrow{\varphi_{k+1}} A_{k+2} \longrightarrow \ldots \tag{1.2}$$

is said to be *exact*, if $\operatorname{im}\varphi_k = \ker \varphi_{k+1}$ for all k. An exact sequence of the form

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{1.3}$$

is called *short exact*. Example: If $I \subseteq A$ is an ideal, then

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0 \tag{1.4}$$

is short exact (ι the natural embedding $I \to A$). If a short exact sequence (1.3) is given, then $\varphi(I)$ is an ideal in A, there is an isomorphism $\psi_/ : B \to A/\varphi(I)$, and the diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & A \arrow[r, "\psi"] \arrow[d, "\mathrm{id}"'] & B \arrow[r] \arrow[d, "\psi_/"'] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota"] & A \arrow[r, "\pi"] & A/I \arrow[r] & 0
\end{tikzcd}$$

is commutative. If for a short exact sequence (1.3) exists $\lambda : B \to A$ with $\psi \circ \lambda = \mathrm{id}_B$, then the sequence is called *split exact*, and λ is called *lift* of ψ. Diagrammatic:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\varphi"] & A \arrow[r, shift left, "\psi"] & B \arrow[l, shift left, "\lambda"] \arrow[r] & 0.
\end{tikzcd} \tag{1.5}$$

Not all short exact sequences are split exact.
Example:

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\iota} C([0,1]) \xrightarrow{\psi} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0 \tag{1.6}$$

with $\psi(f) = (f(0), f(1))$ is an exact sequence. It does not split: Every linear map $\lambda : \mathbb{C} \oplus \mathbb{C} \to C([0,1])$ is determined by its values on the basis elements, $\lambda((1,0)) = f_1,\ \lambda((0,1)) = f_2$. The split condition means $f_1(0) = 1,\ f_1(1) = 0$ and $f_2(0) = 0,\ f_2(1) = 1$. If λ is to be a homomorphism, because of $(1,0)^2 = (1,0)$, we should have $f_1^2 = \lambda((1,0))^2 = \lambda((1,0)^2) = \lambda((1,0)) = f_1$, and analogously $f_2^2 = f_2$. However, a continuous function on a connected space is equal to its square if and only if it is either the constant function 1 or the constant function 0. Both is not the case for f_1 and f_2.
Geometric interpretation: ψ corresponds to the embedding of two points as end points of the interval $[0,1]$. However, it is not possible to map this interval continuously onto the set $\{0,1\}$.

The direct sum $A \oplus B$ of two C*-algebras is the direct sum of the underlying vector spaces, with component-wise defined multiplication and involution, and with the norm $\|(a,b)\| = \max(\|a\|, \|b\|)$. It is again a C*-algebra. There are natural homomorphisms $\iota_A : A \to A \oplus B,\ a \mapsto (a,0)$, $\pi_A : A \oplus B \to A,\ (a,b) \mapsto a$, analogously $\iota_B,\ \pi_B$. Then

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota_A"] & A \oplus B \arrow[r, shift left, "\pi_B"] & B \arrow[l, shift left, "\iota_B"] \arrow[r] & 0. \end{tikzcd} \tag{1.7}$$

is a split exact sequence with lift ι_B. Not all split exact sequences come in this manner from direct sums.
Example (not presented in lecture).
Let

$$0 \longrightarrow A \xrightarrow{\varphi} E \xrightarrow{\psi} B \longrightarrow 0 \tag{1.8}$$

be an exact sequence. Then there exists an isomorphism $\theta : E \to A \oplus B$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"'] & E \arrow[r, "\psi"] \arrow[d, "\theta"'] & B \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "\iota_A"] & A \oplus B \arrow[r, "\pi_B"] & B \arrow[r] & 0 \end{tikzcd}$$

commutative if and only there exists a homomorphism $\nu : E \to A$ such that $\nu \circ \varphi = \mathrm{id}_A$.

Proof. If $\theta : E \to A \oplus B$ makes the diagram commutative, then $\theta_{|\operatorname{im}\varphi}$ is an injective map whose image is $\iota_A(A)$. $\nu := \pi_A \circ \theta_{|\operatorname{im}\varphi} : E \to A$ fulfills $\nu \circ \varphi = \mathrm{id}_A$. If $\nu : E \to A$ with this property is given, put $\theta(e) = (\iota_A \circ \nu(e), \psi(e))$. θ is an isomorphism:
surjective: Let $a \in A$. Then $\varphi(a) \in \ker\psi$, hence $\psi(\varphi(a)) = 0$. But, $\nu(\varphi(a)) = a$, i.e., $\theta(\varphi(a)) = (a, 0)$. On the other hand, as $\pi_B \circ \theta = \psi$ and ψ is surjective, for any $b \in B$ exists $a' \in A$ such that $(a', b) \in \operatorname{im}\theta$. Since $(a', 0) \in \operatorname{im}\theta$, also $(0, b) \in \operatorname{im}\theta$ for any $b \in B$, thus finally all $(a, b) \in \operatorname{im}\theta$.
injective: If $\psi(e) = 0$ with $e \neq 0$ then $e = \varphi(a)$ with $a \neq 0$, and $\nu(e) = \nu \circ \varphi(a) =$

$a \neq 0$, thus $\iota_A \circ \nu(e) \neq 0$ by injectivity of ι_A. Otherwise, $\psi(e) \neq 0$ already means $\theta(e) \neq 0$. ☐

If this condition is satisfied, the upper sequence is isomorphic to the lower one, and thus also split. Counterexample (where the condition is not fulfilled)?

1.1.6 Adjoining a unit

Definition 1.4. *Let A be a $*$-algebra. Put $\tilde{A} = A \oplus \mathbb{C}$ (direct sum of vector spaces) and*

$$(a, \alpha)(b, \beta) := (ab + \beta a + \alpha b, \alpha\beta), \quad (a, \alpha)^* := (a^*, \bar{\alpha}). \tag{1.9}$$

Define $\iota : A \to \tilde{A}$ and $\pi : \tilde{A} \to \mathbb{C}$ by $\iota(a) = (a, 0)$, $\pi(a, \alpha) = \alpha$ (i.e., $\iota = \iota_A$, $\pi = \pi_{\mathbb{C}}$ in the direct sum terminology used above).

Proposition 1.5. *With the operations just introduced, $\tilde{A}$ is a unital $*$-algebra with unit $1_{\tilde{A}} = (0, 1)$. ι is an injective, π a surjective $*$-homomorphism.*

Proof. Straightforward. ☐

Sometimes ι is suppressed, and we write also $\tilde{A} = \{a + \alpha 1 \mid a \in A, \alpha \in \mathbb{C}\}$.

Let now A be a C*-algebra, and let $\|.\|_A$ be the norm on A.

Note that the direct sum norm $\|(a, \alpha)\| = \max(\|a\|, |\alpha|)$ does in general not have the C^*-property (because $A \oplus \mathbb{C}$ does not have the direct sum product). Example: A unital, put $\alpha = 1$, $a = 1_A$, then $\|(a, \alpha)(a^*, \bar{\alpha})\| = \max(\|aa^* + \bar{\alpha}a^* + \alpha a^*\|, |\alpha|^2) = 3$, $\|(a, \alpha)\|^2 = \max(\|a\|^2, |\alpha|^2) = 1$.
Recall that the algebra $B(E)$ of linear operators on a Banach space E is a Banach space (algebra) with norm $\|b\| = \sup_{\|x\| \leq 1} \|b(x)\|$ (see [rs72, Theorem III.2], [d-j73, 5.7]). Note that $(a, \alpha) \mapsto L_a + \alpha \mathrm{id}_A$, where $L_a(b) = ab$ for $a, b \in A$, defines a homomorphism φ of $\tilde{A}$ onto the subspace of all continuous linear operators of the form $L_a + \alpha \mathrm{id}_A$ in $B(A)$. This homomorphism is injective if and only if A is not unital. (exercise) Indeed, let A be not unital, and assume $L_a(b) + \alpha b = 0$ for all $b \in A$. If α would be $\neq 0$ then $-\frac{a}{\alpha}$ would be a left unit for A, thus also a right unit, hence a unit, contradicting the assumed non-unitality. Thus we have $\alpha = 0$, i.e. $ab = 0$ for all $b \in A$. In particular, $aa^* = 0$, hence $\|a^*\|^2 = \|aa^*\| = 0$, i.e., $\|a\| = \|a^*\| = 0$, i.e., $a = 0$. On the other hand, if A is unital, $(1_A, -1)$ is in the kernel of φ.

We have $\|a\| = \|L_a\|$ for $a \in A$: $\|L_a\| \leq \|a\|$ is clear by the definition of the operator norm ($\|L_a\| = \sup_{\|b\| \leq 1} \|ab\| \leq \sup_{\|b\| \leq 1} \|a\|\|b\| = \|a\|$), and $\|a\|^2 = \|aa^*\| = \|L_a(a^*)\| \leq \|L_a\|\|a^*\|$, hence also $\|a\| \leq \|L_a\|$. Thus it makes sense to define for non-unital A a norm on $\tilde{A}$ by transporting the norm of $B(A)$, i.e., we put $\|(a, \alpha)\|_{\tilde{A}} := \|L_a + \alpha \mathrm{id}_A\|$. For unital A, we note that $\tilde{A}$ is as a $*$-algebra isomorphic to $A \oplus \mathbb{C}$ (direct sum of C*-algebras). The isomorphism is given by $(a, \alpha) \mapsto (a + \alpha 1_A, \alpha)$ (easy exercise). As before, we define the norm on $\tilde{A}$ by transport with the isomorphism. Note that $(-1_A, 1)$ is a projector in $A \oplus \mathbb{C}$.

Proposition 1.6. *$\tilde{A}$ is a unital C^*-algebra with norm $\|.\|_{\tilde{A}}$. $\iota(A)$ is a closed ideal in $\tilde{A}$.*

Proof. The additive and multiplicative triangle inequality come from these properties for the norm in $B(A)$ and $A \oplus \mathbb{C}$. Since $\{L_a | a \in A\}$ is closed and thus complete in $B(A)$, and $\{L_a | a \in A\}$ has codimension 1 in $\varphi(\tilde{A})$, the latter is also complete in the nonunital case, and it is obviously complete in the unital case. Also, it is obvious in the unital case that the norm has the C^*-property. To prove the latter for the nonunital case, we define the involution on $\varphi(\tilde{A})$ by transport with φ, i.e.,

$$(L_a + \alpha \mathrm{id}_A)^* := L_{a^*} + \bar{\alpha} \mathrm{id}_A. \tag{1.10}$$

Hence, by this definition $\varphi(\tilde{A})$ is a complete normed $*$-algebra. It remains to show that the C^*-property is satisfied. Let $\epsilon > 0$ and let $x = L_a + \alpha \mathrm{id}_A \in \varphi(\tilde{A})$. By the definition of the operator norm, there exists $b \in A$ with $\|b\| \leq 1$ such that

$$\|x\|^2 = \|L_a + \alpha \mathrm{id}_A\|^2 \leq \|(L_a + \alpha \mathrm{id}_A)(b)\|^2 + \epsilon. \tag{1.11}$$

The right hand side can be continued as follows:

$$\begin{aligned}
&= \|ab + \alpha b\|^2 + \epsilon \\
&= \|(ab + \alpha b)^*(ab + \alpha b)\| + \epsilon \\
&= \|(b^* a^* + \bar{\alpha} b^*)(ab + \alpha b)\| + \epsilon \\
&= \|b^*(L_{a^*} + \bar{\alpha} \mathrm{id}_A)(L_a + \alpha \mathrm{id}_A)(b)\| + \epsilon \\
&\leq \|b^*\| \|(L_{a^*} + \bar{\alpha} \mathrm{id}_A)(L_a + \alpha \mathrm{id}_A)(b)\| + \epsilon \\
&\leq \|b^*\| \|((L_{a^*} + \bar{\alpha} \mathrm{id}_A)(L_a + \alpha \mathrm{id}_A)\| \|b\| + \epsilon \\
&\leq \|x^* x\| + \epsilon.
\end{aligned}$$

Thus we have $\|x\|^2 \leq \|x^* x\| + \epsilon$ for any ϵ, hence $\|x\|^2 \leq \|x^* x\|$. However, also $\|x^* x\| \leq \|x^*\| \|x\|$ ($B(A)$ is a normed algebra). Exchanging the roles of x and x^*, we also obtain $\|x^*\|^2 \leq \|x\| \|x^*\|$, together $\|x\| = \|x^*\|$. Going back to the inequalities, this also gives the C^*-property. □

For both the unital and nonunital case, we have $\tilde{A}/\iota(A) \cong \mathbb{C}$, and the sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \tilde{A} \arrow[r, shift left, "\pi"] & \mathbb{C} \arrow[l, shift left, "\lambda"] \arrow[r] & 0. \end{tikzcd} \tag{1.12}$$

with $\pi : \tilde{A} \to \mathbb{C}$ the quotient map and $\lambda : \mathbb{C} \to \tilde{A}$ given by $\alpha \mapsto (0, \alpha)$, is split exact. Note also that adjoining a unit is functorial: If $\varphi : A \to B$ is a homomorphism of C*-algebras, there is a unique homorphism $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota_A"] \arrow[d, "\varphi"'] & \tilde{A} \arrow[r, "\pi_A"] \arrow[d, "\tilde{\varphi}"'] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & B \arrow[r, "\iota_B"] & \tilde{B} \arrow[r, "\pi_B"] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

commutative. It is given by $\tilde{\varphi}(a, \alpha) = (\varphi(a), \alpha)$. $\tilde{\varphi}$ is unit-preserving, $\tilde{\varphi}(0, 1) = (0, 1)$. If A is a sub-C*-algebra of a unital C*-algebra B whose unit 1_B is not in A, then $\tilde{A}$ is isomorphic to the sub-C*-algebra $A + \mathbb{C}1_B$ of B (exercise).

1.2 Spectral theory

1.2.1 Spectrum

Let A be a unital C*-algebra. Then the spectrum (with respect to A) of $a \in A$ is defined as

$$\mathrm{sp}(a)(=\mathrm{sp}_A(a)) := \{\lambda \in \mathbb{C} \mid a - \lambda 1_A \text{ is not invertible in } A\}. \tag{1.13}$$

Elementary statements about the spectrum, true already for a unital algebra, are:

(i) If $A = \{0\}$ then $\mathrm{sp}(0) = \emptyset$.

(ii) $\mathrm{sp}(\lambda 1_A) = \{\lambda\}$ for $\lambda \in \mathbb{C}$.

(iii) $a \in A$ is invertible if and only if $0 \notin \mathrm{sp}(a)$.

(iv) If $P \in \mathbb{C}[X]$ (polynomial in one variable with complex coefficients), then $\mathrm{sp}(P(a)) = P(\mathrm{sp}(a))$.

(v) If $a \in A$ is nilpotent, then $\mathrm{sp}(a) = \{0\}$ (if $A \neq \{0\}$).

(vi) If $\varphi : A \to B$ is a morphism of unital algebras over $\mathbb{C}$, then $\mathrm{sp}_B(\varphi(a)) \subseteq \mathrm{sp}_A(a)$.

(vii) If $(a, b) \in A \oplus B$ (direct sum of algebras), then $\mathrm{sp}_{A\oplus B}((a, b)) = \mathrm{sp}_A(a) \cup \mathrm{sp}_B(b)$. (Can be generalized to direct products.)

If A is the algebra of continuous complex-valued functions on a topological space, then the spectrum of any element is the set of values of the function. If A is the algebra of endomorphisms of a finite dimensional vector space over $\mathbb{C}$ then the spectrum of an element is the set of eigenvalues.

For a Banach algebra, the spectrum of an element is always a compact subset of $\mathbb{C}$ contained in the ball of radius $\|a\|$,

$$r(a) = \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(a)\} \leq \|a\|. \tag{1.14}$$

Idea of proof: If $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$, hence $1 - \lambda^{-1}a$ is invertible (This uses: if $\|a\| < 1$ then $1 - a$ is invertible, with $(1 - a)^{-1} = 1 + a + a^2 + \ldots$ – Neumann series.) Thus $\lambda \notin \mathrm{sp}(a)$. The spectrum is closed because the set of invertible elements is open (use again the fact stated in parentheses).

The number $r(a)$ is called spectral radius of a. Using complex analysis, one can show that the spectrum is non-empty. The sequence $(\|a^n\|^{1/n})$ is convergent, and $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. If A is not unital, the spectrum of an element $a \in A$ is defined as the spectrum of $\iota(a) \in \tilde{A}$. In this case always $0 \in \mathrm{sp}(a)$ ($(a, 0)(b, \beta) = (ab + \beta a, 0) \neq (0, 1) = 1_{\tilde{A}}$).

Definition 1.7. *An element a of a C*-algebra A is called*

- *normal if $aa^* = a^*a$,*
- *self-adjoint if $a = a^*$,*
- *positive if it is normal and* $\mathrm{sp}(a) \subseteq \mathbb{R}_+ (= [0, \infty[)$,

- *unitary if A is unital and $aa^* = a^*a = 1_A$.*
- *a projector if $a = a^* = a^2$.*

The set of positive elements is denoted by A^+.

The spectrum of a self-adjoint element is contained in $\mathbb{R}$, that of a unitary element is contained in $\mathbb{T}^1 = S^1$ (the unit circle, considered as a subset of $\mathbb{C}$), that of a projector is contained in $\{0, 1\}$ (exercises). An element a of a C*-algebra A is positive if and only if it is of the form $a = x^*x$, for some $x \in A$. For normal elements, the above formula for the spectral radius reduces to $r(a) = \|a\|$. This allows to conclude

Proposition 1.8. *The C^*-norm of a C^*-algebra is unique.*

Proof. $\|a\|'^2 = \|a^*a\|' = r(a^*a) = \|a^*a\| = \|a\|^2$. □

Let us also note that every element is a linear combination of two self-adjoint elements, $a = \frac{1}{2}(a + a^*) + i\frac{1}{2i}(a - a^*)$ (this is the unique decomposition $a = h_1 + ih_2$, with h_1 and h_2 self-adjoint), and also a linear combination of four unitary elements.

The spectrum a priori depends on the ambient C*-algebra. However, if B is a unital C^*-subalgebra of a unital C*-algebra A, whose unit coincides with the unit of A, then the spectrum of an element of B with respect to B coincides with its spectrum with respect to A (exercise, use that the inverse of an element belongs to the smallest C*-algebra containing that element, i.e., the C^*-subalgebra generated by that element). If A is not unital, or if the unit of A does not belong to B, then $\mathrm{sp}_A(b) \cup \{0\} = \mathrm{sp}_B(b) \cup \{0\}$ (exercise).

1.2.2 Continuous functional calculus

Let A be a unital C*-algebra, and let $a \in A$ be normal. Then there is a unique C^*-isomorphism $j : C(\mathrm{sp}(a)) \to C^*(a, 1)$ mapping the identity map of $\mathrm{sp}(a)$ into a. Moreover, this isomorphism maps a polynomial P into $P(a)$ and the complex conjugation $z \mapsto \bar{z}$ into a^*. Therefore one writes $j(f) = f(a)$. One knows that $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$ (spectral mapping theorem).

If $\varphi : A \to B$ is $*$-homomorphism of unital C*-algebras, then $\mathrm{sp}(\varphi(a)) \subseteq \mathrm{sp}(a)$ and $\varphi(f(a)) = f(\varphi(a))$ for $f \in \mathbb{C}(\mathrm{sp}(a))$.

If a C*-algebra is realized as a subalgebra of $B(\mathcal{H})$, the functional calculus is realized for self-adjoint elements in terms of their spectral decompositions: If $a = \int \lambda dE_\lambda$ then $f(a) = \int f(\lambda) dE_\lambda$, where E_λ is the family of spectral measures belonging to a.

If a is a normal element of a non-unital C*-algebra A, then $f(a)$ is a priori in $\tilde{A}$. We have $f(a) \in \iota(A) \simeq A$ if and only if $f(0) = 0$: When $\pi : \tilde{A} \to \mathbb{C}$ is the quotient mapping, we have $\pi(f(a)) = f(\pi(a)) = f(0)$.

Lemma 1.9. *Let $K \subseteq \mathbb{R}$ be compact and non-empty, and let $f \in C(K)$. Let A be a unital C^*-algebra, and let Ω_K be the set of self-adjoint elments of A with spectrum contained in K. Then the induced function*

$$f : \Omega_K \longrightarrow A, \quad a \mapsto f(a), \tag{1.15}$$

is continuous.

Proof. The map $a \mapsto a^n$, $A \to A$ is continuous (continuity of multiplication). Thus every complex polynomial f induces a continuous map $A \to A$, $a \mapsto f(a)$.

Now, let $f \in C(K)$, let $a \in \Omega_K$, and let $\epsilon > 0$. Then there is a complex polynomial g such that $|f(z) - g(z)| < \frac{\epsilon}{3}$ for every $z \in K$. By continuity discussed above, for every ϵ we find $\delta < 0$ such that $\|g(a) - g(b)\| \leq \frac{\epsilon}{3}$ for $b \in A$ with $\|a - b\| \leq \delta$. Since, moreover,

$$\|f(c) - g(c)\| = \|(f - g)(c)\| = \sup\{|(f - g)(z)| \mid z \in \mathrm{sp}(c)\} \leq \frac{\epsilon}{3} \tag{1.16}$$

for $c \in \Omega_K$, we conclude $\|f(a)-f(b)\| = \|f(a)-g(a)+g(a)-g(b)+g(b)-f(b)\| \leq \|f(a-g(a)\| + \|g(a)-g(b)\| + \|g(b)-f(b)\| \leq \epsilon$ for $b \in \Omega_K$ with $\|a-b\| \leq \delta$. □

1.3 Matrix algebras and tensor products

Let A_1, A_2 be C*-algebras. The algebraic tensor product $A_1 \otimes A_2$ is a $*$-algebra with multiplication and adjoint given by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2, \tag{1.17}$$

$$(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*. \tag{1.18}$$

Problem: There may exist different norms with the C^*-property on this $*$-algebra, leading to different C*-algebras under completion (though one can show that all norms with the C^*-property are cross norms, $\|a_1 \otimes a_2\| = \|a_1\|\|a_2\|$). We will restrict to the case where this problem is not there by definition: A C*-algebra is called *nuclear* if for any C*-algebra B there is only one C^*-norm on the algebraic tensor product $A \otimes B$. Examples: finite dimensional, commutative, type I (every non-zero irreducible representation in a Hilbert space contains the compact operators). If one of the tensor factors is nuclear, the unique C^*-norm on the algebraic tensor product coincides with the norm in $B(\mathcal{H})$ under a faithful representation of the completed tensor product.

We will mainly need the following very special situation. Let A be a C*-algebra, and let $M_n(\mathbb{C})$ ($n \in \mathbb{N}$) be the algebra of complex $n \times n$-matrices. Then $A \otimes M_n(\mathbb{C})$ can be identified with $M_n(A)$, the $*$-algebra of $n \times n$-matrices with entries from A, with product and adjoint given according to the matrix structure. The unique C^*-norm on $A \otimes M_n(\mathbb{C}) = M_n(A)$ is defined using any injective $*$-homomorphism $\varphi : A \to B(\mathcal{H})$, and the canonical injective $*$-homomorphism $M_n(\mathbb{C}) \to B(\mathbb{C}^n)$, i.e., $\|a \otimes m\| = \|\varphi(a) \otimes m\|$, where on the right stands the norm in $B(\mathcal{H}) \otimes B(\mathbb{C}^n) = B(\mathcal{H} \otimes \mathbb{C}^n)$. One has the inequality (exercise):

$$\max \|a_{ij}\| \leq \left\| \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \right\| \leq \sum \|a_{ij}\|. \tag{1.19}$$

The following lemma will be needed later. It involves the C*-algebra $C_0(X, A)$, see Exercice 6.

Lemma 1.10. *Let X be a locally compact Hausdorff space and let A be a C*-algebra. Define for $f \in C_0(X)$, $a \in A$ an element $fa \in C_0(X, A)$ by*

$$(fa)(x) = f(x)a. \tag{1.20}$$

Then $\operatorname{span}\{fa \mid f \in C_0(X), a \in A\}$ *is dense in* $C_0(X, A)$.

Proof. Let $X^+ = X \cup \{\infty\}$ be the one-point compactification of X. Then

$$C_0(X, A) = \{f \in C(X^+, A) \mid f(\infty) = 0\}. \tag{1.21}$$

Let $f \in C_0(X, A)$, $\epsilon > 0$. There is an open covering $U_1, \ldots, U_n$ of X^+ such that $\|f(x) - f(y)\| < \epsilon$ if $x, y \in U_k$. (Compactness of X^+, continuity of f.) Choose $x_k \in U_k$, with $x_k = \infty$ if $\infty \in U_k$. Let $(h_k)_{k=1}^n$ be a partition of unity subordinate to the covering (U_k), i.e., $h_k \in C(X^+)$, $\operatorname{supp} h_k \subseteq U_k$, $\sum_{k=1}^n h_k = 1$, $0 \le h_k \le 1$. (Note that every compact Hausdorff space is paracompact.) Then $\|f(x)h_k(x) - f_k(x_k)h_k(x)\| \le \epsilon h_k(x)$ for $x \in X$, $k = 1, \ldots, n$. It follows that $\|f(x) - \sum_{k=1}^n f(x_k)h_k(x)\| \le \epsilon$, for $x \in X$. Put $a_k = f(x_k) \in A$. Then $\sum_{k=1}^n h_k a_k \in \operatorname{span}\{fa \mid f \in C_0(X), a \in A\}$, because $a_k = f(x_k) = 0$ if $\infty \in U_k$, and $\|f - \sum_{k=1}^n h_k a_k\| \le \epsilon$. □

1.4 Examples and Exercises

Exercise 1. If A is a sub-C*-algebra of a unital C*-algebra B whose unit 1_B is not in A, then $\tilde{A}$ is isomorphic to the sub-C*-algebra $A + \mathbb{C}1_B$ of B.

The map $(a, \alpha) \mapsto a + \alpha 1_B$ is the desired isomorphism: It is obviously surjective, and injectivity follows as injectivity of φ in the proof of Proposition 1.6: Let $a + \alpha 1_B = 0$. If $\alpha \ne 0$, then $1_B = -\frac{a}{\alpha} \in A$, contradicting the assumption. Thus $\alpha = 0 = a$. That the mapping is a $*$-homomorphism is straightforward.

Exercise 2. Let A be a unital C*-algebra. Show the following.

(i) Let u be unitary. Then $\operatorname{sp}(u) \subseteq \mathbb{T}$.

(ii) Let u be normal, and $\operatorname{sp}(u) \subseteq \mathbb{T}$. Then u is unitary.

(iii) Let a be self-adjoint. Then $\operatorname{sp}(a) \subseteq \mathbb{R}$.

(iv) Let p be a projection. Then $\operatorname{sp}(p) \subseteq \{0, 1\}$.

(v) Let p be normal with $\operatorname{sp}(p) \subseteq \{0, 1\}$. Then p is a projector.

(i): $\|u\| = 1$, due to $\|u\|^2 = \|u^*u\| = \|1_A\| = 1$. Hence $|\lambda| \le 1$ for $\lambda \in \operatorname{sp}(u)$. By the spectral mapping theorem, $\lambda^{-1} \in \operatorname{sp}(u^{-1}) = \operatorname{sp}(u^*)$. But also $\|u^*\| = 1$, and thus $|\lambda^{-1}| \le 1$, so $|\lambda| = 1$.

(ii): Due to normality, there is a C^*-isomorphism $C(\operatorname{sp}(u)) \to C^*(u, 1)$, mapping $\mathrm{id}_{\operatorname{sp}(u)} \mapsto u$ and $\bar{\mathrm{id}}_{sp(u)} \mapsto u^*$. Hence $1_{\operatorname{sp}(u)} \mapsto u^*u = uu^* = 1 (= 1_A)$.

(iii): $a \in A$ is invertible if and only if a^* is invertible, thus $a - \lambda 1_A$ is invertible if and only if $a^* - \bar{\lambda}$ is invertible. Thus $\lambda \in \operatorname{sp}(a)$ if and only if $\bar{\lambda} \in \operatorname{sp}(a^*)$, and for $a = a^*$ the spectrum is invariant under complex conjugation. The series $\exp(ia) := \sum_{n=0}^\infty \frac{(ia)^n}{n!}$ is absolutely convergent, its adjoint is (due to continuity of the star operation) $\exp(-ia) = \sum_{n=0}^\infty \frac{(-ia)^n}{n!}$ and fulfills $\exp(ia)\exp(-ia) = 1_A = \exp(-ia)\exp(ia)$, so it is a unitary element in $C^*(a, 1)$, which means that $\exp(i\lambda) \in \mathbb{T}$ for $\lambda \in \operatorname{sp}(a)$, i.e., $\lambda \in \mathbb{R}$.

(iv): Let $p = p^* = p^2$. By (iii), $\operatorname{sp}(p)$ is real, and by the spectral mapping theorem we have $\operatorname{sp}(p) = \operatorname{sp}(p)^2$. This means that $\operatorname{sp}(p) \subseteq [0, 1]$. Using the isomorphism $C(\operatorname{sp}(p)) \to C^*(p, 1)$, we have $\mathrm{id}_{\operatorname{sp}(p)} = \mathrm{id}^2_{\operatorname{sp}(p)}$, thus $\operatorname{sp}(p) \subseteq \{0, 1\}$.

(v): Let p be normal, $\operatorname{sp}(p) \subseteq \{0, 1\}$. Then $\mathrm{id}_{\operatorname{sp}(p)} = \bar{\mathrm{id}}_{\operatorname{sp}(p)} = \mathrm{id}^2_{\operatorname{sp}(p)}$, and the same is true for p (using the isomorphism $C(\operatorname{sp}(p)) \to C^*(p, 1)$).

Exercise 3. Let A be a unital C*-algebra, $a \in A$.

(i) a is invertible if and only if aa^* and a^*a are invertible. In that case, $a^{-1} = (a^*a)^{-1}a^* = a^*(aa^*)^{-1}$.

(ii) Let a be normal and invertible in A. Then there exists $f \in C(\mathrm{sp}(a))$ such that $a^{-1} = f(a)$, i.e., a^{-1} belongs to $C^*(a, 1)$.

(iii) Let $a \in A$ be invertible. Then a^{-1} belongs to $C^*(a, 1)$, the smallest unital C^*-subalgebra containing a.

(i): If a^{-1} exists, then also $a^{*-1} = a^{-1*}$ and $(aa^*)^{-1} = a^{*-1}a^{-1}$, $(a^*a)^{-1} = a^{-1}a^{*-1}$. If $(aa^*)^{-1}$ and $(a^*a)^{-1}$ exist, put $b := a^*(aa^*)^{-1}$ and $c := (a^*a)^{-1}a^*$. Then $ab = 1 = ca$ and, multiplying the left of these equalities by c from the left, the right one by b from the right, $cab = c$, $b = cab$. This means $b = c = a^{-1}$.

(ii): a invertible means that $0 \notin \mathrm{sp}(a)$. Thus, the function $\mathrm{id}_{\mathrm{sp}(a)}$ corresponding to a under the isomorphism $C(\mathrm{sp}(a)) \to C^*(a, 1)$ is invertible, and the corresponding inverse is in $C^*(a, 1)$.

(iii): aa^* and a^*a are normal (selfadjoint) and by (i) invertible in A. By (ii) their inverses are in the C^*-subalgebras generated by $\{aa^*, 1\}$ and $\{a^*a, 1\}$, thus also in $C^*(a, 1)$. Again using (i) (considering $C^*(a, 1)$ instead of A), we obtain $a^{-1} \in C^*(a, 1)$.

Exercise 4. Show the uniqueness of the decomposition $a = h_1 + ih_2$, $h_{1,2}$ selfadjoint.

We have $a^* = h_1 - ih_2$, hence $h_1 = \frac{1}{2}(a + a^*)$ and $h_2 = \frac{1}{2i}(a - a^*)$.

Exercise 5. Let $\varphi : A \to B$ be a morphism of unital C*-algebras.

(i) Show that $\mathrm{sp}(\varphi(a)) \subseteq \mathrm{sp}(a)$ for all $a \in A$, and that there is equality if φ is injective.

(ii) Show that $\|\varphi(a)\| \leq \|a\|$, equality if φ is injective.

Let φ be not necessarily injective. If $a - \lambda 1_A$ is invertible, then $\varphi(a - \lambda 1_A) = \varphi(a) - \lambda 1_B$ is invertible (with inverse $\varphi((a - \lambda 1_A)^{-1})$). This shows $\mathbb{C} \setminus \mathrm{sp}(a) \subseteq \mathbb{C} \setminus \mathrm{sp}(\varphi(a))$. Thus we also have $r(\varphi(a^*a)) \leq r(a^*a)$, which gives $\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2$.

Let φ be injective, and let $a \in A$. With the isomorphisms $C(\mathrm{sp}(a^*a)) \to C^*(a^*a, 1)$ and $C(\mathrm{sp}(\varphi(a^*a))) \to C^*(\varphi(a^*a), 1)$, φ gives rise under to an injective C^*-homomorphism $\varphi_a : C(\mathrm{sp}(a^*a)) \to C(\mathrm{sp}(\varphi(a^*a)))$. One shows as in [d-j77, Proof of 1.8.1] that φ_a corresponds to a surjective continuous map $\psi_a : \mathrm{sp}(\varphi(a^*a)) \to \mathrm{sp}(a^*a)$. Now, the pull-back of any surjective continuous map is isometric: If $\psi : Y \to X$ is a surjective map of sets, and if $f : X \to \mathbb{C}$ is a function such that $\sup_{x \in X} |f(x)|$ exists, then $\sup_{x \in X} |f(x)| = \sup_{y \in Y} |f(\psi(y))|$. In our situation, each φ_a is isometric, which, using the above isomorphisms, just amounts to saying that φ is isometric. This proves both desired equalities.

Exercise 6. If A is a C*-algebra, and X is a locally compact Hausdorff space, then let $C_0(X, A)$ denote the set of all continuous maps $f : X \to A$ such that $\|f\| := \sup_{x \in X} \|f(x)\|$ exists and f vanishes at infinity, i.e., $\forall \epsilon > 0\ \exists$ compact $K \subseteq X : \|f(x)\| < \epsilon$ for $x \in X \setminus K$. On $C(X, A)$, introduce operations of a $*$-algebra pointwise. Show that $C_0(X, A)$ is a C*-algebra.

The algebraic properties, the triangle inequalities and the C^* property are easy to verify. The proof of completeness (convergence of Cauchy sequences) is standard (e.g., [d-j73, 7.1.3] or [rs72, Theorem I.23]). The idea is to show that the limit given by pointwise Cauchy sequences is indeed an element of $C_0(X, A)$. The only thing not proven in the above references is vanishing at infinity of the limit. This can be concluded from the following statement: Let $f \in C(X, A)$, $g \in C_0(X, A)$, $\|f - g\| < \epsilon/2$. Then there is a compact $K \subseteq X$ such that $\|f(x)\| < \epsilon$ for $x \in X \setminus K$. Indeed, since $g \in C_0(X, A)$, there is a compact $K \subseteq X$ such that $\|g(x)\| < \epsilon/2$ for $x \in X \setminus K$. Then $\|f(x)\| \leq \|f(x) - g(x)\| + \|g(x)\| < \epsilon/2 + \epsilon/2 = \epsilon$ for $x \in X \setminus K$.

Exercise 7. Let A be a unital C*-algebra, $x \in M_2(A)$. Show that x commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $x = \mathrm{diag}(a, b)$ for some $a, b \in A$. Then a, b are unitary if and only if x is unitary.

Exercise 8. Prove the inequalities (1.19).

Let $a^{(ij)}$ be the element of $M_n(A)$ which has a_{ij} at the intersection of the i-th row with the j-th column and zero at all other places. Let us first show $\|a^{(ij)}\| = \|a_{ij}\|$. In the identification $M_n(A) = A \otimes M_n(\mathbb{C})$ we have $a^{(ij)} = a_{ij} \otimes e_{ij}$, where $e_{ij} \in M_n(\mathbb{C})$ is the ij-th matrix unit. Thus, for an injective $*$-homomorphism $\varphi : A \to B(\mathcal{H})$, we have $\|a^{(ij)}\| = \|\varphi \otimes \mathrm{id}(a^{(ij)}\| = \|\varphi(a_{ij}) \otimes e_{ij}\| = \|\varphi(a_{ij}\|\|e_{ij}\| = \|a_{ij}\|$. Here, we have made use of the following facts: Every injective $*$-homomorphism of C*-algebras is isometric (Exercice 5 (ii)), the norm of a tensor product of operators is the product of the norms of the factors (see e.g. [m-gj90, p. 187]), and $\|e_{ij}\| = 1$ (easy to verify). This is enough to prove the right inequality: $\|(a_{ij})\| = \|\sum_{i,j} a^{(ij)}\| \leq \sum_{i,j} \|a^{(ij)}\| = \sum_{i,j} \|a_{ij}\|$.

For the left inequality, we have

$$\begin{aligned}
\|(a_{ij})\|^2 &= \sup_{\psi \in \mathcal{H} \otimes \mathbb{C}^n, \|\psi\|=1} \|\sum_{i,j} \varphi(a_{ij}) \otimes e_{ij}(\psi)\|^2 \\
&\geq \sup_{\psi = \psi_1 \otimes \psi_2, \|\psi_1\| = \|\psi_2\| = 1} \|\sum_{i,j} \varphi(a_{ij})(\psi_1) \otimes e_{ij}(\psi_2)\|^2. \qquad (1.22)
\end{aligned}$$

Now, choose $\psi_2 = e_k$, e_k an element of the canonical basis of $\mathbb{C}^n$. Then $e_{ij}(e_k) = \delta_{jk} e_i$, and the above inequality can be continued:

$$\begin{aligned}
&\geq \sup_{\|\psi\|=1} \|\sum_i \varphi(a_{ik})(\psi) \otimes e_i\|^2 \\
&= \sup_{\|\psi\|=1} \sum_i \|\varphi(a_{ik})(\psi)\|^2 \qquad (1.23) \\
&\geq \max_i \|\varphi(a_{ik})\|^2 = \max_i \|a_{ik}\|^2.
\end{aligned}$$

(Note that $\|\sum_i \psi_i \otimes e_i\|^2 = \sum_i \|\psi_i\|^2$.) Since this is true for all k, we have the desired inequality.

Exercise 9. Let A be a unital C*-algebra, and let $a \in M_n(A)$ be upper triangular, i.e.,

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}. \qquad (1.24)$$

Show that a has an inverse in the subalgebra of upper triangular elements of $M_n(A)$ if and only if all diagonal elements a_{kk} are invertible in A.

Let all a_{kk} be invertible in A. Then $a_0 := \mathrm{diag}(a_{11}, \dots, a_{nn})$ is invertible in $M_n(A)$ (with inverse $a_0^{-1} = \mathrm{diag}(a_{11}^{-1}, \dots, a_{nn}^{-1})$), and $a = a_0 + N$ with nilpotent $N \in M_n(A)$. We can write $a = a_0 + N = a_0(1 + a_0^{-1}N)$ where in our concrete case $a_0^{-1}N$ is again nilpotent. Since for nilpotent m we have $(1+m)^{-1} = 1 - m + m^2 - m^3 + \ldots \pm m^k$ for a certain $k \in \mathbb{N}$, a is invertible.

Conversely, assume that there exists an inverse b of a that is upper triangular. Then $ab = 1$ and $ba = 1$ give immediately that $b_{kk} = a_{kk}^{-1}$ for $k = 1, \dots, n$.

Note that there are invertible upper triangular matrices, whose diagonal elements are not invertible, and whose inverse is not upper triangular. Example: Let s be the unilateral shift, satisfying $s^*s = 1$. Neither s nor s^* is invertible. Nevertheless, the matrix $\begin{pmatrix} s & 1 \\ 0 & s^* \end{pmatrix}$ has the inverse $\begin{pmatrix} s^* & -1 \\ 1 - ss^* & s \end{pmatrix}$.

Exercise 10. Let A be a C*-algebra, $a, b \in A$. Show that $\left\| \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| = \max\{\|a\|, \|b\|\}$.

Chapter 2

Projections and Unitaries

2.1 Homotopy for unitaries

Definition 2.1. *Let X be a topological space. Then $x, y \in X$ are homotopic in X, $x \sim_h y$ in X, if there exists a continuous map $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$.*

The relation $\sim_h$ is an equivalence relation on X (exercise). $f : t \mapsto f(t) = f_t$ as above is called continuous path from x to y. In a vector space, any two elements are homotopic: Take the path $t \mapsto (1-t)x + ty$.

Definition 2.2. *Let A be a unital C^*-algebra, and let $\mathcal{U}(A)$ denote the group of unitary elements of A. Then $\mathcal{U}_0(A) := \{u \in \mathcal{U}(A) \mid u \sim_h 1_A \text{ in } \mathcal{U}(A)\}$ (connected component of 1_A in $\mathcal{U}(A)$).*

Remark 2.3. If $u_1, u_2, v_1, v_2 \in \mathcal{U}(A)$ with $u_i \sim_h v_j$, $j = 1, 2$, then $u_1 u_2 \sim_h v_1 v_2$. Indeed, if $t \mapsto w_j(t)$ are continuous paths connecting u_j with v_j, then $t \mapsto w_1(t)w_2(t)$ is a continuous path connecting u_1u_2 with v_1v_2 (everything in $\mathcal{U}(A)$).

Lemma 2.4. *Let A be a unital C^*-algebra.*

(i) If $h \in A$ is self-adjoint, then $\exp(ih) \in \mathcal{U}_0(A)$.

(ii) If $u \in \mathcal{U}(A)$ and $\mathrm{sp}(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(A)$.

(iii) If $u, v \in \mathcal{U}(A)$ and $\|u - v\| < 2$, then $u \sim_h v$.

Proof. (i) By the contiuous functional calculus, if $h = h^*$ and f is a continuous function on $\mathbb{R}$ with values in $\mathbb{T}$, then $f(h)^* = \bar{f}(h) = f^{-1}(h)$, i.e., $f(h)$ is unitary. In particular, $\exp(ih)$ is unitary. Now for $t \in [0,1]$ define $f_t : \mathrm{sp}(h) \to \mathbb{T}$ by $f_t(x) := \exp(itx)$. Then, by continuity of $t \mapsto f_t$, the path $t \mapsto f_t(h)$ in $\mathcal{U}(A)$ is continuous, thus $\exp(ih) = f_1(h) \sim_h f_0(h) = 1$.
(ii) If $\mathrm{sp}(u) \neq \mathbb{T}$, there exists $\theta \in \mathbb{R}$ such that $\exp(i\theta) \notin \mathrm{sp}(u)$. Note that $\varphi(\exp(it)) = t$ defines a continuous function φ on $\mathrm{sp}(u)$ with values in the open interval $]\theta, \theta + 2\pi[\subseteq \mathbb{R}$. We have $z = \exp(i\varphi(z))$ for $z \in \mathrm{sp}(u)$. Then $h = \varphi(u)$ is a self-adjoint element of A with $u = \exp(ih)$, and by (i) $u \in \mathcal{U}_0(A)$.
(iii) From $\|u - v\| < 2$ it follows that $\|v^*u - 1\| = \|v^*(u - v)\| < 2$ (since $\|v^*\| = 1$). Thus $-2 \notin \mathrm{sp}(v^*u - 1)$, i.e., $-1 \notin \mathrm{sp}(v^*u)$. Then, by (ii), $v^*u \sim_h 1$, hence $u \sim_h v$ (remark before the lemma). $\square$

Corollary 2.5. *$\mathcal{U}(M_n(\mathbb{C}) = \mathcal{U}(M_n(\mathbb{C}))$, i.e., the unitary group in $M_n(\mathbb{C})$ is connected.*

Proof. Each unitary in $M_n(\mathbb{C})$ has finite spectrum, therefore the assumption of (ii) of Lemma 2.4 is satisfied. □

Lemma 2.6. *(Whitehead) Let A be a unital C^*-algebra, and $u, v \in \mathcal{U}(A)$. Then*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \quad \text{in } \mathcal{U}(M_2(A)). \tag{2.1}$$

In particular,

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{in } \mathcal{U}(M_2(A)). \tag{2.2}$$

Proof. First note that the spectrum of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $\{1, -1\}$ (direct elementary computation). Thus by Lemma 2.4 (ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now write

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Then, by Remark 2.3,

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.4}$$

analogously

$$\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.5}$$

thus $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}$. In particular, $\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$, thus

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.6}$$

□

Proposition 2.7. *Let A be a unital C^*-algebra.*

(i) $\mathcal{U}_0(A)$ is a normal subgroup of $\mathcal{U}(A)$.

(ii) $\mathcal{U}_0(A)$ is open and closed relative to $\mathcal{U}(A)$.

(iii) $u \in \mathcal{U}_0(A)$ if and only if there are finitely many self-adjoint $h_1, \ldots, h_n \in A$ such that

$$u = \exp(ih_1) \cdots \exp(ih_n). \tag{2.7}$$

Proof. (i): First note that $\mathcal{U}_0(A)$ is closed under multiplication by Remark 2.3. In order to show that with $u \in \mathcal{U}_0(A)$ also $u^{-1} \in \mathcal{U}_0(A)$ and $vuv^* \in \mathcal{U}_0(A)$ (for any $v \in \mathcal{U}(A)$), let $t \mapsto w_t$ be a continuous path from 1 to u in $\mathcal{U}(A)$. Then $t \mapsto w_t^{-1}$ and $t \mapsto vw_tv^*$ are continuous paths from 1 to u^{-1} and vuv^* in $\mathcal{U}(A)$.

(ii) and (iii): Let $G := \{\exp(ih_1)\cdots\exp(ih_n) \mid n \in \mathbb{N},\ h_k = h_k^* \in A\}$. By (i) and Lemma 2.4, (i), $G \subseteq \mathcal{U}_0(A)$. Since $\exp(ih)^{-1} = \exp(-ih)$, for $h = h^*$, G is a subgroup of $\mathcal{U}_0(A)$.

G is open relative to $\mathcal{U}(A)$: If $v \in G$ and $u \in \mathcal{U}(A)$ with $\|u - v\| < 2$, then $\|1 - uv^*\| = \|(u - v)\| < 2$, and by Lemma 2.4 (iii) and its proof, $\mathrm{sp}(uv^*) \neq \mathbb{T}$, and, by the proof of Lemma 2.4 (ii), there exists $h = h^* \in A$ such that $uv^* = \exp(ih)$. Thus $u = \exp(ih)v \in G$.

G is closed relative to $\mathcal{U}(A)$: $\mathcal{U}(A) \setminus G$ is a disjoint union of cosets Gu, with $u \in \mathcal{U}(A)$. Each Gu is homeomorphic to G, therefore Gu is open relative to $\mathcal{U}(A)$. Thus G is closed in $\mathcal{U}(A)$.

By the above, G is a nonempty subset of $\mathcal{U}_0(A)$, it is open and closed in $\mathcal{U}(A)$, consequently also in $\mathcal{U}_0(A)$. The latter is connected, hence $G = \mathcal{U}_0(A)$. This proves (ii) and (iii). $\square$

Lemma 2.8. *Let A and B be unital C^*-algebras, and let $\varphi : A \to B$ be a surjective (thus unital) $*$-homorphism.*

(i) $\varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$.

(ii) $\forall u \in \mathcal{U}(B) \exists v \in \mathcal{U}_0(M_2(A))$:

$$\varphi_2(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

with $\varphi_2 : M_2(A) \to M_2(B)$ *the extension of* φ.

(iii) If $u \in \mathcal{U}(B)$ *and there is* $v \in \mathcal{U}(A)$ *with* $u \sim_h \varphi(v)$, *then* $u \in \varphi(\mathcal{U}(A))$.

Proof. Any unital $*$-homomorphism is continuous and maps unitaries into unitaries, hence $\varphi(\mathcal{U}_0(A)) \subseteq \mathcal{U}_0(B)$. Conversely, if $u \in \mathcal{U}_0(B)$, then by Proposition 2.7 (iii) there are self-adjoint $h_j \in B$ such that

$$u = \exp(ih_1)\cdots\exp(ih_n).$$

By surjectivity of φ, there are $a_j \in A$ with $\varphi(a_j) = h_j$. Then $k_j := \frac{a_j + a_j^*}{2}$ are self-adjoint and satisfy $\varphi(k_j) = h_j$. Put

$$v = \exp(ik_1)\cdots\exp(ik_n).$$

Then $\varphi(v) = u$ and $v \in \mathcal{U}_0(A)$ by Proposition 2.7 (iii). This proves (i).

(ii): By Lemma 2.4 we have $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \in \mathcal{U}_0(M_2(A))$. On the other hand, $\varphi_2 : M_2(A) \to M_2(B)$ is a surjective $*$-homomorphism, so (i) proves the desired claim.

(iii): If $u \sim_h \varphi(v)$, then $u\varphi(v^*) \in \mathcal{U}_0(B)$, and, by (i), $u\varphi(v^*) = \varphi(w)$ with $w \in \mathcal{U}_0(A)$. Hence $u = \varphi(wv)$, with $wv \in \mathcal{U}(A)$. $\square$

Definition 2.9. *Let A be a unital C^*-algebra. The group of invertible elements in A is denoted by $GL(A)$. $GL_0(A) := \{a \in GL(A) \mid a \sim_h 1$ in $GL(A)\}$.*

$\mathcal{U}(A)$ is a subgroup of $GL(A)$.

If $a \in A$, then there is a well-defined element $|a| = (a^*a)^{\frac{1}{2}}$, by the continuous functional calculus. $|a|$ is called absolute value of a.

Proposition 2.10. *Let A be a unital C*-algebra.*

(i) If $a \in GL(A)$, then also $|a| \in GL(A)$, and $a|a|^{-1} \in \mathcal{U}(A)$.

(ii) Let $\omega : GL(A) \to \mathcal{U}(A)$ be defined by $\omega(a) = a|a|^{-1}$. Then ω is continuous, $\omega(u) = u$ for $u \in \mathcal{U}(A)$, and $\omega(a) \sim_h a$ in $GL(A)$ for every $a \in GL(A)$.

(iii) If $u, v \in \mathcal{U}(A)$ and if $u \sim_h v$ in $GL(A)$, then $u \sim_h v$ in $\mathcal{U}(A)$.

Proof. (i): If $a \in GL(A)$ then also $a^*, a^*a \in GL(A)$. Hence also $|a| = (a^*a)^{\frac{1}{2}} \in GL(A)$, with $|a|^{-1} = ((a^*a)^{-1})^{\frac{1}{2}}$. Then $a|a|^{-1}$ is invertible and unitary: $|a|^{-1}$ is self-adjoint and $|a|^{-1}a^*a|a|^{-1} = |a|^{-1}|a|^2|a|^{-1} = 1$.

(ii): Multiplication in a C*-algebra is continuous, as well as the map $a \mapsto a^{-1}$ in $GL(A)$. (see [m-gj90, Theorem 1.2.3]) Therefore to show continuity of ω, it is sufficient to show that $a \mapsto |a|$ is continuous. The latter is the composition of $a \mapsto a^*a$ and $h \mapsto h^{\frac{1}{2}}$ (for $h \in A^+$). The first of these maps is continuous by continuity of $*$ and the multiplication. Now it is sufficient to show the continuity of the square root on any bounded $\Omega \subseteq A^+$.This follows from Lemma 1.9, because each such Ω is contained in Ω_K, with $K = [0, R]$, $R = \sup_{h\in\Omega} \|h\|$.

For $u \in \mathcal{U}(A)$ we have $|u| = 1$, hence $\omega(u) = u$.

For $a \in GL(A)$, put $a_t := \omega(a)(t|a|+(1-t)1_A)$, $t \in [0, 1]$. This is a continuous path from $\omega(a) = a_0$ to $a = a_1$. It remains to show that $a_t \in GL(A)$, $t \in [0, 1]$. Since $|a|$ is positive and invertible, there is $\lambda \in]0, 1]$ with $|a| \geq \lambda 1_A$. Then, for each $t \in [0, 1]$, $t|a| + (1-t)1_A \geq \lambda 1_A$. (Properties of positive operators, use the isomorphism $C(\mathrm{sp}(a^*a)) \to C^*(a^*a, 1)$.) Hence $t|a|+(1-t)1_A$ and consequently a_t are invertible.

(iii) If $t \mapsto a_t$ is a continuous path in $GL(A)$ from u to v (unitaries), then $t \mapsto \omega(a_t)$ is such a path in $\mathcal{U}(A)$. $\square$

Remark 2.11. (ii) of the above proposition says that $\mathcal{U}(A)$ is a retract of $GL(A)$. $\omega : GL(A) \to \mathcal{U}(A)$ is the corresponding retraction. (A subspace X of a topological space Y is called retract of Y if there is a continuous $r : Y \to X$ with $x \sim_h r(x)$ in Y $\forall x \in Y$ and $r(x) = x$ $\forall x \in X$.)

Remark 2.12. (ii) also says that $a = \omega(a)|a|$, with unitary $\omega(a)$, for invertible a. This is called the (unitary) polar decomposition of a. For any $a \in A$, there is a polar decomposition $a = v|a|$, with a unique partial isometry v.

Proposition 2.13. *Let A be a unital C*-algebra. Let $a \in GL(A)$, and let $b \in A$ with $\|a - b\| < \|a^{-1}\|^{-1}$. Then $b \in GL(A)$,*

$$\|b^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a - b\|,$$

and $a \sim_h b$ in $GL(A)$.

Proof. We have

$$\|1 - a^{-1}b\| = \|a^{-1}(a - b)\| \leq \|a^{-1}\|\|a - b\| < 1,$$

thus $(a^{-1}b)^{-1} = \sum_{k=0}^{\infty}(1 - a^{-1}b)^k$ is absolutely convergent with norm

$$\|(a^{-1}b)^{-1}\| \leq \sum_{k=0}^{\infty} \|1 - a^{-1}b\|^k = (1 - \|1 - a^{-1}b\|)^{-1}.$$

Thus $b \in GL(A)$ with inverse $b^{-1} = (a^{-1}b)^{-1}a^{-1}$, and

$$\|b^{-1}\|^{-1} \geq \|(a^{-1}b)^{-1}\|^{-1}\|a^{-1}\|^{-1} \geq (1 - \|1 - a^{-1}b\|)\|a^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a-b\|.$$

For the last claim, put $c_t = (1-t)a + tb$ for $t \in [0,1]$. Then $\|a - c_t\| = t\|a-b\| < \|a^{-1}\|^{-1}$, therefeore $c_t \in GL(A)$ by the first part of the proof. □

2.2 Equivalence of projections

Definition 2.14. *The set of projections in a C*-algebra A is denoted by $\mathcal{P}(A)$. A* partial isometry *is a $v \in A$ such that $v^*v \in \mathcal{P}(A)$. If v is a partial isometry, then vv^* is also a projection (exercise). v^*v is called the* support projection*, vv^* the* range projection *of v.*

If v is a partial isometry, put $p = v^*v$ and $q = vv^*$. then

$$v = qv = vp = qvp. \tag{2.8}$$

(exercise).

Lemma 2.15. *The following are equivalence relations on $\mathcal{P}(A)$:*

- *$p \sim q$ if and only if there exists $v \in A$ with $p = v^*v$ and $q = vv^*$ (*Murray-von Neumann equivalence*),*
- *$p \sim_u q$ if and only if there exists $u \in \mathcal{U}(A)$ with $q = upu^*$ (*unitary equivalence*).*

Proof. Transitivity of Murray-von Neumann: Let $p \sim q$ and $q \sim r$, and let v, w be partial isometries such that $p = v^*v$, $q = vv^* = w^*w$, $r = ww^*$. Put $z = wv$. Then $z^*z = v^*w^*wv = v^*qv = v^*v = p$, $zz^* = wvv^*w^* = wqw^* = ww^* = r$, i.e., $p \sim r$. The other claims are checked easily. □

Proposition 2.16. *Let $p, q\mathcal{P}(A)$, A unital. The following are equivalent:*

(i) $\exists u \in \mathcal{U}(\tilde{A}) : q = upu^*$,

(ii) $\exists u \in \mathcal{U}(A) : q = upu^*$,

(iii) $p \sim q$ *and* $1_A - p \sim 1_A - q$.

Proof. Let $f = 1_{\tilde{A}} - 1_A = (-1_A, 1)$. Then $\tilde{A} = A + \mathbb{C}f$ and $fa = af = 0\ \forall a \in A$.

(i) $\Longrightarrow$ (ii): Let $q = zpz^*$ for some $z \in \mathcal{U}(\tilde{A})$. Then $z = u + \alpha f$ for some $u \in A$ and $\alpha \in \mathbb{C}$. It is straightforward to show $u \in \mathcal{U}(A)$ and $q = upu^*$.

(ii) $\Longrightarrow$ (iii): Let $q = upu^*$ for $u \in \mathcal{U}(A)$. Put $v = up$ and $w = u(1_A - p)$. Then

$$v^*v = p, \quad vv^* = q, \quad w^*w = 1_A - p, \quad ww^* = 1_A - q. \tag{2.9}$$

(iii) $\Longrightarrow$ (i): Assume that there are partial isometries v, w satisfying (2.9). Then (2.8) gives by direct calculation $z := v + w + f \in \mathcal{U}(\tilde{A})$, and that $zpz^* = vpv^* = vv^* = q$. □

Note that one could prove (iii) $\Longrightarrow$ (ii) using the unitary $u = v + w \in \mathcal{U}(A)$.

Lemma 2.17. *Let A be a C^*-algebra, $p \in \mathcal{P}(A)$, and $a = a^* \in A$. Put $\delta = \|p - a\|$. Then*

$$\mathrm{sp}(a) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta]. \tag{2.10}$$

Proof. We know $\mathrm{sp}(a) \in \mathbb{R}$ and $\mathrm{sp}(p) \in \{0,1\}$. It suffices to show that for $t \in \mathbb{R}$ the assumption $\mathrm{dist}(t, \{0,1\}) > \delta$ implies $t \notin \mathrm{sp}(a)$. Such a t is not in $\mathrm{sp}(p)$, i.e., $p - t1$ is invertible in $\tilde{A}$, and

$$\|(p - t1)^{-1}\| = \max(|-t|^{-1}, |1-t|^{-1}) = d^{-1}. \tag{2.11}$$

(consider $p - t1$ as an element of $C(\mathrm{sp}(p)) \subseteq \mathbb{C}^2$.) Consequently,

$$\|(p-t1)^{-1}(a - t1) - 1\| = \|(p-t1)^{-1}(a-p)\| \le d^{-1}\delta < 1. \tag{2.12}$$

Thus $(p - t1)^{-1}(a - t1)$ is invertible, hence also $a - t1$ is invertible, i.e., $t \notin \mathrm{sp}(a)$. □

Proposition 2.18. *If $p, q \in \mathcal{P}(A)$, $\|p - q\| < 1$, then $p \sim_h q$.*

Proof. Put $a_t = (1-t)p + tq$, $t \in [0,1]$. Then $a_t = a_t^*$, $t \mapsto a_t$ is continuous, and

$$\min(\|a_t - p\|, \|a_t - q\|) \le \|p - q\|/2 < 1/2. \tag{2.13}$$

Thus by Lemma 2.17 $\mathrm{sp}(a_t) \subseteq K := [-\delta, \delta] \cup [1-\delta, 1+\delta]$, with $\delta = \|p-q\|/2 < 1/2$, i.e., $a_t \in \Omega_K$ in the notation of Lemma 1.9. Then $f : K \to \mathbb{C}$, defined to be zero on $[-\delta, \delta]$ and one on $[1-\delta, 1+\delta]$, is continuous, and $f(a_t)$ is a projection for each $t \in [0,1]$ because $f = f^2 = \bar{f}$. By Lemma 1.9, $t \mapsto f(a_t)$ is continuous, and $p = f(p) = f(a_0) \sim_h f(a_1) = f(q) = q$. □

Proposition 2.19. *Let A be a unital C^*-algebra, $a, b \in A$ selfadjoint. Suppose $b = zaz^{-1}$ for some invertible $z \in A$. Then $b = uau^*$, where $u \in \mathcal{U}(A)$ is the unitary in the polar decomposition $z = u|z|$ of z (see Remark 2.12).*

Proof. $b = zaz^{-1}$ is the same as $bz = za$, and also $z^*b = az^*$. Hence

$$|z|^2 a = z^*za = z^*bz = az^*z = a|z|^2, \tag{2.14}$$

a commutes with $|z|^2$. Thus a commutes with all elements of $C^*(1, |z|^2)$, in particular with $|z|^{-1}$. Therefore,

$$uau^* = z|z|^{-1}au^* = za|z|^{-1}u^* = bz|z|^{-1}u^* = buu^* = b. \tag{2.15}$$

□

Proposition 2.20. *Let A be a C^*-algebra, $p, q \in \mathcal{P}(A)$. Then $p \sim_h q$ in $\mathcal{P}(A)$ if and only if $\exists u \in \mathcal{U}_0(\tilde{A}) : q = upu^*$.*

Proof. Assume $q = upu^*$ for some $u \in \mathcal{U}_0(\tilde{A})$, and let $t \mapsto u_t$ be a continuous path in $\mathcal{U}_0(\tilde{A})$ connecting $1 (= 1_{\tilde{A}})$ and u. Then $t \mapsto u_t p u_t^*$ is a continuous path of projections in A (A is an ideal in $\tilde{A}$).

Conversely, if $p \sim_h q$, then there are $p = p_0, p_1, \ldots, p_n \in \mathcal{P}(A)$ such that $\|p_j - p_{j+1}\| < 1/2$ (the set $\{p_t \mid t \in [0,1]\}$ is compact in the metric space $\mathcal{P}(A)$ and thus totally bounded, cf. [d-j73, 3.16]). Thus it is sufficient to consider only the case $\|p - q\| < 1/2$. The element $z := pq + (1-p)(1-q) \in \tilde{A}$ satisfies

$$pz = pq = zq \tag{2.16}$$

and $\|z-1\| = \|p(q-p)+(1-p)((1-q)-(1-p))\| \leq 2\|p-q\| < 1$ (consider $2p-1$ under the isomorphism $C(\mathrm{sp}(p)) \to C^*(p,1)$), thus z is invertible and $z \sim_h 1$ by Proposition 5.2. If $z = u|z|$ is the unitary polar decomposition of z (Remark 2.12), then from formula (2.16) and Proposition 2.19 $p = uqu^*$. Eventually, it follows from Proposition 2.10 (ii) that $u \sim_h z \sim_h 1$ in $GL(\tilde{A})$, and from Proposition 2.10 (iii) that $u \in \mathcal{U}_0(\tilde{A})$. □

Proposition 2.21. *Let A be a C^*-algebra, $p, q \in \mathcal{P}(A)$.*

(i) $p \sim_h q \implies p \sim_u q$.

(ii) $p \sim_u q \implies p \sim q$.

Proof. (i): Immediate from Proposition 2.20.
(ii): If $upu^* = q$ for $u \in \mathcal{U}(\tilde{A})$, then $v = up \in A$, $v^*v = p$, and $vv^* = q$. □

Proposition 2.22. *Let A be a C^*-algebra, $p, q \in \mathcal{P}(A)$.*

(i) $p \sim q \implies \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ *in* $M_2(A)$.

(ii) $p \sim_u q \implies \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ *in* $M_2(A)$.

Proof. Let $v \in A$ such that $p = v^*v$, $q = vv^*$. Then (2.8) can be used to show that

$$u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}, \quad w = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \in \mathcal{U}(M_2(\tilde{A})). \tag{2.17}$$

Since

$$wu \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^*w^* = w \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} w^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.18}$$

on the other hand

$$wu = \begin{pmatrix} v + (1-q)(1-p) & (1-q)v^* \\ q(1-p) & 1-q+qv^* \end{pmatrix} \in \widetilde{M_2(A)}, \tag{2.19}$$

(i) is proved. Note that $\widetilde{M_2(A)}$ is considered as a unital subalgebra of $M_2(\tilde{A})$ via the map $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \alpha\right) \mapsto \begin{pmatrix} (a,\alpha) & (b,0) \\ (c,0) & (d,\alpha) \end{pmatrix}$, and that one has to check that wu, being a priori in $M_2(\tilde{A})$, is indeed in $\widetilde{M_2(A)}$.

(ii): The assumption means $q = upu^*$ for some $u \in \mathcal{U}(\tilde{A})$. By Lemma 2.6 there is a homotopy $t \mapsto w_t$ in $\mathcal{U}(M_2(\tilde{A}))$ connecting $w_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with $w_0 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. Put $e_t = w_t \operatorname{diag}(p,0) w_t^*$. Then $e_t \in \mathcal{P}(M_2(A))$ (A is an ideal in $\tilde{A}$), $t \mapsto e_t$ is continuous, $e_0 = \operatorname{diag}(p,0)$, and $e_1 = \operatorname{diag}(q,0)$. □

Remark 2.23. Propositions 2.21 and 2.22 say that the three equivalence relations $\sim$, $\sim_u$ and $\sim_h$ are equivalent if one passes to matrix algebras. Otherwise, the implications $p \sim q \implies p \sim_u q$ and $p \sim_u q \implies p \sim_h q$ do not hold:

Let A be a unital C*-algebra containing a non-unitary isometry s, i.e., $s^*s = 1 \neq ss^*$. Example: one-sided shift. Then s^*s and ss^* are projections, and by definition $s^*s \sim ss^*$. On the other hand, $1-s^*s = 0 \nsim 1-ss^* \neq 0$, because from $v^*v = 0$ follows $v = 0$ (C^*-property and thus also $vv^* = 0$. By Proposition 2.16 (iii), s^*s and ss^* cannot be unitarily equivalent.

Example of a unital C*-algebra containing projections p, q with $p \sim_u q$ and $p \nsim_h q$: There exists a unital C*-algebra B such that $M_2(B)$ contains $u \in \mathcal{U}(M_2(B))$ not being homotopic in $\mathcal{U}(M_2(B))$ to any $\mathrm{diag}(v, 1)$, $v \in \mathcal{U}(B)$. Then $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim_u u \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u^*$ in $M_2(B)$, but $p \nsim_h q$. Indeed, if one assumes $p \sim_h q$, then by Proposition 2.20 there is $w \in \mathcal{U}(M_2(B))$ such that $wqw^* = p$. Hence $(wu)p = p(wu)$, and (see exercise) $wu = \mathrm{diag}(a, b)$, with $a, b \in \mathcal{U}(B)$. From Lemma 2.6 and $w \in \mathcal{U}(M_2(B))$ we obtain $u \sim_h wu = \mathrm{diag}(a, b) \sim_h \mathrm{diag}(ab, 1)$, contradicting the original assumption about u.

2.3 Semigroups of projections

Definition 2.24. *Let A be a C^*-algebra, $n \in \mathbb{N}$. Put $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$ and $\mathcal{P}_\infty(A) = \cup_{n=1}^{\infty} \mathcal{P}_n(A)$ (disjoint union).*

Let $M_{m,n}(A)$ be the set of rectangular $m \times n$-matrices with entries from A. The adjoint of such a matrix is defined combining the matrix adjoint with the adjoint in A.

Definition 2.25. *Let $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$. Then $p \sim_0 q$ iff $\exists v \in M_{m,n}(A)$: $p = v^*v, q = vv^*$.*

$\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(A)$ and reduces for $m = n$ to the Murray-von Neumann equivalence on $\mathcal{P}(M_n(A))$.

Definition 2.26. *Define a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ by*

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}. \tag{2.20}$$

If $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$, then $p \oplus q \in \mathcal{P}_{n+m}(A)$.

Proposition 2.27. *Let A be a C^*-algebra, $p, q, r, p', q' \in \mathcal{P}_\infty(A)$.*

(i) $\forall n \in \mathbb{N} : p \sim_0 p \oplus 0_n$ *(0_n the zero of $M_n(A)$),*

(ii) if $p \sim_0 p'$ and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$,

(iii) $p \oplus q \sim_0 q \oplus p$,

(iv) if $p, q \in \mathcal{P}_n(A)$, $pq = 0$, then $p + q \in \mathcal{P}_n(A)$ and $p + q \sim_0 p \oplus q$,

(v) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.

Proof. (i): Let $m, n \in \mathbb{N}$, $p \in \mathcal{P}_m(A)$. Put $u_1 = \begin{pmatrix} p \\ 0 \end{pmatrix} \in M_{m+n,m}(A)$. Then $p = u_1^*u_1 \sim_0 u_1u_1^* = p \oplus 0_n$.
(ii): If $p \sim_0 p'$ and $q \sim_0 q'$, then $\exists v, w : p = v^*v,\ p' = vv^*,\ q = w^*w,\ q' = ww^*$. Put $u_2 = \mathrm{diag}(v, w)$. Then $p \oplus q = u_2^*u_2 \sim_0 u_2u_2^* = p' \oplus q'$.

(iii): Let $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$, and put $u_3 := \begin{pmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{pmatrix}$, with $0_{k,l}$ the zero of $M_{k,l}(A)$. Then $u_3 \in M_{n+m}(A)$, and $p \oplus q = u_3^* u_3 \sim_0 u_3 u_3^* = q \oplus p$.
(iv): If $pq = 0$ then $p+q$ is a projection (exercise). Put $u_4 = \begin{pmatrix} p \\ q \end{pmatrix} \in M_{2n,n}(A)$. Then $p + q = u_4^* u_4 \sim_0 u_4 u_4^* = p \oplus q$.
(v): trivial. □

Definition 2.28.

$$\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim_0 . \tag{2.21}$$

$[p]_\mathcal{D} \in \mathcal{D}(A)$ denotes the equivalence class of $p \in \mathcal{P}_\infty(A)$.

Lemma 2.29. *The formula*

$$[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D} \tag{2.22}$$

defines a binary operation on $\mathcal{D}(A)$ making it an abelian semigroup.

Proof. This is immediate from Proposition 2.27. □

2.4 Examples and Exercises

Exercise 11. Let $\varphi : A \to B$ be a surjective $*$-homomorphism of C*-algebras. If $\varphi(a) = b$, then a is called lift of b.

(i) Any $b \in B$ has a lift $a \in A$ with $\|b\| = \|a\|$.

(ii) Any selfadjoint b has a selfadjoint lift a with $\|b\| = \|a\|$.

(iii) Any positive b has a positive lift a with $\|b\| = \|a\|$.

(iv) A normal element does not in general have a normal lift.

(v) A projection does not in general lift to a projection.

(vi) A unitary does not in general lift to a unitary.

(ii) For a lift x of b, also $a_0 := \frac{x+x^*}{2} = a_0^*$ is a lift of b. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} -\|b\| & t \leq \|b\|, \\ t & -\|b\| \leq t \leq \|b\|, \\ \|b\| & t \geq \|b\|. \end{cases} \tag{2.23}$$

Put $a = f(a_0)$. Then $a = a^*$, $\mathrm{sp}(a) = \{f(t) \mid t \in \mathrm{sp}(a_0)\} \subseteq [-\|b\|, \|b\|]$ (by definition of f), and $\|a\| = r(a) \leq r(b) = \|b\|$. Also, a is a lift of b, $\varphi(a) = \varphi(f(a_0)) = f(\varphi(a_0)) = f(b) = b$, because $f(t) = t$ for $t \in \mathrm{sp}(b)$. Finally, φ is norm-decreasing (as any $*$-homomorphism), thus also $\|b\| = \|\varphi(a)\| \leq \|a\|$, hence $\|b\| = \|a\|$.

(i) For $b \in B$, $y := \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}$ is a self-adjoint element of $M_2(B)$ with $\|y\| = \|b\|$ ($\|y\|^2 = \|y^* y\| = \|\begin{pmatrix} bb^* & 0 \\ 0 & b^* b \end{pmatrix}\| = \max\{\|bb^*\|, \|b^* b\|\} = \|b\|^2$, using

Exercise 10 of Chapter 1). By (ii), there is a self-adjoint lift $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ of y with $\|x\| = \|y\|$. $a = x_{12}$ is a lift of b, and by (1.19) $\|a\| \leq \|x\| = \|y\| = \|b\|$. But also $\|b\| \leq \|a\|$, thus $\|a\| = \|b\|$.

(iii) For a lift x of b, also $a_0 := (x^*x)^{1/2} \geq 0$ is a lift: $\varphi(a_0) = (\varphi(x^*)\varphi(x))^{1/2} = (b^*b)^{1/2} = b$. Put $a = f(a_0)$, with f from (2.23). Then a is normal, $\varphi(a) = b$ ($\varphi(a) = \varphi(f(a_0)) = f(\varphi(a_0)) = f(b) = b$), $\mathrm{sp}(a) \subseteq [0, \|b\|]$. Thus, $a \geq 0$, $\|a\| = \|b\|$.

(iv) Let s be the unilateral shift. Then $s^*s = 1$, $s^*s - ss^* = \mathrm{pr}_{e_0}$ is compact. Let $\pi : B(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}$ (Calkin algebra). Then $\pi(s)$ is normal ($\pi(\mathrm{pr}_{e_0}) = 0$), however, $\pi(s)$ has no lift to a normal operator: There is no normal operator N such that $s - N$ is compact.

(v) Let $A = C([0,1])$, $B = \mathbb{C} \oplus \mathbb{C}$, $\varphi(f) = (f(0), f(1))$. Then $q = (0,1) \in \mathcal{P}(\mathbb{C} \oplus \mathbb{C})$. However, there are no nontrivial projections in $C([0,1])$ ($\varphi(p) = q$ would mean $p(0) = 1,\ p(1) = 0$).

Exercise 12. Let A be a unital C*-algebra,

$$a = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \in M_n(A).$$

Show: $a \in GL_n(A)$, $a \sim_h 1$ in $GL_n(A)$.

The first claim is immediate from Exercise 9 of Chapter 1. For the second claim, write $a = 1 + a_0$. Then $a_t = 1 + ta_0$ is a curve connecting a and 1 in $GL_n(A)$ (again by Exercise 9 of Chapter 1).

Exercise 13. Let A be a C*-algebra, $p, q \in \mathcal{P}(A)$. Write $p \perp q$ if $pq = 0$. The following are equivalent:

(i) $p \perp q$,

(ii) $p + q \in \mathcal{P}(A)$,

(iii) $p + q \leq 1$.

(i) $\implies$ (ii): $p + q$ is self-adjoint, and $(p+q)^2 = p^2 + pq + qp + q^2 = p + q$.

(ii) $\implies$ (iii): $1 - (p+q) = 1 - (p+q) - (p+q) + (p+q)^2 = (1 - (p+q))^2$.

(iii) $\implies$ (i): Use the general implication $a \leq b \implies (c^*ac \leq c^*bc,\ \forall c \in A)$ to conclude $p + q \leq 1 \implies p(p+q)p \leq p^2 = p \implies p + pqp \leq p \implies pqp \leq 0$. On the other hand, $pqp = pqqp \geq 0$, thus $pqqp = pqp = 0$, which is equivalent to $pq = qp = 0$.

More generally, for $p_1, \dots, p_n \in \mathcal{P}(A)$, the following are equivalent:

(i) $p_i \perp p_j$, for all $i \neq j$,

(ii) $p_1 + \ldots + p_n \in \mathcal{P}(A)$,

(iii) $p_1 + \ldots + p_n \leq 1$.

Chapter 3

The K_0-Group for Unital C*-algebras

3.1 The Grothendieck Construction

Lemma 3.1. *Let $(S, +)$ be an abelian semigroup. Then the binary relation $\sim$ on $S \times S$ defined by*

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists z \in S : x_1 + y_2 + z = x_2 + y_1 + z \tag{3.1}$$

is an equivalence relation.

Proof. The relation $\sim$ is clearly symmetric and reflexive. Transitivity: Let $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, i.e., $x_1 + y_2 + z = x_2 + y_1 + z,\ x_2 + y_3 + w = x_3 + y_2 + w$ for some $z, w \in S$. Then $x_1 + y_3 + (y_2 + z + w) = x_2 + y_1 + z + y_3 + w = x_3 + y_1 + (y_2 + z + w)$, i.e., $(x_1, y_1) \sim (x_3, y_3)$. □

Let $G(S) := (S \times S)/\sim$, and $\langle x, y\rangle$ denote the class of (x, y).

Lemma 3.2. *The operation*

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle \tag{3.2}$$

is well-defined and yields an abelian group $(G(S), +)$. Inverse and zero are given by

$$-\langle x, y\rangle = \langle y, x\rangle, \quad 0 = \langle x, x\rangle. \tag{3.3}$$

Proof. Straightforward. □

The group $(G(S), +)$ is called the *Grothendieck group* of S.

For $y \in S$, there is a map $\gamma : S \to G(S),\ x \mapsto \langle x+y, y\rangle$ (Grothendieck map). It is independent of y and a homomorphism of abelian semigroups (additive).

Definition 3.3. *An abelian semigroup $(S, +)$ is said to have the* cancellation property *if from $x + z = y + z$ follows $x = y$ $(x, y, z \in S)$.*

Proposition 3.4. *Let $(S, +)$ be an abelian semigroup.*

(i) If H is an abelian group, $\varphi : S \to H$ additive, then there is a unique group homorphism $\psi : G(S) \to H$ such that $\varphi = \psi \circ \gamma$ (universal property).

(ii) *If $\varphi : S \to T$ is a homomorphism (additive map) of abelian semigroups, then there is a unique group homomorphism $G(\varphi) : G(S) \to G(T)$ such that $\gamma_T \circ \varphi = G(\varphi) \circ \gamma_S$ (functoriality).*

(iii) *$G(S) = \{\gamma_S(x) - \gamma_S(y) \mid x, y \in S\}$.*

(iv) *For $x, y \in S$, $\gamma_S(x) = \gamma_S(y)$ if and only if $\exists z \in S$ such that $x + z = y + z$.*

(v) *Let $(H, +)$ be an abelian group, $\emptyset \neq S \subseteq H$. If S is closed under addition, then $(S, +)$ is an abelian semigroup with the cancellation property, and $G(S)$ is isomorphic to the subgroup H_0 generated by S, with $H_0 = \{x - y \mid x, y \in S\}$.*

(vi) *The map $\gamma_S : S \to G(S)$ is injective if and only if S has the cancellation property.*

Proof. (iii): For $\langle x, y\rangle \in G(S)$ we have $\langle x, y\rangle = \langle x, y\rangle + \langle x+y, x+y\rangle = \langle x+x+y, y+x+y\rangle = \langle x+y, y\rangle + \langle x, x+y\rangle = \langle x+y, y\rangle - \langle x+y, x\rangle = \gamma_S(x) - \gamma_S(y)$.
(iv): If $x + z = y + z$, then by additivity of γ_S $\gamma_S(x) + \gamma_S(z) = \gamma_S(y) + \gamma_S(z)$, hence, since $G(S)$ is a group, $\gamma_S(x) = \gamma_S(y)$. Conversely, let $\gamma_S(x) = \gamma_S(y)$, in particular $\langle x+y, y\rangle = \langle y+x, x\rangle$, i.e., $\exists w \in S : (x+y)+x+w = (y+x)+y+w$. Thus $x + z = y + z$, with $z = x + y + w$.
(v): immediate from (iv).
(i): If ψ exists, it has to satisfy $\psi(\langle x, y\rangle) = \varphi(x) - \varphi(y)$, in order to have $\psi \circ \gamma_S = \varphi$. Then additivity of ψ follows from additivity of φ, and uniqueness follows from (iii). To see that ψ exists, assume $\langle x_1, y_1\rangle = \langle x_2, y_2\rangle$, i.e., $\exists z \in S : x_1 + y_2 + z = x_2 + y_1 + z$. Then $\varphi(x_1) + \varphi(y_2) + \varphi(z) = \varphi(x_2) + \varphi(y_1) + \varphi(z)$ in H, by addivity of φ. Since H is a group, we have $\varphi(x_1) - \varphi(y_1) = \varphi(x_2) - \varphi(y_2)$, which shows that ψ is well-defined by $\psi(\langle x, y\rangle) = \varphi(x) - \varphi(y)$.
(ii): $\gamma_T \circ \varphi : S \to G(T)$ is an additive map into the group $G(T)$, thus by (i) there is a unique group homomorphism $G(\varphi) : G(S) \to G(T)$ such that $\gamma_T \circ \varphi = G(\varphi) \circ \gamma_S$.
(vi): Any non-empty subset of an abelian group that is closed under addition is an abelian semigroup with cancellation property. The inclusion $\iota : S \to H$ is additive and gives by (i) rise to a group homomorphism $\psi : G(S) \to H$ such that $\psi \circ \gamma_S = \iota$, i.e., $\psi(\gamma_S(x)) = x$ for $x \in S$. By (iii), $\psi(G(S)) = \{x - y \mid x, y \in S\} = H_0$. If $\psi(\gamma_S(x) - \gamma_S(y)) = 0$, then $x = y$ and so $\gamma_S(x) - \gamma_S(y) = 0$, i.e., ψ is injective. $\square$

Examples:

- $(\mathbb{N}, +)$ is an abelian semigroup with cancellation property, whose Grothendieck group is isomorphic to $(\mathbb{Z}, +)$.
- Let $(\mathbb{N} \cup \{\infty\}, +)$ be the abelian semigroup whose addition is defined by the usual addition in $\mathbb{N}$ and by $x + \infty = \infty = \infty + \infty$. Then $(\mathbb{N} \cup \{\infty\}, +)$ does not have the cancellation property, and the corresponding Grothendieck group is $\{0\}$. Indeed, from $x + \infty = \infty + \infty$ it does not follow that $x = \infty$, and $\langle x, y\rangle = \langle x, x\rangle$ for any $x, y \in \mathbb{N} \cup \{\infty\}$, because $x + x + \infty = y + x + \infty = \infty$ $\forall x, y \in \mathbb{N} \cup \{\infty\}$.

3.2 Definition of the K_0-group of a unital C*-algebra

Definition 3.5. *Let A be a unital C*-algebra. The $\mathrm{K}_0(A)$ group is defined as the Grothendieck group of the semigroup $\mathcal{D}(A)$:*

$$\mathrm{K}_0(A) \stackrel{def}{=} G(\mathcal{D}(A)).$$

We also define a map $[\cdot]_0 : \mathcal{P}_\infty(A) \to \mathrm{K}_0(A)$ by $[p]_0 = \gamma_{\mathcal{D}(A)}([p]_{\mathcal{D}})$ for $p \in \mathcal{P}_\infty(A)$.

Remark 3.6. Formally, this definition could be made for non-unital C*-algebras as well, but it would not be appropriate, since the resulting K_0-functor would not be half-exact.

3.2.1 Portrait of K_0 — the unital case

We define a binary relation $\sim_s$ on $\mathcal{P}_\infty(A)$ as follows: $p \sim_s q$ iff there exists an $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. The relation $\sim_s$ is called *stable equivalence* and it is easy to verify that it is indeed an equivalence relation. Furthermore, the relation can be defined equivalently as $p \sim_s q$ if and only if $p \oplus 1_n \sim_0 q \oplus 1_n$ for some positive integer n. Indeed, if $p \oplus r \sim_0 q \oplus r$ for $r \in \mathcal{P}_n(A)$, then $p \oplus 1_n \sim_0 p \oplus r \oplus (1_n - r) \sim_0 q \oplus r \oplus (1_n - r) \sim_0 q \oplus 1_n$.

Proposition 3.7. *Let A be a unital C*-algebra. Then*

(i) $\mathrm{K}_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A),\ n \in \mathbb{N}\}$,

(ii) $[p]_0 + [q]_0 = [p \oplus q]_0$ *for* $p, q \in \mathcal{P}_\infty(A)$, *and if p and q are orthogonal then* $[p]_0 + [q]_0 = [p + q]_0$,

(iii) $[0_A]_0 = 0$,

(iv) if $p, q \in \mathcal{P}_n(A)$ *and* $p \sim_h q$ *in* $\mathcal{P}_n(A)$ *then* $[p]_0 = [q]_0$,

(v) $[p]_0 = [q]_0$ *if and only if* $p \sim_s q$ *for* $p, q \in \mathcal{P}_\infty(A)$.

Proof. Straightforward. As an example, we only verify (v). If $[p]_0 = [q]_0$ then by part (iv) of Proposition [3.4] there is an $r \in \mathcal{P}_\infty(A)$ such that $[p]_{\mathcal{D}} + [r]_{\mathcal{D}} = [q]_{\mathcal{D}} + [r]_{\mathcal{D}}$. Hence $[p \oplus r]_{\mathcal{D}} = [q \oplus r]_{\mathcal{D}}$. Thus $p \oplus r \sim_0 q \oplus r$ and consequently $p \sim_s r$. Conversely, if $p \sim_s q$ then there is $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. Then $[p]_0 + [r]_0 = [q]_0 + [r]_0$ by part (ii) above and hence $[p]_0 = [q]_0$ since $\mathrm{K}_0(A)$ is a group. □

3.2.2 The universal property of K_0

Proposition 3.8. *Let A be a unital C*-algebra, let G be an abelian group, and let $\nu : \mathcal{P}_\infty(A) \to G$ be a map satisfying the following conditions:*

(i) $\nu(p \oplus q) = \nu(p) + \nu(q)$,

(ii) $\nu(0_A) = 0$,

(iii) if $p \sim_h q$ *in* $\mathcal{P}_n(A)$ *then* $\nu(p) = \nu(q)$.

Then there exists a unique homomorphism $\mathrm{K}_0(A) \to G$ *such that the diagram*

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[d, "{[\cdot]_0}"'] \arrow[dr] & \\ \mathrm{K}_0(A) \arrow[r] & G \end{tikzcd}$$

is commutative.

Proof. At first we observe that if $p, q \in \mathcal{P}_\infty(A)$ and $p \sim_0 q$ then $\nu(p) = \nu(q)$. Indeed, let $p \in \mathcal{P}_k(A)$, $q \in \mathcal{P}_l(A)$. Take $n \geq \max\{k, l\}$ and put $p' = p \oplus 0_{n-k}$ and $q' = q \oplus 0_{n-l}$. We have $p' \sim_0 p \sim_0 q \sim_0 q'$ and hence $p' \sim q'$. Thus $p' \oplus 0_{3n} \sim_h q' \oplus 0_{3n}$ in $\mathcal{P}_{4n}(A)$ by Proposition 2.2.9. Hence

$$\nu(p) = \nu(p) + (4n - k)\nu(0) = \nu(p' \oplus 0_{3n}) = \nu(q' \oplus 0_{3n}) = \nu(q),$$

as required. Consequently, the map $\mathcal{D}(A) \to G$, $[p]_\mathcal{D} \mapsto \nu(p)$ is well-defined. Clearly, this map is additive. The rest follows from the univesal property of the Grothendieck construction (part (i) of Proposition 3.1.4). $\square$

3.2.3 Functoriality

Now we observe that K_0 is a covariant functor from the category of unital C*-algebras with (not necessarily unital) $*$-homomorphisms to the category of abelian groups.

Let $\varphi : A \to B$ be a (not necessarily unital) $*$-homomorphism between unital C*-algebras. For each n it extends to a $*$-homomorphism $\varphi_n : M_n(A) \to M_n(B)$, and this yields a map $\varphi : \mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)$. Define $\nu : \mathcal{P}_\infty(A) \to \mathrm{K}_0(B)$ by $\nu(p) = [\varphi(p)]_0$. Then ν satisfies the conditions of Proposition 3.8. Thus, there is a homomorphsm $\mathrm{K}_0(\varphi) : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ such that $\mathrm{K}_0(\varphi)([p]_0) = [\varphi(p)]_0$. That is, we have a commutative diagram

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[r, "\varphi"] \arrow[d, "{[\cdot]_0}"'] & \mathcal{P}_\infty(B) \arrow[d, "{[\cdot]_0}"'] \\ \mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(\varphi)"] & \mathrm{K}_0(B) \end{tikzcd}$$

Proposition 3.9. *Let* $\varphi : A \to B$, $\psi : B \to C$ *be* $*$*-homomorphisms between unital C*-algebras. Then*

(i) $\mathrm{K}_0(id_A) = id_{\mathrm{K}_0(A)}$,

(ii) $\mathrm{K}_0(\psi \circ \varphi) = \mathrm{K}_0(\psi) \circ \mathrm{K}_0(\varphi)$.

Proof. By definition, (i) and (ii) hold when applied to $[p]_0$, $p \in \mathcal{P}_\infty(A)$. Then use part (i) of Proposition 3.7. $\square$

3.2.4 Homotopy invariance

Let A, B be C*-algebras. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are *homotopic* $\varphi \sim_h \psi$ if there exist $*$-homomorphisms $\varphi_t : A \to B$ for $t \in [0, 1]$ such that

$\varphi_0 = \varphi$, $\varphi_1 = \psi$, and the map $[0,1] \ni t \mapsto \varphi_t(a) \in B$ is norm contnuous for each $a \in A$.

C*-algebras A and B are *homotopy equivalent* if there exist $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \varphi \sim_h \mathrm{id}_A$ and $\varphi \circ \psi \sim_h \mathrm{id}_B$. In such a case we write $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$.

Proposition 3.10. *Let A, B be unital C*-algebras.*

(i) If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms then $\mathrm{K}_0(\varphi) = \mathrm{K}_0(\psi)$.

(ii) If A is homotopy equivalent via $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ then $\mathrm{K}_0(\varphi) : \mathrm{K}_0(A) \to \mathrm{K}_0(\psi)$ is an isomorphism with $\mathrm{K}_0(\varphi)^{-1} = \mathrm{K}_0(\psi)$.

Proof. Part (i) follows from Proposition 3.7. Part (ii) follows from part (i) and functoriality of K_0 (Proposition 3.9). □

3.3 Examples and Exercises

Example 3.11. $\mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$. Indeed, $\mathcal{D}(\mathbb{C}) \cong (\mathbb{Z}_+, +)$ and the Grothendieck group of $\mathbb{Z}_+$ is $\mathbb{Z}$.

Example 3.12. If $\mathcal{H}$ is an infinite dimensional Hilbert space then $\mathrm{K}_0(\mathcal{B}(\mathcal{H})) = 0$. Indeed, if $\mathcal{H}$ is separable then $\mathcal{D}(\mathcal{B}(\mathcal{H})) \cong \mathbb{Z}_+ \cup \{\infty\}$ with the addition in $\mathbb{Z}_+$ extended by $m + \infty = \infty + m = \infty + \infty = \infty$. The Grothendieck group of this semigroup is 0. The non-separable case is handled similarly.

Exercise 14. If X is a *contractible* compact, Hausdorff space then $\mathrm{K}_0(C(X)) \cong \mathbb{Z}$. Hint: recall that X is contractible if there exists a point $x_0 \in X$ and a continuous map $\alpha : [0,1] \times X \to X$ such that $\alpha(0,x) = x$ and $\alpha(1,x) = x_0$ for all $x \in X$, and use Proposition 3.10 and Example 3.3.1.

Example 3.13 (Traces). Let A be a unital C*-algebra. A bounded linear functional $\tau : A \to \mathbb{C}$ is a *trace* if $\tau(ab) = \tau(ba)$ for all $a, b \in A$. Hence $\tau(p) = \tau(q)$ if p, q are Murray-von Neumann equivalent projections. A trace τ is *positive* if $\tau(a) \geq 0$ for all $a \geq 0$. It is a *tracial state* if it is positive of norm 1.

A trace τ extends to a trace τ_n on $M_n(\mathbb{C})$ by $\tau_n([a_{i,j}]) = \sum_{i=1}^n \tau(a_i)$. Thus τ gives rise to a function $\tau : \mathcal{P}_\infty(A) \to \mathbb{C}$. By the universal property of K_0 this yields a group homomorphism $\mathrm{K}_0(\tau) : \mathrm{K}_0(A) \to \mathbb{C}$ such that $\mathrm{K}_0(\tau)([p]_0) = \tau(p)$. If τ is positive then $\mathrm{K}_0(\tau) : \mathrm{K}_0(A) \to \mathbb{R}$ and $\mathrm{K}_0([p]_0) \in \mathbb{R}_+$ for $p \in \mathcal{P}_\infty(A)$.

Exercise 15. If $n \in \mathbb{Z}_+$ then $\mathrm{K}_0(M_n(\mathbb{C})) \cong \mathbb{Z}$, and the class of a minimal projection is a generator. In fact, let Tr be the standard matrix trace. Then $\mathrm{K}_0(\mathrm{Tr}) : \mathrm{K}_0(M_n(\mathbb{C})) \to \mathbb{Z}$ is an isomorphism.

Exercise 16. Let X be a connected, compact Hausdorff space. Show that there exists a surjective homomorphism

$$\dim : \mathrm{K}_0(C(X)) \to \mathbb{Z}$$

such that $\dim([p]_0) = \mathrm{Tr}(p(x))$.

To this end, identify $M_n(C(X))$ with $C(X, M_n(\mathbb{C}))$. For each $x \in X$ the evaluation at x is a positive trace and hence, by Example 3.3.4 gives rise to a homomorphism from $\mathrm{K}_0(C(X))$ to $\mathbb{R}$. If $p \in \mathcal{P}_\infty(C(X))$ then the function $x \mapsto \mathrm{Tr}(p(x)) \in \mathbb{Z}$ is continuous and locally constant, hence constant since X is connected. Finally, the homomorhism is surjective since $\dim([1]_0) = 1$.

Exercise 17. Let X be a compact Hausdorff space.

(1) By generalizing Exercise 3.3.6, show that there exists a surjective group homomorphism

$$\dim : \mathrm{K}_0(C(X)) \to C(X, \mathbb{Z})$$

such that $\dim([p]_0)(x) = \mathrm{Tr}(p(x))$.

(2) Given $p \in \mathcal{P}_n(C(X))$ and $q \in \mathcal{P}_m(C(X))$ show that $\dim([p]_0) = \dim([q]_0)$ if and only if for each $x \in X$ there exists $v_x \in M_{m,n}(C(X))$ such that $v_x^* v_x = p(x)$ and $v_x v_x^* = q(x)$. Note that in general one cannot choose v_x so that the map $x \mapsto v_x$ be continuous.

(3) Show that if X is *totally disconnected* then the map dim is an isomorphism.

Recall that a space is totally disconnected if it has a basis for topology consisting of sets which are simultaneously open and closed. To prove the claim it sufficies (in view of part (1)) to show that dim is injective. To this end use part (ii) and total disconnectedness of X to find a partition of X into open and closed subsets $X = X_1 \cup X_2 \cup \ldots \cup X_k$ and rectangular matrices $v_1, v_2, \ldots, v_k$ over $C(X)$ such that $||v_i^* v_i - p(x)|| < 1$ and $||v_i v_i^* - q(x)|| < 1$ for all $x \in X_i$. From this deduce that $p \sim_0 q$.

Exercise 18. Let A be a unital C*-algebra and let $\tau : A \to \mathbb{C}$ be a bounded linear functional. Show that the following conditions are equivalent:

(i) $\tau(ab) = \tau(ba)$ for all $a, b \in A$,

(ii) $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$,

(iii) $\tau(uyu^*) = \tau(y)$ for all $y \in A$ and all unitary $u \in A$.

(ii)⇒(iii) Suppose (ii) holds. At first consider $a \geq 0$ and set $x = u|a|^{1/2}$. Then $\tau(uau^*) = \tau(xx^*) = \tau(x^*x) = \tau(a)$. Then use the fact that every element of a C*-algebra can be written as a linear combination of four positive elements.

(iii)⇒(i) Suppose (iii) holds. If $b \in A$ and u is a unitary in A then $\tau(ub) = \tau(u(bu)u^*) = \tau(bu)$. Then use the fact that every element of a unital C*-algebra may be written as a linear combination of four untaries.

Example 3.14. Let Γ be a countable discrete group with infinite conjugacy classes (an ICC group). Let $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be its left regular representation. Let $W^*(\Gamma)$ be the closure of the linear span of $\lambda(\Gamma)$ in the strong operator topology (that is, in the topology of pointwise convergence). It can be shown that there exists a unique tracial state τ on $W^*(\Gamma)$, and that this trace has the following properties:

(i) Two projections p, q are Murray-von Neumann equivalent in $W^*(\Gamma)$ iff $\tau(p) = \tau(q)$.

(ii) $\{\tau(p) : p \in \mathcal{P}(W^*(\Gamma))\} = [0, 1]$.

Deduce that $\mathrm{K}_0(W^*(\Gamma) \cong (\mathbb{R}, +)$. The conslusion of this example remains valid if $W^*(\Gamma)$ is replaced by any factor von Neumann algebra of type II_1.

Example 3.15 (Matrix stability of K_0). Let A be a unital C*-algebra and let n be a positive integer. Then

$$\mathrm{K}_0(A) \cong \mathrm{K}_0(M_n(A)).$$

More specifically, the $*$-homomorphism $\varphi : A \to M_n(A)$, $a \mapsto \mathrm{diag}(a, 0_{n-1})$ induces an isomorphism $\mathrm{K}_0(\varphi) : \mathrm{K}_0(A) \to \mathrm{K}_0(M_n(A))$.

Indeed, we construct inverse to $\mathrm{K}_0(\varphi)$, as follows. For each k let $\gamma_k : M_k(M_n(A)) \to M_{kn}(A)$ be the isomorphism which "erases parentheses". Define $\gamma : \mathcal{P}_\infty(M_n(A)) \to \mathrm{K}_0(A)$ by $\gamma(p) = [\gamma_k(p)]_0$ for $p \in \mathcal{P}_k(M_n(A))$. The universal property of K_0 applied to γ yields a homomorphism $\alpha : \mathrm{K}_0(M_n(A)) \to \mathrm{K}_0(A)$ such that $\alpha([p]_0) = [\gamma(p)]_0$.We claim that $\alpha = \mathrm{K}_0(\varphi)^{-1}$. To this end it sufficies to show that

(i) $\varphi_{kn}(\gamma_k(p)) \sim_0 p$ in $\mathcal{P}_\infty(M_n(A))$ for $p \in \mathcal{P}_k(M_n(A))$, and

(i) $\gamma_k(\varphi_k(p)) \sim_0 p$ in $\mathcal{P}(A)$ for $p \in \mathcal{P}_k(A)$.

Proof of (i). exercise.

Proof of (ii). Let $e_1, \ldots, e_{kn}$ be the standard basis in $\mathbb{C}^{kn}$ and let u be a permutation unitary such that $ue_i = e_{n(i-1)+1}$ for $i = 1, 2, \ldots, k$. Then $p \sim_0 p \oplus 0_{(n-1)k} = u^*\gamma_k(\varphi_k(p))u$ for all projections p in $\mathcal{P}_k(A)$.

Exercise 19. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are *orthogonal* if $\varphi(A)\psi(B) = \{0\}$. Show that if φ and ψ are orthogonal then $\varphi + \psi$ is a $*$-homomorphism and $\mathrm{K}_0(\varphi + \psi) = \mathrm{K}_0(\varphi) + \mathrm{K}_0(\psi)$.

Exercise 20 (Cuntz algebras). Let $\mathcal{H}$ be a Hilbert space, let n be a positive integer bigger than 1, and let $S_1, \ldots, S_n$ be isometries on $\mathcal{H}$ whose range projections add up to the identity. Let $C^*(S_1, \ldots, S_n)$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\{S_1, \ldots, S_n\}$. It was proved by Cuntz in [c-j77] that this C*-algebra is independent of the choice of such isometries. That is, if $T_1, \ldots, T_n$ is another family of isometries whose range projections add up to the identity then there is a $*$-isomorphism $\varphi : C^*(S_1, \ldots S_n) \to C^*(T_1, \ldots, T_n)$ such that $\varphi(S_j) = T_j$ for $j = 1, \ldots, n$. Thus defined C*-algebra is denoted $\mathcal{O}_n$ and called *Cuntz algebra*. It is a simple, unital, separable C*-algebra. Alternatively, $\mathcal{O}_n$ may be defined as the universal C*-algebra generated by elements $S_1, \ldots, S_n$ subject to the relations:

(i) $S_i^* S_i = I$ for $i = 1, \ldots, n$,

(ii) $\sum_{i=1}^n S_i S_i^* = I$.

(1) Let u be a unitary in $\mathcal{O}_n$. There exists a unique unit preserving injective $*$-homomorphism $\lambda_u : \mathcal{O}_n \to \mathcal{O}_n$ (i.e. an *endomorphism* of $\mathcal{O}_n$) such that $\lambda_u(S_j) = uS_j$ for $j = 1, \ldots, n$. Moreover, if φ is an endomorphism of $\mathcal{O}_n$ then $\varphi = \lambda_u$ with $u = \sum_{i=1}^n \varphi(S_i)S_i^*$.

(2) Let σ be an endomorphism of $\mathcal{O}_n$ such that $\sigma(x) = \sum_{i=1}^n s_i x s_i^*$ (the *shift* endomorphism). Then $\mathrm{K}_0(\sigma) : \mathrm{K}_0(\mathcal{O}_n) \to \mathrm{K}_0(\mathcal{O}_n)$ is the multiplication by n, that is $\mathrm{K}_0(\sigma)(g) = ng$ for all $g \in \mathrm{K}_0(\mathcal{O}_n)$. Hint: Use Exercise 3.3.11 and the following fact. If v is an isometry in a unital C*-algebra A then the map $\mu : A \to A$, $\mu(x) = vxv^*$ is a $*$-homomorphism and $\mathrm{K}_0(\mu) = \mathrm{id}$. For the latter observe that $\mu_k : M_k(A) \to M_k(A)$ is given by $\mu_k(y) = v_k y v_k^*$, where $v_k = \mathrm{diag}(v, \ldots, v)$.

(3) Let w be a unitary in $\mathcal{O}_n$ such that $\sigma = \lambda_w$. Then $w \sim_h 1$ in $\mathcal{U}(\mathcal{O}_n)$ and hence $\sigma \sim_h \mathrm{id}$. Consequently, $\mathrm{K}_0(\sigma) = \mathrm{id}_{\mathrm{K}_0(\mathcal{O}_n)}$. Hint: Note that w belongs

to $M = \text{span}\{S_i S_j S_k^* S_m^*\}$, and that M is a C^*-subalgebra of $\mathcal{O}_n$ isomorphic to $M_{n^2}(\mathbb{C})$.

(4) Combining (2) and (3) we get $(n-1)\,\mathrm{K}_0(\mathcal{O}_n) = 0$. Thus, in particular, $\mathrm{K}_0(\mathcal{O}_2) = 0$. In fact, Cuntz showed in [c-j81] that $\mathrm{K}_0(\mathcal{O}_n) \cong \mathbb{Z}_{n-1}$ for all $n = 2, 3, \ldots$

Exercise 21 (Properly infinite algebras). Let A be a unital C*-algebra. A is called *properly infinite* if there exist two projections e, f in A such that $ef = 0$ and $1 \sim e \sim f$. For example, Cuntz algebras are properly infinite. For the reminder of this exercise assume A is properly infinite.

(1) A contains isometries S_1, S_2 whose range projections are orthogonal.

(2) A contains an infinite sequence $\{t_j\}$ of isometries with mutually orthogonal ranges. Hint: take $S_2^k S_1$ for $k = 0, 1, 2, \ldots$

(3) For each natural number n let v_n be an element of $M_{1,n}(A)$ with entries $t_1, \ldots, t_n$. Then $v_n^* v_n = 1_n$ and for $p \in \mathcal{P}_n(A)$ we have $p \sim_0 v_n p v_n^*$, with $v_n p v_n^*$ a projection in A.

(4) Let p, q be projections in A. Set

$$r = t_1 p t_1^* + t_2(1-q)t_2^* + t_3(1 - t_1 t_1^* - t_2 t_2^*)t_3^*.$$

Then r is a projection in A and $[r]_0 = [p]_0 - [q]_0$.

Conclude that

$$\mathrm{K}_0(A) = \{[p]_0 : p \in \mathcal{P}(A)\}.$$

Exercise 22. If A is a separable, unital C*-algebra then $\mathrm{K}_0(A)$ is countable.

Exercise 23. Show that condition (iii) of Proposition 3.8 may be replaced by any of the following three conditions:

(i) $\forall n \forall p, q \in \mathcal{P}_n(A)$ if $p \sim_u q$ then $\nu(p) = \nu(q)$,

(ii) $\forall p, q \in \mathcal{P}_\infty(A)$ if $p \sim_0 q$ then $\nu(p) = \nu(q)$,

(iii) $\forall p, q \in \mathcal{P}_\infty(A)$ if $p \sim_s q$ then $\nu(p) = \nu(q)$.

Exercise 24. Let A be a unital C*-algebra and let $a \in A$ be such that $a \geq 0$ and $||a|| \leq 1$.

(1) Show that

$$p = \begin{pmatrix} a & \sqrt{a - a^2} \\ \sqrt{a - a^2} & 1 - a \end{pmatrix}$$

is a projection in $M_2(A)$.

(2) Show that $p \sim \text{diag}(1, 0)$ in $M_2(A)$. Hint: consider

$$v = \begin{pmatrix} \sqrt{a} & \sqrt{1-a} \\ 0 & 0 \end{pmatrix}$$

(3) Let B be another unital C*-algebra and let $\varphi : A \to B$ be a unit preserving, surjective $*$-homomorphism. Let q be a projection in B. Show that there is

$a \geq 0$ in A such that $||a|| \leq 1$ and $\varphi(a) = q$. Then use this a to define p as in (1) above and show that

$$\varphi(p) = \begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix}$$

Exercise 25 (Partial isometries). Show that for an element S of a C*-algebra the following conditions are equivalent:

(i) S^*S is a projection,

(ii) SS^* is a projection,

(iii) $SS^*S = S$.

(i)⇒(iii) Show $(SS^*S - S)^*(SS^*S - S) = 0$.

An element S satisfying these conditions is called *partial isometry.*

Exercise 26. Let A be a unital C*-algebra, a, b two elements of A, and p, q two projections in A. Show the following.

(i) $abb^*a^* \leq ||b||^2 aa^*$.

(ii) $p \leq q$ if and only if $pq = p$.

(i) Since $||b||^2 - bb^* \geq 0$ there is $x \in A$ such that $||b||^2 - bb^* = xx^*$. Thus

$$||b||^2 aa^* - abb^*a^* = a(||b||^2 - bb^*)a^* = axx^*a^* = (ax)(ax)^* \geq 0.$$

(ii) If $p \leq q$ then $pqp - p = p(q-p)p \geq 0$, and hence $pqp \geq p$. But $pqp \leq ||q||p = p$ (by part (i)). Thus $pqp = p$. Hence

$$(pq - p)(pq - p)^* = (pq - p)(qp - p) = pqp - pqp - pqp + p = 0$$

and consequently $pq - p = 0$.

Exercise 27. Let A be a unital C*-algebra. Then the exact sequence

$$0 \longrightarrow A \xrightarrow{\imath} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

is split exact, with a splitting map $\lambda : \mathbb{C} \longrightarrow \tilde{A}$, and induces a split exact sequence

$$0 \longrightarrow \mathrm{K}_0(A) \xrightarrow{\mathrm{K}_0(\imath)} \mathrm{K}_0(\tilde{A}) \xrightarrow{\mathrm{K}_0(\pi)} \mathrm{K}_0(\mathbb{C}) \longrightarrow 0,$$

with a splitting map $\mathrm{K}_0(\lambda) : \mathrm{K}_0(\mathbb{C}) \longrightarrow \mathrm{K}_0(\tilde{A})$.

Hint: Let $f = 1_{\tilde{A}} - 1_A$, a projection such that $\tilde{A} = A \oplus \mathbb{C}f$ (direct sum of C*-algebras). Let μ be the natural surjection from $\tilde{A}$ onto A and let $\lambda' : \mathbb{C} \to \tilde{A}$ be defined by $\lambda'(t) = tf$. Then we have the following identities: $\mathrm{id}_A = \mu \circ \imath$, $\pi \circ \imath = 0$, $\pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$, $\mathrm{id}_{\tilde{A}} = \imath \circ \mu + \lambda' \circ \pi$, and the maps $\imath \circ \mu$ and $\lambda' \circ \pi$ are orthogonal to one another (see Exercise 3.3.11). The claim follows from these identities and functoriality of K_0.

Example 3.16 (Algebraic definition of K_0). Let R be a unital ring. Recall that $e \in R$ is an *idempotent* if $e^2 = e$. We define $\mathcal{I}(R) = \{e \in R : e^2 = 2\}$, $\mathcal{I}_n(R) = \mathcal{I}(M_n(R))$, $\mathcal{I}_\infty(R) = \bigcup_{n=1}^{\infty} \mathcal{I}_n(R)$. We define a relation $\approx_0$ in $\mathcal{I}_\infty(R)$ as follows. If $e \in \mathcal{I}_n(R)$ and $f \in \mathcal{I}_m(R)$ then $e \approx_0 f$ if and only if there exist $a \in M_{n,m}(R)$ and $b \in M_{m,n}(R)$ such that $e = ab$ and $f = ba$. If this is the case then taking $a' = aba$ and $b = bab$ we may assume that a, b satisfy $aba = a$ and $bab = b$ (this we always assume in what follows). Claim: $\approx_0$ is an equivalence relation. For transitivity, let $e \approx_0 f$, $f \approx_0 g$ be idempotents and let c, d, x, y be matrices such that $e = cd$, $f = dc$, $f = xy$, $g = yx$. Then $(cx)(yd) = e$ and $(yd)(cx) = g$, and hence $e \approx_0 g$. Set $V(R) = \mathcal{I}_\infty(R)/_{\approx_0}$, anddenote the class of e by $[e]_V$.

Define a binary operation $\oplus$ on $\mathcal{I}_\infty(R)$ by $e \oplus f = \mathrm{diag}(e, f)$. This operation is well-defined on equivalence classes of $\approx_0$ and turns $V(R)$ into an abelian semigroup. Define$\mathrm{K}_0(R)$ as the Grothendieck group of $(V(R), \oplus)$.

Now suppose A is a unital C*-algebra. We show hat the two definitions of $\mathrm{K}_0(A)$ coincide. In fact, the two semigoups $\mathcal{D}(A)$ and $V(A)$ are isomorphic. The proof proceeds in three steps.

(1) If $e \in \mathcal{I}_\infty(A)$ ten there exists a $p \in \mathcal{P}_\infty(A)$ such that $e \approx_0 p$. Indeed, let $e \in M_n(A)$, and set $h = 1_n + (e - e^*)(e - e^*)^*$. Then h is invertible and satisfies $eh = ee^*e = he$. Then $p = ee^*h^{-1}$ is a projection. Since $ep = p$ and $pe = e$, $e \approx_0 p$.

(2) If $p, q \in \mathcal{P}_\infty(A)$ then $p \sim_0 q$ if and only if $p \approx_0 q$. Indeed, suppose (after diagonalling adding zeros, if necessary) $p, q \in M_n(A)$ and $a, b \in M_n(A)$ are such that $p = ab$, $q = ba$, $a = aba$ (hence $a = paq$), $b = bab$ (hence $b = qbp$). Then $b^*b = (bab)^*b = (ab)^*b^*b = pb^*b$. It follows that b^*b belongs to the corner C*-algebra $pM_n(A)p$. Since $p = (ab)^*ab = b^*(a^*a)b \le ||a||^2 b^*b$, the element b^*b is invertible in $pM_n(A)p$. Set $v = bp(b^*b)^{-1/2}$. We have $p = v^*v$ (straightforward calculation). In particular, v is a partial isometry. In fact, $b = v|b|$ is the polar decomposition of b in $M_n(A)$. Hence $bb^* = vb^*bv^*$.

It now suffices to show that $q = vv^*$. First note that $v = qv$ (directly follows from the definition of v). Thus

$$vv^* = qvv^*q \le ||v||^2 q = q = baa^*b^* \le ||a||^2 bb^* = ||a||^2 vb^*bv^* \le ||a||^2 ||b||^2 vv^*.$$

That is, vv^* and q are projections satisfying $vv^* \le q \le ||a||^2||b||^2 vv^*$. It follows that $vv^* = q$.

(3) The map $\mathcal{D}(A) \to V(A)$ given by $[p]_{\mathcal{D}} \mapsto [p]_V$ is a semigroup isomorphism. (exercise)

Chapter 4

K_0-Group — the General Case

4.1 Definition of the K_0-Functor

Definition 4.1. Let A be a non-unital C*-algebra and let $\tilde{A}$ be its minimal unitization. We have a split-exact sequence

$$0 \longrightarrow A \longrightarrow \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$

Define $\mathrm{K}_0(A) = \ker(\mathrm{K}_0(\pi))$, where $\mathrm{K}_0(\pi) : \mathrm{K}_0(\tilde{A}) \to \mathrm{K}_0(\mathbb{C})$ is the map induced by π.

Thus, by definition, $\mathrm{K}_0(A)$ is a subgroup of $\mathrm{K}_0(\tilde{A})$ and hence an abelian group. If $p \in \mathcal{P}_\infty(A)$ then $[p]_0 \in \mathrm{K}_0(\tilde{A})$. But $[p]_0 \in \ker \mathrm{K}_0(\pi)$ and hence $[p]_0 \in \mathrm{K}_0(A)$. Thus, just as in the unital case, we have a map $[\cdot]_0 : \mathcal{P}_\infty(A) \to \mathrm{K}_0(A)$.

If A is unital then we can still form direct sum (of C*-algebras) $\tilde{A} = A \oplus \mathbb{C}$. Let π be the natural surjection from $\tilde{A}$ onto $\mathbb{C}$. As shown in Exercise 3.3.19, we have $\mathrm{K}_0(A) = \ker(\mathrm{K}_0(\pi))$. Thus, Definition 4.1 works equally well in the case of a unital C*-algebra.

4.1.1 Functoriality of K_0

Let $\varphi : A \to B$ be a $*$-homomorphism. Then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"'] & \tilde{A} \arrow[r] \arrow[d, "\tilde{\varphi}"'] & \mathbb{C} \arrow[d, equal] \\ B \arrow[r] & \tilde{B} \arrow[r] & \mathbb{C} \end{tikzcd}$$

commutes. Functoriality of K_0 for unital C*-algebras yields a commutative diagram

$$\begin{tikzcd} \mathrm{K}_0(A) \arrow[r] \arrow[d, "\mathrm{K}_0(\varphi)"'] & \mathrm{K}_0(\tilde{A}) \arrow[r] \arrow[d, "\mathrm{K}_0(\tilde{\varphi})"'] & \mathrm{K}_0(\mathbb{C}) \arrow[d, equal] \\ \mathrm{K}_0(B) \arrow[r] & \mathrm{K}_0(\tilde{B}) \arrow[r] & \mathrm{K}_0(\mathbb{C}) \end{tikzcd}$$

and there exists exactly one map $\mathrm{K}_0(\varphi) : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ which completes the diagram. Note that we have $\mathrm{K}_0([p]_0) = [\varphi(p)]_0$ for $p \in \mathcal{P}_\infty(A)$.

Proposition 4.2. *Let $\varphi : A \to B$, $\psi : B \to C$ be $*$-homomorphsms between C*-algebras. Then*

(i) $\mathrm{K}_0(\mathrm{id}_A) = \mathrm{id}_{\mathrm{K}_0(A)}$,

(ii) $\mathrm{K}_0(\psi \circ \varphi) = \mathrm{K}_0(\psi) \circ \mathrm{K}_0(\varphi)$.

Proof. Exercise — use functoriality of K_0 for unital C*-algebras. □

Moreover, it is immediate from the definitions that K_0 of the zero algebra is 0 and K_0 of the zero homomorphism is the zero map.

4.1.2 Homotopy invariance of K_0

Proposition 4.3. *Let A, B be C*-algebras.*

(i) If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms then $\mathrm{K}_0(\varphi) = \mathrm{K}_0(\psi)$.

(ii) If $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ is a homotopy then $\mathrm{K}_0(\varphi)$ and $\mathrm{K}_0(\psi)$ are isomorphisms and inverses of one another.

Proof. (i) Since φ and ψ are homotopic so are they unital extensions $\tilde{\varphi}$ and $\tilde{\psi}$ to $\tilde{A}$, whence $\mathrm{K}_0(\tilde{\varphi}) = \mathrm{K}_0(\tilde{\psi})$ by Proposition 3.10. Then $\mathrm{K}_0(\varphi) = \mathrm{K}_0(\psi)$ being the restrictions of these maps to $\mathrm{K}_0(A)$. Part (ii) follows from part (i) and functoriality of K_0. □

4.2 Further Properties

4.2.1 Portrait of K_0

Let A be a C*-algebra and consider the split-exact sequence

$$0 \longrightarrow A \xrightarrow{\imath} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0,$$

with the splitting map $\lambda : \mathbb{C} \to \tilde{A}$. Define the *scalar map* $s = \lambda \circ \pi : \tilde{A} \to \tilde{A}$, so that $s(a + t1) = t1$. Let $s_n : M_n(\tilde{A}) \to M_n(\tilde{A})$ be the natural extensions of s. The image of s_n is isomorphic to $M_n(\mathbb{C})$, and its elements are called *scalar matrices*. The scalar map is natural in the sense that for any $*$-homomorphism $\varphi : A \to B$ the diagram

$$\begin{tikzcd} \tilde{A} \arrow[r, "s"] \arrow[d, "\tilde{\varphi}"'] & \tilde{A} \arrow[d, "\tilde{\varphi}"] \\ \tilde{B} \arrow[r, "s"] & \tilde{B} \end{tikzcd}$$

commutes.

Proposition 4.4. Let A be a C*-algebra.

(i) $\mathrm{K}_0(A) = \{[p]_0 - [s(p)]_0 : p \in \mathcal{P}_\infty(\tilde{A})\}$.

(ii) If $p, q \in \mathcal{P}_\infty(\tilde{A})$ then the following are equivalent:

(a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$,

(b) there are k, l such that $p \oplus 1_k \sim_0 q \oplus 1_l$ in $\mathcal{P}_\infty(\tilde{A})$,

(c) there are scalar projections r_1, r_2 such that $p \oplus r_1 \sim_0 q \sim_0 r_2$ such that $p \oplus r_1 \sim_0 q \oplus r_2$.

(iii) If $p \in \mathcal{P}_\infty(\tilde{A})$ and $[p]_0 - [s(p)]_0 = 0$ then there is m such that $p \oplus 1_m \sim s(p) \oplus 1_m$.

(iv) If $\varphi : A \to B$ is a $*$-homomorphism then $\mathrm{K}_0(\varphi)([p]_0 - [s(p)]_0) = [\tilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0$ for each $p \in \mathcal{P}_\infty(\tilde{A})$.

Proof. (i) It is clear that $[p]_0 - [s(p)]_0 \in \ker(\mathrm{K}_0(\pi)) = \mathrm{K}_0(A)$. Conversely, let $g \in \mathrm{K}_0(A)$, and let e, f be projections in $M_n(\tilde{A})$ such that $g = [e]_0 - [f]_0$. Put

$$p = \begin{pmatrix} e & 0 \\ 0 & 1_n - f \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}.$$

We have $[p]_0 - [q]_0 = [e]_0 + [1_n - f]_0 - [1_n]_0 = [e]_0 - [f]_0 = g$. As $q = s(q)$ and $\mathrm{K}_0(\pi)(g) = 0$, we also have $[s(p)]_0 - [q]_0 = [s(p)]_0 - [s(q)]_0 = \mathrm{K}_0(s)(g) = (\mathrm{K}_0(\lambda) \circ \mathrm{K}_0(\pi))(g) = 0$. Hence $g = [p]_0 - [s(p)]_0$.

(ii) (a)$\Rightarrow$(c) If $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ then $[p \oplus s(q)]_0 = [q \oplus s(p)]_0$ and hence $p \oplus s(q) \sim_s q \oplus s(p)$ in $\mathcal{P}_\infty(\tilde{A})$. Thus there is n such that $p \oplus s(q) \oplus 1_n \sim_0 q \oplus s(p) \oplus 1_n$, and it suffices to take $r_1 = s(q) \oplus 1_n$ and $r_2 = s(p) \oplus 1_n$.

(c)$\Rightarrow$(b) If r_1, r_2 are scalar projections in $\mathcal{P}_\infty(\tilde{A})$ of rank k and l, respectively, then $r_1 \sim_0 1_k$ and $r_2 \sim_0 1_l$. Thus $p \oplus 1_k \sim_0 q \oplus 1_l$.

(b)$\Rightarrow$(a) We have $[p \oplus 1_k]_0 - [s(p \oplus 1_k)]_0 = [p]_0 + [1_k]_0 - [s(p)]_0 - [1_k]_0 = [p]_0 - [s(p)]_0$ and likewise $[q \oplus 1_l]_0 - [s(q \oplus 1_l)]_0 = [q]_0 - [s(q)]_0$. Thus it suffices to show that $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ whenever $p \sim_0 q$. So let $p = v^*v$ and $q = vv^*$. Then $s(v)$ is a scalar rectangular matrix and $s(p) = s(v)^* s(v)$, $s(q) = s(v)s(v)^*$. Thus $s(p) \sim_0 s(q)$. Consequently $[p]_0 = [q]_0$ and $[s(p)]_0 = [s(q)]_0$.

(iii) If $[p]_0 - [s(p)]_0 = 0$ then $p \sim_s s(p)$ and hence there is m such that $p \oplus 1_m \sim s(p) \oplus 1_m$.

(iv) $\mathrm{K}_0(\varphi)([p]_0 - [s(p)]_0) = \mathrm{K}_0(\tilde{\varphi})([p]_0 - [s(p)]_0) = [\tilde{\varphi}(p)]_0 - [\tilde{\varphi}(s(p))]_0 = [\tilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0$. □

4.2.2 (Half)exactness of K_0

In this section we proof that the K_0 functor is half exact — a property of crucial importence. To this end, we first proof the following technical lemma. Another lemma we need is given in Exercise 4.4.5.

Lemma 4.5. Let $\psi : A \to B$ be a $*$-homomorphism between two C*-algebras, and let $g \in \ker(\mathrm{K}_0(\psi))$.

(i) There is n, a projection $p \in \mathcal{P}_n(\tilde{A})$, and a unitary $u \in M_n(\tilde{B})$ such that

$$g = [p]_0 - [s(p)]_0 \quad \text{and} \quad u\tilde{\psi}(p)u^* = s(\tilde{\psi}(p)).$$

(ii) If ψ is surjective then there is a projection $p \in \mathcal{P}_\infty(\tilde{A})$ such that

$$g = [p]_0 - [s(p)]_0 \quad \text{and} \quad \tilde{\psi}(p) = s(\tilde{\psi}(p)).$$

Proof. (i) By virtue of Proposition 4.4, there is a projection $p_1 \in \mathcal{P}_k(\tilde{A})$ such that $g = [p_1]_0 - [s(p_1)]_0$, and we have $[\tilde{\psi}(p_1)]_0 - [s(\tilde{\psi}(p_1))]_0 = 0$. Thus $\tilde{\psi}(p_1) \oplus 1_m \sim s(\tilde{\psi}(p_1)) \oplus 1_m$ for some m, again by Proposition 4.4. Put $p_2 = p_1 \oplus 1_m$. Then $g = [p_2]_0 - [s(p_2)]_0$ and $\tilde{\psi}(p_2) = \tilde{\psi}(p_1) \oplus 1_m \sim s(\tilde{\psi}(p_1)) \oplus 1_m = s(\tilde{\psi}(p_2))$. Put $n = 2(k+m)$ and $p = p_2 \oplus 0_{k+m} \in \mathcal{P}_n(\tilde{A})$. Clearly, $[p]_0 - [s(p)]_0 = g$. By Proposition 2.2.9, there is a unitary u in $M_n(\tilde{A})$ such that $u\tilde{\psi}(p)u^* = s(\tilde{\psi}(p))$.

(ii) By virtue of part (i), there is n, a projection $p_1 \in \mathcal{P}_n(\tilde{A})$, and a unitary $u \in M_n(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\psi}(p_1)u^* = s(\tilde{\psi}(p_1))$. By Lemma 2.1.8, there exists a unitary $v \in M_{2n}(\tilde{A})$ such that $\tilde{\psi}(v) = \operatorname{diag}(u, u^*)$. Put $p = v \operatorname{diag}(p_1, 0_n) v^*$. Then

$$\tilde{\psi}(p) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \tilde{\psi}(p_1) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} s(\tilde{\psi}(p_1)) & 0 \\ 0 & 0 \end{pmatrix}$$

is a scalar matrix. Thus $s(\tilde{\psi}(p)) = \tilde{\psi}(p)$. Finally, $g = [p]_0 - [s(p)]_0$ since $p \sim_0 p_1$. $\square$

Theorem 4.6. A short exact sequence of C*-algebras

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{4.1}$$

induces an exact sequence

$$\mathrm{K}_0(J) \xrightarrow{\mathrm{K}_0(\varphi)} \mathrm{K}_0(A) \xrightarrow{\mathrm{K}_0(\psi)} \mathrm{K}_0(B).$$

If the sequence (4.1) splits with a splitting map $\lambda : B \to A$, then there is a split-exact sequence

$$0 \longrightarrow \mathrm{K}_0(J) \xrightarrow{\mathrm{K}_0(\varphi)} \mathrm{K}_0(A) \xrightarrow{\mathrm{K}_0(\psi)} \mathrm{K}_0(B) \longrightarrow 0 \tag{4.2}$$

with a splitting map $\mathrm{K}_0(\lambda) : \mathrm{K}_0(B) \to \mathrm{K}_0(A)$.

Proof. Since the sequence (4.1) is exact, functoriality of K_0 yields $\mathrm{K}_0(\psi) \circ \mathrm{K}_0(\varphi) = \mathrm{K}_0(\psi \circ \varphi) = \mathrm{K}_0(0) = 0$. Thus the image of $\mathrm{K}_0(\varphi)$ is contained in the kernel of $\mathrm{K}_0(\psi)$. Conversely, let $g \in \ker(\mathrm{K}_0(\psi))$. Then there is a projection p in $\mathcal{P}_\infty(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $\tilde{\psi}(p) = s(\tilde{\psi}(p))$ by part (ii) of Lemma 4.5. By part (ii) of Exercise 4.4.5, there is an element e in $M_\infty(\tilde{J})$ such that $\tilde{\varphi}(e) = p$. Since $\tilde{\varphi}$ is injective (by part (i) of Exercise 4.4.5), e must be a projection. Hence $g = [\tilde{\varphi}(e)]_0 - [s(\tilde{\varphi}(e))]_0 = \mathrm{K}_0(\varphi)([e]_0 - [s(e)]_0)$ belongs to the image of $\mathrm{K}_0(\varphi)$.

Now suppose the sequence (4.1) is split-exact. The sequence (4.2) is exact at $\mathrm{K}_0(A)$ by part (i) above. Functoriality of K_0 yields $\mathrm{id}_{\mathrm{K}_0(B)} = \mathrm{K}_0(\mathrm{id}_B) = \mathrm{K}_0(\psi) \circ \mathrm{K}_0(\lambda)$ and hence the sequence is exact at $\mathrm{K}_0(B)$. It remains to show that $\mathrm{K}_0(\varphi)$ is injective. Let $g \in \ker(\mathrm{K}_0(\varphi))$. By part (i) of Lemma 4.5 there is n, a projection $p \in \mathcal{P}_n(\tilde{J})$, and a unitary $u \in M_n(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\varphi}(p)u^* = s(\tilde{\varphi}(p))$. Put $v = (\tilde{\lambda} \circ \tilde{\psi})(u^*)u$, a unitary in $M_n(\tilde{A})$ such that $\tilde{\psi}(v) = 1_n$. By Exercise 4.4.5, there is an element $w \in M_n(\tilde{J})$ such that $\tilde{\varphi}(w) = v$. Since $\tilde{\varphi}$ is injective w must be unitary. We have

$$\begin{aligned} \tilde{\varphi}(wpw^*) = v\tilde{\varphi}(p)v^* &= (\tilde{\lambda} \circ \tilde{\psi})(u^*)s(\tilde{\varphi}(p))(\tilde{\lambda} \circ \tilde{\psi})(u) = (\tilde{\lambda} \circ \tilde{\psi})(u^* s(\tilde{\varphi}(p))u) \\ &= (\tilde{\lambda} \circ \tilde{\psi})(\tilde{\varphi}(p)) = s(\tilde{\varphi}(p)) = \tilde{\varphi}(s(p)). \end{aligned}$$

Since $\tilde{\varphi}$ is injective, we conclude that $wpw^* = s(p)$. Thus $p \sim_u s(p)$ in $M_n(\tilde{J})$ and hence $g = 0$. $\square$

4.3 Inductive Limits. Continuity and Stability of K_0

4.3.1 Increasing limits of C*-algebras

Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of C*-algebras such that $A_i \subseteq A_{i+1}$. Then $A_\infty = \bigcup_{i=1}^{\infty} A_n$ is a normed $*$-algebra satisfying all the axioms of a C*-algebra except perhaps of completeness. Let A be the completion of A_∞. Then A is a C*-algebra, called the *increasing limit* of A_n.

4.3.2 Direct limits of $*$-algebras

Let A_i be an infinite sequence of $*$-algebras. Suppose that for each pair $j \leq i$ there is given a $*$-homomorphism $\Phi_{ij} : A_j \to A_i$, and that the following coherence condition holds: $\Phi_{ij} = \Phi_{ik} \circ \Phi_{kj}$ whenever $j \leq k \leq i$, and $\Phi_{ii} = \mathrm{id}$. Let $\prod_i A_i$ be the product $*$-algebra, with coordinate-wise operations inherited from A_i's. Let $\sum_i A_i$ be the $*$-ideal of $\prod_i A_i$ consisting of sequences whose all but finitely many terms are 0, and let $\pi : \prod_i A_i \to \prod_i A_i / \sum_i A_i$ be the canonical surjection. Set

$$A_\infty = \pi\left(\left\{(a_i) \in \prod_i A_i : \exists i_0 \forall i : i \geq i_0 \Rightarrow a_i = \Phi_{ii_0}(a_{i_0})\right\}\right). \tag{4.3}$$

A_∞ is called *direct limit* of the *directed system* $\{A_i, \Phi_{ij}\}$ and denoted $\varinjlim\{A_i, \Phi_{ij}\}$. By definition, A_∞ is a $*$-algebra, and there exist canonical morphisms $\Phi_i : A_i \to A_\infty$ such that $A_\infty = \bigcup_i \Phi_i(A_i)$ and for all $j \leq i$ the following diagram commutes:

$$\begin{tikzcd}
A_j \arrow[dd, "\Phi_{ij}"'] \arrow[dr, "\Phi_j"] & \\
& A_\infty \\
A_i \arrow[ur, "\Phi_i"'] &
\end{tikzcd}$$

Indeed, for $x \in A_j$ define $\Phi_j(x) = \pi((a_i))$, where $a_i = 0$ if $i < j$ and $a_i = \Phi_{ij}(x)$ if $i \geq j$.

The direct limit $A_\infty = \varinjlim\{A_i, \Phi_{ij}\}$ has the following *universal property.* If B is a $*$-algebra and for each i there is a $*$-homomorphism $\Psi_i : A_i \to B$ such that $\Psi_i \circ \Phi_{ij} = \Psi_j$ for every $j \leq i$, then there exists a unique $*$-homomorphism $\Lambda : A_\infty \to B$ such that the diagram

$$\begin{tikzcd}
A_j \arrow[dd, "\Phi_{ij}"'] \arrow[dr, "\Phi_j"'] \arrow[drr, "\Psi_j"] & & \\
& A_\infty \arrow[r, "\Lambda"] & B \\
A_i \arrow[ur, "\Phi_i"] \arrow[urr, "\Psi_i"'] & &
\end{tikzcd}$$

commutes.

Everything from this section may be generalized to the case of directed systems of $*$-algebras over *directed sets* rather than merely sequences. Furthermore, the same construction works for abelian groups (or even monoids) and their homomorphisms rather than $*$-algebras and $*$-homomorphisms.

4.3.3 C*-algebraic inductive limits

Now suppose that each A_i is a C*-algebra rater than just a $*$algebra. By definition, the product $\prod_i A_i$ consists of sequences (a_i) for which $||(a_i)|| = \sup\{||a_i||\}$ is finite. With this norm $\prod_i A_i$ is a C^*-alebra. Let $\sum_i A_i$ be the closure of the ideal of sequences whose all but finitely many terms are 0, and let $\pi : \prod_i A_i \to \prod_i A_i / \sum_i A_i$ be the canonical surjection. We define

$$\begin{aligned}&\varinjlim\{A_i, \Phi_{ij}\} = \\ &\text{the closure of } \pi\left(\left\{(a_i) \in \prod_i A_i : \exists i_0 \forall i : i \geq i_0 \Rightarrow a_i = \Phi_{ii_0}(a_{i_0})\right\}\right).\end{aligned} \tag{4.4}$$

This definition is correct since $*$-homomorphisms between C*-algebras are norm-decreasing. As before, there exist $*$-homomorphisms $\Phi_i : A_i \to A_\infty$ satisfing (4.3.2), and the universal property (4.3.2) holds.

4.3.4 Continuity of K_0

Theorem 4.7. *Let $\{A_i, \Phi_{ij}\}$ be an inductive sequence of C^*-algebras and let $A = \varinjlim\{A_i, \Phi_{ij}\}$. Then $\{\mathrm{K}_0(A_i), \mathrm{K}_0(\Phi_{ij})\}$ is a direct sequence of abelian groups and*

$$\mathrm{K}_0(A) = \mathrm{K}_0(\varinjlim\{A_i, \Phi_{ij}\}) \cong \varinjlim\{\mathrm{K}_0(A_i), \mathrm{K}_0(\Phi_{ij})\}.$$

Proof. W denote by $\Phi_i : A_i \to A = \lim A_i$ the canonical maps. Functoriality of K_0 implies that $\{\mathrm{K}_0(A_i), \mathrm{K}_0(\Phi_{ij})\}$ is a direct sequence of abelian groups. Let $\varphi_i : \mathrm{K}_0(A_i) \to \lim \mathrm{K}_0(A_i)$ be the canonical maps. Since for $j \leq i$ we have $\mathrm{K}_0(\Phi_j) = \mathrm{K}_0(\Phi_i) \circ \mathrm{K}_0(\Phi_{ij})$ by functoriality of K_0, the universal property of $\lim \mathrm{K}_0(A_i)$ yields a unique homomorphism $\varphi : \lim \mathrm{K}_0(A_i) \to \mathrm{K}_0(A)$ such that $\varphi_i = \varphi \circ \varphi_j$ for all $j \leq i$.

$$\begin{tikzcd} \mathrm{K}_0(A_j) \arrow[r, "\varphi_j"] \arrow[d, "\mathrm{K}_0(\Phi_{ij})"'] & \lim \mathrm{K}_0(A_i) \arrow[d, "\varphi"] \\ \mathrm{K}_0(A_i) \arrow[r, "\mathrm{K}_0(\Phi_i)"'] & \mathrm{K}_0(\lim A_i) \end{tikzcd}$$

We must show that φ is injective and surjective.

Injectivity of φ. Since $\lim \mathrm{K}_0(A_i) = \bigcup_i \varphi_i(\mathrm{K}_0(A_i))$, it suffices to show that $\varphi|_{\varphi_j(\mathrm{K}_0(A_j))}$ is injective for all j. That is we must show that if $g \in \mathrm{K}_0(A_j)$ and $\mathrm{K}_0(\Phi_j)(g) = (\varphi \circ \varphi_j)(g) = 0$ in $\mathrm{K}_0(A)$ then $\varphi_j(g) = 0$ in $\lim \mathrm{K}_0(A_i)$. So let $g = [p]_0 - [s(p)]_0$ for some $p \in \mathcal{P}_n(\tilde{A}_j)$. Then $0 = \mathrm{K}_0(\Phi_j)(g) = [\tilde{\Phi}_j(p)]_0 - [s(\tilde{\Phi}_j(p))]_0$ in $\mathrm{K}_0(A)$. Hence there is m and a partial isometry $w \in M_{n+m}(\tilde{A})$ such that

$$ww^* = \tilde{\Phi}_j(p) \oplus 1_m \quad \text{and} \quad w^*w = s(\tilde{\Phi}_j(p)) \oplus 1_m.$$

By Exercise 34, there is $i \geq j$ and $x_i \in M_{n+m}(\tilde{A}_i)$ with $\tilde{\Phi}_i(x_i)$ close enough to w to ensure that

$$||\tilde{\Phi}_i(x_i)\tilde{\Phi}_i(x_i)^* - \tilde{\Phi}_j(p) \oplus 1_m|| < 1/2 \quad \text{and} \quad ||\tilde{\Phi}_i(x_i)^*\tilde{\Phi}_i(x_i) - s(\tilde{\Phi}_j(p)) \oplus 1_m|| < 1/2.$$

Since $\Phi_j = \Phi_i \circ \Phi_{ij}$, Exercise 37 implies that there is $k \geq i$ such that

$$||x_k x_k^* - \tilde{\Phi}_{kj}(p) \oplus 1_m|| < 1/2 \quad \text{and} \quad ||x_k^* x_k - s(\tilde{\Phi}_{kj}(p)) \oplus 1_m|| < 1/2,$$

where $x_k = \tilde{\Phi}_{ki}(x_i)$. By part (ii) of Exercise 38, $\tilde{\Phi}_{kj}(p) \oplus 1_m$ is equivalent to $s(\tilde{\Phi}_j(p)) \oplus 1_m$ in $M_{n+m}(\tilde{A}_m)$. Thus

$$\mathrm{K}_0(\Phi_{kj})(g) = [\tilde{\Phi}_{kj}(p) \oplus 1_m]_0 - [s(\tilde{\Phi}_j(p)) \oplus 1_m]_0 = 0$$

in $\mathrm{K}_0(A_k)$. Consequently, $\varphi_j(g) = (\varphi_k \circ \mathrm{K}_0(\Phi_{kj}))(g) = 0$, as required.

Surjectivity of φ. Consider an element $[p]_0 - [s(p)]_0$ of $\mathrm{K}_0(A)$, for some $p \in \mathcal{P}_k(\tilde{A})$. Take a small $\epsilon > 0$. By Exercise 34, there is n and $b_n \in M_k(\tilde{A}_n)$ such that $||\tilde{\Phi}_n(b_n) - p|| < \epsilon$. Put $a_n = (b_n + b_n^*)/2$ and $a_m = \tilde{\Phi}_{mn}(a_n)$ for $m \geq n$. Each a_m is self-adjoint and $||\tilde{\Phi}_m(a_m) - p|| < \epsilon$. We have $||\tilde{\Phi}_n(a_n - a_n^2)|| < \epsilon(3+\epsilon) < 1/4$ for sufficiently small ϵ. Thus, by Exercise 37, $||a_m - a_m^2|| < 1/4$ for sufficiently large m. By Exercise 38, there is a projection q in $M_k(\tilde{A}_m)$ such that $||a_m - q|| < 1/2$. We have $||\tilde{\Phi}_m(q) - p|| < 1/2 + \epsilon < 1$ and hence $\tilde{\Phi}_m(q)$ and p are equivalent. Thus

$$[p]_0 - [s(p)]_0 = [\tilde{\Phi}_m(q)]_0 - [s(\tilde{\Phi}_m(q))]_0 = \mathrm{K}_0(\Phi_m)([q]_0 - [s(q)]_0).$$

Since $\mathrm{K}_0(\Phi_m) = \varphi \circ \varphi_m$, surjectivity of φ follows. □

4.3.5 Stability of K_0

Proposition 4.8. *Let A be a C*-algebra, and let p be minimal projection in $\mathcal{K}$. The map $\varphi : A \to A \otimes \mathcal{K}$ such that $\varphi(a) = a \otimes p$ induces an isomorphism $\mathrm{K}_0(\varphi) : \mathrm{K}_0(A) \to \mathrm{K}_0(A \otimes \mathcal{K})$.*

Proof. For $n \geq m$ let $\Phi_{nm} : M_m(A) \to M_n(A)$ be the imbedding $\Phi_{nm}(a) = \mathrm{diag}(a, 0_{n-m})$. By Exercise 36, $A \otimes \mathcal{K}$ is isomorphic with the limit of the inductive sequence $\{M_n(A), \Phi_{nm}\}$. We have $\Phi_{n1} = \Phi_{nm} \circ \Phi_{m1}$ and hence $\mathrm{K}_0(\Phi_{n1}) = \mathrm{K}_0(\Phi_{nm}) \circ \mathrm{K}_0(\Phi_{m1})$. Moreover, all the mapsare isomorphism on K_0, by Exercise 4.4.7. Let $\psi_n = \mathrm{K}_0(\Phi_{n1})^{-1}$. Then $\psi_m = \psi_n \circ \mathrm{K}_0(\Phi_{nm})$ for all $n \geq m$. Thus the universal property of direct limits yields a unique homomorphism $\Lambda : \varinjlim\{M_n(A), \Phi_{nm}\} \cong \mathrm{K}_0(A \otimes \mathcal{K}) \to \mathrm{K}_0(A)$ which fits into the commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0(M_m(A)) \arrow[r, "\mathrm{K}_0(\Phi_m)"] \arrow[d, "\mathrm{K}_0(\Phi_{nm})"'] & \mathrm{K}_0(A \otimes \mathcal{K}) \arrow[d, "\Lambda"] \\
\mathrm{K}_0(M_n(A)) \arrow[r, "\psi_n"'] & \mathrm{K}_0(A)
\end{tikzcd}$$

where $\Phi_n : M_n(A) \to A \otimes \mathcal{K}$ are the canonical maps. It follows that Λ is an isomorphism. Furthermore, $\Lambda^{-1} = \mathrm{K}_0(\varphi)$, as required. □

4.4 Examples and Exercises

Example 4.9. Consider the exact sequence

$$0 \longrightarrow C_0((0,1)) \longrightarrow C([0,1]) \xrightarrow{\psi} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

We have $\mathrm{K}_0(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z}^2$ and $\mathrm{K}_0(C([0,1])) \cong \mathbb{Z}$ (since $[0,1]$ is contractible). Thus $\mathrm{K}_0(\psi)$ cannot be *surjective*.

Example 4.10. Let $\mathcal{H}$ be a separable Hilbert space,and let $\mathcal{K}$ be the ideal of compact operators on $\mathcal{H}$. There is an exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{\imath} \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})/\mathcal{K} \longrightarrow 0.$$

We already know that $\mathrm{K}_0(\mathcal{B}(\mathcal{H})) = 0$ and we will see later that $\mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z}$. Thus $\mathrm{K}_0(\imath)$ cannot be *injective*.

Exercise 28. Let $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}$ be the Calkin algebra (corresponding to a separable Hilbert space $\mathcal{H}$), and let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}$ be the natural surjection. Show the following.

(i) If $p \neq 0$ is a projection in $\mathcal{Q}$ then there is a projection $\tilde{p}$ in $\mathcal{B}(\mathcal{H})$ with infinite dimensional range such that $\pi(\tilde{p}) = p$.

(ii) Any two non-zero projections in $\mathcal{Q}$ are Murray-von Neumann equivalent.

(iii) For each positive integer n we have $\mathcal{B}(\mathcal{H}) \cong M_n(\mathcal{B}(\mathcal{H}))$, $\mathcal{K} \cong M_n(\mathcal{K})$, and $\mathcal{Q} \cong M_n(\mathcal{Q})$.

(iv) The semigroup $\mathcal{D}(\mathcal{Q})$ is isomorphic to $\{0, \infty\}$, with $\infty + \infty = \infty$.

(v) $\mathrm{K}_0(\mathcal{Q}) = 0$.

(i) Hint: If p is a projection in $\mathcal{Q}$ then there exists $x = x^*$ in $\mathcal{B}(\mathcal{H})$ such that $\pi(x) = p$. Thus $x^2 - x$ is compact. Let $x^2 - x = \sum_n \lambda_n e_n$ be the spectral decomposition ($0 \neq \lambda_n \in \mathbb{R}$, $\lambda_n \to 0$, $\{e_n\}$ mutually orthogonal projections of finite rank, commuting with x). Correct each xe_n.

Example 4.11. In this example we argue why Definition 3.5 would not be appropriate for non-unital C*-algebras. Nameley, let A be a C^*-alebra (unital or not) and define $K_{00}(A)$ as the Grothendieck group of $\mathcal{D}(A)$. Thus, if A is unital the $K_{00}(A) = \mathrm{K}_0(A)$, but in the non-unital case these two groups may be different. It can be shown that such defined K_{00} is a covariant functor. However, this functor has a serious defect of not being half-exact. Indeed, consider an exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \longrightarrow C(S^2) \longrightarrow \mathbb{C} \longrightarrow 0.$$

We have $\mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$, and it can be shown that $\mathrm{K}_0(S^2) \cong \mathbb{Z}^2$ and $K_{00}(\mathbb{R}^2) = 0$ (for the latter see Exercise 4.3.4 below). Thus K_{00} cannot be half-exact.

Exercise 29. If $0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$ is an exact sequence of C*-algebras then:

(i) $\tilde{\varphi}_n : M_n(\tilde{J}) \to M_n(\tilde{A})$ is injective,

(ii) $a \in M_n(\tilde{A})$ is in the image of $\tilde{\varphi}_n$ if and only if $\tilde{\psi}_n(a) = s_n(\tilde{\psi}_n(a))$.

Exercise 30. Let X be a connected, locally compact but not compact Hausdorff space. Then $K_{00}(C_0(X)) = 0$. To this end show that $\mathcal{P}_\infty(C_0(X)) = \{0\}$, as follows. Identify $M_n(C_0(X))$ with $C_0(X, M_n(\mathbb{C}))$, and let p be a projection in $\mathcal{P}_n(C_0(X))$. As usual, let Tr be the standard trace on $M_n(\mathbb{C})$. The function $x \mapsto \mathrm{Tr}(p(x))$ belongs to $C_0(X, \mathbb{Z})$ and hence it is the zero function, since X is connected and non-compact.

Exercise 31 (Matrix stability of K_0). Let A be a C*-algebra and let n be a positive integer. Then $\mathrm{K}_0(A) \cong \mathrm{K}_0(M_n(A))$. More specifically, the map $\varphi_A : A \to M_n(A)$, $a \mapsto \mathrm{diag}(a, 0_{n-1})$ induces an isomorphism $\mathrm{K}_0(\varphi_A) \to \mathrm{K}_0(M_n(A))$. Indeed, the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi_A"'] & \tilde{A} \arrow[r] \arrow[d, "\varphi_{\tilde{A}}"'] & \mathbb{C} \arrow[r] \arrow[d, "\varphi_{\mathbb{C}}"'] & 0 \\
0 \arrow[r] & M_n(A) \arrow[r] & M_n(\tilde{A}) \arrow[r] & M_n(\mathbb{C}) \arrow[r] & 0
\end{tikzcd}$$

commutes and has split-exact rows. Thus the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r] \arrow[d, "\mathrm{K}_0(\varphi_A)"'] & \mathrm{K}_0(\tilde{A}) \arrow[r] \arrow[d, "\mathrm{K}_0(\varphi_{\tilde{A}})"'] & \mathrm{K}_0(\mathbb{C}) \arrow[r] \arrow[d, "\mathrm{K}_0(\varphi_{\mathbb{C}})"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(M_n(A)) \arrow[r] & \mathrm{K}_0(M_n(\tilde{A})) \arrow[r] & \mathrm{K}_0(M_n(\mathbb{C})) \arrow[r] & 0
\end{tikzcd}$$

commutes and has split-exact rows. Hence the Five Lemma (or an easy diagram chasing) implies that $\mathrm{K}_0(\varphi_A)$ is an isomorphism if both $\mathrm{K}_0(\varphi_{\tilde{A}})$ and $\mathrm{K}_0(\varphi_{\mathbb{C}})$ are. This reduces the proof to the unital case (see Exercise 3.3.10).

Exercise 32. Let A be a C*-algebra, and denote by $\mathrm{Aut}(A)$ the group of $*$-automorphisms of A. If $\alpha \in \mathrm{Aut}(A)$ then $\mathrm{K}_0(\alpha)$ is an automorphism of $\mathrm{K}_0(A)$.

(i) If u is a unitary in $\tilde{A}$ then $\mathrm{Ad}(u) : A \to A$, $a \mapsto uau^*$, is an automorphism of A. Moreover, the map $\mathcal{U}(\tilde{A}) \to \mathrm{Aut}(A)$, $u \mapsto \mathrm{Ad}(u)$ is a group homomorphism, and $\mathrm{Inn}(A) = \{\mathrm{Ad}(u) : u \in \mathcal{U}(\tilde{A})\}$ is a normal subgroup of $\mathrm{Aut}(A)$.

(ii) If $\alpha \in \mathrm{Inn}(A)$ then $\mathrm{K}_0(\alpha) = \mathrm{id}$.

(iii) An $\alpha \in \mathrm{Aut}(A)$ is *approximately inner* if and only if for any finite subset F of A and any $\epsilon > 0$ there is $\beta \in \mathrm{Inn}(A)$ such that $||\alpha(x) - \beta(x)|| < \epsilon$ for all $x \in F$. The collection of all approximately inner automorphisms of A is denoted $\overline{\mathrm{Inn}}(A)$.

Show that if A is separable then α is approximately inner if and only if there is a sequence $\beta_n \in \mathrm{Inn}(A)$ such that $\beta_n(a) \to \alpha(a)$ for each $a \in A$.

(iv) $\overline{\mathrm{Inn}}(A)$ is a normal subgroup of $\mathrm{Aut}(A)$, and $\mathrm{K}_0(\alpha) = \mathrm{id}$ for each $\alpha \in \overline{\mathrm{Inn}}(A)$.

(v) Give examples of automorphisms of C*-algebras which induce non-trivial automorphisms on K_0.

Example 4.12. Let A be a C*-algebra. We define the *cone* CA and the *suspension* SA as follows:

$$\begin{aligned} CA &= \{f : [0,1] \to A : f \text{ continuous and } f(0) = 0\}, \\ SA &= \{f : [0,1] \to A : f \text{ continuous and } f(0) = f(1) = 0\}. \end{aligned}$$

There is a short exact sequence

$$0 \longrightarrow SA \longrightarrow CA \xrightarrow{\pi} A \longrightarrow 0,$$

with $\pi(f) = f(1)$. Furthermore, CA is homotopy equivalent to $\{0\}$. Indeed, with $t \in [0,1]$ set $\varphi_t : CA \to CA$ as $\varphi_t(f)(s) = f(st)$. Then for each $f \in CA$ the map $t \mapsto \varphi_t(f)$ is continuous, and $\varphi_0 = 0$, $\varphi_1 = \mathrm{id}$. We conclude that

$$\mathrm{K}_0(CA) = 0.$$

Example 4.13 (Direct sums). For any two C*-algebras A, B we have

$$\mathrm{K}_0(A \oplus B) \cong \mathrm{K}_0(A) \oplus \mathrm{K}_0(B).$$

More specifically, if i_A and i_B are the inclusions of A and B, respectively, into $A \oplus B$, then $\mathrm{K}_0(i_A) \oplus \mathrm{K}_0(B) : \mathrm{K}_0(A) \oplus \mathrm{K}_0(B) \to \mathrm{K}_0(A \oplus B)$ is an isomorphism. Indeed, let π_A and π_B be the surjections from $A \oplus B$ onto A and B, respectively. The following diagram has exact rows (the bottom one by split-exactness of K_0) and commutes (since $\pi_B \circ i_A = 0$ and $\pi_B \circ i_B = \mathrm{id}_B$):

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\alpha"] \arrow[d, equal] & \mathrm{K}_0(A) \oplus \mathrm{K}_0(B) \arrow[r, "\beta"] \arrow[d, "\mathrm{K}_0(i_A)\oplus \mathrm{K}_0(i_B)"'] & \mathrm{K}_0(B) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(i_A)"'] & \mathrm{K}_0(A \oplus B) \arrow[r, "\mathrm{K}_0(\pi_B)"'] & \mathrm{K}_0(B) \arrow[r] & 0
\end{tikzcd}$$

An easy diagram chasing (or the Five Lemma) implies that $\mathrm{K}_0(i_A) \oplus \mathrm{K}_0(i_B)$ is an isomorphism.

Exercise 33. Let $\{A_i\}$ be a sequence of C*-algebras, and let $a = (a_i) \in \prod_i A_i$. Then

$$||\pi(a)|| = \overline{\lim}||a_i||.$$

In particular, a belongs to $\sum_i A_i$ if and only if $\lim\limits_{i\to\infty} ||a_i|| = 0$.

Exercise 34. Let $A = \varinjlim\{A_i, \Phi_{ij}\}$. To each $x \in A$ and $\epsilon > 0$ there is an arbitrarily large index i and $x_i \in A_i$ such that

$$||x - \Phi_i(x_i)|| < \epsilon.$$

Example 4.14 (UHF algebras). Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of integers $p_n \geq 2$. For $1 \leq j$ define $\Phi_{j+1,j} : \bigotimes_{n=1}^{j} M_{p_n}(\mathbb{C}) \to \bigotimes_{n=1}^{j+1} M_{p_n}(\mathbb{C})$ by $\Phi_{j+1,j}(x) = x \otimes I$. These are unital, injectve $*$-homomorphisms. Then for $1 \leq j \leq i$ define $\Phi_{ij} : \bigotimes_{n=1}^{j} M_{p_n}(\mathbb{C}) \to \bigotimes_{n=1}^{i} M_{p_n}(\mathbb{C})$ by $\Phi_{ij} = \Phi_{i,i-1} \circ \ldots \circ \Phi_{j+1,j}$. The inductive limit $\varinjlim\{A_i, \Phi_{ij}\}$ is called the UHF algebra corresponding to the supernatural number $(p_1 p_2 \cdots)$. These are simple, unital C*-algebras, equipped with a unique tracial state. To learn much more about UHF algebras see [g-j60].

Example 4.15 (AF-algebras). For $n = 1, 2, \ldots$ let A_n be a finite dimensional C*-algebra. Thus, A_n is isomorphic to a direct sum of matrix algebras

$$A_n \cong M_{k_1^n}(\mathbb{C}) \oplus \ldots \oplus M_{k_n^{r(n)}}(\mathbb{C}).$$

For $1 \leq j$ let $\Phi_{j+1,j} : A_j \to A_{j+1}$ be a $*$-homomorphism, and define $\Phi_{ij} = \Phi_{i,i-1} \circ \ldots \circ \Phi_{j+1,j}$. The corresponding inductive limit $\varinjlim\{A_i, \Phi_{ij}\}$ is called an AF-algebra. To learn much more about AF-algebras and about *Bratteli diagrams* which describe them see [b-o72].

Exercise 35 (The compacts). Let $\mathcal{H}$ be a separable (infinite dimensional) Hilbert space. Denote by $\mathcal{F}$ the collection of all finite rank operators in $\mathcal{B}(\mathcal{H})$, and let $\mathcal{K}$ be the norm closure of $\mathcal{F}$ (the C*-algebra of *compact* operators). Show the following.

(i) $\mathcal{F}$ is a two-sided $*$-ideal of $\mathcal{B}(\mathcal{H})$, but $\mathcal{F}$ is not norm closed in $\mathcal{B}(\mathcal{H})$.

(ii) $\mathcal{K}$ is a norm closed, two-sided $*$-ideal of $\mathcal{B}(\mathcal{H})$, and $\mathcal{K} \neq \mathcal{B}(\mathcal{H})$.

(iii) Let $\{\xi_n\}_{n=0}^{\infty}$ be an orthonoral basis of $\mathcal{H}$. For all i, j let E_{ij} be an operator defined by $E_{ij}(v) = \langle v, \xi_j \rangle \xi_i$. Then each E_{ij} is a rank one opeator with domain $\mathbb{C}\xi_j$ and range $\mathbb{C}\xi_i$. In particular, $\{E_{ii}\}$ are mutually orthogonal projections of rank one whose sum of the ranges densely spans the entire space $\mathcal{H}$. Furthermore, for each i we have $E_{ii}\mathcal{K}E_{ii} = \mathbb{C}E_{ii}$ (a projection in a C*-algebra with this property is called *minimal*). The following identities hold:
$$E_{ij}E_{kn} = \delta_{jk}E_{in}, \qquad E_{ij}^* = E_{ji}. \tag{4.5}$$
(A collection of elements of a C*-algebra satisfying (4.5) is called a *system of matrix units*.) Prove that the closed span of $\{E_{ij} : i, j = 0, 1, \ldots\}$ coincides with $\mathcal{K}$.

(iv) Let $\mathcal{H}'$ be another Hilbert space and let $\pi : \mathcal{K} \to \mathcal{H}'$ be a nondegenerate representation (i.e. a $*$-homomorphism such that $\pi(\mathcal{K})\mathcal{H}'$ is dense in $\mathcal{H}'$). Show that there exists a Hilbert space $\mathcal{H}_0$ and a unitary operator $U : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{H}_0$ such that for all $x \in \mathcal{K}$ we have
$$U\pi(x)U^* = x \otimes I_{\mathcal{H}_0}.$$
The dimension of $\mathcal{H}_0$ is called the *multiplicity* of π. Show that π is irreducible if and only if the multiplicity of π is one. Thus, in particular, the compacts admit precisely one (up to unitary equivalence) irreducible representation.

(v) $\mathcal{K}$ is the universal C*-algebra for the relations (4.5).

(vi) $\mathcal{K}$ is a *simple* C*-algebra, in the sense that the only closed, two-sided $*$-ideals of $\mathcal{K}$ are $\{0\}$ and $\mathcal{K}$. (In fact, it can be shown that every norm closed two-sided ideal of a C*-algebra is automatically closed under $*$).

(vii) For each $j = 1, 2, \ldots$ let $\Phi_{j+1,j} : M_j(\mathbb{C}) \to M_{j+1}(\mathbb{C})$ be an imbedding into the upper-left corner, i.e. $\Phi_{j+1,j}(x) = \mathrm{diag}(x, 0)$. As usual, let $\Phi_{ij} = \Phi_{i,i-1} \circ \ldots \circ \Phi_{j+1,j}$ for $j \leq i$. Show that
$$\mathcal{K} \cong \varinjlim\{M_n(\mathbb{C}), \Phi_{ij}\}.$$

Exercise 36. Let A be a C*-algebra. For $n \geq m$ let $\Phi_{nm} : M_m(A) \to M_n(A)$ be the diagonal imbedding $\Phi_{nm}(a) = \mathrm{diag}(a, 0_{n-m})$. Show that the inductive limit of the drected sequence $\{M_n(A), \Phi_{nm}\}$ is isomorphic with $A \otimes \mathcal{K}$.

Exercise 37. Let $\{A_i, \Phi_{ij}\}$ be an inductive sequence of C*-algebras and let $\Phi_i : A_i \to \varinjlim A_i$ be the canonical maps. Then for all n and $a \in A_n$ we have

$$||\Phi_n(a)|| = \lim_{m\to\infty} ||\Phi_{mn}(a)||.$$

Exercise 38. Let A be a C*-algebra.

(i) If $a = a^*$ in A and $||a - a^2|| < 1/4$ then there is a projection $p \in A$ with $||a - p|| < 1/2$.

(ii) Let p be a projection in A, and let a be a self-adjoint eleent in A. Put $\delta = ||a - p||$. Then

$$\mathrm{sp}(a) \subseteq [-\delta, \delta] \cup [1 - \delta, 1 + \delta].$$

(iii) If p, q are projections in A such that there exists an element $x \in A$ with $||x^*x - p|| < 1/2$ and $||xx^* - q|| < 1/2$ then $p \sim q$.

(i) Use Gelfand Theorem.

(ii) Recall that the spectrum of a self-adjoint element consists of real numbers, and that the spectrum of a non-trivial projection is $\{0, 1\}$. Let t be a real number whose distance d to the set $\{0, 1\}$ is greater than δ. It suffices to show that $t - a$ is invertible in $\tilde{A}$. Indeed, for such a t the element $t - p$ is invertible in $\tilde{A}$ and

$$||(t - p)^{-1}|| = \max\{\frac{1}{|-t|}, : \frac{1}{|1 - t|}\} = \frac{1}{d}.$$

Consequently,

$$||(t - p)^{-1}(t - a) - 1|| = ||(t - p)^{-1}(p - a)|| \leq \frac{1}{d}\delta < 1.$$

Thus $(t - p)^{-1}(t - a)$ is invertible, and so is $t - a$.

(iii) Let $\Omega = \mathrm{sp}(x^*x) \cup \mathrm{sp}(xx^*)$. In view of part (ii) of this exercise, Ω is a compact subset of $[0, 1/2) \cup (1/2, 3/2)$. Let $f : \Omega \to \mathbb{R}$ be a continuous function which is 0 on $\Omega \cap [0, 1/2)$ and 1 on $\Omega \cap (1/2, 3/2)$. Then both $f(x^*x)$ and $f(xx^*)$ are projections. We have $||f(x^*x) - p|| \leq ||f(x^*x) - x^*x|| + ||x^*x - p|| < 1/2 + 1/2 = 1$ and, likewise, $||f(xx^*) - q|| < 1$. Thus $f(x^*x) \sim p$ and $f(xx^*) \sim q$ by Proposition 2.2.5. So it suffices to show that $f(x^*x) \sim f(xx^*)$. To this end, first note that $xh(x^*x) = h(xx^*)x$ for every $h \in C(\Omega)$. Indeed, this is obviously true for polynomials, and the general case follows from the Stone-Weierstrass Theorem. Let $g \in C(\Omega)$, $g \geq 0$ be such that $tg(t)^2 = f(t)$ for all $t \in \Omega$. Set $w = xg(x^*x)$. Then

$$\begin{aligned} w^*w &= g(x^*x)x^*xg(x^*x) = f(x^*x), \\ ww^* &= xg(x^*x)^2x^* = g(xx^*)^2xx^* = f(xx^*), \end{aligned}$$

and the claim follows.

Example 4.16. Let A be a unital Banach algebra, and let $a, b \in A$. Then

$$\mathrm{sp}(ab) \cup \{0\} = \mathrm{sp}(ba) \cup \{0\}.$$

Indeed, let $0 \neq \lambda \notin \mathrm{sp}(ab)$ and let $u = (\lambda - ab)^{-1}$. Then $1 - \lambda u + uab = 0$. Put $w = (1/\lambda)(1 + bua)$. We have

$$w(\lambda - ba) = \frac{1}{\lambda}(1 + bua)(\lambda - ba) = 1 - \frac{1}{\lambda}ba + bua - \frac{1}{\lambda}buaba = 1 - \frac{1}{\lambda}b(1 - \lambda u + uab)a = 1.$$

Similarly $(\lambda - ba)w = 1$ and hence $w = (\lambda - ba)^{-1}$. Thus $\lambda \notin \mathrm{sp}(ba)$.

Exercise 39. In the category of abelian groups, consider a direct sequence $A_i = \mathbb{Z}$ with connecting maps $\Phi_{j+1,j}(1) = j$. Show that the corresponding limit is isomorphic to the additive group of $\mathbb{Q}$.

Exercise 40 (Irrational rotation algebras). For an irrational number $\theta \in [0, 1)$ define A_θ as the universal C*-algebra generated by two elements u, v, subject to the relations

$$vu = e^{2\pi i\theta}uv, \qquad uu^* = u^*u = 1 = vv^* = v^*v. \tag{4.6}$$

A_θ is called the *irrational rotation algebra* corresponding to the angle θ. Show the following.

(i) Let $\mathcal{L}^2(\mathbb{T})$ be the Hilbert space of square integrable functions on the circle group (with respect to the probability Haar measure dz). Set $\mathcal{H} = \mathcal{L}^2(\mathbb{T}) \otimes \mathcal{L}^2(\mathbb{T})$, and define two operators $U, V \in \mathcal{B}(\mathcal{H})$ by

$$(U\xi)(z_1, z_2) = z_1\xi(z_1, z_2), \quad (V\xi)(z_1, z_2) = z_2\xi(e^{2\pi i\theta}z_1, z_2).$$

Then U, V satisfy (4.6). Thus, there exists a representation $\pi : A_\theta \to \mathcal{B}(\mathcal{H})$ such that $\pi(u) = U$ and $\pi(v) = V$.

(ii) Let $\mathcal{A}_\theta$ be the dense $*$-subalgebra of A_θ generated by u, v. Then each element of $\mathcal{A}_\theta$ has the form

$$\sum_{n,m\in\mathbb{Z}} \lambda_{n,m}u^n v^m, \qquad \lambda_{n,m} \in \mathbb{C}.$$

(iii) Let ξ_0 be the unit vector in $\mathcal{H}$ such that $\xi_0(z_1, z_2) = 1$, and define $\tau(a) = \langle\pi(a)\xi_0, \xi_0\rangle$, $a \in A_\theta$. Then $\tau(\sum_{n,m\in\mathbb{Z}} \lambda_{n,m}u^n v^m) = \lambda_{0,0}$ and hence $\tau(aa^*) = \tau(a^*a)$ for all $a \in \mathcal{A}_\theta$. Conclude that τ is a tracial state on A_θ.

(iv) For $f, g : \mathbb{T} \to \mathbb{R}$ let

$$p = f(u)v^* + g(u) + vf(u)$$

be a self-adjoint element of A_θ. Use an approximation of f and g by Laurent polynomials to show that

$$\tau(p) = \int_{\mathbb{T}} g(z)dz.$$

(v) Let $\varphi : \mathbb{T} \to \mathbb{T}$ be given by $\varphi(z) = e^{2\pi i\theta} z$. Then $vh(u) = (h \circ \varphi)(u)v$ for all $h \in C(\mathbb{T})$. Show that $p = p^2$ if and only if

$$(f \circ \varphi)f = 0, \quad (g + g \circ \varphi^{-1})f = f, \quad g = g^2 + f^2 + (f \circ \varphi)^2. \tag{4.7}$$

(vi) Let ϵ be such that $0 < \epsilon \leq \theta < \theta + \epsilon \leq 1$, and set

$$g(e^{2\pi it}) = \begin{cases} \epsilon^{-1}t, & 0 \leq t \leq \epsilon, \\ 1, & \epsilon < t \leq \theta, \\ \epsilon^{-1}(\theta + \epsilon - t), & \theta < t \leq \theta + \epsilon, \\ 0, & \theta + \epsilon < t \leq 1, \end{cases}$$

for $t \in [0, 1]$. For such g one an find f such that (4.7) holds, and hence p is a projection. Then $\tau(p) = \theta$. Thus, the homomorphism $\mathrm{K}_0(\tau) : \mathrm{K}_0(A_\theta) \to \mathbb{R}$ contains $\mathbb{Z} \cup \theta\mathbb{Z}$ in its image.

In fact, it can be shown that $\mathrm{K}_0(\tau)$ is an isomorphism of $\mathrm{K}_0(A_\theta)$ onto $\mathbb{Z} \cup \theta\mathbb{Z} \cong \mathbb{Z}^2$.

The definition of A_θ makes sense for rational θ as well. However, the structure of the *rational rotation algebras* is completely different from the irrational ones. Namely, it can be shown that for an irrational θ the C*-algebra A_θ is simple, while for a rational θ the C*-algebra A_θ contains many non-trivial ideals. In the case $\theta = 0$ we have $A_0 \cong C(\mathbb{T}^2)$. Thus the rotation agebras A_θ are considered noncommutative analogues of the torus.

Chapter 5

K_1-Functor and the Index Map

5.1 The K_1 Functor

5.1.1 Definition of the K_1-group

Let A be a unital C*-algebra. We denote

$$\begin{aligned}
\mathcal{U}(A) &= \text{ the group of unitary elements of } A, \\
\mathcal{U}_n(A) &= \mathcal{U}(M_n(A)), \\
\mathcal{U}_\infty(A) &= \bigcup_{n=1}^{\infty} \mathcal{U}_n(A).
\end{aligned}$$

We define a relation $\sim_1$ in $\mathcal{U}_\infty(A)$ as follows. For $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$ we have $u \sim_1 v$ if and only if there exists $k \geq \max\{n, m\}$ such that $\mathrm{diag}(u, 1_{k-n}) \sim_h \mathrm{diag}(v, 1_{k-m})$. Then $\sim_1$ is an equivalence relation in $\mathcal{U}_\infty(A)$ (exercise). We denote by $[u]_1$ the equivalence class of the unitary $u \in \mathcal{U}_\infty(A)$.

Lemma 5.1. *Let A b a unital C*-algebra. Then*

(i) $[u]_1[v]_1 = [\mathrm{diag}(u, v)]_1$ *is a well-defined associative binary operation on* $\mathcal{U}_\infty(A)/_{\sim_1}$*,*

(ii) $[u]_1[v]_1 = [v]_1[u]_1$ *for all* $u, v \in \mathcal{U}_\infty(A)$*,*

(iii) $[u]_1[1_n]_1 = [1_n]_1[u]_1 = [u]_1$ *for all n and all* $u \in \mathcal{U}_\infty(A)$*,*

(iv) if $u, v \in \mathcal{U}_m(A)$ *then* $[u]_1[v]_1 = [uv]_1$*.*

Proof. Exercise — use Lemma 2.1.6. □

By the above lemma, $\mathcal{U}_\infty(A)/_{\sim_1}$ equipped with the multiplication $[u]_1[v]_1 = [\mathrm{diag}(u, v)]_1$ is an abelian group, with $[u]_1^{-1} = [u^*]_1$.

Definition 5.2. *If A is a C*-algebra then we define*

$$\mathrm{K}_1(A) = \mathcal{U}_\infty(\tilde{A})/_{\sim_1},$$

an abelian group with multiplication $[u]_1[v]_1 = [\mathrm{diag}(u, v)]_1$*.*

When A is *unital* then $\mathrm{K}_1(A)$ may be defined simply as $\mathcal{U}_\infty(A)/_{\sim_1}$ (see Exercise 41. Also, instead of using equivalence classes of *unitaries* one could define K_1 with help of equivalence classes of *invertibles* (see Exercise 45). In particular, the polar decomposition $w = u|w|$ yields a well-defined map

$$[\cdot]_1 : \mathrm{GL}_\infty(A) \to \mathrm{K}_1(A)$$

by $[w]_1 = [u]_1 = [w|w|^{-1}]_1$.

Proposition 5.3 (Universal property of K_1)**.** Let A be a C*-algebra, G an abelian group, and $\nu : \mathcal{U}_\infty(\tilde{A}) \to G$ a map such that:

(i) $\nu(\mathrm{diag}(u, v)) = \nu(u) + \nu(v)$,

(ii) $\nu(1) = 0$,

(iii) if $u, v \in \mathcal{U}_n(\tilde{A})$ and $u \sim_h v$ then $\nu(u) = \nu(v)$.

Then there exists a unique homomorphism $\mathrm{K}_1(A) \to G$ making the diagram

$$\begin{tikzcd} \mathcal{U}_\infty(\tilde{A}) \arrow[d, "{[\cdot]_1}"'] \arrow[dr, "\nu"] & \\ \mathrm{K}_1(A) \arrow[r] & G \end{tikzcd}$$

commutative.

Proof. Exercise. □

5.1.2 Properties of the K_1-functor

Let A, B be C*-algebras and let $\varphi : A \to B$ be a $*$-homomorphism. Then φ extends to unital $*$-homomorphisms $\tilde{\varphi}_n : M_n(\tilde{A}) \to M_n(\tilde{B})$ and thus yields a map $\tilde{\varphi} : \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(\tilde{B})$. We define $\nu : \mathcal{U}_\infty(\tilde{A}) \to \mathrm{K}_1(B)$ by $\nu(u) = [\tilde{\varphi}(u)]_1$ and use the univesal property of K_1 to conclude that there exists a unique homomorphism $\mathrm{K}_1(\varphi) : \mathrm{K}_1(A) \to \mathrm{K}_1(B)$ such that $\mathrm{K}_1([u]_1) = [\tilde{\varphi}(u)]_1$ for $u \in \mathcal{U}_\infty(\tilde{A})$.

Proposition 5.4 (Functoriality of K_1)**.** *Let A, B, C be C*-algebras and let $\varphi : A \to B$ and $\psi : B \to C$ be $*$-homomorphisms. Then*

(i) $\mathrm{K}_1(\mathrm{id}_A) = \mathrm{id}_{\mathrm{K}_1(A)}$,

(ii) $\mathrm{K}_1(\psi \circ \varphi) = \mathrm{K}_1(\psi) \circ \mathrm{K}_1(\varphi)$.

Thus K_1 *is a covariant functor.*

Proof. Exercise. □

It is also clear from the definitions that K_1 of the zero algebra and the zero map are zero.

Proposition 5.5 (Homotopy invariance of K_1)**.** *Let A, B be C*-algebras.*

(i) If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms then $\mathrm{K}_1(\varphi) = \mathrm{K}_1(\psi)$.

(ii) If A and B are homotopy equivalent then $\mathrm{K}_1(A) \cong \mathrm{K}_1(B)$. More specifically, if $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ is a homotopy then $\mathrm{K}_1(\varphi)$ and $\mathrm{K}_1(\psi)$ are isomorphisms and inverses of one another.

Proof. Exercise. □

Theorem 5.6 ((Half)exactness of K_1)**.** *If*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\pi} B \longrightarrow 0 \tag{5.1}$$

is an exact sequence of C^*-algebras then the sequence

$$\mathrm{K}_1(J) \xrightarrow{\mathrm{K}_1(\varphi)} \mathrm{K}_1(A) \xrightarrow{\mathrm{K}_1(\pi)} \mathrm{K}_1(B) \tag{5.2}$$

is exact. If the sequence (5.1) is split-exact with a splitting map $\lambda : B \longrightarrow A$, then the sequence

$$0 \longrightarrow \mathrm{K}_1(J) \xrightarrow{\mathrm{K}_1(\varphi)} \mathrm{K}_1(A) \xrightarrow{\mathrm{K}_1(\pi)} \mathrm{K}_1(B) \longrightarrow 0 \tag{5.3}$$

is split-exact with a splitting map $\mathrm{K}_1(\lambda) : \mathrm{K}_1(B) \longrightarrow \mathrm{K}_1(A)$.

Proof. $\mathrm{K}_1(\pi) \circ \mathrm{K}_1(\varphi) = \mathrm{K}_1(\pi \circ \varphi) = \mathrm{K}_1(0) = 0$ by functoriality of K_1, and hence $\mathrm{im}(\mathrm{K}_1(\varphi)) \subseteq \ker(\mathrm{K}_1(\pi))$. For the reverse inclusion, let $u \in \mathcal{U}_n(\tilde{A})$ and $\mathrm{K}_1(\pi)([u]_1) = [1]_1$. Then there is m such that $\mathrm{diag}(\tilde{\pi}(u), 1_n) \sim_h 1_{n+m}$. By Lemma 2.1.8, there is $w \in \mathcal{U}_{n+m}(\tilde{A})$ such that $w \sim_h 1_{n+m}$ and $\tilde{\pi}(w) = \mathrm{diag}(\tilde{\pi}(u), 1_m)$. Thus $[u]_1 = [\mathrm{diag}(u, 1_n)w^*]_1$ and $\tilde{\pi}(\mathrm{diag}(u, 1_n)w^*) = 1_{n+m}$. By Exercise 4.4.5, there is $v \in \mathcal{U}_{n+m}(\tilde{J})$ such that $\tilde{\varphi}(v) = \mathrm{diag}(u, 1_n)w^*$. Thus $[u]_1 \in \mathrm{im}(\mathrm{K}_1(\varphi))$ and consequently $\ker(\mathrm{K}_1(\pi)) \subseteq \mathrm{im}(\mathrm{K}_1(\varphi))$. This shows that the sequence (5.2) is exact.

Now suppose that the sequence (5.1) is split-exact. Then the sequence (5.3) is exact at $\mathrm{K}_1(A)$ by the preceding argument. By functoriality of K_1 we have $\mathrm{K}_1(\pi) \circ \mathrm{K}_1(\lambda) = \mathrm{id}_{\mathrm{K}_1(B)}$, and hence (5.3) is exact at $\mathrm{K}_1(B)$ (and $\mathrm{K}_1(\lambda)$ is a splitting map). It remains to show that $\mathrm{K}_1(\varphi)$ is injective. So let $u \in \mathcal{U}_n(\tilde{J})$ be such that $\mathrm{K}_1(\varphi)([u]_1) = [1]_1$. Then there is m such that $\mathrm{diag}(\tilde{\varphi}(u), 1_m) \sim_h 1_{n+m}$. Let $t \mapsto w_t$ be a continuous path in $\mathcal{U}_{n+m}(\tilde{A})$ connecting $\mathrm{diag}(\tilde{\varphi}(u), 1_m)$ and 1_{n+m}. We would like to apply $\tilde{\varphi}^{-1}$ to w_t to conclude that $\mathrm{diag}(u, 1_m)$ is homotopic to the identity. In general, this is impossible since some of w_t may lie outside the range of $\tilde{\varphi}$. However, in the presence of a splitting map λ we can correct the path w_t by setting $v_t = w_t(\tilde{\lambda} \circ \tilde{\pi})(w_t^*)$. Then v_t is a continuous path in $\mathcal{U}_{n+m}(\tilde{A})$ connecting $\mathrm{diag}(\tilde{\varphi}(u), 1_m)(\tilde{\lambda} \circ \tilde{\pi})(\mathrm{diag}(\tilde{\varphi}(u^*), 1_m))$ and 1_{n+m}. Since $\tilde{\pi}(v_t) = 1_{n+m}$ for all t, Exercise 4.4.5 implies that each v_t is in the image of $\tilde{\varphi}$. Thus $\tilde{\varphi}^{-1}(v_t)$ is a continuous path in $\mathcal{U}_{n+m}(\tilde{J})$ connecting $\mathrm{diag}(u, 1_m)\tilde{\varphi}^{-1}((\tilde{\lambda} \circ \tilde{\pi})(\mathrm{diag}(\tilde{\varphi}(u^*), 1_m)))$ and 1_{n+m}. Since $\tilde{\varphi}^{-1}((\tilde{\lambda} \circ \tilde{\pi})(\mathrm{diag}(\tilde{\varphi}(u^*), 1_m)))$ is a scalar matrix, it is homotopic to the identity. Thus

$$[u]_1 = [\mathrm{diag}(u, 1_m)]_1 = [\mathrm{diag}(u, 1_m)\tilde{\varphi}^{-1}((\tilde{\lambda} \circ \tilde{\pi})(\mathrm{diag}(\tilde{\varphi}(u^*), 1_m)))]_1 = [1]_1,$$

and the map $\mathrm{K}_1(\varphi)$ is injective. □

Proposition 5.7 (Continuity of K_1)**.** *Let $A = \varinjlim\{A_i, \Phi_{ij}\}$ be the inductive limit of a sequence of C^*-algebras, and let $\Phi_i : A_i \to A$ be the canonical maps.*

Let $G = \varinjlim\{\mathrm{K}_1(A_i), \mathrm{K}_1(\Phi_{ij})\}$ be the inductive limit of the corresponding sequence of abelian groups, and let $\varphi_i : \mathrm{K}_1(A_i) \to G$ be the canonical maps. Then there exists an isomorphism $\Lambda : G \to \mathrm{K}_1(A)$ such that for all $i \geq j$ the diagram

$$\begin{tikzcd} \mathrm{K}_1(A_j) \arrow[r, "\varphi_j"] \arrow[d, "\mathrm{K}_1(\Phi_{ij})"'] \arrow[dr] & G \arrow[d, "\Lambda"] \\ \mathrm{K}_1(A_i) \arrow[r, "\mathrm{K}_1(\Phi_i)"'] \arrow[ur] & \mathrm{K}_1(A) \end{tikzcd} \tag{5.4}$$

is commutative.

Proof. The universal property of the direct limit G of the sequence $\{\mathrm{K}_1(A_i), \mathrm{K}_1(\Phi_{ij})\}$ yields a unique homomorphism $\Lambda : G \to \mathrm{K}_1(A)$ making the diagram (5.4) commutative. We must show that Λ is surjective and injective.

Surjectivity. Let $u \in \mathcal{U}_n(\tilde{A})$. By part (ii) of Exercise (48), there is i and $w \in \mathcal{U}_n(\tilde{A}_i)$ such that $||u - \tilde{\Phi}_i(w)|| < 2$. Thus u and $\tilde{\Phi}_i(w)$ are homotopic in $\mathcal{U}_n(\tilde{A})$ by Lemma 2.1.4. Hence

$$[u]_1 = [\tilde{\Phi}_i(w)]_1 = \mathrm{K}_1(\Phi_i)([w]_1) = (\Lambda \circ \varphi_i)([w]_1),$$

and Λ is surjective.

Injectivity. It suffices to show that for each j the restriction of Λ to the image of φ_j is injective. So let $u \in \mathcal{U}_n(\tilde{A}_j)$ be such that $(\Lambda \circ \varphi_j)([u]_1) = \mathrm{K}_1(\Phi_j)([u]_1) = [\tilde{\Phi}_j(u)]_1 = [1]_1$ in $\mathrm{K}_1(A)$. We must show that $\varphi_j([u]_1) = 0$ in G. Indeed, there is m such that $\mathrm{diag}(\tilde{\Phi}_j(u), 1_m) \sim_h 1_{n+m}$ in $\mathcal{U}_{n+m}(\tilde{A})$. By part (iii) of Exercise (48), there is $i \geq j$ such that $\mathrm{diag}(\tilde{\Phi}_{ij}(u), 1_m)$ is homotopic to 1_{n+m}. Thus $[\tilde{\Phi}_{ij}(u)]_1 = [\mathrm{diag}(\tilde{\Phi}_{ij}(u), 1_m)]_1 = [1]_1$. Consequently, $\varphi_j([u]_1) = (\varphi_i \circ \mathrm{K}_1(\tilde{\Phi}_{ij}))([u]_1) = 0$, and Λ is injective. $\square$

Proposition 5.8 (Stability of K_1)**.** *Let A be a C^*-algebra.*

(i) For each $n \in \mathbb{N}$ we have

$$\mathrm{K}_1(A) \cong \mathrm{K}_1(M_n(A)).$$

More specifically, let $\psi : A \to M_n(A)$ be such that $\psi(a) = \mathrm{diag}(a, 0_{n-1})$. Then $\mathrm{K}_1(\psi) : \mathrm{K}_1(A) \to \mathrm{K}_1(M_n(A))$ is an isomorphism.

(ii) Let $\mathcal{K}$ be the C^*-algebra of compact operators. Then

$$\mathrm{K}_1(A) \cong \mathrm{K}_1(A \otimes \mathcal{K}).$$

More specfically, let p be a minimal projection in $\mathcal{K}$ and let $\varphi : A \to A \otimes \mathcal{K}$ be the map such that $\varphi(a) = a \otimes p$. Then $\mathrm{K}_1(\varphi) : \mathrm{K}_1(A) \to \mathrm{K}_1(A \otimes \mathcal{K})$ is an isomorphism.

Proof. (i) Exercise.

(ii) Since

$$A \otimes \mathcal{K} \cong A \otimes (\lim M_n(\mathbb{C})) \cong \lim M_n(A),$$

the claim follows from part (i) and continuity of K_1. $\square$

5.2 The Index Map

5.2.1 Fredholm index

Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space. We denot by $\mathcal{F}$ the algebra of finite rank operators on $\mathcal{H}$ (a two-sided $*$-ideal in $\mathcal{B}(\mathcal{H})$), by $\mathcal{K}$ the C*-algebra of compact operators on $\mathcal{H}$ (the norm closure of $\mathcal{F}$ and the only non-trivial, norm closed, two-sided ideal of $\mathcal{B}(\mathcal{H})$), by $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}$ the Calkin algebra, and by $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}$ the natural surjection.

Theorem 5.9 (Atkinson)**.** *If $F \in \mathcal{B}(\mathcal{H})$ then the following conditions are equivalent.*

(i) Both $\ker(F)$ *and* $\operatorname{coker}(F)$ *are finite dimensional.*

(ii) There exists an operator $G \in \mathcal{B}(\mathcal{H})$ such that both $FG - 1$ and $GF - 1$ are compact.

(iii) The image $\pi(F)$ of F in the Calkin algebra $\mathcal{Q}$ is invertible.

Furthermore, if F satisfies the above conditions then the range of F is closed in $\mathcal{H}$.

Proof. Obviously, conditions (ii) and (iii) are equivalent.

(i)$\Rightarrow$(ii) We first observe that the image of F is a closed subspace of $\mathcal{H}$. Indeed, let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$ of smallest possible dimension such that $\operatorname{im}(F)+\mathcal{H}_0 = \mathcal{H}$. Then $n = \dim(\mathcal{H}_0)$ is finite, since the cokernel of F is finite dimensional. Then the restriction of F to the orthogonal complement of its kernel is a bijection from $\ker(F)^\perp$ onto $\operatorname{im}(F)$, and it extends to a linear bijection $\tilde{F} : \ker(F)^\perp \oplus \mathbb{C}^n \longrightarrow \operatorname{im}(F) + \mathcal{H}_0 = \mathcal{H}$. By the Inverse Mapping Theorem, the inverse of $\tilde{F}$ is continuous. It follows that $\operatorname{im}(F) = \tilde{F}(\ker(F)^\perp)$ is closed in $\mathcal{H}$.

By the preceding argument, F yields a continuous linear bijection from $\ker(F)^\perp$ onto $\operatorname{im}(F)$ — a closed subspace of $\mathcal{H}$. Thus, by the Inverse Mapping Theorem, it has a continuous inverse $G : \operatorname{im}(F) \to \ker(F)^\perp$. Extend G to a bounded linear operator on $\mathcal{H}$ (still denoted G) by setting $G\xi = 0$ for $\xi \in \operatorname{im}(F)^\perp$. Then both $FG - 1$ and $GF - 1$ are finite dimensional and (ii) holds.

(ii)$\Rightarrow$(i) Let K be a compact operator such that $GF = 1 + K$. Then $\ker(F) \subseteq \ker(GF) = \ker(1 + K)$, and $\ker(1 + K)$ is the eigenspace of K corresponding to eigenvalue -1. Since K is compact this eigenspace is finite dimensional and so is the kernel of F. We also have $\operatorname{im}(F) \supseteq \operatorname{im}(FG) = \operatorname{im}(1 + K)$. Since $1 + K$ can be written as an invertible plus a finite rank operator, its range has finite codimenson. Thus $\operatorname{coker}(F)$ is finite dimensional. $\square$

A bounded operator satisfying the conditions of Theorem 5.9 is called *Fredholm*. In particular, any invertible operator in $\mathcal{B}(\mathcal{H})$ is Fredholm. It follows immediately from Theorem 5.9 that if F, T are Fredholm and K is compact then the operators F^*, FT and $F + K$ are Fredholm.

If F, G are Fredholm operators satisfying condition (ii) of Theorem 5.9, then G is called *parametrix* of F.

Definition 5.10 (Fredholm index)**.** *Let F be a Fredholm operator. Then its* Fredholm index *is an integer defined as*

$$\operatorname{Index}(F) = \dim(\ker(F)) - \dim(\operatorname{coker}(F)).$$

Since $\dim(\operatorname{coker}(F)) = \dim(\ker(F^*))$, we have $\operatorname{Index}(F) = -\operatorname{Index}(F^*)$. If F is Fredholm and V is invertible then clearly $\operatorname{Index}(FV) = \operatorname{Index}(VF) = \operatorname{Index}(F)$ and $\operatorname{Index}(V) = 0$.

Let $\{\xi_n : n = 0, 1, \dots\}$ be an orthonormal basis of $\mathcal{H}$. The operator $S \in \mathcal{B}(\mathcal{H})$ such that $S(\xi_n) = \xi_{n+1}$ is called *unilateral shift*. It is a Fredholm operator with index -1. Thus for any positive integer k we have $\operatorname{Index}(S^k) = -k$ and $\operatorname{Index}((S^*)^k) = k$.

In a finite dimensional Hilbert space all operators are compact and hence all operators are Fredholm. The rank-nullity theorem of elementary linear algebra may then be interpreted as saying that every Fredholm operator on a finite dimensional Hilbert space has index 0.

Theorem 5.11 (Riesz)**.** *If F is Fredholm and K is compact then*

$$\operatorname{Index}(F + K) = \operatorname{Index}(F).$$

Proof. We first observe that if R is of finite rank then $\operatorname{Index}(1+R) = 0$. Indeed, let $\mathcal{H}_0 = \operatorname{im}(R) + \ker(R)^\perp$. Then $\mathcal{H}_0$ is finite dimensional, and the restriction of both R and R^* to $\mathcal{H}_0^\perp$ is zero. Thus the index of $1 + R$ coincides with the index of its restriction to $\mathcal{H}_0$ and hence is 0.

Now let K be compact. Find R of finite rank such that $||K - R|| < 1$. Then $V = 1 + (K - R)$ is invertible. Hence

$$\operatorname{Index}(1 + K) = \operatorname{Index}(V + R) = \operatorname{Index}(V(1 + V^{-1}R)) = 0.$$

Let F be a Fredholm operator of index 0. Then there is a finite rank operator R such that R maps bijectively $\ker(F)$ onto $\operatorname{im}(F)^\perp = \ker(F^*)$. Let $V = F + R$. Then V is a continuous linear bijection of $\mathcal{H}$ onto itself and hence it is an invertible operator. Thus if K is compact then

$$\operatorname{Index}(F + K) = \operatorname{Index}(V + (K - R)) = \operatorname{Index}(V(1 + V^{-1}(K - R))) = 0.$$

Finally, let F be an arbitrary Fredholm operator and let K be compact. Then $\operatorname{Index}(F \oplus F^*) = 0$ and hence $\operatorname{Index}((F + K) \oplus F^*) = 0$. Consequenty, $\operatorname{Index}(F + K) = -\operatorname{Index}(F^*) = \operatorname{Index}(F)$. □

We showed in the course of the proof of Theorem 5.11 that if F is a Fredholm operator with index 0 then there exists a finite rank operator R such that $F + R$ is invertible.

Corollary 5.12. *If F, T are Fredholm operators then*

$$\operatorname{Index}(FT) = \operatorname{Index}(F) + \operatorname{Index}(G).$$

Proof. Suppose first that $\operatorname{Index}(F) = 0$, and let R be an operator of finite rank such that $F + R$ is invertible. Then

$$\operatorname{Index}(FT) = \operatorname{Index}(FT + RT) = \operatorname{Index}((F + R)T) = \operatorname{Index}(T).$$

Now suppose that $\mathrm{Index}(F) = k > 0$, and let S be a unilateral shift on $\mathcal{H}$. Then $\mathrm{Index}(F \oplus S^k) = 0$ and hence

$$\mathrm{Index}(FT \oplus S^k) = \mathrm{Index}((F \oplus S^k)(T \oplus 1)) = \mathrm{Index}(T \oplus 1) = \mathrm{Index}(T).$$

Consequently, we have

$$\mathrm{Index}(FT) = -\mathrm{Index}(S^k) + \mathrm{Index}(T) = \mathrm{Index}(F) + \mathrm{Index}(T),$$

as required. □

In particular, if G is a parametrix of F then $\mathrm{Index}(G) = -\mathrm{Index}(F)$.

Proposition 5.13. *The index map is locally constant and continuous in norm.*

Proof. Let F be a Fredholm operator and let G be its parametrix. Let K be compact such that $FG = 1 + K$. It suffices to show that if T is a Fredholm operator with $||T - F|| < 1/||G||$ then $\mathrm{Index}(F) = \mathrm{Index}(T)$. Indeed, the operator $(T - F)G + 1$ is invertible, since its distance from the identity is less than 1. Thus

$$\begin{aligned}\mathrm{Index}(T) + \mathrm{Index}(G) &= \mathrm{Index}(TG)\\ &= \mathrm{Index}((T - F + F)G)\\ &= \mathrm{Index}((T - F)G + 1) + K) = 0.\end{aligned}$$

Thus $\mathrm{Index}(T) = -\mathrm{Index}(G) = \mathrm{Index}(F)$. □

If F, T are two Fredholm operators then we say that they are *homotopic* if there exists a norm continuos path from F to T consisting of Fredholm operators.

Proposition 5.14. *Two Fredholm operators are homotopic if and only if they have the same index.*

Proof. Let F and T be Fredholm operators.

Suppose that F and T are homotopic, and let $t \mapsto V_t$ be a continuous path of Fredholm operators from F to T. Then the map $t \mapsto \mathrm{Index}(V_t)$ is continuous and hence constant.

To show the converse we first observe that every Fredholm operator V with $\mathrm{Index}(V) = 0$ is homotopic to 1. Indeed, there is a finite rank operator such that $V + R$ is invertible. Then $t \mapsto V + tR$ is a path connecting V to an invertible element, and in $\mathcal{B}(\mathcal{H})$ the group of invertibles is path-connected.

Now suppose that $\mathrm{Index}(F) = \mathrm{Index}(T)$. Then both FT^* and T^*T have index 0 and thus are homotopic to 1. Consequently, the operators F, $F(T^*T) = (FT^*)T$ and T are homotopic. □

Let u be a unitary in $M_n(\mathcal{Q})$ and let $U \in M_n(B(\mathcal{H}))$ be such that $\tilde{\pi}(U) = u$. Then U is a Fredholm operator on $\oplus^n \mathcal{H}$. Define a map $\mu : \mathcal{U}_\infty(\mathcal{Q}) \to \mathbb{Z}$ by $\mu(u) = \mathrm{Index}(U)$. It follows from the properties of Fredholm operators that μ satisfies conditions (i)–(iii) of the universal property of K_1. Thus, there exists a homomorphism $\mathrm{Index} : \mathrm{K}_1(\mathcal{Q}) \to \mathbb{Z}$ such that $\mathrm{Index}([u]_1) = \mu(U) = \mathrm{Index}(U)$. It is not difficult to see that Index is an somorphism. Thus $\mathrm{K}_1(\mathcal{Q}) \cong \mathbb{Z}$.

Since $\mathcal{Q}$ is properly infinite, there is no need to go to matrices over $\mathcal{Q}$ and we have $\mathrm{K}_1(\mathcal{Q}) = \{[u]_1 : u \in \mathcal{U}(\mathcal{Q})\}$ (see Exercise 47). Furthermore, every unitary

u in $\mathcal{Q}$ lifts to a *partial isometry* U in $\mathcal{B}(\mathcal{H})$ (Exercise 49). Thus $\dim(\ker(U))$ equals the rank of $1 - U^*U$ and can be identified with the element $[1 - U^*U]_0$ in $\mathrm{K}_0(\mathcal{K})$. Likewise, $\dim(\mathrm{coker}(U))$ equals the rank of $1 - UU^*$ and can be identified with the element $[1 - UU^*]_0$ in $\mathrm{K}_0(\mathcal{K})$. Consequently, we can view the index map as an isomorphism

$$\mathrm{Index} : \mathrm{K}_1(\mathcal{Q}) \to \mathrm{K}_0(\mathcal{K}),$$

such that if U is a partial isometry lift of u then

$$\mathrm{Index}([u]_1) = [1 - UU^*]_0 - [1 - UU^*]_0.$$

5.2.2 Definition of the index map

Let

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{5.5}$$

be an exact sequence of C*-algebras. Let $u \in \mathcal{U}_n(\tilde{B})$. Then there exists a unitary V in $\mathcal{U}_{2n}(\tilde{A})$ such that

$$\tilde{\psi}(V) = \mathrm{diag}(u, u^*). \tag{5.6}$$

Then $\tilde{\psi}(V\,\mathrm{diag}(1_n, 0)V^*) = \mathrm{diag}(1_n, 0)$. Thus there exists a projection P in $\mathcal{P}_{2n}(\tilde{J})$ such that

$$\tilde{\varphi}(P) = V\,\mathrm{diag}(1_n, 0)V^*. \tag{5.7}$$

Since $(\tilde{\psi} \circ \tilde{\varphi})(P) = \mathrm{diag}(1_n, 0)$, it follows that $s(P) = \mathrm{diag}(1_n, 0)$, where s is the scalar map. Then there is a well-defined map

$$\mu : \mathcal{U}_\infty(\tilde{B}) \to \mathrm{K}_0(J) \text{ such that } \mu(u) = [P]_0 - [s(P)]_0.$$

Indeed, suppose that $W \in \mathcal{U}_{2n}(\tilde{A})$ and $Q \in \mathcal{P}_{2n}(\tilde{J})$ are such that $\tilde{\psi}(W) = \mathrm{diag}(u, u^*)$ and $\tilde{\varphi}(Q) = W\,\mathrm{diag}(1_n, 0)W^*$. We must show that $[P]_0 - [s(P)]_0 = [Q]_0 - [s(Q)]_0$ in $\mathrm{K}_0(J)$. Indeed since $\tilde{\psi}(VW^*) = 1_{2n}$ there is $Y \in \mathcal{U}_{2n}(\tilde{J})$ such that $\tilde{\varphi}(Y) = VW^*$. Since

$$\tilde{\varphi}(P) = VW^*\tilde{\varphi}(Q)(VW^*)^* = \tilde{\varphi}(YQY^*),$$

we have $P = YQY^*$ and the claim follows. That is, $\mu : \mathcal{U}_\infty(\tilde{B}) \to \mathrm{K}_0(J)$ is well-defined.

This map μ satisfies conditions (i)–(iii) of Proposition 5.3. We only verify (iii), leaving (i) and (ii) as exercise. So let $u \sim_h w \in \mathcal{U}_n(\tilde{B})$, $U, W \in \mathcal{U}_{2n}(\tilde{A})$, $P, Q \in \mathcal{P}_{2n}(\tilde{J})$ be such that $\tilde{\psi}(U) = \mathrm{diag}(u, u^*)$, $\tilde{\psi}(W) = \mathrm{diag}(w, w^*)$, $\tilde{\varphi}(P) = U\,\mathrm{diag}(1_n, 0)U^*$ and $\tilde{\varphi}(Q) = W\,\mathrm{diag}(1_n, 0)W^*$ (that is, $\{u, U, P\}$ and $\{w, W, Q\}$ satisfy conditions (5.6) and (5.7), respectively). Then $u^*w \sim_h 1_n \sim_h uw^*$ and thus there exist $X, Y \in \mathcal{U}_n(\tilde{A})$ such that $\tilde{\psi}(X) = u^*w$ and $\tilde{\psi}(Y) = uw^*$. Put $Z = U\,\mathrm{diag}(X, Y)$, a unitary in $\mathcal{U}_{2n}(\tilde{A})$. We have $\tilde{\psi}(Z) = \mathrm{diag}(w, w^*)$ and $\tilde{\varphi}(P) = Z\,\mathrm{diag}(1_n, 0)Z^*$. Thus, by the definition of μ, we have $\mu(w) = [P]_0 - [s(P)]_0 = \mu(u)$. The universal property of K_1 now implies that there exists a homomorphism

$$\partial_1 : \mathrm{K}_1(B) \longrightarrow \mathrm{K}_0(J),$$

called the *index map*, such that

$$\partial_1([u]_1) = [P]_0 - [s(P)]_0.$$

5.2.3 The exact sequence

Theorem 5.15. *Let*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be an exact sequence of C^*-algebras. Then the sequence

$$\begin{tikzcd} \mathrm{K}_0(J) \arrow[r, "\mathrm{K}_0(\varphi)"] & \mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(\psi)"] & \mathrm{K}_0(B) \\ \mathrm{K}_1(B) \arrow[u, "\partial_1"] & \mathrm{K}_1(A) \arrow[l, "\mathrm{K}_1(\psi)"] & \mathrm{K}_1(J) \arrow[l, "\mathrm{K}_1(\varphi)"] \end{tikzcd} \tag{5.8}$$

is exact everywhere.

Proof. By virtue of half-exactness of K_0 and K_1, it suffices to prove that $\mathrm{im}(\mathrm{K}_1(\psi)) = \ker(\partial_1)$ and $\mathrm{im}(\partial_1) = \ker(\mathrm{K}_0(\varphi))$.

1. We show $\mathrm{im}(\mathrm{K}_1(\psi)) \subseteq \ker(\partial_1)$. Indeed, if $U \in \mathcal{U}_n(\tilde{A})$ then $\mathrm{diag}(\tilde{\psi}(U), \tilde{\psi}(U)^*)$ lifts to a diagonal unitary $V = \mathrm{diag}(U, U^*)$ and $\tilde{\varphi}(1_n) = V\,\mathrm{diag}(1_n, 0)V^* = \mathrm{diag}(1_n, 0)$. Thus $\partial_1(\mathrm{K}_1(\psi)([U]_1)) = \partial_1([\tilde{\psi}(U)]_1) = [1_n]_0 - [s(1_n)]_0 = 0$.

2. We show $\mathrm{im}(\mathrm{K}_1(\psi)) \supseteq \ker(\partial_1)$. To simplify notation, we identify J with its image in A and thus put $\varphi = \mathrm{id}$. Let $u \in \mathcal{U}_n(\tilde{B})$ be such that $[u]_1 \in \ker(\partial_1)$. By Exercise 53, there is a partial isometry $U \in M_{2n}(\tilde{A})$ such that

$$\tilde{\psi}(U) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$0 = \partial_1([u]_1) = [1_{2n} - U^*U]_0 - [1_{2n} - UU^*]_0 \text{ in } \mathrm{K}_0(J).$$

Thus there is k and $w \in M_{2n+k}(\tilde{J})$ such that

$$w^*w = (1_{2n} - U^*U) \oplus 1_k \quad \text{and} \quad ww^* = (1_{2n} - UU^*) \oplus 1_k.$$

Hence

$$\tilde{\psi}(w^*w) = \tilde{\psi}(ww^*) = \begin{pmatrix} 0 & 0 \\ 0 & 1_{n+k} \end{pmatrix}$$

and $\tilde{\psi}(w)$ is a scalar matrix, since $w \in M_{2n+k}(\tilde{J})$. Consequently,

$$\tilde{\psi}(w) = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix},$$

with z a scalar unitary matrix in $M_{n+k}(\tilde{B})$. In particular, z is homotopic to 1_{n+k} in $\mathcal{U}_{n+k}(\tilde{B})$. Set

$$V = w + \begin{pmatrix} U & 0 \\ 0 & 0_k \end{pmatrix},$$

an element of $M_{2n+k}(\tilde{A})$. By Exercise 56, V is unitary. We have

$$\tilde{\psi}(V) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & 1_{n+k} \end{pmatrix}.$$

Thus $[u]_1 - [\tilde{\psi}(V)]_1 = \mathrm{K}_1(\psi)([V]_1)$.

3. We show $\mathrm{im}(\partial_1) \subseteq \ker(\mathrm{K}_0(\varphi))$. Indeed, let $u \in \mathcal{U}_n(\tilde{B})$ and let $V \in \mathcal{U}_{2n}(\tilde{A})$, $P \in \mathcal{P}_{2n}(\tilde{J})$ be such that (5.6) and (5.7) hold. Then we have

$$\mathrm{K}_0(\varphi)([P]_0 - [s(P)]_0) = [\tilde{\varphi}(P)]_0 - [1_n]_0 = [V \operatorname{diag}(1_n, 0)V^*]_0 - [1_n]_0 = 0.$$

4. We show $\mathrm{im}(\partial_1) \supseteq \ker(\mathrm{K}_0(\varphi))$. Let $g \in \ker(\mathrm{K}_0(\varphi))$. By Lemma 4.5, there is n, a projection $p \in \mathcal{P}_n(\tilde{J})$ and a unitary $w \in \mathcal{U}_n(\tilde{A})$ such that

$$g = [p]_0 - [s(p)]_0 \quad \text{and} \quad w\tilde{\varphi}(p)w^* = s(p).$$

Set $u_0 = \tilde{\psi}(w(1_n - \tilde{\varphi}(p)))$, a partial isometry in $M_n(\tilde{B})$. We have

$$\begin{aligned} u_0^* u_0 &= 1_n - \tilde{\psi}(\tilde{\varphi}(p)), \\ u_0 u_0^* &= 1_n - \tilde{\psi}(s(p)) = u_0^* u_0. \end{aligned}$$

Thus $u = u_0 + (1_n - u_0 u_0^*)$ is unitary in $M_n(\tilde{B})$. We want to show that $g = \partial_1([u]_1)$. To this end, we frst find a lift of $\operatorname{diag}(u, 0_n)$ to a suitable partial isometry in $M_{2n}(\tilde{A})$. Take

$$V_0 = \begin{pmatrix} w(1_n - \tilde{\varphi}(p)) & 0 \\ 0 & s(p) \end{pmatrix},$$

a partial isometry in $M_{2n}(\tilde{A})$ such that

$$\tilde{\psi}(V_0) = \begin{pmatrix} u_0 & 0 \\ 0 & s(p) \end{pmatrix}.$$

Set

$$Z = \begin{pmatrix} 1_n - s(p) & s(p) \\ s(p) & 1_n - s(p) \end{pmatrix},$$

a self-adjoint, unitary scalar matrix, and put $V = ZV_0Z^*$. Then we have

$$\tilde{\psi}(V) = Z\tilde{\psi}(V_0)Z^* = Z \begin{pmatrix} u_0 & 0 \\ 0 & s(p) \end{pmatrix} Z^* = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, by Exercise 53,

$$\begin{aligned} \partial_1([u]_1) &= [\tilde{\varphi}^{-1}(1_{2n} - V^*V)]_0 - [\tilde{\varphi}^{-1}(1_{2n} - VV^*)]_0 \\ &= [\tilde{\varphi}^{-1}(1_{2n} - V_0^*V_0)]_0 - [\tilde{\varphi}^{-1}(1_{2n} - V_0V_0^*)]_0 \\ &= \left[\begin{pmatrix} p & 0 \\ 0 & 1_n - s(p) \end{pmatrix}\right]_0 - \left[\begin{pmatrix} s(p) & 0 \\ 0 & 1_n - s(p) \end{pmatrix}\right]_0 \\ &= [p]_0 - [s(p)]_0 = g. \end{aligned}$$

That is, $g = \partial_1([u]_1)$, as required.

□

5.3 Examples and Exercises

Exercise 41. Let A be a *unital* C*-algebra. We have $\tilde{A} = A \oplus \mathbb{C}f$, where $f = 1_{\tilde{A}} - 1_A$. Define a unital $*$-homomorphism $\mu : \tilde{A} \to A$ by $\mu(a+\lambda f) = a$. As usual, for each n extend μ to a unital $*$-homomorphism $M_n(\tilde{A}) \to M_n(A)$ (still denoted μ). This yields a map $\mu : \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(A)$. Show that there exists an isomorphism $\mathrm{K}_1(A) \to \mathcal{U}_\infty(A)/_{\sim_1}$ making the diagram

$$\begin{tikzcd}
\mathcal{U}_\infty(\tilde{A}) \arrow[r, "\mu"] \arrow[d, "{[\cdot]_1}"'] & \mathcal{U}_\infty(A) \arrow[d] \\
\mathrm{K}_1(A) \arrow[r] & \mathcal{U}_\infty(A)/_{\sim_1}
\end{tikzcd}$$

commutative. To this end, show the following:

(i) $\mu(\mathrm{diag}(u,v)) = \mathrm{diag}(\mu(u), \mu(v))$,

(ii) if $u, v \in \mathcal{U}_n(\tilde{A})$ then $\mu(u) \sim_h \mu(v)$ if and only if $u \sim_h v$,

(iii) if $u, v \in \mathcal{U}_\infty(\tilde{A})$ then $\mu(u) \sim_1 \mu(v)$ if and only if $u \sim_1 v$.

(ii) Let $\mu(u) \sim_h \mu(v)$. By the definition of μ, there exist unitary $u_0, v_0 \in \mathcal{U}_n(\mathbb{C}f)$ such that $u = \mu(u) + u_0$ and $v = \mu(v) + v_0$. Since the unitary group of $M_n(\mathbb{C})$ is path-connected we have $u_0 \sim_h v_0$ in $\mathcal{U}_n(\mathbb{C}f)$. It follows that $u \sim_h v$ in $\mathcal{U}_n(\tilde{A})$.

Exercise 42. Show the following.

(i) $\mathrm{K}_1(\mathbb{C}) = 0$.

(ii) For any two C*-algebras A, B we have $\mathrm{K}_1(A \oplus B) \cong \mathrm{K}_1(A) \oplus \mathrm{K}_1(B)$.

(iii) If A is an AF-algebra (see Example 4.4.14) then $\mathrm{K}_1(A) = 0$.

Example 5.16. If $\mathcal{H}$ is an infinite dimensional Hilbert space then $\mathrm{K}_1(\mathcal{B}(\mathcal{H})) = 0$. Indeed, since $\mathcal{U}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{U}(\mathcal{B}(\oplus^n \mathcal{H}))$, it suffices to show that every unitary in $\mathcal{B}(\mathcal{H})$ is homotopic to the identity. But this follows from the fact that for every unitary u in $\mathcal{B}(\mathcal{H})$ there is a self-adjoint $a \in \mathcal{B}(\mathcal{H})$ such that $u = \exp(ia)$. Indeed, one may take $a = \varphi(u)$, where $\varphi : \mathbb{T} \to [0, 2\pi)$ is a bounded Borel function such that $\varphi(e^{i\theta}) = \theta$.

Exercise 43. Let X be a compact Hausdorff space.

(i) For each n identify $M_n(C(X))$ with $C(X, M_n(\mathbb{C}))$ and define the *determinant* function $\det : M_n(C(X)) \to C(X)$. Show that det maps $\mathcal{U}_\infty(C(X))$ into $\mathcal{U}(C(X))$.

(ii) Let $\langle v \rangle$ denote the class of $v \in \mathcal{U}(C(X))$ in $\mathcal{U}(C(X))/\mathcal{U}_0(C(X))$. Apply the universal property of K_1 to the map $\mathcal{U}_\infty(C(X)) \ni u \mapsto \langle \det(u) \rangle$ to show that there exists a homomorphism $D : \mathrm{K}_1(A) \to \mathcal{U}(C(X))/\mathcal{U}_0(C(X))$ such that $D([u]_1) = \langle \det(u) \rangle$.

(iii) Show that the sequence

$$0 \longrightarrow \ker(D) \longrightarrow \mathrm{K}_1(C(X)) \xrightarrow{D} \mathcal{U}(C(X))/\mathcal{U}_0(C(X)) \longrightarrow 0$$

is split-exact, with a splitting map $\omega : \mathcal{U}(C(X))/\mathcal{U}_0(C(X)) \to \mathrm{K}_1(C(X))$ given by $\omega(\langle u \rangle) = [u]_1$.

(iv) Let $X = \mathbb{T}$. Recall that $\varphi : \mathbb{R} \to \mathbb{T}$, $\varphi(x) = e^{2\pi i x}$, is a covering map. Thus, if $u \in \mathcal{U}(C(\mathbb{T}))$ then there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that $u(e^{2\pi i t}) = e^{2\pi i f(t)}$. If f, g are two such functions then $f - g$ is a constant integer. Thus, there is a well-defined map $\mu : \mathcal{U}(C(\mathbb{T})) \to \mathbb{Z}$ given by $\mu(u) = f(1) - f(0)$ (the *winding number* of u). Show that μ induces an isomorphism of $\mathcal{U}(C(\mathbb{T}))/\mathcal{U}_0(C(\mathbb{T}))$ and $\mathbb{Z}$ such that $\langle u \rangle \mapsto \mu(u)$.

(v) Conclude that there exists a surjective homomorphism from $\mathrm{K}_1(C(\mathbb{T}))$ onto $\mathbb{Z}$. In fact, we will see later that $\mathrm{K}_1(C(\mathbb{T})) \cong \mathbb{Z}$.

Exercise 44. If A is a separable C*-algebra then $\mathrm{K}_1(A)$ is countable.

Exercise 45. Let A be a unital C*-algebra. Replacing unitaries $\mathcal{U}_\infty(A)$ with invertibles $\mathrm{GL}_\infty(A)$ one can repeat the constructions from Section 5.1.1 and define an abelian group $\mathrm{GL}_\infty(A)/_{\sim_1}$. Show that this group is isomorphic to $\mathrm{K}_1(A) = \mathcal{U}_\infty(A)/_{\sim_1}$ (see Exercise 41). Hint: For $w \in \mathrm{GL}_\infty(A)$ let $w = u|w|$ be the *polar decomposition.* Define a map $[\cdot]_1 : \mathrm{GL}_\infty(A) \to \mathrm{K}_1(A)$ by $[w]_1 = [u]_1 = [w|w|^{-1}]_1$ and use Proposition 2.1.10.

Exercise 46. Let A be a non-unital C*-algebra, and let $s : \tilde{A} \to \tilde{A}$ be the scalar map $s(a + t1) = t1$. Define

$$\begin{aligned} \mathcal{U}^+(A) &= \{u \in \mathcal{U}(\tilde{A}) : s(u) = 1\}, \\ \mathcal{U}^+_n(A) &= \{u \in \mathcal{U}(M_n(\tilde{A})) : s_n(u) = 1_n\}, \\ \mathcal{U}^+_\infty(A) &= \bigcup_{n=1}^{\infty} \mathcal{U}^+_n(A). \end{aligned}$$

Proceeding as in Section 5.1.1, one can define an abelian group $\mathcal{U}^+_\infty(A)/_{\sim_1}$. Show that this group is isomorphic to $\mathrm{K}_1(A)$.

Exercise 47. Let A be a unital C*-algebra.

(i) Let u be unitary and let s be an isometry in A. Then $sus^* + (1 - ss^*)$ is unitary and we have

$$\begin{pmatrix} s & 1-ss^* \\ 0 & s^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & 1-ss^* \\ 0 & s^* \end{pmatrix}^* = \begin{pmatrix} sus^* + (1-ss^*) & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $[sus^* + (1 - ss^*)]_1 = [u]_1$.

(ii) Let $u_1, \ldots, u_n$ be unitary elements of A and let $s_1, \ldots, s_n$ be isometries in A with mutually orthogonal range projections. Then

$$u = s_1 u_1 s_1^* + \ldots + s_n u_n s_n^* + (1 - s_1 s_1^* - \ldots - s_n s_n^*)$$

is unitary. Use (i) to show that $[u]_1 = [u_1]_1 [u_2]_1 \ldots [u_n]_1$.

(iii) et $s_1, \ldots, s_n$ be isometres as in (ii). Put

$$V = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

hen V is an isometry in $M_n(A)$. Show that for any unitary $u \in \mathcal{U}_n(A)$ there is a unitary w in A such that

$$VuV^* + (1_n - VV^*) = \operatorname{diag}(w, 1_{n-1})$$

(iv) Let A be *properly infinite* (see Exercise 3.3.13). Show that

$$\mathrm{K}_1(A) = \{[u]_1 : u \in \mathcal{U}(A)\}.$$

Example 5.17. The K_1-functor is not exact. Indeed, for a separable Hilbert space $\mathcal{H}$ the sequence

$$0 \longrightarrow \mathcal{B}(\mathcal{H}) \xrightarrow{\pi} \mathcal{B}(\mathcal{H})/\mathcal{K} \longrightarrow 0$$

of C*-algebras is exact. But $\mathrm{K}_1(\mathcal{B}(\mathcal{H})) = 0$ and we will see later that $\mathrm{K}_1(\mathcal{B}(\mathcal{H})/\mathcal{K}) \cong \mathbb{Z}$. Thus $\mathrm{K}_1(\pi)$ cannot be surjective. Likewise, there is an exact sequence

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\varphi} C([0,1]).$$

But $\mathrm{K}_1(C([0,1])) = 0$ and we will see later that $\mathrm{K}_1(C_0((0,1))) \cong \mathbb{Z}$. Thus $\mathrm{K}_1(\varphi)$ cannot be injective.

Exercise 48. Let $A = \varinjlim\{A_i, \Phi_{ij}\}$ be the inductive limit of a sequence of C*-algebras, and let $\Phi_i : A_i \to A$ be the canonical maps.

(i) For any invertible $y \in \tilde{A}$ and any $\epsilon > 0$ there is arbitrarily large i and invertible $z \in \tilde{A}_i$ such that $||y - \tilde{\Phi}_i(z)|| < \epsilon$.

(ii) For any unitary $u \in \tilde{A}$ and any $\epsilon > 0$ there is arbitrarily large i and unitary $w \in \tilde{A}_i$ such that $||u - \tilde{\Phi}_i(w)|| < \epsilon$.

(iii) If u is unitary in $\tilde{A}_j$ such that $\tilde{\Phi}_j(u) \sim_h 1$ in $\tilde{A}$, then there is arbitrarily large i such that $\tilde{\Phi}_{ij}(u) \sim_h 1$ in $\tilde{A}_i$.

(iv) Parts (i)–(iii) remain valid with $\tilde{A}$ and $\tilde{A}_i$ replaced by $M_n(\tilde{A})$ and $M_n(\tilde{A})$, respectively.

(i) First find k and $x, x' \in \tilde{A}_k$ so that both $||\tilde{\Phi}_k(x) - y||$ and $||\tilde{\Phi}_k(x') - y^{-1}||$ are small. Thus both $||\tilde{\Phi}_k(xx' - 1)||$ and $||\tilde{\Phi}_k(x'x - 1)||$ are small. Then, using Exercise 37, take i large enough so that both $||\tilde{\Phi}_{ik}(xx' - 1)||$ and $||\tilde{\Phi}_{ik}(x'x - 1)||$ are small. Then $z = \tilde{\Phi}_{ik}(x)$ is both left and right invertible, hence invertible, and $\tilde{\Phi}_i(z)$ approximates y.

(ii) This follows from part (i) and continuity of the polar decomposition (see Proposition 2.1.10).

(iii) Let w_t, $t \in [0,1]$, be a continuous path of unitaries in $\tilde{A}$ connecting $w_0 = \tilde{\Phi}_j(u)$ and $w_1 = 1$. By compactness, there are $0 = t_0 < t_1 < \ldots < t_{k+1} = 1$ such that $||w_{t_{r+1}} - w_{t_r}|| < 2$ for all r. Applying repeatedly part (ii), find $m \geq j$ and unitary elements $v_1, \ldots, v_k$ in $\tilde{A}_i$ so close to $w_{t_1}, \ldots, w_{t_k}$, respectively, that all the norms: $||\tilde{\Phi}_j(u) - \tilde{\Phi}_m(v_1)||$, $||\tilde{\Phi}_m(v_k) - 1||$, and $\tilde{\Phi}_m(v_{r+1}) - \tilde{\Phi}_m(v_r)||$ for $r = 1, \ldots, k-1$ are less than 2. Then by Exercise 37, there is arbitrarily large $i \geq m$ such that all the norms $||\tilde{\Phi}_{ij}(u) - \tilde{\Phi}_{im}(v_1)||$, $||\tilde{\Phi}_{im}(v_k) - 1||$, and $\tilde{\Phi}_{im}(v_{r+1}) - \tilde{\Phi}_{im}(v_r)||$ for $r = 1, \ldots, k-1$ are less than 2. Now the claim follows from Lemma 2.1.4.

(iv) Exercise.

Exercise 49. Show that every unitary in the Calkin algebra $\mathcal{Q}$ lifts to a partial isometry in $\mathcal{B}(\mathcal{H})$. In fact, it can be lifted to an isometry or a coisometry.

Exercise 50. Let $\psi : A \to B$ be a surjective $*$-homomorphism of C^*-alebras. Show the following.

(i) For each $b = b^* \in B$ there is $a = a^* \in A$ such that $||a|| = ||b||$ and $\psi(a) = b$.

(i) For each $b \in B$ there is $a \in A$ such that $||a|| = ||b||$ and $\psi(a) = b$.

(i) Take any $t \in A$ with $\psi(t) = b$ and set $x = 1/2(t + t^*)$. Then $x = x^*$ and $\psi(x) = b$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(r) = r$ if $|r| \leq ||b||$ and $|f(r)| = ||b||$ if $|r| \geq ||b||$. Put $a = f(x)$. Then $\psi(a) = \psi(f(x)) = f(\psi(x)) = f(b) = b$, and $||a|| \leq ||b||$. But $||b|| = ||\psi(a)|| \leq ||a||$ since $||\psi|| = 1$. Thus $||a|| = ||b||$.

(ii) Consider $\psi_2 : M_2(A) \to M_2(B)$, and put

$$y = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}.$$

Since $y = y^*$, there is $x = x^* \in M_2(A)$ such that $\psi_2(x) = y$ and $||x|| = ||y|| = ||b||$, by part (i). Let

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

and set $a = x_{12}$. Then $\psi(a) = b$ and $||a|| \leq ||x|| = ||b||$. But $||b|| = ||\psi(a)|| \leq ||a||$ since $||\psi|| = 1$. Thus $||a|| = ||b||$.

Exercise 51. Consider an exact sequence of C*-algebras

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\psi} B \longrightarrow 0,$$

in which we identify J with its image in A. Let u be a unitary in $\mathcal{U}_n(\tilde{B})$. By part (ii) of Exercse 50, there is $a \in \mathcal{U}_n(\tilde{A})$ such that $\tilde{\psi}(a) = u$ and $||a|| = ||u|| = 1$. Then for any continuous function $f : \mathbb{R} \to \mathbb{C}$ we have $af(a^*a) = f(aa^*)a$. Use this to show that

$$V = \begin{pmatrix} a & (1_n - aa^*)^{1/2} \\ -(1_n - a^*a)^{1/2} & a^* \end{pmatrix}$$

is a unitary in $\mathcal{U}_{2n}(\tilde{A})$. Then show that

$$\tilde{\psi}(V) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

and

$$V \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} V^* = \begin{pmatrix} aa^* & -a(1_n - a^*a)^{1/2} \\ -(1_n - a^*a)^{1/2}a^* & 1_n - a^*a \end{pmatrix}.$$

Then write explicitly $\partial_1([u]_1)$.

Exercise 52. Consider an exact sequence of C*-algebras

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\psi} B \longrightarrow 0,$$

in which we identify J with its image in A. Suppose that $u \in \mathcal{U}_n(\tilde{B})$ is such that there exists a *partial isometry* $U \in M_n(\tilde{A})$ with $\tilde{\psi}(U) = u$. Show the following.

(i) The element
$$V = \begin{pmatrix} U & 1_n - UU^* \\ 1_n - U^*U & U^* \end{pmatrix}$$
is unitary in $\mathcal{U}_{2n}(\tilde{A})$ and $\tilde{\psi}(V) = \mathrm{diag}(u, u^*)$.

(ii) Both $1_n - U^*U$ and $1_n - UU^*$ are projections in $M_n(J)$, and
$$\partial_1([u]_1) = [1_n - U^*U]_0 - [1_n - UU^*]_0.$$

Exercise 53. Consider an exact sequence of C*-algebras
$$0 \longrightarrow J \longrightarrow A \xrightarrow{\psi} B \longrightarrow 0,$$
in which we identify J with its image in A. Let $u \in \mathcal{U}_n(\tilde{B})$, and let $a \in M_n(\tilde{A})$ be such that $\tilde{\psi}(a) = u$ and $||a|| = ||u|| = 1$. Put
$$U = \begin{pmatrix} a & 0 \\ (1_n - a^*a)^{1/2} & 0 \end{pmatrix}.$$
Show that $U^*U = \mathrm{diag}(1_n, 0)$, which entails that U is a partial isometry. Then show that
$$\tilde{\psi}(U) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}.$$
Finally, show that
$$\partial_1([u]_1) = [1_{2n} - U^*U]_0 - [1_{2n} - UU^*]_0.$$

Exercise 54. Consider an exact sequence
$$0 \longrightarrow C_0(\mathbb{R}^2) \longrightarrow C(\mathbb{D}) \longrightarrow C(S^1) \longrightarrow 0,$$
with the map $C(\mathbb{D}) \to C(S^1)$ given by the restriction. In the corresponding exact sequence
$$\begin{tikzcd} \mathrm{K}_0(C_0(\mathbb{R}^2)) \arrow[r] & \mathrm{K}_0(C(\mathbb{D})) \arrow[r] & \mathrm{K}_0(C(S^1)) \\ \mathrm{K}_1(C(S^1)) \arrow[u, "\partial_1"] & \mathrm{K}_1(C(\mathbb{D})) \arrow[l] & \mathrm{K}_1(C_0(\mathbb{R}^2)) \arrow[l] \end{tikzcd}$$
we have $\mathrm{K}_1(C(\mathbb{D})) = 0$, since $C(\mathbb{D})$ and $\mathbb{C}$ are homotopy equivalent. Show that the map $\mathrm{K}_0(C(\mathbb{D})) \to \mathrm{K}_0(C(S^1))$ is injective, and conclude that $\partial_1 : \mathrm{K}_1(C(S^1)) \longrightarrow \mathrm{K}_0(C_0(\mathbb{R}^2))$ is an isomorphism. Then calculate $\partial_1([z]_1)$ and thus find a generator of $\mathrm{K}_0(C_0(\mathbb{R}^2))$.

Exercise 55 (Naturality of the index map). Show that every commutative diagram of C*-algebras, with exact rows,
$$\begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram of abelian groups, with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_1(J_1) \arrow[r] \arrow[d] & \mathrm{K}_1(A_1) \arrow[r] \arrow[d] & \mathrm{K}_1(B_1) \arrow[r, "\partial_1"] \arrow[d] & \mathrm{K}_0(J_1) \arrow[r] \arrow[d] & \mathrm{K}_0(A_1) \arrow[r] \arrow[d] & \mathrm{K}_0(B_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{K}_1(J_2) \arrow[r] & \mathrm{K}_1(A_2) \arrow[r] & \mathrm{K}_1(B_2) \arrow[r, "\partial_1"] & \mathrm{K}_0(J_2) \arrow[r] & \mathrm{K}_0(A_2) \arrow[r] & \mathrm{K}_0(B_2) \arrow[r] & 0
\end{tikzcd}$$

Exercise 56. Show that if partial isometries $v_1, \ldots, v_n$ in a unital C*-algebra satisfy

$$v_1 v_1^* + \ldots + v_n v_n^* = 1 = v_1^* v_1 + \ldots + v_n^* v_n$$

then $u = v_1 + \ldots + v_n$ is unitary. Hint: use Exercise 2.4.3.

Chapter 6

Bott periodicity and the Exact Sequence of K-Theory

6.1 Higher K-Groups

6.1.1 The suspension functor

Recall that the *suspension* SA of a C*-algebra A is defined as

$$SA = \{f \in C([0,1], A) : f(0) = f(1) = 0\},$$

and is isomorphic to $C_0((0,1), A) \cong C_0(\mathbb{R}) \otimes A$ (cf. Lemma 1.3.1). If $\varphi : A \to B$ is a $*$-homomorphism between two C*-algebras, then $S\varphi : SA \to SB$, given by $(S\varphi(f))(t) = \varphi(f(t))$ is a $*$-homomorphism between their suspensions. It is not difficult to verif that this yields a covariant functor from the category of C*-algebras into itself.

Proposition 6.1. *The suspension functor S is exact. That is, if*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

is an exact sequence of C-algebras then the sequence*

$$0 \longrightarrow SJ \xrightarrow{S\varphi} SA \xrightarrow{S\psi} SB \longrightarrow 0$$

is exact.

Proof. Exercise. □

6.1.2 Isomorphism of $\mathrm{K}_1(A)$ and $\mathrm{K}_0(SA)$

Let A be a C*-algebra. We define a map

$$\theta_A : \mathrm{K}_1(A) \longrightarrow \mathrm{K}_0(SA),$$

as follows. By Exercise 57, each element of $\mathrm{K}_1(A)$ is represented by a unitary $u \in \mathcal{U}_n(\tilde{A})$ (for some n) such that $s(u) = 1_n$. For such a u we can find a continuous function $v : [0,1] \to \mathcal{U}_{2n}(\tilde{A})$ such that $v(0) = 1_{2n}$, $v(1) = \mathrm{diag}(u, u^*)$ and $s(v(t)) = 1_{2n}$ for all $t \in [0,1]$. We put

$$p = v \begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix} v^*,$$

a projection in $\mathcal{P}_{2n}(\widetilde{SA})$ with $s(p) = \mathrm{diag}(1_n, 0_n)$. We set

$$\theta_A([u]_1) = [p]_0 - [s(p)]_0.$$

Theorem 6.2. *For any C*-algebra A, the map*

$$\theta_A : \mathrm{K}_1(A) \longrightarrow \mathrm{K}_0(SA)$$

is an isomorphism. Furthermore, if B is a C-algebra and $\varphi : A \to B$ is a $*$-homomorphism then the diagram*

$$\begin{tikzcd} \mathrm{K}_1(A) \arrow[r, "\mathrm{K}_1(\varphi)"] \arrow[d, "\theta_A"'] & \mathrm{K}_1(B) \arrow[d, "\theta_B"] \\ \mathrm{K}_0(SA) \arrow[r, "\mathrm{K}_0(S\varphi)"'] & \mathrm{K}_0(SB) \end{tikzcd}$$

is commutative.

Proof. Recall from Example 4.12 the exact sequence

$$0 \longrightarrow SA \longrightarrow CA \xrightarrow{\pi} A \longrightarrow 0, \tag{6.1}$$

where CA is the cone over A. Since CA is homotopy equivalent to $\{0\}$ we have $\mathrm{K}_0(CA) = \mathrm{K}_1(CA) = 0$. Let $\partial_1 : \mathrm{K}_1(A) \to \mathrm{K}_0(SA)$ be the index map associated with the extension (6.1). It follows from Theorem 5.15 that ∂_1 is an isomorphism. Thus, it suffices to identify ∂_1 with θ_A (exercise). □

6.1.3 The long exact sequence of K-theory

For each natural number $n \geq 2$ we define inductively a covariant functor from the category of C*-algebras t the category of abelian groups as follows. $K_n(A) = K_{n-1}(SA)$, and if $\varphi : A \to B$ is a $*$-homomorphism then $K_n(\varphi) = K_{n-1}(S\varphi)$. It is clear that such defined functor is half-exact.

Now suppose that

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{6.2}$$

is an exact sequence of C*-algebras. Then $\partial_1 : \mathrm{K}_1(B) \to \mathrm{K}_0(J)$ is the index map. We define higher index maps

$$\partial_n : K_n(B) \to K_{n-1}(J),$$

as follows. Applying $n-1$ times the suspension functor to sequence (6.2), we get an exact sequence

$$0 \longrightarrow S^{n-1}J \xrightarrow{S^{n-1}\varphi} S^{n-1}A \xrightarrow{S^{n-1}\psi} S^{n-1}B \longrightarrow 0. \tag{6.3}$$

Let $\overline{\partial}_1 : \mathrm{K}_1(S^{n-1}B) \to \mathrm{K}_0(S^{n-1}J)$ be the index map associated with (6.3). By the definition of higher K-functors, we have $K_n(B) = \mathrm{K}_1(S^{n-1}B)$ and $K_{n-1}(J) = \mathrm{K}_1(S^{n-2}J)$. By Theorem 6.2, there is an isomorphism $\theta_{S^{n-2}J} : \mathrm{K}_1(S^{n-2}J) \to \mathrm{K}_0(S^{n-1}J)$. We define

$$\partial_n = \theta^{-1}_{S^{n-2}J} \circ \overline{\partial}_1.$$

Such defined higher index maps have naturality analogous to the one enjoyed by the usual index (cf. Exercise 55).

Proposition 6.3. *Every short exact sequence*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

of C-algebras induces a long exact sequence on K-theory:*

$$\begin{aligned} \dots \xrightarrow{\partial_{n+1}} K_n(J) \xrightarrow{K_n(\varphi)} K_n(A) \xrightarrow{K_n(\psi)} K_n(B) \xrightarrow{\partial_n} \dots \\ \dots \xrightarrow{\partial_1} \mathrm{K}_0(J) \xrightarrow{\mathrm{K}_0(\varphi)} \mathrm{K}_0(A) \xrightarrow{\mathrm{K}_0(\psi)} \mathrm{K}_0(B). \end{aligned}$$

Proof. Exercise. □

This Proposition serves only as an intermediate step towards the fundamental 6-term exact sequence of K-theory. The point is that $K_{n+2} \cong K_n$ (as we will see in the next section), and the apparently infinite sequence from Proposition 6.3 shrinks to a much more useful finite one, which contains only K_0 and K_1.

6.2 Bott Periodicity

In this section, we prove the fundamental result of Bott that $\mathrm{K}_0(A) \cong \mathrm{K}_1(SA)$ for any C*-algebra A. Combined with Theorem 6.2, it says $K_{n+2}(A) \cong K_n(A)$ — the Bott periodicity.

6.2.1 Definition of the Bott map

We begin by defining a Bott map

$$\beta_A : \mathrm{K}_0(A) \longrightarrow \mathrm{K}_1(SA)$$

for unital C*-algebras A, and then reduce the general case to the unital one. So let A be a unital C*-algebra. We use the obvious identification

$$SA = \{f : \mathbb{T} \to A : f \text{ continuous}, f(1) = 0\}.$$

Thus, elements of $M_n(SA)$ may be identified with continuous *loops* $f : \mathbb{T} \to M_n(A)$ such that $f(1) = 0$. It follows that $M_n(\widetilde{SA})$ may be identified with continuous functions $f : \mathbb{T} \to M_n(A)$ such that $f(1) \in M_n(\mathbb{C}1_A)$.

For any natural n and any projection $p \in \mathcal{P}_n(A)$ we define a *projection loop* $f_p : \mathbb{T} \to \mathcal{U}_n(A)$ by

$$f_p(z) = zp + (1_n - p), \quad z \in \mathbb{T}.$$

Clearly, we have $f_p \in \mathcal{U}_n(\widetilde{SA})$. By the universal property of K_0 we get a homomorphism $\beta_A : \mathrm{K}_0(A) \longrightarrow \mathrm{K}_1(SA)$ such that

$$\beta_A([p]_0) = [f_p]_1,$$

called the *Bott map*.

Now if $\varphi : A \to B$ is a unital $*$-homomorphism, then $\widetilde{S\varphi}(f_p)(z) = \varphi(f_p(z)) = f_{\varphi(p)}(z)$ for all $z \in \mathbb{T}$. Hence the diagram is commutative. This is the *naturality* of the Bott map.

Finally, suppose that A does not have a unit. Then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(\tilde{A}) \arrow[r] \arrow[d, "\beta_{\tilde{A}}"'] & \mathrm{K}_0(\mathbb{C}) \arrow[r] \arrow[d, "\beta_{\mathbb{C}}"'] & 0 \\ 0 \arrow[r] & \mathrm{K}_1(SA) \arrow[r] & \mathrm{K}_1(S\tilde{A}) \arrow[r] & \mathrm{K}_1(\mathbb{C}) \arrow[r] & 0 \end{tikzcd} \tag{6.4}$$

with split-exact rows. It follows that there is exactly one map $\beta_A : \mathrm{K}_0(A) \to \mathrm{K}_1(SA)$ which completes the diagram. By Exercise 58, we have

$$\beta_A([p]_0 - [s(p)]_0) = [f_p f_{s(p)}^*]_1.$$

6.2.2 The periodicity theorem

The following teorem is considered a central result of K-theory.

Theorem 6.4. *For any C*-algebra A, the Bott map*

$$\beta_A : \mathrm{K}_0(A) \longrightarrow \mathrm{K}_1(SA)$$

is an isomorphism.

Proof. It suffices to prove the theorem for unital C*-algebras. Indeed, the general case follows from the unital one and (6.4) through a diagram chase. Thus assume A is unital. It will be convenient for us to use the description of $\mathrm{K}_1(SA)$ as the collection of suitable equivalence classes in $GL_\infty(\widetilde{SA})$ (see Exercise 45). We must show that the Bott map $\beta_A : \mathrm{K}_0(A) \longrightarrow \mathrm{K}_1(SA)$ is both surjective and injective.

Surjectivity. We consider the following subsets of $GL_\infty(\widetilde{SA})$:

$$\begin{aligned} GL^n &= \{f : \mathbb{T} \to GL_n(A) : f \text{ continuous and } f(1) \in M_n(\mathbb{C}1_A)\}, \\ LL^n_m &= \{f \in GL^n : f \text{ a Laurent polynomial in } z \text{ with coefficients in} \\ &\qquad M_n(A) \text{ and } \deg(f) \leq m\}, \\ PL^n_m &= \{f \in LL^n_m : f \text{ a polynomial}\}, \\ PRL^n &= \{f_p : p \in \mathcal{P}_n(A)\}. \end{aligned}$$

Elements of GL^n, LL^n_m, PL^n_m and PL^n_1 are called *invertible loops*, *Laurent loops*, *polynomial loops* and *linear loops*, respectively. We have

$$PRL^n \subseteq PL^n_1 \subseteq \bigcup_m PL^n_m \subseteq \bigcup_m LL^n_m \subseteq GL^n$$

and $\mathrm{K}_1(SA) = \{[f]_1 : f \in GL^n,\ n \in \mathbb{N}\}$.

Step 1. $\bigcup_m LL^n_m$ is dense in GL^n. Indeed, $\mathrm{span}\{z^k : k \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$ by the Stone-Weierstrass theorem. Hence $\mathrm{span}\{z^k : k \in \mathbb{Z}\} \otimes A$ is dense in $C(\mathbb{T}) \otimes A$, and this easily implies that Laurent loops are a dense subset of invertible loops.

Step 2. By virtue of Step 1, it suffices to show that the range of β_A contains the equivalence classes of all Laurent loops. But each Laurent loop is a quotient of two polynomal loops. Thus, it suffices to show that the range of β_A contains the classes of all polynomial loops. To this end, we show that for each $n, m \in \mathbb{N}$ there is a continuous map

$$\mu^n_m : PL^n_m \longrightarrow PL^{mn+n}_1$$

such that $\mu^n_m(f) \sim_h \mathrm{diag}(f, 1_{mn})$ within PL^{mn+n}_k for all $f \in PL^n_k,\ k \le m$. Indeed, let $f(z) = \sum_{j=0}^{m} a_j z^j$, with $a_j \in M_n(A)$ for all $j = 0, \dots, m$. For each z, we define

$$\tilde{\mu}^n_m(f)(z) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{m-1} & a_m \\ -z & 1 & 0 & \dots & 0 & 0 \\ 0 & -z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -z & 1 \end{pmatrix},$$

an element of $M_{m+1}(M_n(A))$. (In the above matrix we wrote 1 for 1_n and z for $z1_n$.) Cleary, $\tilde{\mu}^n_m(f)(z) = T_0 + T_1 z$ for some $T_0, T_1 \in M_{mn+n}(A)$, and the map $f \mapsto \tilde{\mu}^n_m(f)$ is continuous. We claim the following:

(i) $\tilde{\mu}^n_m(f)(z)$ is invertible for all z,

(ii) $\tilde{\mu}^n_m(f)(1) \sim_h 1_{mn+n}$,

(iii) $\tilde{\mu}^n_m(f) \sim_h \mathrm{diag}(f, 1_{mn})$.

Once the properties (i)–(iii) are established, we obtain the desired map μ^n_m by setting $\mu^n_m(f) = (\tilde{\mu}^n_m(f)(1))^{-1}\tilde{\mu}^n_m(f)$.

In order to prove properties (i)–(iii), we consider matrices

$$A_m = \begin{pmatrix} 1 & 0 & 0 & \dots & -a_m \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

$$A_{m-1} = \begin{pmatrix} 1 & 0 & \dots & -(a_{m-1} + a_m z) & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$\dots$

$$A_1 = \begin{pmatrix} 1 & -(a_1 + a_2 z + \dots + a_m z^{m-1}) & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

and matrices B_k having 1's on the main diagonal, z in the entry in column k and row $k+1$, and 0's elsewhere. Then we have

$$A_1 A_2 \cdots A_m \tilde{\mu}_m^n(f)(z) = \begin{pmatrix} f(z) & 0 & \dots & 0 & 0 \\ -z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -z & 1 \end{pmatrix}$$

and

$$A_1 A_2 \cdots A_m \tilde{\mu}_m^n(f)(z) B_m B_{m-1} \cdots B_1 = \operatorname{diag}(f(z), 1_{mn}). \tag{6.5}$$

Since $f(z)$ and all of the matrices $A_1, \dots, A_m, B_1, \dots, B_m$ are invertible for all z, (6.5) implies (i). Furthermore, each of the A_j and B_j matrices may be continuously deformed to the identity within the set polynomial loops by multiplying the sole off-diagonal entry with a parameter $t \in [0,1]$. Thus (6.5) implies (ii) and (iii).

Step 3. By virtue of Step 2, it suffices to show that the range of β_A contains the equivalence classes of all linear loops. This will follow if we show that there exists a continuous retraction

$$\nu : PL_1^n \longrightarrow PRL^n$$

such that $\nu(f) \sim_h f$ within PL_1^n for all $f \in PL_1^n$. Indeed, let $f(z) = a_0 + a_1 z$. Then $f(1) = a_0 + a_1$ is an invertible element of $M_n(\mathbb{C}1_A)$, and we can put $g = f(1)^{-1} f$. Then

$$g(z) = 1_n + b(z-1),$$

with $b = (a_0 + a_1)^{-1} a_1$. When $z \neq 1$ we can write

$$g(z) = (1-z)\left(\frac{1}{1-z} 1_n - b\right),$$

and since $g(z)$ is invertible for all $z \in \mathbb{T}$ we see that $1/(1-z) \notin \operatorname{sp}(b)$ if $z \in \mathbb{T}\backslash\{1\}$. Since the function $z \mapsto 1/(1-z)$ maps $\mathbb{T}\backslash\{1\}$ onto the line $\{\lambda \in \mathbb{C} : \Re(\lambda) = 1/2\}$, we see that

$$\operatorname{sp}(b) \subseteq \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \Re(\lambda) = 1/2\}.$$

For $t \in [0,1]$ consider a function

$$g_t(z) = \begin{cases} tz & \text{if } \Re(z) < 1/2, \\ tz + (1-t) & \text{if } \Re(z) > 1/2. \end{cases}$$

Each function g_t is holomorphic on an open neighbourhood of $\operatorname{sp}(b)$ and thus the holomorphic function calculus (see Exercise 59) yields elements $g_t(b) \in M_n(A)$, which depend continuously on the parameter t. Since the image of $g_t(z)$ does not intersect the line $\{\lambda \in \mathbb{C} : \Re(\lambda) = 1/2\}$, the elements

$$h_t(z) = 1_n + g_t(b)(z-1) = (1-z)\left(\frac{1}{1-z} 1_n - g_t(b)\right)$$

are invertible. We have $g_1(z) = z$ and thus $g_1(b) = b$. On the other hand, $g_0(z)^2 = g_0(z)$ and thus $e = g_0(b)$ is an idempotent. Consequently, $t \mapsto h_t$ is a homotopy within PL_1^n between g and the idempotent loop $1_n + e(z-1)$. Now we can deform the idempotent e to a projection, as follows (cf. Exercise 3.16).

Lemma 6.5. *Let B be a unital C^*-algebra. Recall that $\mathcal{I}(B)$ denotes the set of idempotents in B and $\mathcal{P}(B)$ denotes the set of projections (i.e. self-adjoint idempotents) in B. Then we have the following.*

(i) For every idempotent $e \in B$ the element

$$\rho(e) = ee^*(1 + (e - e^*)(e^* - e))^{-1}$$

is a projection.

(ii) The map $\rho : \mathcal{I}(B) \to \mathcal{P}(B)$, defined in (i), is a continuous retraction. In particular, $\rho(e) \sim_h e$ in $\mathcal{I}(B)$ for every idempotent e.

(iii) If $p, q \in \mathcal{P}(B)$ and $p \sim_h q$ in $\mathcal{I}(B)$, then $p \sim_h q$ in $\mathcal{P}(B)$.

Proof. (i) Put $w = 1 + (e - e^*)(e^* - e)$. Then w is positive and invertible, thus $\rho(e) = ee^*w^{-1}$ is well-defined. A straightforward calculation yields $ew = ee^*e = we$ and $e^*w = e^*ee^* = we^*$. Thus $ee^*w = (ee^*)^2 = wee^*$ and $ee^*w^{-1} = w^{-1}ee^*$. This implies that ee^*w^{-1} is self-adjoint and that

$$\rho(e)^2 = ee^*w^{-1}ee^*w^{-1} = (ee^*)^2w^{-2} = ee^*w^{-1} = \rho(e).$$

Whence $\rho(e)$ is a projection.

(ii) Clearly, ρ is a continuous map and $\rho(p) = p$ if p is a projection.

To see that $\rho(e) \sim_h e$ in $\mathcal{I}(B)$, set $u_t = 1 - t(e - \rho(e))$ for $t \in [0, 1]$. Since $\rho(e)e = e$ an $e\rho(e) = \rho(e)$, we have $(e - \rho(e))^2 = 0$. Therefore u_t is invertible with the inverse $u_t^{-1} = 1 + t(e - \rho(e))$. Thus $u_t^{-1}eu_t$ is an idempotent for all $t \in [0, 1]$, and we have

$$e = u_0^{-1}eu_0 \sim_h u_1^{-1}eu_1 = (1 + (e - \rho(e)))e(1 - (e - \rho(e))) = \rho(e).$$

(iii) If $t \mapsto e_t$ is a continuous path in $\mathcal{I}(B)$ from $e_0 = p$ to $e_1 = q$, then $t \mapsto \rho(e_t)$ is a continuous path in $\mathcal{P}(B)$ from $\rho(e_0) = p$ to $\rho(e_1) = q$. □

Let $\rho : \mathcal{I}_n(A) \to \mathcal{P}_n(A)$ be the map defined in Lemma 6.5 (with $B = M_n(A)$). Then $\nu(f) = 1_n + \rho(e)(z - 1)$ yields the desired map $\nu : PL_1^n \longrightarrow PRL^n$.

Injectivity. Let $p, q \in \mathcal{P}_n(A)$ and assume that $\beta_A([p]_0 - [q]_0) = [f_pf_q^*]_1 = [1]_1$ in $\mathrm{K}_1(SA)$. Then, after increasing n if necessary, we have $f_p \sim_h f_q$ in GL^n. It suffices to show that there exists $m \in \mathbb{N}$ such that $\mathrm{diag}(p, 1_m) \sim_h \mathrm{diag}(q, 1_m)$ in $\mathcal{P}_{n+m}(A)$.

As a first step, we observe that there exists a polygonal (i.e. piece-wise linear) homotopy $t \mapsto h_t$ from f_p to f_q such that all h_t are Laurent loops with a uniform bound on both positive and negative degrees. (This follows from the density of Laurent loops in invertible loops via a routine compactness argument — exercise.) Thus there are $m, k \in \mathbb{N}$ such that $z^mh_t \in PL_k^n$ for all k. Since $z^mf_p \sim_h f_{\mathrm{diag}(p,1_m)}$ in PL_m^{m+n} (exercise), we see that $f_{\mathrm{diag}(p,1_m)}$ and $f_{\mathrm{diag}(q,1_m)}$ are homotopic in PL_{m+k}^{m+n}. Let $t \mapsto e_t$ be such a homotopy. Then applying the maps μ_{m+k}^{m+n} and ν, constructed in steps 2 and 3, respectively, of the proof of surjectivity, we get a homotopy $t \mapsto \nu(\mu_{m+k}^{m+n}(e_t)) = f_{p_t}$ from $f_{\mathrm{diag}(p,1_m)}$ to $f_{\mathrm{diag}(q,1_m)}$ in projection loops. Since the map $f_{p_t} \mapsto p_t$ is continuous (exercise), we finally see that $\mathrm{diag}(p, 1_m)$ and $\mathrm{diag}(g, 1_m)$ are homotopic via a path of projections. Consequently, $[p]_0 = [q]_0$ in $\mathrm{K}_0(A)$, as required. □

Combining Theorems 6.2 and 6.4 we get

$$K_j(SA) \cong K_{1-j}(A)$$

for any C*-algebra A and $j = 0, 1$. Thus, for any natural n we have

$$K_{n+2}(A) \cong K_n(A).$$

Furthermore, naturality of the maps θ_* and β_* easily implies that the functors K_{n+2} and K_n are isomorphic.

6.3 The 6-Term Exact Sequence

6.3.1 The 6-term exact sequence of K-theory

With the Bott periodicity theorem at hand, we are now ready to present the 6-term exact sequence of K-theory — a tool of paramount importance in applications. Let

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be an exact sequence of C*-algebras. Applying the suspension functor, we obtain the exact sequence

$$0 \longrightarrow SJ \xrightarrow{S\varphi} SA \xrightarrow{S\psi} SB \longrightarrow 0.$$

Denote by $\partial : \mathrm{K}_1(SB) \to \mathrm{K}_0(SJ)$ the corresponding index map. Let $\theta_J : \mathrm{K}_1(J) \to \mathrm{K}_0(SJ)$ and $\beta_B : \mathrm{K}_0(B) \to \mathrm{K}_1(SB)$ be the isomorphisms from Therems 6.2 and 6.4, respectively. Then the *exponential map*

$$\partial_0 : \mathrm{K}_0(B) \longrightarrow \mathrm{K}_1(J)$$

is defined as the unique homomorphism making the diagram

$$\begin{tikzcd}
\mathrm{K}_0(B) \arrow[r, "\partial_0"] \arrow[d, "\beta_B"'] & \mathrm{K}_1(J) \arrow[d, "\theta_J"] \\
\mathrm{K}_1(SB) \arrow[r, "\partial"'] & \mathrm{K}_0(SJ)
\end{tikzcd}$$

commutative.

Theorem 6.6. *Let*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be an exact sequence of C-algebras. Then the sequence*

$$\begin{tikzcd}
\mathrm{K}_0(J) \arrow[r, "\mathrm{K}_0(\varphi)"] & \mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(\psi)"] & \mathrm{K}_0(B) \arrow[d, "\partial_0"] \\
\mathrm{K}_1(B) \arrow[u, "\partial_1"] & \mathrm{K}_1(A) \arrow[l, "\mathrm{K}_1(\psi)"'] & \mathrm{K}_1(J) \arrow[l, "\mathrm{K}_1(\varphi)"']
\end{tikzcd} \tag{6.6}$$

is exact everywhere.

Proof. By virtue of Theorem 5.15, it suffices to show exactness at $\mathrm{K}_0(B)$ and $\mathrm{K}_1(J)$. It turns out that at this stage this requires nothing more but a diagram chase.

To prove exactness of (6.6) at $\mathrm{K}_0(B)$, consider the commutative (due to naturality of the Bott map) diagram

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(\psi)"] \arrow[d, "\beta_A"'] & \mathrm{K}_0(B) \arrow[r, "\partial_0"] \arrow[d, "\beta_B"'] & \mathrm{K}_1(J) \arrow[d, "\theta_J"] \\
\mathrm{K}_1(SA) \arrow[r, "\mathrm{K}_0(S\psi)"'] & \mathrm{K}_1(SB) \arrow[r, "\partial"'] & \mathrm{K}_0(SJ)
\end{tikzcd}$$

All the vertical arrows are isomorphisms, and the bottom row is exact by Theorem 5.15. Thus the top row is exact.

To prove exactness of (6.6) at $\mathrm{K}_1(J)$, consider the commutative (due to naturality of the θ_* map) diagram

$$\begin{tikzcd}
\mathrm{K}_0(B) \arrow[r, "\partial_0"] \arrow[d, "\beta_B"'] & \mathrm{K}_1(J) \arrow[r, "\mathrm{K}_1(\varphi)"] \arrow[d, "\theta_J"'] & \mathrm{K}_1(A) \arrow[d, "\theta_A"] \\
\mathrm{K}_1(SB) \arrow[r, "\partial"'] & \mathrm{K}_0(SJ) \arrow[r, "\mathrm{K}_0(S\varphi)"'] & \mathrm{K}_0(SA)
\end{tikzcd}$$

All the vertical arrows are isomorphisms, and the bottom row is exact by Theorem 5.15. Thus the top row is exact. ☐

6.3.2 An explicit form of the exponential map

Proposition 6.7. *Let $0 \to J \xrightarrow{\varphi} \to A \xrightarrow{\psi} \to B \to 0$ be exact and let $\partial_0 : \mathrm{K}_0(B) \to \mathrm{K}_1(J)$ be the associated exponential map. Then*

(i) If $p \in \mathcal{P}_n(\tilde{B})$ and $x = x^* \in M_n(\tilde{A})$ such that $\tilde{\psi}(x) = p$ then $\exists! u \in \mathcal{U}_n(\tilde{J})$ such that $\tilde{\varphi}(u) = \exp(2\pi i x)$, and we have

$$\partial_0([p]_0 - [s(p)]_0) = -[u]_1. \tag{6.7}$$

(ii) Suppose that A is unital. If $p \in \mathcal{P}_n(B)$ and $x = x^* \in M_n(A)$ such that $\psi(x) = p$, then $\exists! u \in \mathcal{U}_n(\tilde{J})$ such that $\tilde{\varphi}(u) = \exp(2\pi i x)$, and we have

$$\partial_0([p]_0) = -[u]_1. \tag{6.8}$$

Proof. Part (i) follows from (ii) by a diagram chase. So we prove (ii). For simplicity, assume $J \subseteq A$ and $\varphi = \mathrm{id}$. Suppose A unital then and let $p \in \mathcal{P}_n(B)$. There is $x = x^* \in M_n(A)$ such that $\psi(x) = p$. Then $\psi(\exp(2\pi i x)) = \exp(2\pi i \psi(x)) = \exp(2\pi i p) = 1_n$, hence $\exp(2\pi i x) \in \mathcal{U}_n(\tilde{J})$. We must show that

$$\theta_J([\exp(-2\pi i x)]_1) = (\partial_1 \circ \beta_B)([p]_0), \tag{6.9}$$

where $\partial_1 : \mathrm{K}_1(SB) \to \mathrm{K}_0(SJ)$ is the index map corresponding to $0 \to SJ \to SA \to SB \to 0$. We identify SB with $\{f \in C([0,1], B) | f(0) = f(1) = 0\}$. Thus

$M_k(\widetilde{SB}) = \{f \in C([0,1], M_k(B)) | f(0) = f(1) \in M_k(\mathbb{C}1_B)\}$, and $f_p \in \mathcal{U}(\widetilde{SB})$ is $f_p(t) = e^{2\pi i t} p + 1_n - p, \quad t \in [0,1]$. Let $v \in \mathcal{U}_{2n}(\widetilde{SA})$ be such that $\widetilde{S\psi}(v) = \begin{pmatrix} f_p & 0 \\ 0 & f_p^* \end{pmatrix}$. Then $v : [0,1] \to \mathcal{U}_{2n}(A)$ is a continuous map such that $v(0) = v(1) \in M_2 n(\mathbb{C}1_A)$, and $\psi(v(t)) = \begin{pmatrix} f_p(t) & 0 \\ 0 & f_p^*(t) \end{pmatrix}$. As $f_p(0) = f_p(1) = 1$, we have $v(0) = v(1) = 1_{2n}$. With $x = x^* \in M_n(A)$ a lift of p, put $z(t) = \exp(2\pi i t x)$ for $t \in [0,1]$. $t \mapsto z(t) \in \mathcal{U}_n(A)$ is continuous and $\psi(z(t)) = f_0(t)$. Hence

$$\psi\left(v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix}\right) = 1_{2n}, \quad s\left(v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix}\right) = 1_{2n}. \tag{6.10}$$

Thus $w(t) = v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix}$ is a unitary element in $\mathcal{U}_{2n}(\tilde{J})$. We have $w(0) = 1_{2n}$ and $w(1) = \begin{pmatrix} \exp(-2\pi i x) & 0 \\ 0 & \exp(2\pi i x) \end{pmatrix}$. Thus, by the definition of θ_J, we have

$$\theta_J([\exp(-2\pi i x)]_1) = \left[w\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} w^*\right]_0 - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right]_0. \tag{6.11}$$

We also have

$$w(t)\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} w(t)^* = v(t)\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v(t)^*, \tag{6.12}$$

and the unitary v was chosen so that

$$\widetilde{S\psi}(v) = \begin{pmatrix} f_p & 0 \\ 0 & f_p^* \end{pmatrix}. \tag{6.13}$$

So, by the definition of the index map, we get

$$\partial_1([f_p]_1) = \theta_J([\exp(-2\pi i x]_1). \tag{6.14}$$

□

6.4 Examples and Exercises

Exercise 57. Let A be a C*-algebra. Show that every class in $\mathrm{K}_1(A)$ contains a unitary $u \in \mathcal{U}_n(\tilde{A})$ normalized so that $s(u) = 1_n$, where s is the scalar map.

Exercise 58. Show that if A is a non-unital C*-algebra then for any $p \in \mathcal{P}_n(\tilde{A})$ we have

$$\beta_A([p]_0 - [s(p)]_0) = [f_p f_{s(p)}^*]_1.$$

Exercise 59 (Holomorphic function calculus). Let $\gamma_1, \ldots, \gamma_n$ be a finite collection of continuous and piece-wise continuously differentialble paths $\gamma_k : [a_k, b_k] \to \mathbb{C}$. We assume that each γ_k is closed, i.e. $\gamma_k(a_k) = \gamma_k(b_k)$. A *contour* is a finite collection $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. If f is a piece-wise continuous, complex function defined on $\mathrm{im}(\Gamma) = \bigcup_k \gamma_k([a_k, b_k])$, then there is a well-defined integral

$$\int_\Gamma f(z)dz = \int_{\gamma_1} f(z)dz + \ldots + \int_{\gamma_n} f(z)dz.$$

If $z_0 \notin \mathrm{im}(\Gamma)$ then the *index* of z_0 with respect to Γ is the integer defined as

$$\mathrm{Ind}_\Gamma(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z - z_0}.$$

Let K be a compact subset of an open set $\Omega \subseteq \mathbb{C}$. Then we say that Γ *surrounds* K in Ω if $\mathrm{im}(\Gamma) \subseteq \Omega \setminus K$ and

$$\mathrm{Ind}_\Gamma(z) = \begin{cases} 1, & z \in K, \\ 0, & z \in \mathbb{C} \setminus \Omega. \end{cases}$$

Let A be a unital C*-algebra. If $a \in A$ and Γ is a contour surrounding $\mathrm{sp}(a)$ in an open set Ω, then for every holomorphic function $f : \Omega \to \mathbb{C}$ there is a well-defined Riemann integral

$$f(a) = \frac{1}{2\pi i}\int_\Gamma f(z)(z1_A - a)^{-1} dz.$$

This integral yields a unique element $f(a)$ of A such that for every continuous functional $\varphi : A \to \mathbb{C}$ we have

$$\varphi(f(a)) = \frac{1}{2\pi i}\int_\Gamma f(z)\varphi((z1_A - a)^{-1}) dz.$$

The mapping $f \mapsto f(a)$ is called the *holomorphic function calculus* for a. It has the following properties (see [t-m79, m-gj90, p-g79]).

(i) The map $f \mapsto f(a)$ is a unital algebra homomorphism.

(ii) If g is a holomorphic function on $f(\Omega)$ then $(g \circ f)(a) = g(f(a))$.

(iii) $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$.

(iv) If f_n is a sequence of holomorphic functions on Ω converging almost uniformly to a function g, then g is holomorphic on Ω and

$$||f_n(a) - g(a)|| \longrightarrow 0.$$

Note that the holomorphic function calculus applies to an arbitrary element a of a C*-algebra, not just a self-adjoint one. If a is self-adjoint then the holomorphic function calculus is compatible with the continuous function calculus via the Gelfand transform.

Exercise 60. By virtue of Theorems 6.2 and 6.4, we have

$$\mathrm{K}_0(C_0(\mathbb{R}^{2n})) \cong \mathrm{K}_1(C_0(\mathbb{R}^{2n+1})) \cong \mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z},$$
$$\mathrm{K}_1(C_0(\mathbb{R}^{2n})) \cong \mathrm{K}_0(C_0(\mathbb{R}^{2n+1})) \cong \mathrm{K}_1(\mathbb{C}) = 0,$$

for all $n \in \mathbb{N}$.

Exercise 61. For each natural number $n \geq 1$, find a split-exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^n) \longrightarrow C(S^n) \longrightarrow \mathbb{C} \longrightarrow 0.$$

Then use Exercise 60 and split-exactness of K_* to determine the K-groups of $C(S^n)$ for all spheres S^n.

Exercise 62. By Exercise 61, we have $\mathrm{K}_0(C(\mathbb{T})) \cong \mathrm{K}_1(C(\mathbb{T})) \cong \mathbb{Z}$. Use the isomorphism $C(\mathbb{T}^{n+1}) \cong C(\mathbb{T}) \otimes C(\mathbb{T}^n)$ to find a split-exact sequence

$$0 \longrightarrow SC(\mathbb{T}^n) \longrightarrow C(\mathbb{T}^{n+1}) \longrightarrow C(\mathbb{T}^n) \longrightarrow 0.$$

Then use split-exactness of K_* to determine the K-groups of $C(\mathbb{T}^n)$ for all tori $\mathbb{T}^n$.

Exercise 63. Let

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be an exact sequence of C*-algebras. Show that if every projection in $\mathcal{P}_\infty(\tilde{B})$ lifts to a projection in $\mathcal{P}_\infty(\tilde{A})$ then $\partial_0 : \mathrm{K}_0(B) \to \mathrm{K}_1(J)$ is the zero map.

Exercise 64 (Toeplitz algebra). Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{\xi_n : n = 0, 1, 2, \ldots\}$. Let $S \in \mathcal{B}(\mathcal{H})$, $S(\xi_n) = \xi_{n+1}$ be the unilateral shift. We define the *Toeplitz algebra* $\mathcal{T}$ as the C*-algebra generated by S. It can be shown [c-l67] that $\mathcal{T}$ is the universal C*-algebra for the relation $S^*S = 1$, and that if T is a *proper isometry* on a Hilbert space then there exists a $*$-isomorphism $\mathcal{T} = C^*(S) \to C^*(T)$ such that $T \mapsto S$.

(i) Show that the closed two-sided ideal of $\mathcal{T}$ generated by $1 - SS^*$ coincides with the algebra $\mathcal{K}(\mathcal{H})$.

(ii) Let $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{K}$ be the natural surjection. Show that $\mathcal{T}/\mathcal{K}$ is isomorphic to $C(S^1)$ and $\pi(S)$ may be identified with the generator z. There is an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

(iii) By Exercise 61, $\mathrm{K}_0(C(S^1)) \cong \mathbb{Z}$ (with a generator $[1]_0$) and $\mathrm{K}_1(C(S^1)) \cong \mathbb{Z}$ (with a generator $[z]_1$, the class of the identity map $z \mapsto z$). Calculate $\partial_1([z]_1)$ and show that the index map

$$\partial_1 : \mathbb{Z} \cong \mathrm{K}_1(C(S^1)) \longrightarrow \mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z}$$

is an isomorphism.

(iv) Use (iii) and the exact sequence from Theorem 6.6 to show that

$$\mathrm{K}_0(\mathcal{T}) \cong \mathbb{Z}, \quad \mathrm{K}_1(\mathcal{T}) = 0.$$

Find the generator of $\mathrm{K}_0(\mathcal{T})$.

Exercise 65. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the compact operators on $\mathcal{H}$, and let $S^n \in \mathcal{B}(\mathcal{H})$ be an isometry with cokernel of dimension n, for some natural number n. Let $C^*(S^n, \mathcal{K})$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by S^n and $\mathcal{K}$. Show that there exists an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(S^n, \mathcal{K}) \longrightarrow C(S^1) \longrightarrow 0$$

and determine the K-theory of $C^*(S^n, \mathcal{K})$.

Exercise 66. Let $\mathbb{R}P^2$ be the real projective plane. Find an exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \longrightarrow C(\mathbb{R}P^2) \longrightarrow C(S^1) \longrightarrow 0$$

and determine the K-theory of $C(\mathbb{R}P^2)$.

Exercise 67. Find an exact sequence

$$0 \longrightarrow C(S^1) \otimes C_0(\mathbb{R}^2) \longrightarrow C(S^3) \longrightarrow C(S^1) \longrightarrow 0.$$

Then apply to it the 6-term exact sequence of K-theory and thus calculate in an alternative way the K-groups of the 3-sphere (cf. Exercise 61).

Exercise 68. Let $\mathcal{H}$ be a separable Hilbert space. Consider two operators $T, U \in \mathcal{B}(\mathcal{H})$ such that T is a proper isometry (i.e. $T^*T = 1 \neq TT^*$) and U is a partial unitary on $1 - TT^*$ with full spectrum (i.e. $U^*U = UU^* = 1 - TT^*$ and $\mathrm{sp}(U) = S^1 \cup \{0\}$). Let A be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T and S.

(i) Let J be the closed two-sided ideal of A generated by U. Show that J is isomorphic to $C(S^1) \otimes \mathcal{K}$, with $\mathcal{K}$ the C*-algebra of compact operators.

(ii) Let $\pi : A \to A/J$ be the natural surjection. Show that A/J is generated (as a C*-algebra) by the unitary element $\pi(T)$. Show that $\mathrm{sp}(\pi(T))$ contains the entire unit circle, and thus A/J is isomorphic to $C(S^1)$.

(iii) By (i) and (ii) above, there is an exact sequence

$$0 \longrightarrow C(S^1) \otimes \mathcal{K} \longrightarrow A \longrightarrow C(S^1) \longrightarrow 0.$$

Apply the 6-term exact sequence and calculate the K-theory of A.

Example 6.8. (Mirror-disc-type quantum two-spheres)
Consider, for $p \in]0, 1[$, the $*$-algebra

$$\mathcal{O}(D_p) := \mathbb{C}\langle x, x^* \rangle / J, \tag{6.15}$$

where J is the $*$-ideal generated by $x^*x - pxx^* - (1 - p)$. This is called $*$-algebra of the quantum disc (see [kl93], where a two-parameter family of such quantum discs is considered). It is not hard to see that $\|\rho(x)\| = 1$ in any bounded representation ρ, so that one can form the C^*-closure $C(D_p)$ of $\mathcal{O}(D_p)$. Moreover $\mathcal{O}(D_p)$ is faithfully imbedded in $C(D_p)$. (There is exactly one faithful irreducible representation, up to unitary equivalence.) It is known that $C(D_p)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$, so all the C*-algebras $C(D_p)$ are isomorphic. There is a $*$-homomorphism $\varphi : C(D_p) \to C(S^1)$, sending the generator x to the unitary generator u of $C(S^1)$. Consider for any $q \in]0, 1[$ a second copy $\mathcal{O}(D_q)$, with generator y.

Definition 6.9. *Let $\alpha : \mathcal{O}(S^1) \to \mathcal{O}(S^1)$ denote the $*$-automorphism defined by $u \mapsto u^*$. Define*

$$\mathcal{O}(S^2_{pq}) := \{(f, g) \in \mathcal{O}(D_p) \oplus \mathcal{O}(D_q) \mid \varphi(f) = \alpha \circ \varphi(g)\}. \tag{6.16}$$

This is called the $*$-algebra of the mirror-disc-type quantum two-sphere.

Proposition 6.10.

$$\mathcal{O}(S^2_{pq}) \cong \mathbb{C}\langle C, C^*, D, D^*, E, E^* \rangle / J, \tag{6.17}$$

where the $*$-ideal J is generated by the relations

$$\begin{aligned}
C^*C &= 1 - pD - E, \\
CC^* &= 1 - D - qE, \\
DC &= pCD, \\
EC &= q^{-1}CE, \\
DE &= 0, \\
D &= D^*, \\
E &= E^*.
\end{aligned}$$

The isomorphism is given by $(x, y^*) \mapsto C$, $(1-xx^*, 0) \mapsto D$, $(0, 1-yy^*) \mapsto E$.

Proposition 6.11. *The following is a complete list (up to unitary equivalence) of irreducible $*$-representations of $\mathcal{O}(S^2_{pq})$ in some Hilbert space:*

(i) ρ_+, acting on a separable Hilbert space $\mathcal{H}$ with orthonormal basis $e_0, e_1, \ldots$ according to

$$\begin{aligned}
\rho_+(C)e_k &= \sqrt{1 - p^{k+1}}e_{k+1}, \\
\rho_+(D)e_k &= p^k e_k, \\
\rho_+(E) &= 0.
\end{aligned}$$

(ii) ρ_-, acting on $\mathcal{H}$ by

$$\begin{aligned}
\rho_-(C)e_k &= \sqrt{1 - q^k}e_{k-1}, \\
\rho_-(D)e_k &= 0, \\
\rho_-(E)e_k &= q^k e_k.
\end{aligned}$$

(iii) An S^1-family ρ_μ, acting on $\mathbb{C}$ by

$$\begin{aligned}
\rho_\mu(C) &= \mu, \\
\rho_\mu(D) &= 0, \\
\rho_\mu(E) &= 0.
\end{aligned}$$

One can again show that there is a uniform bound on the norm of the generators for all bounded $*$-representations, so that one can form a C^*-closure $C(S^2_{pq})$ of $\mathcal{O}(S^2_{pq})$ using bounded $*$-representations. $\rho_+ \oplus \rho_-$ is a faithful representation of $\mathcal{O}(S^2_{pq})$ as well as of $C(S^2_{pq})$, so that $\mathcal{O}(S^2_{pq})$ is faithfully imbedded in $C(S^2_{pq})$. Moreover, the closed ideals J_D, J_E generated by D, E are isomorphic to $\mathcal{K}$ ($\rho_+(J_D) = \mathcal{K} = \rho_-(J_E)$), they have zero intersection, and $(\rho_+ \oplus \rho_-)(J_D + J_E) = \mathcal{K} \oplus \mathcal{K}$. Finally, there is an exact sequence

$$0 \to \mathcal{K} \oplus \mathcal{K} \to C(S^2_{pq}) \xrightarrow{\psi} \to C(S^1) \to 0, \tag{6.18}$$

where ψ is defined by $C \mapsto u$, $D \mapsto 0$, $E \mapsto 0$. This exact sequence can be used to compute the K-theory of $C(S^2_{pq})$:

Proposition 6.12. $\mathrm{K}_0(C(S^2_{pq}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $\mathrm{K}_1(C(S^2_{pq})) = 0$.

Proof. With $\mathrm{K}_0(C(S^1)) \cong \mathbb{Z} \cong \mathrm{K}_1(C(S^1))$, $\mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z}$, $\mathrm{K}_1(\mathcal{K}) = 0$, we obtain from the standard six-term exact sequence corresponding to (6.18)

$$0 \to \mathrm{K}_1(C(S^2_{pq}) \to \mathbb{Z} \xrightarrow{\partial} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{K}_0(C(S^2_{pq})) \xrightarrow{\mathrm{K}_0(\psi)} \to \mathbb{Z} \to 0. \tag{6.19}$$

Let us compute the index map ∂. It is determined by its value on the generator $[u]_1 \in \mathrm{K}_1(C(S^1))$,

$$\partial([u]_1) = [1 - b^*b]_0 - [1 - bb^*]_0, \tag{6.20}$$

where $b \in C(S^2_{pq})$ is any partial isometry with $\psi(b) = u$. Identify $C(S^2_{pq}) \cong (\rho_+ \oplus \rho_-)(C(S^2_{pq})$. Then $b = (s, s^*)$, s the one-sided shift, is a continuous function of $(\rho_+ \oplus \rho_-)(C)$ such that

$$b - (\rho_+ \oplus \rho_-)(C) \in \mathcal{K} \oplus \mathcal{K}$$
$$b = (\rho_+(C)|\rho_+(C)|^{-1}, \rho_-(C)|\rho_-(C)|^{-})$$
$$\text{with } |\rho_-(C)|^{-} e_k := \begin{cases} 0 & k = 0 \\ \frac{1}{\sqrt{1-q^k}} e_k & k > 0. \end{cases}$$

Then $b^*b = (s^*s, ss^*) = 1 - p_2$, $bb^* = (ss^*, s^*s) = 1 - p_1$, where $p_2 = (p_{e_0}, 0)$, $p_1 = (0, p_{e_0})$ can be considered as the generators $(0,1)$, $(1,0)$ of $\mathbb{Z} \oplus \mathbb{Z} \cong \mathrm{K}_0(\mathcal{K} \oplus \mathcal{K})$. Then $\partial([u]_1) = [p_2]_0 - [p_1]_0 = (0,1) - (1,0)$, so that ∂ is injective, and we can conclude that $\mathrm{K}_1(C(S^2_{pq})) = 0$. We are left with the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\partial} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{K}_0(j)} \to \mathrm{K}_0(C(S^2_{pq}) \xrightarrow{\mathrm{K}_0(\psi)} \to \mathbb{Z} \to 0. \tag{6.21}$$

As $\mathbb{Z}$ is a free module over itself, this sequence splits, and $\mathrm{K}_0(C(S^2_{pq})) \cong \operatorname{im} \mathrm{K}_0(j) \oplus \mathbb{Z}$. There remains the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\partial} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{K}_0(j)} \to \operatorname{im} \mathrm{K}_0(j) \to 0. \tag{6.22}$$

Here, $\mathrm{K}_0(j)$ is determined by its values on $(1,0)$ and $(0,1)$, however, $(1,0) - (0,1) \in \ker \mathrm{K}_0(j) = \operatorname{im} \partial$, i.e., $\mathrm{K}_0(j)(1,0) = \mathrm{K}_0(j)(0,1)$, consequently $\operatorname{im} \mathrm{K}_0(j) = \{n\, \mathrm{K}_0(j)(1,0) | n \in \mathbb{Z}\} \cong \mathbb{Z}$. It follows that $\mathrm{K}_0(C(S^2_{pq})) \cong \mathbb{Z} \oplus \mathbb{Z}$. $\square$

Chapter 7

Tools for the computation of K-groups

7.1 Crossed products, the Thom-Connes isomorphism and the Pimsner–Voiculescu sequence

7.1.1 Crossed products

Let G be a locally compact abelian group. Then

$$C_c(G) = \{f \in C(G) | \operatorname{supp}(f) \text{ compact}\}$$

is a $*$-algebra with respect to

$$(f * g)(s) = \int_G f(t)g(t-s)dt, \tag{7.1}$$
$$f^*(s) = \overline{f(s^{-1})}, \tag{7.2}$$

where the integration is with respect to the Haar measure. The universal norm on $C_c(G)$,

$$\|f\| = \sup\{\|\pi(f)\| | \pi : C_c(G) \to B(\mathcal{H}) \text{ a } *\text{-representation}\}, \tag{7.3}$$

is well-defined since (one can show that)

$$\|f\| \leq \|f\|_1 = \int_G |f(t)|dt. \tag{7.4}$$

The completion of $C_c(G)$ with respect to $\|.\|$ is the group C*-algebra $C^*(G)$ of G. By Gelfand's theorem, since $C^*(G)$ is abelian, there is a locally compact Hausdorff space Ω such that $C^*(G) \cong C_0(\Omega)$. Ω may be identified with $\hat{G} = \{\chi : G \to \mathbb{T} | \chi \text{ continuous }, \chi(s+t) = \chi(s)\chi(t)\}$, the dual group of G. $\hat{G}$ is equipped with the topology of almost uniform convergence. Every $\chi \in \hat{G}$ yields a multiplicative functional of $C^*(G)$ by

$$\omega_\chi(f) = \int_G \chi(t)f(t)dt. \tag{7.5}$$

Thus we have $C^*(G) \cong C_0(\hat{G})$ via the Gelfand transform. Now suppose that A is a C*-algebra and $\alpha : G \to Aut(A)$ is a homomorphism such that $G \ni t \mapsto \alpha_t(x) \in A$ is continuous $\forall x \in A$. Then (A, G, α) is called a C^*-dynmaical system. The vector space $\{f \in C(G, A) | \operatorname{supp}(f) \text{ compact}\}$ becomes a $*$-algebra with

$$\begin{aligned} (f * g)(s) &= \int_G f(t)\alpha_t(g(s-t))dt, &(7.6)\\ f^*(s) &= \alpha_s(f(s^{-1})^*). &(7.7) \end{aligned}$$

Note that even if both G and A are abelian, this algebra may be noncommutative if the action α is nontrivial. The universal norm $\|.\|$ on this $*$-algebra is defined as the supremum over the norms in all $*$-representations. $A \rtimes_\alpha G$ is by definition the C*-algebraic closure of $A \otimes C_c(G)$ with respect to $\|.\|$. If $\alpha : G \to Aut(A)$ is trivial, i.e., $\alpha_t(x) = x$, $\forall x$, then we have $A \rtimes_\alpha G \cong A \otimes C^*(G) \cong A \otimes C_0(\hat{G})$ (C*-algebra isomorphisms). For a given action $\alpha : G \to Aut(A)$ there exists a canonical *dual action* $\hat{\alpha} : \hat{G} \to Aut(A \rtimes_\alpha G)$ such that

$$\hat{\alpha}_\chi(f)(t) = \langle \chi, t\rangle f(t) \tag{7.8}$$

for $f \in C(G, A)$ with compact support.

Theorem 7.1. *(Takesaki-Takai duality)*

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes \mathcal{K}, \tag{7.9}$$

if G is infinite.

The dual acion is functorial in the follwoing sense: If $\alpha : G \to Aut(A)$ and $\beta : G \to Aut(B)$ are actions and $\rho : A \to B$ is a G-equivariant $*$-homomorphism, then there exists a $*$-homomorphism $\hat{\rho} : A \rtimes_\alpha G \to B \rtimes_\beta G$ such that

$$(\hat{\rho} f)(s) = \rho(f(s)) \tag{7.10}$$

for $f : G \to A$, and ρ is equivariant with respect to $\hat{\alpha}$ and $\hat{\beta}$.

7.1.2 Crossed products by $\mathbb{R}$ and by $\mathbb{Z}$

Theorem 7.2. *(Connes) For any action $\alpha : \mathbb{R} \to Aut(A)$, we have*

$$\mathrm{K}_j(A) \cong \mathrm{K}_{1-j}(A \rtimes_\alpha \mathbb{R}). \tag{7.11}$$

In the special case of a trivial action, we have $\mathrm{K}_j(A) \cong \mathrm{K}_{1-j}(A \otimes C_0(\mathbb{R}))$ (Bott periodicity). Intuitively, the Connes-Thom isomorphism can be explained as follows: "Any action of $\mathbb{R}$ may be continuously deformed to a trivial one. Then the result follows from the Bott periodicity since K-theory is insensitive to continuous deformations". This can be made precise with the help of K-equivalence.

Theorem 7.3. *(Pimsner–Voiculescu) If $\alpha \in \mathrm{Aut}(A)$, then there is an exact sequence*

$$\begin{tikzcd} \mathrm{K}_0(A) \arrow[r, "\mathrm{id}-\mathrm{K}_0(\alpha^{-1})"] & \mathrm{K}_0(A) \arrow[r, "i_0"] & \mathrm{K}_0(A \rtimes_\alpha) \arrow[d] \\ \mathrm{K}_1(A \rtimes_\alpha) \arrow[u] & \mathrm{K}_1(A) \arrow[l, "i_1"] & \mathrm{K}_1(A) \arrow[l, "\mathrm{id}-\mathrm{K}_1(\alpha^{-1})"] \end{tikzcd} \tag{7.12}$$

where i_0, i_1 are the natural imbeddings.

Proof. (idea, Connes) Define $M_\alpha = \{f \in C(\mathbb{R}, M) | f(1) = \alpha(f(0)\}$ (mapping torus of α). $\mathbb{R}$ acts on M_α by $(\beta_t f)(s) = f(s-t)$. By a result of Green,

$$A \rtimes_\alpha \mathbb{Z} \simeq_{\text{Morita}} M_\alpha \rtimes_\beta \mathbb{R}. \tag{7.13}$$

Hence, by the Connes-Thom isomorphism,

$$\mathrm{K}_j(A \rtimes_\alpha \mathbb{Z}) \cong \mathrm{K}_j(M_\alpha \rtimes_\beta \mathbb{R}) \cong \mathrm{K}_{1-j}(M_\alpha). \tag{7.14}$$

Now, there is an exact sequence

$$0 \to SA \to M_\alpha \to A \to 0, \tag{7.15}$$

and the 6-term exact sequence yields

$$\begin{tikzcd}
\mathrm{K}_1(A) \cong \mathrm{K}_0(SA) \arrow[r] & \mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}) \cong \mathrm{K}_0(M_\alpha) \arrow[r] & \mathrm{K}_0(A) \arrow[d, "\partial_1"] \\
\mathrm{K}_1(A) \arrow[u, "\partial_0"] & \mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}) \cong \mathrm{K}_1(M_\alpha) \arrow[l] & \mathrm{K}_1(sA) \cong \mathrm{K}_0(A) \arrow[l]
\end{tikzcd} \tag{7.16}$$

One can calculate the connecting maps as

$$\partial_* = \mathrm{id} - \mathrm{K}_*(\alpha^{-1}). \tag{7.17}$$

□

7.1.3 Irrational rotation algebras

Let us recall that, for $\theta \in \mathbb{R}$, the rotation algebra A_θ is defined to be the universal C*-algebra $C^*(u, v)$ generated by two unitaries u, v such that

$$vu = e^{2\pi i\theta} uv. \tag{7.18}$$

We have seen that there is a trace $\tau : A_\theta \to \mathbb{R}$, and that the image of $\mathrm{K}_0(\tau)$ contains $\mathbb{Z} \cup \theta\mathbb{Z}$. Notice that $C^*(v) \cong C(S^1)$ and that $\alpha_\theta := Ad_v$ is an automorphism of $C(S^1)$ such that

$$\alpha_\theta v = e^{2\pi i\theta} v. \tag{7.19}$$

It can be shown that

$$A_\theta \cong C(S^1) \rtimes_{\alpha_\theta} \mathbb{Z}. \tag{7.20}$$

The Pimsner–Voiculescu sequence is

$$\begin{tikzcd}
\mathrm{K}_0(C(S^1)) \arrow[r, "\mathrm{id}-\mathrm{K}_0(\alpha_\theta^{-1})"] & \mathrm{K}_0(C(S^1)) \arrow[r] & \mathrm{K}_0(A_\theta) \arrow[d] \\
\mathrm{K}_1(A_\theta) \arrow[u] & \mathrm{K}_1(C(S^1)) \arrow[l] & \mathrm{K}_1(C(S^1)) \arrow[l, "\mathrm{id}-\mathrm{K}_1(\alpha_\theta^-1)"]
\end{tikzcd} \tag{7.21}$$

$\mathrm{K}_0(C(S^1))$ is generated by $[1]_0$, hence $\mathrm{id} - \mathrm{K}_0(\alpha_\theta^{-1})$ is the zero map. Likewise, $e^{2\pi i\theta}u \sim_h u$, hence $\mathrm{id} - \mathrm{K}_1(\alpha_\theta^{-1})$ is also the zero map. Consequently, since $K_j(C(S^1)) \cong \mathbb{Z}$, we get

$$\mathrm{K}_0(A_\theta) \cong \mathbb{Z}^2 \cong \mathrm{K}_1(A_\theta). \tag{7.22}$$

Furthermore, $[u]_1, [v]_1$ are generators of $\mathrm{K}_1(A_\theta)$ and $\mathrm{K}_0(\tau) : \mathrm{K}_0(A_\theta) \cong \mathbb{Z}^2 \to \mathbb{Z} \cup \theta\mathbb{Z}$ is an isomorphism.

7.2 The Mayer–Vietoris sequence

The Mayer–Vietoris sequence in the classical case of topological spaces concerns relating the (co)homologies of a space that is glued from two (or more) subspaces to the (co)homologies of the subspaces and the way they are glued together. In the context of differential forms and De Rham cohomologies, it is natural (due to differentiability) to consider open subspaces. In the purely topological setting and in the realm of Gelfand theory for compact spaces, it seems to be more natural (also easier) to consider closed subsets. Thus we are trying to generalize the following situation to a noncommutative setting: There is a compact Hausdorff space X that is the union of two compact subspaces, which have a certain intersection. Diagrammatically:

$$\begin{tikzcd} X & X_1 \arrow[l] \\ X_2 \arrow[u] & X_1 \cap X_2 \arrow[l] \arrow[u] \end{tikzcd} \tag{7.23}$$

where the maps are injections of sets.

Dually, by Gelfand theory there is the following diagram:

$$\begin{tikzcd} C(X) \arrow[r] \arrow[d] & C(X_1) \arrow[d] \\ C(X_2) \arrow[r] & C(X_1 \cap X_2) \end{tikzcd} \tag{7.24}$$

where the maps are the natural restriction maps. In fact, it is almost obvious that $C(X) \cong \{(f_1, f_2) \in C(X_1) \oplus C(X_2) \mid f_1|X_1 \cap X_2 = f_2|X_1 \cap X_2\}$. Thus we are led to consider the following commutative diagram of unital C*-algebras:

$$\begin{tikzcd} A \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & B_1 \arrow[d, "\pi_1"] \\ B_2 \arrow[r, "\pi_2"'] & D \end{tikzcd} \tag{7.25}$$

where $A = \{(b_1, b_2) \in B_1 \oplus B_2 \mid \pi_1(b_1) = \pi_2(b_2)\}$, with π_1, π_2 surjective $*$-homomorphisms, pr_1 and pr_2 the restrictions of the natural projections $B_1 \oplus B_2 \to B_1$ and $B_1 \oplus B_2 \to B_2$ to the subspace $A \subseteq B_1 \oplus B_2$. A is called the pullback of B_1 and B_2 (over D), or the fiber product of B_1 and B_2 (over D), and the diagram (7.25) is called a pull-back diagram. We have

Theorem 7.4. *Corresponding to (7.25), there is a six-term exact sequence*

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "(\mathrm{K}_0(\mathrm{pr}_1),\mathrm{K}_0(\mathrm{pr}_2))"] & \mathrm{K}_0(B_1)\oplus\mathrm{K}_0(B_2) \arrow[r, "\mathrm{K}_0(\pi_2)-\mathrm{K}_0(\pi_1)"] & \mathrm{K}_0(D) \arrow[d] \\
\mathrm{K}_1(D) \arrow[u] & \mathrm{K}_1(B_1)\oplus\mathrm{K}_1(B_2) \arrow[l, "\mathrm{K}_1(\pi_2)-\mathrm{K}_1(\pi_1)"] & \mathrm{K}_1(A)) \arrow[l, "(\mathrm{K}_1(\mathrm{pr}_1),\mathrm{K}_1(\mathrm{pr}_2))"]
\end{tikzcd} \tag{7.26}$$

Proof. (partial, based on [bhms05], which is in turn based on ideas of Atiyah and Hirzebruch, see [ah62]). Define $\hat{A} \subseteq B_1 \oplus B_2 \oplus C([0,1],D)$ by

$$\hat{A} = \{(b_1, b_2, \omega \mid b_1 \in B_1, b_2 \in B_2, \omega(0) = \pi_1(b_1), \omega(1) = \pi_2(b_2)\} \tag{7.27}$$

Put

$$C_0(]0,1[,D) = \{\omega \in C((0,1), D \mid \omega(0) = \omega(1) = 0\}. \tag{7.28}$$

Then the sequence

$$0 \to C_0(]0,1[,D) \to \hat{A} \to B_1 \oplus B_2 \to 0 \tag{7.29}$$

is exact, where the map $C_0(]0,1[,D) \to \hat{A}$ is $\omega \mapsto (0,0,\omega)$, and the map $\hat{A} \to B_1 \oplus B_2$ is $(b_1, b_2, \omega) \mapsto (b_1, b_2)$. Exactness of this sequence at $C_0(]0,1[,D)$ and $\hat{A}$ is obvious, at $B_1 \oplus B_2$ it is due to the fact that $\omega \in C([0,1],D)$ can have any independent values $\omega(0), \omega(1) \in D$ (any two elements in a vector space are homotopic). As $C_0(]0,1[,D)$ is just the suspension of D, we have

$$K_j(C_0(]0,1[,D) \cong K_{1-j}(D), \quad j = 0,1. \tag{7.30}$$

We will show that

$$K_j(\hat{A}) \cong K_j(A), \quad j = 0,1. \tag{7.31}$$

Then (7.30) and (7.31) together allow to conclude that the 6-term exact sequence corresponding to the exact sequence (7.29) has the form

$$\begin{tikzcd}
\mathrm{K}_1(D) \arrow[r] & \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(B_1 \oplus B_2)) \arrow[d] \\
\mathrm{K}_1(B_1 \oplus B_2) \arrow[u] & \mathrm{K}_1(A) \arrow[l] & \mathrm{K}_0(D). \arrow[l]
\end{tikzcd} \tag{7.32}$$

which after a counter-clockwise rotation about one position gives just the claim of the theorem. It remains to prove (7.31). Our goal is to show that the map $i : A \to \hat{A}$, $(b_1, b_2) \mapsto (b_1, b_2, \overline{\pi_1(b_1)})$, where $\overline{\pi_1(b_1)}$ is the constant path at $\pi_1(b_1)$, is a K-isomorphism. Consider the ideal $I_1 := \ker(\mathrm{pr}_1 : A \to B_1) = \{(0, b_2) \in A\} = \{(0, b_2) \in B_1 \oplus B_2 \mid \pi_2(b_2) = 0\} \subseteq A$, being also isomorphic to $\ker \pi_2$ ($I_1 \ni (0, b_2) \mapsto b_2 \in \ker \pi_2$ being the isomorphism). The image of I_1 under i in A is $\hat{I}_1 = \{(0, b_2, \underline{0}) \mid \pi_2(b_2) = 0\}$. $\hat{I}_1$ is isomorphic to I_1, and is also an ideal in $\hat{A}$. Thus we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I_1 \arrow[r] \arrow[d, "j_1"] & A \arrow[r] \arrow[d, "j"] & A/I_1 \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & \hat{I}_1 \arrow[r] & \hat{A} \arrow[r] & \hat{A}/\hat{I}_1 \arrow[r] & 0.
\end{tikzcd} \tag{7.33}$$

Here, j_1 is an isomorphism, and both j and k are injective. Let us show that k is a homotopy equivalence: First let us note that

$$\begin{aligned} \hat{A}/\hat{I}_1 &= \{(b_1, b_2, \omega) \mid \omega = \pi_1(b_1), \omega(1) = \pi_2(b_2)\}/\{(0, b_2, \underline{0}) \mid \pi_2(b_2) = 0\} \\ &\cong \{(b_1, \omega) \mid b_1 \in B_1, \omega(0) = \pi_1(b_1)\} =: \hat{B}_1. \end{aligned}$$

The isomorphism is given by factorizing the map $(b_1, b_2, \omega) \mapsto (b_1, \omega)$, $\hat{A} \to \hat{B}_1$, whose kernel is $\{(0, b_2, \underline{0}) \mid \pi_2(b_2) = 0\}$, and which is obviously surjective. Define

$$\varphi : \hat{B}_1 \to B_1 \cong A/I_1, \quad \psi : b_1 \to \hat{B}_1, \tag{7.34}$$

by

$$\varphi(b_1, \omega) = b_1, \quad \varphi(b_1) = (b_1, \underline{\pi_1(b_1)}). \tag{7.35}$$

Then $\varphi \circ \psi = \mathrm{id}_{B_1}$, $\psi \circ \varphi(b_1, \omega) = (b_1, \underline{\pi_1(b_1)})$, and the homomorphisms $\varphi_t : \hat{B}_1 = \hat{A}/\hat{I}_1 \to \hat{B}_1$ defined by $\varphi_t(b_1, \omega) = (b_1, (1-t)\omega + t\underline{\pi_1(b_1)})$ satisfies $\varphi_0 = \mathrm{id}$, $\varphi_1 = \psi \circ \varphi$. This proves that A/I_1 and $\hat{A}/\hat{I}_1$ are homotopy equivalent and that $\mathrm{K}_j(k)$ are isomorphisms. Thus from the above commutative diagram (7.33) we obtain another commutative diagram by combining two 6-term exact sequences:

$$\begin{tikzcd}
\mathrm{K}_0(I_1) \arrow[rr] \arrow[dr, "\mathrm{K}_0(j_1)"] & & \mathrm{K}_0(A) \arrow[rr] \arrow[d, "\mathrm{K}_0(j)"] & & \mathrm{K}_0(A/I_1) \arrow[dl, "\mathrm{K}_0(k)"'] \arrow[ddd] \\
& \mathrm{K}_0(\hat{I}_1) \arrow[r] & \mathrm{K}_0(\hat{A}) \arrow[r] & \mathrm{K}_0(\hat{A}/\hat{I}_1) \arrow[d] & \\
& \mathrm{K}_1(\hat{A}/\hat{I}_1) \arrow[u] & \mathrm{K}_1(\hat{A}) \arrow[l] & \mathrm{K}_1(\hat{I}_1) \arrow[l] & \\
\mathrm{K}_1(A/I_1) \arrow[uuu] \arrow[ur, "\mathrm{K}_1(k)"] & & \mathrm{K}_1(A) \arrow[ll] \arrow[u, "\mathrm{K}_1(j)"'] & & \mathrm{K}_1(I_1) \arrow[ll] \arrow[ul, "\mathrm{K}_1(j_1)"']
\end{tikzcd} \tag{7.36}$$

The diagram has two exact circles, and since $\mathrm{K}_i(j_1)$ and $\mathrm{K}_i(k)$ are isomorphisms, we obtain from the Five Lemma that also $\mathrm{K}_i(j)$ are isomorphisms. Thus we have proved the desired isomorphism $\mathrm{K}_i(A) \cong \mathrm{K}_i(\hat{A})$. $\square$

Let us describe the connecting morphisms. For the morphism $\mathrm{K}_0(D) \to \mathrm{K}_1(A)$, let $P \in M_n(D)$ be an idempotent. Choose $P_1 \in M_n(B_1)$ and $P_2 \in M_n(B_2)$ such that $\pi_1(P_1) = P = \pi_2(P_2)$. (Here, π_1 and π_2 are the obvious extensions to matrices, which are also surjective.) Then $(e^{2\pi i P_1}, e^{2\pi i P_2}) \in B_1 \oplus B_2$ is in fact in $B_1 \oplus_D B_2$, because $e^{2\pi i P_1} \mapsto e^{2\pi i P} = I_n + (1 - e^{2\pi i})P = I_n$, $e^{-2\pi i P_2} \mapsto e^{-2\pi i P} = I_n + (1 - e^{-2\pi i})P = I_n$. Thus we have constructed the invertible element $(e^{2\pi i P_1}, e^{-2\pi i P_2}) \in M_n(A)$. The so-constructed map $P \mapsto (e^{2\pi i P_1}, e^{-2\pi i P_2})$ defines the desired morphism $\mathrm{K}_0(D) \to \mathrm{K}_1(A)$. If P is assumed to be selfadjoint, then P_1 and P_2 can be chosen to be selfadjoint (by Exercice 11 (ii)). Then the construction gives a unitary in $M_n(A)$. Note that without the minus sign on one side the resulting element $(e^{2\pi i P_1}, e^{2\pi i P_2}) = e^{2\pi i (P_1, P_2)} \in M_n(A)$ is homotopic to the identity (by the homotopy $[0,1] \ni t \mapsto e^{2\pi i (tP_1, tP_2)}$) and leads to a trivial map $\mathrm{K}_0(D) \to \mathrm{K}_1(A)$.

In order to construct the connecting morphism $\mathrm{K}_1(D) \to \mathrm{K}_0(A)$, let θ be an invertible in $M_n(D)$. Think of θ as acting on the right on $D \oplus D \ldots \oplus D$.

Consider the set $M_\theta := \{((v_1,\dots,v_n),(w_1,\dots,w_n)) \in B_1\oplus\cdots\oplus B_1\oplus B_2\oplus\cdots\oplus B_2 \mid (\pi_1(v_1),\dots,\pi_1(v_1))\theta = (\pi_2(w_1),\dots,\pi_2(w_n))\}$. M_θ is a finitely generated projective module over A. The connecting morphism we are looking for is now $[\theta]\mapsto [M_\theta]-[n] : \mathrm{K}_1(D)\to \mathrm{K}_0(A)$, where $[n]$ denotes the class of the free module of rank n over A. (We make use of the correspondence between idempotents and finitely generated projective modules.)

Example 7.5. Consider the circle S^1 as a union of two closed intervals, $S^1 = I\cup I$. Then we have a pull-back diagram

$$\begin{tikzcd} C(S^1) \arrow[r] \arrow[d] & C(I) \arrow[d] \\ C(I) \arrow[r] & \mathbb{C}\oplus\mathbb{C} \end{tikzcd} \tag{7.37}$$

and a corresponding Mayer–Vietoris six-term exact sequence

$$\begin{tikzcd} \mathrm{K}_0(C(S^1)) \arrow[r] & \mathrm{K}_0(C(I))\oplus \mathrm{K}_0(C(I)) \arrow[r] & \mathrm{K}_0(\mathbb{C}\oplus\mathbb{C}) \arrow[d] \\ \mathrm{K}_1(\mathbb{C}\oplus\mathbb{C}) \arrow[u] & \mathrm{K}_1(C(I)\oplus C(I)) \arrow[l] & \mathrm{K}_1(C(S^1)) \arrow[l] \end{tikzcd} \tag{7.38}$$

Let us take for granted that $\mathrm{K}_0(\mathbb{C}) = \mathbb{Z} = \mathrm{K}_0(C(I))$ and $\mathrm{K}_1(\mathbb{C}) = 0 = \mathrm{K}_1(C(I))$. Then the above diagram is reduced to

$$0\to \mathrm{K}_0(C(S^1))\to \mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}\to \mathrm{K}_1(C(S^1))\to 0. \tag{7.39}$$

We have to determine $\mathrm{K}_0(\pi_2)-\mathrm{K}_0(\pi_1) : \mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}$. $\pi_1=\pi_2 : C(I)\to\mathbb{C}\oplus\mathbb{C}$ is the map $f\mapsto (f(0),f(1))$. The generator of $\mathrm{K}_0(C(I))$ is $[1]_0$, so $\mathrm{K}_0(\pi_1)$ is determined by $\mathrm{K}_0(\pi_1)([1]_0) = ([1]_0,[1]_0)$ (where the 1 on the right is $1\in\mathbb{C}$). It follows that $\mathrm{K}_0(\pi_2)-\mathrm{K}_0(\pi_1)$ has on the generators $([1]_0,0)$ and $(0,[1]_0)$ of $\mathrm{K}_0(C(I)\oplus C(I))$ the values $-([1]_0,[1]_0)$ and $([1]_0,[1]_0)$. Thus $\operatorname{im}(\mathrm{K}_0(\pi_2)-\mathrm{K}_0(\pi_1))$ is the diagonal $\Delta\subseteq\mathbb{Z}\oplus\mathbb{Z}$, and $\mathrm{K}_1(C(S^1))\cong\mathbb{Z}\oplus\mathbb{Z}/\Delta\cong\mathbb{Z}$. On the other hand, also $\ker(\mathrm{K}_0(\pi_2)-\mathrm{K}_0(\pi_1)) = \Delta$, because $([1]_0,[1]_0)\mapsto -([1]_0,[1]_0)+([1]_0,[1]_0) = 0$ and $(n[1]_0,m[1]_0)\mapsto (m-n)([1]_0,[1]_0)\neq 0$ for $m\neq n$. So Δ is the image of the injective map $\mathrm{K}_0(C(S^1))\to\mathbb{Z}\oplus\mathbb{Z}$, i.e., $\mathrm{K}_0(C(S^1))\cong\Delta\cong\mathbb{Z}$.

7.3 The Künneth formula

In classical cohomology theory, say for differential forms, the Künneth formula states that the cohomology of a product of two manifolds is the (graded) tensor product of the cohomologies of the two factors,

$$H^*(M\times N)\cong H^*(M)\hat{\otimes} H^*(N). \tag{7.40}$$

For C*-algebras, the product of noncommutative spaces corresponds to the tensor product of the algebras, and the following theorem generalizes the classical Künneth formula:

Theorem 7.6. *Let A, B be C*-algebras, and assume that $K_*(B)$ is torsion-free and that A is separable and type I. Then*

$$K_*(A\otimes B)\cong K_*(A)\hat{\otimes} K_*(B). \tag{7.41}$$

Note that the formula (7.41) explicitly means

$$\begin{aligned} \mathrm{K}_0(A \otimes B) &\cong (\mathrm{K}_0(A) \otimes \mathrm{K}_0(B)) \oplus (\mathrm{K}_1(A) \otimes \mathrm{K}_1(B)), \\ \mathrm{K}_1(A \otimes B) &\cong (\mathrm{K}_0(A) \otimes \mathrm{K}_1(B)) \oplus (\mathrm{K}_1(A) \otimes \mathrm{K}_0(B)). \end{aligned}$$

Note also that there is no question about the kind of tensor product $A \otimes B$, because every separable type I C*-algebra is nuclear . Also, there are more general statements without assumptions about torsion, but still assuming nuclearity of at least one of the factors (see [b-b98].

Chapter 8

K-theory of graph C*-algebras

In this section, we briefly describe K-theory of *graph algebras*. Graph algebras are a natural and far reaching generalization of Cuntz-Krieger algebras. As a good introduction to general theory of graph algebras we recommend [r-i05]. Calculation of their K-theory presented below follows mainly [rs04]. A slightly different argument is given in [dt02]. Both approaches are ultimately based on and inspired by Cuntz's calculation of K-theory of Cuntz-Krieger algebras in [c-j81]. Closely related determination of K-groups of Cuntz-Krieger algebras corresponding to infinite matrices (the so called Exel-Laca algebras) can be found in [el00] and [s-w00]. Both these classes of C*-algebras may be realized as Cuntz-Pimsner algebras (see [k-t03] and [s-w00]), and this allows for yet another approach to their K-theory.

8.1 Universal graph C*-algebras

Let G be a directed graph with

$$\begin{aligned} G^0 &\quad - \quad \text{vertices}, \\ G^1 &\quad - \quad \text{edges}, \\ r, s\colon G^1 \to G^0 &\quad - \quad \text{range and source of an edge}. \end{aligned}$$

Definition 8.1. *The universal C^*-algebra $C^*(G)$ is given by generators*

$$\{p_v \mid v \in G^0\}, \quad \{s_e \mid e \in G^1\},$$

with the following relations:

- *p_v are mutually orthogonal projections i.e. $p_v^2 = p_v^* = p_v$ and $p_v p_w = 0$ for $v \neq w$,*
- *$s_e^* s_e = p_{r(e)}$ and $s_e^* s_f = 0$ for $e \neq f$,*
- *if the set $\{e \mid s(e) = v\}$ is nonempty (v is not a sink) and finite then*

$$p_v = \sum_{\{e \,\mid\, s(e)=v\}} s_e s_e^*,$$

- $s_e s_e^* \leqslant p_{s(e)}$.

Example 8.2. Some known C*-algebras arise in this way.

1. If G is only one vertex, then there is one generator $p = p^2 = p^*$. In this case $C^*(G) = \mathbb{C}$.

Figure 8.1: $\mathbb{C}$

2. G with one vertex and one edge (loop). Generators:

Figure 8.2: $C(S^1)$

$$p = p^2 = p^*, \quad s$$

relations:

$$s^*s = p = ss^*, \quad sp = ps, \quad s = ss^*s.$$

Then

$$C^*(G) = C^*(1, u) = C(S^1), \quad u - \text{unitary}.$$

3. G with two vertices and two edges like on the picture (8.3).

Figure 8.3: Toeplitz algebra $\mathcal{T}$

$$p_v = p_v^2 = p_v^*, \quad p_w = p_w^2 = p_w^*$$
$$s_e^* s_e = p_v, \quad s_f^* s_f = p_w$$
$$p_v = s_e s_e^* + s_f s_f^*.$$

$C^*(G)$ is isomorphic to the Toeplitz algebra - the universal C*-algebra for the relation $s^*s = 1$. The isomorphism is given by $s \mapsto s_e + s_f$.

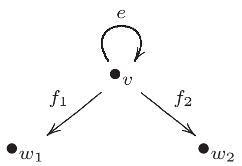


Figure 8.4: $C(S^2_{0\infty})$

4. G with three vertices and three edges like on a picture (8.4).

$$p_v = p_v^2 = p_v^*, \ \ p_{w_i} = p_{w_i}^2 = p_{w_i}^*, \ \ i = 1, 2,$$
$$p_v p_{w_i} = 0, \ \ p_{w_1} p_{w_2} = 0,$$
$$s_e^* s_e = p_v = s_e s_e^* + s_{f_1} s_{f_1}^* + s_{f_2} s_{f_2}^*,$$
$$s_{f_1}^* s_{f_1} = p_{w_1}, \ \ s_{f_2}^* s_{f_2} = p_{w_2}.$$

$C^*(G)$ is isomorphic to the quantum sphere

$$S^2_{0\infty}: \ \ B^*B = 1 - A^2, \ A = A^*, \ BB^* = 1, \ BA = 0$$

and the isomorphism is given by

$$\begin{array}{rcl} A & \mapsto & p_{w_1} - p_{w_2}, \\ B & \mapsto & s_e^* + s_{f_1}^* + s_{f_2}^*. \end{array}$$

We denote this graph by $G_{S^2_{0\infty}}$.

5. In the example (4) we glue the vertices w_1, w_2 into one w obtaining graph G like on a picture (8.5).

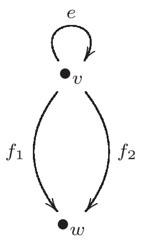


Figure 8.5: $C(\mathbb{R}P^2_q)$

$$p_v = p_v^2 = p_v^*, \ \ p_w = p_w^2 = p_w^*,$$
$$p_v p_w = 0,$$
$$s_e^* s_e = p_v = s_e^* s_e + s_{f_1} s_{f_1}^* + s_{f_2} s_{f_2}^*,$$

$$s_{f_1}^* s_{f_1} = p_w = s_{f_2}^* s_{f_2}.$$

Define $\mathbb{Z}_2$-action on the graph in the example (4).

$$s_e \mapsto -s_e, \quad s_{f_1} \mapsto -s_{f_2}, \quad s_{f_2} \mapsto -s_{f_1}$$

Then

$$p_v \mapsto p_v, \quad p_{w_1} \mapsto p_{w_2}, \quad p_{w_2} \mapsto p_{w_1}.$$

This action corresponds to

$$A \mapsto -A, \quad B \mapsto -B$$

under the identification

$$C^*(G_{S^2_{0\infty}}) \cong C(S^2_{0\infty}).$$

If we take the quotient $C(S^2_{0\infty})/\mathbb{Z}_2$ we obtain $C(\mathbb{R}P^2_q)$ - quantum projective space. On the other hand the quotient of the graph C*-algebra $C^*(G_{S^2_{0\infty}})$ by the defined action is the graph C*-algebra for our graph, which we now can denote $C^*(G_{\mathbb{R}P^2_q})$. The isomorphism is given by

$$p_v \mapsto p_v, \quad p_w \mapsto p_{w_1} + p_{w_2},$$

$$s_e \mapsto s_e s_e, \quad s_{f_1} \mapsto s_e(s_{f_1} + s_{f_2}), \quad s_{f_2} \mapsto s_{f_1} - s_{f_2}.$$

Note that this $\mathbb{Z}_2$ action is not induced from a graph automorphism.

6. G with one vertex and n edges like on the picture (8.6).

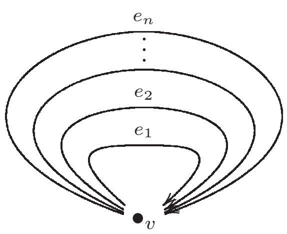


Figure 8.6: Cuntz algebra O_n

$$r(e_k) = s(e_k) = v, \quad k = 1, \dots, n,$$

$$p = s_{e_k}^* s_{e_k} = \sum_{k=1}^{n} s_{e_k} s_{e_k}^*,$$

$$s_{e_k}^* s_{e_{k'}} = 0 \text{ for } k \neq k'.$$

When $p = 1$ then $C^*(G)$ is the Cuntz algebra O_n - the universal C*-algebra for the relations

$$s_k^* s_k = 1, \quad k = 1, \dots n, \quad \sum_{k=1}^{n} s_k s_k^* = 1.$$

$$\bullet_0 \xrightarrow{e_0} \bullet_1 \xrightarrow{e_1} \bullet_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-2}} \bullet_{n-1} \xrightarrow{e_{n-1}} \bullet_n$$

Figure 8.7: $M_n(\mathbb{C})$

7. G with n vertices and $(n-1)$ edges in the straight segment as in the picture (8.7).

$$s(e_k) = k, \;\; r(e_k) = k+1 \text{ for } k = 1, \dots, n-1,$$

$$p_k = s_{e_k} s_{e_k}^*, \;\; p_{k+1} = s_{e_k}^* s_{e_k} \text{ for } k = 1, \dots n-1,$$
$$s_{e_k}^* s_{e_{k'}} = 0 \text{ for } k \neq k'.$$

$C^*(G)$ is the algebra of complex matrices $n \times n$, that is $M_n(\mathbb{C})$.

8. Similarly to the previous example we take straight segment, but infinite in both directions. Vertices are indexed by integers as in the picture (8.8).

$$\cdots \xrightarrow{e_{-2}} \bullet_{-1} \xrightarrow{e_{-1}} \bullet_0 \xrightarrow{e_0} \bullet_1 \xrightarrow{e_1} \bullet_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-2}} \bullet_{n-1} \xrightarrow{e_{n-1}} \bullet_n \xrightarrow{e_n} \cdots$$

Figure 8.8: Compact operators $\mathcal{K}$

$$s(e_k) = k, \;\; r(e_k) = k+1, \;\; k \in \mathbb{Z},$$
$$p_k = s_{e_k} s_{e_k}^*, \;\; p_{k+1} = s_{e_k}^* s_{e_k},$$
$$s_{e_k}^* s_{e_{k'}} = 0 \text{ for } k \neq k'.$$

We obtain algebra of compact operators $\mathcal{K}$, the limit of the algebras in the preceeding example.

9. G with n vertices and n edges forming a cycle as in the picture (8.9).

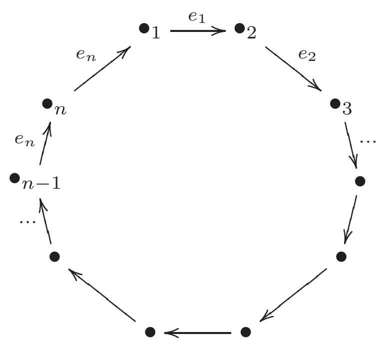


Figure 8.9: $M_n(C(S^1))$

$$s(e_k) = k, \;\; r(e_k) = k+1 \text{ for } k = 1, \ldots, n-1, \;\; r(e_n) = 1,$$

$$p_k = s_{e_k} s_{e_k}^*, \;\; p_{k+1} = s_{e_k}^* s_{e_k},$$
$$s_{e_k}^* s_{e_{k'}} = 0 \text{ for } k \neq k'.$$

We obtain algebra of matrices over the algebra of functions on the circle, $C^*(G) = M_n(C(S^1))$.

10. G with two vertices with loops and connected by one edge.

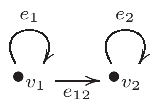


Figure 8.10: $C(\mathrm{SU}_q(2))$

$$p_{v_i} = p_{v_i}^* = p_{v_i}^2, \;\; i = 1, 2, \;\; p_{v_1} p_{v_2} = 0,$$

$$p_{v_1} = s_{e_{11}}^* s_{e_{11}} = s_{e_{11}}^* s_{e_{11}} + s_{e_{12}} s_{e_{11}}^*,$$
$$p_{v_2} = s_{e_{22}}^* s_{e_{22}} = s_{e_{12}}^* s_{e_{12}} = s_{e_{22}} s_{e_{22}}^*,$$
$$s_{e_{11}}^* s_{e_{12}} = 0, \;\; s_{e_{11}}^* s_{e_{22}} = 0, \;\; s_{e_{12}}^* s_{e_{22}} = 0.$$

We obtain C*-algebra for quantum SU(2), that is $C(\mathrm{SU}_q(2)) \cong C(\mathrm{SU}_0(2))$, which is generated by two elements a, b satisfying the relations

$$a^* a + b^* b = 1, \;\; a a^* + q^2 b^* b = 1,$$

$$ab = qba, \;\; ab^* = qb^* a, \;\; b^* b = bb^*.$$

The isomorphism is given by

$$\begin{array}{rcl} a & \mapsto & s_{e_{11}}^* + s_{e_{12}}^*, \\ b & \mapsto & s_{e_{22}}. \end{array}$$

11. The example (10) can be treated as the C*-algebra of the quantum sphere S_q^3. Now we present graph C*-algebra for the quantum sphere S_q^7, which is next generalized to arbitrary odd dimension. We take a graph G with four vertices with loops and each vertex is connected with all vertices with the greater index as in the picture (8.11). The C*-algebra for the quantum sphere S_q^7 is generated by the four elements z_1, z_2, z_3, z_4 satisfying the relations

$$\begin{array}{rcl} z_j z_i & = & q z_i z_j \text{ for } i < j, \\ z_j^* z_i & = & q z_i z_j^* \text{ for } i \neq j, \\ z_1^* z_1 & = & z_1 z_1^* + (1-q^2)(z_2 z_2^* + z_3 z_3^* + z_4 z_4^*), \\ z_2^* z_2 & = & z_2 z_2^* + (1-q^2)(z_3 z_3^* + z_4 z_4^*), \\ z_3^* z_3 & = & z_3 z_3^* + (1-q^2) z_4 z_4^*, \\ z_4^* z_4 & = & z_4 z_4^*, \\ z_1 z_1^* + z_2 z_2^* + z_3 z_3^* + z_4 z_4^* & = & 1. \end{array}$$

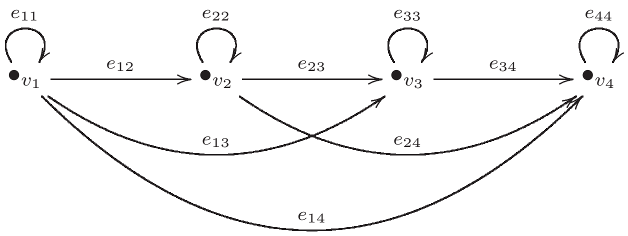


Figure 8.11: $C(S_q^7)$

For $q = 0$ we have the isomorphism $C^*(G) \cong C(S_0^7)$ given by

$$\begin{aligned} z_1 &\mapsto s_{e_{11}} + s_{e_{12}} + s_{e_{13}} + s_{e_{14}}, \\ z_2 &\mapsto s_{e_{22}} + s_{e_{23}} + s_{e_{24}}, \\ z_3 &\mapsto s_{e_{33}} + s_{e_{34}}, \\ z_4 &\mapsto s_{e_{44}}. \end{aligned}$$

12. As in the example (11) we take a graph with n vertices and edge between v_i and v_j if and only if $i \leqslant j$ as in the picture (8.12).

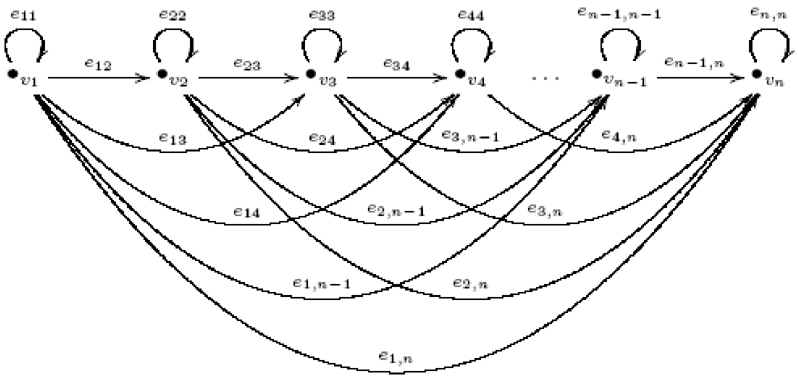


Figure 8.12: $C(S_q^{2n-1})$

$$v_1, \ldots, v_n,$$

$$e_{ij}, \ \ j = i, \ldots, n, \ \ s(e_{ij}) = v_i, \ \ r(e_{ij}) = v_j.$$

The C*-algebra for the quantum sphere S_q^{2n-1} is generated by the n elements $z_1, \dots, z_n$ satisfying the relations

$$\begin{aligned} z_j z_i &= q z_i z_j \text{ for } i < j, \\ z_j^* z_i &= q z_i z_j^* \text{ for } i \neq j, \\ z_i^* z_i &= z_i z_i^* + (1-q^2)\left(\sum_{j>i} z_j z_j^*\right) \text{ for } i = 1, \dots, n, \\ \sum_{i=1}^{n} z_i z_i^* &= 1. \end{aligned}$$

For $q = 0$ we have the isomorphism $C^*(G) \cong C(S_0^{2n-1})$ given by

$$z_i \mapsto \sum_{j=i}^{n} s_{e_{ij}}, \quad i = 1, \dots, n.$$

13. We take a similar graph G to the one in the example (11), but with infinitely many paralell edges $v_i \to v_j$ for $i < j$.

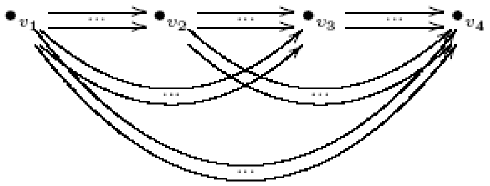


Figure 8.13: $C(\mathbb{C}P_q^3)$

14. We take a similar graph G to the one in the example (12), but with infinitely many paralell edges $v_i \to v_j$ for $i < j$.

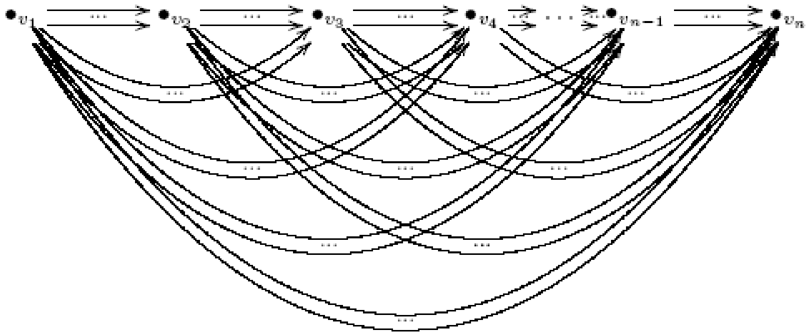


Figure 8.14: $C(\mathbb{C}P_q^{n-1})$

15. If we modify the graph for the quantum sphere S_q^5 by adding two additional vertices w_1, w_2 and edges from each vertex v_1, v_2, v_3 to both of the added ones, then we obtain graph for the sphere S_q^6 as in the picture (8.15).

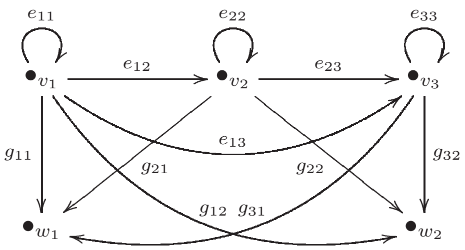


Figure 8.15: $C(S_q^6)$

16. The example (15) can be generalized to arbitrary even dimension just by adding two vertices w_1, w_2 to the graph of the sphere S_q^{2n-1}. We have $n+2$ vertices $v_1, \ldots, v_n$ and w_1, w_2. Edges e_{ij} are from v_i to v_j whenever $i \leqslant j$ and g_{ik} are between v_i and w_k for $k = 1, 2$. More precisely for $i = 1, \ldots, n$ we have

$$s(e_{ij}) = v_i, \;\; r(e_{ij}) = v_j, \;\; j = i, \ldots, n,$$

$$s(g_{ik}) = v_i, \;\; r(g_{ik}) = w_k, \;\; k = 1, 2.$$

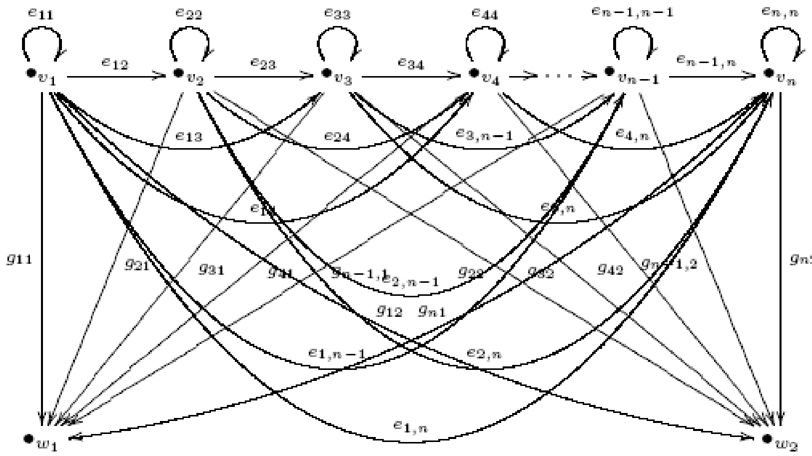


Figure 8.16: $C(S_q^{2n})$

8.2 Computation of K-theory

The main tool for the computation of K-theory groups of the graph C*-algebras is the following

Theorem 8.3. *Let G be a directed graph and let $G_+^0 \subset G^0$ be the collection of vertices that emit at least one and at most finitely many edges. Let $\mathbb{Z}G_+^0$ and*

$\mathbb{Z}G^0$ be the free abelian groups on free generators G^0_+ and G^0. Let $A_G\colon \mathbb{Z}G^0_+ \to \mathbb{Z}G^0$ be the map defined by the formula

$$A_G(v) := \left(\sum_{e\in G^1,\ s(e)=v} r(e)\right) - v.$$

Then

$$\begin{aligned}\mathrm{K}_0(C^*(G)) &\cong \operatorname{coker} A_G\\ \mathrm{K}_1(C^*(G)) &\cong \ker A_G\end{aligned}$$

The proof of this theorem will be postponed to the section (8.3), and now we compute the K-theory groups of the graph C*-algebras for the examples from the Section 8.1.

Example 8.4. 1. $\mathrm{K}_*(\mathbb{C})$

$$\begin{aligned}G^0 &= \{v\}\\ G^0_+ &= \emptyset\end{aligned}$$

$$A_G\colon \emptyset \to \mathbb{Z}$$

In this case A_G is from the empty set, but still we can write

$$\begin{aligned}\mathrm{K}_0(\mathbb{C}) &= \operatorname{coker} A_G &= \mathbb{Z}\\ \mathrm{K}_1(\mathbb{C}) &= \ker A_G &= 0\end{aligned}$$

2. $\mathrm{K}_*(C(S^1))$

$$\begin{aligned}G^0 &= \{v\}\\ G^0_+ &= \{v\}\end{aligned}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z}$$

$$v \mapsto v - v = 0$$

$$\begin{aligned}\mathrm{K}_0(C(S^1)) &= \operatorname{coker} A_G &= \mathbb{Z}\\ \mathrm{K}_1(C(S^1)) &= \ker A_G &= \mathbb{Z}\end{aligned}$$

3. $\mathrm{K}_*(\mathcal{T})$

$$\begin{aligned}G^0 &= \{v, w\}\\ G^0_+ &= \{v\}\end{aligned}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$$

$$v \mapsto v + w - v = w$$

$$\begin{aligned}\mathrm{K}_0(\mathcal{T}) &= \operatorname{coker} A_G &= \mathbb{Z}\\ \mathrm{K}_1(\mathcal{T}) &= \ker A_G &= 0\end{aligned}$$

4. $\mathrm{K}_*(C(S^2_{0\infty}))$

$$\begin{aligned} G^0 &= \{v, w_1, w_2\} \\ G^0_+ &= \{v\} \end{aligned}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$v \quad \mapsto \quad v + w_1 + w_2 - v = w_1 + w_2$$

$$\begin{aligned} \mathrm{K}_0(C(S^2_{0\infty})) &= \operatorname{coker} A_G &= \mathbb{Z} \oplus \mathbb{Z} \\ \mathrm{K}_1(C(S^2_{0\infty})) &= \ker A_G &= 0 \end{aligned}$$

5. $\mathrm{K}_*(C(\mathbb{R}P^2_q))$

$$\begin{aligned} G^0 &= \{v, w\} \\ G^0_+ &= \{v\} \end{aligned}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

$$v \quad \mapsto \quad v + 2w - v = 2w$$

$$\begin{aligned} \mathrm{K}_0(C(\mathbb{R}P^2_q)) &= \operatorname{coker} A_G &= \mathbb{Z} \oplus \mathbb{Z}_2 \\ \mathrm{K}_1(C(\mathbb{R}P^2_q)) &= \ker A_G &= 0 \end{aligned}$$

6. $\mathrm{K}_*(O_n)$

$$\begin{aligned} G^0 &= \{v\} \\ G^0_+ &= \{v\} \end{aligned}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z}$$

$$v \quad \mapsto \quad nv - v = (n-1)v$$

$$\begin{aligned} \mathrm{K}_0(O_n) &= \operatorname{coker} A_G &= \mathbb{Z}_{n-1} \\ \mathrm{K}_1(O_n) &= \ker A_G &= 0 \end{aligned}$$

7. $\mathrm{K}_*(M_n(\mathbb{C}))$

$$\begin{aligned} G^0 &= \{v_1, v_2, \ldots, v_{n-1}\} \\ G^0_+ &= \{v_1, v_2, \ldots, v_n\} \end{aligned}$$

$$A_G\colon \mathbb{Z}^{n-1} \to \mathbb{Z}^n$$

$$v_i \quad \mapsto \quad v_{i+1} - v_i \text{ for } i = 1, \ldots, n-1$$

$$\begin{aligned} \mathrm{K}_0(M_n(\mathbb{C})) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(M_n(\mathbb{C})) &= \ker A_G &= 0 \end{aligned}$$

8. $\mathrm{K}_*(\mathcal{K})$

$$\begin{aligned} G^0 &= \{v_i \mid i \in \mathbb{Z}\} \\ G^0_+ &= \{v_i \mid i \in \mathbb{Z}\} \end{aligned}$$

$$A_G\colon \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \to \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$$

$$v_i \mapsto v_{i+1} - v_i \text{ for } i \in \mathbb{Z}$$

$$\begin{aligned} \mathrm{K}_0(\mathcal{K}) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(\mathcal{K}) &= \ker A_G &= 0 \end{aligned}$$

Remark 8.5. If we take direct product instead of direct sum, then there will be nontrivial kernel.

9. $\mathrm{K}_*(M_n(S^1))$

$$\begin{aligned} G^0 &= \{v_1, v_2, \dots, v_n\} \\ G^0_+ &= \{v_1, v_2, \dots, v_n\} \end{aligned}$$

$$A_G\colon \mathbb{Z}^n \to \mathbb{Z}^n$$

$$\begin{aligned} v_i &\mapsto v_{i+1} - v_i \text{ for } i = 1, \dots, n-1, \\ v_n &\mapsto v_1 - v_n \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(M_n(S^1)) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(M_n(S^1)) &= \ker A_G &= \mathbb{Z} \end{aligned}$$

10. $\mathrm{K}_*(C(\mathrm{SU}_q(2)))$

$$\begin{aligned} G^0 &= \{v_1, v_2\} \\ G^0_+ &= \{v_1, v_2\} \end{aligned}$$

$$A_G\colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

$$\begin{aligned} v_1 &\mapsto v_1 + v_2 - v_1 = v_2, \\ v_2 &\mapsto v_2 - v_2 = 0 \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(C(\mathrm{SU}_q(2))) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(C(\mathrm{SU}_q(2))) &= \ker A_G &= \mathbb{Z} \end{aligned}$$

11. $\mathrm{K}_*(C(S^7_q))$

$$\begin{aligned} G^0 &= \{v_1, v_2, v_3, v_4\} \\ G^0_+ &= \{v_1, v_2, v_3, v_4\} \end{aligned}$$

$$A_G\colon \mathbb{Z}^4 \to \mathbb{Z}^4$$

$$\begin{aligned} v_1 &\mapsto v_1 + v_2 + v_3 + v_4 - v_1 = v_2 + v_3 + v_4 \\ v_2 &\mapsto v_2 + v_3 + v_4 - v_2 = v_3 + v_4 \\ v_3 &\mapsto v_3 + v_4 - v_3 = v_4 \\ v_4 &\mapsto v_4 - v_4 = 0 \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(C(S^7_q)) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(C(S^7_q)) &= \ker A_G &= \mathbb{Z} \end{aligned}$$

12. $\mathrm{K}_*(C(S_q^{2n-1}))$

$$\begin{aligned} G^0 &= \{v_i \mid i = 1, \dots, n\} \\ G^0_+ &= \{v_i \mid i = 1, \dots, n\} \end{aligned}$$

$$A_G \colon \mathbb{Z}^n \to \mathbb{Z}^n$$

$$v_i \mapsto \sum_{j \geqslant i} v_j - v_i = \sum_{j > i} v_j$$

$$\begin{aligned} \mathrm{K}_0(C(S_q^{2n-1})) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(C(S_q^{2n-1})) &= \ker A_G &= \mathbb{Z} \end{aligned}$$

13. $\mathrm{K}_*(C(\mathbb{C}P_q^3))$

$$\begin{aligned} G^0 &= \{v_1, v_2, v_3, v_4\} \\ G^0_+ &= \emptyset \end{aligned}$$

$$A_G \colon \emptyset \to \mathbb{Z}^4$$

$$\begin{aligned} \mathrm{K}_0(C(\mathbb{C}P_q^3)) &= \operatorname{coker} A_G &= \mathbb{Z}^4 \\ \mathrm{K}_1(C(\mathbb{C}P_q^3)) &= \ker A_G &= 0 \end{aligned}$$

14. $\mathrm{K}_*(C(\mathbb{C}P_q^{n-1}))$

$$\begin{aligned} G^0 &= \{v_i \mid i = 1, \dots, n\} \\ G^0_+ &= \emptyset \end{aligned}$$

$$A_G \colon \emptyset \to \mathbb{Z}^n$$

$$\begin{aligned} \mathrm{K}_0(C(\mathbb{C}P_q^{n-1})) &= \operatorname{coker} A_G &= \mathbb{Z}^n \\ \mathrm{K}_1(C(\mathbb{C}P_q^{n-1})) &= \ker A_G &= 0 \end{aligned}$$

15. $\mathrm{K}_*(C(S_q^6))$

$$\begin{aligned} G^0 &= \{v_1, v_2, v_3, w_1, w_2\} \\ G^0_+ &= \{v_1, v_2, v_3\} \end{aligned}$$

$$A_G \colon \mathbb{Z}^3 \to \mathbb{Z}^5$$

$$\begin{aligned} v_1 &\mapsto v_1 + v_2 + v_3 + w_1 + w_2 - v_1 = v_2 + v_3 + w_1 + w_2 \\ v_2 &\mapsto v_2 + v_3 + w_1 + w_2 - v_2 = v_3 + w_1 + w_2 \\ v_3 &\mapsto v_3 + w_1 + w_2 - v_3 = w_1 + w_2 \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(C(S_q^6)) &= \operatorname{coker} A_G &= \mathbb{Z} \oplus \mathbb{Z} \\ \mathrm{K}_1(C(S_q^6)) &= \ker A_G &= 0 \end{aligned}$$

16. $\mathrm{K}_*(C(S_q^{2n}))$

$$\begin{aligned} G^0 &= \{v_1, \dots, v_n, w_1, w_2\} \\ G^0_+ &= \{v_1, \dots, v_n\} \end{aligned}$$

$$A_G \colon \mathbb{Z}^n \to \mathbb{Z}^{n+2}$$

$$v_i \mapsto \sum_{j \geqslant i} v_j + w_1 + w_2 - v_i = \sum_{j > i} v_j + w_1 + w_2$$

$$\begin{aligned} \mathrm{K}_0(C(S_q^{2n})) &= \operatorname{coker} A_G &= \mathbb{Z} \oplus \mathbb{Z} \\ \mathrm{K}_1(C(S_q^{2n})) &= \ker A_G &= 0 \end{aligned}$$

8.3 Idea of proof of the theorem (8.3)

There are seven steps in the proof, which we will sketch here.

1. Gauge action γ.

$$\gamma\colon\ \mathrm{U}(1) = S^1 \to \mathrm{Aut}(C^*(G))$$

$$\begin{aligned} \gamma_z(s_e) &= zs_e, \\ \gamma_z(p_v) &= p_v. \end{aligned}$$

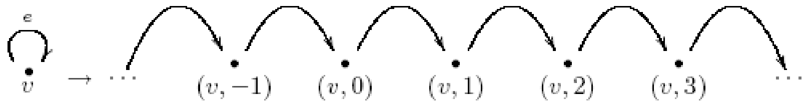
2. $C^*(G) \rtimes_\gamma \mathrm{U}(1) \cong C^*(G \times \mathbb{Z})$.
   We construct the new graph $G \times \mathbb{Z}$

$$\begin{aligned} (G \times \mathbb{Z})^0 &= G^0 \times \mathbb{Z}, \\ (G \times \mathbb{Z})^1 &= G^1 \times \mathbb{Z}. \end{aligned}$$

   It has no loops and

$$s(e,n) = (s(e), n-1), \ \ r(e,n) = (r(e), n).$$

   Each loop is resolved in the infinite segment



3. $C^*(G \times \mathbb{Z})$ is AF.
   It follows that $\mathrm{K}_1(C^*(G \times \mathbb{Z})) = 0$.
4. Dual action $\hat{\gamma}$.

$$\hat{\gamma}\colon \mathbb{Z} \to \mathrm{Aut}(C^*(G) \rtimes_\gamma \mathrm{U}(1))$$

$$\hat{\gamma}_\chi(f)(t) = \langle \chi, t\rangle f(t), \ \text{where } f\colon\ \mathrm{U}(1) \to C^*(G).$$

5. Takesaki-Takai duality.

$$(C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z} \cong C^*(G) \times \mathcal{K}.$$

   From the stability of K_* it follows that

$$\mathrm{K}_*((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z}) \cong \mathrm{K}_*(C^*(G)).$$

6. Pimsner–Voiculescu sequence.

   The Pimsner–Voiculescu sequence is as follows

$$\begin{tikzcd}
\mathrm{K}_0((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \arrow[r, "\mathrm{id}-\mathrm{K}_0(\hat{\gamma}^{-1})"] & \mathrm{K}_0((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \arrow[r] & \mathrm{K}_0((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z}) \arrow[d] \\
\mathrm{K}_1((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z}) \arrow[u] & \mathrm{K}_1((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \arrow[l] & \mathrm{K}_1((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \arrow[l, "\mathrm{id}-\mathrm{K}_1(\hat{\gamma}^{-1})"']
\end{tikzcd}$$

   where the maps are given by the formulas

$$\mathrm{K}_*(C^*(G) \rtimes_\gamma \mathrm{U}(1)) \xrightarrow{\mathrm{id}-\mathrm{K}_*(\hat{\gamma}^{-1})} \mathrm{K}_*(C^*(G) \rtimes_\gamma \mathrm{U}(1)),$$

$$\mathrm{K}_*(C^*(G) \rtimes_\gamma \mathrm{U}(1)) \xrightarrow{\mathrm{id}-\mathrm{K}_*(\beta^{-1})} \mathrm{K}_*((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z}),$$

   and the map $\beta\colon \mathbb{Z} \to \mathrm{Aut}(C^*(G \times \mathbb{Z}))$ is given by

$$\begin{aligned}
\beta_m(p_{(v,n)}) &= p_{(v,n+m)}, \\
\beta_m(s_{(e,n)}) &= s_{(e,n+m)}.
\end{aligned}$$

   Using the preceding computations we can write the sequence as

$$\begin{tikzcd}
\mathrm{K}_0(C^*(G \times \mathbb{Z})) \arrow[r, "\mathrm{id}-\mathrm{K}_0(\hat{\gamma}^{-1})"] & \mathrm{K}_0(C^*(G \times \mathbb{Z})) \arrow[r, "1-\mathrm{K}_0(\beta^{-1})"] & \mathrm{K}_0(C^*(G)) \arrow[d] \\
\mathrm{K}_1(C^*(G)) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

7. Computation of the kernel and cokernel of $1 - \mathrm{K}_0(\hat{\gamma}^{-1})$.

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Part II

Foliations, C*-algebras, and Index Theory

by

Paul F. Baum

Henri Moscovici

Based on the lectures of:

- Paul F. Baum
  (*Mathematics Department, McAllister Building The Pennsylvania State University, University Park, PA 16802, USA*)
  – Chapters 8, 9

- Henri Moscovici
  (*Department of mathematics, The Ohio State University, Columbus, OH 43210, USA*)
  – Chapters 1, 2, 3, 4, 5, 6, 7

With additional lectures by:

- Piotr M. Hajac – Section 7.1

- Tomasz Maszczyk – Section 2.5

Chapter 1

Foliations

1.1 What is a foliation and why is it interesting?

Question 1 (H. Hopf). Is there a completely integrable plane field on S^3? (Plane field - two dimensional subbundle $E \subset TS^3$).

Answer 1 (G. Reeb). Yes, it is a tangent bundle to a 2-dimensional Reeb foliation of S^3, described in the example (1.2(6)).

Question 2 (A. Haefliger). Given a plane subbundle E of TM is it homotopic to an integrable one?

Answer 2 (R. Bott). There exists at least one obstruction; not every subbundle has in its K-theory class as an integrable one.

Roughly speaking, a foliation is the decomposition of a manifold M^n into disjoint family of submanifolds (immersed injectively) of dimension $n-q$, which is locally trivial.

More precisely

Definition 1.1. *A* codimension q foliation *of a manifold M^n is a family $\mathcal{F} = \{L_\alpha\}_{\alpha\in\mathcal{I}}$ of $(n-q)$-dimensional connected, injectively immersed submanifolds that satisfy*

1. $$L_\alpha \cap L_\beta \neq \emptyset \textit{ iff. } \alpha = \beta \textit{ and } \bigcup_{\alpha\in\mathcal{I}} L_\alpha = M.$$

2. *For all $p \in M$ there exist open $U \ni p$ and a diffeomorphism*
$$\varphi\colon U \to \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q,$$
such that for all $\alpha \in \mathcal{I}$
$$\varphi((U \cap L_\alpha)\textit{conn. comp.}) = \{\,\underline{x}; x_{n-q+1} = c_{n-q+1}, \ldots, x_n = c_n\},$$
$$c_j = \textit{ constant}, \quad j = n-q+1, \ldots, n.$$

Example 1.2.

1. Fibrations.

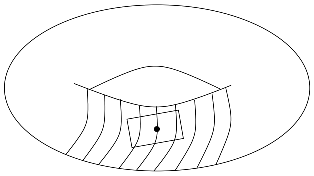


Figure 1.1: $\varphi\colon U \to \mathbb{R}^n$

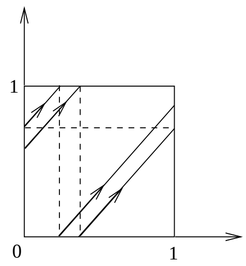


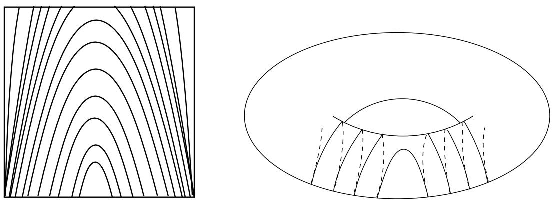
Figure 1.2: Kronecker foliation of $\mathbb{T}$

2. Surjective submersions.

3. The Kronecker foliation of $\mathbb{T} = S^1 \times S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$. Solutions of differential equation $\frac{\partial}{\partial y} = \lambda \frac{\partial}{\partial x}$ with $\lambda = \tan(\theta)$ fixed. If a slope is rational then we get a closed curve – closed leaves of foliation. If $\lambda \notin \mathbb{Q}$ then leaves are dense – they are immersions of $\mathbb{R}$ which is not closed manifold.

   Rough quotient space $M/\mathcal{F}$. Two points are equivalent if and only if they belong to the same leaf. In the Kronecker foliation, when leaves are dense, we get a noncommutative torus.

4. The 1-dimensional Reeb foliation of $\mathbb{T}$.



5. The 2-dimensional Reeb foliation of a solid torus $D^2 \times S^1$.

   In the universal cover $D^2 \times \mathbb{R} \to D^2 \times S^1$

   We rotate these curves along vertical axis and define relation $(x, y, z) \sim (x, y, z+1)$. We have one closed leaf (boundary) and rest are open leaves (images of not closed manifolds).

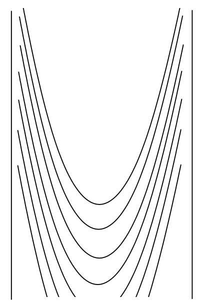


Figure 1.3: Translations of a curve

6. The 2-dimensional Reeb foliation of S^3.

$$S^3 = D^2 \times S^1 \coprod S^1 \times D^2 / \sim$$

$$S^3 = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

   The two tori in above decomposition are

$$\{ x \in S^3 \mid x_1^2 + x_2^2 \leqslant \frac{1}{2}\}$$

$$\{ x \in S^3 \mid x_1^2 + x_2^2 \geqslant \frac{1}{2}\}$$

   We put on each torus Reeb's foliation from preceeding example.

The notion of foliation is interesting for two reasons:

1. the definition is multifaceted
2. it gives rise to an interesting equivalence relation on M, which in turn gives rise to an interesting quotient "space" $M/\mathcal{F}$.

1.2 Equivalent definitions

Definition 1.3 (Manifold reformulation)**.** *There exists covering of M by charts (U_i, φ_i) such that $\varphi(U_i) = V_i \times W_i$, where V_i and W_i are open subsets of $\mathbb{R}^{n-q}$ and $\mathbb{R}^q$, respectively, with the property that if $U_i \cap U_j \neq \emptyset$ then the diffeomorphism*

$$\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is of the form

$$(x, y) \mapsto (h_{ij}(x, y), g_{ij}(y)),\ g_{ij} \colon W_i^\circ \to W_j^\circ.$$

Definition 1.4 (1-cocycle reformulation)**.** *There exists collection* (U_i, f_i, g_{ij}), *where* (U_i) *is a covering of* M, $f_i \colon U_i \to W_i$ *are surjective submersions onto open q-dimensional manifolds,* $g_{ij} \colon f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ *- diffeomorphisms satisfying*

$$f_i = g_{ij} \circ f_j \text{ on } U_i \cap U_j \text{ and } g_{ij} \circ g_{jk} = g_{ik} \text{ on } U_i \cap U_j \cap U_k.$$

Definition 1.5. *Let* $(M, \mathcal{F})$ *be manifold with foliation. The* tangent bundle *to* $\mathcal{F}$ *is*

$$\tau\mathcal{F} := \{\, X \in TM \bigm| X \text{ tangent to a leaf} \,\}.$$

Let $\Gamma(\tau\mathcal{F})$ denote the space of smooth sections of this bundle. Clearly this is an involutive subbundle, i.e.

$$[\Gamma(\tau\mathcal{F}), \Gamma(\tau\mathcal{F})] \subset \Gamma(\tau\mathcal{F}).$$

because this is local property, obvious on charts.

Conversely by theorem of Frobenius we can write

Proposition 1.6. *Any involutive subbundle* $E \subset TM$ *is the tangent bundle to a unique foliation.*

Equivalently we can say

Proposition 1.7. *The ideal* $\mathcal{I}(E)$ *generated by the sections of*

$$\nu\mathcal{F} = \{\, \omega \in T^*M \bigm| \forall X \in \tau\mathcal{F}\ \omega(X) = 0\}$$

is closed under d, *i.e.* $\mathcal{I}(E)$ *is a differential ideal.*

1.3 Holonomy groupoid

Let $x, y \in L \subset M$ be points in a leaf of foliation, $\gamma \colon [0,1] \to M$ - path from x to y contained in L.

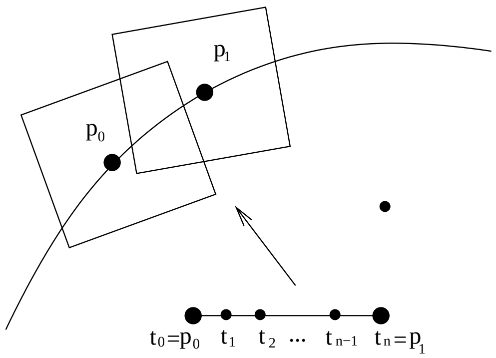


Figure 1.4: Path in the leaf L

Let W -transversal through $\underline{x} = \varphi^{-1}(x_1 = c_1, \ldots, x_{n-q} = c_{n-q})$. If x' is close to x one can copy γ to γ', at least for a while. By the compactness of γ, there exists transversal $T_x \subset W$ such that we reach transversal T_y through y, starting from any $x' \in T_x$, and such that $x' \mapsto y' = \gamma'(1)$ is a diffeomorphism h_γ. We define holonomy of path γ as

$$\mathrm{Hol}(\gamma) := \text{ germ of } h_\gamma \colon \text{ germ of } T_x \to \text{ germ of } T_y$$

Obviously if $\gamma_1 \sim \gamma_2$ are homotopic, then $\mathrm{Hol}(\gamma_1) = \mathrm{Hol}(\gamma_2)$, i.e. holonomy factors through homotopy.

Recall that groupoid is a small category with all arrows invertible.

Definition 1.8. Holonomy groupoid

$$\mathcal{G}(\mathcal{F}) := \{\, (x, \mathrm{Hol}(\gamma), y) \mid \exists \text{ leaf } L \ni x, y, \text{ and path } \gamma \colon [0,1] \to L \text{ from } x \text{ to } y\}$$

with objects

$$\mathcal{G}^0 = M$$

and composition

$$(y, \mathrm{Hol}(\delta), z) \circ (x, \mathrm{Hol}(\gamma), y) = (z, \mathrm{Hol}(\delta \circ \gamma), z).$$

Interpretation:

- $(x, \mathrm{Hol}(\mathrm{const}), x)$ "reflexibility" = unit,
- $(x, \mathrm{Hol}(\gamma), y) = (y, \mathrm{Hol}(\gamma^{-1}), x)$ "symmetry"= inverse,
- $(y, \mathrm{Hol}(\delta), z) \circ (x, \mathrm{Hol}(\gamma), y) = (x, \mathrm{Hol}(\delta \circ \gamma), z)$ "transitivity"= composition.

Let T be a complete transversal to $\mathcal{F}$ i.e. T is an immersed submanifold, transverse to each leaf and intersecting each leaf at least once.

$$\mathcal{G}_T(\mathcal{F}) = \{\, (x, \mathrm{Hol}(\gamma), y) \in \mathcal{G}(\mathcal{F}) \mid x, y \in T\},$$

$$C_c^\infty(\mathcal{G}_T(\mathcal{F})) \hookrightarrow C^*(\mathcal{G}_T(\mathcal{F})),$$

$$(f * g)(\mathrm{Hol}(\gamma)) = \sum_{\mathrm{Hol}(\gamma_1)\,\mathrm{Hol}(\gamma_2) = \mathrm{Hol}(\gamma)} f(\mathrm{Hol}(\gamma_1)) g(\mathrm{Hol}(\gamma_2)).$$

1.4 How to handle "$M/\mathcal{F}$"

$$\text{“}M/\mathcal{F}'' = \text{ groupoid } \mathcal{G}(\mathcal{F})$$

(A) "Homotopy quotient" approach, or equivalently via s. This is similar in spirit to

$$\text{“}M/\Gamma'' \leftrightarrow M \times_\Gamma \mathrm{E}\,\Gamma \to \mathrm{B}\,\Gamma,$$

where Γ is a group.

$$\text{“}M/\mathcal{F}'' \sim \mathrm{B}\,\mathcal{G}(\mathcal{F}) \to \mathrm{B}\,\Gamma_q$$

(B) "Topos" approach, by extending "duality"

$$\text{Topological spaces} \leftrightarrow \text{ Sheaves of sets},$$

and associating a suitably defined topos to $\mathcal{G}(\mathcal{F})$.

(C) Connes noncommutative geometry approach, by extending the duality

$$\text{Topological spaces} \leftrightarrow \text{ Commutative C*-algebras},$$

to include $C^*(\mathcal{G})$, for $\mathcal{G}$-groupoid.

1.5 Characteristic classes

All approaches produce cohomology groups for groupoids, equivalent for (A) & (B), and cyclic homology HC^* for (C), as well as characteristic maps. They are all "huge" and not well understood. The ones which are best understood are the "geometric" characteristic classes.

1. Bott construction a la .

2. Gelfand-Fuks realization.

3. Hopf–cyclic cohomological construction.

Chapter 2

Characteristic classes

2.1 Preamble: Chern–Weil construction of Pontryagin ring

Definition 2.1. *Let$E \to M$ be a real vector bundle. A* connection *on E is a linear operator*

$$\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E) = \Omega^1(M) \otimes \Gamma(E)$$

satisfying following rule

$$\nabla(fs) = df \otimes s + f\nabla(s).$$

A connection ∇ extends to a graded $\Omega(M)$-module map

$$\nabla\colon \Omega^*(M) \otimes \Gamma(E) \to \Omega^*(M) \otimes \Gamma(E) = \Omega^*(M, E), \text{ by}$$

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg\omega}\omega\nabla(s).$$

We can view $\Omega^*(M, E)$ as a module over $\Omega^*(M)$ and then for any $\zeta \in \Omega^*(M, E)$ and any $\omega \in \Omega^*(M)$ we have

$$\begin{aligned}\nabla^2(\omega\zeta) &= \nabla(d\omega\zeta + (-1)^{\partial\omega}\omega\nabla(\zeta))\\ &= (-1)^{\partial\omega+1}d\omega\nabla(\zeta) + (-1)^{\partial\omega}d\omega\nabla(\zeta) + \omega\nabla^2(\zeta)\\ &= \omega\nabla^2(\zeta).\end{aligned}$$

It means that ∇^2 is a local operator - multiplication by an element of the base ring. It follows that

$$\nabla^2(\zeta) = R \cdot \zeta, \quad R \in \Omega^2(M, \mathrm{End}(E)).$$

Definition 2.2. *We call R a* curvature form*, or* curvature of connection ∇*.*

Explicit expression in terms of covariant derivative:

$$X - \text{ vector field } , \nabla_X(s) = \nabla s(X)$$

$$\nabla_X\colon \Gamma(E) \to \Gamma(E).$$

Let $\{X_i\}$ be basis of TM, i.e. linearly independent vector fields, $\{\omega^i\}$ - its dual basis of 1-forms. Then

$$\nabla(s) = \sum_i \omega^i \otimes \nabla_{X_i}(s), \text{ hence}$$

$$\begin{aligned}\nabla^2(s) &= \sum_i d\omega^i \otimes \nabla_{X_i}(s) - \sum_i \omega^i \nabla(\nabla_{X_i}(s)) \\ &= \sum_i d\omega^i \otimes \nabla_{X_i}(s) - \sum_{i,j} \omega^i \wedge \omega^j \nabla_{X_j} \nabla_{X_i} s.\end{aligned}$$

Where the second sum could be written as

$$\sum_{i,j} \omega^i \wedge \omega^j \nabla_{X_j} \nabla_{X_i} s = \sum_{i<j} \omega^i \wedge \omega^j [\nabla_{X_j}, \nabla_{X_i}] s.$$

Write

$$d\omega^i = \sum_{j<k} f^i_{jk} \omega^j \wedge \omega^k,$$

with $f^i_{jk} = d\omega^i(X_j, X_k) = -\omega^i([X_j, X_k])$. With that, we can rewrite first sum as

$$\begin{aligned}\sum_i d\omega^i \otimes \nabla_{X_i}(s) &= -\sum_{j<k} \sum_i \omega^i([X_j, X_k]) \omega^j \wedge \omega^k \otimes \nabla_{X_i}(s) \\ &= -\sum_{j<k} \omega^j \wedge \omega^k \otimes \nabla_{\sum_i \omega^i([X_j, X_k]) X_i}(s) \\ &= -\sum_{j<k} \omega^j \wedge \omega^k \otimes \nabla_{[X_j, X_k]}(s).\end{aligned}$$

We just proved

Lemma 2.3.

$$\nabla^2 s = \sum_{j<k} \omega^j \wedge \omega^k R_{X_j, X_k}(s) = R \cdot s, \textit{ where}$$

$$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \mathrm{End}(E), \textit{ and}$$

$$R = \sum_{j<k} R_{X_j, X_k} \omega^i \wedge \omega^k.$$

For any Lie algebra $\mathfrak{g}$ of a Lie group G, we denote by $\mathcal{I}(\mathfrak{g})$ set of polynomials on $\mathfrak{g}$ which are invariant under adjoint action Ad_G. For

$$P \in \mathrm{Sym}(\mathfrak{g}^* \otimes \ldots \otimes \mathfrak{g}^*)$$

it means that

$$P(\mathrm{Ad}(g)x_1, \ldots, \mathrm{Ad}(g)x_r) = P(x_1, \ldots, x_r), \text{ where}$$

$$\mathrm{Ad}(g)(a) = gag^{-1}.$$

Let $\mathfrak{gl}_n(\mathbb{R})$ be the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$. The set $\mathcal{I}(\mathfrak{gl}_n)$ is in fact ring, and is generated by elements

$$P_{2k}(A) = P_{2k}(A, \ldots, A) = \mathrm{tr}(A^k).$$

Theorem 2.4 (Chern–Weil)**.** *Let $P \in \mathcal{I}(\mathfrak{gl}_n(\mathbb{R}))$ be an invariant polynomial of degree k, R - curvature of connection ∇ on real vector bundle $E \to M$.*

1. *Then $P(R) = P(R, \dots, R) \in \Omega^{2k}(M)$ is closed and its de Rham cohomology class is independent of the connection.*
2. *More precisely, if ∇_0, ∇_1 are two connections, then*

$$P(R_1) - P(R_0) = k \cdot d \int_0^1 P(\alpha, R_t, \dots, R_t) dt,$$

where $\alpha \in \Omega^1(M, \operatorname{End}(E))$ is the difference $\alpha = \nabla_1 - \nabla_0$, and R_t is the curvature of a connection $\nabla_t = (1-t)\nabla_0 + t\nabla_1$.

Proof. It is based on the two lemmas.

Lemma 2.5. *If $\deg(P)$ is odd, then $P(R) = 0$ for any metric connection.*

Proof. By hypothesis we have using Euclidean structure $(E, \langle -, - \rangle)$

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle.$$

This implies

$$\begin{aligned} XY\langle s, t\rangle &= X(\langle \nabla_Y s, t\rangle + \langle s, \nabla_Y t\rangle) \\ &= \langle \nabla_X \nabla_Y s, t\rangle + \langle \nabla_Y s, \nabla_X t\rangle + \langle \nabla_X s, \nabla_Y t\rangle + \langle s, \nabla_X \nabla_Y t\rangle, \end{aligned}$$

and

$$\begin{aligned} [X, Y]\langle s, t\rangle &= \langle [\nabla_X, \nabla_Y] s, t\rangle + \langle s, [\nabla_X, \nabla_Y] t\rangle \\ &= \langle \nabla_{[X,Y]} s, t\rangle + \langle s, \nabla_{[X,Y]} t\rangle. \end{aligned}$$

We can write then

$$\langle R_{X,Y} s, t\rangle + \langle s, R_{X,Y} t\rangle = 0, \text{ i.e.}$$
$$R + R^t = 0, \text{ and } P(R) = P(R^t, \dots, R^t) = (-1)^k P(R).$$

∎

Lemma 2.6. *For $\omega \in \Gamma(M, \operatorname{End}(E))$ one has*

$$d(\operatorname{tr} \omega) = \operatorname{tr}[\nabla, \omega].$$

Proof. Locally, on a chart U we have $\nabla = d + \alpha$, $\alpha \in \Omega^1(U, \operatorname{End}(E))$. Hence

$$[\nabla, \omega] = [d + \alpha, \omega] = d\omega + [\alpha, \omega], \text{ and}$$

$$\operatorname{tr}[\nabla, \omega] = \operatorname{tr} d\omega + \operatorname{tr}[\alpha, \omega] = d(\operatorname{tr} \omega).$$

∎

In particular (Bianchi's identity)

$$d \operatorname{tr}(R^k) = \operatorname{tr}[\nabla, R^k] = \operatorname{tr}[\nabla, \nabla^{2k}] = 0.$$

This gives proof of the first part, because polynomials of the form $\operatorname{tr}(R^k)$ generate $\mathcal{I}(\mathfrak{gl}_n(\mathbb{R}))$.

For the second part, note that if $\nabla_t = (1-t)\nabla_0 + t\nabla_1$, we have

$$\frac{d}{dt}(R_t) = \frac{d}{dt}\left(\nabla_t^2\right) = \frac{d}{dt}(\nabla_t)\nabla_t + \nabla_t \frac{d}{dt}\nabla_t =$$

$$= \left[\frac{d}{dt}\nabla_t, \nabla_t\right] = [\alpha, \nabla_t] = [\nabla_t, \alpha],$$

where $\alpha = \nabla_1 - \nabla_0$. Now

$$\frac{d}{dt}\operatorname{tr}(R_t^k) = \operatorname{tr}\left(\frac{d}{dt}R_t^k\right) = k\operatorname{tr}\left(\frac{dR_t}{dt}R_t^{k-1}\right) =$$

$$= k\operatorname{tr}\left([\nabla_t, \alpha]\nabla_t^{2(k-1)}\right) = k\operatorname{tr}([\nabla_t, \alpha\nabla_t^{2(k-1)}]) = kd\operatorname{tr}(\alpha R_t^{k-1}).$$

□

2.2 Adapted connection and Bott theorem

Let $E \subset TM$ be an involutive subbundle and let $Q = TM/E$ with $\pi\colon TM \to Q$ be the projection.

Definition 2.7. *An* adapted *(or* E-flat*)* connection *on Q is a connection ∇ such that*

$$\nabla_X \pi(Z) = \pi([X, Z]),\ \forall X \in \Gamma(E).$$

This makes sense, since

$$\nabla_{fX}\pi(Z) = \pi([fX, Z]) = -\pi(Z(f)X) + f\pi([X, Z]) = f\nabla_X\pi(Z), \text{ and}$$

$$\nabla_X(f\pi(Z)) = \pi([X, fZ]) = \pi(X(f)Z) + f\pi([X, Z]) = X(f)\pi(Z) + f\nabla_X(\pi(Z)).$$

To construct such a connection, take a decomposition $TM = E \oplus Q$ and set

$$\nabla_X\pi(Z) = \nabla_{X_E}\pi(Z) + \nabla_{X_{E^\perp}}(Z) = \pi([X_E, Z]) + \nabla_{X_{E^\perp}}(Z)$$

where we take an arbitrary connection on $E^\perp$.

Lemma 2.8. *For any adapted connection*

$$R_{X,Y} = 0,\ \ \forall X, Y \in \Gamma(E).$$

Proof.

$$R_{X,Y}\pi(Z) = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})(\pi(Z)) =$$

$$\pi([X, [Y, Z]] - [Y, [X, Z]] - [[X, Y], Z]) = 0.$$

□

Theorem 2.9 (Bott vanishing theorem)**.** *Given $E \subset TM$ which is involutive, we have for $Q = TM/E$, $\dim Q = q$*

$$\operatorname{Pont}^{>2q}(Q) = 0.$$

Proof. Let
$$P_{2k}(A) := \mathrm{tr}(A^k).$$
Then for
$$R = \sum_{i<j} R_{X_i,X_j}\omega^i \wedge \omega^j$$
we have
$$P_{2k}(R) = \mathrm{tr}(R^k) = \sum \mathrm{tr}(R_{X_{i_1},X_{j_1}},\dots,R_{X_{i_{2k}},X_{j_{2k}}})\omega^{i_1}\wedge\omega^{j_1}\wedge\ldots\wedge\omega^{i_{2k}}\wedge\omega^{j_{2k}}.$$
If $k > q$, at least one pair belongs to E, otherwise
$$\omega^{i_1}\wedge\ldots\wedge\omega^{i_{2k}} = 0.$$
□

Remark 2.10.
$$\mathrm{Pont}(Q) = \mathrm{Pont}(TM \ominus E),$$
hence the above is a restriction of $[E] \in \mathrm{K}^0(M)$.

2.3 The Godbillon–Vey class

Let $\mathcal{F}$ be a codimension q foliation of M^n, $E = \tau\mathcal{F}$, $Q = TM/E$. First, assume that $\mathcal{F}$ is *transversaly orientable* i.e. $\Lambda^q Q$ has nowhere zero section (giving trivialization $\Lambda^q Q \cong M \times \mathbb{R}$).

Lemma 2.11. *Let Ω be nonvanishing section of $\Lambda^q Q$. Then*
$$d\Omega = \alpha \wedge \Omega \tag{2.1}$$
for some $\alpha \in \Omega^1(M, \mathrm{End}(E))$.

Proof. It suffices to prove (2.1) locally, then patch by partition of unity.

On a chart U, choose a basis $\omega_1,\dots,\omega_q \in \mathcal{I}(E)$ such that
$$\Omega = \omega_1\wedge\ldots\wedge\omega_q,$$
$$d\omega_i = \sum_{j=1}^q \alpha_{ij}\wedge\omega_j$$
Then
$$d\Omega = \sum_{i=1}^q (-1)^i\omega_1\wedge\ldots\wedge d\omega_i\wedge\ldots\wedge\omega_q =$$
$$= \sum_{i=1}^q (-1)^i\omega_1\wedge\ldots\wedge\left(\sum_{j=1}^q \alpha_{ij}\wedge\omega_j\right)\wedge\ldots\wedge\omega_q$$
Only $\alpha_{ii}\wedge\omega_i$ can contribute to the sum, so
$$d\Omega = \left(\sum_{i=1}^q \alpha_{ii}\right)\wedge\Omega.$$
□

Lemma 2.12. *For all α as above $(d\alpha)^{q+1} = 0$.*

Proof.

$$0 = d^2\Omega = d\alpha \wedge \omega - \alpha \wedge d\Omega = d\alpha \wedge \Omega + \alpha \wedge \alpha \wedge \Omega = d\alpha \wedge \Omega.$$

Write $d\alpha$ using basis of 2-forms extending $\{\omega_1, \dots, \omega_q\}$

$$d\alpha = \sum_{1 \leqslant i < j \leqslant n} f_{ij} \omega_i \wedge \omega_j.$$

Now take exterior product with $\Omega = \omega_1 \wedge \ldots \wedge \omega_q$

$$\sum_{1 \leqslant i < j \leqslant n} f_{ij} \omega_i \wedge \omega_j \wedge \omega_1 \wedge \ldots \wedge \omega_q = 0.$$

If at least one of $i, j \in \{1, \dots, q\}$ then corresponding summand is 0. Hence

$$\sum_{q+1 \leqslant i < j \leqslant n} f_{ij} \omega_i \wedge \omega_j \wedge \omega_1 \wedge \ldots \wedge \omega_q = 0,$$

so

$$f_{ij} = 0 \text{ for } q + 1 \leqslant i < j \leqslant n.$$

Now we can write

$$d\alpha = \sum_{i<j;\text{ at least one } \leqslant q} f_{ij} \omega_i \wedge \omega_j = \sum_{j=1}^{q} \alpha_j \wedge \omega_j \in \Gamma(E),$$

and

$$(d\alpha)^{q+1} = \sum f_{i_1 j_1} \ldots f_{i_{q+1} j_{q+1}} \omega_{i_1} \wedge \omega_{j_1} \wedge \ldots \wedge \omega_{i_{q+1}} \wedge \omega_{j_{q+1}} = 0.$$

□

We just proved that form $\eta = \alpha \wedge (d\alpha)^q$ is closed.

Lemma 2.13. *The class*

$$[\eta] \in \mathrm{H}^{2q+1}(M, \mathbb{R})$$

is independent on all choices involved in definition.

Proof. First assume that $\Omega' = f\Omega$ for $f > 0$ everywhere. Then

$$d\Omega' = f d\Omega + df \Omega = f\alpha \wedge \Omega + df \wedge \Omega = \alpha \wedge \Omega' + \frac{df}{f} \wedge \Omega' =$$

$$= (\alpha + d(\log f)) \wedge \Omega' = \alpha' \wedge \Omega'.$$

Hence

$$\Omega' \wedge (d\Omega')^q = (\alpha + d(\log f)) \wedge (d\alpha)^q = \alpha \wedge (d\alpha)^{+} d(\log(f)(d\alpha)^q),$$

so η and $\eta' = \alpha' \wedge (d\alpha')$ differ by boundary.

Now assume that $d\Omega = \alpha' \wedge \Omega$, $\beta = \alpha - \alpha'$ sucht that $\beta \wedge \Omega = 0$. Hence $\beta \in \Gamma(E)$, and recall that also $d\alpha, d\alpha' \in \Gamma(E)$. Then we have

$$\eta' = \alpha' \wedge (d\alpha')^q = (\alpha + \beta) \wedge ((d\alpha)^q + d\beta \wedge \sigma)$$

with

$$\sigma = \sum_{i=0}^{q-1} c_i (d\alpha^i) \wedge (d\beta)^{q-i-1} \in \Gamma(E)^{q-1}, \quad \text{and } d\sigma = 0.$$

Then

$$\alpha' \wedge (d\alpha')^q = \alpha \wedge (d\alpha)^q + \alpha \wedge d\beta \wedge \sigma + \beta \wedge (d\alpha)^q + \beta \wedge d\beta \wedge \sigma,$$

where the last two summands belong to $\Gamma(E)^{q+1} = 0$, so in fact we have

$$\alpha' \wedge (d\alpha')^q = \alpha \wedge (d\alpha)^q + \alpha \wedge d\beta \wedge \sigma =$$

$$= \alpha \wedge (d\alpha)^q + \alpha \wedge d(\beta \wedge \sigma) = \alpha \wedge (d\alpha)^q - d(\alpha \wedge \beta \wedge \sigma) + d\alpha \wedge \beta\sigma,$$

where the last summand is from $\Gamma(E)^{q+1} = 0$. Again we see, that $\eta' - \eta$ is a boundary. □

Definition 2.14. *The class* $\mathrm{gv}(\mathcal{F}) := [\eta] \in \mathrm{H}^{2q+1}(M;\mathbb{R})$ *is called the* Godbillon-Vey class *of a manifold with foliation* $(M, \mathcal{F})$.

Remark 2.15. Nonorientable case. Lift $\mathcal{F}$ to $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ = orientable double covering with γ= the generator of $\mathbb{Z}/2$. Replacing $\widetilde{\Omega}$ by $\frac{1}{2}(\widetilde{\Omega} - \gamma^*\widetilde{\Omega}) \neq 0$ if needed, we can always assume $\gamma^*(\widetilde{\Omega}) = -\widetilde{\Omega}$. Then $d\widetilde{\Omega} = \widetilde{\alpha} \wedge \widetilde{\Omega}$, and $d(\gamma^*\widetilde{\Omega}) = \gamma^*(\widetilde{\alpha}) \wedge \gamma^*(\widetilde{\Omega})$. Hence $d\widetilde{\Omega} = \gamma^*(\widetilde{\alpha}) \wedge \widetilde{\Omega}$, and $\frac{1}{2}(\widetilde{\alpha} + \gamma^*(\widetilde{\alpha}))$ drops down to M.

2.4 Nontriviality of Godbillon–Vey class

On $G = \mathrm{SL}(2,\mathbb{R})$, with $TG \cong G \times \mathfrak{g}$, ($\mathfrak{g}$ - Lie slgebra of G = traceless matrices) take the foliation given by the subbundle E generated by the left invariant vector fields corresponding to

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$[X, H] = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2X.$$

The third basis element is

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with

$$[Y, H] = 2Y, \quad [X, Y] = H.$$

Take the dual basis $\{\zeta, \eta, \chi\}$ of $\mathfrak{g}^*$ and extend them as left-invariant 1-forms. Then η defines $\mathcal{F}$ (i.e. $E = \ker \eta$). One has

$$\begin{aligned} d\chi &= a\chi \wedge \zeta + b\chi \wedge \eta + c\zeta \wedge \eta, \\ b &= d\chi(H, Y) = -\chi([H, Y]) = 2\chi(Y) = 0, \\ c &= d\chi(X, Y) = -\chi([X, Y]) = -\chi(H) = -1, \\ a &= d\chi(H, X) = \chi([X, H]) = -2\chi(X) = 0, \end{aligned}$$

hence

$$\begin{aligned} d\chi &= -\zeta \wedge \eta, \\ d\zeta &= -2\chi \wedge \zeta, \\ d\eta &= 2\chi \wedge \eta. \end{aligned}$$

The last implies

$$\alpha = 4\chi \wedge d\chi = -4\chi \wedge \zeta \wedge \eta.$$

The form α drops down to $M = \Gamma \setminus G$ for any Γ cocompact giving a volume form, hence

$$[\alpha_\Gamma] = \text{ generator of } \mathrm{H}^3(M;\mathbb{R}).$$

More precisely, let Σ_g be the Riemann surface of genus $g \geqslant 2$. Then its universal cover is the upper half plane

$$\mathbb{H} = \mathrm{SL}(2,\mathbb{R}) / \mathrm{SO}(2),$$

on which $\Gamma = \pi_1(\Sigma_g)$ acts by Mobius transformation

$$\Gamma \subset \mathrm{PSL}(2,\mathbb{R}), \quad z \mapsto \frac{az+b}{cz+d}.$$

Let $\tilde{\Gamma}$ be the double cover of Γ. Then $\tilde{\Gamma}$ is cocompact. Morover $M \cong S^1\Sigma_g$ (unit tangent bundle), hence

$$\begin{aligned} [\alpha_\Gamma]([M]) &= 4 \int_{S^1\Sigma_g} \zeta \wedge \eta \wedge \chi \\ &= 4\pi \int_{\Sigma_g} \zeta \wedge \eta = 4\pi \mathrm{Area}(\Sigma_g) \\ &= -4\pi \int_{\Sigma_g} K d\sigma \\ &= -8\pi^2(2-2g). \end{aligned}$$

2.5 Foliations with rigid Godbillon-Vey class

Let $\mathcal{F}$ be a foliation of codimension one on a manifold X. It can be described

1. locally by the compatible system of submersions

$$X \supset U_i \xrightarrow{x_i} \mathbb{R},$$

   when the leaves are level sets of functions x_i, or

2. globally by the nonvanishing 1-form ω_0 with values in the orientation sheaf (the sheaf of sections of the flat bundle associated by the sign of the determinant) of the normal bundle Q of $\mathcal{F}$, whose kernel is equal to the tangent distribution F of $\mathcal{F}$.

Definition 2.16. *We say that $\mathcal{F}$ is given a* transversal projective structure *if there are given functions x_i as above, whose domains cover X and which are related on intersections $U_i \cap U_j$ by fractional linear transformations*

$$x \mapsto \frac{ax+b}{cx+d}.$$

The Frobenius condition of integrability of F yields the existence of 1-form ω_1 such that

$$d\omega_0 = \omega_1 \wedge \omega_0.$$

Then the 3-form $\omega_1 \wedge d\omega_1$ represents the Godbillon-Vey class $\mathrm{gv}(\mathcal{F})$ in the De Rham cohomology of X. It is an invariant of C^1-diffeomorphisms [r-g88], the concordance equivalence of foliations and for oriented closed 3-folds the Godbillon-Vey number is invariant under foliated cobordisms [t-w72]. However, in contrast to primary characteristic classes, it can vary under continuous deformations, which means that it is not a homotopy invariant [t-w72]. This suggests that not being a robust invariant it can be related with more subtle geometric structure of the foliation. First indication in this direction was the *rigidity theorem* [h-yxx],[t-w72] saying that under a deformation in the class of foliations admitting a transversal projective structure the Godbillon-Vey class does not change. The main result of [**?**] is the following generalization of that result on deformations of foliations admitting a transversal projective structure, but taken in the class of infinitesimal deformations with arbitrary singularities and not necessarily satisfying the transversal projective structure assumption, and somewhat surprising inversion of this generalization on closed oriented threefolds.

Theorem 2.17. *(Thm 4, [m-t99]) Let $\mathcal{F}$ be a foliation of codimension one on a manifold X.*

1. *If $\mathcal{F}$ admits a transversal projective structure and the topological vector space $H^3(X, \mathbb{R})$ is separated, then $\mathrm{gv}(\mathcal{F})$ is topologically rigid.*
2. *If X is closed oriented, the topological vector space $\mathrm{H}^2(X, \overline{\mathcal{T}_{X/\mathcal{F}}})$ is separowalna and $\mathrm{gv}(\mathcal{F})$ is topologically rigid, then $\mathcal{F}$ admits a transversal projective structure.*

Intuitively, if one sees the Godbillon-Vey number as a function on the moduli space of foliations of codimension one on a closed oriented threefold, then the set of critical points of this function consists precisely of foliations admitting a transversal projective structure. Proof of this theorem, and even its formulation, needs some modification of the theory of deformations of foliations introduced by Heitsch [h-lxx], Hamilton [h-r77] and developed by Desolneux-Moulis [dmxx], so to take singular perturbation of a smooth foliation into account. They are described by distributional sections of sheaves naturally related with a foliation. Among such sheaves the following ones are important to us: the sheaf $\mathcal{O}_{X/\mathcal{F}}$ of functions locally constant along the leaves of the foliation $\mathcal{F}$, transversal tangent sheaf $\mathcal{T}_{X/\mathcal{F}}$, its $\mathcal{O}_{X/\mathcal{F}}$-dual sheaf $\Omega^1_{X/\mathcal{F}}$ and their tensor products balanced over $\mathcal{O}_{X/\mathcal{F}}$. We denote sheaves of distributional sections of these sheaves using the symbol $\overline{(-)}$, e.g. $\overline{\mathcal{T}_{X/\mathcal{F}}}$.

Example 2.18. Define the family $\mathcal{F}(t)$ of singular foliations on $\mathbb{R}^2$ defined by the differentiable family of nonvanishing smooth 1-currents

$$\omega_0(t) := dx + \left(\int_0^t f(s, \cdot)ds\right) dy.$$

Then the value of $\omega_0(t)$ at $t = 0$, $\omega_0 := \omega_0(t) = dx$ is smooth, but the class of its derivative, as a current depending on t, at $t = 0$ modulo ideal generated by

ω_0

$$\delta(\omega_0) := \left(\frac{d}{dt}\omega_0(t)\right)\big|_{t=0} = f(\cdot)dy \mod (\omega_0)$$

can be arbitrarily singular. To illustrate this phenomenon, let us take

$$\omega_0(t)(x,y) := dx + \left(\int_0^t \frac{e^{-(y/s)^2}}{\sqrt{\pi}s} ds\right) dy.$$

Then leaves are level sets of the function

$$x(t) := x + \int_0^t \left(\frac{1}{\sqrt{\pi}} \int_0^{y/s} e^{-u^2} du\right) ds.$$

One can see that

1. for $t = 0$ leaves are vertical lines
2. the common singular locus for all $t \neq 0$ is equal to $S = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$ and singularities of leaves are cuspidal.

In this way we obtain a deformation of the foliation by vertical lines in the family of singular foliations. Let j_S be the current corresponding to the singular locus S. Computing the first derivative of $\omega_0(t)$ at $t = 0$ we obtain

$$\delta(\omega_0) = j_S \mod (\omega_0).$$

This means that the infinitesimal variation of this smooth foliation in this family describing adding microscopic cusps has distributional values. Note that topological type of the singular foliation is unchanged under this deformation. This is an example of an *infinitesimal singular deformation* of a smooth foliation.

In general, infinitesimal topological deformation theory of smooth foliations is described by means of sheaf cohomology [m-t99], where the sheaf in question is the sheaf $\overline{\mathcal{T}_{X/\mathcal{F}}}$ of distributional local sections of the transversal tangent sheaf $\mathcal{T}_{X/\mathcal{F}}$. Theory of infinitesimal singular deformations of foliations is based on the following identifications.

Proposition 2.19. *(Prop. 2, [m-t99]) There exist natural bijections:*

1. *between the space of global sections $\mathrm{H}^0(X, \overline{\mathcal{T}_{X/\mathcal{F}}})$ and the space of infinitesimal transversal singular automorphisms of $\mathcal{F}$,*
2. *between the cohomology space $\mathrm{H}^1(X, \overline{\mathcal{T}_{X/\mathcal{F}}})$ and the space of infinitesimal singular deformations $\mathcal{F}$ up to infinitesimal singular conjugations.*

It turns out that the formal differential calculus analogical to that one from the above example makes sense also for the Godbillon-Vey class in the De Rham cohomology. This gives rise to the notion of the *universal variation of the Godbillon-Vey class under infinitesimal singular deformations*, which is a continuous linear map

$$\mathrm{H}^1(X, \overline{\mathcal{T}_{X/\mathcal{F}}}) \xrightarrow{\delta\,\mathrm{gv}(\mathcal{F})} \mathrm{H}^3(X, \mathbb{R})$$

between cohomology spaces equipped with appropriate canonical topologies [m-t99]. We say that a given foliation has *topologically rigid* Godbillon-Vey class, if the universal variation of the Godbillon-Vey class under infinitesimal topological deformations is zero. Sheaf cohomology can be used to study relations between characteristic classes of foliations and the transversal projective structure because there exists a cohomological obstruction to the existence of such a structure.

Theorem 2.20. *(Thm 2, [m-t99]) There exists an invariant*

$$i_2(\mathcal{F}) \in \mathrm{H}^1(X, (\Omega^1_{X/\mathcal{F}})^{\otimes 2}),$$

which vanishes if and only if the foliation $\mathcal{F}$ admits a transversal projective structure. If $i_2(\mathcal{F}) = 0$, then the set of transversal projective structures on $\mathcal{F}$ is an affine space whose space of vectors equals $\mathrm{H}^0(X, (\Omega^1_{X/\mathcal{F}})^{\otimes 2})$.

To proceed further, we will apply the following useful canonical nondegenerate pairings between cohomology with values in the above and other similar sheaves.

Theorem 2.21. *(Thm 1, [m-t99]) Let $\mathcal{F}$ be a foliation of dimension n and of codimension m on a manifold$X$. For every locally free $\mathcal{O}_{X/\mathcal{F}}$-module $\mathcal{E}$ of finite rank there exist natural nondegenerate pairings*

$$\mathrm{H}^k(X, \mathcal{H}om_{\mathcal{O}_{X/\mathcal{F}}}(\mathcal{E}, \omega_{X/\mathcal{F}})) \otimes \mathrm{H}^{n-k}_c(X, \mathcal{E}) \to \mathbb{R},$$

$$\mathrm{H}^k_c(X, \mathcal{H}om_{\mathcal{O}_{X/\mathcal{F}}}(\mathcal{E}, \omega_{X/\mathcal{F}})) \otimes \mathrm{H}^{n-k}(X, \mathcal{E}) \to \mathbb{R}.$$

Now, the following fact is crucial for the proof of the main theorem.

Theorem 2.22. *(Thm 3, Prop. 3, [m-t99]) On a manifold X of dimension $n+1$, the variation of the Godbillon-Vey class $\mathrm{gv}(\mathcal{F})$ of a foliation $\mathcal{F}$ of codimension one*

$$\mathrm{H}^1(X, \overline{\mathcal{T}_{X/\mathcal{F}}}) \xrightarrow{\delta\, \mathrm{gv}(\mathcal{F})} \mathrm{H}^3(X, \mathbb{R})$$

and the Yoneda product with the obstruction $i_2(\mathcal{F})$ to the existence of a transversal projective structure on $\mathcal{F}$

$$\mathrm{H}^{n-1}_c(X, (\Omega^1_{X/\mathcal{F}})^{\otimes 2} \otimes or_X) \xleftarrow{i_2(\mathcal{F})\cup} \mathrm{H}^{n-2}_c(X, or_X)$$

are (up to the factor $(-1)^{n+1}/2$) adjoint one to each other with respect to Poincaré duality

$$\mathrm{H}^3(X, \mathbb{R}) \otimes \mathrm{H}^{n-2}_c(X, or_X) \to \mathbb{R}$$

and the duality

$$\mathrm{H}^1(X, \overline{\mathcal{T}_{X/\mathcal{F}}}) \otimes \mathrm{H}^{n-1}_c(X, (\Omega^1_{X/\mathcal{F}})^{\otimes 2} \otimes or_X) \to \mathbb{R}.$$

The last theorem allows us to derive vanishing of one of these maps from vanishing of the other one, provided appropriate topological cohomology spaces are separated. Especially interesting in this context is separatedness of the space $\mathrm{H}^2(X, \overline{\mathcal{T}}_{X/\mathcal{F}})$, which is an analog of the space occuring in the Kodaira-Spencer theory of deformations of complex structures as a receptor of obstructions to deformations. The following theorem gives a sufficient criterion to separatedness of $\mathrm{H}^2(X, \overline{\mathcal{T}_{X/\mathcal{F}}})$. Denoting $\mathrm{H}^k(X) := \mathrm{H}^k(X, \overline{\mathcal{T}_{X/\mathcal{F}}})$ we have

Theorem 2.23. *(Thm 5, [m-t99]) Let I be a linearly ordered set and let $X = \bigcup_{i\in I} X_i$ be an open covering of the manifold X such that for all $i < j < k$*

$$X_i \cap X_j \cap X_k = \emptyset$$

and for all $i \in I$ spaces $\mathrm{H}^2(X_i)$ are separated. If at least one of the conditions below 1) $\mathrm{H}^1(X_i) = 0$ and $\mathrm{H}^1(X_i \cap X_j)$ - separated, 2) $\mathrm{H}^1(X_i \cap X_j) = 0$, is fulfilled for all $i < j$, then $\mathrm{H}^2(X)$ is separated.

2.6 Naturality under transversality

Let $\phi\colon N \to M$, $E \subset TM$ integrable subbundle, $\mathcal{F}$- codimension q foliation, $\tau\mathcal{F} = E$. If $V \to M$ is a vector bundle, then for each invariant polynomial $P \in \mathcal{I}(\mathfrak{gl}_q(\mathbb{R}))$ of degree k, we have a class $P(V) \in \mathrm{H}^{2k}(M;\mathbb{R})$. It behaves naturally with respect to pullback

$$\begin{tikzcd} \phi^*(V) \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\ N \arrow[r, "\phi"] & M \end{tikzcd}$$

$$P(\phi^*(V)) = \phi^*(P(V)).$$

By Bott vanishing Theorem 2.9, all classes for $Q = TM/E$ are 0 if $k > q$. The Godbillon–Vey class $\mathrm{gv}(M,\mathcal{F}) \in \mathrm{H}^{2q+1}(M;\mathbb{R})$ is a nontrivial invariant.

Definition 2.24. *We say that ϕ is transversal to E (or to $\mathcal{F}$), $\phi \pitchfork E$, if for each $x \in N$*

$$T_{\phi(x)}M = \phi_*(T_xN) \oplus E_{\phi(x)}.$$

Equivalently

$$\pi \circ \phi_{*x}\colon T_xN \to T_{\phi(x)}M/E$$

is surjective.

Lemma 2.25. *$\widetilde{E} := \phi_*^{-1}(E)$ is involutive, hence defining a foliation $\widetilde{\mathcal{F}} = \phi^{-1}(\mathcal{F})$, whose leaves are the connected components of $\phi^{-1}(L)$, $L \subset \mathcal{F}$.*

Proof. (Short) Let $E = \tau\mathcal{F}$ be given by a cocycle $\{(U_i, f_i, g_{ij}) \mid i,j \in I\}$, $f_i\colon U_i \to \mathbb{R}^q$ submersions, $g_{ij}\colon f_j(U_i\cap U_j) \xrightarrow{\cong} f_i(U_i \cap U_j)$. Then $\{(\phi^{-1}(U_i), f_i \circ \phi, g_{ij}) \mid i,j \in I\}$ define $\widetilde{\mathcal{F}}$. □

Proof. (More useful) Any map ϕ can be decomposed as a composition

$$N \xrightarrow{\mathrm{id}\times\phi} N \times M \xrightarrow{\mathrm{pr}_M} M,$$

$$x \mapsto (x, \phi(x)); \quad (x,y) \mapsto y.$$

It is sufficient to prove the lemma for

(a) $\mathrm{id}\times\phi$ - injective immersion,

(b) pr_M - projection.

For each map in this composition the statement is obvious.

(a) $\widetilde{E} = E \cap TN$,

(b) $\widetilde{E} = TN \oplus E$.

$\square$

Definition 2.26. *A* characteristic class for foliation $\mathcal{F}$ *is an assignment*

$$(M, \mathcal{F}) \mapsto \gamma(M, \mathcal{F}) \in \mathrm{H}^*(M; \mathbb{R})$$

such that if $\phi\colon N \to M$ *is transversal to* $\mathcal{F}$*, then*

$$\gamma(N, \phi^*(\mathcal{F})) = \phi^*(\gamma(M, \mathcal{F})).$$

Example 2.27. If $(M, \mathcal{F})$ is transversally oriented, i.e. there exists nowhere zero section Ω of $\Lambda^q Q$, then we have Godbillon–Vey class. On local chart U

$$\Omega = \omega_1 \wedge \ldots \wedge \omega_q, \quad \{\omega_1, \ldots, \omega_q\} - \text{ generators of } \Gamma(E\big|_U),$$

$$d\Omega = \alpha \wedge \Omega, \quad \mathrm{gv}(M, \mathcal{F}) = [\alpha \wedge (d\alpha)^q] \in \mathrm{H}^{2q+1}(M; \mathbb{R}).$$

For $\phi\colon N \to M$, $\{\phi^*(\omega_1), \ldots, \phi^*(\omega_q)\}$ is the set of generators of for $\Gamma(\phi^*(E)\big|_{\phi^{-1}(U)})$ and therefore

$$d\phi^*(\Omega) = \phi^*(d\Omega) = \phi^*(\alpha) \wedge \phi^*(\Omega),$$

and thus

$$\mathrm{gv}(N, \phi^*(\mathcal{F})) = \phi^*(\alpha) \wedge (d\phi^*(\alpha))^q = \phi^*(\alpha \wedge (d\alpha)^q) = \phi^*(\mathrm{gv}(M, \mathcal{F})).$$

Example 2.28. Pontryagin classes are characteristic classes of for foliation, since for $P \in \mathcal{I}^k(\mathfrak{gl}_q(\mathbb{R}))$ we have

$$P(\phi^*(\mathcal{F})) = \phi^*(P(\mathcal{F})),$$

where $P(\mathcal{F}) = P(Q)$ for $Q = TM/\tau\mathcal{F}$.

2.7 Transgressed classes

Let $(M, \mathcal{F})$ be a manifold with foliation, ∇_0, ∇_1 two connections on $Q = TM/E$, $E = \tau\mathcal{F}$. Then

$$\nabla_1 - \nabla_0 = \alpha \in \Omega^1(M, \mathrm{End}(E)).$$

Let $\nabla_t := t\nabla_1 + (1-t)\nabla_0$ be linear homotopy between connections, and R_0, R_1, R_t corresponding curvatures. Then by the theorem of (2.4) for $P \in \mathcal{I}^k(\mathfrak{gl}_q(\mathbb{R}))$

$$P(R_1) - P(R_0) = dTP(\nabla_1, \nabla_0), \text{ where}$$

$$TP(\nabla_1, \nabla_0) := k \int_0^1 P(\alpha, R_t, \ldots, R_t) dt.$$

Let $\nabla_1 = \nabla^\flat$ be the E-flat connection (or Bott connection) (Definition 2.7), i.e.

$$\nabla^\flat_X(\pi(Y)) = \pi([X, Y]), \quad \forall X \in \Gamma(E), \pi\colon TM \to TM/E = Q.$$

The corresponding curvature satisfies (Lemma 2.8)

$$R^\flat(X_1, X_2) = 0, \quad \forall X_1, X_2 \in \Gamma(E).$$

As a second connection ∇_0 we take metric (or Riemannian) connection $\nabla^\sharp$, i.e.

$$X\langle s_1, s_2\rangle = \langle \nabla^\sharp_X s_1, s_2\rangle + \langle s_1, \nabla^\sharp_X s_2\rangle,$$

for $s_1, s_2 \in \Gamma(Q)$. Then

- $P(R^\flat) = 0$ if $k > q$, by Bott theorem (2.9),
- $P(R^\sharp) = 0$ if k is odd, by lemma (2.5).

In particular for $k > q$ odd form $TP(\nabla^\flat, \nabla^\sharp)$ is closed, $dTP(\nabla^\flat, \nabla^\sharp) = 0$, so

$$TP(M, \mathcal{F}) := [TP(\nabla^\flat, \nabla^\sharp)] \in \mathrm{H}^{2k-1}(M, \mathbb{R}).$$

Definition 2.29. *We call $TP(M, \mathcal{F})$ a* transgressed class*.*

Proposition 2.30. *For foliation $\mathcal{F}$ on a manifold M and $P \in \mathcal{I}^k(\mathfrak{gl}_q(\mathbb{R}))$, $k > q = \dim TM/\tau\mathcal{F}$, class $[TP(M, \mathcal{F})] \in \mathrm{H}^{2k-1}(M; \mathbb{R})$ is independent of choices $\nabla^\flat$ and $\nabla^\sharp$, and therefore is an invariant of foliation.*

Proof. Let ${}^i\nabla^\flat$, ${}^i\nabla^\sharp$, $i = 0, 1$ be two different choices of connections, and let

$${}^t\nabla^\flat := \psi(t)^1\nabla^\flat + (1 - \psi(t))^0\nabla^\flat,$$

$${}^t\nabla^\sharp := \psi(t)^1\nabla^\sharp + (1 - \psi(t))^0\nabla^\sharp,$$

where in both cases $\psi\colon [0, 1] \to [0, 1]$ is a smooth function such that $\psi \equiv 0$ near 0 and $\psi \equiv 1$ near 1.

Now take the bundle $\widetilde{E} = E \oplus \mathbb{R}$ on $M \times \mathbb{R}$ (as a integrable bundle of foliation on $M \times \mathbb{R}$). On the quotient $\mathrm{pr}_M^*(Q)$ we define the connections $\widetilde{\nabla^\flat}$ and $\widetilde{\nabla^\sharp}$.

$$\begin{tikzcd}
\mathrm{pr}_M^*(Q) \arrow[r, equal] & T(M \oplus \mathbb{R})/\widetilde{E} \arrow[d] & Q = TM/\tau\mathcal{F} \arrow[d] \\
& M \times \mathbb{R} \arrow[r, "\mathrm{pr}_M"] & M
\end{tikzcd}$$

Sections of bundles over $M \times \mathbb{R}$ can be represented as follows

$$\Gamma(T(M \times \mathbb{R})) = \{f(x, s)Y + g(x, s)\frac{\partial}{\partial s} \bigm| Y \in \Gamma(TM), f, g \in C^\infty(M \times \mathbb{R})\}.$$

$$\Gamma(\mathrm{pr}_M^*(Q)) = \{f(x, s)\pi(Y) \bigm| Y \in \Gamma(TM), \pi\colon TM \to Q, f \in C^\infty(M \times \mathbb{R})\}$$

It suffices to define

$$\widetilde{\nabla}_{(X, \frac{\partial}{\partial t})}(\pi(Y)) :=^s \nabla_X(\pi(Y)).$$

for $\widetilde{\nabla} = \widetilde{\nabla^\flat}$ or $\widetilde{\nabla^\sharp}$.

We have

$$\widetilde{\nabla}_X(f(x,s)\pi(Y)) = X(f)\pi(Y) + f^s\nabla_X(\pi(Y)),$$
$$\widetilde{\nabla}_{\frac{\partial}{\partial s}}(f(x,s)\pi(Y)) = \frac{\partial f}{\partial s}\pi(Y),$$

where ${}^s\nabla^\flat = s^0\nabla^\flat + (1-s)^0\nabla^\flat$, ${}^s\nabla^\sharp = s^0\nabla^\sharp + (1-s)^0\nabla^\sharp$. Using inclusions $i_s\colon M \to M\times\mathbb{R}$, $i_s(x) = (x,s)$, we can write

$$i_0^*(\widetilde{R^\flat}) =^0 R^\flat, \quad i_1^*(\widetilde{R^\flat}) =^1 R^\flat$$

and analogously for $\nabla^\sharp, R^\sharp$. Similarly

$$i_0^*(\widetilde{\alpha}) =^0 \alpha, \quad i_1^*(\widetilde{\alpha}) =^1 \alpha$$

for corresponding differences ${}^0\alpha =^0 \nabla^\flat -^0 \nabla^\sharp$ and ${}^1\alpha =^1 \nabla^\flat -^1 \nabla^\sharp$. Hence

$$i_0^*(TP(\widetilde{\nabla}^\flat, \widetilde{\nabla}^\sharp)) = TP({}^0\nabla^\flat, {}^0\nabla^\sharp), \text{ and}$$
$$i_1^*(TP(\widetilde{\nabla}^\flat, \widetilde{\nabla}^\sharp)) = TP({}^1\nabla^\flat, {}^1\nabla^\sharp).$$

Note that $\widetilde{\nabla}^\flat$ is $\widetilde{E}$-flat, and $\widetilde{\nabla}^\sharp$ is Riemannian for $\mathrm{pr}_M^*(Q)$.

The proof is completed by the elementary lemma (homotopy invariance of de Rham cohomology)

Lemma 2.31. *Let $\omega \in \Omega^k(M\times\mathbb{R})$, $d\omega = 0$. Then $i_1^*(\omega) - i_0^*(\omega)$ is exact.*

Proof. We can write

$$\omega = \pi^*(\alpha)\wedge f(x,t)dt + g(x,t)\pi^*(\beta),$$

with $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$.

One has

$$\begin{aligned}\mathcal{L}_{\partial_t}(\omega) &= d\iota_{\partial t} + \iota_{\partial t} d\omega = \mathcal{L}_{\partial_t}(\omega) = d((-1)^{k-1} f(x,t)\,\mathrm{pr}_M^*(\alpha)) \\ &= (-1)^{k-1} f(x,t) d\,\mathrm{pr}_M^*(\alpha) + \mathrm{pr}_M^*(\alpha)\wedge d_x f + \mathrm{pr}_M^*(\alpha)\wedge \partial_t f dt,\end{aligned}$$

where $\partial_t := \frac{\partial}{\partial t}$. On the other hand

$$\begin{aligned}\mathcal{L}_{\partial_t}\big|_{s=t_0}(\omega) &= \frac{\partial}{\partial s}\big|_{s=t_0}(i_s(\mathrm{pr}_M^*(\alpha)\wedge f(x,t)dt + g(x,t)\,\mathrm{pr}_M^*(\beta))) \\ &= \partial_t f(x,t)\big|_{t_0}\mathrm{pr}_M^*(\alpha)\wedge dt + \partial_t g(x,t)\big|_{t_0}\mathrm{pr}_M^*(\beta).\end{aligned}$$

Comparing both sides one gets

$$\begin{aligned}\partial_t g(x,t)\wedge \mathrm{pr}_M^*(\beta) &= (-1)^{k-1}(f(x,t)d\,\mathrm{pr}_M^*(\alpha) + d_x f(x,t)\wedge \mathrm{pr}_M^*(\alpha)) \\ &= (-1)^{k-1} d_x(f(x,t)\,\mathrm{pr}_M^*(\alpha)).\end{aligned}$$

Hence

$$g(x,1)\,\mathrm{pr}_M^*(\beta) - g(x,0)\,\mathrm{pr}_M^*(\beta) = (-1)^{k-1} d_x\left(\int_0^1 f(x,t)dt\cdot \mathrm{pr}_M^*(\alpha)\right),$$

so

$$i_1^*(\omega) - i_0^*(\omega) = d\left((-1)^{k-1}\int_0^1 f(x,t)dt\cdot\alpha\right).$$

□

□

Proposition 2.32. *For any $P \in \mathcal{I}^k(\mathfrak{gl}_n(\mathbb{R}))$ with $k > q$ odd, $TP(M\mathcal{F})$ is a characteristic class.*

Proof. It is sufficient to prove the naturality in two special cases

1. $i \colon N \to M$ is injective immersion,
2. $p \colon N \times M \to M$ a projection.

Case. 1 We have $i^*(E) = E \cap TN$, $i^*(Q) = Q\big|_N$, hence $\nabla^\flat$, $\nabla^\sharp$ restrict to the same kind of connections. Thus one has

$$TP(N, i^*(\mathcal{F})) = i^*(TP(M, \mathcal{F})).$$

Case. 2 We lift $\nabla^\flat$, $\nabla^\sharp$ to the same kind of connections on $N \times M$. $\widetilde{R}_t = p^*(R_t)$, $\widetilde{\alpha} = p^*(\alpha)$.

□

Definition 2.33. *Two vector bundles $E_0, E_1 \subset TM$ of* codim $= q$ *are transversaly homotopic if there exists $\widetilde{E} \subset T(M \times \mathbb{R})$ of* codim $= q$*, such that*

1. *$\widetilde{E}$ is involutive,*
2. *$\widetilde{E}$ is transversal to $M \times \{0\}$ and $M \times \{1\}$,*
3. *$i_0^*(\widetilde{E}) = E_0$ and $i_1^*(\widetilde{E}) = E_1$.*

Proposition 2.34. *The class $TP(M, \mathcal{F})$ depends only on transverse homotopy class of foliation $\mathcal{F}$.*

Chapter 3

Weil algebras

3.1 The truncated Weil algebras and characteristic homomorphism

The set of invariant polynomials $\mathcal{I}(\mathfrak{gl}_q(\mathbb{R}))$ is generated by $P_{2k}(A) := \operatorname{tr}(A^k)$, $A \in \mathfrak{gl}_q(\mathbb{R})$. Alternatively we have

$$\det(I + tA) = \sum_{i=0}^{q} c_i(A)t^i.$$

Coefficients $c_i(A)$ are symmetric functions of eigenvalues. If

$$A \sim \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_q \end{pmatrix}$$

then

$$\det(I + tA) = (1 + t\lambda_1)(1 + t\lambda_2)\ldots(1 + t\lambda_q) =$$

$$= 1 + t(\lambda_1 + \lambda_2 + \ldots + \lambda_q) + t^2(\sum \lambda_i\lambda_j) + \ldots + t^q\lambda_1\lambda_2\ldots\lambda_q.$$

$$c(A) := \det(I + A) = 1 + c_1(A) + \ldots + c_q(A),$$

$$c(A \oplus B) = c(A)c(B).$$

The set $\mathcal{I}(\mathfrak{gl}_q(\mathbb{R}))$ can be presented as polynomial ring

$$\mathcal{I}(\mathfrak{gl}_q(\mathbb{R})) = \mathbb{R}[c_1, \ldots, c_q].$$

For manifold with foliation $(M, \mathcal{F})$, $Q = TM/E$, $E = \tau\mathcal{F}$, we have

$$c_k(R^\flat) = 0, \ \ \forall k > q.$$

Moreover for each $P \in \mathbb{R}^k[c_1, \ldots, c_q]$, $k > q$

$$P(R^\flat) = 0 \in \Omega^{2k}(M).$$

Define

$$\mathbb{R}[c_1,\dots,c_q]_q := \mathbb{R}[c_1,\dots,c_q]/(\text{weight } > 2q), \ \ \deg(c_i)=2i.$$

For any connection ∇ on E we have a map

$$\lambda_E(\nabla)\colon \mathbb{R}[c_1,\dots,c_q] \to \Omega^\bullet(M),$$

$$\lambda_E(\nabla)(P) := P(\nabla^2).$$

Proposition 3.1. *1. $\lambda_E(\nabla^\flat)$ annihilates all polynomials of degree $> q$, so it induces a map*

$$\lambda_E(\nabla^\flat)\colon \mathbb{R}[c_1,\dots,c_q]_q \to \Omega^\bullet(M).$$

2. $\lambda_E(\nabla^\sharp)$ annihilates all polynomials of odd degree, in particular

$$\lambda_E(\nabla^\sharp)(c_{2i-1}) = 0.$$

3. There is a third map

$$T\lambda_E(\nabla^\flat,\nabla^\sharp)\colon \mathbb{R}[c_1,\dots,c_q] \to \Omega^*(M)$$

satisfying

$$dT\lambda_E(\nabla^\flat,\nabla^\sharp)(P) = \lambda_E(\nabla^\flat)(P) - \lambda_E(\nabla^\sharp)(P).$$

In particular

$$dT\lambda_E(\nabla^\flat,\nabla^\sharp)(c_{2i-1}) = \lambda(\nabla^\flat)(c_{2i-1}).$$

This can be summarized in the following cochain complex. First form a differential graded algebra (DGA)

$$WO_q := \Lambda\langle u_1,u_3,\dots,u_{2l-1}\rangle \otimes \mathbb{R}[c_1,\dots,c_q]_q,$$

where the first algebra in the tensor product is an exterior algebra generated by elements u_{2i-1} of degree $4i-3$, and l is maximal integer such that $2l-1 \leqslant q$. Generators of second algebra c_j have degree $2j$, and this is a quotient of polynomial algebra by the ideal of polynomials of degree $> q$ (weight $> 2q$). Now define $d\colon WO_q \to WO_q$ as the differenital of degree 1 given on generators by the formula

$$du_{2i-1} = c_{2i-1}, \ \ 1 \leqslant i \leqslant l,$$

$$dc_j = 0, \ \ 1 \leqslant i \leqslant q.$$

Definition 3.2. *Define a map $\lambda_E\colon WO_q \to \Omega^\bullet(M)$ by*

$$\lambda_E(u_{2i-1}) := T\lambda_E(\nabla^\flat,\nabla^\sharp)(c_{2i-1}),$$

$$\lambda_E(c_j) := \lambda_E(\nabla^\flat)(c_j), \ \ 1 \leqslant j \leqslant q.$$

Then $\lambda_E\colon WO_q \to \Omega^\bullet(M)$ is a map of DGA's, hence it induces a map

$$\lambda_E^*\colon \ \mathrm{H}^*(WO_q) \to \mathrm{H}^*(M;\mathbb{R})$$

of cohomology algebras.

We call λ_E^* a characteristic map in analogy to

$$\chi_E\colon \mathrm{H}^*(\mathrm{B}\,\mathrm{GL}_n(\mathbb{R})) = \mathcal{I}(\mathfrak{gl}_n(\mathbb{R})) \to \mathrm{H}^*(M;\mathbb{R})$$

for a n-dimesional vector bundle $E \to M$.

Theorem 3.3 (Bott)**.** *1. λ_E^* depends only on E, and not on the choice of connections.*

2. λ_E^* is natural, i.e. for $\phi\colon N \to M$, $\phi \pitchfork \mathcal{F}$, one has

$$\lambda_{\phi^*(E)}^* = \phi^* \circ \lambda_E^*.$$

3. λ_E^* depends only on the transverse homotopy class of E (def. (2.33)).

Proof. Theorem has essentially been proved.

1. This has been proved in proposition (2.30).

2. This has been proved in proposition (2.32).

3. The same proof as in proposition (2.30) and lemma (2.31) with $\widetilde{\nabla}_t$ on $M \times I$ inducing ∇_t^0 on E_0 and ∇_t^1 on E_1.

□

Example 3.4 (WO_1 and Godbillon–Vey class)*.* For $q = 1$ we have

$$WO_1 = \Lambda\langle u_1\rangle \otimes \mathbb{R}[c_1]_1,$$

hence $\{1, u_1, c_1, u_1c_1\}$ form a $\mathbb{R}$-basis and $du_1 = c_1$, $dc_1 = 0$. Clearly

$$\mathrm{H}^0(WO_1) = \mathbb{R}\cdot 1,$$

$$\mathrm{H}^1(WO_1) = 0,$$
$$\mathrm{H}^2(WO_1) = 0,$$
$$\mathrm{H}^3(WO_1) = \mathbb{R}\cdot u_1c_1.$$

Let (M, E) be a manifold with codim $= 1$ foliation $\mathcal{F}$, $\tau\mathcal{F} = E$, and assume that $Q = TM/E$ is trivializable (i.e. E transversaly oriented).

$$\lambda_E(c_1) = \lambda_E(\nabla^\flat)(c_1),$$

$$\lambda_E(u_1) = T\lambda_E(\nabla^\flat, \nabla^\sharp)(c_1).$$

Let $\Omega \in \Omega^1(M)$ be the orientation form of Q^*, so $E = \ker\Omega$. Let Z be a vector field with $\Omega(Z) = 1$, which gives trivialization of Q. Then

$$TM = E \oplus \mathbb{R}Z.$$

Let Ω be defined by

$$\Omega(X) = 0, \quad \text{for } X \in E,$$

$$\Omega(Z) = 1.$$

Then
$$d\Omega = \alpha \wedge \Omega, \ \ \alpha \in \Omega^1(M).$$
Form α defines a Bott connection by
$$\nabla^\flat(\pi(Z)) = -\alpha \otimes \pi(Z),$$
$$\nabla^\flat_X(\pi(Z)) = -\alpha(X)(\pi(Z)) = \pi([X,Z]).$$
Indeed, one has for all $X \in E$
$$d\Omega(X,Z) = -\Omega([X,Z]) = -\Omega(\pi([X,Z])), \text{ and}$$
$$\alpha \wedge \Omega(X,Z) = \alpha(X)\Omega(Z) - \alpha(Z)\Omega(X) = \alpha(X).$$
Thus
$$\alpha(X) = -\Omega(\pi([X,Z])).$$
Godbillon–Vey class is a class of $\alpha \wedge d\alpha$ in $\mathrm{H}^3(M;\mathbb{R})$. One the other hand one has
$$(\nabla^\flat)^2(\pi(Z)) = \nabla^\flat(-\alpha \otimes \pi(Z)) = -d\alpha \otimes \pi(Z) + \alpha \wedge \alpha \otimes \pi(Z) =$$
$$= d\alpha \otimes \pi(Z),$$
hence
$$R^\flat = d\alpha, \text{ so}$$
$$\lambda_E(c_1) = d\alpha.$$
Define a Riemannian connection on Q by
$$\nabla^\sharp_X(\pi(Z)) = 0, \ \ \forall X \in E,$$
$$\nabla^\sharp_Z(\pi(Z)) = 0, \text{ where } ||Z|| = 1.$$
Then $\nabla^\flat - \nabla^\sharp = -\alpha \in \Omega^1(M, \mathrm{End}(Q)) = \Omega^1(M)$, hence
$$\lambda_E(u_1) = T\lambda_E(\nabla^\flat, \nabla^\sharp)(c_1) = -\alpha.$$
This implies
$$\lambda_E(u_1 c_1) = \alpha \wedge d\alpha = \mathrm{gv}(M, \mathcal{F}).$$
Proposition 3.5. *If $E = \tau\mathcal{F}$ is of* codim $= q$*, transversally oriented, then*
$$\lambda_E(u_1 c_1^q) = \mathrm{gv}(E).$$
Proof. We have nonvanishing form $\Omega \in \Gamma((Q^*)^q)$. Locally it can be written as
$$\Omega = \omega_1 \wedge \ldots \wedge \omega_q,$$
with $\{\omega_1, \ldots, \omega_q\}$- generators of $\Gamma(E)$. Write
$$d\omega_i = \sum_j \alpha_{ij} \wedge \omega_j,$$

and define $\nabla^\flat \colon \Gamma(Q) \to \Gamma(T^*M \otimes Q)$ by

$$\nabla^\flat(\pi(Z_i)) = -\sum_j \alpha_{ji} \otimes \pi(Z_j),$$

where $\{Z_1, \dots, Z_q\}$ is a dual basis to $\{\omega_1, \dots, \omega_q\}$ on a complement of E. One has for all $X \in E$

$$d\omega_i(X, Z_k) = \sum_j (\alpha_{ij}(X)\omega_j(Z_k) - \alpha_{ij}(Z_k)\omega_j(X)).$$

But

$$d\omega_i(X, Z_k) = -\omega_i([X, Z_k]) = \pi([X, Z_k]),$$

and on the right hand side we have only $\alpha_{ik}(X)$, so

$$\pi([X, Z_k]) = \sum_i \alpha_{ik}(X)\pi(Z_i),$$

while

$$\nabla^\flat_X(\pi(Z_k)) = -\sum_j \alpha_{jk}(X)\pi(Z_j) = \pi([X, Z_k]),$$

hence it is a Bott connection. Its curvature is

$$(\nabla^\flat)^2(\pi(Z_i)) = -\sum_j \nabla^\flat(\alpha_{ij} \otimes \pi(Z_j)) =$$

$$= -\sum_j d\alpha_{ji} \otimes \pi(Z_j) + \sum_j \alpha_{ji}(-\sum_k \alpha_{kj} \otimes \pi(Z_k)) =$$

$$= -\sum_k (d\alpha_{ki} - \sum_j \alpha_{kj} \wedge \alpha_{ji})\pi(Z_k),$$

i.e.

$$R = d\alpha - \alpha \wedge \alpha.$$

This implies

$$c_1(R) = \operatorname{tr}(d\alpha) - \operatorname{tr}(\alpha \wedge \alpha) = \operatorname{tr}(d\alpha) = d(\operatorname{tr}\alpha),$$

hence

$$c_1(R)^q = d(\operatorname{tr}\alpha)^q.$$

Take Riemannian connection given by an orthogonal matrix form

$$\nabla^\sharp(\pi(Z_i)) = \sum_j \beta_{ij} \otimes \pi(Z_j).$$

Now

$$(\nabla^\flat - \nabla^\sharp)(\pi(Z_i)) = \sum_j (\alpha_{ij} + \beta_{ij}) \otimes \pi(Z_j),$$

hence

$$\nabla^\flat - \nabla^\sharp = -\alpha - \beta, \ \ \operatorname{tr}\beta = 0$$

so the transgressed form is

$$Tc_1(\alpha + \beta) = \operatorname{tr}\alpha.$$

Now

$$\operatorname{gv}(E) = [\operatorname{tr}\alpha \wedge (\operatorname{tr}(d\alpha))^q] = [u_1 c_1(R)^q].$$

□

3.2 W_q and framed foliations

Definition 3.6. *Differential graded algebra* W_q

$$W_q := \Lambda\langle u_1, \dots, u_q\rangle \otimes \mathbb{R}[c_1, \dots, c_q]_q$$

$$du_i = c_i, \quad dc_i = 0, \ \forall i = 1, \dots, q.$$

These algebras are useful for foliation $(M, \mathcal{F})$ with Q trivializable, when one can transgress to a flat Riemannian connection and get

$$\mu_E \colon W_q \to \Omega^\bullet(M),$$

$$\mu_E(u_i) := T\lambda_E(\nabla^\flat, \nabla^{\sharp,0})(c_i),$$

$$\mu_E(c_i) := \lambda_E(\nabla^\flat)(c_i).$$

Notation: for $\underbrace{i_1 < \dots < i_r}_{I}$, $\underbrace{j_1 \leqslant \dots \leqslant j_s}_{J}$ we denote

$$u_I c_J = u_{i_1} \dots u_{i_r} c_{j_1} \dots c_{j_r}.$$

Proposition 3.7. *The elements*

(a)

$$1 \cup \{\, u_I c_J \;\big|\; |J| \leqslant q, i_1 + |J| > q, i_1 \leqslant j_1 \}$$

form a basis of $\mathrm{H}^*(W_q)$.

(b)

$$1 \cup \{\, u_I c_J \;\big|\; i_k \ \textit{odd}\ , |J| < q, i_1 + |J| > q, \ \textit{and} \ \begin{Bmatrix} \textit{if } r = 0 \textit{ then all } j_k \textit{ even,} \\ \textit{if } r \neq 0 \textit{ then } i_1 \leqslant \min_{odd}\{j_k\} \end{Bmatrix}$$

form a basis of $\mathrm{H}^*(WO_q)$.

Proof. (sketch)

Ad.(a)

$$\begin{aligned} d(u_I c_J) &= \sum_{k=1}^{r} (-1)^{k-1} u_{i_1} \dots du_{i_k} \dots u_{i_r} c_J \\ &= \sum_{k=1}^{r} (-1)^{k-1} u_{i_1} \dots \widehat{u_{i_k}} \dots u_{i_r} c_{i_k} c_J = 0, \end{aligned}$$

because $\deg c_{i_k} c_J \geqslant 2(|J| + i_1) > 2q$.

Ad.(b) If $r = 0$ then $d(c_J) = 0$. The case $r \neq 0$ is treated as above.

□

Consequences of (a) for $\mathrm{H}^*(W_q)$.

1.

$$\begin{aligned} \deg(u_I c_J) &= (2i_1 - 1) + \dots + (2i_r - 1) + (2j_1 + \dots + 2j_s) \\ &\leqslant 2(1 + \dots + q) - q + 2|J| \leqslant q(q+1) - q + 2q = q^2 + 2q, \end{aligned}$$

hence $\mathrm{H}^m(W_q) = 0$, for $m > q^2 + 2q$.

2. On the other hand
$$\deg(u_I c_J) \geqslant 2|J| \geqslant 2q,$$
hence $\mathrm{H}^m(W_q) = 0$, for $1 \leqslant m < 2q$. With a little more work we can eliminate $m = 2q$ which can occur only if $|I|$ even.

3. The product structure is trivial.

4. In $\mathrm{H}^{2q+1}(W_q)$ the classes $u_1 c_1^{\alpha_1} \dots c_k^{\alpha_k}$ with $\sum_{i=1}^k \alpha_i = q$ are linearly independent

Similar conclusions hold for $\mathrm{H}^*(WO_q)$:

1. $\mathrm{H}^m(WO_q) = 0$, for $m > q^2 + 2q$.

2. For $m \leqslant 2q$ one gets the Pontryagin classes
$$\{1, p_1, \dots, p_{[\frac{q}{2}]}\}.$$

3. The product structure is trivial in “high degree”.

4. In $\mathrm{H}^{2q+1}(WO_q)$ the classes $u_1 c_1^{\alpha_1} \dots c_k^{\alpha_k}$ with $\sum_{i=1}^k \alpha_i = q$ are linearly independent.

Chapter 4

Gelfand-Fuks cohomology

4.1 Cohomology of Lie algebras

Recall the formula for the exterior derivation $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$

$$
\begin{aligned}
d\omega(X_0, \ldots, X_p) &= \sum_{i=0}^{p} (-1)^i X_i \omega(X_0, \ldots, \widehat{X_i}, \ldots, X_p) \\
&+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p).
\end{aligned}
$$

$$\mathrm{H}^*(\Omega^\bullet(M), d) = \mathrm{H}^*_{dR}(M; \mathbb{R}).$$

We can view $\Omega^\bullet(M)$ as a $C^\infty(M)$ linear homomorphisms

$$\Omega^\bullet(M) \cong \mathrm{Hom}_{C^\infty(M)}(\Lambda^\bullet V_M, C^\infty(M)),$$

where V_M is a Lie algebra of vector fields on M with

$$[X, Y] = XY - YX.$$

More general context consists of

- $\mathfrak{g}$ - a Lie algebra of finite dimension over a field k,
- A - $\mathfrak{g}$-module
- Cochains $C^\bullet(\mathfrak{g}; A) := \mathrm{Hom}_k(\Lambda^\bullet \mathfrak{g}, A)$ with differential

$$d\colon C^p(\mathfrak{g}; A) \to C^{p+1}(\mathfrak{g}; A),$$

  given by the same formula as above.
- Cohomology
$$\mathrm{H}^*(\mathfrak{g}; A) := \mathrm{H}^*(C^\bullet(\mathfrak{g}; A), d).$$

Relative Lie algebra cohomology is defined as follows. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie subalgebra. Define relative cochains as

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) := \{c \in C^\bullet(\mathfrak{g}; A) \mid \iota_X c = 0 \text{ and } \iota_X dc = 0 \ \forall X \in \mathfrak{h}\}.$$

By definition it is a subcomplex and its cohomology is

$$\mathrm{H}^*(\mathfrak{g}, \mathfrak{h}; A) := \mathrm{H}^*(C^\bullet(\mathfrak{g}, \mathfrak{h}; A), d).$$

Since

$$\mathcal{L}_X = d\iota_X + \iota_X d,\ \mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega = 0,$$

alternatively we can put

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) := \{c \in C^\bullet(\mathfrak{g}; A) \mid c \text{ basic i.e. } \iota_X c = 0 \text{ and } \mathcal{L}_X c = 0\ \forall X \in \mathfrak{h}\}.$$

One has

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) = \mathrm{Hom}_k(\Lambda^\bullet(\mathfrak{g}/\mathfrak{h}), A)^{\mathfrak{h}}.$$

Slightly more generally, if H is a Lie group with $\mathfrak{h} = \mathrm{Lie}(H)$, acting on $\mathfrak{g}$ and A such that, the differential of the action on $\mathfrak{g}$ is $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$, then

$$C^\bullet(\mathfrak{g}, H; A) := \{c \in \mathrm{Hom}_H(\Lambda^\bullet\mathfrak{g}, A) \mid \iota_X c = 0\ \forall X \in \mathfrak{h}\},$$

and its cohomology is

$$\mathrm{H}^*(\mathfrak{g}, H; A).$$

Example 4.1. Let $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{R})$. Its complexification is $\mathfrak{g}_{\mathbb{C}} := \mathfrak{gl}_n(\mathbb{C})$. We have

$$\mathrm{H}^*(\mathfrak{g}_{\mathbb{C}}) = \mathrm{H}^*(\mathfrak{g}) \otimes \mathbb{C}.$$

Also one has for $\mathfrak{u}_n := \mathrm{Lie}(U(n))$

$$\mathrm{H}^*(\mathfrak{gl}_n(\mathbb{R})) = \mathrm{H}^*(\mathfrak{u}_n) = \Lambda\langle u_1, u_3, \ldots, u_{2l+1}\rangle, l = \left[\frac{n}{2}\right].$$

Furthermore for $g \in U(n)$ and k odd

$$d\,\mathrm{tr}((g^{-1}dg)^k) = -\,\mathrm{tr}((g^{-1}dg)^{k+1}) = 0.$$

The class $u_k := [\mathrm{tr}((g^{-1}dg)^k)]$ is called a Chern-Simons class.

4.2 Gelfand-Fuks cohomology

Let V_M be the algebra of vector fields on a manifold M, that is $\Gamma(TM)$. C^∞ topology on V_M is given by C^∞ convergence on compacta of the local components (which are functions), and their derivatives.

$$X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i},\ f^i \in C^\infty(M).$$

Definition 4.2. *Define the Gelfand-Fuks cohomology as the cohomology of the algebra V_M continuous with respect to the C^∞ topology on V_M*

$$\mathrm{H}^*_{GF}(V_M) := \mathrm{H}^*_{cont}(V_M; \mathbb{R}).$$

Here $C^\bullet_{cont}(V_M;\mathbb{R})$ are continuous functionals on V_M with respect to C^∞ topology.

The remarkable fact [Gelfand-Fuks] is that H^*_{GF} is finite dimensional. An important step in the proof of this is played by an algebra of formal vector fields on M

$$\mathfrak{A}_n := \left\{ X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i \in \mathbb{R}[[x^1,\dots,x^n]] \right\}.$$

The dual algebra of vector fields

$$V_M^* := \mathrm{Hom}_{cont}(V_M,\mathbb{R})$$

consists of distributions with compact support. The notion of support makes sense for the cochains

$$C^\bullet_{cont}(V_M,\mathbb{R}) := \Lambda^\bullet V_M^*$$

and is preserved by

$$d\colon \Lambda^\bullet V_M^* \to \Lambda^{\bullet+1} V_M^*.$$

In particular one can take for $p_0 \in M$ the subcomplex

$$\Lambda^\bullet V^*_{M,p_0} := \text{ distributions supported at } p_0.$$

Then V^*_{M,p_0} is a real vector space spanned by ∇_{p_0} and its partial derivatives

$$X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$$

$$X \mapsto (-1)^{|\alpha|} \frac{\partial^{|\alpha|} f^i}{\partial x^\alpha}.$$

They only depend on the jet of X at p_0. Thus we are dealing with the continuous Lie algebra complex of

$$\mathfrak{A}_n := \left\{ X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i \in \mathbb{R}[[x^1,\dots,x^n]] \right\}.$$

with the $\mathcal{I}$-adic topology (since the elements of the dual depend on finite set).

In $\mathfrak{A}_n^*$ we have following forms

$$\begin{aligned}
\theta^i(X) &:= f^i(0),\ 1 \leqslant i \leqslant n,\\
\theta^i_j(X) &:= -\frac{\partial f^i}{\partial x^j}\Big|_{x=0},\ 1 \leqslant i,j \leqslant n,\\
\theta^i_{jk}(X) &:= \frac{\partial^2 f^i}{\partial x^j \partial x^k}\Big|_{x=0},\ 1 \leqslant i,j,k \leqslant n,
\end{aligned}$$

and generally for multiindex $\alpha = (\alpha_1,\dots,\alpha_n)$

$$\theta^i_\alpha := (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha}\Big|_{x=0}.$$

We make $\Lambda^\bullet \mathfrak{A}_n^*$ into a complex by defining the differential

$$d\omega(X_0,\dots,X_n) := \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_n).$$

1. The elements
$$\{\theta^i_\alpha \mid 1 \leqslant i \leqslant n, \alpha \in (\mathbb{Z}_+)^n\}$$
span $C^1(\mathfrak{A}_n) = \mathfrak{A}_n^*$, hence generate all of
$$C^\bullet(\mathfrak{A}_n) = \bigoplus_{k=0}^{\infty} \Lambda^k \mathfrak{A}_n^*.$$
Note that $\theta^i_\alpha = \theta^i_\beta$ if $\alpha = \beta$ as an unordered sets.

2. The Lie derivative
$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\theta^i = \theta^i_j, \text{ and}$$
$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\mathcal{L}\left(\frac{\partial}{\partial x^k}\right)\theta^i = \theta^i_{jk}, \text{ etc.}$$
Indeed
$$\begin{aligned}\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\theta^i(X) &= \left(\frac{d}{dt}\big|_{t=0}\tau^j_t\theta^i\right)(X)\\ &= \theta^i\left(\frac{d}{dt}\big|_{t=0}\tau^j_{-t}(X)\right)\\ &= \frac{d}{dt}\big|_{t=0} f^i(x^1, \ldots, x^j - t, \ldots, x^n)\\ &= -\frac{\partial f^i}{\partial x^i}\big|_{x=0} = \theta^i_j(X).\end{aligned}$$
In general
$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\theta^i_\alpha = \theta^i_{\alpha\cup j}$$
Since
$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0,$$
we have
$$\left[\mathcal{L}\left(\frac{\partial}{\partial x^i}\right), \mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\right] = 0,$$
whence

3. $$C^1(\mathfrak{A}_n) \cong \mathbb{R}\left[\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right]\{\theta^1, \ldots, \theta^n\}$$
i.e. is a free module with n generators over the polynomial ring in n generators.

Proposition 4.3. *We have following identities in $C^\bullet(\mathfrak{A}_n)$*

1.
$$d\theta^i + \sum_j \theta^i_j \wedge \theta^j = 0,$$

2.
$$d\theta^i_k + \sum_j \left(\theta^i_{jk} \wedge \theta^j + \theta^i_j \wedge \theta^j_k\right) = 0,$$

3.
$$d\theta^i_{kl} + \sum_j \left(\theta^i_{jkl} \wedge \theta^j + \theta^i_{jk} \wedge \theta^j_l + \theta^i_{jl} \wedge \theta^j_k + \theta^i_j \wedge \theta^j_{kl}\right) = 0.$$

Proof.
$$d\theta^i(X,Y) = \underbrace{X\theta^i(Y) - Y\theta^i(X)}_{=0} - \theta^i([X,Y]) = -\theta^i([X,Y]),$$

where $X = \sum_j f^j \frac{\partial}{\partial x^j}$, $X = \sum_j g^k \frac{\partial}{\partial x^k}$.

$$\begin{aligned}[X,Y] &= \sum_{j,k} \left(f^j \frac{\partial g^k}{\partial x^j}\frac{\partial}{\partial x^k} - g^k \frac{\partial f^j}{\partial x^k}\frac{\partial}{\partial x^j}\right) \\ &= \sum_k \left(\sum_j \left(f^j \frac{\partial g^k}{\partial x^j} - g^j \frac{\partial f^k}{\partial x^j}\right)\right)\frac{\partial}{\partial x^k}.\end{aligned}$$

Hence
$$d\theta^i(X,Y) = \sum_j \left(\underbrace{f^j \frac{\partial g^i}{\partial x^j} - g^j \frac{\partial f^i}{\partial x^j}}_{=0} - f^j \frac{\partial g^i}{\partial x^j} + g^j \frac{\partial f^i}{\partial x^j}.\right)$$

On the other hand
$$\begin{aligned}\theta^i_j \wedge \theta^j(X,Y) &= \theta^i_j(X)\theta^j(Y) - \theta^i_j(Y)\theta^j(X) \\ &= \sum_j \left(-\frac{\partial f^i}{\partial x^j} g^j + \frac{\partial g^i}{\partial x^j} f^j\right).\end{aligned}$$

This proves (1). To obtain (2) we apply $\mathcal{L}\left(\frac{\partial}{\partial x_k}\right)$, and applying $\mathcal{L}\left(\frac{\partial}{\partial x_l}\right)$ to (2) we obtain (3) etc. These equations completely determine differential d. $\square$

Denote
$$R^i_j := d\theta^i_j + \sum_k \theta^i_k \wedge \theta^k_j \in C^2(\mathfrak{A}_n) = \Lambda^2 \mathfrak{A}_n^*.$$

Then equation (2) becomes

2'
$$R^i_j = -\sum_k \theta^i_{jk} \wedge \theta^k.$$

Proposition 4.4.

1.
$$R^i_j \wedge \theta^j = 0,$$

2.
$$dR^i_j = \sum_k \left(R^i_k \wedge \theta^k_j - \theta^i_k \wedge R^k_j\right).$$

Proof. From (2’)

$$R^i_j \wedge \theta^j = -\sum_k \theta^i_{jk} \wedge \theta^k \wedge \theta^j = 0$$

since $\theta^i_{jk} = \theta^i_{kj}$.

From (2)

$$\begin{aligned}
dR^i_j &= \sum_k \left(d\theta^i_k \wedge \theta^k_j - \theta^i_k \wedge d\theta^k_j \right) \\
&= \sum_k \left(-\sum_l (\theta^i_{lk} \wedge \theta^l + \theta^i_l \wedge \theta^l_k) \wedge \theta^k_j + \sum_l \theta^i_k \wedge (\theta^k_{lj} \wedge \theta^l + \theta^k_l \wedge \theta^l_j) \right) \\
&= \sum_{k,l} \left(R^i_k \wedge \theta^k_j - \theta^i_l \wedge \theta^l_k \wedge \theta^k_j + \theta^i_k \wedge R^k_j + \theta^i_k \wedge \theta^k_l \wedge \theta^l_j \right) \\
&= \sum_k \left(R^i_k \wedge \theta^k_j - \theta^i_k \wedge R^k_j \right).
\end{aligned}$$

□

Corollary 4.5. *The subalgebra $\widetilde{W_n} := \mathbb{R}\{\theta^i_j, R^i_j\}$ is closed under d and finite dimensional.*

Proof. Finite dimension follows from (2’). □

4.3 Some “soft” results

We describe the grading on an algebra $\mathfrak{A}_n$.

$$\mathfrak{A}_n = \left\{ X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i(x) = \sum_\alpha c^i_\alpha x^\alpha \in \mathbb{R}[[x_1, \dots, x_n]], \alpha = (\alpha_1, \dots, \alpha_n) \right\}.$$

$$\mathfrak{A}_n = \mathbb{R}^n \oplus \mathfrak{gl}_n(\mathbb{R}) \oplus \dots$$

One has

$$[x^i \frac{\partial}{\partial x^j}, x^k \frac{\partial}{\partial x^l}] = \delta^k_j x^i \frac{\partial}{\partial x^l} - \delta^i_l x^k \frac{\partial}{\partial x^j},$$

To see grading we take $E = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} \in \mathfrak{A}_n$. Then

$$[E, X] = \sum_j \sum_i \left(x^i \frac{\partial f^j}{\partial x^i} - f^j \right) \frac{\partial}{\partial x^j}$$

and if $f^j = c^j_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $|\alpha| = r$, then

$$\begin{aligned}
\left[E, c^j_\alpha x^\alpha \frac{\partial}{\partial x^j} \right] &= \left[\sum_i x^i \frac{\partial}{\partial x^i}, c^j_\alpha x^\alpha \frac{\partial}{\partial x^j} \right] \\
&= \sum_i \alpha_i x^\alpha \frac{\partial}{\partial x^j} - \sum_i x^\alpha \delta^i_j \frac{\partial}{\partial x^i} \\
&= (|\alpha| - 1) x^\alpha \frac{\partial}{\partial x^j}.
\end{aligned}$$

Thus each monomial is an eigenvector for E, and we can write $\mathfrak{A}_n$ as a sum of eigenspaces

$$\mathcal{L}_E(x^\alpha \frac{\partial}{\partial x^j}) = (|\alpha| - 1)x^\alpha \frac{\partial}{\partial x^j},$$
$$\mathfrak{A}_n^{(p)} := \{X \in \mathfrak{A}_n \mid \mathcal{L}_E(X) = pX\},$$
$$\mathfrak{A}_n = \bigoplus_{p=-1}^{\infty} \mathfrak{A}_n^{(p)}, \quad E\big|_{\mathfrak{A}_n^{(p)}} = p \cdot \mathrm{Id}.$$

It is a grading, i. e.

$$[\mathfrak{A}_n^{(p)}, \mathfrak{A}_n^{(q)}] \subset \mathfrak{A}_n^{(p+q)}.$$

We have a dual grading on the Gelfand-Fuks complex $C^\bullet(\mathfrak{A}_n) = \Lambda^\bullet \mathfrak{A}_n^*$. One has the Lie derivative

$$\mathcal{L}_E \colon \mathfrak{A}_n^* \to \mathfrak{A}_n^*.$$
$$\mathcal{L}_E = d\iota_E + \iota_E d,$$

The dual grading on $\mathfrak{A}_n^*$ can be described as

$$(\mathfrak{A}_n^*)^{(p)} := \{\omega \in \mathfrak{A}_n^* \mid \mathcal{L}_E(\omega) = -p\omega\}.$$

This induces a grading on G-F complex

$$C^m(\mathfrak{A}_n)^{(p)} = (\Lambda^m \mathfrak{A}_n^*)^{(p)} = \bigoplus \Lambda^{k_{-1}}(\mathfrak{A}_n^*)^{(-1)} \otimes \Lambda^{k_0}(\mathfrak{A}_n^*)^{(0)} \otimes \ldots \otimes \Lambda^{k_r}(\mathfrak{A}_n^*)^{(r)},$$

where

$$k_{-1} + k_0 + \ldots = m, \quad -k_{-1} + k_1 + 2k_2 + \ldots + rk_r = p.$$

We have $\mathcal{L}_E d = d\mathcal{L}_E$ (so $\mathcal{L}_E$ is a map of complexes). We can restrict to degree p

$$\mathcal{L}_E\big|_{C^\bullet(\mathfrak{A}_n)^{(p)}} = -p \cdot \mathrm{Id}$$

Proposition 4.6.

$$\dim \mathrm{H}^*_{GF}(\mathfrak{A}_n) < \infty, \ \forall n \geqslant 0,$$
$$\mathrm{H}^m_{GF}(\mathfrak{A}_n) = 0, \ \forall m > n^2 + 2n.$$

Proof. One has

$$\mathcal{L}_E(\omega) = d\iota_E(\omega) + \iota_E d\omega$$

so any $\omega \in C^m(\mathfrak{A}_n)^{(p)}$ with $p \neq 0$ such that $d\omega = 0$ is exact, since then

$$d\iota_E(\omega) = \mathcal{L}_E(\omega) = -p\omega.$$

This gives on cohomology

$$\mathrm{H}^m_{GF}(\mathfrak{A}_n) = \mathrm{H}^m_{GF}(\mathfrak{A}_n)^{(0)} := \mathrm{H}^m(C^\bullet(\mathfrak{A}_n)^{(0)}),$$

where

$$C^m(\mathfrak{A}_n)^{(0)} = (\Lambda^m \mathfrak{A}_n^*)^{(0)} = \bigoplus \Lambda^{k_{-1}}(\mathfrak{A}_n^*)^{(-1)} \otimes \Lambda^{k_0}(\mathfrak{A}_n^*)^{(0)} \otimes \ldots \otimes \Lambda^{k_r}(\mathfrak{A}_n^*)^{(r)},$$

$$-k_{-1} + k_1 + 2k_2 + \ldots + rk_r = 0,$$
$$k_{-1} + k_0 + k_1 + \ldots + k_r = m.$$

Since
$$\dim \mathfrak{A}_n^{(-1)} = \dim \mathbb{R}^n = n \implies k_{-1} \leqslant n,$$
$$\dim \mathfrak{A}_n^{(0)} = n^2 \implies k_0 \leqslant n^2.$$
Furthermore
$$k_1 \leqslant n, k_2 \leqslant \frac{n}{2}, \ldots, k_n \leqslant 1.$$
Hence
$$\dim C^m(\mathfrak{A}_n)^{(0)} < \infty \text{ for } m \geqslant 0,$$
$$C^m(\mathfrak{A}_n)^{(0)} = 0 \text{ for } m > n^2 + 2n.$$

□

Example 4.7. For $n = 1$ we have following
$$k_1 + 2k_2 + \ldots k_r = k_{-1},$$
$$k_{-1} + k_0 + k_1 + \ldots + k_r \leqslant 3.$$
This gives
$$k_1 \leqslant 1, k_2 \leqslant \frac{1}{2} \text{ etc.} \implies k_2 = \ldots = k_r = 0.$$
The dual algebra
$$\mathfrak{A}_n^* \cong \underbrace{\mathbb{R}\theta^1}_{\mathrm{deg}=-1} \oplus \underbrace{\mathbb{R}\theta_1^1}_{deg=0} \oplus \underbrace{\mathbb{R}\theta_{11}^1}_{deg=1} \oplus \ldots$$
If $k_{-1} = 0$ then $k_1 = k_2 = \ldots = 0$ hence the only one allowed is
$$\Lambda^\bullet(\mathfrak{A}_1^*)^{(0)} = \mathbb{R} \oplus \mathbb{R}\theta_1^1.$$
For $k_{-1} = 1$ we have $k_1 = 1$ and
$$\underbrace{\Lambda^1(\mathfrak{A}_1^*)^{(-1)}}_{=\mathbb{R}\theta^1} \otimes \underbrace{\Lambda^\bullet(\mathfrak{A}_1^*)^{(0)}}_{=\mathbb{R}\oplus\mathbb{R}\theta_1^1} \otimes \underbrace{\Lambda^1(\mathfrak{A}_1^*)^{(1)}}_{=\mathbb{R}\theta_{11}^1}$$
Thus we need only to look at the subcomplex
$$\mathbb{R}\{1, \theta_1^1, \theta^1 \wedge \theta_{11}^1, \underbrace{\theta^1 \wedge \theta_1^1 \wedge \theta_{11}^1}_{=\theta_1^1\wedge R_1^1}\}$$
because $R_1^1 = d\theta_1^1 = -\theta_{11}^1 \wedge \theta^1 \neq 0$, so the cohomology is
$$\mathrm{H}_{GF}^* = \underbrace{\mathbb{R}}_{\dim=0} \oplus \underbrace{\mathbb{R}(\theta_1^1 \wedge R_1^1)}_{\dim=3}.$$

4.4 Spectral sequences

The algebra generated by $\{\theta_j^i, R_j^i\}$ is closed under the differential d, so we have a subcomplex
$$(\mathbb{R}\{\theta_j^i, R_j^i\}, d) =: (\widetilde{W_n}, d) \subset (C^\bullet(\mathfrak{A}_n), d).$$
where
$$\mathbb{R}\{\theta_j^i, R_j^i\} \cong \Lambda^\bullet \mathfrak{gl}_n(\mathbb{R})^* \otimes S_n(\mathfrak{gl}_n(\mathbb{R})^*)$$

Theorem 4.8. *The inclusion*

$$(\widetilde{W_n}, d) \hookrightarrow (C^\bullet(\mathfrak{A}_n), d)$$

is a quasi-isomorphism (induces isomorphism on cohomology).

The proof uses Hochschild–Serre spectral sequence, which we describe next.

4.4.1 Exact couples

Assume we have an exact sequence of the form

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

It is called an *exact couple*. Define

$$d\colon B \to B,\ \ d := jk,\ \ d^2 = jkjk = 0,\ \text{and}$$

$$\mathrm{H}(B) := \ker d / \operatorname{im} d.$$

Now we can form derived couple taking

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] & \\ & B' \arrow[ul, "k'"] \arrow[r, equal] & \mathrm{H}(B) \end{tikzcd}$$

where

- $A' := i(A)$,
- $B' := \mathrm{H}(B)$,
- $i'(a') = i(a') = i(i(a))$,
- $j'(a') = [j(a)]$ for $a' = i(a)$,
- $k'([b]) = k(b)$.

Check this definitions for independence of representatives. The derived couple is again exact couple.

4.4.2 Filtered complexes

Let $(C^\bullet, d)$ be a filtered complex i.e. there is a sequence of subcomplexes

$$C^\bullet = C_0^\bullet \supset C_1^\bullet \supset C_2^\bullet \supset \dots$$

Let

$$A := \bigoplus_{p\in\mathbb{Z}} C_p, \quad B := \bigoplus_{p\in\mathbb{Z}} C_p/C_{p+1}$$

Inclusions $C_{p+1} \hookrightarrow C_p$ induce exact sequence

$$0 \to A \xrightarrow{i} A \xrightarrow{B} \to 0,$$

a long exact sequence of homology

$$\dots \mathrm{H}(A) \xrightarrow{i_*} \mathrm{H}(A) \xrightarrow{j_*} \mathrm{H}(B) \xrightarrow{k_*} A \to \dots,$$

and an exact couple

$$\begin{tikzcd}
A_1 := \mathrm{H}(A) \arrow[rr, "i_*"] & & \mathrm{H}(A) \arrow[dl, "j_*"] \\
& \mathrm{H}(B) =: B_1 \arrow[ul, "k_*"] &
\end{tikzcd}$$

4.4.3 Illustration of convergence

Consider simple case, filtration of a complex $\mathrm{H}(C^\bullet)$

$$\dots = C_{-2} = C_{-1} = C_0 \supset C_1 \supset C_2 \supset 0 = \dots$$

$$\begin{tikzcd}
\dots = & C_{-2} = & C_{-1} = \arrow[dl] & C_0 \supset \arrow[dl] & C_1 \supset \arrow[dl] & C_2 = \arrow[dl] & 0 = \arrow[dl] & \cdots \\
\dots = & C_{-2} = & C_{-1} = & C_0 \supset & C_1 \supset & C_2 = & 0 = & \cdots
\end{tikzcd}$$

Here

$$B = \dots \oplus 0 \oplus 0 \oplus C_0/C_1 \oplus C_1/C_2 \oplus C_2 \oplus 0 \oplus \dots$$

Taking homology we get sequences

$$\mathrm{H}(C^\bullet) = \mathrm{H}(C_0) \leftarrow \mathrm{H}(C_1) \leftarrow \mathrm{H}(C_2) \leftarrow 0 \leftarrow \dots$$

$$A_1 := \bigoplus_{p\in\mathbb{Z}} \mathrm{H}(C_p)$$

$$\mathrm{H}(C^\bullet) = \mathrm{H}(C_0) \supset i_* \mathrm{H}(C_1) \leftarrow i_* \mathrm{H}(C_2) \leftarrow 0 \leftarrow \dots$$

$$A_2 := \bigoplus_{p\in\mathbb{Z}} i_* \mathrm{H}(C_p)$$

$$\mathrm{H}(C^\bullet) = \mathrm{H}(C_0) \supset i_* \mathrm{H}(C_1) \supset i_* i_* \mathrm{H}(C_2) \leftarrow 0 \leftarrow \dots$$

$$A_3 := \bigoplus_{p\in\mathbb{Z}} i_* i_* \mathrm{H}(C_p).$$

When we reach the stage in wich all maps become inclusions, process is stationary i.e.

$$A_3 = A_4 = \dots$$

$$\begin{tikzcd}
A_3 \arrow[rr, "i"] & & A_3 \arrow[dl, "j"] \\
& B_3 \arrow[ul, "k"] \arrow[r, equal] & \mathrm{H}(A_3)
\end{tikzcd}$$

where i is inclusion, $\operatorname{im} k = \ker i = 0$ so $k = 0$. This means that also

$$B_3 = B_4 = \dots$$

since $d = kj = 0$

4.4.4 Hochschild–Serre spectral sequence

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra of a Lie algebra $\mathfrak{g}$.

$$C^\bullet(\mathfrak{g}; M) = \mathrm{Hom}(\Lambda^\bullet \mathfrak{g}, M), \quad d\colon C^\bullet(\mathfrak{g}; M) \to C^{\bullet+1}(\mathfrak{g}; M)$$

$$\begin{aligned} d\omega(X_0, X_1, \dots, X_r) = &\sum_i (-1)^i X_i \omega(X_0, \dots, \widehat{X_i}, \dots, X_r) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_r). \end{aligned}$$

Define the filtration on the above complex by

$$F^p C^{p+q}(\mathfrak{g}; M) := \{\omega \in C^{p+q} \mid \iota_{X_1} \dots \iota_{X_q} \omega = 0 \ \forall X_1, \dots, X_q \in \mathfrak{h}\}.$$

This means that we can associate with $\omega \in F^p C^{p+q}$ an element

$$\phi(\omega) \in \mathrm{Hom}(\Lambda^q \mathfrak{h}, \mathrm{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), M))$$

given by the formula

$$\phi(\omega)(X_1, \dots, X_q)(\underbrace{\widehat{Y_1}, \dots, \widehat{Y_p}}_{\text{classes}}) = \omega(X_1, \dots, X_q, Y_1, \dots, Y_p).$$

Then

$$\ker \phi = F^{p+1} C^{p+q},$$

Hence there is a spectral sequence with

$$\begin{aligned} E_0^{p,q} &\cong C^q(\mathfrak{h}; \mathrm{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), M)), \ d_0 = d, \\ E_1^{p,q} &\cong \mathrm{H}^q(\mathfrak{h}; \mathrm{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), M)), \\ E_2^{p,0} &\cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{h}; M), \\ E_\infty^* &\implies \mathrm{H}^*(\mathfrak{g}; M) \end{aligned}$$

Now we are ready to prove that the inclusion

$$i\colon \widetilde{W_n} \hookrightarrow C^\bullet(\mathfrak{A}_n)$$

induces an isomorphism

$$\mathrm{H}^*(\widetilde{W_n}, d) \cong \mathrm{H}^*_{GF}(\mathfrak{A}_n)$$

that is theorem (4.8).

Proof. Both $\widetilde{W_n}$ and $C^\bullet(\mathfrak{A}_n)$ are filtered differential graded algebras, and their associated spectral sequences converge to $\mathrm{H}^*(\widetilde{W_n})$ and respectively to $\mathrm{H}^*_{GF}(\mathfrak{A}_n)$. On the other hand i induces isomorphism on the level of E_1.

First $\widetilde{W_n}$ is graded by

$$\widetilde{W_n}^p = \bigoplus_{r+2s=p} \Lambda^r \langle \theta^i_j \rangle \otimes S^s_n[R^i_j]$$

and then

$$F^p \widetilde{W_n}^{p+q} := \{\omega \in \widetilde{W_n}^{p+q} \mid \iota_{X_0} \dots \iota_{X_q} \omega = 0 \ \forall X_0, \dots, X_q \in \mathfrak{A}_n^{(0)}\}$$

Fact 4.9.

$$E_0^{p,q} \cong \begin{cases} 0, & p \text{ odd or } p > 2n, \\ C^q(\mathfrak{A}_n^{(0)}; S_n^{\frac{p}{2}}[R_j^i]), & p \text{ even and } p \leqslant 2n. \end{cases}$$

$$E_1^{p,q} \cong \begin{cases} 0, & p \text{ odd or } p > 2n, \\ \mathrm{H}_{GF}^q(\mathfrak{A}_n^{(0)}; S_n^{\frac{p}{2}}[R_j^i]), & p \text{ even and } p \leqslant 2n. \end{cases}$$

The filtration on $C^\bullet(\mathfrak{A}_n) = \bigoplus_p C^p(\mathfrak{A}_n)$ is the Hochschild–Serre filtration relative to $\mathfrak{A}_n^{(0)}$.

$$F^p C^{p+q}(\mathfrak{A}_n) = \begin{cases} C^{p+q}(\mathfrak{A}_n), & p \leqslant 0 \\ \{\omega \in C^{p+q}(\mathfrak{A}_n) \mid \iota_{X_0} \dots \iota_{X_q} \omega = 0 \ \forall X_0, \dots, X_q \in \mathfrak{A}_n^{(0)}\}, & p > 0, q \geqslant 0. \end{cases}$$

Fact 4.10.

$$E_1^{p,q} \cong \mathrm{H}_{GF}^q(\mathfrak{A}_n^{(0)}; F^p C^p(\mathfrak{A}_n)).$$

It is a filtration, so

$$[\mathfrak{A}_n^{(0)}, \mathfrak{A}_n^{(p)}] \subset \mathfrak{A}_n^{(p)}$$

and we have an action of $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{A}_n^{(0)}$ on $\mathfrak{A}_n^{(p)}$ for each p. Since $\mathfrak{A}_n^{(0)}$ acts semisimply on the coefficients one gets further

$$E_1^{p,q} \cong \mathrm{H}_{GF}^q\left(\mathfrak{A}_n^{(0)}, \left(\Lambda^p(\mathfrak{A}_n^{(0)})\right)^*\right) \cong \mathrm{H}_{GF}^q(\mathfrak{A}_n^{(0)}; B^p),$$

where

$$B^p := \{\omega \in C^p(\mathfrak{A}_n) \mid \iota_X \omega = 0 = \mathcal{L}_X \omega \ \forall X \in \mathfrak{A}_n^{(0)}\}$$

are the *basic* elements with respect to $\mathfrak{A}_n^{(0)}$. Note that if $Y = Y_s^r = X^r \frac{\partial}{\partial x^s}$

$$\iota_Y R_j^i = -\iota_Y(\theta_{jk}^i \wedge \theta^k) = 0,$$

whence the map

$$E_1^{p,q}(\widetilde{W_n}) \to E_1^{p,q}(C^\bullet(\mathfrak{A}_n)).$$

Lemma 4.11. *The inclusion $i\colon \widetilde{W_n} \hookrightarrow C^\bullet(\mathfrak{A}_n)$ induces an isomorphism between the $\mathfrak{A}_n^{(0)}$-basic elements of $\widetilde{W_n}$ and $C^\bullet(\mathfrak{A}_n)$.*

Proof. Elementary invariance theory to eliminate the form θ_α^i with $|\alpha| > 2$. □

□

Again let

$$W_n = \Lambda\langle u_1, \dots, u_n\rangle \otimes S_n[c_1, \dots, c_n]$$

$$\deg(u_i) = 2i - 1,\ \deg(c_i) = 2i,\ du_i = c_i,\ dc_i = 0.$$

$$\widetilde{W_n} = \Lambda\langle\theta_j^i\rangle \otimes S_n[R_j^i]$$

Proposition 4.12. *The map*

$$c_i \mapsto c_i(R),\ R = (R_j^i)$$

has an extension to a map of complexes $W_n \to \widetilde{W_n}$. Any such extension induces isomorphism in cohomology

$$\mathrm{H}^*(W_n) \xrightarrow{\cong} \mathrm{H}^*(\widetilde{W_n}).$$

For example if $n = 1$ we have

$$c_1 \mapsto c_1(R) = R_1^1,$$
$$u_1 \mapsto \theta_1^1.$$

Proof.

$$E_1^{0,2q-1}(\widetilde{W_n}) = \mathrm{H}^{2q-1}(\mathfrak{gl}_n(\mathbb{R});\mathbb{R}) \ni u_j,$$

where u_j is a generator for $j = 1, \ldots, n$. Now each u_j has a representative $[w_j]$ such that

$$w_j \in F^0\widetilde{W_n}^{2q-1},\ dw_j = c_j \in F^{2q}\widetilde{W_n}^{2q}$$

thus giving a basic element of $\widetilde{W_n}$ in

$$E_1^{2q,0} \cong S^q(R_j^i)_{inv}.$$

The basic elements of $\widehat{W_n}$ form an algebra isomorphic to $\mathbb{R}[c_1, \ldots, c_n]$.

The extension is given by

$$u_j \mapsto w_j,$$
$$c_j \mapsto d\omega_j.$$

Filtering W_n by the ideals F^pW_n generated by polynomials of degree at least p in the c_i's one obtains a morphism of complexes compatible with filtrations, which induces isomorphism on the level of E_1. □

In the relative case $\mathfrak{o}_n \subset \mathfrak{gl}_n(\mathbb{R}) = \mathfrak{A}_n^{(0)}$ gives actions of $\mathfrak{o}_n$ on $\widetilde{W_n}$ and $C^\bullet(\mathfrak{A}_n)$. Passing to the subalgebras of $\mathfrak{o}_n$-basic elements, then restricting the filtrations one obtains isomorphisms

$$\mathrm{H}^*(WO_n) \cong \mathrm{H}^*(\widetilde{W_n}, \mathfrak{o}_n) \cong \mathrm{H}^*_{GF}(\mathfrak{A}_n, \mathfrak{o}_n),$$

where

$$WO_n = \Lambda\langle u_1, u_3, \ldots u_k\rangle \otimes S_n[c_1, \ldots, c_n],$$
$$du_{2j-1} = c_{2j},\ dc_j = 0.$$

Corollary 4.13. *Any class in* $\mathrm{H}^*(\mathfrak{A}_n)$ *(respectively* $\mathrm{H}^*(\mathfrak{A}_n, \mathfrak{o}_n)$*) has a representative which depends only on the second jet.*

Chapter 5

Characteristic maps and Gelfand-Fuks cohomology

5.1 Jet groups

Definition 5.1. *Let $x \in \mathbb{R}^n$ and let $f\colon U \to \mathbb{R}^n$ be a C^∞-function. Then $j^k_x(f)$ is an equivalence class with respect to*

$$f \sim_k g \quad \text{if and only if} \quad \frac{\partial^{|\alpha|} f}{\partial x^\alpha}\Big|_x = \frac{\partial^{|\alpha|} g}{\partial x^\alpha}\Big|_x,\ \forall |\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant k.$$

Then

$$G_k(n) := \{ j^k_0(f) \mid f \text{ local diffeomorphism of } \mathbb{R}^n,\ f(0) = 0 \}$$

is a Lie group under composition

$$j^k_0(f) \circ j^k_0(g) := j^k_0(f \circ g).$$

Identifying with polynomial representatives

$$j^k_0(f) \cong \{ \sum_{1 \leqslant |\alpha| \leqslant k} a^j_\alpha x^\alpha \in \mathcal{P}^k_0[x_1, \ldots, x_n] \mid 1 \leqslant j \leqslant n \}$$

Then $j^k_0(f) \in G_k(n)$ means $a^j_\alpha \in \mathrm{GL}_n(\mathbb{R})$.

One has a sequence of projections

$$G_\infty(n) := \ldots \to G_{k+1}(n) \to G_k(n) \to \ldots \to G_1(n).$$

If $h = f \circ g$

$$h^i(x^1, \ldots, x^n) = f^i(g^1(x^1, \ldots, x^n), \ldots, g^n(x^1, \ldots, x^n))$$

$$c^i_k := \frac{\partial h^i}{\partial x^k}\Big|_0 = \sum_l \frac{\partial f^i}{\partial x^l}\Big|_0 \frac{\partial g^l}{\partial x^k}\Big|_0 = \sum_l a^i_l b^l_k.$$

$$c^i_{jk} := \frac{\partial^2 h^i}{\partial x^j \partial x^k}\Big|_0 = \sum_{l,s} \frac{\partial^2 f^i}{\partial x^s \partial x^l}\Big|_0 \frac{\partial g^s}{\partial x^j}\Big|_0 \frac{\partial g^l}{\partial x^k}\Big|_0 + \sum_l \frac{\partial f^i}{\partial x^l}\Big|_0 \frac{\partial^2 g^l}{\partial x^j \partial x^k}\Big|_0$$

so
$$c^i_{jk} = \sum_{l,s} a^i_{sl} b^s_j b^l_k + \sum_l a^i_l b^l_{jk}$$
etc. In particular $\ker(G_2(n) \to G_1(n))$ has multipllication
$$c^i_{jk} = a^i_{jk} + b^i_{jk}.$$
In general
$$N_k(n) := \ker(G_k(n) \to G_1(n))$$
is a vector space equipped with a polynomial multiplication which implies that $N_k(n)$ is a nilpotent Lie subgroup, and
$$G_k(n) = G_1(n) \ltimes N_k(n).$$
$$\mathfrak{g}_k(n) := \mathrm{Lie}(G_k(n)) \cong \{j^k_0 X \mid X = \sum_i \frac{\partial}{\partial x^i},\ X(0) = 0\}$$
with the bracket
$$[j^k_0(X), j^k_0(Y)] = -j^k_0([X,Y]).$$

5.2 Jet bundles

Definition 5.2. *Let M^n be a C^∞-manifold. The* jet bundle *on M*
$$J^k(M) := \{j^k_0(f) \mid f \colon U \subset \mathbb{R}^n \to M \text{ local diffeomorphism at } 0 \in U\}.$$

It has a tautological C^∞-structure modelled on
$$J^k(\mathbb{R}^n) = \mathcal{P}_k(n) \cong \text{ polynomial jets}$$
Again one has a sequence of natural projections
$$J^\infty(M) := \ldots \to J^{k+1}(M) \to J^k(M) \to \ldots \to J^1(M) \to M,$$
which are principal bundles with structure groups
$$G_\infty(n) := \ldots \to G_{k+1}(n) \to G_k(n) \to \ldots \to G_1(n).$$

$J^1(M) = F(M) \to M$ is a frame bundle with the structure group $\mathrm{GL}_n(\mathbb{R}) = G_1(n)$.

There is a natural (commuting with Diff_M) map
$$\mathfrak{A}_n \xrightarrow{\cong} T_{j^\infty_0(\phi)} J^\infty(M)$$
For
$$X \in \mathfrak{A}_n,\ X = \sum_i f^i \frac{\partial}{\partial x^i}$$
and a 1-parameter family ψ_t of local diffeomorphism of $\mathbb{R}^n$ such that
$$\psi_t(0) = 0,\ \ \psi_0 = \mathrm{Id},\ \ X = j^\infty_0 \left(\frac{d\psi_t}{dt}\Big|_{t=0} \right),$$

we have a curve in a manifold of jets $j_0^\infty(\psi_t)$. For a local diffeomorphism $\phi\colon \mathbb{R}^q \to M^n$ we have a curve passing through ϕ

$$j_0^\infty\left(\frac{d}{dt}(\phi\circ\psi_t)\big|_{t=0}\right)$$

and

$$X = \frac{d}{dt}j_0^\infty(\psi_t)\big|_{t=0} = j_0^\infty\left(\frac{d\psi_t}{dt}\big|_{t=0}\right).$$

Let $u = j_0^\infty(\phi) \in J^\infty(M)$, and define

$$\widetilde{X_u} := j_0^\infty\left(\frac{d}{dt}\phi\circ\psi_t\big|_{t=0}\right) = \frac{d}{dt}(\phi\circ\psi_t)\big|_{t=0} \in T_u J^\infty(M),\quad \phi\circ\psi_t\big|_{t=0} = \phi.$$

The map

$$\mathfrak{A}_n \to T_u J^\infty(M),\quad X \mapsto \widetilde{X_u}$$

is natural i.e. it commutes with the action of the diffeomorphisms

$$\begin{tikzcd} & T_{j_0^\infty(\rho\circ\phi)}J^\infty(M) \\ \mathfrak{A}_n \arrow[ur, "\cong"] \arrow[r, "\cong"] & T_{j_0^\infty(\phi)}J^\infty(M) \arrow[u, "\rho^*"'] \end{tikzcd}$$

Proposition 5.3. *We have a natural isomorphism of differential graded algebras*

$$(C^\bullet(\mathfrak{A}_n), d) \xrightarrow{\cong} (\Omega^\bullet(J^\infty(M))^{\mathrm{Diff}_M}, -d).$$

Proof. We take for $u = j_0^\infty(\phi)$

$$\widetilde{\omega_u}(\widetilde{X_u}^1, \ldots, \widetilde{X_u}^p) := \omega(X_1, \ldots, X^p).$$

$$[\widetilde{X}, \widetilde{Y}] := -\widetilde{[X,Y]}.$$

In particular if we set for a basis $\{\theta^i_\alpha\}$ of $\mathfrak{A}_n^*$

$$\widetilde{\theta}^i_\alpha(\widetilde{X_u}) = \frac{\partial^{|\alpha|} f^i}{\partial x^\alpha}\big|_{x=0} = (-1)^{|\alpha|}\theta^i_\alpha(X)$$

then they satisfy the same differential equations as θ^i_α. □

Example 5.4. In local coordinates $(v_1, \ldots, v_n)$ around $u = j_0^\infty(\phi)$

$$\left\{ v_i\big|_u, v^i_j := \frac{\partial(v^i\circ\phi)}{\partial x^j}\big|_u, v^i_{jk} := \frac{\partial^2(v^i\circ\phi)}{\partial x^j\partial x^k}\big|_u, \ldots, v^i_\alpha = \frac{\partial^{|\alpha|}(v^i\circ\phi)}{\partial x^\alpha}\big|_u \right\}$$

one has

$$dv^i_\alpha = \sum_{\beta+\gamma=\alpha} v^i_{\beta[k]}\widetilde{\theta}^k_\gamma,\quad \beta[k] := (\beta_1, \ldots, \beta_k + 1, \ldots, \beta_n).$$

5.3 Characteristic map for foliation

Let $(M, \mathcal{F})$ be a manifold with foliation, which we can describe by a 1-cycle with values in Γ_q given by the following data

1. an open cover $M = \bigcup_\alpha U_\alpha$,
2. $\forall \alpha$ there is a submersion $f_\alpha \colon U_\alpha \to V_\alpha \in \mathbb{R}^q$,
3. $\forall x \in U_\alpha \cap U_\beta$ there is a local diffeomorphism $g_{\alpha\beta} \colon V_\alpha \to V_\beta$ (neighbourhoods of $f_\alpha(x)$ and $f_\alpha(x)$ rspectively) such that $f_\beta = g_{\beta\alpha} \circ f_\alpha$ near x.

Then

$$f_\alpha^*(J^\infty(V_\alpha)) \to U_\alpha, \text{ and } f_\beta^*(J^\infty(V_\beta)) \to U_\beta$$

can be identified over $U_\alpha \cap U_\beta$ via $j_0^\infty(g_{\beta\alpha})$, giving the principal $G^k(q)$-bundles over M:

$$J^\infty(\mathcal{F}) := \ldots \to J^{k+1}(\mathcal{F}) \to J^k(\mathcal{F}) \to \ldots \to J^2(\mathcal{F}) \to J^1(\mathcal{F}) \to M.$$

This are jet bundles of "transverse local diffeomorphisms". In particular $J^1(\mathcal{F})$ is a principal $\mathrm{GL}_q(\mathbb{R})$-bundle associated to the transverse bundle $Q(\mathcal{F}) = TM/\mathcal{F}$ - bundle of transverse frames.

The forms θ^i_κ on $J^\infty(V_\alpha)$ are invariant under Diff hence they also define forms on $J^\infty(\mathcal{F})$. They are the "canonical forms" on $J^\infty(\mathcal{F})$.

The characteristic homomorphisms

$$\chi_{GF} \colon C^\bullet(\mathfrak{A}_q) \to \Omega^\bullet(J^\infty(\mathcal{F}))$$

is defined by sending ω to the lift to M of the Diff-invariant forms $\widetilde{\omega}_\alpha$ on V_α. It is a homomorphism of DGA's inducing

$$\chi^*_{GF} \colon \mathrm{H}^*_{GF}(\mathfrak{A}_q) \to \mathrm{H}^*(J^\infty(\mathcal{F})) \cong \mathrm{H}^*(J^1(\mathcal{F})).$$

Remark 5.5 (Bott vanishing theorem revisited). Any E-flat (Bott) connection (def. (2.7)) $\nabla^\flat$ on Q is given by a $\mathfrak{gl}_n(\mathbb{R})$-valued form on $J^1(\mathcal{F})$ which is of the form $\omega^i_j = s^*(\widetilde{\theta}^i_j)$ for some $\mathrm{GL}_n(\mathbb{R})$ -equivariant section $s \colon J^1(\mathcal{F}) \to J^2(\mathcal{F})$. Then its curvature form

$$\Omega^i_j = s^*(R^i_j) \implies \Omega^i_j \wedge \omega^j = s^*(R^i_j \wedge \theta^j) = 0$$

hence

$$\Omega^{i_1}_{j_1} \wedge \ldots \wedge \Omega^{i_p}_{j_p} = 0, \ \forall p > q.$$

Assume the normal bundle $Q = Q(\mathcal{F})$ is trivializable and choose a global section $s \colon M \to \mathcal{F}$. Then the diagram

$$\begin{tikzcd}
\mathrm{H}^*_{GF}(\mathfrak{A}_q) \arrow[r, "s^*\circ\chi^*_{GF}"] & \mathrm{H}^*(M) \arrow[r, "\mathrm{pr}^*", "\cong"'] & \mathrm{H}^*(J^1(\mathcal{F})) \\
& \mathrm{H}^*(W_q) \arrow[u, "\cong"'] \arrow[ur, "\mu^*_E"'] &
\end{tikzcd}$$

is commutative.

Passing to the relative subcomplex one gets

$$\chi_{GF}^{rel}\colon C^\bullet(\mathfrak{A}_n, O(n)) \to \Omega^\bullet(J^\infty/O(n))$$

which induces

$$\chi_{GF}^{rel}\colon \mathrm{H}^*(\mathfrak{A}_n, O(n)) \to \mathrm{H}^*(J^1(\mathcal{F})/O(n)) \xrightarrow{\cong} \mathrm{H}^*(M).$$

The isomorphism

$$\sigma^*\colon \mathrm{H}^*(J^1(\mathcal{F})/O(n)) \to \mathrm{H}^*(M)$$

is implemented by a metric on Q (i.e. a section $\sigma\colon M \to J^1(\mathcal{F})/O(n)$). Then the diagram

$$\begin{tikzcd}
\mathrm{H}^*(\mathfrak{A}_n, O(n)) \arrow[rr, "\chi_{GF}^{rel*}"] & & \mathrm{H}^*(M) \\
& \mathrm{H}^*(WO_n) \arrow[ul] \arrow[ur, "\chi_E^*"'] &
\end{tikzcd}$$

is again commutative.

Chapter 6

Index theory and noncommutative geometry

6.1 Classical index theorems

Let (M, g) be a Riemannian manifold, g-metric. Index theorems describe properties of geometric elliptic operators in terms of topological characteristic classes. For a selfadjoint elliptic operator $D = D^*$

$$\mathrm{Index}(D) := \dim\ker D - \dim\operatorname{coker} D \in \mathbb{Z}$$

We will give a few examples of index theorems.

Take the de Rham complex $\Omega^\bullet(M)$ with

$$d\colon \Omega^i(M) \to \Omega^{i+1}(M)$$

and its adjoint

$$d^*\colon \Omega^i(M) \to \Omega^{i-1}(M).$$

One has even/odd grading on forms ($\gamma = (-1)^{\deg}$), and the operator

$$d + d^*\colon \Omega^{ev} \to \Omega^{odd}$$

is selfadjoint elliptic operator. Furthermore

$$\mathrm{Index}(d + d^*)^{ev} = \dim\ker(d + d^*)^{ev} - \dim\operatorname{coker}(d + d^*)^{ev}$$

and

$$\ker(d + d^*) = \mathrm{H}^*_{dR}(M; \mathbb{R}),$$

$$\ker(d + d^*)^{ev} = \mathrm{H}^{ev}_{dR}(M; \mathbb{R}), \quad \operatorname{coker}(d + d^*)^{odd} = \mathrm{H}^{odd}_{dR}(M; \mathbb{R}).$$

This means

$$\mathrm{Index}(d + d^*) = \dim\mathrm{H}^{ev}(M; \mathbb{R}) - \dim\mathrm{H}^{odd}(M; \mathbb{R}) = \chi(M)$$

- the Euler characteristic of a manifold M.

Theorem 6.1 (Gauss-Bonnet)**.**

$$\chi(M) = \operatorname{Index}(d+d^*)^{ev} = \int_M \operatorname{Pf}(R),$$

where $\operatorname{Pf}(M)$ *is a Pffafian i.e. the square root of the determinant, and* R *- a curvature.*

This theorem gives topological constraints on Gaussian curvature, for if $n = 2$ one has $\operatorname{Pf}(R) = K$. The right hand side depends on the metric, while on the left we have topological invariant.

In the example above lets take different grading. Assume that $\dim M = 4n$. Take a Hodge star operator

$$*\colon \Omega^k(M) \to \Omega^{4n-k}.$$

One has $*^2 = (-1)^{k(4n-k)}$ so it gives rise to another grading γ on $\Omega^\bullet(M)$. It splits the complex into $\Omega^-(M)$ and $\Omega^+(M)$ (negative and positive eigenspaces). Furthermore

$$\operatorname{Index}(d+d^*)^+ = \dim \mathrm{H}^{2n}(M)^+ - \dim \mathrm{H}^{2n}(M) = \sigma(M)$$

- the signature of M i.e. a signature of bilinear form

$$\mathrm{H}^{2n}(M) \times \mathrm{H}^{2n}(M) \to \mathbb{R}, \ \ (\alpha,\beta) \mapsto \int_M \alpha \wedge \beta.$$

On the other side

Theorem 6.2 (Hirzebruch signature thm.)**.**

$$\sigma(M) = \operatorname{Index}(d+d^*) = \int_M L(R), \ \ L(R) := (\det)^{\frac{1}{2}} \left(\frac{\frac{R}{2}}{\tanh \frac{R}{2}} \right)$$

as a formal series. $L(R)$ *is a* L*-genus of a manifold.*

$L(R)$ is a combination of Pontryagin classes which depends on a metric structure of a manifold.

Let E be a holomorphic Hermitian bundle on a manifold M. One has an operator $\bar\partial_E \oplus \bar\partial_E^*$ on $\Omega^{0,\bullet} \otimes \Gamma(E)$. Its index

$$\operatorname{Index}(\bar\partial_E \oplus \bar\partial_E^*) = \chi(E)$$

- the Euler characteristic of a bundle E. On the other hand

Theorem 6.3 (Riemann-Roch-Hirzebruch)**.**

$$\chi(E) = \operatorname{Index}(\bar\partial_E \oplus \bar\partial_E^*) = \int_M \operatorname{td}(M)\operatorname{ch}(E),$$

where the Todd class of M *and Chern character of* E *are given by*

$$\operatorname{td}(M) = \det \frac{R^{hol}}{e^{R^{hol}} - 1}, \ \ \operatorname{ch}(E) = \operatorname{Tr}(e^{F_E}).$$

The most general example one has for Dirac operator $\not{D}$. One has a grading $\not{D}^+$, $\not{D}^-$ from Spin-bundle.

$$\text{Index}\,\not{D} = \dim\ker\not{D} - \dim\operatorname{coker}\not{D} = S(M)$$

- the spinor number of a manifold M. On the other side

Theorem 6.4 (Atiyah-Singer)**.**

$$S(M) = \text{Index}\,\not{D} = \int_M \widehat{A}(R), \quad \widehat{A}(R) := \det{}^{\frac{1}{2}}\left(\frac{\frac{R}{2}}{\sinh\frac{R}{2}}\right)$$

$\widehat{A}(R)$ is another combination of Pontryagin classes. Together with Lichnerowicz theorem it gives constraints on scalar curvature.

We can summarize above theorems in the table

Elliptic operator and grading	Analitic index	Index formula
$d + d*$ on $\Omega^*(M)$ $\gamma = (-1)^{\deg}$	Euler characteristic $\chi(M)$	Chern-Gauss-Bonnet $\int_M \mathrm{Pf}(R)$ $\mathrm{Pf}(R) = \det^{\frac{1}{2}}(R)$
Corollaries	Topological constraints on Gaussian curvature	
$d + d*$ on $\Omega^*(M)$ $\gamma = *^{?\,\deg}$	Signature number $\sigma(M)$	Hirzerbruch's theorem $\int_M \mathrm{L}(R)$ $\mathrm{L}(R) = \det^{\frac{1}{2}} \frac{R/2}{\tanh(R/2)}$
Corollaries	Homotopy invariance of L-genus	
M spin-manifold, $\not{D}$ Dirac operator on $S(M)$ γ from spin-bundle	Spinor number $S(M)$	Atiyah-Singer theorem $\int_M \widehat{\mathrm{A}}(R)$ $\widehat{\mathrm{A}}(R) = \det^{\frac{1}{2}} \frac{R/2}{\sinh(R/2)}$
Corollaries	Topological constraints on scalar curvature (with Lichnerowicz theorem)	
M Kähler manifold E holomorphic bundle $\bar{\partial}_E \oplus \bar{\partial}_E^*$ on $\Omega^{0,*}(M) \otimes \Gamma(E)$	Euler characteristic $\chi(E)$	Riemann-Roch-Hirzerbruch theorem $\int_M \mathrm{td}(M)\,\mathrm{ch}(E)$ $\mathrm{td}(M) = \det \frac{R^{hol}}{e^{R^{hol}}-1}$ $\mathrm{ch}(E) = \mathrm{Tr}(e^{F_E})$
Corollaries	Dimension of space of holomorphic sections (with Kodaira vanishing theorems)	

6.2 General formulation and proto-index formula

Let A be a C*-algebra and $\mathfrak{A}$ its dense subalgebra such that if $a \in \mathfrak{A}$ has an inverse $a^{-1} \in A$, then $a^{-1} \in \mathfrak{A}$

Example 6.5. M - closed manifold, $A = C(M)$, $\mathfrak{A} = C^\infty(M)$. Then

$$\mathrm{K}^*(M) = \mathrm{K}_*(C(M)) = \mathrm{K}_*(C^\infty(M)),$$

(via Serre-Swan theorem) where the right hand side has algebraic definition (purely for $* = \mathit{even}$ and almost for $* = \mathit{odd}$).

In general

$$\mathrm{K}_0(\mathfrak{A}) := \mathrm{Idemp}(M_\infty(\mathfrak{A}))/\sim\, \cong \pi_1(\mathrm{GL}_\infty(\mathfrak{A})),$$

where $\sim$ is some equivalence relation,

$$\mathrm{K}_1(\mathfrak{A}) := \mathrm{GL}_\infty(\mathfrak{A})/\,\mathrm{GL}_\infty(\mathfrak{A})^0 \cong \pi_0(\mathrm{GL}_\infty(\mathfrak{A})),$$

where $\mathrm{GL}_\infty(\mathfrak{A})^0$ is a group of connected components. For the definition of $\mathrm{K}_1(\mathfrak{A})$ we need a topology on $\mathfrak{A}$. We can replace $\mathrm{GL}_\infty(\mathfrak{A})$ by $U_\infty(\mathfrak{A})$ (unitary matrices). From Bott periodicity $\mathrm{K}_2(\mathfrak{A}) = \mathrm{K}_0(A)$ and so on.

What is the dual (homology) theory ? K-homology.

Assume $A \subset B(\mathcal{H})$ (bounded operators on Hilbert space $\mathcal{H}$). Let $F = F^* \in A$, Fredholm operator, such that

$$[F, A] \subset \mathcal{K}(\mathcal{H}), \text{ (compact operators)},$$

and moreover

$$[F, \mathfrak{A}] \subset \mathcal{L}^p(\mathcal{H}), \text{ (Schatten class)}$$

for some $p \geqslant 1$. The triple $(\mathfrak{A}, \mathcal{H}, F)$ is a *p-summable Fredholm module.* Together with grading γ such that

$$\gamma^2 = \mathrm{Id},\ \gamma = \gamma^*,\ \gamma a = a\gamma \ \forall\, a \in \mathfrak{A},$$

$$\gamma F + F\gamma = 0,$$

the quadruple $(\mathfrak{A}, \mathcal{H}, \gamma, F)$ is a K-cycle. The Hilbert space $\mathcal{H}$ decomposes into positive and negative eigenspaces of γ

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

and there is a decomposition of F

$$F = \begin{pmatrix} 0 & F^+ \\ F^- & 0 \end{pmatrix}.$$

Lemma 6.6. *Let F be bounded selfadjoint involution on $\mathcal{H}$ (i.e. $F^2 = \mathrm{Id}$). Then*

1. *If $e^2 = e \in \mathfrak{A}$ then*
$$F_e := eFe$$
is a Fredholm operator.
2. *If $g \in \mathrm{GL}_1(\mathfrak{A})$ and $P = \frac{1+F}{2}$ then*
$$F_g := PgP$$
is a Fredholm operator.

Proof.

Ad. 1
$$F_e^2 = eFeFe = e([F,e] + eF)Fe$$
which is a sum of e and compact operator on $e\mathcal{H}e$.

Ad. 2
$$F_g F_{g^{-1}} = PgPg^{-1}P = Pg([P,g^{-1}] + g^{-1}P)P$$
which is a sum of P and compact operator on $P\mathcal{H}P$.

□

If $e^2 = e \in M_N(\mathfrak{A}) = \mathfrak{A} \otimes M_N(\mathbb{C})$ then we can form
$$\mathcal{H}_N := \mathcal{H} \otimes \mathbb{C}^N, \quad F_N := F \otimes \mathrm{Id}.$$
For an idempotent e, assignment
$$(F,e) \mapsto \mathrm{Index}(F_e^+) \in \mathbb{Z}$$
extends to a pairing
$$\mathrm{K}^0(\mathfrak{A}) \times \mathrm{K}_0(\mathfrak{A}) \to \mathbb{Z}.$$
Similarly for $g \in \mathrm{GL}_1(\mathfrak{A})$, assignment
$$(P,g) = \left(\frac{1+F}{2}, g\right) \mapsto \mathrm{Index}(F_g) \in \mathbb{Z}$$
extends to a pairing
$$\mathrm{K}^1(\mathfrak{A}) \times \mathrm{K}_1(\mathfrak{A}) \to \mathbb{Z}.$$

Lemma 6.7 (Well known)**.** *Let P, Q be bounded operators on a Hilbert space $\mathcal{H}$, such that*
$$\mathrm{Id} - QP, \mathrm{Id} - PQ \in \mathcal{L}^p.$$
Then P, Q are Fredholm operatos and
$$\mathrm{Index}(P) = \mathrm{Tr}((\mathrm{Id} - QP)^n) - \mathrm{Tr}((\mathrm{Id} - PQ)^n), \ \forall\, n \geqslant p.$$

Proposition 6.8. *Assume $[F,\mathfrak{A}] \in \mathcal{L}^p$ (that is $(\mathfrak{A},\mathcal{H},F)$ is p-summable Fredholm module). Then*

1. *In the graded case, that is given $\gamma\colon \mathcal{H} \to \mathcal{H}$, one has for all projections e*
$$\mathrm{Index}(F_e^+) = (-1)^m \,\mathrm{Tr}(\gamma e [F,e]^{2m}), \ \forall\, 2m \geqslant p.$$

2. *In the ungraded case one has for all $g \in \mathrm{GL}_1(\mathfrak{A})$*
$$\mathrm{Index}(F_g) = \frac{1}{2^{2m+1}} \,\mathrm{Tr}(g[F,g^{-1}])^{2m+1}, \ \forall\, 2m \geqslant p.$$

Proof. In the graded case

$$\operatorname{Index}(F_e^+) = \operatorname{Tr}(\gamma P_{\ker F_e}) = \operatorname{Tr}(\gamma(e - F_e^2)^m) = \operatorname{Tr}(\gamma(e - eFeFe)^m)$$

for $2m = n \geqslant p$. Now as above

$$\begin{aligned} e - eFeFe &= -e[F,e]Fe \\ &= -e[F,e]([F,e] + eF) \\ &= -e[F,e][F,e] - \underbrace{e[F,e]e}_{=0} F \\ &= -e[F,e]^2 = [F,e]^2 e \end{aligned}$$

since

$$[F,e] = [F,e^2] = [F,e]e + e[F,e].$$

Thus

$$\operatorname{Tr}(\gamma(e - eFeFe)^m) = (-1)^m \operatorname{Tr}(\gamma(e[F,e]^2)^m) = (-1)^m \operatorname{Tr}(\gamma e([F,e])^{2m}).$$

In the ungraded case one has

$$\operatorname{Index}(F_g) = \operatorname{Tr}((P - Pg^{-1}PgP)^m) - \operatorname{Tr}((P - PgPg^{-1}P)^m)$$

for m sufficiently large. Furthermore

$$\begin{aligned} P - Pg^{-1}PgP &= P + P([P,g^{-1}] - Pg^{-1})gP \\ &= P[P,g^{-1}]gP \\ &= -P[P,g^{-1}]([P,g] - Pg) \\ &= -P[P,g^{-1}][P,g] + \underbrace{P[P,g^{-1}]P}_{=0} g \end{aligned}$$

because

$$P^2 = P \implies [g^{-1},P]P + P[g^{-1},P] = [g^{-1},P] \implies P[P,g^{-1}]P = 0.$$

Hence

$$\operatorname{Tr}((P - Pg^{-1}PgP)^m) = (-1)^m \operatorname{Tr}(P([P,g^{-1}][P,g])^m).$$

Writing again

$$\begin{aligned} [P,g^{-1}] &= P[P,g^{-1}] + [P,g^{-1}]P, \\ [P,g] &= P[P,g] + [P,g]P \end{aligned}$$

one has

$$P[P,g^{-1}][P,g] = P[P,g^{-1}][P,g]P = [P,g^{-1}][P,g]P.$$

Therefore

$$\begin{aligned} \operatorname{Tr}((P - Pg^{-1}PgP)^m) &= (-1)^m \operatorname{Tr}(P([P,g^{-1}][P,g])^m) \\ &= (-1)^m \operatorname{Tr}\left(\frac{1+F}{2}\left(\frac{1}{2}[F,g^{-1}]\frac{1}{2}[F,g]\right)^m\right) \\ &= \frac{(-1)^m}{2^{2m+1}}\left(\operatorname{Tr}(([F,g^{-1}][F,g])^m) + \operatorname{Tr}(F([F,g^{-1}][F,g])^m)\right). \end{aligned}$$

Changing g to g^{-1} one gets

$$\mathrm{Tr}((P-PgPg^{-1}P)^m) = \frac{(-1)^m}{2^{2m+1}}\left(\mathrm{Tr}(([F,g][F,g^{-1}])^m) + \mathrm{Tr}(F([F,g][F,g^{-1}])^m)\right).$$

Noting that

$$[F,g^{-1}][F,g] = (-g^{-1}[F,g]g^{-1})(-g[F,g^{-1}]g) = g[F,g][F,g^{-1}]g$$

one has

$$\mathrm{Tr}(([F,g^{-1}][F,g])^m) = \mathrm{Tr}(([F,g][F,g^{-1}])^m).$$

Now

$$([F,g^{-1}][F,g])^m = (-g^{-1}[F,g^{-1}]g^{-1}[F,g])^m = (-1)^m(g^{-1}[F,g])^{2m},$$

hence

$$\mathrm{Index}(F_g) = \frac{1}{2^{2m+1}}(\mathrm{Tr}(F(g^{-1}[F,g])^{2m}) - \mathrm{Tr}(F(g[F,g^{-1}])^{2m})).$$

The second term can be written as

$$\begin{aligned}\mathrm{Tr}(F(g[F,g^{-1}])^{2m}) &= \mathrm{Tr}(F([F,g]g^{-1})^{2m})\\ &= \mathrm{Tr}(Fg(g^{-1}[F,g]g^{-1}g)^{2m}g^{-1})\\ &= \mathrm{Tr}(g^{-1}Fg(g^{-1}[F,g])^{2m}).\end{aligned}$$

So the difference gives

$$\begin{aligned}\mathrm{Index}(F_g) &= \frac{1}{2^{2m+1}}\,\mathrm{Tr}((F-g^{-1}Fg)(g^{-1}[F,g])^{2m})\\ &= \frac{1}{2^{2m+1}}\,\mathrm{Tr}(g^{-1}[g,F](g^{-1}[F,g])^{2m})\\ &= \frac{1}{2^{2m+1}}\,\mathrm{Tr}((g^{-1}[F,g])^{2m+1})\\ &= \frac{1}{2^{2m+1}}\,\mathrm{Tr}((g[F,g^{-1}])^{2m+1}).\end{aligned}$$

□

6.3 Multilinear reformulation: cyclic homology (Connes)

Observe that if $T \in \mathcal{L}^1$ then

$$\mathrm{Tr}(\gamma T) = \frac{1}{2}\,\mathrm{Tr}(\gamma F[F,T]).$$

Indeed

$$\mathrm{Tr}(\gamma F[F,T]) = \mathrm{Tr}(\gamma(T - FTF)) = \mathrm{Tr}(\gamma T) + \mathrm{Tr}(\gamma T)$$

since $F\gamma + \gamma F = 0$.

Both formulas in proposition (6.8) can be obtained from multilinear forms $\tau \in \mathrm{Hom}(\mathfrak{A}^{\otimes n+1}, \mathbb{C})$.

$$\tau_F(a^0, a^1, \dots, a^n) = \begin{cases} \mathrm{Tr}(\gamma F[F, a^0][F, a^1] \dots [F, a^n]) & n \text{ even } > p-1, \\ \mathrm{Tr}(F[F, a^0][F, a^1] \dots [F, a^n]) & n \text{ odd } > p-1. \end{cases}$$

The first comes from (using graded commutators)

$$\begin{aligned}\mathrm{Tr}(\gamma F[F, a^0[F, a^1] \dots [F, a^n]]) &= \mathrm{Tr}(\gamma F[F, a^0][F, a^1] \dots [F, a^n]) \\ &\quad + \sum_{i=1}^{n} \mathrm{Tr}(\gamma F a^0 [F, a^1] \dots [F, [F, a^i]] \dots [F, a^n]),\end{aligned}$$

where the terms in the sum are 0 because

$$[F, [F, a]] = F[F, a] + [F, a]F = a - FaF + FaF - a = 0.$$

For anti-commutation reasons, the first expression vanishes for n odd, while the second expression vanishes for n even.

Element $\phi \in \mathrm{Hom}(\mathfrak{A}^{\otimes n+1}, \mathbb{C})$ ic cyclic if

$$\phi(a^n, a^0, \dots, a^{n-1}) = (-1)^n \phi(a^0, a^1, \dots, a^n)$$

i. e. $\lambda_n \phi = \mathrm{Id}$ for cyclic operator $\lambda_n^{n+1} = \mathrm{Id}$. One has

$$\begin{aligned} b\tau_F(a^0, a^1, \dots, a^{n+1}) &= \sum_{i=0}^{n} \tau_F(a^0, \dots, a^i a^{i+1}, \dots, a^{n+1}) \\ &\quad + (-1)^{n+1} \tau_F(a^{n+1} a^0, a^1, \dots, a^n) \\ &= \sum_{i=1}^{n} (-1)^i \,\mathrm{Tr}(F[F, a^0] \dots [F, a^i a^{i+1}] \dots [F, a^n]) \\ &\quad + (-1)^{n+1}\, \mathrm{Tr}(F[F, a^{n+1} a^0][F, a^1] \dots [F, a^n]). \end{aligned}$$

Now

$$[F, a^i a^{i+1}] = [F, a^i] a^{i+1} + a^i [F, a^{i+1}].$$

Because of the alternating signs, terms cancel pairwise if $n+1$ is even

$$\begin{aligned} &\mathrm{Tr}(F[F, a^0] a^1 [F, a^2] \dots [F, a^{n+1}]) \\ &+ \mathrm{Tr}(F a^0 [F, a^1][F, a^2] \dots [F, a^{n+1}]) \\ &- \mathrm{Tr}(F[F, a^0][F, a^1] a^2 \dots [F, a^{n+1}]) \\ &- \mathrm{Tr}(F[F, a^0] a^1 [F, a^2] \dots [F, a^{n+1}]) \\ &+ \dots + \\ +(-1)^{n+1} &\mathrm{Tr}(F[F, a^{n+1}] a^0 [F, a^1] \dots [F, a^{n+1}]) \\ +(-1)^{n+1} &\mathrm{Tr}(F a^{n+1} [F, a^0][F, a^1] \dots [F, a^{n+1}]). \end{aligned}$$

Hence for odd n

$$b\tau_F = 0.$$

For even n

$$\mathrm{Tr}(\gamma F[F, a^n][F, a^1] \dots [F, a^{n-1}]) = \mathrm{Tr}(F[F, a^n][F, a^0] \dots [F, a^{n-1}]) =$$

$$- \operatorname{Tr}(F[F,a^0]\dots[F,a^n]).$$

This leads to the definition of *cyclic cohomology*, a homology of complex

$$(C_\lambda^\bullet(\mathfrak{A}), b), \;\; C_\lambda^n(\mathfrak{A}) = \operatorname{Hom}_{cont}(\mathfrak{A}^{\otimes n+1}, \mathbb{C})$$

for locally convex algebra $\mathfrak{A}$ (with continuous multiplication).

The fact that $n \mapsto n+2$ leaves formulas in proposition (6.8) unchanged is related to the periodicity operator

$$S\colon \operatorname{HC}_\lambda^n(\mathfrak{A}) \mapsto \operatorname{HC}_\lambda^{n+2}(\mathfrak{A})$$

which in turn is an arrow in Connes long exact sequence

$$\dots \xrightarrow{S} \operatorname{HC}_\lambda^n(\mathfrak{A}) \xrightarrow{I} \operatorname{HH}^n(\mathfrak{A}) \xrightarrow{B} \operatorname{HC}_\lambda^{n-1}(\mathfrak{A}) \xrightarrow{S} \operatorname{HC}^{n+1}(\mathfrak{A}) \xrightarrow{I} \dots$$

For $\mathfrak{A} = C^\infty(M)$, $\partial M = 0$

$$\tau(f^0, f^1, \dots, f^n) = \int_M f^0 df^1 \wedge \dots \wedge df^n$$

From Leibniz rule and Stokes theorem

$$b\tau = 0, \;\; \lambda(\tau) = \tau.$$

If $\omega \in \Omega^{n-k}(M)$ then

$$\tau_\omega(f^0, \dots, f^k) := \int_M f^0 df^1 \wedge \dots \wedge df^k \wedge \omega, \;\; d\omega = 0.$$

If C-k-current

$$\tau_C(f^0, \dots, f^k) = \langle C, f^0 df^1 \wedge \dots \wedge df^k \rangle, \;\; dC = 0.$$

Theorem 6.9 (Connes)**.**

$$\begin{array}{ccccccc}
\operatorname{HC}_\lambda^q(\mathfrak{A}) \xrightarrow{\cong} \ker d_q^+ & \oplus & \operatorname{H}_{q-2}^{dR}(M;\mathbb{C}) & \oplus & \operatorname{H}_{q-4}^{dR}(M;\mathbb{C}) & \oplus & \cdots \\
\Big\downarrow S \qquad\quad \Big\downarrow & & \Big\downarrow \cong & & \Big\downarrow \cong & & \\
\operatorname{HC}_\lambda^{q+2}(\mathfrak{A}) \xrightarrow{\cong} \ker d_{q+2}^+ & \oplus & \operatorname{H}_q^{dR}(M;\mathbb{C}) & \oplus & \operatorname{H}_{q-2}^{dR}(M;\mathbb{C}) & \oplus & \cdots
\end{array}$$

where the inclusion $\ker d_q^+ \hookrightarrow \operatorname{HC}_\lambda^q(\mathfrak{A})$ *is*

$$C \mapsto \phi_C(f^0, f^1, \dots, f^q) = \langle C, f^0 df^1 \wedge \dots \wedge df^q \rangle.$$

Compatibility considerations lead to the following normalization for the *Connes-Chern character* of a K-cycle F over $\mathfrak{A}$ of Schatten dimension p.

- For n odd $> p-1$

$$\tau_n(a^0, a^1, \dots, a^n) = (-1)^{\frac{n-1}{2}} \frac{n}{2}\left(\frac{n}{2} - 1\right)\dots\frac{1}{2} \operatorname{Tr}(F[f,a^0][F,a^1]\dots[F,a^n]),$$

$$S\tau_n = \tau_{n+2}$$

- For n even $> p-1$

$$\tau_n(a^0, a^1, \ldots, a^n) = \left(\frac{n}{2}\right)! \frac{1}{2} \operatorname{Tr}(\gamma F[f, a^0][F, a^1] \ldots [F, a^n]),$$

$$S\tau_n = \tau_{n+2}$$

Homological Chern character is a homomorphism

$$\mathrm{ch}_*\colon \mathrm{K}_*(M) \to \mathrm{H}_*^{dR}(M; \mathbb{C})$$

It is a special case of the Connes- Chern character for an algebra

$$\mathrm{ch}^* \, \mathrm{K}^*(\mathfrak{A}) \to \mathrm{HP}^*(\mathfrak{A})$$

if one takes $\mathfrak{A} = C^\infty(M)$. For a cocycle $(\mathfrak{A}, \mathcal{H}, F)$ representing an element in K-homology one has

$$\mathrm{ch}^*(\mathfrak{A}, \mathcal{H}, F) := [\phi^n],$$

where ϕ^n is the following cocycle

$$\phi^n(a^0, a^1, \ldots, a^n) = \operatorname{Tr}(\gamma a^0[F, a^0] \ldots [F, a^n])$$

for n even.

$$S[\phi^n] = [\phi^{n+2}]$$

For a Dirac operator D we can take $F = D|D|^{-1}$ and then

$$\mathrm{ch}_*(D) = \widehat{A}(M) = (\det)^{\frac{1}{2}} \left(\frac{\frac{R}{2}}{\sinh \frac{R}{2}} \right)$$

If γ is a gradation on $\mathcal{H}$ i.e.

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

then

$$\operatorname{Index}(D^+) = \operatorname{Tr}(\gamma e^{-tD^2}), \ t > 0$$

$$D^2 = \begin{pmatrix} D^- D^+ & 0 \\ 0 & D^+ D^- \end{pmatrix}.$$

For $t \to 0^+$ function $\operatorname{Tr}(\gamma e^{-tD^2})$ has an expansion

$$c_0 + c_1 t + c_2 t^2 + \ldots,$$

where

$$c_0 = \int_M \omega_\delta(D)$$

and $\omega_\delta(D)$ is called the local index formula.

6.4 Connes cyclic homology

$\mathrm{HC}^*(\mathfrak{A})$ is defined as the cohomology of a complex $(C_\lambda(\mathfrak{A}), b)$. A *cycle* representing an element in $\mathrm{HC}^*(\mathfrak{A})$ is a triple

$$(\Omega, d, \int),$$

where (Ω, d) is a differential graded algebra

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n, \quad d^2 = 0, \quad \text{(finite length)},$$

and $\int$ is a *closed graded trace* $\int \Omega^n \to \mathbb{C}$ i.e.

$$\int \omega_1\omega_2 = (-1)^{|\omega_1||\omega_2|} \int \omega_2\omega_1 \text{ (graded trace)},$$

$$\int d\omega = 0 \text{ (closed)}.$$

Using homomorphism $\rho\colon \mathfrak{A} \to \Omega^0$ we can write a character of $(\Omega, d, \int)$

$$\tau(a^0, a^1, \dots, a^n) = \int a^0 da^1 \dots da^n.$$

It is a cyclic cocycle.

Define a *chain* as a triple $(\Omega, \partial\Omega, \int)$, where $\partial\Omega \subset \Omega$, $\dim \Omega = n$, $\dim \partial\Omega = n-1$, and d preserves $\partial\Omega$. There is given a surjective homomorphism $r\colon \Omega \to \partial\Omega$ of degree 0 (restriction to the boundary) and

$$\int d\omega = 0, \ \forall \ \omega \quad \text{such that } r(\omega) = 0.$$

A *boundary* of such chain is a cycle $(\partial\Omega, d, \int')$, where for $\omega' \in \partial\Omega^{n-1}$

$$\int' \omega' := \int d\omega, \quad \text{for } r(\omega) = \omega'.$$

Two cycles Ω_1, Ω_2 are *cobordant*, $\Omega_1 \sim \Omega_2$ if and only if there exists a chain $(\Omega, \partial\Omega, \int)$ such that

$$\partial\Omega = \Omega_1 \oplus \widetilde{\Omega_2}$$

where $(\widetilde{\Omega_2}, d, \widetilde{\int})$ is a cycle in which $\widetilde{\int}\omega = -\int \omega$.

Theorem 6.10.

$$\Omega_1 \sim \Omega_2 \textit{ iff. } \tau_2 - \tau_1 = B_0\phi \in \operatorname{im} B_0$$

where the operator B_0 is defined as follows.

$$B_0\phi(a^0, a^1, \dots, a^n) = \phi(1, a^0, \dots, a^n) - (-1)^{n+1}\phi(a^0, \dots, a^n, 1).$$

The operator B is then equal to AB_0, where A is the cyclic antisymmetrization

$$(A\phi)(a^0, a^1, \dots, a^n) := \sum_{i=0}^{n} (-1)^{ni}\phi(a^i, a^{i+1}, \dots, a^{i-1}).$$

The Connes exact sequence

$$\dots \xrightarrow{B} \mathrm{HC}_\lambda^{n-2}(\mathfrak{A}) \xrightarrow{S} \mathrm{HC}_\lambda^{n}(\mathfrak{A}) \xrightarrow{I} \mathrm{H}^n(\mathfrak{A}) \xrightarrow{B} \mathrm{HC}_\lambda^{n-1}(\mathfrak{A}) \xrightarrow{S}$$

starts with $\mathrm{HC}_\lambda^0(\mathfrak{A}) = \mathrm{H}^0(\mathfrak{A})$. Thus if there is an algebra homomorphism $\mathfrak{A} \to \mathfrak{A}'$ which induces isomorphism on Hochshild cohomology, then it also induces isomorphism on cyclic homology.

We can form a bicomplex $(C^{n,m}, b, B)$ with $b^2 = 0$, $B^2 = 0$, $bB + Bb = 0$, and $C^{n,m} = C^{n-m}(\mathfrak{A}) = \mathfrak{A}^{\otimes n-m+1}$. The homology of the total complex is then cyclic homology.

6.5 An alternate route, via the Families Index Theorem

Set up: $(\mathfrak{A}, \mathcal{H}, D)$, $D = D^*$ unbounded with

$$[D, \mathfrak{A}] \subset B(\mathcal{H}), \quad (1 + D^2) \in \mathcal{L}^p$$

In fact we shall assume that D is invertible with $D^{-1} \in \mathcal{L}^p$. The bounded version of this K-cycle is given by $(\mathfrak{A}, \mathcal{H}, F)$, where $F = D|D|^{-1}$ is a phase.

On $\mathfrak{A}$ one has a norm

$$|||a||| := ||a|| + ||[D, a]||, \text{ for } a \in \mathfrak{A}.$$

Let $\mathcal{V} = \mathcal{V}(\mathfrak{A})$ be the span of "vector potentials", that is

$$\mathcal{V} := \left\{ A = \sum_i a_i[D, b_i] \mid a_i, b_i \in \mathfrak{A},\ A = A^* \right\}.$$

Let $\mathcal{U} = \mathcal{U}(\mathfrak{A})$ be the gauge group, that is

$$\mathcal{U} = \mathcal{U}(\mathfrak{A}) := \{u \in \mathrm{GL}_1(\mathfrak{A}) \mid u^*u = uu^* = 1\},$$

acting on $\mathcal{V}$ by (affine action)

$$u \cdot A := u[D, u^*] + uAu^* = u(D + A)u^* - D.$$

Denoting $D_A := D + A$ one has

$$D_{u\cdot A} = uD_Au^*.$$

Fact 6.11. *D_A has the same dimension as D and $D_A^* = D_A$. Also $\ker D_A = \ker(\mathrm{Id} + D^{-1}A)$, hence is finite dimensional.*

Let

$$\mathcal{V}_{inj} := \{A \in \mathcal{V} \mid D_A \text{ injective }\} \subset \mathcal{V}$$

It is an open subset with respect to $|||\cdot|||$. For $A \in \mathcal{V}_{inj}$ operator D_A is invertible with

$$D_A^{-1} = (1 + D^{-1}A)^{-1}D^{-1} \in \mathcal{L}^p.$$

Graded trivial vector bundle over $\mathcal{V}_{inj}$

$$\widetilde{\mathcal{H}}^{\pm} := \mathcal{V}_{inj} \times \mathcal{H}^{\pm}.$$

Superconnection is an operator $d + \widetilde{D}$, where

$$\widetilde{D}\colon \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}, \quad \text{is in the fiber } \widetilde{D}_A = D_A\colon \mathcal{H}^{\pm} \to \mathcal{H}^{\pm}.$$

Curvature

$$\mathcal{R} := (\gamma d + \widetilde{D})^2 = \gamma d\widetilde{D} + \widetilde{D}d + \widetilde{D}^2 = \underbrace{[\gamma d, \widetilde{D}]}_{=:\widetilde{D}'} + \widetilde{D}^2.$$

Explicit expression of $\widetilde{D}' = [d, \widetilde{D}] \in \Omega^1(\mathcal{V}_{inj}, \widetilde{\mathcal{H}})$:

$$d\colon \Omega^p(\mathcal{V}_{inj}, \widetilde{\mathcal{H}}) \to \Omega^{p+1}(\mathcal{V}_{inj}, \widetilde{\mathcal{H}})$$

$$(d\omega)(\widetilde{X}_0, \ldots, \widetilde{X}_{p+1}) = \sum_{i=0}^{p} \widetilde{X}_i \omega(\widetilde{X}_0, \ldots, \widehat{\widetilde{X}_i}, \ldots, \widetilde{X}_p)$$

(commutators vanish), where

$$\widetilde{X}_A f := \frac{d}{dt}\Big|_{t=0} f(A + tX), \quad X \in \mathcal{V}.$$

One has with $F\colon \mathcal{V}_{inj} \to B(\mathcal{H})$, $F(A) := D + A$

$$\gamma d(\widetilde{D}\omega) = \gamma dF \wedge \omega,$$

Hence

$$\widetilde{D}'(\omega) = dF \wedge \omega, \quad dF_A(\widetilde{X}_A) = X,$$

$$\widetilde{D}'(\omega)_A(X_0, \ldots, X_{p+1}) = \sum_{i=0}^{r} (-1)^i \underbrace{X_i}_{\in B(\mathcal{H})} \underbrace{\omega_A(X_0, \ldots, \widehat{X_i}, \ldots, X_p)}_{\in \mathcal{H}}$$

(Super) Chern form

$$\Omega_t^{(n)} := \operatorname{Tr}\left(\gamma e^{-(t\widetilde{D}' + t^2\widetilde{D}^2)}\right)^{(n)} = \operatorname{Tr}\left(\gamma e^{-\mathcal{R}_t^2}\right)^{(n)} =$$

$$= (-t)^n \int_{\Delta_n} \operatorname{Tr}\left(e^{-s_1 t^2 \widetilde{D}^2} \widetilde{D}' e^{-(s_1 - s_2)t^2\widetilde{D}^2} \widetilde{D}' \ldots e^{-(s_n - s_{n-1})t^2\widetilde{D}^2} \widetilde{D}' e^{-(1-s_n)t^2\widetilde{D}^2}\right) d\mathbf{s},$$

where

$$d\mathbf{s} := ds_1 ds_2 \ldots ds_n,$$

and the integration is over a simplex

$$\Delta_n := \{0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n \leqslant 1 \mid s_1 + s_2 + \ldots + s_n = 1\}$$

One has

$$\frac{d}{ds}(e^{s(A+B)} e^{-sB}) = e^{s(A+B)} A e^{-sB}$$

$$e^{u(A+B)} = e^{uB} + \int_0^u e^{s(A+B)} A e^{(u-s)B} ds.$$

6.6 Index theory for foliations

Let $(M^m, \mathcal{F})$ be a foliated manifold. To define an index in noncommutative geometry we have to complete definitions of the following tasks

1. transverse coordinates,
2. analog of elliptic operator,
3. index pairing between K-theory and K-homology.

Foliation can be described using 1-cocycle (V_i, f_i, g_{ij}), where

$$f_i \colon V_i \to U_i \subset \mathbb{R}^n, \;\; n = \operatorname{codim}\mathcal{F} \text{ are surjective submersions,}$$

$$\text{and } g_{ij} \colon f_j(V_i \cap V_j) \to f_i(V_i \cap V_j) \text{ are diffeomorphisms such that}$$

$$g_{ij} \circ g_{jk} = g_{ik}.$$

Above cocycle gives a grupoid $\Gamma = \{g_{ij}\}$ which leads to the algebra of foliation

$$\mathfrak{A}_\Gamma := C_c^\infty(FM) \rtimes \Gamma$$

$$f u_\phi \cdot g u_\psi = f g \phi^{-1} u_{\phi\psi}, \;\; \phi, \psi \in \Gamma.$$

where $FM = J^1(M)$ is a frame bundle. This gives a transverse coordinates. The advantage in working with frame bundle is that FM has a natural volume form. It is paralelizable (i.e. TFM is trivial). One has a principal bundle

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{R}) & = & FM \\ \big\downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

One has vertical vector fields Y_i^j coming from the $\mathrm{GL}_n(\mathbb{R})$ action, and when chooses a connection, also horizontal vector fields X_k. Let $\{\theta^k, \omega_j^i\}$ be the dual basis of differential forms. Then

$$\Lambda\omega_j^i \wedge \Lambda\theta^k$$

is an invariant volume form.

For our second task we have to give up ellipticity. Consider a quotient bundle

$$\begin{array}{ccc} FM/\operatorname{SO}(n) & = & PM \\ \big\downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

The fiber PM_x is the space of all Euclidean structures on T_xM

$$\langle \zeta, \eta \rangle = \langle a\zeta, a\eta \rangle, \;\; a \in \mathrm{SO}(n).$$

Section of PM are all Riemannian metrics on TM. Let

$$\mathcal{V} \subset TPM = \ker \pi_*$$

be the vertical subbundle (vectors tangent to the fibers). On the quotient $\mathrm{GL}_n(\mathbb{R})/\mathrm{SO}(n)$ there is a metric, and determines a metric on $\mathcal{V}$.

$$\begin{array}{ccc} TPM/\mathcal{V} & & \mathcal{N} \\ \big\downarrow & & \\ PM & & \end{array}$$

The horizontal bundle $\mathcal{N}$ has a tautological Riemannian structure. Indeed, $p \in PM$ is an Euclidean structure for $T_{\pi(p)}M$, and $\mathcal{N}_p$ is identified with $T_{\pi(p)}M$ by π_*.

The bundle TPM has a decomposition into vertical and horizontal part, $TPM = \mathcal{V} \oplus \mathcal{N}$. The Hilbert space

$$L^2(\Lambda T^* PM, \mathrm{Vol}_P)$$

where Vol_P is a volume form induced by canonical volume form on FM, decomposes also as a tensor product of corresponding Hilbert spaces

$$L^2(\Lambda T^* PM) = L^2(\Lambda \mathcal{V}^*) \otimes L^2(\Lambda \mathcal{N}^*).$$

On this two parts we have operators

- On $L^2(\Lambda \mathcal{V}^*)$ with vertical differential d_V

$$Q_V := i(d_V + d_V^*)(d_V - d_V^*) = -i(d_V d_V^* + d_V^* d_V)$$

- On $L^2(\Lambda \mathcal{N}^*)$ with horizontal differential d_H

$$Q_H := d_H + d_H^*$$

On the whole $L^2(\Lambda T^* PM)$ we put $Q = Q_V \oplus \gamma_V Q_H$, where γ_V is the grading of the vertical signature. Operator $Q = Q^*$ is called *hypoeliptic signature operator*. We have a spectral triple $(\mathfrak{A}_\Gamma, \mathcal{H}, D)$, where D is determined by the equation $Q = D|D|$.

For $a \in \mathfrak{A}$ $[D, a] \in B(\mathcal{H})$ and $(1 + D^2)^{-\frac{1}{2}} \in \mathcal{L}^p(\mathcal{H})$ for $p = \dim \mathcal{V} + 2n$, where $\dim M = n$. The K-cycle $(\mathfrak{A}, \mathcal{H}, D)$ gives an element in $\mathrm{K}^*_{\mathrm{Diff}_M}(\mathfrak{A})$ (Diff_M-equivariant K-cycle). Its character $\mathrm{ch}_*(D) \in \mathrm{HC}_*(\mathfrak{A}_\Gamma)$ can be expressed in terms of residues of spectrally defined zeta-functions, and is given by a cocycle $\{\phi_n\}$ in the (b, B)-bicomplex of $\mathfrak{A}_\Gamma$ whose components are of the following form

$$\mathrm{Res}_{s=0} \mathrm{Tr}(a^0 [a^1, D]^{(k_1)} \dots [a^n, D]^{(k_n)} |D|^{-n-2|k|-s})$$

which we denote by

$$\int \mathrm{Tr}(a^0 [a^1, D]^{(k_1)} \dots [a^n, D]^{(k_n)} |D|^{-n-2|k|-s})$$

$$\phi_n(a^0, \dots, a^n) = \sum_{\mathbf{k}} c_{n,\mathbf{k}} \int a^0 [Q, a^1]^{(k_1)} \dots [Q, a^n]^{(k_n)} |Q|^{-n-2|k|}$$

Chapter 7

Hopf–cyclic cohomology

7.1 Preliminaries

7.1.1 Cyclic cohomology in abelian category

Our task is to understand cup product for Hopf–cyclic cohomology with coefficients, that is mapping

$$\mathrm{HC}^m_H(C;M) \otimes \mathrm{HC}^n_H(A;M) \to \mathrm{HC}^{m+n}(A;M).$$

Concider a category $\mathcal{C}$, with finite sets $[n] := \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}$ as objects, and morphism which preserve order. To describe a cyclic structure we introduce following morphisms

- Face
$$[n-1] \xrightarrow{\delta_i} [n], \;\; 0 \leqslant i \leqslant n,$$
- injection which misses i.

- Degeneracy
$$[n+1] \xrightarrow{\sigma_j} [n], \;\; 0 \leqslant j \leqslant n,$$
- surjection which sends both j and $j+1$ to j.

- Cyclic operator
$$[n] \xrightarrow{\tau_n} [n]$$
- cyclic shift to the right.

The morphism above satisfy following identities, which we can group to obtain succesive complications of our category.

- Presimplicial simplicial category.
$$\mathrm{Mor}(\mathcal{C}) := \{\delta_i^{(n)} \mid 0 \leqslant i \leqslant n, \; n \in \mathbb{N}\},$$
with
$$\delta_j \delta_i = \delta_i \delta_j, \; j > i.$$

- Simplicial category.

$$\mathrm{Mor}(\mathcal{C}) := \{\delta_i^{(n)},\ \sigma_j^{(m)} \mid 0 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant m,\ n, m \in \mathbb{N}\},$$

  with additional identities

$$\sigma_j\sigma_i = \sigma_i\sigma_{j+1},\ i \leqslant j,$$

$$\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1}, & i < j, \\ \mathrm{id}_{[n]}, & i \in \{j, j+1\}, \\ \delta_{i-1}\sigma_j, & i > j+1 \end{cases}$$

- Precyclic category.

$$\mathrm{Mor}(\mathcal{C}) := \{\delta_i^{(m)},\ \tau_n \mid 0 \leqslant i \leqslant m,\ m, n \in \mathbb{N}\},$$

  with the identities as for presimlicial category and

$$\tau_n^{n+1} = \mathrm{id}_{[n]},$$

$$\tau_n\delta_i = \delta_{i-1}\tau_{n-1},\ 1 \leqslant i \leqslant n.$$

- Cyclic Category.

$$\mathrm{Mor}(\mathcal{C}) := \{\delta_i^{(m)},\ \sigma_j^{(l)},\ \tau_n \mid 0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant l,\ m, l, n \in \mathbb{N}\},$$

  with all above identieties and

$$\tau_n\sigma_0 = \sigma_n\tau_{n+1}^2,$$

$$\tau_n\sigma_j = \sigma_{j-1}\tau_{n+1},\ 1 \leqslant j \leqslant n.$$

Now, let $\mathcal{A}$ be an abelian category, and $F\colon \mathbb{C} \to \mathcal{A}$ a functor. It means that we have a sequence of objects, and morphisms

$$A_n \xrightarrow{\delta_i} A_n \xrightarrow{\tau_n} A_n \xleftarrow{\sigma_i} A_{n+1}.$$

Define

$$b_n := \sum_{i=0}^{n}(-1)^i\delta_i,\ \ b'_n := \sum_{i=0}^{n-1}(-1)^i\delta_i,$$

$$\lambda_n := (-1)^n\tau_n,\ \ n \in \mathbb{N}.$$

These morphisms satisfy the following identities

$$b_{n+1}b_n = 0,\ \ (1-\lambda_n)b_n = b'_n(1-\lambda_{n-1}).$$

Consider a diagram

$$\begin{tikzcd}
\ker_{n+1} \arrow[r] & A_{n+1} \arrow[r, "1-\lambda_{n+1}"] & A_{n+1} \\
\ker_n \arrow[r] \arrow[u, "\overline{b_{n+1}}"] & A_n \arrow[r, "1-\lambda_n"] \arrow[u, "b_{n+1}"] & A_n b'_{n+1} \arrow[u] \\
\ker_{n-1} \arrow[r] \arrow[u, "\overline{b_n}"] & A_{n-1} \arrow[r, "1-\lambda_{n-1}"] \arrow[u, "b_n"] & A_{n-1} b'_n \arrow[u]
\end{tikzcd}$$

The composition $\overline{b_{n+1}b_n} = 0$, so we have a complex

$$\begin{tikzcd}
\ker_{n-1} \arrow[r, "\overline{b_n}"] & \ker_n \arrow[r, "\overline{b_{n+1}}"] & \ker_{n+1} \\
\ker\operatorname{coker} b_n \arrow[ur] \arrow[rr, "\exists!\phi_n"] & & \ker \overline{b_{n+1}} \arrow[ul]
\end{tikzcd}$$

Define the cyclic homology of the complex $(A_\bullet, b_n)$ as the cokernel of the unique map ϕ_n

$$\mathrm{HC}^n(F) := \mathrm{HC}^n(A_\bullet) := \operatorname{coker}\phi_n.$$

Define another operator

$$N_n := \sum_{i=0}^{n} (\lambda_n)^i,\ n \in \mathbb{N}.$$

Now one can form a bicomplex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
A_2 \arrow[u, "b_3"] \arrow[r, "1-\lambda_2"] & A_2 \arrow[u, "-b_3'"] \arrow[r, "N_2"] & A_2 \arrow[u, "b_3"] \arrow[r, "1-\lambda_2"] & A_2 \arrow[u, "-b_3"] \arrow[r, "N_2"] & A_2 \arrow[u, "b_3"] \arrow[r] & \cdots \\
A_1 \arrow[u, "b_2"] \arrow[r, "1-\lambda_1"] & A_1 \arrow[u, "-b_2'"] \arrow[r, "N_1"] & A_1 \arrow[u, "b_2"] \arrow[r, "1-\lambda_1"] & A_1 \arrow[u, "-b_2'"] \arrow[r, "N_1"] & A_1 \arrow[u, "b_2"] \arrow[r] & \cdots \\
A_0 \arrow[u, "b_1"] \arrow[r, "1-\lambda_0"] & A_0 \arrow[u, "-b_1'"] \arrow[r, "N_0"] & A_0 \arrow[u, "b_1"] \arrow[r, "1-\lambda_0"] & A_0 \arrow[u, "-b_1'"] \arrow[r, "N_0"] & A_0 \arrow[u, "b_1"] \arrow[r] & \cdots
\end{tikzcd}$$

Then the cohomology of the total complex is the cyclic homology of the functor $F\colon \mathcal{C} \to \mathcal{A}$

$$\mathrm{HC}^n(F) = \mathrm{H}^n(\operatorname{Tot} A_{\bullet\bullet}).$$

7.1.2 Hopf algebras

Summary of notations.

- Coalgebra (C, Δ, ϵ)

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes\mathrm{id}"] \\
C \otimes C \arrow[r, "\mathrm{id}\otimes\Delta"] & C \otimes C \otimes C
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\epsilon\otimes\mathrm{id}"] \\
C \otimes C \arrow[r, "\mathrm{id}\otimes\epsilon"] & C
\end{tikzcd}$$

- Comodule (M, Δ_R)

$$\begin{tikzcd} M \arrow[r, "\Delta_R"] \arrow[d, "\Delta_R"'] & M \otimes C \arrow[d, "\Delta_R \otimes \mathrm{id}"] \\ M \otimes C \arrow[r, "\mathrm{id} \otimes \Delta_R"] & M \otimes C \otimes C \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "\Delta_R"] \arrow[dr, "\mathrm{id}"] & M \otimes C \arrow[d, "\epsilon \otimes \mathrm{id}"] \\ & M \end{tikzcd}$$

- Bicomodule (M, Δ_L, Δ_R)

$$\begin{tikzcd} M \arrow[r, "\Delta_R"] \arrow[d, "\Delta_L"'] & M \otimes C \arrow[d, "\Delta_L \otimes \mathrm{id}"] \\ M \otimes C \arrow[r, "\mathrm{id} \otimes \Delta_R"] & C \otimes M \otimes C \end{tikzcd}$$

- Hopf algebra $(H, m, 1, \Delta, \epsilon, S)$, where
  - $(H, m, 1)$ algebra,
  - (H, Δ, ϵ) coalgebra,
  - Δ, ϵ are algebra homomorphisms,
  - Convoloution product $f * g$

$$f * g \colon H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{m} H,$$

  - Antipode S

$$S * \mathrm{id} = 1\epsilon = \mathrm{id} * S.$$

Properties of S:

- if exists, it is unique,
- it is an antialgebra map: $S(ab) = S(b)S(a)$,
- it is an anticoalgebra map: $\Delta \circ S = (S \otimes S) \circ \Delta^{op}$,
- if there exists S^{-1}, it has the above properties and satisfies

$$S^{-1} *_{cop} \mathrm{id} = 1\epsilon = \mathrm{id} *_{cop} S^{-1}.$$

Sweedler notation:

$$\Delta h = \sum_i a_i \otimes b_i =: h^{(1)} \otimes h^{(2)}.$$

If we treat multiple tensor products as trees, then we can forget how the tree was constructed.

$$\begin{aligned}\Delta^2 h &= h^{(1)(1)} \otimes h^{(1)(2)} \otimes h^{(2)} \\ &= h^{(1)} \otimes h^{(2)(1)} \otimes h^{(2)(2)} \\ &= h^{(1)} \otimes h^{(2)} \otimes h^{(3)}.\end{aligned}$$

$$\Delta_R(m) = m^{(0)} \otimes m^{(1)}, \;\; \Delta_L(m) = m^{(-1)} \otimes m^{(0)}.$$

7.1.3 Motivation for Hopf–cyclic cohomology

If D is a Dirac operator, E idempotent, then there exists an index pairing

$$\langle \mathrm{ch}^*(D), ch_*(E)\rangle =: \mathrm{Index}(D_E).$$

For the transverse geometry of a codim $= n$ foliation

$$\mathrm{ch}^*(D)(a_0, \dots, a_m) = \mathrm{tr}_\delta(a_0 h_1(a_1) \dots h_m(a_m)),$$

where $h_i \in \mathcal{H}_n$ - the universal Hopf algebra for codim $= n$ foliations, $\delta\colon H \to k$-character, tr_δ - δ-invariant trace.

$$\mathcal{H}_n \otimes A \to A$$

$$h(ab) = h^{(1)}(a)h^{(2)}(b), \;\; 1_H(a) = a.$$

In particular

$$\Delta(g) = g \otimes g \text{ (group-like element)} \implies g(ab) = g(a)g(b),$$

$$\Delta x = x \otimes 1 + 1 \otimes x \text{ (primitive element)} \implies x(ab) = x(a)b + ax(b).$$

One has

$$\begin{aligned}\mathrm{tr}_\delta(a_0 h_1(a_1) \dots h_m(a_m)) &= (-1)^m \,\mathrm{tr}_\delta(a_m h_1(a_0) \dots h_m(a_{m-1})) \\ &= (-1)^m \,\mathrm{tr}_\delta(h_1(a_0) \dots h_m(a_{m-1})a_m).\end{aligned}$$

In particular

$$\begin{aligned}\mathrm{tr}_\delta(h(a)) &= \delta(h)\,\mathrm{tr}_\delta(a), \\ \mathrm{tr}_\delta(h(a)b) &= \mathrm{tr}_\delta(h^{(1)}(a)(h^{(2)}S(h^{(3)}))(b)) \\ &= \mathrm{tr}_\delta(h^{(1)}(a)h^{(2)}(S(h^{(3)}))(b))) \\ &= \mathrm{tr}_\delta(h^{(1)}(aS(h^{(2)})(b))) \\ &= \delta(h^{(1)})\,\mathrm{tr}_\delta(aS(h^{(2)})(b)) \\ &= \mathrm{tr}_\delta(a(\delta * S)(h)(b)).\end{aligned}$$

Hence

$$\mathrm{tr}_\delta(a_0 h_1(a_1) \dots h_m(a_m)) = (-1)^m \,\mathrm{tr}_\delta(a_0(\delta * S)(h_1)(h_2(a_1) \dots h_m(a_{m-1})a_m))$$

Denote

$$h_1 \otimes \dots \otimes h_m = (-1)^m (\delta * S)(h_1)(h_2 \otimes \dots h_m \otimes 1) =: (-1)^m \tau_m(h_1 \otimes \dots \otimes h_m).$$

For an element $\sigma \in \mathcal{H}_n$ such that $\Delta\sigma = \sigma \otimes \sigma$, $\delta(\sigma) = 1$

$$\mathrm{tr}^\sigma_\delta(ab) = \mathrm{tr}^\sigma_\delta(b\sigma(a))$$

which implies

$$\begin{aligned}
\tau_m(h_1 \otimes \ldots \otimes h_m) &= (\delta * S)(h_1)(h_2 \otimes \ldots \otimes h_m \otimes \sigma) \\
(-1)^m \mathrm{tr}_\delta(h_1(a_0) \underbrace{h_2(a_1) \ldots h_m(a_{m-1})a_m}_{b}) &= (-1)^m \mathrm{tr}_\delta(a_0 \underbrace{(\delta * S)(h_1)}_{\tilde{h}} \\
(h_2(a_1) \ldots h_m(a_{m-1})a_m)) &= (-1)^m \mathrm{tr}_\delta(a_0 \tilde{h}(b)). \\
(-1)^m(\delta * S)(h_1)(h_2 \otimes \ldots \otimes h_m \otimes 1) &= \lambda_m(h_1 \otimes \ldots \otimes h_m).
\end{aligned}$$

Now one has to check that $\tau_m^{m+1} = \mathrm{id}$. For $m = 1$

$$\begin{aligned}
\tau_1^2(h) &= \tau_1((\delta * S)(h)\sigma) = \delta(h^{(1)})(\delta * S)(S(h^{(2)})\sigma)\sigma \\
&= \delta(h^{(1)})\delta(S(h^{(3)}))\sigma^{-1}S^2(h^{(2)})\sigma \\
&= \sigma^{-1}(\delta * S^2 * \delta^{-1})(h)\sigma \\
&= h
\end{aligned}$$

Denote

$$S^\sigma_\delta(h) := (\delta * S)(h)\sigma.$$

Now from $(\tau_1)^2 = (S^\sigma_\delta)^2 = \mathrm{id}$ one can deduce after computation that for all m $\tau_m^{m+1} = \mathrm{id}$ (Connes–Moscovici). This yields a new cyclic complex

$$(H^{\otimes m}, \delta_i, \sigma_j, \tau_m)_{m\in\mathbb{N}}$$

for any Hopf algebra H equipped with modular pair in involution (MPII) (δ, σ). For example, if $S^2 = \mathrm{id}$, then $(\epsilon, 1)$ is a modular pair in involution.

Example 7.1. Let $H = \mathcal{H}_1$ be an universal algebra for codim $= 1$ foliations. First take a Lie algebra $\mathfrak{h}_1$ with generators $X, Y, \lambda_n,\ n \in \mathbb{N}$ satisfying

$$\begin{aligned}
[Y, X] &= X, \\
[X, \lambda_n] &= \lambda_{n+1}, \\
[Y, \lambda_n] &= n\lambda_n, \\
[\lambda_n, \lambda_m] &= 0\ \ \forall\, n, m \geqslant 1.
\end{aligned}$$

Then form an universal enveloping algebra $\mathcal{H}_1 := U(\mathfrak{h}_1)$. The coproduct on $\mathcal{H}_1$ id uniquely determined by

$$\begin{aligned}
\Delta(X) &= X \otimes 1 + 1 \otimes X + \lambda_1 \otimes Y, \\
\Delta(Y) &= Y \otimes 1 + 1 \otimes Y, \\
\Delta(\lambda_1) &= \lambda_1 \otimes 1 + 1 \otimes \lambda_1.
\end{aligned}$$

The counit

$$\epsilon(X) = \epsilon(Y) = \epsilon(\lambda_1) = 0.$$

The antipode

$$\begin{aligned}
S(X) &= -X + \lambda_1 Y, \\
S(Y) &= -Y, \\
S(\lambda_1) &= -\lambda_1.
\end{aligned}$$

Now take $\sigma = 1$,

$$\delta(X) = 0,$$
$$\delta(Y) = -1,$$
$$\delta(\lambda_1) = 0.$$

One has to check that

$$\delta(h^{(1)})S^2(h^{(2)})\delta(S(h^{(3)})) = h.$$

On generators

$$Y^{(1)} \otimes Y^{(2)} \otimes Y^{(3)} = Y \otimes 1 \otimes 1 + 1 \otimes Y \otimes 1 + 1 \otimes 1 \otimes Y,$$

$$\delta(Y) + S^2(Y) - \delta(Y) = Y.$$

Similarly for λ_1.

$$X^{(1)} \otimes X^{(2)} \otimes X^{(3)} =$$

$$= X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X + 1 \otimes \lambda_1 \otimes Y + \lambda_1 \otimes Y \otimes 1 + \lambda_1 \otimes 1 \otimes Y,$$

$$\begin{aligned} S^2(X) + \underbrace{\delta(S(X))}_{=0} - S^2(\lambda_1)\delta(Y) &= S(-X + \lambda_1 Y) + \lambda_1 \\ &= X \underbrace{-\lambda_1 Y + S(Y)S(\lambda_1)}_{=[Y,\lambda_1]=\lambda_1} + \lambda_1 \\ &= X + \lambda_1 - \lambda_1 = X. \end{aligned}$$

Thus $(\delta, 1)$ is a modular pair in involution.

7.1.4 Hopf–cyclic cohomology with coefficients

Motivation:

- Short proof of
$$\tau_1^2 = \mathrm{id} \implies \tau_n^{n+1} = \mathrm{id}.$$
- Constructive common denominator for all known cyclic theories.
- Non-trivial coefficients are geometrically desired and occur in "real life" in the number theory work of Connes–Moscovici.

Simplicial structure in coalgebra case:

$$\mathcal{C}^n(C, M) := M \otimes C \otimes C^{\otimes n}, \;\; n \in \mathbb{N},$$

C is an H-module coalgebra

$$\Delta(hc) = h^{(1)}c^{(1)} \otimes h^{(2)}c^{(2)}, \;\; \epsilon(hc) = \epsilon(h)\epsilon(c).$$

M is a C-bimodule

$$\Delta_R(m \otimes c) = (m \otimes c^{(1)}) \otimes c^{(2)},$$
$$\Delta_L(m \otimes c) = m^{(-1)}c^{(1)} \otimes (m^{(0)} \otimes c^{(2)}).$$

The standard example yields

$$\begin{aligned}\delta_i(m\otimes c_0\otimes\ldots\otimes c_{n-1}) &= m\otimes c_0\ldots\otimes c_i^{(1)}\otimes c_i^{(2)}\otimes\ldots\otimes c_{n-1},\\ \delta_n(m\otimes c_0\otimes\ldots\otimes c_{n-1}) &= m^{(0)}\otimes c_0^{(2)}\otimes c_1\otimes\ldots\otimes c_{n-1}\otimes m^{(-1)}c_0^{(1)},\\ \sigma_i(m\otimes c_0\otimes\ldots\otimes c_{n+1}) &= m\otimes c_0\otimes\ldots\otimes \epsilon(c_{i+1})\otimes\ldots\otimes c_{n+1}.\end{aligned}$$

Simplicial structure in algebra case:

$$\mathcal{C}^n(A,M) := \mathrm{Hom}(M\otimes A\otimes A^{\otimes n},k),\quad n\in\mathbb{N}.$$

A is an H-module algebra

$$h(ab)=(h^{(1)}a)(h^{(2)}b),\quad h1=\epsilon(h).$$

M is aleft H-comodule

$$\mathrm{Hom}(M\otimes A\otimes A^{\otimes n},k)\cong \mathrm{Hom}(A^{\otimes n},\mathrm{Hom}(M\otimes A,k)).$$

$M\otimes A$ is an A-bimodule

$$(m\otimes a)b=m\otimes ab,\quad b(m\otimes a)=m^{(0)}\otimes(S^{-1}(m^{(-1)})b)a$$

The standard example yields

$$\begin{aligned}(\delta_i f)(m\otimes a_0\otimes\ldots\otimes a_n) &= f(m\otimes a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n),\\ (\delta_n f)(m\otimes a_0\otimes\ldots\otimes a_n) &= f(m^{(0)}(S^{-1}(m^{(-1)})a_n)a_0\otimes\ldots\otimes a_{n-1}),\\ (\sigma_i f)(m\otimes a_0\otimes\ldots\otimes a_n) &= f(m\otimes a_0\otimes\ldots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\ldots\otimes a_n).\end{aligned}$$

Paracyclic structures:
For $\{\mathcal{C}^n(A,M)\}_{n\in\mathbb{N}}$

$$(\tau_n f)(m\otimes a_0\otimes\ldots\otimes a_n)=f(m^{(0)}(S^{-1}(m^{(-1)})a_n)\otimes a_0\otimes\ldots\otimes a_{n-1}).$$

For $\{\mathcal{C}^n(C,M)\}_{n\in\mathbb{N}}$

$$\tau_n(m\otimes c_0\otimes\ldots\otimes c_n)=m^{(0)}\otimes c_1\otimes\ldots\otimes c_n\otimes m^{(-1)}c_0.$$

Invariant complexes:

$$\begin{gathered}\mathcal{C}^n_H(A,M) := \mathrm{Hom}_H(M\otimes A^{\otimes n+1},k),\\ M\in{}^H\mathbf{M}_H,\quad (m\otimes\tilde{a})h=mh^{(1)}\otimes S(h^{(2)})\tilde{a},\quad k=k_\epsilon\\ \mathcal{C}^n_H(C,M) := M\otimes_H C^{\otimes n+1},\\ M\in{}^H\mathbf{M}_H,\quad h(c_0\otimes\ldots c_n)=h^{(1)}c_0\otimes\ldots\otimes h^{(n+1)}c_n.\end{gathered}$$

To describe cyclic structure we need

Definition 7.2. *We say that a bimodule $M\in{}^H\mathbf{M}_H$ is* stable *if and only if*

$$\forall\, m\in M\ \ m^{(0)}m^{(-1)}=m.$$

It is anti-Yetter-Drinfeld *if and only if*

$$\Delta_L(mh)=S(h^{(3)})m^{(-1)}h^{(1)}\otimes m^{(0)}h^{(2)},\ \ \forall\, m,h.$$

Theorem 7.3. *If M is a stable anti-Yetter-Drinfeld module (SAYD), then the formulas for δ_i, σ_i and τ_n define cyclic structures on $\mathcal{C}^n_H(A, M)$ and $\mathcal{C}^n_H(C, M)$.*

Shortly

- anti-Yetter-Drinfeld $\implies$ τ_n is well defined,
- stability $\implies$ $\tau_n^{n+1} = \mathrm{id}$.

Proof. First we check that τ_n is well defined, that is

$$\tau_n(mh \otimes c_0 \otimes \ldots \otimes c_n) = \tau_n(m \otimes h(c_0 \otimes \ldots \otimes c_n)),$$

$$(mh)^{(0)} \otimes_H (c_1 \otimes \ldots \otimes c_n \otimes (mh)^{-1} c_0) = m^{(0)} \otimes_H (h^{(2)}(c_1 \otimes \ldots \otimes c_n) \otimes m^{(-1)} h^{(1)} c_0),$$

hence it suffices to prove the following identity

$$(mh)^{(0)} \otimes_H (1 \otimes (mh)^{(-1)}) = m^{(0)} \otimes_H (h^{(2)} \otimes m^{(-1)} h^{(1)}).$$

Take

$$M \otimes_H (H. \otimes H.) \text{ (diagonal structure)}$$

and morphism

$$H. \otimes H. \xrightarrow{\Phi} H. \otimes H \text{ (multiplication on the first term)}$$

$$\Phi(h \otimes k) = h^{(1)} \otimes S(h^{(2)})k,$$
$$\Phi^{-1}(h \otimes k) = h^{(1)} \otimes h^{(2)}k.$$

Now

$$\Phi^{(-1)}(l(h \otimes k)) = \Phi^{-1}(lh \otimes k) = l\Phi^{-1}(h \otimes k).$$

Consider

$$M \otimes_H (H. \otimes H.) \xrightarrow{\mathrm{id} \otimes_H \Phi} M \otimes_H (H. \otimes H) \cong M \otimes H.$$

$$(mh)^{(0)} \otimes (mh)^{(-1)} = m^{(0)} h^{(2)} \otimes S(h^{(3)}) m^{(-1)} h^{(1)}.$$

-anti-Yetter-Drinfeld condition.

$$\begin{aligned}
\tau_n^{n+1}(m \otimes_H c_0 \otimes \ldots \otimes c_n) &= \tau_n^n(m^{(0)} \otimes_H c_1 \otimes \ldots \otimes c_n \otimes m^{(-1)} c_0) \\
&= m^{(0)} \otimes m^{(-1)}(c_0 \otimes \ldots \otimes c_n) \\
&= m^{(0)} m^{(-1)} \otimes c_0 \otimes \ldots \otimes c_n \\
&= m \otimes_H c_0 \otimes \ldots \otimes c_n,
\end{aligned}$$

where in the last equality we used stability of M. □

7.1.5 Special cases

1. Connes–Moscovici construction.

$$C = H, \quad M =^{\sigma} k_{\delta}$$

Then ${}^{\sigma}k_{\delta}$ is SAYD iff. (δ, σ) is MPII. Let F be the isomorphism

$$F\colon k \otimes_H (H_{.} \otimes H_{.}^{\otimes n}) \xrightarrow{\cong} H^{\otimes n}.$$

Then for $\tilde{f} \in H^{\otimes n}$

$$\begin{aligned}
\tau_n(h_1 \otimes \dots h_n) &= (F \circ \widetilde{\tau_n} \circ F^{-1})(\tilde{h}) \\
&= (F \circ \widetilde{\tau_n})(1 \otimes_H \widetilde{\Phi^{-1}}(1 \otimes \tilde{h})) \\
&= F(1 \otimes_H (\tilde{h} \otimes \sigma)) \\
&= 1 \otimes_H \widetilde{\Phi}(h_1 \otimes \dots \otimes h_n \otimes \sigma) \\
&= 1 \otimes_H h_1^{(1)} \otimes S(h_1^{(2)})(h_2 \otimes \dots \otimes h_n \otimes \sigma) \\
&= \delta(h_1^{(1)}) S(h_1^{2})(h_2 \otimes \dots \otimes h_n \otimes \sigma).
\end{aligned}$$

2. $\mathrm{tr}_{\delta}^{\sigma} \in \mathrm{HC}_H^0(A;^{\sigma} k_{\delta})$

3. Characteristic map of Connes–Moscovici

$$\mathrm{HC}_H^m(H;^{\sigma} k_{\delta}) \otimes \mathrm{HC}_H^0(A;^{\sigma} k_{\delta}) \to \mathrm{HC}^m(A),$$

$$h_1 \otimes \dots \otimes h_m \mapsto ((a_0 \otimes \dots \otimes a_m) \mapsto \mathrm{tr}_{\delta}^{\sigma}(a_0 h_1(a_1) \otimes h_m(a_m)))$$

4. The $n > 0$ and $\dim M > 1$ already applied in Connes–Moscovici work on number theory.

5. $\mathrm{HC}_k^m(A; k) = \mathrm{HC}^m(A)$

6. Twisted cyclic homology

$$\mathrm{HC}^*_{k[\sigma, \sigma^{-1}]}(A;^{\sigma} k_{\epsilon}).$$

Lemma 7.4.

$${}^{\sigma}k_{\delta} \textit{ is SAYD} \Longleftrightarrow (\delta, \sigma) \textit{ is MPII.}$$

Proof.

$$m^{(0)} m^{(-1)} = m \Leftrightarrow 1 \cdot \sigma = \delta(\sigma) = 1,$$

$$(mh)^{(-1)} \otimes (mh)^{(0)} = S(h^{(3)}) m^{(-1)} h^{(1)} \otimes m^{(0)} h^{(2)}$$

$$\sigma \delta(h) = S(h^{(3)}) \sigma h^{(1)} \delta(h^{(2)})$$

$$L(h) = R(h) \Leftrightarrow (L *_{op} S^{-1})(h) = (R *_{op} S^{-1})(h)$$

$$L(h^{(2)}) S^{(-1)}(h^{(1)}) = R(h^{(2)}) S^{(-1)}(h^{(1)})$$

$$\tilde{S}_{\delta}^{\sigma}(h) = \sigma\delta(h^{(2)}) S^{(-1)}(h^{(1)}) = S(h^{(2)}) \sigma \delta(h^{(1)}) =: S_{\delta}^{\sigma}(h)$$

By direct computation

$$\tilde{S}^\sigma_\delta \circ S^\sigma_\delta = \mathrm{id} = S^\sigma_\delta \circ \tilde{S}^\sigma_\delta, \text{ i.e.}$$
$$\tilde{S}^\sigma_\delta = (S^\sigma_\delta)^{-1}.$$

Therefore

$$\mathrm{AYD} \Leftrightarrow (S^\sigma_\delta)^{-1} = S^\sigma_\delta$$
$$(S^\sigma_\delta)^2 = \mathrm{id} \text{ (involution condition).}$$

□

7.2 The Hopf algebra $\mathcal{H}_n$

Let the manifold M^n be affine flat (the $\mathbb{R}^n$ or the disjoint union of $\mathbb{R}^n$). The frame bundle is then trivial with $FM \cong M \times \mathrm{GL}_n(\mathbb{R})$. In local coordinates (x^μ) for $x \in U \subset M$, we can view the frame coordinates x^μ, y^μ_j as a 1-jet of a map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$

$$\phi(t) = x + yt, \;\; x, t \in \mathbb{R}^n, \;\; y \in \mathrm{GL}_n(\mathbb{R}),$$

where $(yt)^\mu = \sum_i y^\mu_i t^i$ for $t = (t^i) \in \mathbb{R}^n$.

We endow it with the trivial connection, given by the matrix-valued 1-form $\omega = (\omega^i_j)$, where

$$\omega^i_j := \sum_\mu (y^{-1})^i_\mu dy^\mu_j = (y^{-1}dy)^i_j$$

The corresponding basic horizontal fields on FM are

$$X_k = \sum_\mu y^\mu_k \partial_\mu, \;\; k = 1, \ldots, n, \;\; \partial_\mu = \frac{\partial}{\partial x^\mu}.$$

Denote by θ^k be the canonical form of the frame bundle

$$\theta^k := \sum_\mu (y^{-1})^k_\mu dx^\mu = (y^{-1}dx)^k, \;\; k = 1, \ldots, n.$$

Then let

$$Y^j_i = \sum_\mu y^\mu_i \partial^j_\mu, \;\; i, j = 1, \ldots, n, \;\; \partial^j_\mu := \frac{\partial}{\partial y^\mu_j}$$

be the fundamental vertical vector fields associated to the standard basis of $\mathfrak{gl}_n(\mathbb{R})$ and generating the canonical right action of $\mathrm{GL}_n(\mathbb{R})$ on FM. At each point of FM, $\{X_k, Y^j_i\}$ and $\{\theta^k, \omega^i_j\}$ form bases of the tangent and cotangent space, dual to each other

$$\langle \omega^i_j, Y^l_k \rangle = \delta^i_k \delta^l_j, \;\; \langle \omega^i_j, X_k \rangle = 0,$$

$$\langle \theta^i, Y^l_k \rangle = 0, \;\; \langle \theta^i, X_j \rangle = \delta^i_j.$$

The group of diffeomorphism $\mathrm{Diff}_M = \mathrm{Diff}_{\mathbb{R}^n}$ acts on FM by the natural lift of the tautological action to the frame level

$$\widetilde{\varphi}(x, y) := (\varphi(x), \varphi'(x)y)$$

where $\varphi'(x)$ is Jacobi matrix $\varphi'(x)^i_j = \frac{\partial \varphi^i}{\partial x^j}$.

Viewing Diff_M as a discrete group we form the crossed product algebra

$$\mathfrak{A}_M := C_c^\infty(FM) \rtimes \mathrm{Diff}_M$$

As a vector space, it is spanned by monomials of the form fu^*_φ, where $f \in C^\infty(FM)$ and u^*_φ stands for φ^{-1}. The product is given by

$$f_1 u^*_{\varphi_1} \cdot f_2 u^*_{\varphi_2} = f_1(f_2 \circ \widetilde{\varphi}_1) u^*_{\varphi_2 \varphi_1}.$$

Since the right action of $\mathrm{GL}_n(\mathbb{R})$ on FM commutes with the action of Diff_M, at the Lie algebra level one has

$$u_\varphi Y_i^j u^*_\varphi = Y_i^j.$$

This allows to promote the vertical vector fields to derivations of $\mathfrak{A}_M$. Indeed, setting

$$Y_i^j(fu^*_\varphi) = Y_i^j(f)u^*_\varphi$$

the extended operators satisfy the derivation rule

$$Y_i^j(ab) = Y_i^j(a)b + aY_i^j(b), \quad a, b \in \mathfrak{A}_M.$$

We shall also prolong the horizontal vector fields to linear transformations $X_k \in \mathcal{L}(\mathfrak{A}_M)$ in similar fashion

$$X_k(fu^*_\varphi) = X_k(f)u^*_\varphi.$$

The resulting operators are no longer Diff_M-invariant. They satisfy

$$u_\varphi X_k u^*_\varphi = X_k - \gamma^i_{jk}(\varphi^{-1})Y_i^j,$$

where $\varphi \mapsto \gamma^i_{jk}(\varphi)$ is a group 1-cocycle on Diff_M with values in $C^\infty(FM)$. Specifically

$$\gamma^i_{jk}(\varphi)(x,y) = \sum_\mu (y^{-1} \cdots \varphi'(x)^{-1} \cdot \partial_\mu \cdot y)^i_j y^\mu_k$$

The above expression comes from the pull-back formula for the connection

$$\widetilde{\varphi}^*(\omega^i_j) = \omega^i_j + \gamma^i_{jk}(\varphi)\theta^k.$$

Now one uses the fact that $\{\theta^k, (\widetilde{\varphi}^{-1})^*(\omega^i_j)\}$ is the dual basis to $\{u_\varphi X_k u^*_\varphi, Y_i^j\}$.

As a consequence, the operators $X_k \in \mathcal{L}(\mathfrak{A}_M)$ are no longer derivations of $\mathfrak{A}_M$, but satisfy a non-symmetric Leibniz rule

$$X_k(a,b) = X_k(a)b + aX_k(b) + \delta^i_{jk}(a)Y_i^j(b), \quad a, b \in \mathfrak{A}_M,$$

where the linear operators $\delta^i_{jk} \in \mathcal{L}(\mathfrak{A}_M)$ are defined by

$$\delta^i_{jk}(fu^*_\varphi) = \gamma^i_{jk} f u^*_\varphi.$$

These are derivations, i.e.

$$\delta^i_{jk}(ab) = \delta^i_{jk}(a)b + a\delta^i_{jk}(b).$$

The operators $\{X_k, Y_j^i\}$ satisfy the commutation relations of the group of affine transformations of $\mathbb{R}^n$

$$[Y_i^j, Y_k^l] = \delta_k^j Y_i^l - \delta_i^l Y_k^j,$$

$$[Y_i^j, X_k] = \delta_k^j X_i,$$
$$[X_k, X_l] = 0.$$

The succesive commutators of the operators δ_{jk}^i with the X_l's yield new generations of

$$\delta_{jk|l_1\ldots l_r}^i := [X_{l_r}, \ldots [X_{l_1}, \delta_{jk}^i] \ldots],$$

which involve multiplication by higher order jets of diffeomorphisms

$$\delta_{jk|l_1\ldots l_r}^i (f u_\varphi^*) = \gamma_{jk|l_1\ldots l_r}^i f u_\varphi^*, \text{ where}$$

$$\delta_{jk|l_1\ldots l_r}^i := X_{l_r} \ldots X_{l_1}(\gamma_{jk}^i).$$

They commute among themselves

$$[\delta_{jk|l_1\ldots l_r}^i, \delta_{j'k'|l'_1\ldots l'_r}^{i'}] = 0.$$

It can be checked that the order of $\{j, k\}$ and $\{l_1, \ldots, l_r\}$ does not matter - in any case we get the same operator.

The commutators between Y_μ^λ's and δ_{jk}^i's can be obtained from explicit expression of the cocycle γ, by computing its derivatives in the direction of the vertical vector fields. One obtains

$$[Y_\mu^\lambda, \delta_{jk}^i] = \delta_j^\lambda \delta_{\mu k}^i + \delta_k^\lambda \delta_{j\mu}^i - \delta_\mu^i \delta_{jk}^\lambda$$

By induction

$$[Y_\mu^\lambda, \delta_{j_1 j_2|j_3\ldots j_r}^i] = \sum_{s=0}^{r} \delta_{j_s}^\lambda \delta_{j_1 j_2|j_3\ldots j_{s-i}\mu j_{s+1}\ldots j_r}^i - \delta_\mu^i \delta_{j_1 j_2|j_3\ldots j_r}^\lambda.$$

Definition 7.5. *Let $\mathcal{H}_n$ be the universal enveloping algebra of the Lie algebra $\mathfrak{h}_n$ with basis*

$$\{X_\lambda, Y_\nu^\mu, \delta_{jk|l_1\ldots l_r}^i \mid 1 \leqslant \lambda, \mu, \nu, i \leqslant n,\ 1 \leqslant j \leqslant k \leqslant n,\ 1 \leqslant l_1 \leqslant \ldots \leqslant l_r \leqslant n\}$$

and the following presentation

$$\begin{aligned}
[X_k, X_l] &= 0,\\
[Y_i^j, Y_k^l] &= \delta_k^j Y_i^l - \delta_i^l Y_k^j,\\
[Y_i^j, X_k] &= \delta_k^j X_i,\\
[X_{l_r}, \delta_{jk|l_1\ldots l_{r-1}}^i] &= \delta_{jk|l_1\ldots l_r}^i,\\
[Y_\nu^\lambda, \delta_{j_1 j_2|j_3\ldots j_r}^i] &= \sum_{s=0}^{r} \delta_{j_s}^\lambda \delta_{j_1 j_2|j_3\ldots j_{s-i}\nu j_{s+1}\ldots j_r}^i - \delta_\nu^i \delta_{j_1 j_2|j_3\ldots j_r}^\lambda,\\
[\delta_{jk|l_1\ldots l_r}^i, \delta_{j'k'|l'_1\ldots l'_r}^{i'}] &= 0.
\end{aligned}$$

We shall endow $\mathcal{H}_n := U(\mathfrak{h}_n)$ with a canonical Hopf structure, which is non-commutative, and therefore different from the standard structure of a universal enveloping algebra.

Proposition 7.6.

1. *The formulae*

$$\begin{aligned}\Delta X_k &= X_k \otimes 1 + 1 \otimes X_k + \delta^i_{jk} \otimes Y^j_i,\\ \Delta Y^j_i &= Y^j_i \otimes 1 + 1 \otimes Y^j_i,\\ \Delta \delta^i_{jk} &= \delta^i_{jk} \otimes 1 + 1 \otimes \delta^i_{jk},\end{aligned}$$

uniquely determine a coproduct $\Delta\colon \mathcal{H}_n \to \mathcal{H}_n \otimes \mathcal{H}_n$, which makes $\mathcal{H}_n$ a bialgebra with respect to the product $m\colon \mathcal{H}_n \otimes \mathcal{H}_n \to \mathcal{H}_n$ and the counit $\varepsilon\colon \mathcal{H}_n \to \mathbb{C}$ inherited from $U(\mathfrak{h}_n)$.

2. *The formulae*

$$\begin{aligned}S(X_k) &= -X_k + \delta^i_{jk} Y^j_i,\\ S(Y^j_i) &= -Y^j_i,\\ S(\delta^i_{jk}) &= -\delta^i_{jk},\end{aligned}$$

uniquely determine an anti-homomorphism $S\colon \mathcal{H}_n \to \mathcal{H}_n$, which provides the antipode that turns $\mathcal{H}_n$ into a Hopf algebra.

The notation is justified while one proves that the subalgebra of $\mathcal{L}(\mathfrak{A}_M)$ generated by the linear operators $\{X_k, Y^i_j, \delta^i_{jk} \mid i,j,k = 1,\dots,n\}$ is isomorphic to the algebra $\mathcal{H}_n$. The action of $\mathcal{H}_n$ turns $\mathfrak{A}_n$ into a left $\mathcal{H}_n$-module algebra. Morover to any element $h^1 \otimes \ldots \otimes h^p \in \mathcal{H}^p_n$ we can associate a multilinear differential operator T acting on $\mathfrak{A}_M$ as follows

$$T(h^1 \otimes \ldots \otimes h^p)(a^1, \ldots, a^p) = h^1(a_1) \ldots h^p(a_p).$$

The linearization $T\colon T\mathcal{H}^p_n \to \mathcal{L}(\mathfrak{A}^{\otimes p}_M, \mathfrak{A}_M)$ of this assignment is injective for each $p \in \mathbb{N}$.

Chapter 8

Bott periodicity and index theorem

8.1 Bott periodicity

In one of its forms the Bott periodicity theorem ([b-r59]) can be stated as

$$\pi_j(\mathrm{GL}(n,\mathbb{C})) = \begin{cases} 0 & j \text{ even,} \\ \mathbb{Z} & j \text{ odd} \end{cases}$$

for $j = 0, 1, 2, \ldots, 2n-1$. This is the original form of Bott theorem. It has reformulation, for example in the language of topological K-theory or the K-theory of C*-algebras where it gives an isomorphism

$$\mathrm{K}_j(A) \cong \mathrm{K}_j(A \otimes C_0(\mathbb{R}^2)).$$

The homotopy groups are constructed as follows. We take maps from the sphere

$$S^j \xrightarrow{f} X$$

preserving base points $p_0 \in S^j$, $x_0 \in X$, i.e. $f(p_0) = x_0$. On the set of homotopy classes of such maps we give a group structure by composing with the map contracting the equator of the sphere S^j making it a wedge of two copies of S^j.

$$S^j \to S^j \vee S^j \xrightarrow{f \vee g} X.$$

The sphere S^n can be described as

$$S^n = \{(t_1, t_2, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid t_1^2 + \ldots + t_{n+1}^2 = 1\},$$

however if n is odd we can embed it in the complex space $\mathbb{C}^n$

$$S^{2r-1} = \{(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{C}^r \mid \sum_{j=1}^{r} \overline{\lambda_j}\lambda_j = 1\}.$$

For a map $f\colon S^n \to S^n$ define degree $\deg(f) \in \mathbb{Z}$ as follows. On homology f induces a map

$$f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}) \to \mathrm{H}_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$$

Then for each $u \in \mathrm{H}_n(S^n; \mathbb{Z})$

$$f_*(u) = \deg(f)u.$$

We shall define for $j = 1, 3, 5, \ldots, 2n-1$ a homomorphism of abelian groups

$$\beta\colon \pi_j(\mathrm{GL}(n, \mathbb{C})) \to \mathbb{Z}.$$

First consider $j = 2n-1$ and a map

$$f\colon S^{2n-1} \to \mathrm{GL}(n, \mathbb{C}).$$

For $p \in S^{2n-1}$

$$f(p) = \begin{bmatrix} \lambda_{11}(p) & \lambda_{12}(p) & \ldots & \lambda_{1n}(p) \\ \lambda_{21}(p) & \lambda_{22}(p) & \ldots & \lambda_{2n}(p) \\ \vdots & \vdots & & \vdots \\ \lambda_{n1}(p) & \lambda_{n2}(p) & \ldots & \lambda_{nn}(p) \end{bmatrix}$$

We take first column and divide it by the norm i.e.

$$f_1(p) := (\lambda_{11}(p), \lambda_{21}(p), \ldots, \lambda_{n1}(p))/(\sum_{j=1}^{n} \overline{\lambda_{ji}(p)}\lambda_{ji}(p))^{\frac{1}{2}}.$$

This gives a map $f_1\colon S^{2n-1} \to S^{2n-1}$. Now define

$$\beta(f) := \frac{\deg(f_1)}{(n-1)!} \in \mathbb{Z}.$$

It os a part of Bott theorem that this number actually is an integer.

For all n the unitary subgroup $U(n) \in \mathrm{GL}(n, \mathbb{C})$ is a maximal compact subgroup, and the inclusion induces a homotopy equivalence. From the fibering

$$U(n-1) \to U(n) \to S^{2n-1}$$

and the homotopy exact sequence we get that for $0 < j < 2n-1$ the homotopy

$$\pi_j(U(n-1)) \cong \pi_j(U(n)).$$

Lemma 8.1. *If $j = 2r-1$ with $1 \leqslant r < n$, then*

$$f\colon S^{2r-1} \to GL(n, \mathbb{C})$$

is homotopic to a map

$$\tilde{f}\colon S^{2r-1} \to \mathrm{GL}(n, \mathbb{C})$$

of the form

$$\tilde{f}(p) = \begin{bmatrix} \lambda_{11}(p) & \lambda_{12}(p) & \ldots & \lambda_{1r}(p) & 0 & 0 & \ldots & 0 \\ \lambda_{21}(p) & \lambda_{22}(p) & \ldots & \lambda_{2r}(p) & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_{r1}(p) & \lambda_{r2}(p) & \ldots & \lambda_{rr}(p) & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Define

$$\tilde{f}_1 \colon S^{2r-1} \to S^{2r-1}$$

$$\tilde{f}_1(p) := (\lambda_{11}(p), \lambda_{21}(p), \ldots, \lambda_{r1}(p))/(\sum_{j=1}^{r} \overline{\lambda_{ji}(p)}\lambda_{ji}(p))^{\frac{1}{2}}.$$

and as before

$$\beta(f) := \frac{\deg(\tilde{f}_1)}{(r-1)!} \in \mathbb{Z}.$$

This number is an integer, which is part of the

Theorem 8.2. *For $j = 1, 3, 5, \ldots, 2n-1$*

$$\beta \colon \pi_j(\mathrm{GL}(n, \mathbb{C})) \to \mathbb{Z}$$

is an isomorphism.

8.2 Elliptic operators

Let X be a C^∞-manifold (Hausdorff, second countable, finite dimensional, without boundary), and $E \to X$ a complex C^∞-vector bundle. For each $p \in X$, E_p is a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} E_p < \infty$.

By $C^\infty(X, E)$ we denote a $\mathbb{C}$-vector space of all C^∞ sections of E,

$$\dim_{\mathbb{C}} C^\infty(X, E) = \infty.$$

If E^0, E^1 are two vector bundles on X, then an *elliptic differential operator* (or elliptic $\psi D0$) is a $\mathbb{C}$-linear map

$$D \colon C^\infty(X, E^0) \mapsto C^\infty(X, E^1)$$

which is differential operator (or an $\psi D0$) and which satisfies a condition called *ellipticity*.

Example 8.3. Laplacian on $X = \mathbb{R}^n$, $E^0 = E^1 = X \times \mathbb{C}$

$$D = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

This operator is elliptic of order 2, because polynomial

$$\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$$

has no real zeros except $(0, 0, \ldots, 0)$.

Example 8.4. Cauchy-Riemann operator on $X = \mathbb{R}^2$, $E^0 = E^1 = X \times \mathbb{C}$

$$D = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}.$$

This operator is elliptic of order 1, because polynomial $\xi_1 + i\xi_2$ has no real zeroes except $(0, 0)$. If we denote

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

then $\bar{\partial} f = 0$ if and only if f is holomorphic.

Let
$$D\colon C^\infty(X,E^0)\to C^\infty(X,E^1)$$
be a differential operator of order r. To each $p\in X$ and $\xi\in T_p^*X=\mathrm{Hom}_{\mathbb{R}}(T_pX,\mathbb{R})$ we shall associate a map of $\mathbb{C}$-vector spaces
$$\sigma(\xi,D)\colon E_p^0\to E_p^1.$$
To do this, given $v\in E_p^0$ and $\xi\in T_p^*X$, choose

1. Section $s\in C^\infty(X,E^0)$ with $s(p)=v$,
2. C^∞-function $f\colon X\to\mathbb{R}$ with $f(p)=0$ and $(df)(p)=\xi$.

Lemma 8.5. *$D(f^rs)(p)$ depends only on D, ξ, v; and does not depend on the choice of s and f.*

Set
$$\sigma(\xi,D)v:=D(f^rs)(p),$$
$$\sigma(\xi,D)\colon E_p^0\to E_p^1.$$

Definition 8.6. *A differential operator of order r*
$$D\colon C^\infty(X,E^0)\to C^\infty(X,E^1)$$
is elliptic *if whenever $p\in X$ and $0\neq\xi\in T_p^*X$, then $\sigma(\xi,D)\colon E_p^0\to E_p^1$ is an isomorphism of $\mathbb{C}$-vector spaces.*

Example 8.7. Let X be a manifold. We consider complex valued differential forms on X i.e. elements of
$$\Lambda^j(T_{\mathbb{C}}^*X),\ \ T_{\mathbb{C}}^*X:=T^*X\otimes_{\mathbb{R}}\mathbb{C},\ \ j=0,1,2,\dots.$$
The de Rham operator
$$C^\infty(X,\Lambda^j(T_{\mathbb{C}}^*X))\xrightarrow{d}C^\infty(X,\Lambda^{j+1}(T_{\mathbb{C}}^*X))$$
is a differential operator of order 1. For $p\in X$, $\xi\in T_p^*X$, $v\in(\Lambda^j(T_{\mathbb{C}}^*X))_p$ choose a form ω such that
$$\omega(p)=v.$$
Then choose a function $f\colon X\to\mathbb{R}$ such that
$$(df)(p)=\xi.$$
We have
$$d(f\omega)=df\wedge\omega+fd\omega$$
$$d(f\omega)_p=(df\wedge\omega)_p\text{ because }f(p)=0,$$
and thus the map $\sigma(\xi,d)$ is given by
$$v\mapsto\xi\wedge v\text{ because }(df)(p)=\xi.$$
This operator is not elliptic. However if we take
$$\bigoplus_j C^\infty(X,\Lambda^{2j}(T_{\mathbb{C}}^*X))\xrightarrow{d+d^*}\bigoplus_j C^\infty(X,\Lambda^{2j-1}(T_{\mathbb{C}}^*X)),$$

where $d^*\colon \Lambda^j(T^*_{\mathbb{C}}X) \to \Lambda^{j-1}(T^*_{\mathbb{C}}X)$ is formal adjoint to d, then $d+d^*$ is elliptic, and $\sigma(\xi, d+d^*)$ is given by

$$v \mapsto \xi \wedge v + \iota(\xi)v,$$

where $\iota(\xi)$ is a contraction of form by ξ.

Lemma 8.8. *If X is compact and D is elliptic, then*

$$\dim_{\mathbb{C}}(\ker D) < \infty, \text{ and } \dim_{\mathbb{C}}(\operatorname{coker} D) < \infty.$$

Definition 8.9. *If X is compact and D is elliptic, then*

$$\operatorname{Index}(D) := \dim_{\mathbb{C}}(\ker D) - \dim_{\mathbb{C}}(\operatorname{coker} D).$$

Theorem 8.10 (Atiyah-Singer)**.** *If X is compact and D is elliptic, then*

$$\operatorname{Index}(D) = (\textit{topological formula}).$$

Example 8.11. Toeplitz operator

$$X = S^1 = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^2 + a_2^2 = 1\}$$

Take a trivial bundles $E^0 = E^1 = S^1 \times \mathbb{C}$. Sections of these are just smooth functions on S^1

$$C^\infty(S^1, E^0) = C^\infty(S^1, E^0) = C^\infty(S^1).$$

Any $u \in C^\infty(S^1)$, $u\colon S^1 \to \mathbb{C}$, has a Fourrier series

$$u = \sum_{n=-\infty}^{n=\infty} a_n e^{in\theta}, \;\; a_n \in \mathbb{C}.$$

We have a decomposition

$$\begin{aligned} C^\infty(S^1) &= C^\infty_+(S^1) \oplus C^\infty_-(S^1) \\ C^\infty_+(S^1) &= \big\{u \in C^\infty(S^1) \mid a_n = 0 \;\forall\, n < 0\big\} \\ C^\infty_-(S^1) &= \big\{u \in C^\infty(S^1) \mid a_n = 0 \;\forall\, n \geqslant 0\big\} \end{aligned}$$

Denote the projection

$$P\colon C^\infty(S^1) \to C^\infty_+(S^1),$$

$$P\left(\sum_{n=-\infty}^{n=\infty} a_n e^{in\theta}\right) = \sum_{n=0}^{n=\infty} a_n e^{in\theta}.$$

Fix a C^∞ function $\alpha\colon S^1 \to \mathbb{C}$ and define

$$T_\alpha(u) = P(\alpha u), \;\; \alpha(u)(\lambda) = (\alpha\lambda)u(\lambda), \;\; u \in C^\infty(S^1), \;\; \lambda \in S^1.$$

Define operator

$$D_\alpha\colon C^\infty(S^1) \to C^\infty(S^1)$$

using decomposition $C^\infty(S^1) = C^\infty_+(S^1) \oplus C^\infty_-(S^1)$

$$D_\alpha = \begin{bmatrix} T_\alpha & 0 \\ 0 & I \end{bmatrix}$$

$$D_\alpha u = \begin{cases} T_\alpha u & u \in C^\infty_+(S^1) \\ u & u \in C^\infty_-(S^1) \end{cases}$$

Proposition 8.12. *1. $D_\alpha\colon C^\infty(S^1) \to C^\infty(S^1)$ is a pseudo-differential operator ($\psi D0$),*

2. D_α is elliptic iff. $\alpha(\lambda) \neq 0$ for all $\lambda \in S^1$,

3. if $\alpha(\lambda) \neq 0$ for all $\lambda \in S^1$, then

$$\mathrm{Index}(D_\alpha) = -(\textit{winding number})(\alpha) = -\frac{1}{2\pi i}\int_{S^1} \frac{d\alpha}{\alpha}.$$

Remark 8.13. The winding number is also present in the Bott periodicity theorem. Indeed

$$\pi_1(\mathrm{GL}(n, \mathbb{C})) \cong \mathbb{Z}$$

and the isomorphism is given by winding number of

$$S^1 \mapsto \mathrm{GL}(n, \mathbb{C}) \xrightarrow{\det} \mathbb{C}^*.$$

Example 8.14. Classical Riemann-Roch theorem. Let X be connected Riemann surface, i.e compact connected complex analitic manifold with $\dim_{\mathbb{C}} X = 1$. The genus of X is a number of holes which is equal to

$$g = \frac{1}{2} \operatorname{rank} \mathrm{H}_1(X; \mathbb{Z}).$$

Assume we are given a complex analitic line bundle L on X. For each $p \in X$, L_p is a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} L_p = 1$. The degree $\deg(L)$ of this bundle can be defined as follows. Choose any meromorphic section u of L. Then the order of u at $p \in X$ is defined as

$$\mathrm{ord}_p(u) = \begin{cases} 0 & \text{if } p \in X \text{ is neither a zero nor a pole of } u, \\ n & \text{if } p \in X \text{ is a zero of order } n \text{ of } u, \\ n & \text{if } p \in X \text{ is a pole of order } n \text{ of } u. \end{cases}$$

Then

$$\deg(L) := \sum_{p \in X} \mathrm{ord}_p(u).$$

Lemma 8.15. $\deg(L)$ *does not depend on the choice of meromorphic section* u.

Remark 8.16. Another way to describe the degree is to evaluate first Chern class of bundle L on the fundamental class of the base X

$$\deg(L) = \langle c_1(L), [X] \rangle \in \mathbb{Z}.$$

Consider operator

$$\bar\partial\colon C^\infty(X, L) \to C^\infty(X, L \otimes \Lambda^{0,1} T^*_{\mathbb{C}} X)$$

given for $s = f\alpha$ by

$$\bar\partial s = \frac{\partial f}{\partial \bar z} \otimes d\bar z$$

$$z = x + iy, \;\; d\bar z = dx - idy, \;\; \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

and $C^\infty(X, L \otimes \Lambda^{0,1} T^*_{\mathbb{C}} X)$ are complex valued 1-forms of type $(0,1)$. Then $u \in C^\infty(X, L)$ is holomorphic iff. $\bar\partial u = 0$.

Theorem 8.17 (Riemann-Roch).

$$\mathrm{Index}(\bar\partial) = \deg(L) - g + 1.$$

8.2.1 Pseudodifferenital operators

When we consider non compact manifolds the Atiyah-Singer index theorem must be stated using elliptic pseudodifferential operators.

Let $U \in \mathbb{R}^n$ be an open subset, $m \in \mathbb{Z}$. Define a subspace

$$S^m(U) \subset C^\infty(U \times \mathbb{R}^n)$$

$$\phi \in S^m(U), \quad \phi \colon U \times \mathbb{R}^n \to \mathbb{C}, \quad (x, \xi) \mapsto \phi(x, \xi),$$

by the condition

Function $\phi \in S^m(U)$ if and only if for every compact subset $\Delta \subset U$ and for all multiindices α, β there exists constant $C_{\phi,\alpha,\beta,\Delta}$ with

$$\left| D_x^\beta D_\xi^\alpha \phi(x,\xi) \leqslant C_{\phi,\alpha,\beta,\Delta}(1 + |\xi|)^{m-|\alpha|}. \right|$$

Constant $C_{\phi,\alpha,\beta,\Delta}$ depends on $\phi, \alpha, \beta, \Delta$,

$$D_\xi^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \left(\frac{1}{i}\frac{\partial}{\partial \xi_2}\right)^{\alpha_2} \cdots \left(\frac{1}{i}\frac{\partial}{\partial \xi_n}\right)^{\alpha_n},$$

$$|\alpha| = \sum_{j=1}^n \alpha_j.$$

Now define a subspace

$$S_0^m(U) \subset S^m(U)$$

Function $\phi \in S_0^m(U)$ if and only if

$$\lim_{\lambda \to \infty} \frac{\phi(x, \lambda, \xi)}{\lambda^m}$$

exists.

For $\phi \in S^m(U)$ set

$$\sigma_\phi(x, \xi) = \lim_{\lambda \to \infty} \frac{\phi(x, \lambda, \xi)}{\lambda^m}.$$

Then σ_ϕ is a C^∞ function defined on $U \times (\mathbb{R}^n \times \{0\})$

$$\sigma_\phi \colon U \times (\mathbb{R}^n - \{0\}) \to \mathbb{C}$$

and σ_ϕ is homogeneous of degree m in ξ. We take it as a symbol of the following operator

$$P_\phi \colon C_C^\infty(U) \to C_C^\infty(U)$$

$$P_\phi(x) = \frac{1}{2\pi} \int \phi(x, \xi) \hat{f}(\xi) e^{\langle x, \xi \rangle} d\xi$$

8.3 Topological formula of Atiyah-Singer

Let X be compact hausdorff topological space, E $\mathbb{C}$-vector bundle on X. To describe the topological index formula one has to introduce Chern character $\mathrm{ch}(E)$ and the Todd class $\mathrm{td}(E)$, both being elements of $\bigoplus_j \mathrm{H}^j(X;\mathbb{Q})$.

For line bundle $L \to X$

$$\mathrm{ch}(E) = e^{c_1(L)} = 1 + c_1(L) + \frac{c_1^2(L)}{2} + \dots$$

For a sum of a line bundles $E = L_1 \oplus L_2 \oplus \dots \oplus L_n$

$$\begin{aligned}\mathrm{ch}(L_1 \oplus L_2 \oplus \dots \oplus L_n) &= e^{c_1(L_1)} + e^{c_1(L_2)} + \dots + e^{c_1(L_n)} \\ &= \mathrm{ch}(L_1) + \mathrm{ch}(L_2) + \dots + \mathrm{ch}(L_n).\end{aligned}$$

General formula can be obtained using above and splitting principle. Just as Chern character is based on a function e^x, the Todd class is based on a function

$$\begin{aligned}\frac{x}{1-e^{-x}} &= \frac{x}{1 - [1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \dots]} \\ &= \frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \dots.\end{aligned}$$

Now for a line bundle $L \to X$ we have

$$\mathrm{td}(L) = \frac{c_1(L)}{1 - e^{-c_1(L)}}$$

and for a sum of line bundles

$$\mathrm{td}(L_1 \oplus L_2 \oplus \dots \oplus L_n) = \mathrm{td}(L_1) \cup \mathrm{td}(L_2) \cup \dots \cup \mathrm{td}(L_n).$$

Let $E^0, E^1 \to X$ be vector bundles on X, $\psi\colon E^0 \to E^1$ a vector bundle map.

$$\mathrm{supp}(\psi) := \{\, p \in X \mid \psi\colon E^0_p \to E^1_p \text{ is not an isomorphism of } \mathbb{C}\text{-vector spaces}\}$$

Assume $\mathrm{supp}(\psi)$ is compact. Then

$$\mathrm{ch}(\psi) \in \mathrm{H}^{2j}_c(X;\mathbb{Q}).$$

Example 8.18. Let M be a compact C^∞-manifold with no boundary. For a pair of vector bundles

$$\begin{tikzcd} E^0 \arrow[rr] \arrow[dr] & & E^1 \arrow[dl] \\ & M & \end{tikzcd}$$

let

$$D\colon C^\infty(M, E^0) \to C^\infty(M, E^1)$$

be an elliptic operator. The cotangent bundle

$$\pi\colon T^*M \to M, \ \ \pi(T^*_pM) = p$$

induces pullback bundles on T^*M

$$\begin{tikzcd} \pi^*E^0,\ \pi^*E^1 \arrow[r] \arrow[d] & E^0,\ E^1 \arrow[d] \\ T^*M \arrow[r, "\pi"] & M \end{tikzcd}$$

The symbol of D is a map of vector bundles

$$\sigma\colon \pi^*E^0 \to \pi^*E^1$$

and σ has compact support. Then

$$\mathrm{ch}(\sigma) \in \mathrm{H}_c^{2j}(T^*M;\mathbb{Q})$$

and

$$\mathrm{Index}(D) = (ch(\sigma) \cup \pi^*\,\mathrm{td}(TM \otimes_{\mathbb{R}} \mathbb{C}))[T^*M].$$

In the proof of index theorem one uses

Lemma 8.19. *Let A be an abelian group. Let $\varphi\colon A \to \mathbb{Z}$ and $\tau\colon A \to \mathbb{Z}$ be homomorphisms. Assume that φ is an isomorphism. Assume also that there exists $a \in A$, with $a \neq 0$ and $\varphi(a) = \tau(a)$. Then $\varphi = \tau$.*

Now we shall describe apriopriate abelian group A.

Definition 8.20. *A* symbol datum *is a 4-tuple (M, F^0, F^1, σ) such that*

1. *M is a C^∞-manifold, finite dimensional, Hausdorff, second countable, with $\pi_0(M)$ finite, and with no boundary,*
2. *F^0, F^1 are complex vector bundles on T^*M,*
3. *σ is a vector bundle map $F^0 \to F^1$ with* $\mathrm{supp}(\sigma)$ *compact.*

On a set of such 4-tuples we will define an equivalence relation $\sim$, and then put

$$A := \{(M, F^0, F^1, \sigma)\}/\sim$$
$$(M, F^0, F^1, \sigma) + (W, E^0, E^1, \theta) = (M \cup W, F^0 \cup E^0, F^1 \cup E^1, \sigma \cup \theta).$$

Now the two homomorphisma which are mentioned in the Lemma 8.19 are as follows

$$\varphi\colon A \to \mathbb{Z}$$
$$\varphi(M, F^0, F^1, \sigma) := (ch(\sigma) \cup \pi^*\,\mathrm{td}(TM \otimes_{\mathbb{R}} \mathbb{C}))[T^*M],$$
$$\tau\colon A \to \mathbb{Z}$$
$$\tau(M, F^0, F^1, \sigma) = \mathrm{Index}(D),$$

where D is any elliptic operator on M whose symbol datum is (M, F^0, F^1, σ).

Remark 8.21. If M is non-compact then D will be an elliptic pseudodifferential operator (ΨDO) on M which is trivial at infinity.

The equivalence relation $\sim$ betwen symbol data is described in five steps

1. isomorphism,
2. homotopy of σ,
3. direct sum - disjoint union,
4. excision,
5. vector bundle modification.

8.3.1 Isomorphism

.

4-tuples (M, F^0, F^1, σ) and (W, E^0, E^1, θ) are isomorphic

$$(M, F^0, F^1, \sigma) \cong (W, E^0, E^1, \theta)$$

if and only if there exists a diffeomorphism

$$h\colon M \to W$$

such that one can assign in a continuous way, to each $\xi \in T^*W$ an isomorphisms of vector spaces

$$\eta^0_\xi\colon E^0_\xi \xrightarrow{\cong} F^0_{h'\xi},$$

$$\eta^1_\xi\colon E^1_\xi \xrightarrow{\cong} F^1_{h'\xi},$$

with commutativity in the diagram

$$\begin{tikzcd} E^0_\xi \arrow[r, "\eta^0_\xi"] \arrow[d, "\theta"'] & F^0_{h'\xi} \arrow[d, "\sigma"] \\ E^1_\xi \arrow[r, "\eta^1_\xi"] & F^1_{h'\xi} \end{tikzcd}$$

where

$$h'\colon T^*W \to T^*M$$

is the map of cotangent bundles induced by $h\colon M \to W$.

8.3.2 Homotopy of σ

.

We consider homotopies between symbol data (M, F^0, F^1, σ) such that M, F^0, F^1 is fixed, and for $0 \leqslant t \leqslant 1$ we have family of symbols σ_t. Then

$$(M, F^0, F^1, \sigma_0) \sim (M, F^0, F^1, \sigma_1).$$

Furthermore the set of $(\sigma, t) \in T^*M \times [0, 1]$ such that

$$\sigma_t(\xi)\colon F^0_\xi \to F^1_\xi$$

is not an isomorphism of $\mathbb{C}$-vector spaces, is compact.

8.3.3 Direct sum - disjoint union

.

Let (M, F^0, F^1, σ) and (M, E^0, E^1, θ) be two symbol data with the same M. Then

$$(M, F^0, F^1, \sigma) \cup (M, F^0, F^1, \theta) \sim (M, F^0 \oplus E^0, F^1 \oplus E^1, \sigma \oplus \theta).$$

8.3.4 Excision

. Let (M, F^0, F^1, σ) be a symbol data. Recall that

$$\operatorname{supp}(\sigma) = \{ \xi \in T^*M \mid \sigma(\xi) \colon F^0_\xi \to F^1_\xi \text{ is not an isomorphism of } \mathbb{C} \text{ vector spaces}\}.$$

Denote by π the projection $T^*M \to M$. Let $U \subset M$ be an open set with

$$\pi(\operatorname{supp}(\sigma)) \subset U.$$

Then

$$(M, F^0, F^1, \sigma) \sim (U, F^0|_{T^*U}, F^1|_{T^*U}, \sigma|_{T^*U}).$$

8.3.5 Vector bundle modification

.

Let (M, F^0, F^1, σ) be a symbol data and E any C^∞-vector bundle on M. Then we describe another 4-tuples $(E, -, -, -)$ which will be equivalent to the given one. First we give a basic example of symbol datum.

Example 8.22. For each $n = 1, 3, 5, \ldots$ w shall define a symbol datum $\mathbb{R}^{n\wedge}$. For $n = 1$ we take

$$\mathbb{R}^{1\wedge} = (\mathbb{R}, (T^*\mathbb{R}) \times \mathbb{C}, (T^*\mathbb{R}) \times \mathbb{C}, \cdot)$$

One has

$$T^*\mathbb{R} = \mathbb{R} \times \mathbb{R} = \mathbb{C}$$

$$(t_1, t_2 dx) \leftrightarrow t_1 + it_2, \quad t_1, t_2 \in \mathbb{R}$$

and $\cdot$ denotes multiplication on the second coordinate

$$(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_1 \cdot \lambda_2).$$

For $n > 1$ we put

$$\mathbb{R}^{n\wedge} := \mathbb{R}^{1\wedge} \times \mathbb{R}^{1\wedge} \times \ldots \times \mathbb{R}^{1\wedge}.$$

More explicitly

$$\mathbb{R}^{n\wedge} = (\mathbb{R}^n, (T^*\mathbb{R}^n) \times \Lambda^{ev}\mathbb{C}^n, (T^*\mathbb{R}^n) \times \Lambda^{odd}\mathbb{C}^n, \wedge + \iota),$$

where

$$T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n,$$

$$\Lambda^{ev}\mathbb{C}^n = \bigoplus_j \Lambda^{2j}\mathbb{C}^n, \quad \Lambda^{odd}\mathbb{C}^n = \bigoplus_j \Lambda^{2j+1}\mathbb{C}^n$$

$$\wedge + \iota \colon \mathbb{C}^n \times \Lambda^{ev}\mathbb{C}^n \to \mathbb{C}^n \times \Lambda^{odd}\mathbb{C}^n$$

$$(v, w) \mapsto (v, v \wedge w + \iota(v)w), \quad v \in \mathbb{C}^n, \ w \in \Lambda^{ev}\mathbb{C}^n.$$

Now in the special case of trivial bundle $E = M \times \mathbb{R}^n$ we have

$$(M, F^0, F^1, \sigma) \sim (M, F^0, F^1, \sigma) \times \mathbb{R}^{n\wedge}.$$

However this construction has enough naturality, so that it can be done even when E is not trivial. Let $E \to M$ be smooth complex vector bundle. Then we have

$$\begin{array}{ccc} E & & T^*E \\ \downarrow & & \downarrow{\scriptstyle\rho} \\ M & & T^*M \end{array}$$

and T^*E is a $\mathbb{C}$-vector bundle on T^*M.

Set

$$\Lambda^{ev} := \bigoplus_j \Lambda^{2j}(T^*E), \quad \Lambda^{odd} := \bigoplus_j \Lambda^{2j+1}(T^*E)$$

and then form a symbol datum

$$(E, \rho^*[(F^0 \hat{\otimes} \Lambda^{ev}) \oplus (F^1 \hat{\otimes} \Lambda^{odd})], \rho^*[(F^1 \hat{\otimes} \Lambda^{ev}) \oplus (F^0 \hat{\otimes} \Lambda^{odd})], \sigma \#(\wedge + \iota))$$
$$\sim (M, F^0, F^1, \sigma).$$

In the formula above we use external tensor product of vector bundles and external tensor product of symbols, which we describe next. For a pair of vector bundles $E \to X$, $F \to Y$ their external tensor product is a bundle

$$\begin{array}{c} E \hat{\otimes} F \\ \downarrow \\ X \times Y \end{array}$$

with fiber

$$(E \hat{\otimes} F)_{(x,y)} = E_x \otimes_{\mathbb{C}} F_y.$$

Then the external product of symbol data is defined as follows

$$(M, F^0, F^1, \sigma) \times (W, E^0, E^1, \theta) :=$$
$$M \times W, (F^0 \hat{\otimes} E^0) \oplus (F^1 \hat{\otimes} E^1), (F^1 \hat{\otimes} E^0) \oplus (F^0 \hat{\otimes} E^1), \sigma \# \theta)$$
$$T^*(M \times W) = T^*M \times T^*W,$$
$$\sigma \# \theta = \begin{bmatrix} \sigma \hat{\otimes} I_{E^0} & -I_{F^1} \hat{\otimes} \theta^* \\ I_{F^0} \hat{\otimes} \theta & \sigma^* \hat{\otimes} I_{E^1}, \end{bmatrix}$$

where I is the identity map.

$$\mathrm{supp}(\sigma \# \theta) = \mathrm{supp}(\sigma) \cup \mathrm{supp}(\theta).$$

Now we can put

$$A := \{(M, F^0, F^1, \sigma)\} / \sim .$$

A is an abelian group with the addition defined as

$$(M, F^0, F^1, \sigma) + (W, E^0, E^1, \theta) = (M \cup W, F^0 \cup E^0, F^1 \cup E^1, \sigma \cup \theta),$$

the inverse

$$-(M, F^0, F^1, \sigma) = (M, F^1, F^0, \sigma^*),$$

and the identity being any datum (M, F^0, F^1, σ) with $\mathrm{supp}(\sigma) = \emptyset$, for example

$$(M, F, F, \mathrm{id}).$$

Now we can state and proof the

Theorem 8.23 (Atiyah-Singer)**.** *Let $E^0, E^1 \to M$ be smooth $\mathbb{C}$-vector bundles on a smooth manifold M. For any elliptic pseudodifferential operator*

$$D \colon C^\infty(M, E^0) \to C^\infty(M, E^1),$$

with symbol datum

$$(T^*M, \pi^*E^0, \pi^*E^1, \sigma), \quad \pi \colon T^*M \to M$$

$$\mathrm{Index}(D) = (ch(\sigma) \cup \pi^* \operatorname{td}(TM \otimes_{\mathbb{R}} \mathbb{C}))[T^*M].$$

Remark 8.24. We do not assume that M is compact. If it is so, then one can use elliptic differential operator D.

Proof. (An outline) By the lemma (8.19) it is sufficient to show that the two maps

$$\varphi \colon A \to \mathbb{Z}, \quad \tau \colon A \to \mathbb{Z}$$

given by the formulas

$$\varphi(M, F^0, F^1, \sigma) := (ch(\sigma) \cup \pi^* \operatorname{td}(TM \otimes_{\mathbb{R}} \mathbb{C}))[T^*M],$$

$$\tau(M, F^0, F^1, \sigma) = \mathrm{Index}(D),$$

satisfy assumptions of the lemma, and therefore $\varphi = \tau$. To do this we have to check that each of them is

1. well defined, that is compatible with the equivalence relation $\sim$,
2. integer valued,
3. additive, that is homomorbism of abelian groups.

Moreover for φ we have to check that it is "1-1" and "onto". Finally that there exists nonzero element of A on which both agree.

It is easy to check that φ is well defined, additive, and $\varphi(\mathbb{R}^{1\wedge}) = 1$, that is φ is onto. It follows from the naturality of the Chern character and the Todd class used in the formula. The more difficult part is to check that it is integer valued and one to one. To prove that it is so one needs a

Lemma 8.25. *Any symbol datum (M, F^0, F^1, σ) is equivalent to a symbol datum whose manifold is $\mathbb{R}^n$*

$$(M, F^0, F^1, \sigma) \sim (\mathbb{R}^n, G^0, G^1, \eta)$$

Proof. (An outline) Embed M into $\mathbb{R}^n$ for sufficiently large n in sauch way that M is a closed subset and C^∞ manifold of $\mathbb{R}^n$.

Next step is to use the normal bundle ν of M in $\mathbb{R}^n$ and do vector bundle modification by ν

$$(M, F^0, F^1, \sigma) \sim (\nu, -, -, -).$$

Now ν is an open subset of $\mathbb{R}^n$ and one can do excision "in reverse"

$$(\nu, -, -, -) \sim (\mathbb{R}^n, -, -, -).$$

□

Any vector bundle on $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ is trivial so we can assume that

$$(M, F^0, F^1, \sigma) \sim (\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta).$$

Furthermore we can assume that $l \geqslant n$, for if $l < n$ let $r := n - l$, and then

$$(\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta) \sim$$
$$(\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta) \cup (\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \mathrm{id}) \sim$$
$$(\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^{l+r}, (T^*\mathbb{R}^n) \times \mathbb{C}^{l+r}, \eta \oplus \mathrm{id}).$$

Thus we can assume that

$$(M, F^0, F^1, \sigma) \sim (\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta), \;\; l \geqslant n.$$
$$T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

Mapping η can be considered as

$$\eta \colon \mathbb{R}^n \to M_l(\mathbb{C}) = \{\, l \times l \text{ matrices } [\lambda_{ij}] \mid \lambda_{ij} \in \mathbb{C}\}.$$

There extist a compact set $\Delta \in \mathbb{R}^n$ with

$$\eta(\xi) \in \mathrm{GL}(l, \mathbb{C}) \;\; \forall\, \xi \in \mathbb{R}^{2n} - \Delta.$$

Making an evident homotopy (if necessary) of η we may assume

$$\eta(\xi) \in \mathrm{GL}(l, \mathbb{C}) \;\; \forall\, ||\xi|| \geqslant 1.$$

Then

$$\eta|_{S^{2n-1}} \colon S^{2n-1} \to \mathrm{GL}(l, \mathbb{C}), \;\; l \geqslant n.$$
$$\varphi((\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta)) = \beta(\eta|_{S^{2n-1}})$$

for β defined in section (8.1), and we have

$$\beta(\eta|_{S^{2n-1}}) \in \mathbb{Z},$$

so φ is integer valued.

Suppose now that

$$\beta(\eta|_{S^{2n-1}}) = 0,$$

then

$$[\eta|_{S^{2n-1}}] = 0 \text{ in } \pi_{2n-1}(\mathrm{GL}(l, \mathbb{C})).$$

By making a homotopy of η w obtain

$$\tilde{\eta} \colon \mathbb{R}^{2n} \to M_l(\mathbb{C})$$

with

$$\tilde{\eta}(\mathbb{R}^{2n}) \in \mathrm{GL}(l, \mathbb{C}).$$

Such $\tilde{\eta}$ in an abelian group A is equal to 0, so this proves that φ is one to one.

Now for (M, F^0, F^1, σ) let D be any elliptic pseudodifferential operator whose symbol (up to homotopy of σ) is σ.

$$\tau(M, F^0, F^1, \sigma) = \mathrm{Index}(D) = \dim\ker D - \dim\operatorname{coker} D.$$

It is obvious that it is integer valued. Also it is easy to check that τ is a homomorphism of abelian groups. The difficult part is to check that it is well defined, that is index does not change, when we do any of five steps defining equivalence relation $\sim$ on symbol data. $\square$

8.4 Index theorem for families of operators

Let $\mathcal{H}$ be a Hilbert space, $T\colon \mathcal{H} \to \mathcal{H}$ a Fredholm operator (has finite dimensional kernel and cokernel). The space of Fredholm operators we denote $\mathcal{F}(\mathcal{H}) \subset B(\mathcal{H})$.

Theorem 8.26. *For compact space X*

$$[X, \mathcal{F}(\mathcal{H})] \cong \mathrm{K}^0(X).$$

We describe only a map from the homotoy classes $[X, \mathcal{F}(\mathcal{H})]$ to $\mathrm{K}^0(X)$. Let $p \in X$, $f\colon X \to \mathcal{F}(\mathcal{H})$

$$f(p) \in \mathcal{F}(\mathcal{H})$$

Each $f(p)$ has finite dimensional kernel and cokernel. Let $\mathcal{N}(f(p))$ be the nullspace of $f(p)$ and $\mathcal{R}(f(p))$ its range. Then we have mappings

$$p \mapsto \mathcal{N}(f(p)),$$

$$p \mapsto \mathcal{R}(f(p))^{\perp},$$

If the dimensions of spaces $\mathcal{N}(f(p))$, $\mathcal{R}(f(p))^{\perp}$ are locally constant functions on X, then we have two vector bundles N, $R^{\perp}$ over X, both subbundles of infinite dimensional vector bundle $X \times \mathcal{H}$. The formal difference of isomorphism classes

$$[N] - [R^{\perp}]$$

is an element of K-theory of X. The contruction needs to be modified if the dimensions of $\mathcal{N}(f(p))$ or $\mathcal{R}(f(p))^{\perp}$ are not locally constant functions on X.

Let W, X be smooth manifolds without boundary, X compact. Suppose we are given submersion $\pi\colon W \to X$ and fibers of π are compact submanifolds. Suppose also that we have an elliptic differential operator on each fiber. Then we can form a kernel bundle and cokernel bundle which formal difference is an element of $\mathrm{K}^0(X)$. This element we call an index for a given family of operators.

We will give an idea of proof of the Bott periodicity theorem stated as

Theorem 8.27. *There is an isomorphism*

$$\beta\colon\ \mathrm{K}^0(X) \to \mathrm{K}^0(X \times \mathbb{R}^2)$$

$$\begin{tikzcd} E^0 \arrow[rr] \arrow[dr] & & E^1 \arrow[dl] \\ & M & \end{tikzcd} \quad \otimes \quad \begin{tikzcd} \mathbb{R}^2 \times \mathbb{C} \arrow[rr] \arrow[dr] & & \mathbb{R}^2 \times \mathbb{C} \arrow[dl] \\ & \mathbb{R}^2 & \end{tikzcd}$$

The map β is a multiplication by some element $b \in \mathrm{K}^0(\mathbb{R}^2) \cong \mathrm{K}^0(S^2)$. There is an isomorphism

$$\mathrm{K}^0(X \times \mathbb{R}^2) \to \widetilde{\mathrm{K}}(X \times S^2)$$

(X is without distinguished point). Consider a complex vector bundle on $X \times S^2$

$$\begin{tikzcd} E \arrow[d] \\ X \times S^2 \end{tikzcd}$$

For each $p \in X$ we have a vector bundle on S^2

$$\begin{tikzcd} E_p \arrow[r, equal] \arrow[d] & E|_{\{p\}\times S^2} \\ S^2 & \end{tikzcd}$$

The Dirac operator on S^2 can be represented as

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

We are interested only in

$$D_+ \colon C^\infty(S^2, L^+) \to C^\infty(S^2, L^-)$$

Where $L \to S^2$ is a line bundle. We can tensor the Dirac operator with the bundle E on S^2 and obtain for each $p \in X$

$$D_+ \otimes E_p \colon C^\infty(S^2, L^+ \otimes E_p) \to C^\infty(S^2, L^- \otimes E_p).$$

This gives a family of elliptic operators parametrized by X. It has an index in $\mathrm{K}^0(X)$, so we have defined a map

$$\alpha\colon\ \mathrm{K}^0(X \times \mathbb{R}^2) \to \mathrm{K}^0(X).$$

It can be proved that it is an inverse of β.

We list properties of maps β and α.

$$\beta\colon\ \mathrm{K}^0(X) \to \mathrm{K}^0(X \times \mathbb{R}^2)$$

1. it is functorial in X
2. it is $\mathrm{K}^0(X)$-module homomorphism
3. For $X = \mathrm{pt}$
$$\beta\colon\ \mathrm{K}^0(\mathrm{pt}) \to \mathrm{K}^0(\mathbb{R}^2) \cong \mathbb{Z}$$
$$\beta(1) = b.$$

$$\alpha\colon\ \mathrm{K}^0(X \times \mathbb{R}^2) \to \mathrm{K}^0(X).$$

1. it is functorial in X
2. it is $\mathrm{K}^0(X)$-module homomorphism
3. For $X = \mathrm{pt}$
$$\alpha\colon\ \mathrm{K}^0(\mathbb{R}^2) \to \mathrm{K}^0(\mathrm{pt}) \cong \mathbb{Z}$$
$$\alpha(b) = 1.$$

After proving above properties it is clear that β is an isomorphism and α is its inverse.

Chapter 9

Clifford algebras and Dirac operators

9.1 The Dirac operator of $\mathbb{R}^n$

First we consider n even. We shall construct matrices

$$E_1, E_2, \ldots, E_n, \quad n = 2r$$

each E_j being $2^r \times 2^r$ matrix of complex numbers. In fact each entry will be in $\{0, 1, -1, i, -i\}$.

Properties of E_j

1. $E_j^* = -E_j$,
2. each E_j is block anti-diagonal
$$E_j = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$$
and each block has size $2^{r-1} \times 2^{r-1}$,
3. $E_j^2 = -I_{2^r}$,
4. $E_j E_k + E_k E_j = 0$ for $j \neq k$,
5. $$i^r E_1 E_2 \ldots E_n = \begin{bmatrix} I_{2^{r-1}} & 0 \\ 0 & -I_{2^{r-1}} \end{bmatrix}$$

We will proceed by induction on n even. For $n = 2$ we take

$$E_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Suppose we have $E_1, E_2, \ldots, E_n$ of size $2^r \times 2^r$. Then we put first n matrices of size $2^{r+1} \times 2^{r+1}$ as

$$\begin{bmatrix} 0 & E_1 \\ E_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & E_2 \\ E_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & E_n \\ E_n & 0 \end{bmatrix}$$

and two additional matrices

$$\begin{bmatrix} 0 & -I_{2^r} \\ I_{2^r} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & iI_{2^{r-1}} & 0 \\ 0 & 0 & 0 & iI_{2^{r-1}} \\ iI_{2^{r-1}} & 0 & 0 & 0 \\ 0 & iI_{2^{r-1}} & 0 & 0 \end{bmatrix}.$$

Example 9.1. For $n = 4$ we have

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

For n odd, $n = 2r + 1$, we define matrices $E_1, E_2, \ldots, E_r$ satisfying

1. $E_j^* = -E_j$,
2. $E_j^2 = -I_{2^r}$,
3. $E_jE_k + E_kE_j = 0$ for $j \neq k$,
4. $i^{r+1}E_1E_2 \ldots E_n = I_{2^r}$.

First if $n = 1$ we set

$$E_1 = [-i].$$

Then for $n = 2r + 1$ we use $2r$ matrices $E_1, E_2, \ldots, E_{n-1}$ as for the even case and as the last one we put

$$\begin{bmatrix} -iI_{2^{r-1}} & 0 \\ 0 & iI_{2^{r-1}} \end{bmatrix}.$$

From $E_1, E_2, \ldots, E_n$ we obtain:

1. The Dirac operator of $\mathbb{R}^n$ (described above)
2. The Bott generator vector bundle on S^n (n even)
3. The spin representation of $\mathrm{Spin}^c(n)$

9.1.1 Dirac operator

Now we can define the *Dirac operator of* $\mathbb{R}^n$. For each n we set

$$D := \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j}.$$

Example 9.2. For $n = 1$ we have Dirac operator of $\mathbb{R}$

$$D = -i\frac{\partial}{\partial x}.$$

For $n = 2$

$$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \frac{\partial}{\partial x_2}$$

.

For $n = 2r$ and $n = 2r+1$ D is an unbounded operator on the Hilbert space

$$\underbrace{L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)}_{2^r}.$$

D is a first order elliptic differential operator on

$$\underbrace{C_c^\infty(\mathbb{R}^n) \oplus C_c^\infty(\mathbb{R}^n) \oplus \ldots \oplus C_c^\infty(\mathbb{R}^n)}_{2^r}$$

With this domain D is symmetric (that is D is formally self-adjoint) and D is essentially self-adjoint (that is D has unique self-adjoint extension). For n even

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

where D_- is the formal adjoint of D_+.

We will descirbe these notions in a general context. Let $\mathcal{H}$ be Hilbert space. An *unbounded operator* on $\mathcal{H}$ is a pair $(\mathcal{D}, T)$ such that

1. $\mathcal{D} \subset \mathcal{H}$ is a vector subspace of $\mathcal{H}$,
2. $\mathcal{D}$ is dense in $\mathcal{H}$,
3. $T\colon \mathcal{D} \to \mathcal{H}$ is a $\mathbb{C}$-linear map,
4. $(\mathcal{D}, T)$ is closeable, i.e. the closure of graph(T) in $\mathcal{H} \oplus \mathcal{H}$ is the graph of a $\mathbb{C}$-linear map
$$P(\overline{\text{graph}(T)}) \to \mathcal{H}$$
$$P(u, v) = u.$$

An unbounded operator $(\mathcal{D}, T)$ is *symmetric* if and only if

$$\langle Tu, v\rangle = \langle u, Tv\rangle \;\; \forall\, u, v \in \mathcal{D}.$$

For an unbounded operator $(\mathcal{D}, T)$ on $\mathcal{H}$ let

$$\mathcal{D}(T^*) := \{u \in \mathcal{H} \mid v \mapsto \langle u, Tv\rangle \text{ extends from } \mathcal{D} \text{ to } \mathcal{H} \text{ extends}$$
$$\text{to be a bounded linear functional on } \mathcal{H}\}$$

For $u \in \mathcal{D}(T^*)$ and $v \in \mathcal{H}$ there exists

$$T^*\colon \mathcal{D}(T^*) \to \mathcal{H}$$

such that

$$\langle u, Tv\rangle = \langle T^*u, v\rangle.$$

Now $(\mathcal{D}, T)$ is *self-adjoint* if and only if

$$(\mathcal{D}, T) = (\mathcal{D}(T^*), T^*).$$

Remark 9.3. Symmetric operator needs not to be self-adjoint, but a self-adjoint operator is symmetric.

Example 9.4. Take $C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ and

$$\begin{aligned} \mathcal{D} &= \left\{u \in L^2(\mathbb{R}) \mid -i\tfrac{du}{dx} \in L^2(\mathbb{R}) \text{ in the distribution sense}\right\} \\ &= \{u \in L^2(\mathbb{R}) \mid x\hat{u} \in L^2(\mathbb{R})\}, \end{aligned}$$

where $\hat{u}$ is the Fourier transform of u and

$$x\colon \mathbb{R} \to \mathbb{R}, \;\; x(t) = t, \;\; \forall\, t \in \mathbb{R}.$$

Then $(C_c^\infty(\mathbb{R}), -i\frac{d}{dx})$ has unique self-adjoint extension $(\mathcal{D}, -i\frac{d}{dx})$.

Let D be Dirac operator of $\mathbb{R}^n$, $n = 2r$ or $2r + 1$.

$$\begin{aligned} \Omega^1(\mathbb{R}^n) &= \{C^\infty \text{ 1-forms on } \mathbb{R}^n\} \\ &= \{f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n \mid f_j\colon \mathbb{R}^n \to \mathbb{C}, \; j = 1, 2, \ldots, n\} \end{aligned}$$

$\Omega^1(\mathbb{R}^n)$ acts on

$$\underbrace{C_c^\infty(\mathbb{R}^n) \oplus C_c^\infty(\mathbb{R}^n) \oplus \ldots \oplus C_c^\infty(\mathbb{R}^n)}_{2^r}$$

in the following way. Let

$$\omega = f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n,$$

$$s = (s_1, s_2, \ldots, s_{2^r}), \;\; s_l\colon \mathbb{R}^n \to \mathbb{C}, \;\; l = 1, 2, \ldots, 2^r.$$

Then

$$\omega s = \sum_{j=1}^n f_j E_j s.$$

There is following Leibniz rule for D

$$D(fs) = (df)s + f(Ds),$$

$$f\colon \mathbb{R}^n \to \mathbb{C}, \;\; f \in C^\infty(\mathbb{R}^n), \;\; df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j.$$

If M is C^∞-manifold, compact or non-compact, with or without boundary, $\dim M = M$, then the *Dirac operator of* M is an elliptic operator which is locally like the Dirac operator of $\mathbb{R}^n$.

9.1.2 Bott generator vector bundle

Let W be finite dimensional $\mathbb{C}$-vector space,

$$T \in \mathrm{Hom}_{\mathbb{C}}(W, W), \;\; T^2 = -I.$$

Then eigenvalues of T are $\pm i$ and there is decomposition

$$\begin{aligned} W &= W_i \oplus W_{-i}, \\ W_i &= \{v \in W \mid Tv = iv\} \\ W_{-i} &= \{v \in W \mid Tv = -iv\} \end{aligned}$$

Assume that n is even, $S^n \subset \mathbb{R}^{n+1}$

$$S^n = \{(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{R}^n \mid a_1^2 + a_2^2 + \ldots + a_{n+1}^2 = 1\}.$$

We have a map

$$\begin{aligned} S^n &\to M(2^r, \mathbb{C}) \\ (a_1, a_2, \ldots, a_{n+1}) &\mapsto a_1E_1 + a_2E_2 + \ldots + a_{n+1}E_{n+1} =: F. \end{aligned}$$

From the properties of E_j we obtain

$$\begin{aligned} F^2 &= (a_1E_1 + a_2E_2 + \ldots + a_{n+1}E_{n+1})^2 \\ &= (-a_1^2 - a_2^2 - \ldots - a_{n+1}^2)I \\ &= -I \end{aligned}$$

so the eigenvalues of F are $\pm i$.

The Bott generator vector bundle β on S^n is given by

$$\begin{aligned} \beta_{(a_1,a_2,\ldots,a_{n+1})} &:= \text{i-eigenspace of } F \\ &= \{v \in \mathbb{C}^{2^r} \mid F(v) = iv\} \end{aligned}$$

For n even and $S^n \subset \mathbb{R}^{n+1}$ there is an isomorphism

$$\begin{array}{cccc} \mathrm{K}^0(S^n) = & \mathbb{Z} & \oplus & \mathbb{Z} \\ & 1 & & \beta \end{array}$$

where $1 = S^n \times \mathbb{C}$.

9.2 Spin representation and Spin^c

Let G be a topological group, Hausdorff and paracompact, X topological space Hausdorff and paracompact. A *principal G-bundle* on X is a pair (P, π) where

1. P is a Hausdorff and paracompact topological space with given continuous (right) action of G
$$\begin{aligned} P \times G &\to P \\ (p, g) &\mapsto pg \end{aligned}$$
2. $\pi\colon P \to X$ is a continuous map, mapping P onto X

such that given any $x \in X$, there exists an open subset U of X with $x \in U$ and a homeomorphism

$$\varphi\colon U \times G \to \pi^{-1}(U)$$

with

$$\begin{aligned} \pi\varphi(u, g) &= u && \forall\, (u, g) \in U \times G \\ \varphi(u, g_1g_2) &= \varphi(u, g_1)g_2 && \forall\, (u, g_1, g_2) \in U \times G \times G \end{aligned}$$

Such $\varphi\colon U \times G \to \pi^{-1}(U)$ is referred to as a local trivialization.

Two principal G-bundles (P,π) and (Q,θ) are isomorphic if there exists a G-equivariant homeomorphism $f\colon P \to Q$ with commutativity in the diagram

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Q \arrow[dl, "\theta"] \\ & X & \end{tikzcd}$$

Let G, H be two topological groups and let (P,π), (G,θ) be a principal G-bundle and H-bundle on X. A homomorphism of principal bundles from (P,π) to (Q,θ) is a pair (η,ρ) such that

1. ρ is a homomorphism of topological groups $\rho\colon G \to H$
2. $P \to Q$ is a continuous map with commutativity in the diagrams

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Q \arrow[dl, "\theta"] \\ & X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \times G \arrow[r, "\eta\times\rho"] \arrow[d] & Q \times H \arrow[d] \\ P \arrow[r, "\eta"] & Q \end{tikzcd}$$

$$\pi p = \theta(\eta p) \qquad \eta(pg) = (\eta p)(\rho g)$$

A homomorphism of principal bundles on X will be denoted $\eta\colon P \to Q$ and $\rho\colon G \to H$ will be referred to as homomorphism of topological groups underlying η.

Lemma 9.5. *Let $\eta\colon P \to Q$ be a homomorphism of principal bundles on X with underlying homomorphism of topological groups $\rho\colon G \to H$. Then for any $x \in X$ there exists an open subset U of X with $x \in U$ and local trivializations*

$$\varphi\colon U \times G \to \pi^{-1}(U)$$

$$\psi\colon U \times H \to \theta^{-1}(U)$$

such that the diagram

$$\begin{tikzcd} U \times G \arrow[r, "\varphi"] \arrow[d, "\mathrm{Id}_U\times\eta"'] & \pi^{-1}(U) \arrow[d, "\eta"] \\ U \times H \arrow[r, "\psi"] & \theta^{-1}(U) \end{tikzcd}$$

commutes.

Example 9.6. Let E be $\mathbb{R}$-vector bundle on X, $\dim_{\mathbb{R}}(E_p) = n$ for all $p \in X$. Denote

$\Delta(E) := \{(p, v_1, v_2, \ldots, v_n) \mid p \in X,\ v_1, v_2, \ldots, v_n \text{ form a vector space basis for } E_p\}$

$\Delta(E)$ is topologized by

$$\Delta(E) \subset \underbrace{E \oplus E \oplus \ldots \oplus E}_{n}.$$

Define an action

$$\Delta(E) \times \mathrm{GL}(n, \mathbb{R}) \to \Delta(E)$$
$$((p, v_1, v_2, \ldots, v_n), [a_{ij}]) \mapsto (p, w_1, w_2, \ldots, w_n),$$
$$w_j = \sum_{i=1}^{n} a_{ij} v_i, \ \ [a_{ij}] \in \mathrm{GL}(n, \mathbb{R})$$

and a map

$$\theta\colon \Delta(E) \to X,$$
$$\theta(p, v_1, v_2, \ldots, v_n) = p.$$

Then $(\Delta(E), \theta)$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle on X.

For $n \geqslant 3$

$$\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$$

and $\mathrm{Spin}(n)$ is the unique non-trivial 2-fold cover of $\mathrm{SO}(n)$. It is a compact connected Lie group.

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[d] & & \\
\mathrm{SO}(n) & \subset & \mathrm{GL}(n, \mathbb{R})
\end{tikzcd}$$

There is an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

The group $\mathbb{Z}/2\mathbb{Z}$ embeds in the $\mathrm{Spin}(n)$ and S^1 as the $\{1, -1\}$. We define

$$\mathrm{Spin}^c(n) := S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(n).$$

Then there is an exact sequence

$$1 \to S^1 \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \to 1$$

$\mathrm{Spin}^c(n)$ is a compact connected Lie group

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[d] & & \\
\mathrm{Spin}^c(n) \arrow[d] & & \\
\mathrm{SO}(n) & \subset & \mathrm{GL}(n, \mathbb{R})
\end{tikzcd}$$

Example 9.7. For $n = 1$

$$\mathrm{Spin}(1) = \mathbb{Z}/2\mathbb{Z}, \ \ \mathrm{SO}(1) = 1$$
$$\mathrm{Spin}^c(1) = S^1$$
$$\rho\colon S^1 \to \mathrm{pt}.$$

For $n = 2$

$$\mathrm{Spin}(2) = S^1 = \mathrm{SO}(2)$$

$$\mathrm{Spin}(2) \to \mathrm{SO}(2)$$

$$\zeta \mapsto \zeta^2$$

and

$$\mathrm{Spin}^c(2) = S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(2)$$

$$\rho(\lambda, \zeta) = \zeta^2.$$

Remark 9.8. Since $\mathrm{SO}(n) \subset \mathrm{GL}(n, \mathbb{R})$ we can view the standard map $\mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ as $\mathrm{Spin}^c(n) \to \mathrm{GL}(n, \mathbb{R})$.

Definition 9.9. *A* Spin^c datum *for an $\mathbb{R}$-vector bundle $E \to X$ is a homomorphism of principal bundles*

$$\eta \colon P \to \Delta(E),$$

where P is a principal Spin$^c(n)$*-bundle on X $(n = \dim_{\mathbb{R}}(E_p))$ and the homomorphism of topological groups underlying η is the standard map*

$$\rho \colon \mathrm{Spin}^c(n) \to \mathrm{GL}(n, \mathbb{R}).$$

Two Spin^c data $\eta \colon P \to \Delta(E)$, $\eta' \colon P' \to \Delta(E)$ are isomorphic if there exists an isomorphism $f \colon P \to P'$ of principal Spin$^c(n)$-bundles on X with commutativity in the diagram

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\eta"'] & & P' \arrow[dl, "\eta'"] \\ & \Delta(E) & \end{tikzcd} \qquad \eta = \eta' \circ f.$$

Two Spin^c data $\eta \colon P \to \Delta(E)$, $\eta' \colon P' \to \Delta(E)$ are homotopic if there exists a principal Spin$^c(n)$-bundle Q on X and a continuous map

$$\Phi \colon Q \times [0, 1] \to \Delta(E)$$

such that

1. For $t \in [0, 1]$ each
$$\Phi_t = \Phi(-, t) \colon Q \to \Delta(E)$$
is a Spin^c data.

2. $$\Phi_0 \colon Q \to \Delta(E) \text{ is isomorphic to } \eta \colon P \to \Delta(E)$$
$$\Phi_1 \colon Q \to \Delta(E) \text{ is isomorphic to } \eta' \colon P \to \Delta(E)$$

Definition 9.10. *A* Spin$^c(n)$-structure *for E is an equivalence class of* Spin$^c(n)$ *data, where the equivalence relation is homotopy.*

A Spin^c structure for an $\mathbb{R}$-bundle E determines an orientation of E. Let $w_1(E), w_2(E), \dots$ be the Stiefel–Whitney classes of E, $w_j(E)\,\mathrm{H}^j(X;\mathbb{Z}/2\mathbb{Z})$-Cech cohomology. Then E is orientable if and only if $w_1(E) = 0$.

A *spin manifold* is a smooth manifold M, $\dim M = n$, for which the structure group of the tangent bundle TM has been lifted from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{Spin}(n)$. Such lifting is possible if and only if

$$\begin{array}{l} w_1(M) = 0, \quad w_1(M) \in \mathrm{H}^1(M;\mathbb{Z}/2\mathbb{Z}) \\ \text{and} \\ w_2(M) = 0, \quad w_2(M) \in \mathrm{H}^2(M;\mathbb{Z}/2\mathbb{Z}). \end{array}$$

A Spin^c *manifold* is a smooth manifold M, $\dim M = n$, for which the structure group of the tangent bundle TM has been lifted from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{Spin}^c(n)$. Such lifting is possible if and only if

$$\begin{array}{ll} w_1(M) = 0, & w_1(M) \in \mathrm{H}^1(M;\mathbb{Z}/2\mathbb{Z}) \\ \text{and} & \\ w_2(M) \text{ is in the image of} & \mathrm{H}^2(M;\mathbb{Z}) \to \mathrm{H}^2(M;\mathbb{Z}/2\mathbb{Z}). \end{array}$$

Various well known structures on a manifold M make M into Spin^c manifold

$$\begin{tikzcd}
 & \text{(complex analitic)} \arrow[d, Rightarrow] \\
\text{(symplectic)} \arrow[r, Rightarrow] & \text{(almost complex)} \arrow[d, Rightarrow] \\
\text{(contact)} \arrow[r, Rightarrow] & \text{(stably almost complex)} \arrow[d, Rightarrow] \\
\mathrm{Spin} \arrow[r, Rightarrow] & \mathrm{Spin}^c \arrow[d, Rightarrow] \\
 & \text{(oriented)}
\end{tikzcd}$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds. Spin^c structures behave very much like orientations. For example, an orientation on two of three $\mathbb{R}$ vector bundles in a short exact sequence determine an orientation on the third vector bundle. Analogous assertions are true for Spin^c structures.

Lemma 9.11 (Two-out-of-Three Lemma)**.** *Let*

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of $\mathbb{R}$ vector bundles on X. If Spin^c structures are given for any two of E', E, E'' then a Spin^c structure is determined for the third.

Corollary 9.12. *If M is a Spin^c manifold with boundary ∂M, then ∂M is in canonocal way a Spin^c manifold.*

Proof. There is an exact sequence

$$0 \to T\partial M \to TM|_{\partial M} \to \partial M \times \mathbb{R} \to 0$$

$\square$

Remark 9.13. If E is orientable ($w_1(E) = 0$), then the set of all possible orientations of E is in 1-1 correspondence with $\mathrm{H}^0(X; \mathbb{Z}/2\mathbb{Z})$. If E is Spin^c-able ($w_1(E) = 0$ and $w_2(E) \in \mathrm{im}(\mathrm{H}^2(X;\mathbb{Z}) \to \mathrm{H}^2(X;\mathbb{Z}/2\mathbb{Z}))$), then the set of all possible Spin^c-structures for E is then in 1-1 correspondence with $\mathrm{H}^0(X;\mathbb{Z}/2\mathbb{Z}) \times \mathrm{H}^2(X;\mathbb{Z})$.

9.2.1 Clifford algebras and spinor systems

Let V be a finite dimensional $\mathbb{R}$-vector space, $\langle -, - \rangle$ a positive defninite, symmetric, bilinear $\mathbb{R}$-valued inner product on V. We can form a tensor algebra

$$\mathcal{T}V := \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

with multiplication given by composing the tensors, and then define *Clifford algebra*

$$\mathrm{Cl}(V) := \mathcal{T}V/(v \otimes v + \langle v, v \rangle \cdot 1)$$

where $(v \otimes v + \langle v, v \rangle \cdot 1)$ denotes the two-sided ideal in $\mathcal{T}V$ generated by all elements of the form

$$v \otimes v + \langle v, v \rangle \cdot 1, \;\; v \in V, \;\; 1 \in \mathbb{R}.$$

As a vector space over $\mathbb{R}$ $\mathrm{Cl}(V)$ is canonically isomorphic to the exterior algebra

$$\Lambda^* V = \mathbb{R} \oplus V \oplus \Lambda^2 V \oplus \dots \Lambda^n V, \;\; n = \dim_{\mathbb{R}} V.$$

Let $e_1, e_2, \dots, e_n$ be an orthonormal basis of V. The monomials

$$e_1^{\epsilon_1} e_2^{\epsilon_2} \dots e_n^{\epsilon_n}, \;\; \epsilon_j \in \{0, 1\}$$

form a vector space basis of $\mathrm{Cl}(V)$. The canonical isomorphism of $\mathbb{R}$-vector spaces

$$\mathrm{Cl}(V) \to \Lambda^* V$$

is given by

$$e_1^{\epsilon_1} e_2^{\epsilon_2} \dots e_n^{\epsilon_n} \mapsto e_1^{\epsilon_1} \wedge e_2^{\epsilon_2} \wedge \dots \wedge e_n^{\epsilon_n}.$$

This isomorphism does not depend on the choice of orthonormal basis of V.

$$\dim_{\mathbb{R}}(\mathrm{Cl}(V)) = 2^n, \;\; n = \dim_{\mathbb{R}} V.$$

In $\mathrm{Cl}(V)$ we have following identities

$$e_j^2 = -1, \;\; j = 1, 2, \dots, n,$$

$$e_i e_j + e_j e_i = 0, \;\; i \neq j.$$

We can introduce $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Cl}(V)$ in the following way

$$\mathrm{Cl}(V) = (\mathrm{Cl}(V))_0 \oplus (\mathrm{Cl}(V))_1,$$

where $(\text{Cl}(V))_0$ is an $\mathbb{R}$-vector space spanned by $e_1^{\epsilon_1} e_2^{\epsilon_2} \dots e_n^{\epsilon_n}$ with $\epsilon_1 + \epsilon_2 + \dots + \epsilon_n$ even, and $(\text{Cl}(V))_1$ is an $\mathbb{R}$-vector space spanned by $e_1^{\epsilon_1} e_2^{\epsilon_2} \dots e_n^{\epsilon_n}$ with $\epsilon_1 + \epsilon_2 + \dots + \epsilon_n$ odd. This $\mathbb{Z}/2\mathbb{Z}$-grading does not depend on the choice of orthonormal basis of V.

Take $\mathbb{R}^n$ with the usual inner product

$$S^{n-1} \subset \mathbb{R}^n \subset \text{Cl}(\mathbb{R}^n).$$

The elements of S^{n-1} are invertible in $\text{Cl}(\mathbb{R}^n)$. Let $\text{Pin}(n)$ be the subgroup of the invertible elements of $\text{Cl}(\mathbb{R}^n)$ generated by S^{n-1}. Then

$$\text{Spin}(n) = \text{Pin}(n) \cap (\text{Cl}(\mathbb{R}^n))_0$$

$$\rho\colon \text{Spin}(n) \to \text{SO}(n)$$

$$(\rho g)(x) = gxg^{-1}, \quad g \in S^{n-1}, \quad x \in \mathbb{R}^n.$$

For $n \geqslant 3$ this is the unique non-trivial 2-fold covering space of $\text{SO}(n)$.

Consider complexification

$$\mathbb{C}\text{l}(V) := \mathbb{C} \otimes_{\mathbb{R}} \text{Cl}(V).$$

Then $\mathbb{C}\text{l}(V)$ is a C*-algebra with

$$v^* = -v$$

for

$$v \in V \subset \text{Cl}(V) \subset \mathbb{C}\text{l}(V).$$

Let

$$\mathbb{C}\text{l}(\mathbb{R}^n) := \mathbb{C}_{\mathbb{R}}\, \text{Cl}(\mathbb{R}^n),$$

$$\text{Spin}^c(n) = S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \text{Spin}(n) \subset \mathbb{C}\text{l}(\mathbb{R}^n).$$

Then $\text{Spin}^c(n)$ is a subgroup of the group of unitary elements of the C*-algebra $\mathbb{C}\text{l}(\mathbb{R}^n)$.

Let us now choose an orthogonal basis $e_1, e_2, \dots, e_n$ for even-dimensional $\mathbb{R}$-vector space V, $n = 2n = \dim_{\mathbb{R}}(V)$. Recall $2^r \times 2^r$ matrices $E_1, E_2, \dots, E_n$ defined in the beginning of the chapter and then define a mapping

$$\mathbb{C}\text{l}(V) \to M(2^r, \mathbb{C})$$

$$e_j \mapsto E_j, \quad j = 1, 2, \dots, n.$$

This gives an isomorphism of C*-algebras $\mathbb{C}\text{l}(V)$ and $M(2^r, \mathbb{C})$. For an odd dimension $n = 2r + 1$ recall $2^r \times 2^r$ matrices $E_1, E_2, \dots, E_n$ and define two mappings

$$\varphi_+\colon \mathbb{C}\text{l}(V) \to M(2^r, \mathbb{C})$$

$$\varphi_+(e_j) = E_j, \quad j = 1, 2, \dots, n,$$

$$\varphi_-\colon \mathbb{C}\text{l}(V) \to M(2^r, \mathbb{C})$$

$$\varphi_-(e_j) = -E_j, \quad j = 1, 2, \dots, n.$$

Then

$$\varphi_+ \oplus \varphi_-\colon \mathbb{C}\text{l}(V) \to M(2^r, \mathbb{C}) \oplus M(2^r, \mathbb{C})$$

is an isomorphism of C*-algebras.

Remark 9.14. This isomorphisms are non-canonical since they depend on the choice of an orthonormal basis for V.

Let E be an $\mathbb{R}$-vector bundle on X. Assume given an inner product $\langle -, - \rangle$ for E. Then define $\mathbb{C}\mathrm{l}(E)$ as a bundle of C*-algebras over X whose fiber at $p \in X$ is $\mathbb{C}\mathrm{l}(E_p)$.

Definition 9.15. *An* Hermitian module over $\mathbb{C}\mathrm{l}(E)$ *is a complex vector bundle F on X with a $\mathbb{C}$-valued inner product $(-,-)$ and a module structure*

$$\mathbb{C}\mathrm{l}(E) \otimes F \to F$$

such that

1. *$(-,-)$ makes F_p into a finite dimensional Hilbert space,*
2. *for each $p \in X$, the module map*
$$\mathbb{C}\mathrm{l}(E_p) \to \mathcal{L}(F_p)$$
is a unital homomorphism of C-algebras.*

Remark 9.16. Of course all structures here are assumed to be continuous. If X is a smooth manifold then we could take everything to be smooth.

If E is oriented define a section ω of $\mathbb{C}\mathrm{l}(E)$ as follows. Given $p \in X$, choose a positively oriented orthonormal basis $e_1, e_2, \ldots, e_n$ of E_p. For n even, $n = 2r$, set

$$\omega(p) = i^r e_1 e_2 \ldots e_{2r}.$$

For $n = 2r + 1$ odd

$$\omega(p) = i^{r+1} e_1 e_2 \ldots e_{2r+1}.$$

Then $\omega(p)$ does not depend on the choice of positively oriented orthonormal basis. In $\mathbb{C}\mathrm{l}(E_p)$ we have

$$(\omega(p))^2 = 1.$$

If n is odd, then $\omega(p)$ is in the center of $\mathbb{C}\mathrm{l}(E_p)$. Note that to define ω, E must be oriented. Reversing the orientation will change ω to $-\omega$.

Definition 9.17. *Let E be an $\mathbb{R}$-vector bundle on X. A* Spinor system *for E is a triple $(\epsilon, \langle -, - \rangle, F)$ such that*

1. *ϵ is an orientation of E,*
2. *$\langle -, - \rangle$ is an inner product for E,*
3. *F is an Hermitian module over $\mathbb{C}\mathrm{l}(E)$ with each F_p an irreducible module over $\mathbb{C}\mathrm{l}(E_p)$,*
4. *if $n = \dim_{\mathbb{R}}(E_p)$ is odd, then $\omega(p)$ acts identically on F_p.*

Remark 9.18. The irreducibillity of F_p in (3) is equivalent to $\dim_{\mathbb{C}}(F_p) = 2^r$, where $n = 2r$ or $n = 2r + 1$. In (4) note that $\omega(p)^2 = 1$ so for n odd $\omega(p)$ is in the center of $\mathbb{C}\mathrm{l}(E_p)$. Hence irreducibility of F_p implies that $\omega(p)$ acts either by I or $-I$ on F_p. Thus (4) normalizes the matter by requiring that $\omega(p)$ acts as I. When $n = \dim_{\mathbb{R}}(E_p)$ is even no such normalization is made.

If $(\epsilon, \langle -,- \rangle, F)$ is a Spinor system for E, then F is referred to as the *Spinor bundle.*

Suppose that $n = \dim_{\mathbb{R}}(E_p)$ is even. Let F_p^+ (F_p^+) be the $+1$ (-1) eigenspace of $\omega(p)$. We have a direct sum decomposition

$$F = F^+ \oplus F^-,$$

where F^+, F^- are $\frac{1}{2} -$ Spin *bundles.* F^+ (F^-) is a vector bundle of positive (negative) spinors.

Assume we have right and left actions of the group G on topological spaces X, Y

$$X \times G \to X$$

$$G \times Y \to Y$$

Then

$$X \times_G Y := X \times Y/\sim, \quad (xg, y) \sim (x, gy).$$

Example 9.19. Let E be an $\mathbb{R}$-vector bundle on X. Then

$$\Delta(E) \times_{\mathrm{GL}(n,\mathbb{R})} \cong E$$

$$((p, v_1, v_2, \ldots, v_n), (a_1, a_2, \ldots, a_n)) \mapsto a_1 v_1 + a_2 v_2 + \ldots + a_n v_n.$$

Let E be an $\mathbb{R}$-vector bundle on X. A Spin^c datum

$$\eta \colon P \to \Delta(E)$$

determines a Spinor system $(\epsilon, \langle -,- \rangle, F)$ for E. For $p \in X$, given orientation ϵ, and inner product $\langle -,- \rangle$, an $\mathbb{R}$-basis $v_1, v_2, \ldots, v_n$ of E_p is positively oriented and orthonormal if and only if

$$(v_1, v_2, \ldots, v_n) \in \mathrm{im}(\eta).$$

The Spinor bundle for $n = 2r$ or $n = 2r + 1$

$$F = P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}.$$

We have to describe how $\mathrm{Spin}^c(n)$ acts on $\mathbb{C}^{2^r}$. For n odd $\mathrm{Spin}^c(n)$ has an irreducible representation known as its spin representation

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^r, \mathbb{C}), \quad n = 2r + 1.$$

For n even $\mathrm{Spin}^c(n)$ has two irreducible representations known as its $\frac{1}{2} -$ Spin representations

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C}),$$

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C}), \quad n = 2r.$$

The direct sum

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C}) \oplus \mathrm{GL}(2^{r-1}, \mathbb{C}) \subset \mathrm{GL}(2^r, \mathbb{C}),$$

of these representations is the spin representation of $\mathrm{Spin}^c(n)$.

Consider $\mathbb{R}^n$ with its usual inner product and usual orthonormal basis $e_1, e_2, \ldots, e_n$

$$\varphi \colon \mathbb{C}\mathrm{l}(\mathbb{R}^n) \to M(2^r, \mathbb{C})$$

$$\varphi(e_j) = E_j, \quad j = 1, 2, \ldots, n.$$

There is a canonical inclusion

$$\mathrm{Spin}^c(n) \subset \mathbb{C}\mathrm{l}(\mathbb{R}^n)$$

and φ restricted to $\mathrm{Spin}^c(n)$ maps $\mathrm{Spin}^c(n)$ to $2^r \times 2^r$ unitary matrices

$$\mathrm{Spin}^c(n) \to \mathrm{U}(2^r) \subset \mathrm{GL}(n, \mathbb{C}).$$

This is *Spin representation* of $\mathrm{Spin}^c(n)$ and $\mathrm{Spin}^c(n)$ acts on $\mathrm{GL}(2^r, \mathbb{C})$ acts on $\mathbb{C}^{2^r}$ via this representation.

Let M be smooth manifold, possibly ∂M non-empty, TM the tangent bundle of M. Then

$$\begin{pmatrix} \text{Spin}^c \text{ datum for } TM \\ \eta\colon P \to \Delta(TM) \end{pmatrix}$$
$$\downarrow$$
$$\begin{pmatrix} \text{Spinor system for } TM \\ (\epsilon, \langle -, - \rangle, F) \end{pmatrix}$$
$$\downarrow$$
$$\begin{pmatrix} \text{Dirac operator} \\ D\colon \mathrm{smooth}_c(M, F) \to \mathrm{smooth}_c(M, F) \end{pmatrix}$$

where F is the Spinor bundle on M and $\mathrm{smooth}_c(M, F)$ are its smooth sections with compact support.

The Dirac operator

$$D\colon \mathrm{smooth}_c(M, F) \to \mathrm{smooth}_c(M, F)$$

is such that

1. D is $\mathbb{C}$-linear
$$D(s_1 + s_2) = Ds_1 + Ds_2,$$
$$D(\lambda s) = \lambda Ds, \quad s_1, s_2, s \in \mathrm{smooth}_c(M, F), \quad \lambda \in \mathbb{C}.$$

2. If $f\colon M \to \mathbb{C}$ is a smooth function, then
$$D(fs) = (df)s + f(Ds).$$

3. If $s_1, s_2 \in \mathrm{smooth}_c(M, F)$ then
$$\int_M (Ds_1(x), s_2(x))dx = \int_M (s_1(x), Ds_2(x))dx$$

4. If $\dim M$ is even, then D is off-diagonal
$$F = F^+ \oplus F^-$$
$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

$D\colon \mathrm{smooth}_c(M,F) \to \mathrm{smooth}_c(M,F)$ is an elliptic first-order differential operator. It can be viewed as an unbounded operator on the Hilbert space $L^2(M,F)$ with the scalar product

$$(s_1, s_2) := \int_M (s_1(x), s_2(x))dx.$$

Moreover it is a symmetric operator.

One proves existence of D by constructing it locally and patching together with a smooth partition of unity. The uniqueness of D is obtained by the fact that if D_0, D_1 satisfy conditions (1)-(4) above, then

$$D_0 - D_1\colon F \to F$$

is a vector bundle map, hence D_0, D_1 differ by lower order terms.

Example 9.20. Let n be even, $S^n \subset \mathbb{R}^{n+1}$, D-Dirac operator of S^n, F-Spinor bundle of S^n, $F = F^+ \oplus F^-$.

$$D\colon \mathrm{smooth}_c(S^n, F) \to \mathrm{smooth}_c(S^n, F)$$

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

$$D^+\colon \mathrm{smooth}_c(S^n, F^+) \to \mathrm{smooth}_c(S^n, F^-)$$

Then

$$\mathrm{Index}(D^+) := \dim_{\mathbb{C}}(\ker D^+) - \dim_{\mathbb{C}}(\operatorname{coker} D^+).$$

Theorem 9.21.

$$\mathrm{Index}(D^+) = 0.$$

We can tensor D^+ with the Bott generator vector bundle β from section (9.1.2)

$$D^+_\beta\colon \mathrm{smooth}_c(S^n, F^+ \otimes \beta) \to \mathrm{smooth}_c(S^n, F^- \otimes \beta).$$

Then we have

Theorem 9.22.

$$\mathrm{Index}(D^+_\beta) = 1.$$

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Part III

Dirac operators and spectral geometry

by

Joseph C. Vàrilly

Based on the lectures of:

- Joseph C. Vàrilly
  (*Departamento de Matematicas, Universidad de Costa Rica, 2060 San Jose, Costa Rica*)
  – Chapters 1, 2, 3, 4, 5, 6, 7, 8.

Introduction and Overview

Noncommutative geometry asks: "What is the geometry of the Quantum World?"

Quantum field theory considers aggregates of "particles", which are of two general species, "bosons" and "fermions". These are described by solutions of (relativistic) wave equations:

- Bosons: Klein–Gordon equation, $(\Box + m^2)\phi(x) = \rho_b(x)$ – "source term";
- Fermions: Dirac equation, $(i\partial\!\!\!/ - m)\psi(x) = \rho_f(x)$ – "source term";

where $x = (t, \vec{x}) = (x^0, x^1, x^2, x^3)$; $\Box = -\partial^2/\partial t^2 + \partial^2/\partial\vec{x}^2$; and $\partial\!\!\!/ = \sum_{\mu=0}^{3} \gamma^\mu\, \partial/\partial x^\mu$. In order that $\partial\!\!\!/$ be a "square root of $\Box$", we need $(\gamma^0)^2 = -1$, $(\gamma^j)^2 = +1$ for $j = 1, 2, 3$ and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$. Thus, the γ^μ must be matrices; in fact there are four (4×4) matrices satisfying these relations.

Point-like measurements are often ruled out by quantum mechanics; thus we replace *points* $x \in M$ by *coordinates* $f \in C(M)$. The metric distance on a Riemannian manifold (M, g) can be computed in two ways:

$$\begin{aligned} d_g(p,q) &:= \inf\{\, \text{length}(\gamma\colon [0,1] \to M) : \gamma(0) = p,\ \gamma(1) = q \,\} \\ &= \sup\{\, |f(p) - f(q)| : f \in C(M), \|D\!\!\!\!/, f\| \leq 1 \,\}, \end{aligned}$$

where $D\!\!\!\!/$ is a Dirac operator with positive-definite signature (all $(\gamma^\mu)^2 = +1$) if it exists, so the Dirac operator specifies the metric. $D\!\!\!\!/$ is an (unbounded) operator on a Hilbert space $\mathcal{H} = L^2(M, S)$ of "square-integrable spinors" and $C^\infty(M)$ also acts on $\mathcal{H}$ by multiplication operators with $\|[D\!\!\!\!/, f]\| = \|\operatorname{grad} f\|_\infty$.

Noncommutative geometry generalizes $(C^\infty(M), L^2(M,S), D\!\!\!\!/)$ to a *spectral triple* of the form $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a "smooth" algebra acting on a Hilbert space $\mathcal{H}$, D is an (unbounded) selfadjoint operator on $\mathcal{H}$, subject to certain conditions: in particular that $[D, a]$ be a bounded operator for each $a \in \mathcal{A}$. The tasks of the geometer are then:

1. To describe (metric) differential geometry in an operator language.
2. To reconstruct (ordinary) geometry in the operator framework.
3. To develop new geometries with noncommutative coordinate algebras.

The long-term goal is to geometrize quantum physics at very high energy scales, but we are still a long way from there.

The general program of these lectures is as follows.

(A) The classical theory of spinors and Dirac operators in the Riemannian case.

(B) The operational toolkit for noncommutative generalization.

(C) Reconstruction: how to recover differential geometry from the operator framework.

(D) Examples of spectral triples with noncommutative coordinate algebras.

Chapter 1

Clifford algebras and spinor representations

Here are a few general references on Clifford algebras, in reverse chronological order: [fgv01, 2001], [f-t00, 1997/2000], [bgv92, 1992], [lm89, 1989] and [abs64, 1964]. (See the bibliography for details.)

1.1 Clifford algebras

We start with (V, g), where $V \cong \mathbb{R}^n$ and g is a *nondegenerate symmetric bilinear form*. If $q(v) = g(v, v)$, then $2g(u, v) = q(u + v) - q(u) - q(v)$. Thus g is determined by the corresponding "quadratic form" q.

Definition 1.1. *The* Clifford algebra $\mathrm{Cl}(V, g)$ *is an algebra (over* $\mathbb{R}$*) generated by the vectors* $v \in V$ *subject to the relations* $uv + vu = 2g(u, v)1$ *for* $u, v \in V$.

The existence of this algebra can be seen in two ways. First of all, let $\mathcal{T}(V)$ be the tensor algebra on V, that is, $\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} V^{\otimes n}$. Then

$$\mathrm{Cl}(V, g) := \mathcal{T}(V)/\,\mathrm{Ideal}\langle u \otimes v + v \otimes u - 2g(u, v)\,1 : u, v \in V\rangle. \tag{1.1}$$

Since the relations are *not* homogeneous, the $\mathbb{Z}$-grading of $\mathcal{T}(V)$ is lost, only a $\mathbb{Z}_2$-grading remains:

$$\mathrm{Cl}(V, g) = \mathrm{Cl}^0(V, g) \oplus \mathrm{Cl}^1(V, g).$$

The second option is to define $\mathrm{Cl}(V, g)$ as a subalgebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$ generated by all expressions $c(v) = \varepsilon(v) + \iota(v)$ for $v \in V$, where

$$\varepsilon(v) \colon u_1 \wedge \cdots \wedge u_k \mapsto v \wedge u_1 \wedge \cdots \wedge u_k$$

$$\iota(v) \colon u_1 \wedge \cdots \wedge u_k \mapsto \sum_{j=1}^{k} (-1)^{j-1} g(v, u_j) u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_k.$$

Note that $\varepsilon(v)^2 = 0$, $\iota(v)^2 = 0$, and $\varepsilon(v)\iota(u) + \iota(u)\varepsilon(v) = g(v, u)\,1$. Thus

$$c(v)^2 = g(v, v)\,1 \quad \text{for all} \quad v \in V,$$
$$c(u)c(v) + c(v)c(u) = 2g(u, v)\,1 \quad \text{for all} \quad u, v \in V.$$

Thus these operators on $\Lambda^\bullet V$ do provide a representation of the algebra (1.1).

Dimension count: suppose $\{e_1, \dots, e_n\}$ is an orthonormal basis for (V, g), i.e., $g(e_k, e_k) = \pm 1$ and $g(e_j, e_k) = 0$ for $j \neq k$. Then the $c(e_j)$ anticommute and thus a basis for $\mathrm{Cl}(V, g)$ is $\{c(e_{k_1}) \dots c(e_{k_r}) : 1 \leq k_1 < \cdots < k_r \leq n\}$, labelled by $K = \{k_1, \dots, k_r\} \subseteq \{1, \dots, n\}$. Indeed,

$$c(e_{k_1}) \dots c(e_{k_r}) \colon 1 \mapsto e_{k_1} \wedge \cdots \wedge e_{k_r} \equiv e_K \in \Lambda^\bullet V$$

and these are linearly independent. Thus the dimension of the subalgebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$ generated by all $c(v)$ is just $\dim \Lambda^\bullet V = 2^n$. Now, a moment's thought shows that in the abstract presentation (1.1), the algebra $\mathrm{Cl}(V, g)$ is generated as a vector space by the 2^n products $e_{k_1} e_{k_2} \dots e_{k_r}$, and these are linearly independent since the operators $c(e_{k_1}) \dots c(e_{k_r})$ are linearly independent in $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$. Therefore, this representation of $\mathrm{Cl}(V, g)$ is faithful, and $\dim \mathrm{Cl}(V, g) = 2^n$.

The so-called "symbol map":

$$\sigma : a \mapsto a(1) : \mathrm{Cl}(V, g) \to \Lambda^\bullet V$$

is inverted by a "quantization map":

$$Q : u_1 \wedge u_2 \wedge \cdots \wedge u_r \longmapsto \frac{1}{r!} \sum_{\tau \in S_r} (-1)^\tau \, c(u_{\tau(1)}) c(u_{\tau(2)}) \dots c(u_{\tau(r)}). \tag{1.2}$$

To see that it is an inverse to σ, one only needs to check it on the products of elements of an orthonormal basis of (V, g).

From now, we write uv instead of $c(u)c(v)$, etc., in $\mathrm{Cl}(V, g)$.

1.2 The universality property

Chevalley [c-c54] has pointed out the usefulness of the following property of Clifford algebras, which is an immediate consequence of their definition.

Lemma 1.2. *Any $\mathbb{R}$-linear map $f \colon V \to A$ (an $\mathbb{R}$-algebra) that satisfies*

$$f(v)^2 = g(v, v) \, 1_A \quad \textit{for all} \quad v \in V$$

extends to an unique unital $\mathbb{R}$-algebra homomorphism $\tilde{f} \colon \mathrm{Cl}(V, g) \to A$.

Proof. There is really nothing to prove: $\tilde{f}(v_1 v_2 \dots v_r) := f(v_1) f(v_2) \dots f(v_r)$ gives the uniqueness, provided only that this recipe is well-defined. But observe that

$$\begin{aligned} \tilde{f}(uv + vu - 2g(u, v)1) &= \tilde{f}((u + v)^2 - u^2 - v^2) - 2g(u, v) \, \tilde{f}(1) \\ &= [q(u + v) - q(u) - q(v) - 2g(u, v)] \, 1_A = 0. \end{aligned}$$

□

Here are a few applications of universality that yield several useful operations on the Clifford algebra.

1. *Grading*: take $A = \mathrm{Cl}(V, g)$ itself; the linear map $v \mapsto -v$ on V extends to an *automorphism* $\chi \in \mathrm{Aut}(\mathrm{Cl}(V, g))$ satisfying $\chi^2 = \mathrm{id}_A$, given by

$$\chi(v_1 \dots v_r) := (-1)^r \, v_1 \dots v_r.$$

This operator gives the $\mathbb{Z}_2$-grading

$$\mathrm{Cl}(V, g) =: \mathrm{Cl}^0(V, g) \oplus \mathrm{Cl}^1(V, g).$$

2. *Reversal*: take $A = \mathrm{Cl}(V, g)^{\mathrm{op}}$, the opposite algebra. Then the map $v \mapsto v$, considered as the inclusion $V \hookrightarrow A$, extends to an *antiautomorphism* $a \mapsto a^!$ of $\mathrm{Cl}(V, g)$, given by $(v_1 v_2 \dots v_r)^! := v_r \dots v_2 v_1$.

3. *Complex conjugation*: the *complexification* of $\mathrm{Cl}(V, g)$ is $\mathrm{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C}$, which is isomorphic to $\mathrm{Cl}(V^{\mathbb{C}}, g^{\mathbb{C}})$ as a $\mathbb{C}$-algebra. Now take A to be $\mathrm{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C}$ and define $f : v \mapsto \bar{v} : V^{\mathbb{C}} \to V^{\mathbb{C}} \hookrightarrow A$ (a real-linear map). It extends to an *antilinear automorphism* of A. Note that Lemma 1.2 guarantees $\mathbb{R}$-linearity, but not $\mathbb{C}$-linearity, of the extension even when A is a $\mathbb{C}$-algebra.

4. *Adjoint*: Also, $a^* := (\bar{a})^!$ is an *antilinear involution* on $\mathrm{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C}$.

5. *Charge conjugation*: $\kappa(a) := \chi(\bar{a}) : v_1 \dots v_r \mapsto (-1)^r \bar{v}_1 \dots \bar{v}_r$ is an antilinear automorphism of $\mathrm{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C}$.

Notation. We write $\mathbb{C}\mathrm{l}(V) := \mathrm{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C}$ to denote the complexified Clifford algebra. Up to isomorphism, this is independent of the signature of the symmetric bilinear form g, because all complex nondegenerate bilinear forms are congruent.

1.3 The trace

Proposition 1.3. *There is an unique trace $\tau : \mathbb{C}\mathrm{l}(V) \to \mathbb{C}$ such that $\tau(1) = 1$ and $\tau(a) = 0$ for a odd.*

Proof. If $\{e_1, \dots, e_n\}$ is an orthonormal basis for (V, g), then

$$\tau(e_{k_1} \dots e_{k_{2r}}) = \tau(-e_{k_2} \dots e_{k_{2r}} e_{k_1}) = -\tau(e_{k_1} \dots e_{k_{2r}}) = 0.$$

(Here we have moved e_{k_1} to the right by anticommutation, and returned it to the left with the trace property.) Thus, if $a = \sum_{K \text{ even}} a_{k_1 \dots k_{2r}} e_{k_1} \dots e_{k_{2r}}$ lies in $\mathbb{C}\mathrm{l}^0(V)$, then $\tau(a) = a_\emptyset$. We will check that $a_\emptyset$ does not depend on the orthonormal basis used. Suppose $e'_j = \sum_{k=1}^n h_{kj} e_j$, with $H^t H = 1_n$, is another orthonormal basis. Then

$$e'_i e'_j = (\vec{h}_i \cdot \vec{h}_j)\, 1 + \sum_{k<l} c_{ij}^{kl}\, e_k e_l,$$

but $\vec{h}_i \cdot \vec{h}_j = [H^t H]_{ij} = 0$ for $i \neq j$. Next, the matrix of $e_k e_l \mapsto e'_i e'_j$ is $H \wedge H$, of size $\binom{n}{2}$, that is also orthogonal, so $e'_i e'_j e'_r e'_s$ has zero scalar part in the $e_k e_l e_p e_q$-expansion; and so on: the same is true for expressions $e'_{j_1} \dots e'_{j_{2r}}$ by induction. Thus $\tau(a) = a_\emptyset$ does not depend on $\{e_1, \dots, e_n\}$. ☐

Remark 1.4. At this point, it was remarked that for existence of the trace, one could use the restriction of the (normalized) trace on $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\Lambda^\bullet V^{\mathbb{C}})$, in which $\mathbb{C}\mathrm{l}(V)$ is embedded. True enough: although one must see why odd elements must have trace zero. For that, it is enough to note that if $a \in \mathbb{C}\mathrm{l}^1(V)$, then $c(a)$ takes even [respectively, odd] elements of the $\mathbb{Z}$-graded algebra $\mathrm{End}_{\mathbb{C}}(V^{\mathbb{C}})$ to odd [respectively, even] elements; thus, in any basis, the matrix of $c(a)$ will have only zeroes on the diagonal, so that $\mathrm{tr}(c(a)) = 0$. Nonetheless, Proposition 1.3 is useful in that it establishes the uniqueness of the trace.

Now $\mathbb{C}\mathrm{l}(V)$ is a *Hilbert space* with scalar product

$$\langle a \mid b\rangle := \tau(a^* b).$$

1.4 Periodicity

Write $\mathrm{Cl}_{pq} := \mathrm{Cl}(\mathbb{R}^{p+q}, g)$, where g has signature (p,q), and the orthonormal basis is written as $\{e_1, \dots, e_p, \varepsilon_1, \dots, \varepsilon_q\}$, where $e_1^2 = \cdots = e_p^2 = 1$ and $\varepsilon_1^2 = \cdots = \varepsilon_q^2 = -1$. For example,

$$\begin{aligned}
&\mathrm{Cl}_{10} = \mathbb{R} \oplus \mathbb{R};\\
&\mathrm{Cl}_{01} = \mathbb{C}, \quad \text{with} \quad \varepsilon_1 = i;\\
&\mathrm{Cl}_{20} = M_2(\mathbb{R}), \quad \text{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\\
&\mathrm{Cl}_{02} = \mathbb{H}, \quad \text{with} \quad \varepsilon_1 = i,\ \varepsilon_2 = j,\ \varepsilon_1\varepsilon_2 = k.
\end{aligned}$$

Lemma 1.5 ("(1,1)-periodicity")**.** $\mathrm{Cl}_{p+1,q+1} \cong \mathrm{Cl}_{pq} \otimes M_2(\mathbb{R})$.

Proof. Take $V = \mathbb{R}^{p+q+2}$, $A = \mathrm{Cl}_{pq} \otimes M_2(\mathbb{R})$. Define $f\colon V \to A$ on basic vectors by

$$\begin{aligned}
f(e_r) &:= e_r \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad r = 1, \dots, p,\\
f(\varepsilon_s) &:= \varepsilon_s \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad s = 1, \dots, q,\\
f(e_{p+1}) &:= 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\\
f(\varepsilon_{q+1}) &:= 1 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{1.3}$$

Thus $f(e_k)^2 = +1$, $f(\varepsilon_l)^2 = -1$ in all cases, and all $f(e_k)$, $f(\varepsilon_l)$ anticommute. This entails that f extends by linearity to a linear map satisfying $f(v)^2 = g(v,v)\,1$ for all $v \in V$. Hence there exists a homomorphism $\tilde{f}\colon \mathrm{Cl}_{p+1,q+1} \to A$, which is surjective since the right hand sides of (1.3) generate A as an $\mathbb{R}$-algebra. It is an isomorphism, because the dimensions over $\mathbb{R}$ are equal. ☐

Lemma 1.6. $\mathrm{Cl}^0_{p+1,q} \cong \mathrm{Cl}_{qp}$.

Proof. Define $f\colon \mathbb{R}^{q+p} \to \mathrm{Cl}^0_{p+1,q}$ on basic vectors by

$$\begin{aligned}
f(e_r) &:= \varepsilon_r e_{p+1}, \qquad r = 1, \dots, q,\\
f(\varepsilon_s) &:= e_s e_{p+1}, \qquad s = 1, \dots, p.
\end{aligned}$$

Then

$$\begin{aligned}
f(e_r)^2 &= \varepsilon_r e_{p+1} \varepsilon_r e_{p+1} = -\varepsilon_r^2 e_{p+1}^2 = -\varepsilon_r^2 = +1,\\
f(\varepsilon_s)^2 &= e_s e_{p+1} e_s e_{p+1} = -e_s^2 e_{p+1}^2 = -e_s^2 = -1,
\end{aligned}$$

and all $f(e_r)$, $f(\varepsilon_s)$ anticommute. The rest of the proof is like that of the previous Lemma. ☐

Lemma 1.7. $\mathrm{Cl}_{p+4,q} \cong \mathrm{Cl}_{pq} \otimes M_2(\mathbb{H}) \cong \mathrm{Cl}_{p,q+4}$.

Proof. We will prove the first isomorphism. Take $A = \mathrm{Cl}_{pq} \otimes M_2(\mathbb{H})$; define $f\colon \mathbb{R}^{p+4+q} \to A$ by

$$f(e_r) := e_r \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \qquad r = 1, \dots, p,$$
$$f(\varepsilon_s) := \varepsilon_s \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \qquad s = 1, \dots, q,$$

and on the remaining four basic vectors, define

$$f(e_{p+1}) := 1 \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad f(e_{p+2}) := 1 \otimes \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix},$$
$$f(e_{p+3}) := 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad f(e_{p+4}) := 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \square$$

Corollary 1.8 ("(+8)-periodicity")**.** $\mathrm{Cl}_{p+8,q} \cong \mathrm{Cl}_{pq} \otimes M_{16}(\mathbb{R}) \cong \mathrm{Cl}_{p,q+8}$.

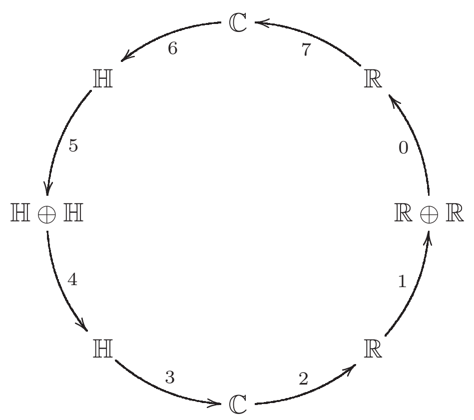
Proof. This reduces to $M_2(\mathbb{H}) \otimes_{\mathbb{R}} M_2(\mathbb{H}) \cong M_{16}(\mathbb{R})$, that in turn reduces to $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$, which is left as an exercise. $\square$

All Cl_{pq} are given, up to $M_N(\mathbb{R})$ tensor factors, by Cl_{p0} for $p = 1, \dots, 8$:

$$\begin{aligned}
\mathrm{Cl}_{10} &= \mathbb{R} \oplus \mathbb{R} \\
\mathrm{Cl}_{20} &= M_2(\mathbb{R}) \\
\mathrm{Cl}_{30} &= M_2(\mathbb{C}) \\
\mathrm{Cl}_{40} &= M_2(\mathbb{H}) \\
\mathrm{Cl}_{50} &= M_2(\mathbb{H}) \oplus M_2(\mathbb{H}) \\
\mathrm{Cl}_{60} &= M_4(\mathbb{H}) \\
\mathrm{Cl}_{70} &= M_8(\mathbb{C}) \\
\mathrm{Cl}_{80} &= M_{16}(\mathbb{R})
\end{aligned} \tag{1.4}$$

Two algebras Cl_{10} and Cl_{50} are direct sums of simple algebras, and the others are simple. We could also define $\mathrm{Cl}_{00} = \mathbb{R}$ (the base field), so that Corollary 1.8 holds even when $p = q = 0$.

Those eight algebras Cl_{p0} can be arranged on a "spinorial clock", which is taken from Budinich and Trautman's book [bt88].



If $p-q \equiv m \bmod 8$, then Cl_{pq} is of the form $A \otimes M_N(\mathbb{R})$, where A is the diagram entry at the head of the arrow labelled m. Moreover, Lemma 1.6 says that the even subalgebra Cl^0_{pq} is of the same kind, where A is now the diagram entry at the tail of the arrow labelled m. The matrix size N is easily determined from the real dimension, in each case. In this way, the spinorial clock displays the full classification of real Clifford algebras.

1.5 Chirality

From now on, $n = 2m$ for n even, $n = 2m+1$ for n odd. We take $\mathbb{C}\mathrm{l}(V) \cong \mathrm{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C}$ with g always positive definite.

Suppose $\{e_1, \dots, e_n\}$ is an *oriented* orthonormal basis for (V, g). If $e'_k = \sum_{j=1}^n h_{jk}\, e_j$ with $H^t H = 1_n$, then $e'_1 \dots e'_n = (\det H)\, e_1 \dots e_n$, and $\det H = \pm 1$. We restrict to the oriented case $\det H = +1$, so the expression $e_1 e_2 \dots e_n$ is independent of $\{e_1, e_2, \dots, e_n\}$. Thus

$$\gamma := (-i)^m e_1 e_2 \dots e_n$$

is well-defined in $\mathbb{C}\mathrm{l}(V)$. Now

$$\gamma^* = i^m e_n \dots e_2 e_1 = (-i)^m (-1)^m (-1)^{n(n-1)/2} e_1 e_2 \dots e_n = (-1)^m (-1)^{n(n-1)/2} \gamma,$$

and

$$\frac{n(n-1)}{2} = \left\{ \begin{matrix} m(2m-1), & n \text{ even} \\ (2m+1)m, & n \text{ odd} \end{matrix} \right\} \equiv m \bmod 2,$$

so $\gamma^* = \gamma$. But also $\gamma^*\gamma = (e_n \dots e_2 e_1)(e_1 e_2 \dots e_n) = (+1)^n = 1$, so γ is "unitary". Hence $\gamma^2 = 1$, so $\frac{1+\gamma}{2}$, $\frac{1-\gamma}{2}$ are "orthogonal projectors" in $\mathbb{C}\mathrm{l}(V)$.

Since $\gamma e_j = (-1)^{n-1} e_j \gamma$, we get that if n is odd, then γ is *central* in $\mathbb{C}\mathrm{l}(V)$; and for n even, γ anticommutes with V, but is central in the even subalgebra $\mathrm{Cl}^0(V)$. Moreover, when n is even and $v \in V$, then $\gamma v \gamma = -v$, so that $\gamma(\cdot)\gamma = \chi \in \mathrm{Aut}(\mathbb{C}\mathrm{l}(V))$.

Proposition 1.9. *The centre of* $\mathbb{C}\mathrm{l}(V)$ *is* $\mathbb{C}1$ *if* n *is even; and it is* $\mathbb{C}1 \oplus \mathbb{C}\gamma$ *if* n *is odd.*

Proof. Denote this centre by $Z(\mathbb{C}\mathrm{l}(V))$.

Even case: $a \in Z(\mathbb{C}\mathrm{l}(V))$ implies $\gamma a \gamma = a\gamma^2 = a$, so a lies in $\mathrm{Cl}^0(V)$.

If $a = \sum_{K \text{ even}} a_K\, e_{k_1} \dots e_{k_r}$, then $0 = a - e_j a e_j = \sum_{K \text{ even},\, j \in K} 2a_K e_{k_1} \dots e_{k_r}$, so $a_K = 0$ if $j \in K$. Since this holds for any j, we conclude that $a = a_\emptyset\, 1 = \tau(a)\, 1$. Therefore $Z(\mathbb{C}\mathrm{l}(V)) \cong \mathbb{C}1$ when n is even.

Odd case: If $a = a_0 + a_1$ (even + odd) lies in $Z(\mathbb{C}\mathrm{l}(V))$, then

$$0 = [a, v] = \underbrace{[a_1, v]}_{\text{even}} + \underbrace{[a_0, v]}_{\text{odd}} \quad \text{for all} \quad v \in V,$$

so $[a_0, v] = [a_1, v] = 0$ for all $v \in V$. In particular, $a_0 \in Z(\mathrm{Cl}^0(V)) \cong \mathbb{C}1$, and thus $a_0 = \tau(a)\, 1$.

Also, $a_1\gamma$ is even and central, so $a_1\gamma = \tau(a\gamma)\, 1$ and $a_1 = \tau(a\gamma)\, \gamma$. Thus $Z(\mathbb{C}\mathrm{l}(V)) = \mathbb{C}1 \oplus \mathbb{C}\gamma$ when n is odd. ∎

1.6 Spin$^{\mathrm{c}}$ and Spin groups

Let v be a unit vector, $g(v,v) = 1$. Then $v^2 = 1$ in $\mathrm{Cl}(V,g)$, so $v = v^*$ and $v^*v = vv^* = 1$ in $\mathbb{C}\mathrm{l}(V)$. If $w = \lambda v \in V^{\mathbb{C}}$ with $|\lambda| = 1$, then $ww^* = w^*w = 1$ in $\mathbb{C}\mathrm{l}(V)$ also. Now

$$\begin{aligned}\langle wa \mid wb\rangle &= \tau(a^*w^*wb) = \tau(a^*b) = \langle a \mid b\rangle,\\ \langle aw \mid bw\rangle &= \tau(w^*a^*bw) = \tau(ww^*a^*b) = \tau(a^*b) = \langle a \mid b\rangle,\end{aligned}$$

so $a \mapsto wa$, $a \mapsto aw$ are unitary operators in $\mathcal{L}(\mathbb{C}\mathrm{l}(V))$.

⟦Exercise: Conversely, if $u \in \mathbb{C}\mathrm{l}(V)$ and $a \mapsto ua$ and $a \mapsto au$ are both unitary, then $u^*u = uu^* = 1$.⟧

If $v, x \in V$, with $g(v,v) = 1$, then

$$-vxv^{-1} = -vxv = (xv - 2g(v,x))v = x - 2g(v,x)v \ \underline{\in V}.$$

This is a reflection of x in the hyperplane orthogonal to v. For $w = \lambda v$, $|\lambda| = 1$ we also get $-wxw^{-1} = -\lambda\bar{\lambda}vxv^{-1} = -vxv^{-1}$, which is the same as above. If $a = w_1 \dots w_r$ is a product of unit vectors in $V^{\mathbb{C}}$, then

$$\chi(a)xa^{-1} = (-1)^r w_1 \dots w_r x w_r^{-1} \dots w_1^{-1}$$

is a product of r reflections of $x \in V$. If $r = 2k$ is *even*, and $a = w_1 \dots w_{2k}$, then $axa^{-1} \in V$ after k rotations. Thus $\phi(a)\colon x \mapsto axa^{-1}$ lies in $\mathrm{SO}(V) = \mathrm{SO}(V,g)$.

Definition 1.10. *The set of all even products of unitary vectors,*

$$\mathrm{Spin}^{\mathrm{c}}(V) := \{\, u = w_1 \dots w_{2k} : w_j \in V^{\mathbb{C}},\ w_j^*w_j = 1,\ k = 0,1,\dots,m \,\},$$

is a group *included in* $\mathrm{Cl}^0(V)$*, and* $\phi\colon \mathrm{Spin}^{\mathrm{c}}(V) \to \mathrm{SO}(V)$ *is a group homomorphism.*

The inverse of $u = w_1 \dots w_{2k}$ is $u^{-1} = u^* = \bar{w}_{2k} \dots \bar{w}_1$.

Suppose $u \in \ker\phi$, which means that $uxu^{-1} = x$ for all $x \in V$. Thus $\ker\phi \subset Z(\mathbb{C}\mathrm{l}(V))$ for n even, and $\ker\phi \subset Z(\mathrm{Cl}^0(V))$ for n odd; in both cases, u lies in $\mathbb{C}1$. It follows that $\ker\phi \cong \{\, \lambda \in \mathbb{C} : |\lambda| = 1 \,\} = \mathbb{T} = \mathrm{U}(1)$. Therefore, there is a *short exact sequence* (SES) of groups:

$$1 \to \mathbb{T} \to \mathrm{Spin}^{\mathrm{c}}(V) \xrightarrow{\phi} \mathrm{SO}(V) \to 1. \tag{1.5}$$

If $u = w_1 \dots w_{2k} \in \mathrm{Spin}^{\mathrm{c}}(V)$ with $w_j = \lambda_j v_j$ where $\lambda_j \in \mathbb{T}$ and $v_j \in V$, then $u^! = w_{2k} \dots w_1$, and $u^!u = \lambda_1^2\lambda_2^2 \dots \lambda_n^2 \in \mathbb{T}$. Thus, $u^!u$ is central, so $(u_1u_2)^!u_1u_2 = u_2^!u_1^!u_1u_2 = u_1^!u_1u_2^!u_2$, so that $u \mapsto u^!u$ is a *homomorphism* $\nu\colon \mathrm{Spin}^{\mathrm{c}}(V) \to \mathbb{T}$, which restricts to $\mathbb{T} \subset \mathrm{Spin}^{\mathrm{c}}(V)$ as $\lambda \mapsto \lambda^2$. The combined $(\phi,\nu)\colon \mathrm{Spin}^{\mathrm{c}}(V) \to \mathrm{SO}(V) \times \mathbb{T}$ is a homomorphism with kernel $\{\pm 1\}$.

Definition 1.11. $\mathrm{Spin}(V) := \ker\nu \le \mathrm{Spin}^{\mathrm{c}}(V)$.

Indeed $\mathrm{Spin}(V)$ is included in (the even part of) the real Clifford algebra $\mathrm{Cl}^0(V,g)$:

$$u^*u = 1,\ u^!u = 1 \implies u^* = u^! \implies \bar{u} = u \implies u \in \mathrm{Cl}^0(V,g).$$

The SES (1.5) now becomes

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \xrightarrow{\phi} \mathrm{SO}(V) \to 1, \tag{1.6}$$

so that ϕ is a double covering of $\mathrm{SO}(V)$. Furthermore, $\mathrm{Spin}^c(V) \cong \mathrm{Spin}(V) \times_{\mathbb{Z}_2} \mathbb{T}$.

Example 1.12. Case $n = 2$: We write $\mathrm{Spin}(n) \equiv \mathrm{Spin}(\mathbb{R}^n)$. It is easy to check that

$$\begin{aligned} \mathrm{Spin}(V) &= \{\, \alpha + \beta e_1 e_2 : \alpha, \beta \in \mathbb{R},\ \alpha^2 + \beta^2 = 1 \,\} \\ &= \{\, u = \cos\frac{\psi}{2} + \sin\frac{\psi}{2} e_1 e_2 : -2\pi < \psi \le 2\pi \,\} \cong \mathbb{T}. \end{aligned}$$

We compute

$$\begin{aligned} ue_1u^{-1} &= \Big(\cos\frac{\psi}{2} + \sin\frac{\psi}{2} e_1e_2\Big) e_1 \Big(\cos\frac{\psi}{2} - \sin\frac{\psi}{2} e_1e_2\Big) = (\cos\psi)\, e_1 - (\sin\psi)\, e_2, \\ ue_2u^{-1} &= \Big(\cos\frac{\psi}{2} + \sin\frac{\psi}{2} e_1e_2\Big) e_2 \Big(\cos\frac{\psi}{2} - \sin\frac{\psi}{2} e_1e_2\Big) = (\sin\psi)\, e_1 + (\cos\psi)\, e_2, \end{aligned}$$

so that

$$\phi(u) = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \in \mathrm{SO}(2),$$

which is (nontrivial) double covering of the circle.

Example 1.13. $\mathrm{Spin}(3) \cong \mathrm{SU}(2) = \{\text{unit quaternions}\}$ in $\mathrm{Cl}^0_{30} \cong \mathrm{Cl}_{02} \cong \mathbb{H}$, and $\phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is the adjoint representation of $\mathrm{SU}(2)$.

Example 1.14. $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ in $\mathrm{Cl}^0_{40} \cong \mathrm{Cl}_{03} \cong \mathbb{H} \oplus \mathbb{H}$. If $u = (q, p)$ with $q, p \in \mathrm{SU}(2)$, then $\phi(u)$ becomes $x \mapsto qxp^{-1}$ for $x \in \mathbb{H} \cong \mathbb{R}^4$, and this map lies in $\mathrm{SO}(4)$. If $\phi(u) = 1_{\mathbb{H}}$, then $1 \mapsto qp^{-1}$, so $p = q$, and $x \mapsto qxq^{-1} = x$, so q is central; hence $p = q = \pm 1$ and ϕ is indeed a double covering of $\mathrm{SO}(4)$.

1.7 The Lie algebra of Spin(V)

Recall the linear isomorphism $Q\colon \Lambda^\bullet V \to \mathrm{Cl}(V, g)$, inverse to $\sigma\colon a \mapsto c(a)1$. Write

$$b = Q(u \wedge v) = \tfrac{1}{2}(uv - vu) = uv + g(u, v)\, 1 \in \mathrm{Cl}^0(V, g).$$

Note in passing that $b^! = \tfrac{1}{2}(vu - uv) = -b$.

Although the algebra $\mathrm{Cl}(V, g)$ is not $\mathbb{Z}$-graded, it is $\mathbb{Z}$-*filtered*: we may write $\mathrm{Cl}^{\le k}(V, g)$ to denote the vector subspace generated by products of *at most* k vectors from V. With that notation, the subspace $Q(\Lambda^2 V)$ may also be described as the set of all even elements $b \in \mathrm{Cl}^{\le 2}(V, g)$ with $\tau(b) = 0$.

For $x \in V$, we compute

$$[b, x] = [uv, x] = uvx + uxv - uxv - xuv = 2g(v, x)u - 2g(u, x)v \ \underline{\in V},$$

so $\operatorname{ad} b\colon V \to V$. Also

$$[b, b'] = \tfrac{1}{2}[b, u'v' - v'u'] = \tfrac{1}{2}[b, u']v' + \tfrac{1}{2}u'[b, v'] - \tfrac{1}{2}[b, v']u' - \tfrac{1}{2}v'[b, u']$$

so that $[b, b'] \in \mathrm{Cl}^{\leq 2}(V, g)$ with $\tau([b, b']) = 0$. Hence $[b, b'] \in Q(\Lambda^2 V)$, and this is a *Lie algebra*. Next,

$$g(y, [b, x]) = 2g(v, x)g(y, u) - 2g(u, x)g(y, v) = -g([b, y], x),$$

so that $\operatorname{ad} b$ is *skewsymmetric*: thus $\operatorname{ad} b \in \mathfrak{so}(V)$. By the Jacobi identity,

$$[[b, b'], x] = [b, [b', x]] - [b', [b, x]] \quad \text{for all} \quad x \in V,$$

and so $\operatorname{ad}([b, b']) = [\operatorname{ad} b, \operatorname{ad} b']$. Thus, $\operatorname{ad}\colon Q(\Lambda^2 V) \to \mathfrak{so}(V)$ is a Lie algebra homomorphism.

If $\operatorname{ad} b = 0$, so that $[b, x] = 0$ for all $x \in V$, then $b \in Z(\mathrm{Cl}^0(V)) \cong \mathbb{C}1$. But $\tau(b) = 0$ then implies $b = 0$, so ad is injective. Since $\dim \Lambda^2 V = n(n-1)/2 = \dim \mathfrak{so}(V)$, we see that $\operatorname{ad}\colon Q(\Lambda^2 V) \to \mathfrak{so}(V)$ is a *Lie algebra isomorphism*.

There is an important formula for the inverse of ad. For $A \in \mathfrak{so}(V)$, define

$$\dot\mu(A) = \frac{1}{4} \sum_{j,k=1}^{n} g(e_j, Ae_k)\, e_j e_k = \frac{1}{2} \sum_{j<k} g(e_j, Ae_k)\, e_j e_k. \tag{1.7}$$

Since $\tau(\dot\mu(A)) = 0$, we get $\dot\mu(A) \in Q(\Lambda^2 V)$. Also

$$\begin{aligned}
[\dot\mu(A), e_r] &= \frac{1}{4} \sum_{j,k} g(e_j, Ae_k)\big(e_j \underbrace{\{e_k, e_r\}}_{\delta_{kr}} - \underbrace{\{e_j, e_r\}}_{\delta_{jr}} e_k\big) \\
&= \frac{1}{2} \sum_j g(e_j, Ae_r)\, e_j - \frac{1}{2} \sum_k g(e_r, Ae_k)\, e_k = \sum_j g(e_j, Ae_r)\, e_j \\
&= Ae_r,
\end{aligned}$$

where we have used the anticommutator notation $\{X, Y\} := XY + YX$. Hence $\operatorname{ad}(\dot\mu(A)) = A \in \mathfrak{so}(V)$.

Now consider $u = \exp b := 1 + \sum_{k \geq 1} \frac{1}{k!}\, b^k \in \mathrm{Cl}^0(V, g)$ for $b \in Q(\Lambda^2 V)$. Then $u^* u = u^! u = \exp(-b) \exp b = 1$ since $b^! = -b$. Also, u is unitary and even, and if $x \in V$ then

$$\begin{aligned}
uxu^{-1} &= \sum_{k,l \geq 0} \frac{1}{k!\,l!}\, b^k x (-b)^l \\
&= \sum_{r \geq 0} \frac{1}{r!} \sum_{k=0}^{r} \binom{r}{k} b^k x (-b)^{r-k} \\
&= \sum_{r \geq 0} \frac{1}{r!} (\operatorname{ad} b)^r (x) \ \underline{\in V},
\end{aligned}$$

and thus $u = \exp(b)$ lies in Spin(V). When $b = \dot\mu(A)$, we get $\phi(\exp(b)) = \exp(\operatorname{ad} b) = \exp(A)$, and it is known that $\exp\colon \mathfrak{so}(V) \to \mathrm{SO}(V)$ is surjective (a property of compact connected matrix groups).

Now $\exp(Q(\Lambda^2 V))$ is a subset of Spin(V) covering all of SO(V). If we can show that $-1 = \exp c$ for some c, then $-\exp b = (\exp b)(\exp c) = \exp(b + c)$, provided that c, b *commute*. If $b = \dot\mu(A)$, we can express the skewsymmetric matrix A as a direct sum of 2×2 skewsymmetric blocks in a suitable orthonormal basis:

$$A = \begin{pmatrix} 0 & * & & & & & & & \\ * & 0 & & & & & & & \\ & & 0 & * & & & & & \\ & & * & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & * & & \\ & & & & & * & 0 & & \\ & & & & & & & \ddots & \end{pmatrix}.$$

That is, we can choose the (oriented) orthonormal basis $\{e_1, \dots, e_n\}$ so that

$$b = \tfrac{1}{2}g(e_1, Ae_2)\, e_1e_2 + \tfrac{1}{2}g(e_3, Ae_4)\, e_3e_4 + \cdots + \tfrac{1}{2}g(e_{2r-1}, Ae_{2r})\, e_{2r-1}e_{2r}$$

with $r \leq m$. Now this particular e_1e_2 commutes with b: $(e_1e_2)b = b(e_1e_2)$; take $c := \pi e_1e_2$. Then $\exp c = \exp(\pi e_1e_2) = \cos\pi + \sin\pi e_1e_2 = -1$. We have shown that $\exp\colon Q(\Lambda^2 V) \to \mathrm{Spin}(V)$ is surjective.

Note that $t \mapsto \exp(te_1e_2)$, for $0 \leq t \leq \pi$, is a path in $\mathrm{Spin}(V)$ from $+1$ to -1. Since $\pi_1(\mathrm{SO}(V)) \cong \mathbb{Z}_2$ for $n \geq 3$, the double covering $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ is nontrivial. We get an important consequence.

Corollary 1.15. $\mathrm{Spin}(n)$ *is simply connected, for* $n \geq 3$. □

1.8 Orthogonal complex structures

Suppose that $n = 2m$ is even, $V \cong \mathbb{R}^{2m}$. Then V can be identified with $\mathbb{C}^m$, *but not canonically.*

Definition 1.16. *An operator* $J \in \mathrm{End}_{\mathbb{R}} V$ *is called an* orthogonal complex structure, *written* $J \in \mathcal{J}(V, g)$, *if*

(a) $J^2 = -1$ *in* $\mathrm{End}_{\mathbb{R}} V$*;*

(b) $g(Ju, Jv) = g(u, v)$ *for all* $u, v \in V$*.*

Then also $g(Ju, v) = -g(Ju, J^2v) = -g(u, Jv)$, so that J is skewsymmetric with respect to g: $J^t = -J$. Note that (b) says that $J^tJ = 1$.

We can now make V a $\mathbb{C}$-module by setting $iv := Jv$, that is,

$$(\alpha + i\beta)v := \alpha v + \beta Jv \quad \text{for all} \quad \alpha, \beta \in \mathbb{R}.$$

We define a hermitian scalar product on V by

$$\langle u \mid v \rangle_J := g(u, v) + i\, g(Ju, v)$$

Note that $\langle Ju \,|\, v\rangle_J = -i\langle u \,|\, v\rangle_J$ and $\langle u \,|\, Jv\rangle_J = +i\langle u \,|\, v\rangle_J$ (check it!). We denote the resulting m-dimensional complex Hilbert space by V_J.

If $\{u_1, \dots, u_m\}$ is an orthonormal basis for $V_J := (V, \langle\cdot \mid \cdot\rangle_J)$, then

$$\{u_1, Ju_1, \dots, u_m, Ju_m\}$$

is an orthonormal oriented basis for V (over $\mathbb{R}$). The orientation may or may not be compatible with the given one on V.

Exercise 1.17. *If $2m = 4$, show that all such J can be parametrized by two disjoint copies of $\mathbb{S}^2$, one for each orientation.*

If $J^2 = -1$, $J^t J = 1$ and if $h \in \mathrm{O}(n) = \mathrm{O}(V, g)$ is an orthogonal linear transformation, then $K := hJh^{-1}$ is also an orthogonal complex structure. In that case,

$$\begin{aligned}\langle hu \mid hv \rangle_K &= g(hu, hv) + i\, g((hJh^{-1})hu, hv) \\ &= g(u, v) + i\, g(Ju, v) = \langle u \mid v \rangle_J,\end{aligned}$$

so that $h\colon V_J \to V_K$ is unitary. Thus $hJh^{-1} = J$ if and only if $h \in \mathrm{U}(V_J) \cong \mathrm{U}(m)$. In short: $\mathrm{O}(n) = \mathrm{O}(2m)$ acts transitively on $\mathcal{J}(V, g)$ with isotropy subgroups isomorphic to $\mathrm{U}(m)$. Hence, as a manifold,

$$\mathcal{J}(V, g) \approx \mathrm{O}(2m)/\, \mathrm{U}(m).$$

Those J which are compatible with orientation on V form one component (of two), homeomorphic to $\mathrm{SO}(2m)/\, \mathrm{U}(m)$.

We may complexify V to get $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$. Take

$$W_J := \{\, v - iJv \in V^{\mathbb{C}} : v \in V \,\} = \tfrac{1}{2}(1 - iJ)V = P_J V.$$

This is an isotropic subspace for the symmetric bilinear form $g^{\mathbb{C}}$ on $V^{\mathbb{C}}$.

$$g(u - iJu, v - iJv) = g(u, v) - ig(u, Jv) - ig(Ju, v) - g(Ju, Jv) = 0.$$

The conjugate subspace

$$\overline{W}_J = \{\, v + iJv : v \in V \,\} = \tfrac{1}{2}(1 + iJ)V = P_J V$$

satisfies $W_J \oplus \overline{W}_J \cong V^{\mathbb{C}}$, an orthogonal direct sum for the hermitian scalar product

$$\langle\!\langle w \mid z \rangle\!\rangle := 2\, g(\bar{w}, z) \quad \text{for } w, z \in V^{\mathbb{C}}.$$

Note that $P_J^2 = P_J$ and $P_J = P_J^*$ with respect to this product. We say that W_J is a *polarization* of $V^{\mathbb{C}}$. Also $P_J\colon V_J \to W_J$ is an unitary isomorphism.

Conversely: given a splitting $V = W \oplus \overline{W}$, orthogonal with respect to $\langle\!\langle \cdot \mid \cdot \rangle\!\rangle$, write $w =: u - iv$ for $w \in W$, with $u, v \in V$; then $J_W\colon u \mapsto v$ lies in $\mathcal{J}(V, g)$, and $W_{J_W} = W$ (exercise). Thus the correspondence $J \leftrightarrow W_J$ is bijective.

1.9 Irreducible representations of $\mathbb{C}l(V)$

We continue to suppose that $n = 2m$ is even.

Definition 1.18. *The (fermionic)* Fock space *corresponding to $J \in \mathcal{J}(V, g)$ is defined as*

$$\mathcal{F}_J(V) := \Lambda^\bullet W_J,$$

with hermitian scalar product given by

$$\langle\!\langle w_1 \wedge \cdots \wedge w_k \mid z_1 \wedge \cdots \wedge z_l \rangle\!\rangle := \delta_{kl} \det\big[\langle\!\langle w_i \mid z_j \rangle\!\rangle\big]. \tag{1.8}$$

This is a complex Hilbert space of dimension 2^m. Choose and fix a unit vector $\Omega \in \Lambda^0 W_J$: it is unique up to a factor $\lambda \in \mathbb{T}$. For $w \in W_J$ (so that $\bar{w} \in \overline{W}_J = W_J^\perp$), we write

$$\begin{aligned}
&\varepsilon(w) : z_1 \wedge \cdots \wedge z_k \mapsto w \wedge z_1 \wedge \cdots \wedge z_k, \\
&\iota(\bar{w}) : z_1 \wedge \cdots \wedge z_k \mapsto \sum_{j=1}^{k} (-1)^{k-1} \langle\!\langle w \mid z_j \rangle\!\rangle \, z_1 \wedge \cdots \wedge \widehat{z_j} \wedge \cdots \wedge z_k.
\end{aligned}$$

For $v \in V$, write $w = \frac{1}{2}(v - iJv) = P_J v \in W_J$ and define

$$c_J(v) := \varepsilon(w) + \iota(\bar{w}) = \varepsilon(P_J v) + \iota(P_{-J} v).$$

Then

$$c_J^2(v) := \langle\!\langle w \mid w \rangle\!\rangle \, 1 = \langle v \mid v \rangle_J \, 1 = g(v, v) \, 1,$$

so that $c_J \colon V \to \operatorname{End}_{\mathbb{C}}(\mathcal{F}_J V) \equiv \mathcal{L}(\mathcal{F}_J V)$. That is to say, c_J is a *representation* of $\mathbb{C}l(V)$ on the Hilbert space $\mathcal{F}_J V$.

Note that we complexify the representation of $\mathrm{Cl}(V, g)$, given by universality. One can check that

$$c_J(w) = \varepsilon(w) \quad \text{if } w \in W_j; \quad c_J(\bar{z}) = \iota(\bar{z}) \quad \text{if } \bar{z} \in \overline{W}_J.$$

From (1.8) and the properties of determinants, it is easy to check that the operators $\varepsilon(w)$ and $\iota(\bar{w})$ are adjoint to one another, that is, $\varepsilon(w)^\dagger = \iota(\bar{w})$ for $w \in W_J$; in particular, $c_J(v)^\dagger = c_J(v)$ for $v \in V$. (This is a consequence of our choice of g to have positive definite signature: were we to have taken g to be negative definite, as in done in many books, then the operators $c_J(v)$ would have been skewadjoint.) More generally, we get $c_J(a)^\dagger = c_J(a^*)$ for $a \in \mathbb{C}l(V)$: we say that c_J is a *selfadjoint representation* of the $*$-algebra $\mathbb{C}l(V)$ on the Fock space $\mathcal{F}_J(V)$.

Now, if $T \in \mathcal{L}(\mathcal{F}_J(V))$ *commutes* with $c_J(V^{\mathbb{C}})$, then in particular $\iota(\bar{z})T\Omega = T\iota(\bar{z})\Omega = T(0) = 0$ for $\bar{z} \in \overline{W}_J$. Therefore $T\Omega \in \Lambda^0 W_J$, i.e., $T\Omega = t\Omega$ for some $t \in \mathbb{C}$. Now

$$T(w_1 \wedge \cdots \wedge w_k) = T\varepsilon(w_1) \dots \varepsilon(w_k)\Omega = \varepsilon(w_1) \dots \varepsilon(w_k) T\Omega = t \, w_1 \wedge \cdots \wedge w_k$$

for $w_1, \dots, w_k \in W_J$. Thus $T = t\,1 \in \mathcal{L}(\Lambda^\bullet W_J)$. By Schur's lemma, the representation c_J is *irreducible*.

Suppose $K \in \mathcal{J}(V, g)$ with $K = hJh^{-1}$ for $h \in \mathrm{O}(2m)$. Then $hJ = Kh$, $hP_{\pm J} = P_{\pm K} h$, and so $c_K(hv) = (\Lambda^\bullet h) \, c_J(v)$. By universality again, we get $c_K \circ \Lambda^\bullet h = \Lambda^\bullet h \circ c_J$, so that the irreducible representations c_K and c_J are *equivalent*.

The Fock space is $\mathbb{Z}_2$-*graded* as $\Lambda^{\mathrm{even}} W_J \oplus \Lambda^{\mathrm{odd}} W_J$. What operator determines its $\mathbb{Z}_2$-grading? In fact, this operator is $c_J(\gamma)$. To see that, write $\gamma = (-1)^m e_1 e_2 \dots e_{2m}$, where $e_{2j} = Je_{2j-1}$ for $j = 1, \dots, m$. If $z_1 := P_J e_1 = \frac{1}{2}(e_1 - ie_2)$ we get

$$\bar{z}_1 z_1 - z_1 \bar{z}_1 = \tfrac{1}{4}(e_1 + ie_2)(e_1 - ie_2) - \tfrac{1}{4}(e_1 - ie_2)(e_1 + ie_2) = -e_1 e_2.$$

With $z_j := P_J e_{2j-1} = \frac{1}{2}(e_{2j-1} - ie_{2j})$, this gives

$$\gamma = (\bar{z}_1 z_1 - z_1 \bar{z}_1) \dots (\bar{z}_m z_m - z_m \bar{z}_m) \quad \text{in } \mathbb{C}l^0(V).$$

Now $c_J(\bar{z}_j z_j - z_j \bar{z}_j) = \iota(\bar{z}_j)\varepsilon(z_j) - \varepsilon(z_j)\iota(\bar{z}_j)$ is the operator

$$z_{k_1} \wedge \cdots \wedge z_{k_r} \longmapsto \begin{cases} -z_{k_1} \wedge \cdots \wedge z_{k_r} & \text{if } j \in K \\ +z_{k_1} \wedge \cdots \wedge z_{k_r} & \text{if } j \notin K. \end{cases}$$

Thus $c_J(\gamma)$ acts as $(-1)^k$ on $\Lambda^k W_J$: this is indeed the $\mathbb{Z}_2$-grading operator.

Finally, the *odd case* is treated as follows. Let $U := \mathbb{R}\text{-}\operatorname{span}\{e_1, \dots, e_{2m}\} \leq V$. Then $\mathbb{C}l(U) \cong \mathbb{C}l^0(V)$ via $u \mapsto i\, u e_{em+1}$, extended to $\mathbb{C}l(U)$. Now $\mathcal{F}_J(U)$ is an irreducible $\mathbb{C}l^0(V)$-module, while $Z(\mathbb{C}l(V)) = \mathbb{C}1 \oplus \mathbb{C}\gamma$. Since $\gamma^2 = 1$, we can extend the action of $\mathbb{C}l^0(V)$ on $\mathcal{F}_J(U)$ to the full $\mathbb{C}l(V)$ by setting either $c_J(\gamma) := +1$ or $c'_J(\gamma) := -1$ on $\mathcal{F}_J(U)$.

These representations c_J, c'_J are *inequivalent*, since $T(1) = (-1)T$ is not possible unless $T = 0$, using Schur's lemma again. Thus $\mathbb{C}l(V)$ has two irreducible Fock representations of dimension 2^m in the odd case.

Proposition 1.19. *The Fock representations yield all irreducible representations of* $\mathbb{C}l(V)$*. If* $\dim_{\mathbb{R}} V = 2m$*, the irreducible representation is unique up to equivalence; if* $\dim_{\mathbb{R}} V = 2m + 1$*, there are exactly two such representations.*

Proof. We have already described and classified the Fock representations. It remains to show that this list is complete.

We have seen that up to tensoring with a matrix algebra $M_N(\mathbb{R})$, the real Clifford algebras occur in eight species. The periodicity of *complex* Clifford algebras is much simpler, and may be obtained from (1.4) by complexifying each algebra found there. Since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$, we obtain directly that

$$\mathbb{C}l(\mathbb{R}^{2m}) \cong M_{2^m}(\mathbb{C}) \quad \text{and} \quad \mathbb{C}l(\mathbb{R}^{2m+1}) \cong M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}). \tag{1.9}$$

From this it is clear that, when $\dim V$ is even, $\mathbb{C}l(V)$ is a simple matrix algebra and therefore all irreducible representations are equivalent and arise from matrix multiplication on a minimal left ideal, whose dimension is 2^m. Similar arguments in the odd case show that there are at most two inequivalent representations of $\mathbb{C}l(V)$. Thus the Fock representations we have constructed account for all of them: there are no others. $\square$

1.10 Representations of $\mathrm{Spin}^{\mathrm{c}}(V)$

We obtain representations of the *group* $\mathrm{Spin}^{\mathrm{c}}(V)$ by restriction of the irreducible representations of $\mathbb{C}l(V)$.

$$\mathrm{Spin}^{\mathrm{c}}(V) = \{\, w_1 w_2 \dots w_{2k} : w_i \in V^{\mathbb{C}},\ w_i^* w_i = 1 \,\}.$$

We have to check whether these restrictions are irreducible or not.

Even case, $n = 2m$: γ belongs to $\mathrm{Spin}^{\mathrm{c}}(V)$ and is central there, so $c_J(\gamma)$ commutes with $c_J(\mathrm{Spin}^{\mathrm{c}}(V))$. Thus the group representation reduces over $\Lambda^\bullet W_J = \Lambda^{\mathrm{even}} W_J \oplus \Lambda^{\mathrm{odd}} W_J$: there are two subrepresentations. Since

$$w_1 \wedge \cdots \wedge w_{2k} = \varepsilon(w_1) \dots \varepsilon(w_{2k}) = c_J(w_1) \dots c_J(w_{2k})\, \Omega,$$

we get at once that $c_J(\mathrm{Spin}^c(V))\,\Omega = \Lambda^{\mathrm{even}} W_J$: the "even" subrepresentation is irreducible.

If $w_1, w_2 \in W_J$ are unit vectors, then

$$c_J(w_2 \bar{w}_1)\, w_1 = \varepsilon(w_2)\iota(\bar{w}_1)\, w_1 = \varepsilon(w_2)\,\Omega = w_2.$$

From there we soon conclude that $c_J(\mathrm{Spin}^c(V))\, w_1 = \Lambda^{\mathrm{odd}} W_J$: the "odd" subrepresentation is also irreducible.

Are these subrepresentations equivalent? *No*: for suppose $R \colon \Lambda^{\mathrm{even}} W_J \to \Lambda^{\mathrm{odd}} W_J$ intertwines both subrepresentations. Then in particular $Rc_J(\gamma) = c_J(\gamma)R$ means that $R(+1) = (-1)R : \Lambda^{\mathrm{even}} W_J \to \Lambda^{\mathrm{odd}} W_J$, so that $R = 0$.

Conclusion: The algebra representation c_J of $\mathbb{C}\mathrm{l}(V)$ restricts to a group representation c_J of $\mathrm{Spin}^c(V)$ which is the direct sum of two inequivalent irreducible subrepresentations, if $\dim V$ is even.

Odd case, $n = 2m+1$: There are two irreducible representations c_J and c'_J of $\mathbb{C}\mathrm{l}(V)$ on $\mathcal{F}_J(U)$, but they *coincide* on $\mathbb{C}\mathrm{l}^0(V)$: in this case, γ is odd. Declaring $c_J(\gamma)$ to be, say, $+1$ on $\mathcal{F}(U)$, we get for $w_1, \dots, w_{2k+1} \in W_J$:

$$\begin{aligned} w_1 \wedge \cdots \wedge w_{2k} &= c_J(w_1 \dots w_{2k})\,\Omega, \\ w_1 \wedge \cdots \wedge w_{2k+1} &= c_J(w_1 \dots w_{2k+1}\gamma)\,\Omega, \end{aligned}$$

so that in this case, $\Lambda^\bullet W_J$ is an irreducible representation if $\dim V$ is odd.

Conclusion: The two algebra representations c_J and c'_J of $\mathbb{C}\mathrm{l}(V)$ restrict to the same group representation c_J of $\mathrm{Spin}^c(V)$ which is already irreducible, if $\dim V$ is odd.

Chapter 2

Spinor modules over compact Riemannian manifolds

2.1 Remarks on Riemannian geometry

Let M be a *compact* C^∞ manifold *without boundary*, of dimension n. Compactness is not crucial for some of our arguments (although it may be for others), but is very convenient, since it means that the algebras $C(M)$ and $C^\infty(M)$ are *unital*: the unit is the constant function 1. For convenience we use the function algebra $A = C(M)$ —a commutative C*-algebra— at the beginning. We will change to $\mathcal{A} = C^\infty(M)$ later, when the differential structure becomes important.

Any A-module (or more precisely, a "symmetric A-bimodule") which is *finitely generated and projective* is of the form $\mathcal{E} = \Gamma(M, E)$ for $E \to M$ a (complex) vector bundle. Two important cases are

$$\begin{aligned} \mathfrak{X}(M) = \Gamma(M, T_\mathbb{C}M) &= \text{ (continuous) vector fields on } M; \\ \mathcal{A}^1(M) = \Gamma(M, T^*_\mathbb{C}M) &= \text{ (continuous) 1-forms on } M. \end{aligned}$$

These are *dual* to each other: $\mathcal{A}^1(M) \cong \operatorname{Hom}_A(\mathfrak{X}(M), A)$, where Hom_A means "A-module maps" commuting with the action of A (by multiplication).

Definition 2.1. *A* Riemannian metric *on M is a symmetric bilinear form*

$$g\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C(M)$$

such that:

1. *$g(X, Y)$ is a real function if X, Y are real vector fields;*
2. *g is $C(M)$-*bilinear*: $g(fX, Y) = g(X, fY) = f\,g(X, Y)$, if $f \in C(M)$;*
3. *$g(X, X) \geq$ for X real, with $g(X, X) = 0 \implies X = 0$ in $\mathfrak{X}(M)$.*

The second condition entails that g is given by a continuous family of symmetric bilinear maps $g_x \colon T^{\mathbb{C}}_x M \times T^{\mathbb{C}}_x M \to \mathbb{C}$ or $g_x \colon T_x M \times T_x M \to \mathbb{R}$; the latter version is positive definite.

Fact 2.2. *Riemannian metrics always exist (in abundance).*

Since each g_x is positive definite, there are "musical isomorphisms" between $\mathfrak{X}(M)$ and $\mathcal{A}^1(M)$, as A-modules

$$\mathfrak{X}(M) \underset{\alpha^\sharp \leftarrow \alpha}{\overset{X \to X^\flat}{\rightleftarrows}} \mathcal{A}^1(M) \quad \text{given by} \quad \begin{cases} X^\flat(Y) := g(X, Y), \\ \ \alpha(Y) =: g(\alpha^\sharp, Y). \end{cases}$$

They are mutually inverse, of course. In fact, they can be used to transfer the metric form $\mathfrak{X}(M)$ to $\mathcal{A}^1(M)$:

$$g(\alpha, \beta) := g(\alpha^\sharp, \beta^\sharp), \quad \text{for } \alpha, \beta \in \mathcal{A}^1(M).$$

One should perhaps write $g^{-1}(\alpha, \beta)$ —as is done in [fgv01]— since in local coordinates $g_{ij} := g(\partial/\partial x^i, \partial/\partial x^j)$ and $g^{rs} := g(dx^r, dx^s)$ have inverse matrices: $[g^{rs}] = [g_{ij}]^{-1}$.

If $f \in C^1(M)$, the *gradient* of f is $\operatorname{grad} f := (df)^\sharp$, so that

$$g(\operatorname{grad} f, Y) = df(Y) := Yf.$$

2.2 Clifford algebra bundles

More generally a real vector bundle $E \to M$ is a *Euclidean bundle* if, with $\mathcal{E} = \Gamma(M, E^{\mathbb{C}})$, there is a symmetric A-bilinear form $g \colon \mathcal{E} \times \mathcal{E} \to A = C(M)$ such that

1. $g(s, t) \in C(M; \mathbb{R})$ when s, t lie in $\Gamma(M, E)$ —the real sections;
2. $g(s, s) \geq 0$ for $s \in \Gamma(M, E)$, with $g(s, s) = 0 \implies s = 0$.

By defining $(s \mid t) := g(s^*, t)$, we get a *hermitian pairing* with values in A:

- $(s \mid t)$ is A-linear in t;
- $(t \mid s) = \overline{(s \mid t)} \in A$;
- $(s \mid s) \geq 0$, with $(s \mid s) = 0 \implies s = 0$ in $\mathcal{E}$;
- $(s \mid ta) = (s \mid t)\, a$ for all $s, t \in \mathcal{E}$ and $a \in A$.

These properties make $\mathcal{E}$ a (right) C^*-module over A, with C^*-norm given by

$$\|s\|_{\mathcal{E}} := \sqrt{\|(s \mid s)\|_A} \quad \text{for } s \in \mathcal{E}.$$

For each $x \in M$, we can form $\mathbb{C}\mathrm{l}(E_x) := \mathrm{Cl}(E_x, g_x) \otimes_{\mathbb{R}} \mathbb{C}$. Using the linear isomorphisms $\sigma_x \colon \mathbb{C}\mathrm{l}(E_x) \to (\Lambda^\bullet E_x)^{\mathbb{C}}$, we see that these are fibres of a vector bundle $\mathbb{C}\mathrm{l}(E) \to M$, isomorphic to $(\Lambda^\bullet E)^{\mathbb{C}} \to M$ as $\mathbb{C}$-vector bundles (but not as algebras!). Under $(\kappa\lambda)(x) := \kappa(x)\lambda(x)$, the sections of $\mathbb{C}\mathrm{l}(E)$ also form an algebra $\Gamma(M, \mathbb{C}\mathrm{l}(E))$. It has an A-valued pairing

$$(\kappa \mid \lambda) \colon x \mapsto \tau(\kappa(x)^* \lambda(x)).$$

By defining $\|\kappa\| := \sup_{x \in M} \|\kappa(x)\|_{\mathbb{C}\mathrm{l}(E_x)}$, this becomes a C*-algebra.

Lemma 2.3. *If g, h are two different "metrics" on $\mathcal{E} = \mathcal{A}^1(M)$, the corresponding C^*-algebras*

$$B_g := \Gamma(M, \mathrm{Cl}(T^*M, g) \otimes_{\mathbb{R}} \mathbb{C}) \quad \textit{and} \quad B_h := \Gamma(M, \mathrm{Cl}(T^*M, h) \otimes_{\mathbb{R}} \mathbb{C})$$

are isomorphic.

Proof. We compose $\alpha \mapsto \alpha^{\sharp_g} : \mathcal{A}^1(M) \to \mathfrak{X}(M)$ and $X \mapsto X^{\flat_h} : \mathfrak{X}(M) \to \mathcal{A}^1(M)$ to get an A-linear isomorphism $\rho\colon \mathcal{A}^1(M) \to \mathcal{A}^1(M)$. Now

$$h(\bar\alpha, \rho(\alpha)) = \bar\alpha(\alpha^{\sharp_g}) = g(\bar\alpha, \alpha) \geq 0, \quad \text{for all} \quad \alpha \in \mathcal{A}^1(M). \tag{2.1}$$

At each $x \in M$, the $\mathbb{C}$-vector space $T_x^{\mathbb{C}}(M)$ may be regarded as a Hilbert space with scalar product $\langle \alpha_x \mid \beta_x \rangle_h := h_x(\bar\alpha_x, \beta_x)$, and now (2.1) says that each $\rho_x \in \mathrm{End}_{\mathbb{C}}(T_x^{\mathbb{C}} M)$ is a positive operator with a positive square root σ_x: we thereby obtain an A-linear isomorphism $\sigma\colon \mathcal{A}^1(M) \to \mathcal{A}^1(M)$ such that $\rho = \sigma^2$. We may regard σ as an injective A-linear map from $\mathcal{A}^1(M)$ into the algebra B_h; when $\alpha \in \mathcal{A}^1(M)$ is real, we get

$$\sigma(\alpha)^2 = h(\sigma(\alpha), \sigma(\alpha))\, 1 = h(\alpha, \rho(\alpha))\, 1 = g(\alpha, \alpha)\, 1.$$

By a now familiar argument, applied to each σ_x separately, we may extend σ to an A-linear unital *-algebra homomorphism $\tilde\sigma\colon B_g \to B_h$. Exchanging g and h gives an inverse homomorphism $\tilde\sigma^{-1}\colon B_h \to B_g$. Since any unital $*$-homomorphism between C*-algebras is automatically norm-decreasing and thus continuous, we conclude that $\tilde\sigma\colon B_g \to B_h$ is an isomorphism of C*-algebras. □

Definition 2.4. *A* Clifford module *over (M, g) is a finitely generated projective A-module, with $A = C(M)$, of the form $\mathcal{E} = \Gamma(M, E)$ for E a (complexified) Euclidean bundle, together with an A-linear homomorphism $c\colon B \to \Gamma(M, \mathrm{End}\, E)$, where $B := \Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$ is the Clifford algebra bundle generated by $\mathcal{A}^1(M)$, such that*

$$(s \mid c(\kappa)t) = (c(\kappa^*)s \mid t) \quad \textit{for all} \quad s, t \in \mathcal{E},\ \kappa \in B.$$

Example 2.5. Take $\mathcal{E} = \mathcal{A}^\bullet(M) = \Gamma(M, (\Lambda^\bullet T^*M)^{\mathbb{C}})$ —all differential forms on M— with $c(\alpha)\colon \omega \mapsto \varepsilon(\alpha)\omega + \iota(\alpha^\sharp)\omega$ for $\omega \in \mathcal{A}^\bullet(M)$ and $\alpha \in \mathcal{A}^1(M)$ real. Then $\mathcal{E}$ is indeed a Clifford module, but it is a rather large one: it may have nontrivial submodules. The goal of the next subsection is top explore how some minimal submodules may be constructed.

2.3 The existence of Spin^c structures

Suppose $n = 2m + 1 = \dim M$ is odd. Then the fibres of B are semisimple but not simple: $\mathbb{C}\mathrm{l}(T_x^*M) \cong M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$. We shall restrict to the even subalgebras, $\mathbb{C}\mathrm{l}^0(T_x^*M) \cong M_{2^m}(\mathbb{C})$, by demanding *that $c(\gamma)$ act as the identity in all cases.* Then we may adopt the convention that

$$c(\kappa) := c(\kappa\gamma) \quad \text{when } \kappa \text{ is odd.}$$

Notice here that $\kappa\gamma$ is even; and $c(\gamma) = c(\gamma^2) = +1$ is required for consistency of this rule.

We take $A = C(M)$, but for B we now take

$$B := \begin{cases} \Gamma(M, \mathbb{C}\mathrm{l}(T^*M)), & \text{if } \dim M \text{ is even,} \\ \Gamma(M, \mathbb{C}\mathrm{l}^0(T^*M)), & \text{if } \dim M \text{ is odd.} \end{cases} \tag{2.2}$$

The fibres of these bundles are *central simple algebras* of finite dimension 2^{2m} in all cases.

We classify the algebras B as follows. Taking

$$\underline{B} := \begin{cases} \{\, B_x = \mathbb{C}\mathrm{l}(T_x^*M) : x \in M \,\}, & \text{if } \dim M \text{ is even} \\ \{\, B_x = \mathbb{C}\mathrm{l}^0(T_x^*M) : x \in M \,\}, & \text{if } \dim M \text{ is odd} \end{cases}$$

to be the collection of fibres, we can say that $\underline{B}$ is a "continuous field of simple matrix algebras", which moreover is *locally trivial*. There is an invariant

$$\delta(\underline{B}) \in \mathrm{H}^3(M;\mathbb{Z})$$

for such fields, found by Karrer [k-g63] and in more generality —allowing the compact operators $\mathcal{K}$ as an infinite-dimensional simple matrix algebra— by Dixmier and Douady [d-j69].

Here is a (rather pedestrian) sketch of how $\delta(\underline{B})$ is constructed:

If $x \in M$, take $p_x \in B_x$ to be a *projector of rank one*, that is,

$$p_x = p_x^* = p_x^2 \quad \text{and} \quad \operatorname{tr} p_x = 1.$$

On the left ideal $S_x := B_x p_x$, we introduce a hermitian scalar product

$$\langle a_x p_x \mid b_x p_x \rangle := \operatorname{tr}(p_x a_x^* b_x p_x). \tag{2.3}$$

Notice that the recipe

$$|a_x p_x\rangle\langle b_x p_x| : c_x p_x \mapsto (a_x p_x)(b_x p_x)^*(c_x p_x) = (a_x p_x b_x^*)(c_x p_x)$$

identifies $\mathcal{L}(S_x)$ —or $\mathcal{K}(S_x)$ in the infinite-dimensional case— with B_x, since the two-sided ideal $\operatorname{span}\{\, a_x p_x b_x^* : a_x, b_x \in B_x \,\}$ equals B_x by simplicity.

By local triviality, this can be done locally with varying x. If $\{U_i\}$ is a "good" open cover[1] of M, we get local fields $\underline{S}_i = \{\, S_{i,x} : x \in U_i \,\}$ with isomorphisms $\underline{\theta}_i \colon \mathcal{L}(\underline{S}_i) \to \underline{B}\big|_{U_i}$ of fields of simple C*-algebras. On nonempty intersections $U_{ij} := U_i \cap U_j$, we get $*$-algebra isomorphisms $\underline{\theta}_i^{-1}\underline{\theta}_j \colon \mathcal{L}(\underline{S}_j) \to \mathcal{L}(\underline{S}_i)$, so there are fields of unitary maps $\underline{u}_{ij} \colon \underline{S}_j \to \underline{S}_i$ such that $\underline{\theta}_i^{-1}\underline{\theta}_j = \underline{u}_{ij}(\cdot)\underline{u}_{ij}^{-1}$.

On $U_{ijk} := U_i \cap U_j \cap U_k$, we see that $(\operatorname{Ad}\underline{u}_{ij})(\operatorname{Ad}\underline{u}_{jk}) = \operatorname{Ad}\underline{u}_{ik}$, and so

$$\underline{u}_{ij}\underline{u}_{jk} = \underline{\lambda}_{ijk}\underline{u}_{ik},$$

where $\underline{\lambda}_{ijk} \colon U_{ijk} \to \mathbb{T}$ are *scalar* maps. We may now check that $\underline{\lambda}_{jkl}\underline{\lambda}_{ikl}^{-1}\underline{\lambda}_{ijl} = \underline{\lambda}_{ijk}$ on U_{ijkl}. Thus $\underline{\lambda}$ is a *Čech 2-cocycle*, and its Čech cohomology class lies in $\check{\mathrm{H}}^2(M;\mathbb{T}) \cong \mathrm{H}^3(M;\mathbb{Z})$. We may go one more step in order to exhibit this

[1] The word *good* has a precise technical meaning: namely, that all nonempty finite intersections of open sets of the cover are both connected and simply connected. On Riemannian manifolds, good open covers may always be formed using geodesically convex balls.

isomorphism: if we write $\underline{\lambda}_{ijk} = \exp(2\pi i \underline{f}_{ijk})$ —we can take logarithms since U_{ijk} is simply connected— then

$$a_{ijkl} := \underline{f}_{ijk} - \underline{f}_{ijl} + \underline{f}_{ikl} - \underline{f}_{jkl}$$

takes values in $\mathbb{Z}$ (and since each U_{ijkl} is connected, these will be constant functions); thus, these a_{ijkl} form a $\mathbb{Z}$-valued 3-cocycle, a. Finally, one may check that its class $[a] \in \mathrm{H}^3(M;\mathbb{Z})$ is independent of all choices made so far. We define $\delta(\underline{B}) := [a]$, which is called the *Dixmier–Douady class* of $\underline{B}$.

Suppose now that the Hilbert spaces $S_x \cong \mathbb{C}^{2^m}$ can be chosen *globally* for $x \in M$ —not just locally for $x \in U_i$— that is, they are fibres of a vector bundle $S \to M$ (that may b gifted with a Hermitian metric) such that $\mathcal{L}(S_x) \cong B_x$, for $x \in M$, via a single field of isomorphisms $\underline{\theta}\colon \underline{\mathcal{L}(S)} \to \underline{B}$ such that $\underline{\theta}_i = \underline{\theta}\big|_{U_i}$ for each U_i. Then $\underline{u}_{ij} = \underline{\theta}_i^{-1}\underline{\theta}_j = \mathrm{id}$ over U_{ij}, and so $\underline{\lambda}_{ijk} = 1$ over U_{ijk}, and $a_{ijkl} = 0$ over each U_{ijkl}; hence $\delta(\underline{B}) = [a] = 0$ in $\mathrm{H}^3(M;\mathbb{Z})$.

Conversely if $\delta(\underline{B}) = 0$, so that $[\underline{\lambda}]$ is trivial in $\check{\mathrm{H}}^2(M;\mathbb{T})$, i.e., $\underline{\lambda}$ is a 2-coboundary, then there are maps $\underline{\nu}_{ij}\colon U_{ij} \to \mathbb{T}$ such that $\underline{\lambda}_{ijk} = \underline{\nu}_{ij}\underline{\nu}_{ik}^{-1}\underline{\nu}_{jk}$ on U_{ijk}. Setting $\underline{v}_{ij} := \underline{\nu}_{ij}^{-1}\underline{u}_{ij}$, we get local fields of unitaries such that $\underline{v}_{ij}\underline{v}_{jk} = \underline{v}_{ik}$ on each U_{ijk}. These $\underline{v}_{ij}\colon \underline{S}_j \to \underline{S}_i$ are therefore *transition functions* for a (Hermitian) vector bundle $S \to M$ such that $\underline{S}\big|_{U_i} \cong \underline{S}_i$ for each U_i. Let $\mathcal{S} := \Gamma(M,S)$ denote the A-module of sections of this bundle. Now the pointwise isomorphisms $B_x \cong \operatorname{End} S_x$, for each $x \in M$, imply that $B \cong \operatorname{End}_A S$ as A-modules, and indeed as C*-algebras. We summarize all this in the following Proposition.

Proposition 2.6. *Let (M,g) be a compact Riemannian manifold. With $A = C(M)$ and B the algebra of Clifford sections given by (2.2), the Dixmier–Douady class vanishes, i.e., $\delta(\underline{B}) = 0$, if and only if there is a finitely generated projective A-module $\mathcal{S}$, carrying a selfadjoint action of B by A-linear operators, such that* $\operatorname{End}_A(S) \cong B$.

2.4 Morita equivalence for (commutative) unital algebras

Definition 2.7. *If A, B are* unital $\mathbb{C}$*-algebras, we say that they are* Morita equivalent *if there is a B-A-bimodule $\mathcal{E}$ and an A-B-bimodule $\mathcal{F}$, such that $\mathcal{E} \otimes_A \mathcal{F} \cong B$ as B-B-bimodules and $\mathcal{F} \otimes_B \mathcal{E} \cong A$ as A-A-bimodules. We say that such an $\mathcal{E}$ is an "equivalence bimodule".*

In general, we may choose $\mathcal{F} \cong \mathcal{E}^\sharp := \operatorname{Hom}_A(\mathcal{E}, A)$ to be the "dual" right A-module with a specified action of B. We can then identify $\operatorname{End}_A \mathcal{E} \cong B$.

Fact 2.8. $\operatorname{End}_A \mathcal{E} \cong B$ *whenever $\mathcal{E}$ is an equivalence B-A-bimodule.*

Fact 2.9. *Since A, B are unital, each $\mathcal{E}$ is finitely generated and projective (and full).*

Remark 2.10. There is a C*-version, due to Rieffel, whereby all bimodules are provided with compatible A-valued and B-valued Hermitian pairings. This becomes nontrivial in the more general context of nonunital algebras. The full story is told in [rw98]. We shall not need this machinery in the unital case: remember that M is taken to be compact.

Notation. For isomorphism classes $[\mathcal{E}]$ of bimodules, we form the set

$$\mathrm{Mrt}(B, A) := \{\, [\mathcal{E}] : \mathcal{E} \text{ is a } B\text{-}A\text{-equivalence bimodule} \,\}.$$

In the case $B = A$, we write $\mathrm{Pic}(A) := \mathrm{Mrt}(A, A)$; this is called the "Picard group" of A.

We call an A-bimodule $\mathcal{E}$ *symmetric* if the left and right actions are the same: $a \triangleright x = x \triangleleft a$ for $x \in \mathcal{E}$ and $a \in A$. When A is commutative, a symmetric A-A-bimodule can be called, more simply, an "A-module" —as we have already been doing. Even when A is commutative, an A-A-equivalence bimodule $\mathcal{E}$ need not be symmetric. Indeed, suppose $\phi, \psi \in \mathrm{Aut}(A)$. Then we define ${}_\phi\mathcal{E}_\psi$ to be the same vector space $\mathcal{E}$, but with the bi-action of A on $\mathcal{E}$ twisted as follows:

$$a_1 \triangleright a_0 \triangleleft a_2 := \phi(a_1)\, a_0\, \psi(a_2)$$

For $\phi = \psi = \mathrm{id}$, this is the original A-A-bimodule (when either $\phi = \mathrm{id}$ or $\psi = \mathrm{id}$, we shall not write that subscript). In particular, we can apply this twisting to $\mathcal{E} = A$ itself.

Lemma 2.11. *If A is a unital algebra, there exists an A-A-bimodule isomorphism $\theta\colon A \to {}_\phi A$ if and only if ϕ is inner.*

Proof. If $\theta\colon A \to {}_\phi A$ is an A-bimodule isomorphism, then

$$\phi(a)\theta(1) = \theta(a) = \theta(1)a$$

so that $\phi(a) = uau^{-1}$, where $u = \theta(1)$ is invertible. ∎

Thus the "outer automorphism group" $\mathrm{Out}(A) := \mathrm{Aut}(A)/\mathrm{Inn}(A)$ classifies the asymmetric A-bimodules. When A is commutative, so that $\mathrm{Inn}(A)$ is trivial, this is just $\mathrm{Aut}(A)$.

Recall that

$$\mathrm{Aut}(C(M)) \cong \mathrm{Homeo}(M), \qquad \mathrm{Aut}(C^\infty(M)) \cong \mathrm{Diff}(M),$$

where $\phi(f)\colon x \mapsto f(\phi^{-1}x)$ for $f \in C(M)$. We shall write $\mathrm{Pic}_A(A)$, following [bw04], to denote the isomorphism classes of symmetric A-bimodules. (This repairs an oversight in [fgv01, Chap. 9], which did not distinguish between $\mathrm{Pic}(A)$ and $\mathrm{Pic}_A(A)$, as was pointed out to me by Henrique Bursztyn.)

Fact 2.12. $\mathrm{Pic}(A) \cong \mathrm{Pic}_A(A) \rtimes \mathrm{Aut}(A)$ *as a semidirect product of groups, with product given by* $([\mathcal{E}], \phi) \cdot ([\mathcal{F}], \psi) = ([{}_\psi\mathcal{E}_\psi \otimes_A \mathcal{F}], \phi \circ \phi)$.

The proof is not difficult, but we refer to the paper [bw04].

Lemma 2.13. *For $A = C(M)$ or $C^\infty(M)$,* $\mathrm{Pic}_A(A) \cong \mathrm{H}^2(M; \mathbb{Z})$.

Proof. Since invertible A-modules $\mathcal{L}$ are given by $\mathcal{L} = \Gamma(M, L)$ —either continuous or smooth sections, respectively— where $L \to M$ are $\mathbb{C}$-line bundles; and these are classified by the first Chern class $c_1(L) \in \mathrm{H}^2(M; \mathbb{Z})$, obtained from $[\underline{\lambda}] = [\underline{\lambda}_{ij}] \in \check{\mathrm{H}}^1(M; \mathbb{T}) \cong \mathrm{H}^2(M; \mathbb{Z})$. Indeed, here $\mathcal{L}^\sharp = \Gamma(M, L^*)$, where $L^* \to M$ is the dual bundle and $\mathcal{L} \otimes_A \mathcal{L}^\sharp = \Gamma(M, L \otimes L^*) \cong \Gamma(M, M \times \mathbb{C}) \cong C(M)$ or $C^\infty(M)$, respectively.

The group operation in $\mathrm{Pic}_A(A)$ is $[\mathcal{L}_1] \cdot [\mathcal{L}_2] = [\mathcal{L}_1 \otimes_A \mathcal{L}_2]$: since $\mathcal{L}_1 \otimes_A \mathcal{L}_2 \cong \Gamma(M, L_1 \otimes L_2)$, it is again a module of sections for a $\mathbb{C}$-line bundle. ∎

2.5 Classification of spinor modules

In this section, $A = C(M)$ and as before, $B = \Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$ or $B = \Gamma(M, \mathbb{C}\mathrm{l}^0(T^*M))$, according as the dimension of M is even or odd.

Consider now the set $\mathrm{Mrt}(B, A)$ of isomorphism classes of B-A-bimodules: we have seen that $\delta(B) = 0$ if and only if $\mathrm{Mrt}(B, A)$ is nonempty. We shall assume from now on that indeed $\delta(B) = 0$, so that there exists at least one B-A-bimodule $\mathcal{S} = \Gamma(M, S)$ —continuous sections, for the moment— such that at each $x \in M$, S_x is an irreducible representation of the simple algebra B_x. Therefore, any such $\mathcal{S}$ has a partner $\mathcal{S}^\sharp = \mathrm{Hom}_A(\mathcal{S}, A)$ such that $\mathcal{S} \otimes_A \mathcal{S}^\sharp \cong B$ and $\mathcal{S}^\sharp \otimes_B \mathcal{S} \cong A$: in other words, $\mathcal{S}$ is an equivalence B-A-bimodule, and its isomorphism class $[\mathcal{S}]$ is an element of $\mathrm{Mrt}(B, A)$.

Since $\mathcal{S}^\sharp \cong \Gamma(M, S^*)$ where $S^* \to M$ is the dual vector bundle to $S \to M$, we can write this equivalence fibrewise: $\mathcal{S}_x \otimes_{\mathbb{C}} \mathcal{S}_x^* = \mathrm{End}_{\mathbb{C}}(\mathcal{S}_x) \cong B_x$ and then $\mathcal{S}_x^* \otimes_{B_x} \mathcal{S}_x \cong \mathbb{C}$, for $x \in M$.

Lemma 2.14. $\mathrm{Mrt}(B, A)$ *is a principal homogeneous space for the group* $\mathrm{Pic}_A(A)$, *when* $\delta(B) = 0$.

Proof. There is a right action of $\mathrm{Pic}_A(A)$ on $\mathrm{Mrt}(B, A)$, given by $[\mathcal{S}] \cdot [\mathcal{L}] := [\mathcal{S} \otimes_A \mathcal{L}]$. We say that the spinor module $\mathcal{S}$ is "twisted" by the invertible A-module $\mathcal{L}$.

If $\mathcal{S} \otimes_A \mathcal{S}^\sharp \cong B$ and $\mathcal{S}^\sharp \otimes_B \mathcal{S} \cong A$, then for $\mathcal{S}^1 := \mathcal{S} \otimes_A \mathcal{L}$ we get

$$\mathcal{S}_1 \otimes_A \mathcal{S}_1^\sharp = \mathcal{S} \otimes_A \mathcal{L} \otimes_A \mathcal{L}^\sharp \otimes_A \mathcal{S}^\sharp \cong \mathcal{S} \otimes_A A \otimes_A \mathcal{S}^\sharp \cong \mathcal{S} \otimes_A \mathcal{S}^\sharp \cong B,$$
$$\mathcal{S}_1^\sharp \otimes_B \mathcal{S}_1 = \mathcal{L}^\sharp \otimes_A \mathcal{S}^\sharp \otimes_B \mathcal{S} \otimes_A \mathcal{L} \cong \mathcal{L}^\sharp \otimes_A A \otimes_A \mathcal{L} \cong \mathcal{L}^\sharp \otimes_A \mathcal{L} \cong A.$$

Thus $\mathcal{S}_1$ is again an equivalence B-A-bimodule. Moreover, under the natural isomorphism

$$\mathrm{Hom}_B(\mathcal{S}', \mathcal{S}) \otimes_A \mathcal{L} \cong \mathrm{Hom}_B(\mathcal{S}', \mathcal{S} \otimes_A \mathcal{L}) \ : \ F \otimes l \mapsto [s' \mapsto F(s') \otimes l],$$

we see that $\mathrm{Hom}_B(\mathcal{S}, \mathcal{S} \otimes_A \mathcal{L}) \cong \mathcal{S}^\sharp \otimes_B \mathcal{S} \otimes_A \mathcal{L} \cong A \otimes_A \mathcal{L} = \mathcal{L}$, so that the right action of $\mathrm{Pic}_A(A)$ is *free*. On the other hand, the identification

$$\mathcal{S} \otimes \mathrm{Hom}_B(\mathcal{S}, \mathcal{S}') \cong \mathrm{Hom}_B(\mathrm{End}_A(\mathcal{S}), \mathcal{S}') \ : \ s \otimes F \mapsto [G \mapsto (F \circ G)(s)]$$

yields, for $\mathcal{L} := \mathrm{Hom}_B(\mathcal{S}, \mathcal{S}')$, the isomorphism

$$\mathcal{S} \otimes_A \mathcal{L} \cong \mathrm{Hom}_B(B, \mathcal{S}') \cong \mathcal{S}',$$

so that the action of $\mathrm{Pic}_A(A)$ is *transitive*. □

To proceed, we explain how B acts on $\mathcal{S}^\sharp = \mathrm{Hom}_A(\mathcal{S}, A)$. The spinor module $\mathcal{S}$ carries an A-valued hermitian pairing (2.3) given by the local scalar products defined in the construction of $\mathcal{S}$, that may be written

$$(\psi \mid \phi) \ : \ x \mapsto \langle \psi_x \mid \phi_x \rangle, \quad \text{for } x \in M. \tag{2.4}$$

We can identify elements of $\mathcal{S}^\sharp$ with "bra-vectors" $\langle\psi|$ using this pairing, namely, we define $\langle\psi|$ to be the map $\phi \mapsto (\psi \,|\, \phi) \in A$. Since A is unital, there is a "Riesz theorem" for A-modules showing that all elements of $\mathcal{S}^\sharp$ are of this form. Now the left B-action is defined by

$$b \, \langle\psi| := \langle\psi| \circ \chi(b^!).$$

Recall that $b \mapsto \chi(b!)$ is a linear antiautomorphism of B.

Remark 2.15. In these notes, there are many inner products. As a convention, angle brackets $\langle\cdot\mid\cdot\rangle$ take values in $\mathbb{C}$ —we shall call them *scalar products* to emphasize this— while round brackets $(\cdot\mid\cdot)$ take values in an algebra A —we use the word *pairing* to signal that.

Lemma 2.16. *Let $\mathcal{L}_{\mathcal{S}} := \mathrm{Hom}_B(\mathcal{S}^\sharp, \mathcal{S})$ be the A-module for which $\mathcal{S}^\sharp \otimes_A \mathcal{L}_{\mathcal{S}} \cong \mathcal{S}$. Then $\mathcal{L}_{\mathcal{S}\otimes_A\mathcal{L}} \cong \mathcal{L}_{\mathcal{S}} \otimes_A \mathcal{L} \otimes_A \mathcal{L}$, so that the twisting $[\mathcal{S}] \mapsto [\mathcal{S}\otimes_A \mathcal{L}]$ on $\mathrm{Mrt}(B, A)$ induces a translation by $[\mathcal{L}\otimes_A\mathcal{L}] = 2[\mathcal{L}]$ on $\mathrm{Pic}_A(A)$.*

Proof. First observe that, since $\mathcal{L}$ is an invertible A-module, the dual of $\mathcal{S}\otimes_A\mathcal{L}$ is isomorphic to $\mathcal{S}^\sharp\otimes_A\mathcal{L}^\sharp$. The commutativity of the group $\mathrm{Pic}_A(A)$ the shows that

$$\begin{aligned}(\mathcal{S}\otimes_A\mathcal{L})^\sharp\otimes_A(\mathcal{L}_{\mathcal{S}}\otimes_A\mathcal{L}\otimes_A\mathcal{L}) &\cong \mathcal{S}^\sharp\otimes_A\mathcal{L}^\sharp\otimes_A(\mathcal{L}\otimes_A\mathcal{L}_{\mathcal{S}}\otimes_A\mathcal{L}) \\ &\cong \mathcal{S}^\sharp\otimes_A\mathcal{L}_{\mathcal{S}}\otimes_A\mathcal{L}\cong\mathcal{S}\otimes_A\mathcal{L},\end{aligned}$$

and the freeness of the action now implies the result. ∎

Thus, the "mod 2 reduction" $j_*[\mathcal{L}_{\mathcal{S}}] \in \mathrm{H}^2(M;\mathbb{Z}_2)$, coming from the short exact sequence of abelian groups $0\to\mathbb{Z}\xrightarrow{\times 2}\mathbb{Z}\xrightarrow{j}\mathbb{Z}_2\to 0$, is independent of $[\mathcal{S}]$. Indeed, it defines an invariant $\kappa[B]\in\mathrm{H}^2(M;\mathbb{Z})$. This is clear, when one takes into account the corresponding long exact sequence in Čech cohomology and the governing assumption that $\delta(B) = 0$:

$$\cdots\to\mathrm{H}^1(M;\mathbb{Z}_2)\xrightarrow{\partial}\mathrm{H}^2(M;\mathbb{Z})\xrightarrow{(\times 2)_*}\mathrm{H}^2(M;\mathbb{Z})\xrightarrow{j_*}\mathrm{H}^2(M;\mathbb{Z}_2)\xrightarrow{\partial}\mathrm{H}^3(M;\mathbb{Z})\to\cdots \tag{2.5}$$

Remark 2.17. It can be shown that $\kappa(B) = w_2(TM) = w_2(T^*M)$, the familiar second *Stiefel–Whitney class* of the tangent (or cotangent) bundle. See, for instance, the original papers of Karrer [k-g63] and Plymen [p-rj86], and the lecture notes by Schröder [s-h00].

What is the meaning of the condition $\kappa(B) = 0$? It means that, by replacing any original choice of $\mathcal{S}$ by a suitably twisted $\mathcal{S}\otimes_A\mathcal{L}$, we can arrange that $\mathcal{L}_{\mathcal{S}}$ is trivial, i.e. $\mathcal{L}_{\mathcal{S}}\cong A$, or better yet, that

$$\mathcal{S}^\sharp\cong\mathcal{S}\quad\text{as } B\text{-}A\text{-bimodules.}$$

We now reformulate this condition in terms of a certain antilinear operator C; later on, in the context of spectral triples, we shall rename it to J.

Proposition 2.18. *There is a B-A-bimodule isomorphism $\mathcal{S}^\sharp\cong\mathcal{S}$ if and only if there is an antilinear endomorphism C of $\mathcal{S}$ such that*

(a) $C(\psi\, a) = C(\psi)\,\bar a$ for $\psi\in\mathcal{S}$, $a\in A$;

(b) $C(b\,\psi) = \chi(\bar b)\,C(\psi)$ for $\psi\in\mathcal{S}$, $b\in B$;

(c) C is antiunitary *in the sense that $(C\phi\mid C\psi) = (\psi\mid\phi)\in A$, for $\phi,\psi\in\mathcal{S}$;*

(d) $C^2 = \pm 1$ on $\mathcal{S}$ whenever M is connected.

Proof. Ad (a): We provisionally define C by $C(\psi) := T\langle\psi|$, where $T\colon \mathcal{S}^\sharp \to S$ is the given B-A-bimodule isomorphism. Now since $\langle\psi a| = \langle\psi|\,\bar{a}$ because $(\psi\, a \mid \phi) = \bar{a}\,(\psi \mid \phi) = (\psi \mid \phi)\,\bar{a} = (\psi \mid \phi\,\bar{a})$ —since A is commutative— we get $C(\psi\, a) = T(\langle\psi|\,\bar{a}) = T\langle\psi|\,\bar{a} = C(\psi)\,\bar{a}$.

Ad (b): The formula for the B-action on $\mathcal{S}^\sharp$, and the relation $(b\psi \mid \phi) = (\psi \mid b^*\phi)$ give

$$C(b\,\psi) = T\langle b\,\psi| = T(\langle\psi| \circ b^*) = T(\chi(b^!)\,\langle\psi|) = \chi(\bar{b})\,T\langle\psi|.$$

Ad (c): The pairing $(\phi|C\psi)$ is antilinear (and bounded) in both ϕ and ψ, and thus of the form $(\psi\,|\,\chi)$ for some $\chi \in \mathcal{S}$, by the aforementioned "Riesz theorem". Thus we get an adjoint map to C, namely the antilinear map $C^\dagger\colon \phi \mapsto \chi$ —obeying the rule for transposing antilinear operators, i.e., $(\psi \mid C^\dagger\phi) = (\phi \mid C\psi)$. Next, notice that $C^\dagger C$ is an A-linear bijective endomorphism of $\mathcal{S}$, that commutes with each $b \in B$:

$$\begin{aligned}(\phi \mid C^\dagger C b\psi) &= (Cb\psi \mid C\phi) = (\chi(\bar{b})C\psi \mid C\phi)\\ &= (C\psi \mid \chi(b^!)C\phi) = (C\psi \mid Cb^*\phi)\\ &= (b^*\phi \mid C^\dagger C\psi) = (\phi \mid bC^\dagger C\psi).\end{aligned}$$

Therefore $C^\dagger C \in \mathrm{End}_B(\mathcal{S}) \cong A$, i.e. there is an invertible $a \in A$ such that $C^\dagger C = a1_{\mathcal{S}}$. If $a \neq 1$, we can now *redefine* $C \mapsto a^{-\frac{1}{2}}C$, keeping (a) and (b), so with the redefinition $C(\psi) := a^{-\frac{1}{2}}T\langle\psi|$, we get $C^\dagger C = 1$, i.e., $(C\phi \mid C\psi) = (\psi \mid C^\dagger C\phi) = (\psi \mid \phi)$.

Ad (d): Finally, C^2 is A-linear, and $C^2\,b = C\,\chi(\bar{b})\,C = b\,C^2$ for $b \in B$, so $C^2 = u1_{\mathcal{S}}$ with $u \in A$. From the relations

$$\begin{aligned} uC = C^3 = Cu &= \bar{u}C \quad \text{by antilinearity of } C,\\ \bar{u}u\,1_{\mathcal{S}} = (C^\dagger)^2C^2 &= C^{-2}C^2 = 1_{\mathcal{S}},\end{aligned}$$

we get $u = \bar{u}$ and hence $u^2 = 1$. Thus $u \in A = C(M)$ takes the values ± 1 only, so $u = \pm 1$ when M is connected. (More generally, C^2 lies in $\mathrm{H}^0(M, \mathbb{Z}_2)$.) $\square$

The antilinear operator $C\colon \mathcal{S} \to \mathcal{S}$, which becomes an *antiunitary* operator on a suitable Hilbert-space completion of $\mathcal{S}$, is called the *charge conjugation*. It exists if and only if $\kappa(B) = 0$.

What, then, are spin^c and spin structures on M? We choose on M a metric (without losing generality), and also an *orientation* ε, which organizes the action of B, in that a change $\varepsilon \mapsto -\varepsilon$ induces $c(\gamma) \mapsto -c(\gamma)$, which either

(i) reverses the $\mathbb{Z}_2$-grading of $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, in the even case; or

(ii) changes the action on $\mathcal{S}$ of each $c(\alpha)$ to $-c(\alpha)$, for $\alpha \in \mathcal{A}^1(M)$, in the odd case —recall that $c(\alpha) := c(\alpha\gamma)$ in the odd case.

Definition 2.19. *Let (M, ε) be a compact boundaryless orientable manifold, together with a chosen orientation ε. Let $A = C(M)$ and let B be specified as before (in terms of a fixed but arbitrary Riemannian metric on M). If $\delta(B) = 0$ in $\mathrm{H}^3(M;\mathbb{Z})$, a* spin$^{\mathbf{c}}$ structure *on (M, ε) is an isomorphism class $[\mathcal{S}]$ of equivalence B-A-bimodules.*

If $\delta(B) = 0$ and if $\kappa(B) = 0$ in $\mathrm{H}^2(M;\mathbb{Z}_2)$, a pair $(\mathcal{S}, C)$ give data for a spin structure, when $\mathcal{S}$ is an equivalence B-A-bimodule such that $\mathcal{S}^\sharp \cong \mathcal{S}$, and C is a *charge conjugation operator on $\mathcal{S}$. A* spin structure *on (M, ε) is an isomorphism class of such pairs.*

Remark 2.20. There is an alternative treatment, given in many books, that defines spin^c or spin structures using *principal G-bundles* for $G = \mathrm{Spin}^c(\mathbb{R}^n)$ or $G = \mathrm{Spin}(\mathbb{R}^n)$ respectively. The equivalence of the two approaches is treated in [p-rj86] and [s-h00].

Atiyah, Bott and Shapiro [abs64] called a spin^c structure a "K-orientation", for reasons which may be obvious to K-theorists. At any rate, it is a finer invariant than the orientation class $[\varepsilon]$, provided it exists.

In the long cohomology exact sequence there is a boundary homomorphism

$$\mathrm{H}^2(M; \mathbb{Z}_2) \xrightarrow{\partial} \mathrm{H}^3(M; \mathbb{Z}).$$

By examining the definitions of the various Čech cocycles that we have obtained so far, one can show that $\delta(B) = \partial(\kappa(B))$.

Remark 2.21. It is known that $\delta(B) = 0$ for $\dim M \leq 4$: manifolds of dimensions $1, 2, 3, 4$ always carry spin^c structures. There are 5-dimensional manifolds for which $\delta(B) \neq 0$; the best-known is the homogeneous space $\mathrm{SU}(3)/\mathrm{SO}(3)$. A homotopy-theoretic proof of the obstruction for this example in given in [f-t00].

A complex manifold has a natural orientation and a natural spin^c structure coming from its complex structure. Thus $\mathbb{C}P^m$ come with a spin^c structure, for all m. However, it is known that $\mathbb{C}P^m$ admits spin structures if and only if m is odd: therefore, $\mathbb{C}P^2$ is a 4-dimensional manifold without spin structures.

2.6 The spin connection

We now leave the topological level and introduce differential structure. Thus we replace $A = C(M)$ by $\mathcal{A} = C^\infty(M)$, and continuous sections Γ_{cont} by smooth sections Γ_{smooth}. Thus $\mathcal{S} = \Gamma(M, S)$ will henceforth denote the $\mathcal{A}$-module of *smooth* spinors.

Our treatment of Morita equivalence of *unital* algebras passes without change to the smooth level. We can go back with the functor $- \otimes_{C^\infty(M)} C(M)$, if desired.

Definition 2.22. *A* connection *on a (finitely generated projective) $\mathcal{A}$-module $\mathcal{E} = \Gamma(M, E)$ is a $\mathbb{C}$-linear map $\nabla\colon \mathcal{E} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E} = \Gamma(M, T^*M \otimes E) \equiv \mathcal{A}^1(M, E)$, satisfying the Leibniz rule*

$$\nabla(fs) = df \otimes s + f\, \nabla s.$$

It extends to an odd derivation of degree +1 on $\mathcal{A}^\bullet(M) \otimes_{\mathcal{A}} \mathcal{E} = \Gamma(M, \Lambda^\bullet T^*M \otimes E) \equiv \mathcal{A}^\bullet(M, E)$ with grading inherited from that of $\mathcal{A}^\bullet(M)$, leaving $\mathcal{E}$ trivially graded, so that $\nabla(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^{|\omega|} \omega \wedge \nabla\sigma$ for $\omega \in \mathcal{A}^\bullet(M)$, $\sigma \in \mathcal{A}^\bullet(M, E)$.

Employing the usual contraction of vector fields with forms in $\mathcal{A}^\bullet(M)$, namely,

$$\iota_X \omega(Y_1, \dots, Y_k) := \omega(X, Y_1, \dots, Y_k) \quad \text{for} \quad \omega \in \mathcal{A}^{k+1}(M),$$

extended to $\mathcal{A}^\bullet(M) \otimes_\mathcal{A} \mathcal{E}$ as $\iota_X \otimes \mathrm{id}_\mathcal{E}$ —but still written ι_X— we get operators ∇_X on $\mathcal{A}^\bullet(M, E)$ of degree 0 by defining

$$\nabla_X := \iota_X \circ \nabla + \nabla \circ \iota_X.$$

This is $\mathcal{A}$-linear in X. Moreover, if $\omega \in \mathcal{A}^\bullet(M)$ and $s \in \mathcal{E}$, one can check that $\nabla_X(\omega \otimes s) = \mathcal{L}_X\omega \otimes s + \omega\, \nabla_X s$, where $\mathcal{L}_X = \iota_X\, d + d\, \iota_X$ is the Lie derivative of forms with respect to X.

Exercise 2.23. *Verify that* $\nabla_X(\iota_Y\sigma) = \iota_Y(\nabla_X\sigma) + \iota_{[X,Y]}\sigma$ *for* $\sigma \in \mathcal{A}^\bullet(M, E)$.

Exercise 2.24. *If* $\mathcal{E} = \mathfrak{X}(M) = \Gamma(M, TM)$, *then show that*

$$\nabla_X Y - \nabla_Y X - [X, Y] = \iota_Y \iota_X \nabla\theta,$$

where $\theta \in \mathcal{A}^1(M, TM)$ *is the* fundamental 1-form *defined by* $\iota_X\theta := X$. *We say that* ∇ *is* torsionfree *if* $\nabla\theta = 0$ *in* $\mathcal{A}^2(M, TM)$.

Exercise 2.25. *Show that* $\iota_Y\iota_X\nabla^2 = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, *where the degree* $+2$ *operator* ∇^2 *on* $\mathcal{A}^\bullet(M, E)$ *is the* curvature *of* ∇.

We mention two natural constructions for connections, on tensor products of $\mathcal{A}$-modules and on dual $\mathcal{A}$-modules. Firstly, if $\nabla' \colon \mathcal{F} \to \mathcal{A}^1(M) \otimes_\mathcal{A} \mathcal{F}$ is another connection in another $\mathcal{A}$-module, then

$$\widetilde{\nabla}(s \otimes t) := \nabla s \otimes t + s \otimes \nabla' t$$

(extended by linearity, as usual) makes $\widetilde{\nabla}$ a connection on $\mathcal{E} \otimes_\mathcal{A} \mathcal{F}$.

Next, if $\mathcal{E}^\sharp = \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A})$, then the *dual connection* $\nabla^\sharp$ on $\mathcal{E}^\sharp$ is determined by

$$\begin{aligned} d(\zeta(s)) &=: (\nabla^\sharp\zeta)(s) + \zeta(\nabla s) \quad \text{in } \mathcal{A}^1(M); \quad \text{or equivalently} \\ X(\zeta(s)) &=: (\nabla^\sharp_X\zeta)(s) + \zeta(\nabla_X s) \quad \text{in } \mathcal{A}, \quad \text{for } X \in \mathfrak{X}(M), \end{aligned}$$

whenever $\zeta \in \mathcal{E}^\sharp$ and $s \in \mathcal{E}$.

Definition 2.26. *If* $\mathcal{E}$ *an* $\mathcal{A}$*-module equipped with an* $\mathcal{A}$*-valued Hermitian pairing, we say that a* connection ∇ *on* $\mathcal{E}$ *is* Hermitian *if*

$$\begin{aligned} (\nabla s \mid t) + (s \mid \nabla t) &= d\,(s \mid t), \quad \textit{or, in other words,} \\ (\nabla_X s \mid t) + (s \mid \nabla_X t) &= X\,(s \mid t), \quad \textit{for any } \text{real } X \in \mathfrak{X}(M). \end{aligned}$$

If ∇, ∇' are connections on $\mathcal{E}$, then $\nabla' - \nabla$ is an $\mathcal{A}$-module map: $(\nabla' - \nabla)(fs) = f(\nabla' - \nabla)s$, so that *locally*, over $U \subset M$ for which $E\big|_U \to U$ is trivial, we can write

$$\nabla = d + \alpha, \quad \text{where } \alpha \in \mathcal{A}^1(U, \mathrm{End}\, E).$$

Fact 2.27. *On* $\mathfrak{X}(M) = \Gamma(M, TM)$ *there is, for each Riemannian metric* g*, a* unique torsion-free connection that is compatible with g*:*

$$g(\nabla X, Y) + g(X, \nabla Y) = d(g(X, Y)) \quad \textit{for } X, Y \in \mathfrak{X}(M), \quad \textit{or}$$

$$g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = Z(g(X, Y)) \quad \textit{for } X, Y, Z \in \mathfrak{X}(M).$$

The explicit formula for this connection is

$$\begin{aligned}2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y))\\ + g(Y,[Z,X]) + g(Z,[X,Y]) - g(X,[Y,Z]).\end{aligned} \tag{2.6}$$

It is called *Levi-Civita connection* associated to g. (The proof of existence consists in showing that the right hand side of this expression is $\mathcal{A}$-linear in Y and Z, and obeys a Leibniz rule with respect to X, so it gives a connection; and uniqueness is obtained by checking that metric compatibility and torsion freedom make the right hand side automatic.)

The *dual connection* on $\mathcal{A}^1(M)$ will also be called the "Levi-Civita connection". At the risk of some confusion, we shall use the same symbol ∇ for both of these Levi-Civita connections.

Local formulas From now on, we assume that $U \subset M$ is an open chart domain over which the tangent and cotangent bundles are trivial. Local coordinates are functions $x^1, \dots, x^n \in C^\infty(U)$, and we denote $\partial_j := \partial/\partial x^j \in \mathfrak{X}(M)\big|_U$ for the local basis of vector fields; by definition, their Lie brackets vanish: $[\partial_i, \partial_j] = 0$. We define the *Christoffel symbols* $\Gamma^k_{ij} \in C^\infty(U)$ by

$$\nabla_{\partial_i}\partial_j =: \Gamma^k_{ij}\,\partial_k, \quad \text{or} \quad \nabla\partial_j =: \Gamma^k_{ij}\,dx^i \otimes \partial_k.$$

The explicit expression (2.6) for the Levi-Civita connection reduces to a local formula over U, namely

$$\Gamma^k_{ij} := \tfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}); \qquad \text{here } [g^{rs}] = [g_{ij}]^{-1}. \tag{2.7}$$

Notice that $\Gamma^k_{ji} = \Gamma^k_{ij}$; this is because of torsion freedom.

Dually, the coefficients of the Levi-Civita connection on 1-forms are $-\Gamma^k_{ij}$ (note the change of sign):

$$\nabla_{\partial_i}(dx^k) = -\Gamma^k_{ij}\,dx^j, \quad \text{or} \quad \nabla(dx^k) = -\Gamma^k_{ij}\,dx^i \otimes dx^j.$$

Since the Riemannian metric gives a concept of (fibrewise) orthogonality on the tangent and cotangent bundles, we can select *local orthonormal bases*:

$$\begin{aligned}&\{E_1, \dots, E_n\} \quad \text{for } \mathfrak{X}(M)\big|_U = \Gamma(U, TM): \quad g(E_\alpha, E_\beta) = \delta_{\alpha\beta};\\ &\{\theta^1, \dots, \theta^n\} \quad \text{for } \mathcal{A}^1(M)\big|_U = \Gamma(U, T^*M): \quad g(\theta^\alpha, \theta^\beta) = \delta^{\alpha\beta}.\end{aligned}$$

We rewrite the Christoffel symbols in these local bases:

$$\nabla E_\alpha =: \widetilde{\Gamma}^\beta_{i\alpha}\,dx^i \otimes E_\beta, \qquad \nabla\theta^\beta = -\widetilde{\Gamma}^\beta_{i\alpha}\,dx^i \otimes \theta^\alpha.$$

Metric compatibility means that, for each fixed i, the $\widetilde{\Gamma}^\bullet_{i\bullet}$ are *skewsymmetric matrices*:

$$\widetilde{\Gamma}^\beta_{i\alpha} + \widetilde{\Gamma}^\alpha_{i\beta} = -g(\nabla_{\partial_i}\theta^\beta, \theta^\alpha) - g(\theta^\beta, \nabla_{\partial_i}\theta^\alpha) = -\partial_i(\delta^{\alpha\beta}) = 0.$$

Thus $\widetilde{\Gamma}$ lies in $\mathcal{A}^1(U, \mathfrak{so}(T^*M)) \cong \mathcal{A}^1(U) \otimes_{\mathbb{R}} \mathfrak{so}(\mathbb{R}^n)$.

Definition 2.28. *On a spinor module $\mathcal{S} = \Gamma(M, S)$, a* spin$^{\mathbf{c}}$-connection *is any* Hermitian *connection $\nabla^S : \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$ which is compatible with the action of B in the following way:*

$$\begin{aligned}\nabla^S(c(\alpha)\psi) &= c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi \quad \text{for } \alpha \in \mathcal{A}^1(M),\ \psi \in \mathcal{S}; \quad \text{or} \\ \nabla^S_X(c(\alpha)\psi) &= c(\nabla_X\alpha)\psi + c(\alpha)\nabla^S_X\psi \quad \text{for } \alpha \in \mathcal{A}^1(M),\ \psi \in \mathcal{S},\ X \in \mathfrak{X}(M),\end{aligned} \tag{2.8}$$

where $\nabla\alpha$ and $\nabla_X\alpha$ refer to the Levi-Civita connection *on $\mathcal{A}^1(M)$.*

If $(\mathcal{S}, C)$ are data for a spin structure, we say ∇^S is a spin connection *if, moreover, each $\nabla_X : \mathcal{S} \to \mathcal{S}$ commutes with C whenever X is real.*

Before discussing existence, let us look first at local formulas. We thus write "$\nabla = d - \widetilde{\Gamma}$ locally" for the Levi-Civita connection, with an implicit choice of local orthonormal bases of 1-forms. We recall that there are isomorphisms of Lie algebras

$$\dot{\mu} : \mathfrak{so}(T^*_x M) \to Q(\Lambda^2 T^*_x M) \equiv \mathfrak{spin}(T^*_x M)$$

with the property that $\operatorname{ad}(\dot{\mu}(A)) = A$ for $A \in \mathfrak{so}(T^*_x M)$; in other words, $[\dot{\mu}(A), v] = Av$ for $v \in T^*_x M$ —this is a commutator for the Clifford product in $\mathrm{Cl}(T^*_x M, g_x)$. On the chart domain U, we can apply $\dot{\mu}$ to $\widetilde{\Gamma}$ fibrewise; this means that

$$[\dot{\mu}(\widetilde{\Gamma}), c(\alpha)] = c(\widetilde{\Gamma}\,\alpha)$$

for $\alpha \in \mathcal{A}^1(M)$ with support in U, $\widetilde{\Gamma} \in \Gamma(U, \operatorname{End} T^*M)$ is mapped to $\dot{\mu}(\widetilde{\Gamma}) \in \Gamma(U, \operatorname{End} S)$, and $c(\alpha)$ again denotes the Clifford action action of α on $\mathcal{S}\big|_U = \Gamma(U, S)$.

In this way we get the local expression of a connection,

$$\nabla^S := d - \dot{\mu}(\widetilde{\Gamma}), \quad \text{acting on} \quad \mathcal{S}\big|_U. \tag{2.9}$$

Suppose we take $\alpha \in \mathcal{A}^1(M)$ with support in U, and $\psi \in \mathcal{S}\big|_U$. Then

$$\begin{aligned}\nabla^S(c(\alpha)\psi) &= d(c(\alpha)\psi) - \dot{\mu}(\widetilde{\Gamma})c(\alpha)\psi \\ &= c(d\alpha)\,\psi + c(\alpha)\,d\psi - \dot{\mu}(\widetilde{\Gamma})c(\alpha)\psi \\ &= c(\alpha)(d\psi - \dot{\mu}(\widetilde{\Gamma})\psi) + (c(d\alpha) - [\dot{\mu}(\widetilde{\Gamma}), c(\alpha)]\psi) \\ &= c(\alpha)\nabla^S\psi + c(d\alpha - \widetilde{\Gamma}\,\alpha)\psi \\ &= c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi.\end{aligned} \tag{2.10}$$

Thus $\nabla^S := d - \dot{\mu}(\Gamma)$ provides a local solution to the existence of $\nabla^S : \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$ satisfying the Leibniz rule:

$$\nabla^S(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi.$$

Physicists like to write $\gamma^\alpha := c(\theta^\alpha)$ for a given local orthonormal basis of $\mathcal{A}^1(M)$ —so that the γ^α are *fixed* matrices. For convenience, we also write $\gamma_\beta = \delta_{\alpha\beta}\,\gamma^\alpha$ also (in the Euclidean signature, which we are always using here); in other words, $\gamma_\beta = \gamma^\beta$ but with its index lowered for use with the Einstein summation convention. Thus the Clifford relations are just

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta^{\alpha\beta}, \quad \text{for } \alpha, \beta = 1, \ldots, n.$$

The formula (1.7) for $\dot\mu$ can now be rewritten as

$$\dot\mu(\widetilde\Gamma) = -\tfrac14 \widetilde\Gamma^{\beta}_{\bullet\alpha}\, \gamma^\alpha \gamma_\beta.$$

A more sensible notation arrives by introducing matrix-valued functions $\omega_1, \dots, \omega_n \in \Gamma(U, \operatorname{End} T^*M)$ as follows:

$$\omega_i := -\tfrac14 \widetilde\Gamma^{\beta}_{i\alpha}\, \gamma^\alpha \gamma_\beta.$$

Let us look at the calculation (2.10) again, after contracting with a vectorfield X. We get

$$\begin{aligned}[\nabla^S_X, c(\alpha)]\, \psi &= [\mathcal{L}_X - X^i \dot\mu(\omega_i), c(\alpha)]\, \psi \\ &= \big(c(\mathcal{L}_X \alpha) - X^i\, c(\omega_i \alpha)\big)\, \psi = c(\nabla_X \alpha)\, \psi.\end{aligned}$$

Thus the local coefficients of ∇^S_X are $-\frac14 X^i \widetilde\Gamma^{\beta}_{i\alpha}\, \gamma^\alpha\gamma_\beta$, for $X \in \mathfrak{X}(M)$.

Now suppose S comes from a *spin* structure on M. Since $C(\psi a) = (C\psi)\bar a$ for $a \in \mathcal{A} = C^\infty(M)$, the operator C acts locally (as a field of antilinear conjugations $C_x \colon S_x \to S_x$); and since $C(b) = \chi(\bar b)C$ for $b \in B$, we get, for $\alpha, \beta = 1, \dots, n$:

$$C\gamma^\alpha\gamma^\beta = Cc(\theta^\alpha)c(\theta^\beta) = c(\theta^\alpha)c(\theta^\beta)C = \gamma^\alpha\gamma^\beta C.$$

Thus $\nabla^S_X C - C\nabla^S_X$ vanishes over U, provided $X\big|_U$ is real.

Suppose that ∇' is another local connection defined on $\mathcal{S}\big|_U$ and satisfying the Leibniz rule (2.10) there. Then

$$\nabla' - \nabla^S = \beta \in \mathcal{A}^1(U, \operatorname{End} S)$$

and $c(\kappa)\beta\psi = \beta c(\kappa)\psi$ for all $\kappa \in \mathcal{B}\big|_U$. Thus β_x is a scalar matrix in $\operatorname{End}(S_x)$, for each $x \in U$. To fix β, we ask that both ∇' and ∇^S be *Hermitian* connections; this entails that each β_x is skew-hermitian:

$$\langle \beta_x\phi_x \mid \psi_x\rangle + \langle \phi_x \mid \beta_x\psi_x\rangle = 0 \quad \text{for } x \in U,$$

so this scalar is purely imaginary. On the other hand, if ∇'_X, like ∇^S_X, commutes with C whenever the coefficients X^i of X are real functions, then this scalar must be purely real. Therefore, $\beta = 0$.

Proposition 2.29. *If $(\mathcal{S}, C)$ are data for a spin structure on M, then there is a unique Hermitian spin connection $\nabla^S \colon \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$, such that*

$$\nabla^S(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi, \quad \textit{for } \alpha \in \mathcal{A}^1(M),\ \psi \in \mathcal{S},$$

and such that $\nabla^S_X C = C\nabla^S_X$ for $X \in \mathfrak{X}(M)$ real.

Proof. We have shown that ∇^S exists locally with the recipe (2.9) on any chart domain. This recipe gives a local Hermitian connection since $\dot\mu(\widetilde\Gamma)$ is skewadjoint —because the representation c is selfadjoint— and it commutes with C. Any other local connection with these properties must coincide with (2.9) over U.

Furthermore, on overlaps $U_1 \cap U_2$ of chart domains, we have shown that $\beta := \nabla^S|_{U_1} - \nabla^S|_{U_2} \in \mathcal{A}^1(U_1 \cap U_2, \operatorname{End} S)$ vanishes. Therefore, the local expressions can be assembled into a globally defined spin connection. ∎

Remark 2.30. If $\mathcal{S}$ is only a spinor module for a spin^c structure, then the uniqueness argument for the local spin connection fails. We can only conclude that $\nabla^S|_{U_1} - \nabla^S|_{U_2} = i(\alpha_1 - \alpha_2) \otimes 1_{\operatorname{End} S}$, where $\alpha_1 \in \mathcal{A}^1(U_1)$ and $\alpha_2 \in \mathcal{A}^1(U_2)$ are real 1-forms. We may be able to patch these "gauge potentials" to get a connection ∇ of a line bundle $\mathcal{L}^\sharp = \Gamma(M, L^*)$. Then one can show that on $\mathcal{S} \otimes \mathcal{L}$, there is a connection $\nabla^{S,\alpha}$ that satisfies the Leibniz rule above, and hermiticity. These are "spin^c connections" for the twisted spin^c structures.

If ∇ is any connection on an $\mathcal{A}$-module $\mathcal{E} = \Gamma(M, E)$, then

$$\nabla^2(fs) = \nabla(df \otimes s + f\,\nabla s) = \bigl(d(df) \otimes s - df\,\nabla s\bigr) + \bigl(df\,\nabla s + f\,\nabla^2 s\bigr) = f\,\nabla^2 s,$$

for $f \in \mathcal{A}$, so that ∇^2 is tensorial: $\nabla s = R\,s$ for a certain 2-form $R \in \mathcal{A}^2(M, \operatorname{End} E)$, the *curvature* of ∇. For the Levi-Civita connection, a local calculation gives

$$\nabla^2 \alpha = (d - \widetilde{\Gamma})(d\alpha - \widetilde{\Gamma}\,\alpha) = -d(\widetilde{\Gamma}\,\alpha) - \widetilde{\Gamma}\,d\alpha + \widetilde{\Gamma}(\widetilde{\Gamma}\,\alpha) = (-d\widetilde{\Gamma} + \widetilde{\Gamma} \wedge \widetilde{\Gamma})\,\alpha,$$

which yields the local expression for the Riemannian curvature tensor:

$$R\big|_U = -d\widetilde{\Gamma} + \widetilde{\Gamma} \wedge \widetilde{\Gamma} \in \mathcal{A}^2(U, \mathfrak{so}(T^*M)).$$

Likewise, the curvature R^S of the *spin connection* is locally given by

$$\dot{\mu}(R) = -d\dot{\mu}(\widetilde{\Gamma}) + \dot{\mu}(\widetilde{\Gamma}) \wedge \dot{\mu}(\widetilde{\Gamma}) \in \mathcal{A}^2(U, \operatorname{End} S).$$

One can check these formulas to get more familiar expressions by computing $R(X,Y) = \iota_Y \iota_X R$ and likewise $R^S(X,Y)$, for $X, Y \in \mathfrak{X}(M)$.

2.7 Epilogue: counting the spin structures

A spin structure on (M, ε) is an equivalence class of pairs $(\mathcal{S}, C)$, but what can be said about the equivalence relation?

First, $\mathcal{S}$ has a class $[\mathcal{S}] \in \operatorname{Mrt}(B, A)$: these are classified by $\mathrm{H}^2(M; \mathbb{Z})$. If $(\mathcal{S}_1, C_1)$ is another spin structure, then $C_1 \colon \mathcal{S}_1^\sharp \to \mathcal{S}_1$ comes from a B-A-bimodule isomorphism $T_1 \colon \mathcal{S}_1 \to \mathcal{S}_1$. But now $\mathcal{S}_1 \cong \mathcal{S} \otimes \mathcal{L}$ for some $\mathcal{L}$, where $[\mathcal{L}] \in \mathrm{H}^2(M; \mathbb{Z})$ is well defined. Thus we get

$$\begin{tikzcd}
\mathcal{S}^\sharp \arrow[r, "- \otimes_A \mathcal{L}^\sharp"] \arrow[d, "T"'] & \mathcal{S}^\sharp \otimes_A \mathcal{L}^\sharp \arrow[r, equal] \arrow[d, "T_1"] & (\mathcal{S} \otimes_A \mathcal{L})^\sharp \\
\mathcal{S} \arrow[r, "- \otimes_A \mathcal{L}"] & \mathcal{S} \otimes_A \mathcal{L} &
\end{tikzcd}$$

and therefore $(\mathcal{S}_1, C_1) \sim (\mathcal{S}, C)$ if this diagram commutes. Now

$$\begin{aligned}
\mathcal{S}_1 &\cong \mathcal{S}_1^\sharp \otimes_A \operatorname{Hom}_B(\mathcal{S}_1^\sharp, \mathcal{S}_1) \cong \mathcal{S}_1^\sharp \otimes_A \operatorname{Hom}_B(\mathcal{S}^\sharp \otimes_A \mathcal{L}^\sharp, \mathcal{S} \otimes_A \mathcal{L}) \\
&\cong \mathcal{S}_1^\sharp \otimes_A \mathcal{L} \otimes_A \operatorname{Hom}_B(\mathcal{S}^\sharp, \mathcal{S}) \otimes_A \mathcal{L} \cong \mathcal{S}_1^\sharp \otimes_A \mathcal{L} \otimes_A \mathcal{L},
\end{aligned}$$

since $\operatorname{Hom}_B(\mathcal{S}^\sharp, \mathcal{S})$ is trivial: the existence of T shows that $[\mathcal{S}^\sharp] = [\mathcal{S}]$ in $\operatorname{Mrt}(B, A)$. The conclusion is that $\mathcal{S}_1 \cong \mathcal{S}_1^\sharp \otimes_A \mathcal{L} \otimes_A \mathcal{L}$. Thus $\mathcal{S}_1$ is also selfdual if and only if $\mathcal{L} \otimes_A \mathcal{L}$ is trivial: $(\times 2)_*[\mathcal{L}] = 0$ in $\mathrm{H}^2(M; \mathbb{Z})$. But, using the long exact sequence (2.5), we find that $\ker(\times 2)_* = \operatorname{im}\{\partial \colon \mathrm{H}^1(M; \mathbb{Z}_2) \to \mathrm{H}^2(M; \mathbb{Z})\}$.

Conclusion: Those $[\mathcal{S} \otimes_A \mathcal{L}] \in \operatorname{Mrt}(B, A)$ for which $\mathcal{L} \otimes_A \mathcal{L}$ is trivial, but $\mathcal{L}$ is not, i.e., the distinct spin structures on (M, ε), are classified by $\mathrm{H}^1(M, \mathbb{Z}_2)$.

Remark 2.31. The group $\mathrm{H}^1(M, \mathbb{Z}_2)$ is known to classify *real* line bundles over M. If a twist by $\mathcal{L}$ exchanges the spinor modules for two spin structures, there is an antilinear automorphism of $\mathcal{L}$ which matches the two charge conjugation operators, and the part of $\mathcal{L}$ fixed by this automorphism comprises the sections of the corresponding $\mathbb{R}$-line bundle over M.

Chapter 3

Dirac operators

Suppose we are given a compact oriented (boundaryless) Riemannian manifold (M, ε) and a spinor module with charge conjugation $(\mathcal{S}, C)$, together with a Riemannian metric g, so that the Clifford action $c\colon \mathcal{B} \to \operatorname{End}_{\mathcal{A}}(\mathcal{S})$ has been specified. We can also write it as $\hat{c} \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{S}, \mathcal{S})$ by setting $\hat{c}(\kappa \otimes \psi) := c(\kappa)\,\psi$.

Definition 3.1. *Using the inclusion $\mathcal{A}^1(M) \hookrightarrow \mathcal{B}$ —where in the odd dimensional case this is given by $c(\alpha) := c(\alpha\gamma)$, as before— we can form the composition*

$$\not{D} := -i\,\hat{c} \circ \nabla^S \tag{3.1}$$

where

$$\mathcal{S} \xrightarrow{\nabla^S} \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S} \xrightarrow{\hat{c}} \mathcal{S},$$

so that $\not{D}\colon \mathcal{S} \to \mathcal{S}$ is $\mathbb{C}$-linear. This is the Dirac operator *associated to $(\mathcal{S}, C)$ and g.*

The $(-i)$ is included in the definition to make $\not{D}$ symmetric (instead of skewsymmetric) as an operator on a Hilbert space, because we have chosen g to be positive definite, that is, $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = +2\,\delta^{\alpha\beta}$. Historically, $\not{D}$ was introduced as $-i\gamma^\mu\delta_\mu = \gamma^\mu p_\mu$ where the p_μ are components of a 4-momentum, but in the Minkowskian signature.

Using local (coordinate or orthonormal) bases for $\mathfrak{X}(M)$ and $\mathcal{A}^1(M)$, we get nicer formulas:

$$\not{D}\psi = -i\,\hat{c}(\nabla^S\psi) = -i\,c(dx^j)\nabla^S_{\partial_j}\psi = -i\,\gamma^\alpha\nabla^S_{E_\alpha}\psi. \tag{3.2}$$

The essential algebraic property of $\not{D}$ is the *commutation relation*:

$$[\not{D}, a] = -i\,c(da), \quad \text{for all} \quad a \in \mathcal{A} = C^\infty(M). \tag{3.3}$$

Indeed,

$$\begin{aligned}
[\not{D}, a]\,\psi &= -i\,\hat{c}(\nabla^S(a\psi)) + ia\,\hat{c}(\nabla^S\psi) \\
&= -i\,\hat{c}(\nabla^S(a\psi) - a\,\nabla^S\psi) \\
&= -i\,\hat{c}(da \otimes \psi) = -i\,c(da)\,\psi, \quad \text{for } \psi \in \mathcal{S}.
\end{aligned}$$

3.1 The metric distance property

As an operator, we can make sense of $[\not{D}, a]$ by conferring on $\mathcal{S}$ the structure of a Hilbert space: if we write $\det g := \det[g_{ij}]$ for short, then

$$\nu_g := \sqrt{\det g}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \mathcal{A}^n(M)$$

is the Riemannian volume form (for the given orientation ε and metric g). In the notation, we assume that all local charts are consistent with the given orientation, which just means that $\det[g_{ij}] > 0$ in any local chart. The scalar product on $\mathcal{S}$ is then given by

$$\langle \phi \mid \psi \rangle := \int_M (\phi \mid \psi)\, \nu_g \quad \text{for } \phi, \psi \in \mathcal{S}.$$

On completion in the norm $\|\psi\| := \sqrt{\langle \psi \mid \psi \rangle}$, we get the Hilbert space $\mathcal{H} := L^2(M, S)$ of L^2-spinors on M.

Using the gradient $\operatorname{grad} a := (da)^\sharp \in \mathfrak{X}(M)$, we can compute

$$\begin{aligned}
\|[\not{D}, a]\|^2 = \|c(da)\|^2 &= \sup_{x \in M} \|c_x(da(x))\|^2 \\
&= \sup_{x \in M} g_x(d\bar{a}(x), da(x)) \quad \text{with } g_x \text{ on } \ (T_x^* M)^{\mathbb{C}} \\
&= \sup_{x \in M} g_x\big(\operatorname{grad} \bar{a}\big|_x, \operatorname{grad} a\big|_x\big) \quad \text{using the dual } g_x \text{ on } \ (T_x M)^{\mathbb{C}} \\
&=: \sup_{x \in M} \big\|\operatorname{grad} a\big|_x\big\|^2 =: \|\operatorname{grad} a\|_\infty^2.
\end{aligned}$$

Classically, we compute distances on a (connected) Riemannian manifold by the formula

$$d(x, y) := \inf\{\, \operatorname{length}(\gamma) : \gamma \colon [0,1] \to M;\ \gamma(0) = x,\ \gamma(1) = y \,\},$$

with the infimum taken over all piecewise-smooth paths γ in M from x to y. For $a \in C^\infty(M)$, we then get

$$\begin{aligned}
a(y) - a(x) = a(\gamma(1)) - a(\gamma(0)) &= \int_0^1 \frac{d}{dt}[a(\gamma(t))]\, dt \\
&= \int_0^1 \dot{\gamma}(a)\big|_{\gamma(t)}\, dt = \int_0^1 da(\dot{\gamma})\big|_{\gamma(t)}\, dt = \int_0^1 da_{\gamma(t)}(\dot{\gamma}(t))\, dt \\
&= \int_0^1 g_{\gamma(t)}\big(\operatorname{grad} a\big|_{\gamma(t)}, \dot{\gamma}(t)\big)\, dt,
\end{aligned}$$

and we can estimate this difference by

$$\begin{aligned}
|a(y) - a(x)| &\le \int_0^1 \big|\operatorname{grad} a\big|_{\gamma(t)}\big|\, |\dot{\gamma}(t)|\, dt \\
&\le \|\operatorname{grad} a\|_\infty \int_0^1 |\dot{\gamma}(t)|\, dt = \|\operatorname{grad} a\|_\infty \operatorname{length}(\gamma) \\
&= \|[\not{D}, a]\| \operatorname{length}(\gamma).
\end{aligned}$$

Thus

$$\sup\{\, |a(y) - a(x)| : a \in C(M),\ \|[D\!\!\!/, a]\| \leq 1 \,\} \leq \inf_\gamma \operatorname{length}(\gamma) =: d(x, y). \quad (3.4)$$

In this supremum, we can use $a \in C(M)$ not necessarily smooth; a need only be continuous with $\operatorname{grad} a$ (ν-essentially) bounded. Since we have obtained $|a(y) - a(x)| \leq \|\operatorname{grad} a\|_\infty d(x, y)$, we see that a need only be *Lipschitz* on M —with respect to the distance d— with Lipschitz constant $\leq \|\operatorname{grad} a\|_\infty$. In fact, this is the best general Lipschitz constant: fix $x \in M$, and set $a_x(y) := d(x, y)$. This function lies in $C(M)$, and $|a_x(y) - a_x(z)| \leq d(y, z)$ by the triangle inequality for d. Since $\|\operatorname{grad} a_x\|_\infty = 1$ by a local geodesic calculation, we see that $a = a_x$ makes the inequality in (3.4) sharp:

$$\begin{aligned} d(x, y) &= \sup\{\, |a(y) - a(x)| : \|\operatorname{grad} a\|_\infty \leq 1 \,\} \\ &= \sup\{\, |a(y) - a(x)| : a \in C(M),\ \|[D\!\!\!/, a]\| \leq 1 \,\}, \end{aligned} \quad (3.5)$$

so that $D\!\!\!/$ determines the Riemannian distance d, which in turn determines the metric g. (The Myers–Steenrod theorem of differential geometry says that g is uniquely determined by its distance function d.)

Example 3.2. Take $M = \mathbb{S}^1$ ($n = 1$, $m = 0$, $2^m = 1$). The trivial line bundle is a spinor bundle, with $\mathcal{S} = C^\infty(\mathbb{S}^1) = \mathcal{A}$, and C is just the complex conjugation K of functions. With the *flat* metric on $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$, we can identify $\mathcal{S}$ with the set of smooth 1-periodic functions on $\mathbb{R}$, so both ∇ and ∇^S are trivial since $\Gamma^1_{11} = 0$. Therefore,

$$D\!\!\!/ = -i\,\frac{d}{d\theta}$$

is the Dirac operator in this case. Thus $[D\!\!\!/, f] = -if'$ for $f \in \mathcal{A}$, and for $\alpha, \beta \in [0, 1]$, we get

$$|f(\beta) - f(\alpha)| = \left| \int_\alpha^\beta f'(\theta)\, d\theta \right| \leq \int_\alpha^\beta |f'(\theta)|\, d\theta \leq |\beta - \alpha| \quad \text{whenever } \|f'\|_\infty \leq 1.$$

Using $f_\alpha(\beta) := |\beta - \alpha|$ for $\alpha - \frac{1}{2} \leq \beta \leq \alpha + \frac{1}{2}$ wrapped around $\mathbb{R}/\mathbb{Z}$, we get a Lipschitz function making the inequality sharp. Thus $d(\alpha, \beta) = |\beta - \alpha|$ provided $|\beta - \alpha| \leq \frac{1}{2}$: this is just the *arc length* on the circle of circumference 1.

More generally, the formula for $d(x, y)$ yields the length of the minimal geodesic from x to y, provided y is closer to x, than the "cut-locus" of x.

3.2 Symmetry of the Dirac operator

We now regard $D\!\!\!/$ as an operator on $L^2(M, S)$, defined initially on the dense domain $\mathcal{S} = \Gamma_{\text{smooth}}(M, S)$.

Proposition 3.3. *$D\!\!\!/$ is* symmetric*: that is, whenever $\phi, \psi \in \mathcal{S}$, the following equality holds:*

$$\langle D\!\!\!/\phi \mid \psi \rangle = \langle \phi \mid D\!\!\!/\psi \rangle.$$

Proof. We compute the pairings $(D\!\!\!/\phi \mid \psi)$ and $(\phi \mid D\!\!\!/\psi)$, which take values in $\mathcal{A} = C^\infty(M)$. We need a formula for the *divergence* of a vector field: $\mathcal{L}_X \nu_g =: (\operatorname{div} X)\, \nu_g$, so that

$$\int_M (\operatorname{div} X)\, \nu_g = \int_M \mathcal{L}_X(\nu_g) = \int_M \iota_X(d\nu_g) + d(\iota_X \nu_g) = \int_M d(\iota_X \nu_g) = 0$$

by Stokes' theorem (remember that M has no boundary). This formula is

$$\operatorname{div} X = \partial_j X^j + \Gamma^j_{jk} X^k = dx^j(\nabla_{\partial_j} X),$$

as can easily be checked; on the right hand side we use the Levi-Civita connection on $\mathfrak{X}(M)$. Now we abbreviate $c^j := c(dx^j) \in \Gamma(U, \operatorname{End} S)$, for $j = 1, \dots, n$. Then we compute the difference of $\mathcal{A}$-valued pairings:

$$\begin{aligned} i(\phi \mid D\!\!\!/\psi) - i(D\!\!\!/\phi \mid \psi) &= (\phi \mid c^j\, \nabla^S_{\partial_j} \psi) + (c^j\, \nabla^S_{\partial_j} \phi \mid \psi) \\ &= (\phi \mid \nabla^S_{\partial_j} c^j\, \psi) - (\phi \mid c(\nabla_{\partial_j} dx^j)\psi) + (\nabla^S_{\partial_j} \phi \mid c^j\, \psi) \\ &= \partial_j(\phi \mid c^j\, \psi) - (\phi \mid c(\nabla_{\partial_j} dx^j)\psi). \end{aligned}$$

Here we have used the Leibniz rule for ∇^S, the selfadjointness of c^j since dx^j is a real local 1-form, and the hermiticity of ∇^S.

By duality, the map $\alpha \mapsto (\phi \mid c(\alpha)\psi)$, which takes 1-forms to functions, defines a vector field $Z_{\phi\psi}$ —because $\mathfrak{X}(M) = \operatorname{End}_{C^\infty(M)}(\mathcal{A}^1(M), C^\infty(M))$— so the right hand side becomes

$$\partial_j(dx^j(Z_{\phi\psi})) - (\nabla_{\partial_j} dx^j)(Z_{\phi\psi}) = dx^j(\nabla_{\partial_j} Z_{\phi\psi}) = \operatorname{div} Z_{\phi\psi},$$

where we have used the Leibniz rule for the dual Levi-Civita connections on $\mathcal{A}^1(M)$ and on $\mathfrak{X}(M)$, respectively. Thus

$$(\phi \mid D\!\!\!/\psi) - (D\!\!\!/\phi \mid \psi) = -i \operatorname{div} Z_{\phi\psi}$$

which has integral zero. $\square$

3.3 Selfadjointness of the Dirac operator

If T is a densely defined operator on a Hilbert space $\mathcal{H}$, its *adjoint* T^* has domain

$$\operatorname{Dom} T^* := \{\, \phi \in \mathcal{H} : \exists\, \chi \in \mathcal{H} \text{ with } \langle T\psi \mid \phi\rangle = \langle \psi \mid \chi\rangle \text{ for all } \psi \in \operatorname{Dom} T \,\}$$

and then $T^*\phi := \chi$, of course, so that the formula $\langle T\psi \mid \phi\rangle = \langle \psi \mid T^*\phi\rangle$ holds. If T is *symmetric*, then clearly $\operatorname{Dom} T \subseteq \operatorname{Dom} T^*$ with $T^* = T$ on $\operatorname{Dom} T$: that is, T^* is an *extension* of T to a larger domain.

The second adjoint $T^{**} =: \overline{T}$ is called the *closure* of T (symmetric operators always have this closure), where the domain of the closure is

$$\begin{aligned} \operatorname{Dom} \overline{T} := \{\, \psi \in \mathcal{H} : \exists\, \phi \in \mathcal{H} &\text{ and a sequence } \{\psi_n\} \subset \operatorname{Dom} T, \\ &\text{such that } \psi_n \to \psi \text{ and } T\psi_n \to \phi \text{ in } \mathcal{H} \,\} \end{aligned}$$

In other words, the graph of $\overline{T}$ in $\mathcal{H} \oplus \mathcal{H}$ is the closure of the graph of T. And then, of course, we put $\overline{T}\psi := \phi$. When T is symmetric, we get

$$\operatorname{Dom} T \subseteq \operatorname{Dom} \overline{T} \subseteq \operatorname{Dom} T^*.$$

Definition 3.4. *We say that T is* selfadjoint *if $T = T^*$; thus T is symmetric and closed. Otherwise, we say that T is* essentially selfadjoint *if it is symmetric and its closure $\overline{T}$ is selfadjoint.*

Remark 3.5. Selfadjoint operators have *real spectra*: $\mathrm{sp}(T) \subseteq \mathbb{R}$. This is crucial: an unbounded operator that is merely symmetric may have non-real elements in its spectrum. Moreover, selfadjoint operators obey the *spectral theorem*: $T = \int_{\mathbb{R}} \lambda\, dE_T(\lambda)$, where E_T is a "projector-valued measure" on Borel subsets of $\mathbb{R}$ with support in $\mathrm{sp}(T)$.

The main result of this chapter is that the Dirac operator on a compact Riemannian spin manifold is essentially selfadjoint. This was proved by Wolf in 1973; he actually showed the result also for noncompact manifolds which are *complete* with respect to the Riemannian distance given by the metric [w-ja73]. In his proof, completeness is needed to establish that closed geodesic balls are compact; that proof is also given in the book by Friedrich [f-t00]. For simplicity, we deal here only with the compact case.

Theorem 3.6. *Let (M, g) be a compact boundaryless Riemannian spin manifold. The Dirac operator $D\!\!\!/$ is essentially selfadjoint on its original domain $\mathcal{S}$.*

Proof. There is a natural norm on $\mathrm{Dom}\, D\!\!\!/^*$, given by

$$|||\psi|||^2 := \|\psi\|^2 + \|D\!\!\!/^*\psi\|^2.$$

We claim that $\mathcal{S} = \Gamma_{\mathrm{smooth}}(M, S)$ is dense in $\mathrm{Dom}\, D\!\!\!/^*$ for this norm. Using a finite partition of unity $f_1 + \cdots + f_r = 1$ with each $f_i \in \mathcal{A}$ supported in a chart domain U_i over which $S|_{U_i} \to U_i$ is trivial, it is enough to show that any $f_i\phi$, with $\phi \in \mathrm{Dom}\, D\!\!\!/^*$, can be approximated in the $|||\cdot|||$-norm by elements of $\Gamma_{\mathrm{smooth}}(U_i, S)$. Thus we can suppose that $\mathrm{supp}\,\phi \subset U_i$, and regard $\phi \in L^2(U_i, S)$ as a 2^m-tuple of functions $\phi = \{\phi_k\}$ with each $\phi_k \in L^2(U_i, \nu_g)$.

Previous formulas now show that

$$\begin{aligned}\langle D\!\!\!/^*\phi \mid \psi\rangle = \langle \phi \mid D\!\!\!/\psi\rangle &= \int_M (\phi \mid c^j\, \nabla^S_{\partial_j}\psi) = \int (c^j\phi \mid \nabla^S_{\partial_j}\psi) \\ &= \int \big[\partial_j(c^j\,\phi \mid \psi) - (\nabla^S_{\partial_j} c^j\phi \mid \psi)\big]\, \nu_g \\ &= \int_M \big[-(c^j\phi \mid \psi)\,(\mathrm{div}\,\partial_j) - (\nabla^S_{\partial_j} c^j\phi \mid \psi)\big]\, \nu_g\end{aligned}$$

after an integration by parts, so that $D\!\!\!/^*$ is given by the formula $D\!\!\!/^* = -(\nabla^S_{\partial_j} + \mathrm{div}\,\partial_j)\, c(dx^j)$, as a vector-valued distribution on U_i; in particular, it is also a differential operator (the difference $D\!\!\!/^* - D\!\!\!/$ will soon be seen to vanish).

Now, if $\{h_r\}$ is a smooth delta-sequence, then for large enough r we can convolve both ϕ and $D\!\!\!/^*\phi$ with h_r, while remaining supported in U_i —the convolution is defined after pulling back functions on the chart domain U_i to an fixed open subset of $\mathbb{R}^n$. Thus we find that $\phi * h_r \to \phi$ and $D\!\!\!/^*(\phi * h_r) \to D\!\!\!/^*\phi$ in $L^2(U_i, \nu_g)^{2^m}$, so that $|||\phi * h_r - \phi||| \to 0$. But the spinors $\phi * h_r$ are smooth since the h_r are smooth, so we conclude that $\mathcal{S}$ is $|||\cdot|||$-dense in $\mathrm{Dom}\, D\!\!\!/^*$.

But now $D\!\!\!/^*(\phi * h_r) = D\!\!\!/(\phi * h_r)$ since $\mathcal{S} = \mathrm{Dom}\, D\!\!\!/$, so we have shown that ϕ lies in $\mathrm{Dom}\, \overline{D\!\!\!/}$ and that $\overline{D\!\!\!/}\phi = D\!\!\!/^*\phi$. Thus $\mathrm{Dom}\, \overline{D\!\!\!/} = \mathrm{Dom}\, D\!\!\!/^*$, and it follows that $\overline{D\!\!\!/} = D\!\!\!/^{**} = D\!\!\!/^*$: which establishes that $\overline{D\!\!\!/}$ is selfadjoint. ∎

3.4 The Schrödinger–Lichnerowicz formula

If $E \to M$ is any smooth vector bundle with connection ∇^E on $\mathcal{E} = \Gamma(M, E)$, we can consider not only $\nabla^E \colon \mathcal{E} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$, but also the connection $\nabla^{E'} :=$

$\nabla \otimes 1 + 1 \otimes \nabla^E$ on the tensor product bundle $\mathcal{E}' = \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$; here ∇ is once again the Levi-Civita connection on $\mathcal{A}^1(M)$. Their composition is an operator $\nabla^{E'} \circ \nabla^E$ from $\mathcal{E}$ to $\mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$; using the metric g on $\mathcal{A}^1(M)$ we can take the trace over the first two factors, ending up with a *Laplacian*:

$$\Delta^E := -\operatorname{Tr}_g(\nabla^{E'} \circ \nabla^E) : \mathcal{E} \to \mathcal{E}. \tag{3.6}$$

The minus sign is a convention to yield a positive operator (instead of a negative one) [bgv92]. Locally, this means:

$$\Delta^E = -g^{ij}(\nabla^E_{\partial_i} \nabla^E_{\partial_j} - \Gamma^k_{ij} \nabla^E_{\partial_k}).$$

Definition 3.7. *In particular, when $E = M \times \mathbb{C}$ is the trivial line bundle, we get the "scalar Laplacian"*

$$\Delta = -g^{ij}(\partial_i \partial_j - \Gamma^k_{ij} \partial_k), \tag{3.7}$$

also known as the "Laplace–Beltrami operator" on $\mathcal{A} = C^\infty(M)$. Likewise, when $E = S$, we get the spinor Laplacian *for a spin manifold.*

Before examining the relation between the Dirac operator and the spinor Laplacian, we collect a few well-known formulas for the *Riemann curvature tensor*, R. These can be found in many places, for instance [bgv92]; perhaps the best reference is Milnor's little book [m-jw63].

The square of the Levi-Civita connection on $\mathfrak{X}(M)$ is $C^\infty(M)$-linear, so it is given by $\nabla^2 X = R(X)$, where $R \in \mathcal{A}^2(M, \operatorname{End} TM)$. In local coordinates, its components are $R_{ijkl} := g(\partial_i, R(\partial_k, \partial_l)\, \partial_j)$.

Taking a trace over the first and third indices, we get the *Ricci tensor*, whose components are $R_{jl} := g^{ik} R_{ijkl}$. The trace of the Ricci tensor is the *scalar curvature* (or "curvature scalar") $s := g^{jl} R_{jl} = g^{jl} g^{ik} R_{ijkl} \in C^\infty(M)$. Under exchange of indices, R has the following skewsymmetry and symmetry relations:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}; \qquad R_{ijkl} = R_{klij}.$$

The (first) *Bianchi identity* says that the cyclic sum over three indices vanishes:

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0.$$

Moreover, the Ricci tensor is symmetric: $R_{jl} = R_{lj}$.

The formula in the next Proposition is generally attributed to Lichnerowicz [l-a63, 1963], but was anticipated by Schrödinger in a little-known paper [s-e32, 1932].

Proposition 3.8. *Let (M, g) be a compact Riemannian spin manifold with spinor module $\mathcal{S}$. Then*

$$\not{D}^2 = \Delta^S + \tfrac{1}{4} s \tag{3.8}$$

as an operator on $\mathcal{S}$,

Proof. It is enough to prove the equality when applied to spinors ψ supported in a chart domain, so we may use local coordinate formulas. Since $\not{D} = -i\, c^j\, \nabla^S_{\partial_j}$, we get

$$\begin{aligned} \not{D}^2 &= -c^i\, \nabla^S_{\partial_i} c^j\, \nabla^S_{\partial_j} = -c^i c^j\, \nabla^S_{\partial_i} \nabla^S_{\partial_j} - c^i c(\nabla_{\partial_i} dx^k)\, \nabla^S_{\partial_k} \\ &= -c^i c^j\, (\nabla^S_{\partial_i} \nabla^S_{\partial_j} - \Gamma^k_{ij}\, \nabla^S_{\partial_k}), \end{aligned}$$

and from $\Gamma^k_{ij} = \Gamma^k_{ji}$ (torsion freedom) and the Clifford relation $c^i c^j + c^j c^i = 2g^{ij}$, we get

$$\begin{aligned} D\!\!\!/^2 &= -g^{ij}(\nabla^S_{\partial_i}\nabla^S_{\partial_j} - \Gamma^k_{ij}\nabla^S_{\partial_k}) - \tfrac12 c^i c^j\,[\nabla^S_{\partial_i}, \nabla^S_{\partial_j}] \\ &= \Delta^S - \tfrac12 c^i c^j\,[\nabla^S_{\partial_i}, \nabla^S_{\partial_j}]. \end{aligned}$$

Since $[\partial_i, \partial_j] = 0$, the commutator $[\nabla^S_{\partial_i}, \nabla^S_{\partial_j}]$ is a spin-curvature term:

$$[\nabla^S_{\partial_k}, \nabla^S_{\partial_l}] = R^S(\partial_k, \partial_l) = \tfrac14 R_{ijkl}\,c^i c^j,$$

because the curvature R^S of ∇^S is given by $\dot\mu(R)$. Hence,

$$D\!\!\!/^2 - \Delta^S = -\tfrac18 R_{ijkl}\,c^k c^l c^i c^j = \tfrac18 R_{jikl}\,c^k c^l c^i c^j. \tag{3.9}$$

Since R_{jikl} has cyclic sum zero in the indices i, k, l, we can also skewsymmetrize $c^k c^l c^i = c(dx^k)\,c(dx^l)\,c(dx^i)$. It is a simple exercise to check that

$$c^k c^l c^i = Q(dx^k \wedge dx^l \wedge dx^i) + g^{li}c^k - g^{ki}c^l + g^{kl}c^i,$$

If we now skewsymmetrize the right hand side of (3.9) in the indices i, k, l — which does not change its value— the Q-term contributes zero to the result. Also, the term $g^{kl}c^i c^j = g^{lk}c^i c^j$ contributes zero, while $g^{li}c^k c^j$ and $-g^{ki}c^l c^j$ contribute equally. Thus,

$$\begin{aligned} D\!\!\!/^2 - \Delta^S &= \tfrac14 R_{ijkl}g^{ik}c^l c^j = \tfrac14 R_{jl}c^l c^j = \tfrac18 R_{jl}(c^l c^j + c^j c^l) \\ &= \tfrac14 R_{jl}g^{jl} = \tfrac14 s. \end{aligned}$$ □

One consequence of the formula (3.8) is a famous "vanishing theorem" of Lichnerowicz.

Corollary 3.9. *If $s(x) \geq 0$ for all $x \in M$, and $s(x_0) > 0$ at some point $x_0 \in M$, then* $\ker D\!\!\!/ = \{0\}$.

Proof. Suppose that $\psi \in \mathcal{S}$ satisfies $D\!\!\!/\psi = 0$. Then

$$0 = \|D\!\!\!/\psi\|^2 = \langle\psi \mid D\!\!\!/^2\psi\rangle = \langle\psi \mid \Delta^S\psi\rangle + \int_M \tfrac14 s\,(\psi \mid \psi)\,\nu_g. \tag{3.10}$$

Now it is easy to check that, after an integration by parts over M and discarding a divergence term,

$$\langle\psi \mid \Delta^S\psi\rangle = g^{ij}\,\langle\nabla^S_{\partial_i}\psi \mid \nabla^S_{\partial_j}\psi\rangle. \tag{3.11}$$

Since the matrix $[g^{ij}]$ is positive definite, this (by the way) shows that Δ^S is a positive operator; and since $s \geq 0$, both terms on the right hand side of (3.10) are nonnegative; so they must both vanish, since their sum is zero.

Moreover, (3.11) shows that $\langle\psi \mid \Delta^S\psi\rangle = 0$ implies $\nabla^S\psi = 0$. This in turn implies that $\partial_j\,(\psi \mid \psi) = (\nabla^S_{\partial_j}\psi \mid \psi) + (\psi \mid \nabla^S_{\partial_j}\psi)$ vanishes for each j, so that $k := (\psi \mid \psi)$ is a constant function. But now (3.10) reduces to $0 = k\int_M s\,\nu_g$, which entails $k = 0$ and then $\psi = 0$. □

We saw by example (Appendix A.2) that on $\mathbb{S}^2$, the Dirac operator for the round metric has spectrum $\operatorname{sp}(D\!\!\!/) = \mathbb{N} \setminus \{0\}$: here $s \equiv 2$ and $\ker D\!\!\!/ = \{0\}$. Thus there are no "harmonic spinors" on $\mathbb{S}^2$.

3.5 The spectral growth of the Dirac operator

Since $D\!\!\!/^2 = \Delta^S + \frac{1}{4}s$, and Δ^S is closely related to the Laplacian Δ on the (compact, boundaryless) Riemannian manifold (M, g), the general features of $\mathrm{sp}(D\!\!\!/)$ may be deduced from those of $\mathrm{sp}(\Delta)$.

We require two main properties of Laplacians on compact Riemannian manifolds without boundary. Recall that

$$\Delta = -\operatorname{Tr}_g(\nabla^{T^*M\otimes T^*M} \circ \nabla^{T^*M}) = -g^{ij}\,(\partial_i\,\partial_j - \Gamma^k_{ij}\,\partial_k)$$

is the local expression for the Laplacian (which depends on g through the Levi-Civita connection and g^{ij}). Thus Δ is a second order differential operator on $C^\infty(M)$.

Fact 3.10. *The Laplacian Δ extends to a positive selfadjoint operator on $L^2(M, \nu_g)$ —also denoted by Δ— and $(1 + \Delta)$ has a compact inverse.*

To make Δ selfadjoint, we must complete $C^\infty(M)$ to a larger domain, by defining

$$|\!|\!| f |\!|\!|^2 := \langle f \mid (1+\Delta)f\rangle = \|f\|^2 + g^{ij}\,\langle \partial_i f \mid \partial_j f\rangle,$$

where $\langle f \mid f\rangle := \int_M |f|^2\,\nu_g$. Taking $\operatorname{Dom}\Delta := \{\, f \in L^2(M,\nu_g) : |\!|\!| f |\!|\!| < \infty \,\}$, Δ becomes selfadjoint and $(1+\Delta)^{-1} \colon L^2(M,\nu_g) \to (\operatorname{Dom}\Delta, |\!|\!|\cdot|\!|\!|)$ is bounded. Then one shows that the inclusion $(\operatorname{Dom}\Delta, |\!|\!|\cdot|\!|\!|) \hookrightarrow L^2(M,\nu_g)$ is a compact operator (by Rellich's theorem); and $(1+\Delta)^{-1}$, as a bounded operator on $L^2(M,\nu_g)$, is then the composition of these two, so it is also compact.

Corollary 3.11. *Δ has discrete (point) spectrum of finite multiplicity.*

Proof. Since $(1+\Delta)^{-1}$ is compact, its spectrum —except for 0— consists only of eigenvalues of finite multiplicity. Therefore, the same is true of $1+\Delta$, and of Δ itself. Indeed,

$$\mathrm{sp}((1+\Delta)^{-1}) = \left\{ \frac{1}{1+\lambda_0}, \frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2}, \dots \right\}$$

with $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ being the list of eigenvalues of Δ in increasing order. These are *counted with multiplicity*: an eigenvalue of multiplicity r appears exactly r times on the list. This $\lambda_k \to \infty$, since $(1+\lambda_k)^{-1} \to 0$, as $k \to \infty$. □

As a convention, when A is a compact positive operator, we write $\lambda_k(A)$ to denote the k-th eigenvalue of A in decreasing order (with multiplicity): $\lambda_0(A) \ge \lambda_1(A) \ge \cdots$; on the other hand, if A is an unbounded positive selfadjoint operator with compact inverse, we write the eigenvalues in increasing order, as we did for Δ.

Fact 3.12 (Weyl's theorem)**.** *The counting function for* $\mathrm{sp}(\Delta)$ *is*

$$N_\Delta(\lambda) := \#\{\, \lambda_k(\Delta) : \lambda_k(\Delta) \le \lambda \,\}.$$

For large λ, the following asymptotic estimate holds:

$$N_\Delta(\lambda) \sim C_n \operatorname{Vol}(M)\, \lambda^{n/2} \quad \textit{as } \lambda \to \infty,$$

where $n = \dim M$*, and* $\mathrm{Vol}(M) = \int_M \nu_g$ *is the total volume of the manifold* M*. The constant* C_n*, that depends only on the dimension* n*, is*

$$C_n = \frac{\Omega_n}{n(2\pi)^n} = \frac{1}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)},$$

where $\Omega_n = \mathrm{Vol}(\mathbb{S}^{n-1}) = 2\,\pi^{n/2}/\Gamma(\frac{n}{2})$.

We shall not prove Weyl's theorem, in particular why the number of eigenvalues (up to λ) is proportional to $\mathrm{Vol}(M)$, but we shall compute the constant by considering an example. For a simple and clear exposition of the proof, we recommend Higson's ICTP lectures [h-n04].

Example 3.13. Take $M = \mathbb{T}^n = \mathbb{R}^n/bZ^n$ to be the n-torus with unit volume. Identify $C^\infty(\mathbb{T}^n)$ with the smooth periodic functions on the unit cube $[0,1]^n$. For the flat metric on $\mathbb{T}^n$, and local coordinates $t = (t^1, \dots, t^n)$, we get

$$\Delta = -\Big(\frac{\partial}{\partial t^1}\Big)^2 - \Big(\frac{\partial}{\partial t^2}\Big)^2 - \cdots - \Big(\frac{\partial}{\partial t^n}\Big)^2.$$

Thus we find eigenfunctions, labelled by $r = (r_1, \dots, r_n) \in \mathbb{Z}^n$:

$$\phi_r := e^{2\pi i r\cdot t}, \quad \text{for } t \in [0,1]^n.$$

Since $\{\, \phi_r : r \in \mathbb{Z}^n \,\}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$, these are a complete set of eigenfunctions, and therefore

$$\mathrm{sp}(\Delta) = \{\, 4\pi^2\,|r|^2 : r \in \mathbb{Z}^n \,\}.$$

If $B(0;R)$ is the ball of radius R, centered at $0 \in \mathbb{R}^n$, then

$$\begin{aligned} N_\Delta(\lambda) &= \#\{\, r \in \mathbb{Z}^n : 4\pi^2|r|^2 \le \lambda \,\} \\ &\sim \mathrm{Vol}(B(0;\sqrt{\lambda/4\pi^2})) = \Big(\frac{\lambda}{4\pi^2}\Big)^{n/2}\mathrm{Vol}(B(0;1)) \\ &= \frac{\lambda^{n/2}}{(2\pi)^n}\frac{\Omega_n}{n} = \frac{\Omega_n}{n(2\pi)^n}\,\lambda^{n/2}, \quad \text{as } \lambda \to \infty, \end{aligned}$$

using

$$\begin{aligned} \mathrm{Vol}(B(0;1)) &= \int_{B(0;1)} dx^1 \wedge \cdots \wedge dx^n \\ &= \int_0^1 \Big(\int_{\mathbb{S}^{n-1}} \nu\Big) r^{n-1}\,dr \\ &= \Omega_n \int_0^1 r^{n-1}\,dr = \frac{\Omega_n}{n}. \end{aligned}$$

For the spinor Laplacian Δ^S, a similar estimate holds, but with C_n replaced by $2^m C_n$ (recall that in the flat torus case with untwisted spin structure, $\mathcal{S} \cong C^\infty(\mathbb{T}^n) \otimes \mathbb{C}^{2^m}$). Now by Lichnerowicz' formula, $D\!\!\!\!/^2$ differs from Δ^S by a bounded multiplication operator $\frac{1}{4}s$, thus $N_{D\!\!\!/^2}(\lambda) \sim N_{\Delta^S}(\lambda)$ as $\lambda \to \infty$, hence

$$N_{D\!\!\!/^2}(\lambda) \sim \frac{2^m\,\Omega_n}{n(2\pi)^n}\,\mathrm{Vol}(M)\,\lambda^{n/2}, \quad \text{as } \lambda \to \infty.$$

Consider the positive operator $|\not{D}| := (\not{D}^2)^{1/2}$; remember that μ is an eigenvalue for $|\not{D}|$ if and only if μ^2 is an eigenvalue for $\not{D}^2$ (with the same multiplicity). We arrive at the following estimate.

Corollary 3.14.

$$N_{|\not{D}|}(\lambda) \sim \frac{2^m\,\Omega_n}{n(2\pi)^n}\,\mathrm{Vol}(M)\,\lambda^n, \quad \text{as } \lambda \to \infty. \qquad \square$$

Example 3.15. For $M = \mathbb{S}^2$, with $n = 2$, we have seen (in Appendix A.2) that

$$\begin{aligned} \mathrm{sp}(\not{D}) &= \{\, \pm(l + \tfrac{1}{2}) : l + \tfrac{1}{2} \in \mathbb{N} + \tfrac{1}{2} \,\}, \quad \text{with multiplicities } 2l+1 \\ &= \{\, \pm k : k = 1, 2, 3, \dots \}, \quad \text{with multiplicities } 2k. \end{aligned}$$

Therefore

$$N_{|\not{D}|}(\lambda) = \sum_{1 \le k \le \lambda} 4k = 2\lfloor\lambda\rfloor(\lfloor\lambda\rfloor + 1) \sim 2\lambda(\lambda+1) \sim 2\lambda^2, \quad \text{as } \lambda \to \infty.$$

Now $C_2 = \dfrac{\Omega_2}{2(2\pi)^2} = \dfrac{2\pi}{8\pi^2} = \dfrac{1}{4\pi}$ and $2C_2 = \dfrac{1}{2\pi}$ for spinors. Therefore $2C_2\,\mathrm{Area}(\mathbb{S}^2)\,\lambda^2 = 2\lambda^2$, so

$$\mathrm{Area}(\mathbb{S}^2) = \frac{1}{C_2} = 4\pi.$$

In other words, Weyl's theorem allows us to deduce the area of the 2-sphere $\mathbb{S}^2$ from (the knowledge of the circumference of the circle $\Omega_2 = 2\pi$ and) the growth of the spectrum of the Dirac operator on $\mathbb{S}^2$.

Chapter 4

Spectral Growth and Dixmier Traces

4.1 Definition of spectral triples

We start with the definition of the main concept in noncommutative geometry.

Definition 4.1. *A (unital)* **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ *consists of:*

- *an* algebra $\mathcal{A}$ *with an involution* $a \mapsto a^*$*, equipped with a faithful representation on:*
- *a* Hilbert space $\mathcal{H}$*; and also*
- *a* selfadjoint operator D *on* $\mathcal{H}$*, with dense domain* $\operatorname{Dom} D \subset \mathcal{H}$*, such that* $a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ *for all* $a \in \mathcal{A}$*,*

satisfying the following two conditions:

- *the operator* $[D, a]$*, defined initially on* $\operatorname{Dom} D$*, extends to a* bounded operator *on* $\mathcal{H}$*, for each* $a \in \mathcal{A}$*;*
- *D has* compact resolvent*:* $(D - \lambda)^{-1}$ *is compact, when* $\lambda \notin \operatorname{sp}(D)$*.*

For now, and until further notice, all spectral triples will be defined over unital algebras. The compact-resolvent condition must be modified if $\mathcal{A}$ is nonunital: as well as enlarging $\mathcal{A}$ to a unital algebra, we require only that the products $a(D - \lambda)^{-1}$, for $a \in \mathcal{A}$ and $\lambda \notin \operatorname{sp}(D)$, be compact operators.

Example 4.2. Let (M, ε) be an oriented compact boundaryless manifold which is spin, i.e. admits spin structures, and $(\mathcal{S}, C)$ be data for a specific spin structure. Choose a Riemannian metric g on M (which allows us to define ∇ and ∇^S) and let $D\!\!\!\!/ = -i\,\hat{c} \circ \nabla^S$ be the corresponding Dirac operator, extended to be a selfadjoint operator on $L^2(M, S)$. Then $(C^\infty(M), L^2(M, S), D\!\!\!\!/)$ is a spectral triple. Here $[D\!\!\!\!/, a] = -i\,c(da)$ is a bounded operator on spinors, with $\|[D\!\!\!\!/, a]\| = \|\operatorname{grad} a\|_\infty$, for $a \in C^\infty(M)$. We know by now that $(D\!\!\!\!/^2 + 1)^{-1} = (D\!\!\!\!/ - i)^{-1}(D\!\!\!\!/ + i)^{-1}$ is compact, so $(D\!\!\!\!/ \pm i)^{-1}$ is compact. We refer to these spectral triples as "standard commutative examples".

Note that, if $\lambda, \mu \notin \operatorname{sp}(D)$, then $(D-\lambda)^{-1} - (D-\mu)^{-1} = (\lambda-\mu)(D-\lambda)^{-1}(D-\mu)^{-1}$ —this is the famous "resolvent equation"— since

$$(D-\lambda)\big((D-\lambda)^{-1} - (D-\mu)^{-1}\big)(D-\mu) = (D-\mu) - (D-\lambda) = \lambda - \mu.$$

Thus $(D-\lambda)^{-1}$ is compact if and only if $(D-\mu)^{-1}$ is compact, so we need only to check this condition for *one* value of λ. In the same way, we get the following useful result.

Lemma 4.3. *D has compact resolvent if and only if* $(D^2+1)^{-1}$ *is compact.*

Proof. We may take $\lambda = -i$, since the selfadjointness of D implies that $\pm i \notin \operatorname{sp}(D)$. Thus, D has compact resolvent if and only if $(D+i)^{-1}$ is compact. Let $T = (D+i)^{-1}$; then the proof reduces to the well-known result that a bounded operator T is compact if and only if T^*T is compact. □

By the spectral theorem $(D^2+1)^{1/2} - |D| = f(D)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is the continuous function $f(\lambda) := \sqrt{\lambda^2+1} - |\lambda| = \frac{1}{\sqrt{\lambda^2+1}+|\lambda|}$; and $0 < f(\lambda) \leq 1$ for all $\lambda \in \mathbb{R}$, so that $\|f(D)\| \leq 1$. Or more precisely: the operator $f(D) := (D^2+1)^{1/2} - |D|$, defined initially on $\operatorname{Dom} D$, extends to a bounded operator on $\mathcal{H}$, of norm at most 1.

In many arguments to come, we shall employ $|D|$ and $|D|^{-1}$ as if we knew that $\ker D = \{0\}$. However, even if $\ker D \neq \{0\}$, we can always replace $|D|$ by $(D^2+1)^{1/2}$ and $|D|^{-1}$ by $(D^2+1)^{-1/2}$, at the cost of some extra calculation.

4.2 Logarithmic divergence of spectra

If A is a positive selfadjoint operator with compact resolvent, let $\{\, \lambda_k(A) : k \in \mathbb{N} \,\}$ be its eigenvalues listed in increasing order, $\lambda_0(A) \leq \lambda_1(A) \leq \lambda_2(A) \leq \cdots$ (an eigenvalue of multiplicity r occurs exactly r times in the list). The *counting function* $N_A(\lambda)$, defined for $\lambda > 0$, is the number of eigenvalues not exceeding λ:

$$N_A(\lambda) := \#\{\, k \in \mathbb{N} : \lambda_k(A) \leq \lambda \,\}.$$

If A is invertible (i.e., if $\lambda_0(A) > 0$), we can define the "zeta function"

$$\zeta_A(s) := \operatorname{Tr} A^{-s} = \sum_{k \geq 0} \lambda_k(A)^{-s}, \quad \text{for } s > 0,$$

where we understand that $\zeta_A(s) = +\infty$ when A^{-s} is not traceless. For real s, $\zeta_A(s)$ is a nonnegative decreasing function.

It is actually more useful to consider *finite partial sums.*

Notation. If $T \in \mathcal{K}(\mathcal{H})$ is any compact operator, and if $k \in \mathbb{N}$, let $s_k(T)$, called the k-th **singular value** of T, be the k-th eigenvalue of the compact positive operator $|T| := (T^*T)^{1/2}$, where these are listed in decreasing order, with multiplicity. Thus $s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots$ and each singular value occurs only finitely many times in the list, namely, the finite multiplicity of the that eigenvalue of $|T|$; therefore, $s_k(T) \to 0$ as $k \to \infty$. Note that $s_0(T) = \|T\|$ since $s_0(T)^2$ is the largest eigenvalue of T^*T, so that $s_0(T)^2 = \|T^*T\| = \|T\|^2$. For each $N \in \mathbb{N}$, write

$$\sigma_N(T) := \sum_{k=0}^{N-1} s_k(T).$$

We shall see later that for many spectral triples, the counting function of the positive (unbounded) operator $|D|$ has polynomial growth: for some n, one can verify an asymptotic relation $N_{|D|}(\lambda) \sim C'_n \lambda^n$. In that case we can take $A := |D|^{-n}$, which is compact. Then the number of eigenvalues of A that are $\geq \varepsilon$ equals $N_{|D|}(\lambda)$ for $\lambda = 1/\varepsilon$. This suggests heuristically that for N close to $N_{|D|}(1/\varepsilon)$, the N-th eigenvalue is roughly C/ε for some constant C, so that $\sigma_N(|D|^{-n}) = O(\log N)$. We now check this condition in a few examples.

Example 4.4. We estimate $\sigma_N(|\not{D}|^{-s})$ for $s > 0$, where $\not{D}$ is the Dirac operator on the sphere $\mathbb{S}^2$ with its spin structure and its rotation-invariant metric. We know that the eigenvalues of $|\not{D}|$ are $k = 1, 2, 3, \dots$ with respective multiplicities $2(2k) = 4, 8, 12, \dots$. For $r = 1, 2, 3, \dots$, let

$$N_r := \sum_{k=1}^{r} 4k = 2r(r+1) \sim 2r^2 \quad \text{as } r \to \infty,$$

so that $\log N_r \sim 2 \log r$ as $r \to \infty$. Next,

$$\sigma_{N_r}(|\not{D}|^{-s}) = \sum_{k=1}^{r} 4k(k^{-s}) = 4 \sum_{k=1}^{r} k^{1-s},$$

and thus

$$\frac{\sigma_{N_r}(|\not{D}|^{-s})}{\log N_r} \sim \frac{4}{2\log r} \sum_{k=1}^{r} k^{1-s} \sim \frac{2}{\log r} \int_1^r t^{1-s}\, dt \quad \text{as } r \to \infty,$$

by the "integral test" of elementary calculus. There are three cases to consider:

- If $s < 2$, then $\dfrac{2}{\log r} \displaystyle\int_1^r t^{1-s}\, dt = \dfrac{r^{2-s} - 1}{2 - s}$ diverges as $r \to \infty$;
- if $s > 2$, then $\dfrac{2}{\log r} \displaystyle\int_1^r t^{1-s}\, dt \to 0$ as $r \to \infty$; while
- if $s = 2$, then $\dfrac{\sigma_{N_r}(|\not{D}|^{-s})}{\log N_r} \sim \dfrac{2 \log r}{\log_r} \to 2.$

Finally, note that if $N_{r-1} \leq N \leq N_r$, then

$$\frac{\sigma_{N_{r-1}}(|\not{D}|^{-s})}{\log N_r} \leq \frac{\sigma_N(|\not{D}|^{-s})}{\log N} \leq \frac{\sigma_{N_r}(|\not{D}|^{-s})}{\log N_{r-1}},$$

while $\log N \sim \log N_{r-1} \sim \log N_r \sim 2 \log r$ as $r \to \infty$. Thus

$$\lim_{N\to\infty} \frac{\sigma_N(|\not{D}|^{-s})}{\log N} = \lim_{r\to\infty} \frac{\sigma_{N_r}(|\not{D}|^{-s})}{\log N_r} = \begin{cases} +\infty & \text{if } s < 2, \\ 2 & \text{if } s = 2, \\ 0 & \text{if } s > 2. \end{cases}$$

We express this result by saying that for $s = 2$, "the spectrum of $|\not{D}|^{-2}$ diverges logarithmically". There is precisely one exponent, namely $s = 2$, for which this limit is neither zero nor infinite.

Exercise 4.5. *Do the same calculation for $\not{D}$ on the torus $\mathbb{T}^n$, whose spectrum we know: show that the spectrum of $|\not{D}|^{-s}$ diverges logarithmically if and only if $s = n = \dim \mathbb{T}^n$.*

4.3 Some eigenvalue inequalities

Let $\mathcal{H}$ be a separable (infinite-dimensional) Hilbert space, and denote by $B(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the ideal of compact operators on $\mathcal{H}$. Each $T \in \mathcal{K}$ has a polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2} \in \mathcal{K}$, and $U \in B(\mathcal{H})$ is a partial isometry. This factorization is unique if we require that $U = 0$ on $\ker|T|$, since U must map the range of $|T|$ isometrically onto the range of T.

The spectral theorem yields an orthonormal family $\{\psi_k\}$ in $\mathcal{H}$, such that

$$|T| = \sum_{k\geq 0} s_k(T)\,|\psi_k\rangle\langle\psi_k|, \qquad T = \sum_{k\geq 0} s_k(T)\,|U\psi_k\rangle\langle\psi_k|.$$

(If $|T|$ is invertible, this is an orthonormal basis for $\mathcal{H}$. Otherwise, we can adjoin an orthonormal basis for $\ker|T|$ to the family $\{\psi_k\}$.) Since $\phi_k := U\psi_k$ gives another orthonormal family, any $T \in \mathcal{K}$ has an expansion of the form

$$T = \sum_{k\geq 0} s_k(T)\,|\phi_k\rangle\langle\psi_k|, \tag{4.1}$$

for some pair of orthonormal families $\{\phi_k\}$, $\{\psi_k\}$.

If V_1, V_2 are *unitary* operators on $\mathcal{H}$, we can then write

$$V_1TV_2 = \sum_{k\geq 0} s_k(T)\,|V_1\phi_k\rangle\langle V_2^*\psi_k|,$$

and conclude that $s_k(V_1TV_2) = s_k(T)$ for each k, and hence that

$$\sigma_N(V_1TV_2) = \sigma_N(T).$$

Therefore, *any* norm $|||T|||$ that is built from the sequence $\{\, s_k(T) : k \in \mathbb{N}\,\}$ is unitarily invariant, that is, $|||V_1TV_2||| = |||T|||$ for V_1, V_2 unitary.

Example 4.6. If $\|T\|$ is the usual operator norm on $\mathcal{K}$, then

$$\|T\| = \|T^*T\|^{1/2} = \|\,|T|\,\| = \sup_{k\geq 0} s_k(T) = s_0(T).$$

A compact operator T is called *trace-class*, and we write $T \in \mathcal{L}^1 = \mathcal{L}^1(\mathcal{H})$, if the following series converges:

$$\|T\|_1 := \operatorname{Tr}|T| = \sum_{k\geq 0} s_k(T) = \lim_{N\to\infty} \sigma_N(T).$$

For $1 < p < \infty$, there are *Schatten classes* $\mathcal{L}^p = \mathcal{L}^p(\mathcal{H})$ consisting of operators for which the following norm is finite:

$$\|T\|_p = \biggl(\sum_{k\geq 0} s_k(T)^p\biggr)^{1/p}.$$

There are strict inclusions $\mathcal{L}^1 \subset \mathcal{L}^r \subset \mathcal{L}^p \subset \mathcal{K}$ for $1 < r < p < \infty$.

Soon, we shall introduce a "Dixmier trace class" $\mathcal{L}^{1+}(\mathcal{H})$, with yet another norm built from singular values, such that $\mathcal{L}^1 \subset \mathcal{L}^{1+} \subset \mathcal{L}^p$ for $p > 1$.

Much is known about the singular values of compact operators. For instance, the following relation holds, for $T \in \mathcal{K}$:

$$s_k(T) = \inf\{\, \|T(1-P)\| : P = P^2 = P^*,\ \dim P(\mathcal{H}) \leq k \,\}. \tag{4.2}$$

This comes from a well-known minimax principle: see [rs72], for instance. The infimum is indeed attained at the projector Q of rank k whose range is $Q(\mathcal{H}) := \operatorname{span}\{\psi_0, \dots, \psi_{k-1}\}$, when T is given by (4.1), since $T(1-Q) = T - TQ = \sum_{j \geq k} s_j(T) |\phi_j\rangle\langle\psi_j|$ is an operator with norm $\|T - TQ\| = s_k(T)$.

Lemma 4.7. *If $T \in \mathcal{K}$, then*

$$\sigma_N(T) = \sup\{\, \|TP\|_1 : P = P^2 = P^*,\ \operatorname{rank} P = N \,\}. \tag{4.3a}$$

If A is a positive *compact operator, then it is also true that*

$$\sigma_N(A) = \sup\{\, \operatorname{Tr}(PAP) : P = P^2 = P^*,\ \operatorname{rank} P = N \,\}. \tag{4.3b}$$

Proof. If P is a projector of finite rank N, then $(TP)^*(TP) = PT^*TP$ and thus $|TP|$ has finite rank $\leq N$, so $\|TP\|_1 = \sum_{k=0}^{N-1} s_k(TP) = \sigma_N(TP)$. From the formula (4.2), it follows that $0 \leq A \leq B$ in $\mathcal{K}$ implies $s_k(A) \leq s_k(B)$ for each $k \in \mathbb{N}$, and in particular, since $0 \leq PT^*TP \leq T^*T$, we get $s_k(TP) \leq s_k(T)$. Thus $\sigma_N(TP) \leq \sigma_N(T)$ also. We conclude that the right hand side of (4.3a) is $\leq \sigma_N(T)$.

If we write T in the form (4.1) and then choose Q, as before, to be the projector with range $\operatorname{span}\{\psi_0, \dots, \psi_{k-1}\}$, then $|TQ| = \sum_{j=0}^{k-1} s_j(T)\, |\psi_j\rangle\langle\psi_j|$ and thus $\|TQ\|_1 = \sigma_N(T)$.

When $A \in \mathcal{K}$ is positive, and P is a projector of rank n, then $\operatorname{Tr}(PAP) = \operatorname{Tr}(AP) \leq \|AP\|_1 \leq \sigma_N(A)$. To see that the supremum in (4.3b) is attained, we can write $A = \sum_{k \geq 0} s_k(A)\, |\psi_k\rangle\langle\psi_k|$ and note that $QA = AQ = QAQ$, so that AQ is also a positive operator. It then follows that $\operatorname{Tr}(QAQ) = \operatorname{Tr}(AQ) = \|AQ\|_1 = \sigma_N(A)$. $\square$

Corollary 4.8. *Each σ_N is a norm on $\mathcal{K}$: $\sigma_N(S+T) \leq \sigma_N(S) + \sigma_N(T)$ for $S, T \in \mathcal{K}$.*

Proof. This follows from $\|SP + TP\|_1 \leq \|SP\|_1 + \|TP\|_1$ for $P = P^2 = P^*$, $\operatorname{rank} P = N$. $\square$

Lemma 4.9. *If $T \in \mathcal{K}$, then*

$$\sigma_N(T) = \inf\{\, \|R\|_1 + N\,\|S\| : R, S \in \mathcal{K} \textit{ with } R + S = T \,\}.$$

Proof. If $T = U|T|$, then $|T| = U^*T$ (by the details of polar decomposition, this is true even though U might not be unitary), so $T = R + S$ implies $U^*T = U^*R + U^*S$; thus, we can suppose that $T \geq 0$.

If we now split $T =: R + S$, then $\sigma_N(T) \leq \sigma_N(R) + \sigma_N(S) \leq \|R\|_1 + \sigma_N(S)$, while

$$\sigma_N(S) = \sum_{0 \leq k < N} s_k(S) \leq \sum_{0 \leq k < N} s_0(S) = N\,\|S\|.$$

For $T = \sum_{k\geq 0} s_k(T)\,|\psi_k\rangle\langle\psi_k|$, we consider the special splitting into positive operators,

$$\widetilde{R} := \sum_{0\leq k<N} (s_k(T) - s_N(T))\,|\psi_k\rangle\langle\psi_k|, \qquad \widetilde{S} := T - \widetilde{R}.$$

Then $\|\widetilde{R}\|_1 = \sigma_N(T) - N\,s_N(T)$, while $\|\widetilde{S}\| = s_N(T)$ by inspection. ☐

The triangle inequality in Corollary 4.8 is not good enough for our needs: our goal is get an *additive* functional, rather than just a subadditive one. The next step is to extract from (4.3b) a sort of "wrong-way triangle inequality", at least for *positive compact operators*.

Lemma 4.10. *If $A \geq 0$, $B \geq 0$ are positive compact operators, and if $M, N \in \mathbb{N}$, then*

$$\sigma_{M+N}(A+B) \geq \sigma_M(A) + \sigma_N(B).$$

Proof. From (4.3b) we obtain $\sigma_M(A) = \sup\{\,\mathrm{Tr}(PAP) : P = P^2 = P^*,\ \mathrm{rank}\,P = M\,\}$ and $\sigma_N(B) = \sup\{\,\mathrm{Tr}(P'BP') : P' = P'^2 = P'^*,\ \mathrm{rank}\,P' = N\,\}$. Now $\mathrm{rank}(P+P') = \dim(P\mathcal{H} + P'\mathcal{H}) \leq M+N$, so if P'' is any projector of rank $M+N$ whose range includes the subspace $P\mathcal{H}+P'\mathcal{H}$, then $P \leq P''$ and $P' \leq P''$ as operators. Therefore,

$$\mathrm{Tr}(PAP) + \mathrm{Tr}(P'BP') \leq \mathrm{Tr}(P''AP'') + \mathrm{Tr}(P''BP'') = \mathrm{Tr}(P''(A+B)P''),$$

so that $\sigma_M(A)+\sigma_N(B) \leq \sup_{P''} \mathrm{Tr}(P''(A+B)P'') \leq \sigma_{2N}(A+B)$. (Notice how this argument requires additivity of the trace: it would not have worked with $\|\cdot\|_1$ instead of Tr, hence the restriction to the case of positive operators.) ☐

Corollary 4.11. *If $A, B \in \mathcal{K}$ with $A \geq 0$, $B \geq 0$, then*

$$\sigma_N(A+B) \leq \sigma_N(A) + \sigma_N(B) \leq \sigma_{2N}(A+B). \qquad ☐$$

We see that the functional $A \mapsto \sigma_N(A)/\log N$ is not far from being additive functional on the positive cone $\mathcal{K}_+$. But to get a truly additive functional, we must try to take the limit $N \to \infty$, and here things become more interesting.

4.4 Dixmier traces

It is a bit awkward to deal with the index N of $\sigma_N(A)$ as a discrete variable, but we can fix this by a simple linear interpolation.

If $N \leq \lambda \leq N+1$, so that $\lambda = N+t$ with $0 \leq t \leq 1$, we put

$$\sigma_\lambda(A) := (1-t)\sigma_N(A) + t\sigma_{N+1}(A).$$

Note that $\sigma_\lambda(A+B) \leq \sigma_\lambda(A) + \sigma_\lambda(B)$ now holds for all $\lambda \geq 0$: every σ_λ is a norm on $\mathcal{K}$.

Exercise 4.12. *Check that $\sigma_\lambda(A+B) \leq \sigma_\lambda(A) + \sigma_\lambda(B) \leq \sigma_{2\lambda}(A+B)$, for $A, B \geq 0$ in $\mathcal{K}$, also holds for all $\lambda \geq 0$.*

Definition 4.13. *The* Dixmier ideal $\mathcal{L}^{1+} = \mathcal{L}^{1+}(\mathcal{H}) = \mathcal{L}^{1,\infty}(\mathcal{H})$ *is defined to be*

$$\mathcal{L}^{1+} := \left\{ T \in \mathcal{K} : \sup_{\lambda \geq e} \frac{\sigma_\lambda(T)}{\log \lambda} < \infty \right\}.$$

(The e here is by convention: any constant > 1 would do. Also, the notation $\mathcal{L}^{1+}$ is not universally accepted: some authors prefer the clumsier notation $\mathcal{L}^{1,\infty}$, or even $\mathcal{L}^{(1,\infty)}$, which comes from the historical origin of these operator ideals in real interpolation theory: see [c-a94, IV.C] for that.)

Since each σ_λ is a norm on $\mathcal{K}$, so also is this supremum whenever it is finite. Thus $\mathcal{L}^{1+}$ has a natural (¡unitarily invariant!) norm

$$\|T\|_{1+} := \sup_{\lambda \geq e} \frac{\sigma_\lambda(T)}{\log \lambda} \quad \text{for } T \in \mathcal{L}^{1+}.$$

As stated, the norm depends on the chosen constant e, but the ideal $\mathcal{L}^{1+}(\mathcal{H})$ does not.

Note that $T \in \mathcal{K}$ is *traceclass* if and only if $\sigma_\lambda(T)$ is bounded (by $\|T\|_1$, for instance) without need for the factor $(1/\log \lambda)$. Thus $\mathcal{L}^1(\mathcal{H}) \subset \mathcal{L}^{1+}(\mathcal{H})$.

Remark 4.14. If the bounded function $\sigma_\lambda(T)/\log \lambda$ is actually *convergent* as $\lambda \to \infty$, or equivalently, if $\sigma_N(T)/\log N$ converges as $N \to \infty$, then clearly

$$\lim_{N\to\infty} \frac{\sigma_N(T)}{\log N} = \lim_{\lambda\to\infty} \frac{\sigma_\lambda(T)}{\log \lambda} \leq \|T\|_{1+}.$$

We get an *additive* functional defined on $\mathcal{L}^{1+}$ in three more steps. First, we dampen the oscillations in $\sigma_\lambda(T)/\log \lambda$ by taking a *Cesàro mean* with respect to the logarithmic measure on an interval $[\lambda_0, \infty)$ for some $\lambda_0 > e$. For definiteness, we choose $\lambda_0 = 3$. Our treatment closely follows the appendix of the local-index paper of Connes and Moscovici [c-m95].

Definition 4.15. *For $\lambda \geq 3$, we set*

$$\tau_\lambda(T) := \frac{1}{\log \lambda} \int_3^\lambda \frac{\sigma_u(T)}{\log u} \frac{du}{u}, \quad \textit{for } T \in \mathcal{L}^{1+}(\mathcal{H}). \tag{4.4}$$

Exercise 4.16. *Check the triangle inequality $\tau_\lambda(S+T) \leq \tau_\lambda(S) + \tau_\lambda(T)$ for $\lambda \geq 3$.*

Lemma 4.17 (Connes–Moscovici)**.** *If $A \geq 0$, $B \geq 0$ in $\mathcal{L}^{1+}(\mathcal{H})$, then*

$$\tau_\lambda(A) + \tau_\lambda(B) - \tau_\lambda(A+B) = O\left(\frac{\log\log\lambda}{\log\lambda}\right) \quad \textit{as } \lambda \to \infty.$$

Proof. First of all, it is clear that $\dfrac{\sigma_u(A+B)}{\log u} \leq \|A\|_{1+} + \|B\|_{1+}$ for $\lambda \geq e$. Next,

$$\begin{aligned}
\tau_\lambda(A) + \tau_\lambda(B) - \tau_\lambda(A+B) &\leq \frac{1}{\log\lambda} \int_3^\lambda \left(\frac{\sigma_{2u}(A+B)}{\log u} - \frac{\sigma_u(A+B)}{\log u} \right) \frac{du}{u} \\
&= \frac{1}{\log\lambda} \int_6^{2\lambda} \left(\frac{\sigma_u(A+B)}{\log(u/2)} - \frac{\sigma_u(A+B)}{\log u} \right) \frac{du}{u} \\
&\quad - \frac{1}{\log\lambda} \left(\int_3^\lambda - \int_6^{2\lambda} \right) \frac{\sigma_u(A+B)}{\log u} \frac{du}{u}.
\end{aligned}$$

The second term can be rewritten as

$$\frac{1}{\log\lambda}\left(\int_3^6 - \int_6^{2\lambda}\right)\frac{\sigma_u(A+B)}{\log u}\,\frac{du}{u}.$$

Since $\int_3^6 \frac{du}{u} = \int_\lambda^{2\lambda}\frac{du}{u} = \log 2$, we get an estimate of $\frac{2\log 2}{\log\lambda}\|A+B\|_1$. For the first term, we compute

$$\begin{aligned}\frac{1}{\log\lambda}\int_6^{2\lambda}\frac{\sigma_u(A+B)}{\log u}\left(\frac{\log u}{\log(u/2)}-1\right)\frac{du}{u} &\leq \frac{\|A+B\|_{1+}}{\log\lambda}\int_3^\lambda\left(\frac{\log 2u}{\log u}-1\right)\frac{du}{u}\\ = \frac{\|A+B\|_{1+}}{\log\lambda}\log 2\int_3^\lambda\frac{du}{u\log u} &< \frac{\|A+B\|_{1+}}{\log\lambda}\log 2\,(\log\log\lambda). \qquad \square\end{aligned}$$

Since the failure of additivity of τ_λ vanishes as $\lambda\to\infty$, the second step is to quotient out by functions vanishing at infinity. For that we consider the "corona" C*-algebra

$$B_\infty := \frac{C_b([3,\infty))}{C_0([3,\infty)}.$$

The function $\lambda\mapsto\tau_\lambda(A)$, for $A\geq 0$ in $\mathcal{L}^{1+}$, lies in $C_b([3,\infty))$, and its image $\tau(A)$ in B_∞ defines an additive map, that is,

$$\tau(A+B)=\tau(A)+\tau(B)\quad\text{for } A\geq 0,\ B\geq 0\quad\text{in } \mathcal{L}^{1+}.$$

The final step is to compose this map with a state on B_∞.

Definition 4.18. *For $A\geq 0$ in $\mathcal{L}^{1+}$, let $\tau(A)\in(B_\infty)_+$ denote the image, under the quotient map $C_b([3,\infty))\to B_\infty$, of the bounded function $\lambda\mapsto\tau_\lambda(A)$. This yields an additive map between positive cones, $\tau\colon(\mathcal{L}^{1+})_+\to(B_\infty)_+$. Since the "four positive parts" of any operator in $\mathcal{L}^{1+}$ also lie in $\mathcal{L}^{1+}$, as is easily checked, this map extends in the obvious way to a positive linear map $\tau\colon\mathcal{L}^{1+}\to B_\infty$. Moreover, τ is invariant under unitary conjugation, i.e., $\tau(UAU^*)=\tau(A)$ for each unitary $U\in B(\mathcal{H})$.*

For each state $\omega\colon B_\infty\to\mathbb{C}$, *we can now define a* Dixmier trace Tr_ω *on* $\mathcal{L}^{1+}(\mathcal{H})$ *by*

$$\mathrm{Tr}_\omega T := \omega(\tau(T)).$$

Since $\mathrm{Tr}_\omega(UAU^*)=\mathrm{Tr}_\omega(A)$ *for positive* $A\in\mathcal{L}^{1+}(\mathcal{H})$ *and unitary* $U\in B(\mathcal{H})$, *each such positive linear functional on* $\mathcal{L}^{1+}(\mathcal{H})$ *is indeed a trace.*

This definition has a *drawback*: since B_∞ is a non- separable C*-algebra, there is no way to *exhibit* even one such state. However, Dixmier traces are still computable in a special case: if $\lim_{\lambda\to\infty}\tau_\lambda(T)$ exists, then $\tau(T)$ coincides with the image of a constant function in B_∞, and since the state ω is normalized, $\omega(1)=1$, the value $\omega(\tau(T))$ equals this limit:

$$\mathrm{Tr}_\omega T = \lim_{\lambda\to\infty}\tau_\lambda(T)$$

is independent of ω, provided that the limit exists. Such operators are called *measurable*. When this happens, we shall suppress the label ω and write $\mathrm{Tr}^+ T$ for the common value of all Dixmier traces.

The use of the Cesàro mean (4.4) simplifies the original definition that Dixmier [d-j66] gave of these traces. A detailed analysis of these (and other related) functionals was made recently by Lord, Sedaev and Sukochev [lss05], who called them "Connes–Dixmier traces". As an unexpected consequence of their work, they have shown that a *positive* operator $A \in \mathcal{L}^{1+}(\mathcal{H})$ is measurable if *and only if* the original sequence $\{\, \sigma_N(A)/\log N : N \in \mathbb{N} \,\}$ is already convergent. Thus it is not necessary to compute $\tau_\lambda(A)$, since

$$\operatorname{Tr}^+ A = \lim_{N\to\infty} \frac{\sigma_N(A)}{\log N} \quad \text{for positive, measurable } A \in \mathcal{L}^{1+}.$$

Chapter 5

Symbols and Traces

5.1 Classical pseudodifferential operators

In order to develop a symbol calculus for Dirac operators and their powers, we shall temporarily restrict our attention to a single chart domain $U \subset M$, over which the cotangent bundle is trivial: $T^*M\big|_U \cong U \times \mathbb{R}^n$. If P is an operator on $C^\infty(M)$, or more generally, on a space of sections $\Gamma(M, E)$ of a vector bundle $E \to M$, and if $\{\phi_1, \dots, \phi_m\}$ is a finite partition of unity in $C^\infty(M)$, then $P(f) = \sum_{i,j=1}^m \phi_i P(\phi_j f)$, so we may as well consider operators which are defined on a single chart domain of M. At some later stage, we must ensure that the important properties of such operators are globally defined, independently of the choice of local coordinates.

We will work, then, in *local coordinates* $(x^1, \dots, x^n)$ over a chart domain U; the local coordinates of the cotangent bundle $T^*M\big|_U$ are

$$(x, \xi) = (x^1, \dots, x^n, \xi_1, \dots, \xi_n), \quad \text{where } \xi \in T_x^*M.$$

Let $E \to M$ be a vector bundle of rank r. We assume (without loss of generality) that the vector bundle E is also trivial over U, so we can identify $\Gamma(U, \operatorname{End} E)$ with $U \times M_r(\mathbb{C})$.

A *differential operator* acting on (smooth) local sections $f \in \Gamma(U, E)$ is an operator P of the form

$$P = \sum_{|\alpha| \le d} a_\alpha(x)\, D^\alpha, \quad \text{with } a_\alpha \in \Gamma(U, \operatorname{End} E),$$

where we use the notation $D^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}$, and $D_j := -i\, \partial/\partial x^j$, the positive integer d is the *order* of P.

The local coordinates allow us to identify U with an open subset of $\mathbb{R}^n$. The coefficients a_α are matrix-valued functions $U \to M_r(\mathbb{C})$.

By a Fourier transformation, we can write, for $f \in C_c^\infty(U, \mathbb{R}^r)$,

$$\begin{aligned} Pf(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi}\, p(x,\xi)\, \hat f(\xi)\, d^n\xi \\ &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi}\, p(x,\xi)\, f(y)\, d^n y\, d^n\xi, \end{aligned} \tag{5.1}$$

where $p(x, \xi)$ is a *polynomial* of order d in the ξ-variable, called the (complete) *symbol* of P. (Clearly, this symbol depends on the choice of local coordinates.) Here

$$K_p(x, y) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}\, p(x, \xi)\, f(y)\, d^n\xi \tag{5.2}$$

is the *kernel* of P, as an integral operator: the inverse Fourier transform of $p(x, \xi)$.

For the Dirac operator $D\!\!\!/$, we can use the local expression of the spin connection to write $D\!\!\!/ = -i\, c(dx^j)\, \nabla^S_{\partial_j} = -i\, c(dx^j)(\partial_j + \omega_j(x))$, so the corresponding symbol is

$$p(x, \xi) = c(dx^j)(\xi_j - i\, \omega_j(x)). \tag{5.3}$$

This is a first-order polynomial in the ξ_j variables, so that $D\!\!\!/$ is a first order differential operator. The *leading term* in $p(x, \xi)$ —the part that is homogeneous in ξ_j of degree one— is $c(dx^j)\xi_j = c(\xi_j\, dx^j) = c(\xi)$, where $\xi = \xi_j\, dx^j$ can be regarded as an element of $\mathcal{A}^1(U)$.

More generally, a *pseudodifferential* operator P is given locally by an integral of the form (5.1), where the symbol $p(x, \xi)$ need no longer be a polynomial. In that case, we must specify certain classes of symbols for which these integrals make sense.

Definition 5.1. *The vector space $S^d(U)$ of (scalar)* symbols of order $\leq d$, *consists of functions $p \in C^\infty(U \times \mathbb{R}^n)$ such that, for any compact $K \subset U$, and any multiindices $\alpha, \beta \in \mathbb{N}^n$, there exists a constant $C_{K\alpha\beta}$ such that*

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{K\alpha\beta}(1 + |\xi|^2)^{\frac{1}{2}(d-|\alpha|)} \quad \textit{for all } x \in K, \xi \in \mathbb{R}. \tag{5.4}$$

Here D_x^β and D_ξ^α denote derivatives in the x^i variables and in the ξ_j variables, respectively. We use $(1 + |\xi|^2)^{\frac{1}{2}}$ instead of $|\xi|$ to avoid problems at $\xi = 0$.

In the same way, we define matrix-valued symbols of order $\leq d$ as smooth functions $p\colon U \times \mathbb{R}^n \to M_r(\mathbb{C})$ satisfying the same norm estimates, but with the absolute value $|\cdot|$ on the left hand side of (5.4) replaced by a matrix norm in $M_r(\mathbb{C})$. By a small abuse of notation, we shall write $p \in S^d(U)$ also in the matrix-valued case.

When $p(x, \xi)$ is a *polynomial* in ξ, of order at most d, we can isolate its homogeneous parts:

$$p(x, \xi) = \sum_{j=0}^{d} p_{d-j}(x, \xi), \quad \text{where} \quad p_{d-j}(x, t\xi) = t^{d-j}\, p_{d-j}(x, \xi) \quad \text{for } t > 0.$$

Definition 5.2. *More generally, an element $p \in S^d(U)$ is called a* classical symbol *if we can find a sequence of terms $p_d(x, \xi)$, $p_{d-1}(x, \xi)$, $p_{d-2}(x, \xi)$,..., with $p_{d-j}(x, t\xi) = t^{d-j}\, p_{d-j}(x, \xi)$ for $t > 0$, such that for each $k = 0, 1, 2, \ldots$,*

$$p - \sum_{j=0}^{k-1} p_{d-j} \in S^{d-k}(U).$$

When this is possible, we write

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{d-j}(x, \xi), \tag{5.5}$$

and regard this series as an asymptotic development of the symbol p. This expansion does not determine $p(x,\xi)$ uniquely: a symbol in $\bigcap_{k\in\mathbb{N}} S^{d-k}(U)$ is called "smoothing", and smoothing symbols are exactly those symbols whose asymptotic expansion is zero.

Definition 5.3. *A* classical pseudodifferential operator *of order d, over $U \subset \mathbb{R}^n$, is an operator P defined by (5.1), for which $p(x,\xi)$ is a classical symbol in $S^d(U)$ whose leading term $p_d(x,\xi)$ does not vanish. This leading term is called the* principal symbol *of P, and we also denote it by $\sigma^P(x,\xi) := p_d(x,\xi)$.*

We need a formula for the symbol of the composition of two classical pseudodifferential operators ("classical ΨDOs", for short). It is not clear *a priori* when and if two such operators are composable: we remit to [t-me96], for instance, for the full story on compositions (and adjoints) of classical pseudodifferential operators, and for the justification of the following formula.

If P is a classical ΨDOs of order d_1 with symbol $p \in S^{d_1}(U)$, and if Q is a classical ΨDO of orders d_2 with symbol $p \in S^{d_2}(U)$, then the symbol $p \circ q$ of the composition PQ lies in $S^{d_1+d_2}(U)$ and its asymptotic development is given by

$$(p \circ q)(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x,\xi)\, D_x^\alpha q(x,\xi). \tag{5.6}$$

To find the terms $(p \circ q)_{d_1+d_2-j}(x,\xi)$ of the symbol expansion, one must substitute (5.5) for both p and q into the right hand side of (5.6) and rearrange a finite number of terms. For the case $j = 0$, one need only use $\alpha = 0$ —since D_ξ^α lowers the order by $|\alpha|$— and in particular, the principal symbols compose easily:

$$(p \circ q)_{d_1+d_2}(x,\xi) = p_{d_1}(x,\xi)\, q_{d_2}(x,\xi).$$

The composition formula is valid for both scalar-valued and matrix-valued symbols, provided the matrix size r is the same for both operators.

Exercise 5.4. *If P and Q are classical ΨDOs with scalar-valued symbols, show that the principal symbol of $[P,Q] = PQ - QP$ is $-i\,\{\sigma^P, \sigma^Q\}$, where $\{\cdot,\cdot\}$ is the of functions:*

$$-i\,\{\sigma^P(x,\xi), \sigma^Q(x,\xi)\} = -i \sum_{j=1}^{n} \frac{\partial\sigma^P}{\partial\xi_j}\frac{\partial\sigma^Q}{\partial x^j} - \frac{\partial\sigma^Q}{\partial\xi_j}\frac{\partial\sigma^P}{\partial x^j}.$$

Conclude that the order of $[P,Q]$ is $\leq d_1 + d_2 - 1$. What can be said about the order of $[P,Q]$ if P and Q have matrix-valued symbols of size $r > 1$?

Suppose U and V are open subsets of $\mathbb{R}^n$ and that $\phi\colon U \to V$ is a diffeomorphism. If P is a ΨDO over U, then $\phi_*P\colon f \mapsto P(\phi^*f)\circ\phi^{-1}$ is a ΨDO over V, as can be verified by an explicit change-of-variable calculation. If P is classical, then so also is ϕ_*P. If p^ϕ denotes the symbol of ϕ_*P, we find that the *principal symbols* are related by

$$p_d(x,\xi) = p_d^\phi(\phi(x), \phi'(x)^{-t}\xi),$$

where $\phi'(x)^{-t}$ is the contragredient matrix (inverse transpose) to $\phi'(x)$.

This is the change-of-variable rule for the cotangent bundle. The conclusion is that, for any *scalar* ΨDO P that we may be able to define over a compact manifold M, the complete symbol $p(x,\xi)$ will depend on the local coordinates for a given chart of M, but the leading term $p_d = \sigma^P$ will make sense as an element of $C^\infty(T^*M)$ —i.e., a function on the total space of the cotangent bundle. (The subleading terms $p_{d-j}(x,\xi)$, for $j \geq 1$, will not be invariant under local coordinate changes.)

When P is defined on sections of a vector bundle $E \to M$ of rank r, the principal symbol σ^P becomes a section of the bundle $\pi^*(\operatorname{End} E) \to T^*M$, i.e., the pullback of $\operatorname{End} E \to M$ via the cotangent projection $\pi\colon T^*M \to M$.

For the *Dirac operator* $\not{D}$, which is a first-order differential operator on $\Gamma(M,S)$, we get $\sigma^{\not{D}} \in \Gamma(T^*M, \pi^*(\operatorname{End} S))$. From (5.3), we get at once

$$\sigma^{\not{D}}(x,\xi) = c(\xi_j\, dx^j) = c(\xi).$$

Since taking the principal symbol is a multiplicative procedure, we also obtain

$$\sigma^{\not{D}^2}(x,\xi) = (\sigma_{\not{D}}(x,\xi))^2 = c(\xi)^2 = g(\xi,\xi)\, 1_{2^m}.$$

(Here we use the handy notation 1_r for the $r \times r$ identity matrix.) Notice that the principal symbol of Δ^S is also $g(\xi,\xi)\, 1_{2^m}$, since $\not{D}^2 - \Delta^S = \frac{1}{4}s$ is a term of order zero (it is independent of the ξ_j variables), thus $\not{D}^2$ and Δ^S have the same principal symbol.

Note that $\sigma^{\not{D}^2}(x,\xi)$ only vanishes when $\xi = 0$, that is, on the zero section of T^*M.

Definition 5.5. *A ΨDO P is called* elliptic *if $\sigma_P(x,\xi)$ is invertible when $\xi \neq 0$, i.e., off the zero section of T^*M.*

In particular, $\not{D}$, $\not{D}^2$, Δ, Δ^S are all elliptic differential operators.

5.2 Homogeneity of distributions

We now wish to pass from the symbol p of a classical ΨDO, with a given symbol expansion

$$p(x,\xi) = \sum_{j=0}^{N-1} p_{d-j}(x,\xi) + r_N(x,\xi), \qquad r_N \in S^d(U).$$

to the operator kernel (5.2), by taking an inverse Fourier transform. However, the terms in this expansion may give divergent integrals when $y = x$. Therefore, we first need to look more closely at the inverse Fourier transforms of negative powers of $|\xi|$.

Assume that $n \geq 2$, for the rest of this section.

Definition 5.6. *Let $\lambda \in \mathbb{R}$. A function $\phi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is* homogeneous of degree $\boldsymbol{\lambda}$, *or "λ-homogeneous", if*

$$\phi(t\,\xi) = t^\lambda\, \phi(\xi) \quad \text{for all} \quad t > 0,\ \xi \neq 0.$$

Thus if $\xi = r\omega$ with $r = |\xi| > 0$ and $\omega = \xi/|\xi| \in \mathbb{S}^{n-1}$, we can write $\phi(\xi) = r^\lambda\, \psi(\omega)$ for some $\psi\colon \mathbb{S}^{n-1} \to \mathbb{C}$.

We can extend this definition to (tempered) distributions on $\mathbb{R}^n$. Write ϕ_t for the dilation of ϕ by the scale factor t, that is, $\phi_t(\xi) := \phi(t\xi)$, so that the λ-homogeneity condition can be written as $\phi_t = t^\lambda \phi$ for $t > 0$.

The change-of variables formula for functions,

$$\int_{\mathbb{R}^n} u(t\xi)\, \phi(\xi)\, d^n\xi = \int_{\mathbb{R}^n} t^{-n}\, u(\eta)\, \phi(\eta/t)\, d^n\eta,$$

suggests the following definition of homogeneity.

Definition 5.7. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution on $\mathbb{R}^n$. For $t > 0$, the* dilation u_t *of u by the scale factor t is defined by*

$$\langle u_t, \phi\rangle := t^{-n}\langle u, \phi_{1/t}\rangle, \quad \textit{for } \phi \in \mathcal{S}(\mathbb{R}^n).$$

We say that u is homogeneous of degree $\boldsymbol{\lambda}$ *if $u_t = t^\lambda u$ for all $t > 0$.*

Example 5.8. The Dirac δ is homogeneous of degree $-n$, since for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \delta_t, \phi\rangle = t^{-n}\,\langle \delta, \phi_{1/t}\rangle = t^{-n}\,\phi_{1/t}(0) = t^{-n}\,\phi(0) = t^{-n}\,\langle \delta, \phi\rangle.$$

Suppose now that u is a smooth function on $\mathbb{R}^n \setminus \{0\}$, such that

$$u(\xi) = r^\lambda\, v(\omega), \quad \text{for} \quad \xi = r\omega,\ r = |\xi| > 0,\ \omega \in \mathbb{S}^{n-1}.$$

We would like *to extend it to a (tempered) distribution* on the whole $\mathbb{R}^n$. There are several cases to consider.

Case 1 If $\lambda > 0$, then just put $u(0) := 0$. In this case, u extends to $\mathbb{R}^n$ as a homogeneous *function*.

Case 2 If $-n < \lambda \leq 0$, then $u(0)$ may not exist, but $u(\xi)$ is *locally integrable* near 0, so $\langle u, \phi\rangle$ is defined. Indeed, if $B = B(0; 1)$ and 1_B is its indicator function, and if σ denotes the usual volume form on $\mathbb{S}^{n-1}$, then

$$\begin{aligned}\langle u, 1_B\rangle := \int_B u(\xi)\, d^n\xi &= \int_{\mathbb{S}^{n-1}} v(\omega)\,\sigma \int_0^1 r^\lambda (r^{n-1}\, dr) \\ &= C\int_0^1 r^{\lambda+n-1}\, dr < \infty, \quad \text{since } \lambda + n - 1 > -1.\end{aligned}$$

Case 3 Suppose $\lambda = -n$, and that $\int_{\mathbb{S}^{n-1}} v(\omega)\,\sigma = 0$.

We define a distribution $\mathrm{P}u$ by the following trick. Let $f\colon [0, \infty) \to \mathbb{R}$ be a *cutoff function*, such that:

$$f(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq 1, \end{cases}$$

and f decreases smoothly from 1 to 0 on $[\frac{1}{2}, 1]$. Replace the test function ϕ by $\phi(\xi) - \phi(0)f(|\xi|)$, and put

$$\langle \mathrm{P}u, \phi\rangle := \int_{\mathbb{R}^n} u(\xi)\big(\phi(\xi) - \phi(0)\, f(|\xi|)\big)\, d^n\xi. \tag{5.7}$$

If $g(t)$ is another cutoff function with the same properties, the right hand side of this formula changes by

$$\int_{\mathbb{R}^n} u(\xi)\phi(0)\big(f(r) - g(r)\big)\, d^n\xi = \phi(0) \int_{\mathbb{S}^{n-1}} v(\omega)\, \sigma \int_{1/2}^{1} \big(f(r) - g(r)\big)\, \frac{dr}{r} = 0,$$

since $u(\xi)\, d^n\xi = r^{-n} v(\omega)\, \sigma\, r^{n-1}\, dr = v(\omega)\, \sigma\, dr/r$ by homogeneity. Thus $\langle \mathrm{P}u, \phi\rangle$ is independent of the cutoff chosen. Indeed, since

$$\int_{|\xi|>\varepsilon} u(\xi) f(|\xi|)\, d^n\xi = \int_{\varepsilon}^{1} f(r)\, \frac{dr}{r} \int_{\mathbb{S}^{n-1}} v(\omega)\, \sigma = 0,$$

for any $\varepsilon > 0$, we get another formula for $\mathrm{P}u$:

$$\langle \mathrm{P}u, \phi\rangle = \lim_{\varepsilon\downarrow 0} \int_{|\xi|>\varepsilon} u(\xi)\phi(\xi)\, d\xi.$$

Therefore, $\mathrm{P}u$ is just the "Cauchy principal part" of u at $\xi = 0$.

Lemma 5.9. *When u is a $(-n)$-homogeneous function on $\mathbb{R}^n \setminus \{0\}$ whose integral over $\mathbb{S}^{n-1}$ vanishes, its principal-part extension $\mathrm{P}u$ is a homogeneous distribution of degree $(-n)$.*

Proof. For each $t > 0$, we observe that

$$\begin{aligned}\langle (\mathrm{P}u)_t, \phi\rangle = t^{-n}\langle \mathrm{P}u, \phi_{1/t}\rangle &= t^{-n} \lim_{\varepsilon\downarrow 0} \int_{|\xi|>\varepsilon} u(\xi)\, \phi(\xi/t)\, d\xi \\ &= t^{-n} \lim_{\varepsilon\downarrow 0} \int_{|\eta|>\varepsilon/t} u(\eta)\, \phi(\eta)\, d\eta = t^{-n}\langle \mathrm{P}u, \phi\rangle. \qquad \square\end{aligned}$$

Case 4 Consider the function $u(\xi) := |\xi|^{-n}$ for $\xi \neq 0$. (By averaging $v(\omega)$ over $\mathbb{S}^{n-1}$, one can see that any smooth $(-n)$-homogeneous function on $\mathbb{R}^n \setminus \{0\}$ is a linear combination of $|\xi|^{-n}$ and a function in Case 3.

We can try the cutoff regularization, anyway. Let $R_f u$ be given by the recipe of (5.7):

$$\langle R_f u, \phi\rangle := \int_{\mathbb{R}^n} u(\xi)\big(\phi(\xi) - \phi(0)\, f(|\xi|)\big)\, d^n\xi. \tag{5.8}$$

However, in the present case, $R_f u$ is *not* homogeneous!

Lemma 5.10. *If $\delta\colon \phi \mapsto \phi(0)$ is the Dirac delta, and if $u(\xi) := |\xi|^{-n}$ for $\xi \neq 0$, then*

$$(R_f u)_t - t^{-n}\, R_f u = (\Omega_n\, t^{-n} \log t)\, \delta. \tag{5.9}$$

Proof. We compute $\langle (R_f u)_t - t^{-n} R_f u, \phi\rangle$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$. Since $u(\xi) := |\xi|^{-n}$ and $f(|\xi|)$ are both rotation-invariant, we can first integrate over $\mathbb{S}^{n-1}$, so we may suppose that ϕ is radial: $\phi(\xi) = \psi(|\xi|)$ for some $\psi\colon [0,\infty) \to \mathbb{C}$. Then

$$\begin{aligned}\langle R_f u, \phi\rangle &= \int_{\mathbb{R}^n} r^{-n}(\psi(r) - \psi(0)\, f(r))\, \sigma\, r^{n-1}\, dr \\ &= \Omega_n \int_0^\infty \big(\psi(r) - \psi(0)\, f(r)\big)\, \frac{dr}{r} \\ &= \Omega_n \int_0^\infty \left(\psi\Big(\frac{r}{t}\Big) - \psi(0)\, f\Big(\frac{r}{t}\Big)\right) \frac{dr}{r}, \quad \text{for any } t > 0.\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle (R_f u)_t - t^{-n} R_f u, \phi\rangle &= t^{-n}\langle R_f u, \phi_{1/t} - \phi\rangle \\
&= \Omega_n \phi(0)\, t^{-n} \int_0^\infty \left(f\left(\frac{r}{t}\right) - f(r)\right) \frac{dr}{r} \\
&= \Omega_n \phi(0)\, t^{-n} \int_0^\infty \int_r^{r/t} f'(s)\, ds\, \frac{dr}{r} \\
&= \Omega_n \phi(0)\, t^{-n} \int_0^\infty \int_{st}^{s} \frac{dr}{r}\, f'(s)\, ds \\
&= \Omega_n \phi(0)\, t^{-n} (-\log t) \int_{1/2}^1 f'(s)\, ds = \Omega_n \phi(0)\, t^{-n} \log t.
\end{aligned}$$

The extra $\log t$-term measures the failure of homogeneity of the regularization $R_f u$. □

Case 5 $u(\xi) = |\xi|^{-n-j}$ for $j = 1, 2, 3, \dots$.

Any cutoff function f gives a regularization by "Taylor subtraction", as follows:

$$\langle \widetilde{R}_f u, \phi\rangle := \int_{\mathbb{R}^n} |\xi|^{-n-j} \left(\phi(\xi) - \sum_{|\alpha|\le j} \frac{i^\alpha}{\alpha!} D^\alpha \phi(0) \xi^\alpha\, f(|\xi|)\right) d^n\xi.$$

Again one finds that $\widetilde{R}_f u$ is not homogeneous, by a straightforward calculation along the lines of the previous Lemma. This can be simplified a little by the following observation [fgv01]. One can find constants c_α for $|\alpha| \le j$, such that the modified regularization $R_f u := \widetilde{R}_f u - \sum_{|\alpha|<j} c_\alpha\, D^\alpha \delta$ has a "failure of homogeneity" of the form

$$(R_f u)_t - t^{-n-j}\, R_f u = t^{-n-j} \log t \left(\sum_{|\alpha|=j} c_\alpha\, D^\alpha \delta \right).$$

That completes our study of the extensions of homogeneous functions to distributions on $\mathbb{R}^n$. We need a remark about their Fourier transforms. Recall that the Fourier transformation $\mathcal{F}$ preserves the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and by duality it also preserves $\mathcal{S}'(\mathbb{R}^n)$. If u is a λ-homogeneous function on $\mathbb{R}^n \setminus \{0\}$, its Fourier transform is $\mathcal{F}u(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi}\, u(x)\, d^n x$, thus

$$(\mathcal{F}u)_t(\xi) = \int_{\mathbb{R}^n} e^{itx\xi}\, u(x)\, d^n x = t^{-n} \int_{\mathbb{R}^n} e^{-iy\xi}\, u(y/t)\, d^n y = t^{-n-\lambda}\, \mathcal{F}u(\xi).$$

It follows that $\mathcal{F}$, and also the inverse transformation $\mathcal{F}^{-1}$, take homogeneous functions (or distributions) of degree λ to homogeneous functions (or distributions) of degree $(-n-\lambda)$.

5.3 The Wodzicki residue

Now we return to the symbol expansion of a classical ΨDO P, of integral order $d \in \mathbb{Z}$, with

$$p(x,\xi) = \sum_{j=0}^{N-1} p_{d-j}(x,\xi) + r_N(x,\xi),$$

where $r_N \in S^{d-N}(U)$, and $p_{d-j}(x, t\xi) = t^{d-j}\, p_{d-j}(x, \xi)$. Now apply $\mathcal{F}_2^{-1}$, the inverse Fourier transform in the second variable, to this sum, to get the integral kernel

$$k_P(x, y) = \sum_{j=0}^{N-1} h_{j-d-n}(x, x-y) + (\mathcal{F}_2^{-1} r_N)(x, x-y).$$

If $N > n + d$, then $r_N \in S^{d-N}(U)$ is integrable in ξ, so the term $\mathcal{F}_2^{-1} r_N(x, z)$ is bounded as $z \to 0$. For the terms $h_{j-d-n}(x, z)$, there are 3 cases, which may give singularities. So *before* applying $\mathcal{F}_2^{-1}$ to $p_{d-j}(x, \xi)$, we must *regularize* $p_{d-j}(x, \xi)$ to $R_f p_{d-j}(x, \xi)$ by using a suitable cutoff:

$$p(x, \xi) = \sum_{j=0}^{N-1} R_f p_{d-j}(x, \xi) + s_N(x, \xi),$$

with s_N integrable. Now take $h_{j-d-n} := \mathcal{F}_2^{-1}(R_f p_{d-j})$.

Case 1 Suppose $d - j > -n$. Then $k := j - d - n < 0$, and $R_f p_{d-j}(x, \xi)$ is homogeneous of degree greater than $-n$, so $h_k(x, z)$ is homogeneous of degree k. These terms have no failure of homogeneity.

Before examining the other two cases, we return to the context of functions on $\mathbb{R}^n \setminus \{0\}$, and look first at $w_0(z) := (2\pi)^n \mathcal{F}^{-1}(R_f |\xi|^{-n})$. Since (5.9) holds with $u(\xi) = |\xi|^{-n}$ for $\xi \neq 0$, and since $(2\pi)^n \mathcal{F}^{-1}(\delta) = 1$, we get

$$t^{-n} w_0(z/t) - t^{-n} w_0(z) = \Omega_n\, t^{-n} \log t \quad \text{for } t > 0,$$

or more simply,

$$w_0(z/t) - w_0(z) = \Omega_n \log t. \tag{5.10}$$

Notice that $C = w_0(z/|z|)$ is a constant, because w_0 is rotation-invariant. Substituting $t := |z|$ in (5.10) gives

$$w_0(z) = C - \Omega_n \log |z|, \tag{5.11}$$

so that w_0 "diverges logarithmically". We can suppress the constant term if we replace $R_f |\xi|^{-n}$ by $R_f |\xi|^{-n} - C\delta$, since we must then subtracting the constant C from the inverse Fourier transform.

For $j = 1, 2, \dots$, we define $w_j(z) := (2\pi)^n \mathcal{F}^{-1}(R_f |\xi|^{-n-j})$. A similar analysis shows that $w_j(z) = q_j(z) - r_j(z) \log |z|$, where both q_j and r_j are homogeneous of degree $j > 0$. In this case, $w_j(z)$ remains bounded as $z \to 0$.

We now return to the examination of the terms h_{j-d-n} in the integral kernel $k(x, y)$.

Case 2 Suppose $d - j < -n$. Then $k = j - d - n > 0$, and we find that $h_{j-d-n}(x, z)$ remains bounded as $z \to 0$.

Case 3 Consider the case $d - j = -n$. Then we get $h_0(x, z) = -u_0(x) \log |z|$, after possibly subtracting a term depending only on x. We have proved the following result.

Proposition 5.11. *If P is a classical pseudodifferential operator of integral order d, then its kernel has the following form near the diagonal:*

$$k_P(x,y) = \sum_{-d-n\le k<0} h_k(x, x-y) - u_0(x)\log|x-y| + O(1), \tag{5.12}$$

where each $h_k(x,\cdot)$ is homogeneous of negative degree k, $u_0(x)$ is independent of $x-y$, and $O(1)$ stands for a term which remains bounded as $y \to x$. $\square$

To compute $u_0(x)$, the coefficient of logarithmic divergence, we change coordinates by a local diffeomorphism $\psi(x)$. Note that

$$\log|\psi(x)-\psi(y)| \sim \log|\psi'(x)\cdot(x-y)| \sim \log|x-y| \quad \text{as } y \to x,$$

while $k_P(x,y) \mapsto k_P(\psi(x),\psi(y))\,L(x,y)$, where $L(x,y) \to |\det\psi'(x)|$ as $y \to x$, by the change of variables formula for $|d^ny|$. (We use a 1-density, not an oriented volume form, to do integration; however, if we agree to fix an orientation on M and use only coordinate changes that preserve the orientation, for which $\det\psi'(x) > 0$ at each x, then we need not make this distinction). Thus the log-divergent term transforms as follows:

$$-u_0(x)\log|x-y| \mapsto -u_0(\psi(x))|\det\psi'(x)|\log|x-y|.$$

For the case of scalar pseudodifferential operators, this is all we need. In the general case of operators acting on sections of a vector bundle $E \to M$, we replace $u_0(x) \in \operatorname{End} E_x$ by its matrix trace $\operatorname{tr} u_0(x) \in \mathbb{C}$. The previous formula then says that *the 1-density* $\operatorname{tr} u_0(x)\,|d^nx|$ *is invariant under local coordinate changes.*

Now, when we regularize $p_{-n}(x,\xi)$ to obtain this 1-density after applying $\mathcal{F}_2^{-1}$, we can first subtract the homogeneous "principal part", at each $x \in U$, since this will not change the coefficient of logarithmic divergence. This subtraction is done by replacing $p_{-n}(x,\xi)$ by its average over the sphere $|\xi| = 1$ in the cotangent space T_x^*M. That is to say, we get the same $u_0(x)$ if we replace $p_{-n}(x,\xi)$ by $\Omega_n^{-1}|\xi|^{-n}\int_{|\omega|=1} p_{-n}(x,\omega)\,\sigma$. On applying (5.11) (with $C = 0$) at each x, we conclude that

$$\operatorname{tr} u_0(x) = \int_{|\omega|=1} \operatorname{tr} p_{-n}(x,\omega)\,\sigma.$$

Definition 5.12. *The* Wodzicki residue density *of a classical ΨDO P, acting on sections of a vector bundle $E \to M$, is well defined by the local formula*

$$\operatorname{wres}_x P := \left(\int_{|\omega|=1} \operatorname{tr} p_{-n}(x,\omega)\,\sigma\right)|d^nx|, \quad \text{at } x \in M.$$

The Wodzicki residue *of P is the integral of this 1-density:*

$$\operatorname{Wres} P := \int_M \operatorname{wres}_x P = \int_M \left(\int_{|\omega|=1} \operatorname{tr} p_{-n}(x,\omega)\,\sigma\right)|d^nx|.$$

We shall now show that Wres is a *trace* on the algebra of classical pseudodifferential operators on M acting on a given vector bundle.

We begin with another important property of homogeneous functions on $\mathbb{R}^n \setminus \{0\}$. We shall make use of the *Euler vector field* on this space:

$$R = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j} = r \frac{\partial}{\partial r}.$$

Notice that h is λ-homogeneous if and only if $Rh = \lambda h$, since $Rh(r\omega) = r\frac{\partial}{\partial r}(r^\lambda\, h(\omega)) = \lambda r^\lambda\, h(\omega) = \lambda h(r\omega)$.

Lemma 5.13. *If $\lambda \neq -n$, any λ-homogeneous function h on $\mathbb{R}^n \setminus \{0\}$ is a finite sum of derivatives.*

Proof. It is enough to notice that

$$\sum_{j=1}^{n} \frac{\partial}{\partial \xi_j}(\xi_j\, h(\xi)) = nh(\xi) + Rh(\xi) = (n+\lambda)h(\xi),$$

which implies $h = \frac{1}{n+\lambda} \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j}(\xi_j\, h)$. □

Lemma 5.14. *If $h\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is $(-n)$-homogeneous, with $n > 1$, then h is a finite sum of derivatives if and only if $\int_{\mathbb{S}^{n-1}} h\,\sigma = 0$.*

Proof. Suppose first that $\int_{\mathbb{S}^{n-1}} h\,\sigma = 0$. Since $h(\xi) = r^{-n}\, h(\omega)$, h is determined by its restriction to $\mathbb{S}^{n-1}$, and the hypothesis says that $\langle 1 \,|\, h\rangle = 0$ in $L^2(\mathbb{S}^{n-1}, \sigma)$. Thus $h \in (\mathbb{C}1)^\perp = (\ker \Delta)^\perp = \operatorname{im} \Delta$, where Δ is the Laplacian on the sphere $\mathbb{S}^{n-1}$ (which is a Fredholm operator on the Hilbert space $L^2(\mathbb{S}^{n-1}, \sigma)$, with closed range). Thus the equation $h = \Delta g$ has a unique (and C^∞, since Δ is elliptic) solution g on $\mathbb{S}^{n-1}$. Extend g to $\mathbb{R}^n \setminus \{0\}$ by setting $g(r\omega) := r^{-n+2}\, g(\omega)$ for $0 < r < +\infty$. Since the Laplacian on $\mathbb{R}^n$ is

$$\Delta_{\mathbb{R}^n} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\, \Delta,$$

we get $\Delta_{\mathbb{R}^n} g = h$, and thus $h = \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j}\left(-\frac{\partial g}{\partial \xi_j}\right)$ is a finite sum of derivatives.

Suppose instead that $h(\xi) = \frac{\partial f}{\partial \xi_1}$ for $\xi \neq 0$, where f is $(-n+1)$-homogeneous. Let σ' be the volume form on $\mathbb{S}^{n-2}$, and notice that

$$\int_{\mathbb{S}^{n-2}} \int_{\mathbb{R}} \frac{\partial f}{\partial \xi_1}(\xi_1, \omega')\, d\xi_1 \wedge \sigma' = \int_{\mathbb{S}^{n-2}} [f(+\infty, \omega') - f(-\infty, \omega')]\, \sigma' = 0$$

since $f(\xi_1, \omega') \to 0$ as $\xi_1 \to \pm\infty$, by homogeneity. Thus we must show that

$$\int_{\mathbb{S}^{n-1}} h\,\sigma = \int_{\mathbb{R}\times\mathbb{S}^{n-2}} h\, d\xi_1 \wedge \sigma'.$$

By Stokes' theorem, we must show that the difference is the integral of the zero n-form on the tube T, whose oriented boundary is $(\mathbb{R} \times \mathbb{S}^{n-2}) - \mathbb{S}^{n-1}$. (Picture a ball stuck in a cylinder of radius 1; T is the region inside the cylinder but outside the ball.) Consider the $(n-1)$-form

$$\tilde{\sigma} = \sum_{j=1}^{n} (-1)^{j-1} \xi_j\, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \in \mathcal{A}^{n-1}(\mathbb{R}^n \setminus \{0\}).$$

If $i\colon \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is the inclusion, then $\sigma = i^*\tilde{\sigma}$. Now $\tilde{\sigma} = \iota_R \nu$, where $\nu = d\xi_1 \wedge \cdots \wedge d\xi_n$, and R is the Euler vector field on $\mathbb{R}^n \setminus \{0\}$. Since $Rh = -nh$ by homogeneity, we find that

$$\begin{aligned} d(h\tilde{\sigma}) &= dh \wedge \tilde{\sigma} + h\, d\tilde{\sigma} = dh \wedge \tilde{\sigma} + nh\,\nu \\ &= dh \wedge \iota_R\nu - (Rh)\,\nu = dh \wedge \iota_R\nu - \iota_R(dh)\nu \\ &= -\iota_R(dh \wedge \nu) = \iota_R(0) = 0. \end{aligned}$$

Thus $h\tilde{\sigma}$ is a closed form on $\mathbb{R}^n \setminus \{0\}$, that restricts to $h\sigma$ on $\mathbb{S}^{n-1}$ and to $h\, d\xi_1 \wedge \sigma'$ on $\mathbb{R} \times \mathbb{S}^{n-2}$, therefore

$$\int_{\mathbb{S}^{n-1}} h\,\sigma - \int_{\mathbb{R}\times\mathbb{S}^{n-2}} h\, d\xi_1 \wedge \sigma' = \int_{\partial T} h\,\sigma' = \int_T d(h\tilde{\sigma}) = 0,$$

by Stokes' theorem. Thus $\int_{\mathbb{S}^{n-1}} h\,\sigma = 0$, as required. □

Proposition 5.15. Wres *is a trace on the algebra of classical pseudodifferential operators acting on a fixed vector bundle* $E \to M$.

Proof. We must show that $\operatorname{Wres}([P,Q]) = 0$, for all classical pseudodifferential operators P, Q on M. First we consider the scalar case, where $E = M$. Let $p(x,\xi)$, $q(x,\xi)$ be the complete symbols of P, Q respectively, in some coordinate chart of M. Then if $r(x,\xi)$ is the complete symbol of $[P,Q]$, we know that

$$r(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha p\, D_x^\alpha q - D_\xi^\alpha q\, D_x^\alpha p). \tag{5.13}$$

In particular, the *principal* symbol of $R = [P,Q]$ comes from the terms with $|\alpha| = 1$ in this expansion:

$$\sigma_R(x,\xi) = -i\sum_{j=1}^n \biggl(\frac{\partial p}{\partial \xi_j}\,\frac{\partial q}{\partial x^j} - \frac{\partial q}{\partial \xi_j}\,\frac{\partial p}{\partial x^j}\biggr) = -i\sum_{j=1}^n \frac{\partial}{\partial \xi_j}\biggl(p\,\frac{\partial q}{\partial x^j}\biggr) - \frac{\partial}{\partial x^j}\biggl(p\,\frac{\partial q}{\partial \xi_j}\biggr).$$

In like manner, when $\alpha_j = 2$ and the other $\alpha_i = 0$, we get the terms

$$\begin{aligned} &-\frac{1}{2}\biggl(\frac{\partial^2 p}{\partial \xi_j^2}\,\frac{\partial^2 q}{\partial (x^j)^2} - \frac{\partial^2 q}{\partial \xi_j^2}\,\frac{\partial^2 p}{\partial (x^j)^2}\biggr) \\ &\qquad = -\frac{1}{2}\frac{\partial}{\partial \xi_j}\biggl(\frac{\partial p}{\partial \xi_j}\,\frac{\partial^2 q}{\partial (x^j)^2} - p\,\frac{\partial^3 q}{\partial \xi_j \partial (x_j)^2}\biggr) - \frac{1}{2}\frac{\partial}{\partial x^j}\biggl(\frac{\partial p}{\partial x^j}\,\frac{\partial^2 q}{\partial \xi_j^2} - p\,\frac{\partial^3 q}{\partial \xi_j^2 \partial x_j}\biggr). \end{aligned}$$

By induction, all terms in the expansion (5.13) that contribute to $r_{-n}(x,\xi)$ are finite sums of derivatives.

In the general case, if $p(x,\xi) = [p_{kl}(x,\xi)]$ and $q(x,\xi) = [q_{kl}(x,\xi)]$ are square matrices, the same argument applies to the sums

$$\sum_{k,l}(D_\xi^\alpha p_{kl}\, D_x^\alpha q_{lk} - D_\xi^\alpha q_{lk}\, D_x^\alpha p_{kl}), \quad \text{for each } \alpha \in \mathbb{N}^n,$$

that contribute to the expansion of $\operatorname{tr} r_{-n}(x,\xi)$. Thus, $\operatorname{tr} r_{-n}(x,\xi)$ is a finite sum of derivatives in the variables x^j and ξ_j. Write

$$\operatorname{tr} r_{-n}(x,\xi) = \sum_{j=1}^n \frac{\partial f_j}{\partial x^j} + \frac{\partial g_j}{\partial \xi_j},$$

where $f_j(x,\xi)$, $g_j(x,\xi)$ vanish outside $K \times \mathbb{R}^n$ for some compact subset $K \subset U$ of a coordinate chart of M. (This can be guaranteed by first writing $P = \sum_r \psi_r P$ and $Q = \sum_r \psi_r Q$ for a suitable partition of unity $\{\psi_r\}$ on M.) Then

$$F_j(x) := \int_{|\xi|=1} f_j(x,\xi)\,\sigma_\xi$$

has $\operatorname{supp} F_j \subset K$, so that

$$\int_{|\xi|=1} \frac{\partial f_j}{\partial x^j}\,\sigma = \frac{\partial F_j}{\partial x^j}, \quad \text{and} \quad \int_U \frac{\partial F_j}{\partial x^j}\,|d^n x| = 0.$$

By construction, $\operatorname{tr} r_{-n}(x,\xi)$, and each $\frac{\partial g_j}{\partial \xi_j}(x,\xi)$ also, are $(-n)$-homogeneous in ξ. Lemma 5.14 now implies that

$$\operatorname{Wres}([P,Q]) = \int_M \left(\int_{|\xi|=1} \left(\sum_{j=1}^n \frac{\partial g_j}{\partial \xi_j} \right) \sigma \right) d^n x = 0. \qquad \square$$

To show that the trace is unique (up to constants) when $n > 1$, let T be any trace on the algebra of classical pseudodifferential operators. Again we suppose that all symbols are supported in a coordinate chart $U \subset M$, and we note that the formulas for composition of symbols give the commutation relations

$$[x^j, f] = i\,\frac{\partial f}{\partial \xi_j}, \qquad [\xi_j, f] = -i\,\frac{\partial f}{\partial x^j}.$$

By Lemmas 5.13 and 5.14, $T(P)$ thus depends only on the homogeneous term $\operatorname{tr} p_{-n}(x,\xi)$, of degree $-n$, and moreover $T(P) = 0$ if $\int_{|\xi|=1} \operatorname{tr} p_{-n}(x,\xi)\,\sigma = 0$. We can replace $\operatorname{tr} p_{-n}(x,\xi)$ with $|\xi|^{-n} \int_{|\xi|=1} \operatorname{tr} p_{-n}(x,\xi)\,\sigma$, without changing $T(P)$. Now $f \mapsto T(f(x)\,|\xi|^{-n})$ is a linear functional on $C_c^\infty(U)$ that kills derivatives with respect to each x^j, so it is a multiple of the Lebesgue integral:

$$T(f) = C \int_U f(x)\,|d^n x| \quad \text{for some} \quad C \in \mathbb{C}.$$

Therefore,

$$T(P) = C \int_U \int_{|\xi|=1} \operatorname{tr} p_{-n}(x,\xi)\,\sigma\,|d^n x| = C\,\operatorname{Wres}(P).$$

Example 5.16. If (M,g) is a compact Riemannian spin manifold with Dirac operator $D\!\!\!/$, then

$$\operatorname{Wres} |D\!\!\!/|^{-n} = 2^m\,\Omega_n\,\operatorname{Vol}(M).$$

Proof. Recall that the principal symbol of $|D\!\!\!/|$ is $\sigma^{D\!\!\!/}(x,\xi) = c(\xi)$, and that of $D\!\!\!/^2$ (or of Δ^S) is $\sigma^{D\!\!\!/^2}(x,\xi) = c(\xi)^2 = g(\xi,\xi)\,1_{2^m}$. (Recall that 1_{2^m} means the identity matrix of size 2^m, which is the rank of the spinor bundle.) Thus the principal symbol of $|D\!\!\!/|^{-n}$ is $\sigma^{|D\!\!\!/|^{-n}}(x,\xi) = g(\xi,\xi)^{-n/2}\,1_{2^m}$. This is homogeneous of degree $-n$, so that $p_{-n}(x,\xi)$ is actually the principal symbol when $P = |D\!\!\!/|^{-n}$. Therefore, $\operatorname{tr} p_{-n}(x,\xi) = 2^m\,g(\xi,\xi)^{-n/2}$.

Now $g(\xi,\xi) = g^{ij}\,\xi_i \xi_j$ in local coordinates on T^*M. To compute its integral over the Euclidean sphere $|\xi| = 1$ [rather than over the ellipsoid $g(\xi,\xi) = 1$], we make a change of coordinates $x \mapsto y = \psi(x)$, and we note that $(x, \xi) \mapsto (y, \eta)$ where $\xi = \psi'(x)^t \, \eta$. We can choose ψ such that $\psi'(x) = [g^{ij}(x)]^{1/2}$, a positive-definite $n \times n$ matrix, in which case $g^{ij}(x) \, \xi_i \xi_j = \delta^{kl} \, \eta_k \eta_l = |\eta|^2$. Now $\operatorname{tr} p_{-n}(y, \eta) = 2^m \, |\eta|^{-n}$, so $\int_{|\eta|=1} \operatorname{tr} \sigma_{-n}(y, \eta) = 2^m \, \Omega_n$, and the Wodzicki residue density is

$$\operatorname{wres}_x |D\!\!\!/|^{-n} = 2^m \, \Omega_n \, |d^n y| = 2^m \, \Omega_n \, \det \psi'(x) \, |d^n x|.$$

But $\det \psi'(x)) = \sqrt{\det g(x)}$ by construction, so we arrive at

$$\operatorname{wres}_x |D\!\!\!/|^{-n} = 2^m \, \Omega_n \sqrt{\det(g_x)} \, |d^n x| = 2^m \, \Omega_n \, \nu_g.$$

Integrating this over M gives $\operatorname{Wres} |D\!\!\!/|^{-n} = 2^m \, \Omega_n \, \operatorname{Vol}(M)$, as claimed. $\square$

What we have gained? We no longer need the full spectrum of the Dirac operator: its *principal symbol* is enough to give the Wodzicki residue.

5.4 Dixmier trace and Wodzicki residue

There is a third method of computing the logarithmic divergence of the spectrum of $D\!\!\!/$, by means of residue calculus applied to powers of pseudodifferential operators. We shall give (only) a brief outline of what is involved.

Suppose that H is an *elliptic* pseudodifferential operator on $\Gamma(M, E)$ that extends to a positive selfadjoint operator (also denoted here by H) on the Hilbert space $L^2(M, E)$, which is defined as the completion of $\{ \, s \in \Gamma(M, E) : \int_M (s \mid s) \, \nu_g < \infty \, \}$ in the norm $\|\psi\| := \sqrt{\langle \psi \mid \psi \rangle}$, where

$$\langle \phi \mid \psi \rangle := \int_M (\phi \mid \psi) \, \nu_g$$

is the scalar product introduced in Section 3.1. We have in mind the example $H = |D\!\!\!/| = (D\!\!\!/^2)^{1/2}$ or else $H = (D\!\!\!/^2 + 1)^{1/2}$, in case $\ker D\!\!\!/ \neq \{0\}$.

Since M is compact, the operator H on $L^2(M, E)$ is known to be Fredholm [t-me96], thus $\ker H$ is finite dimensional. We can define its powers H^{-s}, for $s \in \mathbb{C}$, by holomorphic functional calculus:

$$H^{-s} := \frac{1}{2\pi i} \oint_\Gamma \lambda^{-s} (\lambda - H)^{-1} \, d\lambda$$

where Γ is a contour that winds once anticlockwise around the spectrum of H, excluding 0 to avoid the branch point of λ^{-s}. (We define $H^{-s}\psi := 0$ for $\psi \in \ker H$.)

By applying the same Cauchy integrals to the complete symbol of H, one can show that H^{-s} is pseudodifferential, and obtain much information about its integral kernel. This was first done by Seeley [s-rt67]. He found that the following properties hold.

- If H has order $d > 0$, then for $\Re s > n/d$, H^{-s} is traceless and $\zeta_H(s) := \operatorname{Tr} H^{-s}$ is holomorphic on this open half-plane.
- For $x \neq y$, the function $s \mapsto K_{H^{-s}}(x, y)$ extends from the half-plane $\Re s > n/d$ to all of $\mathbb{C}$, as an entire function.

- For $x = y$, the function $s \mapsto K_{H^{-s}}(x,x)$ can be continued to a meromorphic function on $\mathbb{C}$, with possible poles only at $\{\, s = (n-k)/d : k = 0,1,2,\dots \,\}$.
- The residues at these poles are computed by integrating certain symbol terms over the sphere $|\xi| = 1$ in $T^*_x M$.

Later on, Wodzicki [w-m84] made a deep study of the spectral asymptotics of these operators, and in particular found that at $s = n/d$, the operator $H^{-n/d}$ is of order $(-n)$, and the residue at this pole depends only on its principal symbol; in fact,

$$\operatorname{Res}_{s=n/d} K_{H^{-s}}(x,x)\,|d^n x| = \frac{1}{d(2\pi)^n}\operatorname{wres}_x H^{-n/d}.$$

Corollary 5.17. *If A is a positive elliptic ΨDO of order $(-n) = -\dim M$ on $L^2(M,E)$, then $s \mapsto \operatorname{Tr} A^s$ is convergent and holomorphic on $\{\, s \in \mathbb{C} : \Re s > -1 \,\}$, it continues meromorphically to $\mathbb{C}$ with a (simple) pole at $s = 1$, and*

$$\operatorname{Res}_{s=1}(\operatorname{Tr} A^s) = \frac{1}{n(2\pi)^n}\operatorname{Wres} A. \qquad \square$$

(For the proof, one applies Seeley's theory to $H = A^{-1}$.)

A basic result in noncommutative geometry is *Connes' trace theorem* of 1988 [c-a88], which shows that this residue is actually a Dixmier trace.

Theorem 5.18 (Connes)**.** *If A is a positive elliptic ΨDO of order $(-n) = -\dim M$ on $\mathcal{H} = L^2(M,E)$, the operator A lies in the Dixmier trace class $\mathcal{L}^{1+}(\mathcal{H})$, it is "measurable", i.e., $\operatorname{Tr}_\omega A =: \operatorname{Tr}^+ A$ is independent of ω, and the following equalities hold:*

$$\operatorname{Tr}^+ A = \operatorname{Res}_{s=1}(\operatorname{Tr} A^s) = \frac{1}{n(2\pi)^n}\operatorname{Wres} A. \qquad \square$$

We omit the proof, but a few comments can be made. In view of what was already said, it is enough to establish the first equality. The elliptic operator $H = A^{-1}$, of order n, has compact resolvent [t-me96], so that A itself is compact (we ignore any finite-dimensional kernel). If the eigenvalues of A are $\lambda_k = s_k(A)$ (listed in decreasing order), the first equality reduces to the following known theorem on divergent series:

Proposition 5.19 (Hardy)**.** *Suppose that $\lambda_k \downarrow 0$ as $k \to \infty$, that $\sum_{k=1}^\infty \lambda_k^s < \infty$ for $s > 1$, and that $\lim_{s\downarrow 1}(s-1)\sum_{k=1}^\infty \lambda_k^s = C$ exists. Then $\frac{1}{\log N}\sum_{k=1}^N \lambda_k \to C$ as $N \to \infty$.*

For a proof of the Proposition, see [fgv01, pp. 294–295].

Next note that both $\operatorname{Tr}^+ A$ and $\operatorname{Wres} A$ are bilinear in A, so we can weaken the positivity hypothesis when comparing them. (There are other zeta-residue formulas available which are bilinear in A, but we do not go into that here.)

Corollary 5.20. *If A is a linear combination of positive elliptic pseudodifferential operators of order $(-n)$, then $A \in \mathcal{L}^+$, A is measurable, and $\operatorname{Tr}^+ A = \frac{1}{n(2\pi)^n}\operatorname{Wres} A$.*

Chapter 6

Spectral triples: General Theory

6.1 The Dixmier trace revisited

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, whose algebra $\mathcal{A}$ is unital. We continue to assume, for convenience, that $\ker D = \{0\}$, so that D^{-1} is a compact operator on $\mathcal{H}$. Suppose now that $|D|^{-p} \in \mathcal{L}^{1+}$, for some $p \geq 1$. Then the functional on $\mathcal{A}$ given by $a \mapsto \operatorname{Tr}_\omega(a\,|D|^{-p})$, for some particular ω, is our candidate for a "noncommutative integral". To see why that should be so, we first examine the commutative case.

Proposition 6.1. *If M is a compact boundaryless n-dimensional spin manifold, with Riemannian metric g and Dirac operator $D\!\!\!/$, then for each $a \in C^\infty(M)$ the operator $a\,|D\!\!\!/|^{-n}$ is measurable, and*

$$\operatorname{Tr}^+(a\,|D\!\!\!/|^{-n}) = C_n \int_M a\,\nu_g,$$

where C_n is a constant depending only on n, namely,

$$C_n = \frac{2^m\,\Omega_n}{n(2\pi)^n}, \quad \textit{that is,} \quad \begin{cases} C_{2m} = \dfrac{1}{m!(2\pi)^m} & \textit{if } n = 2m, \\ C_{2m+1} = \dfrac{1}{(2m+1)!!\,\pi^{m+1}} & \textit{if } n = 2m+1. \end{cases}$$

Proof. We know that $|D\!\!\!/|^{-n}$ is a ΨDO with principal symbol $\sigma^{|D\!\!\!/|^{-n}}(x,\xi) = g_x(\xi,\xi)^{-n/2}\,1_{2^m}$, a scalar matrix of size $2^m \times 2^m$. As a multiplication operator on $L^2(M,S)$, a is a ΨDO of order 0, with principal symbol $\sigma^a(x,\xi) = a(x)\,1_{2^m}$. Thus $a|D\!\!\!/|^{-n}$ is of order $-n$, with $\operatorname{tr}\sigma^{a|D\!\!\!/|^{-n}}(x,\xi) = 2^m\,a(x)\,g_x(\xi,\xi)^{-n/2}$.

Now by the trace theorem, we find, using the calculation in Example 5.16,

$$\begin{aligned}
\operatorname{Tr}^+(a|\not{D}|^{-n}) &= \frac{1}{n(2\pi)^n}\operatorname{Wres}(a|\not{D}|^{-n}) \\
&= \frac{2^m}{n(2\pi)^n}\int_M a(x)\left(\int_{|\xi|=1} g_x(\xi,\xi)^{-n/2}\,\sigma\right)|d^n x| \\
&= \frac{2^m\,\Omega_n}{n(2\pi)^n}\int_M a(x)\sqrt{\det g_x}\,|d^n x| \\
&= \frac{2^m\,\Omega_n}{n(2\pi)^n}\int_M a(x)\,\nu_g.
\end{aligned}$$

□

Therefore, the functional $a \mapsto \operatorname{Tr}^+(a\,|\not{D}|^{-n})$ is just the usual integral with respect to the Riemannian volume form, expect for the normalization constant. Therefore, it can be adapted to more general spectral triples as a "noncommutative integral".

However, in the noncommutative case, it is not obvious that $a \mapsto \operatorname{Tr}^+(a\,|\not{D}|^{-n})$ will be itself a trace. Why should $\operatorname{Tr}^+(ab\,|\not{D}|^{-n})$ be equal to $\operatorname{Tr}^+(ba\,|\not{D}|^{-n}) = \operatorname{Tr}^+(a\,|\not{D}|^{-n}\,b)$? To check this tracial property of the noncommutative integral, we need the Hölder inequality for Dixmier traces.

Fact 6.2 (Horn's inequality)**.** *If $T, S \in \mathcal{K}$ and $n \in \mathbb{N}$, then*

$$\sigma_n(TS) \leq \sum_{k=0}^{n-1} s_k(T)\,s_k(s). \tag{6.1}$$

Proposition 6.3. (a) *If $T \in \mathcal{L}^{1+}$ and S is a bounded operator on $\mathcal{H}$, then for any Dixmier trace Tr_ω, the following inequality holds:*

$$\operatorname{Tr}_\omega |TS| \leq (\operatorname{Tr}_\omega |T|)\,\|S\|. \tag{6.2a}$$

(b) *Let $1 < p < \infty$ and $q = p/(p-1)$, so that $\frac{1}{p} + \frac{1}{q} = 1$, and let $T, S \in \mathcal{K}$ be such that $|T|^p, |S|^q \in \mathcal{L}^{1+}$. Then for any Tr_ω, we get*

$$\operatorname{Tr}_\omega |TS| \leq (\operatorname{Tr}_\omega |T|^p)^{1/p}\,(\operatorname{Tr}_\omega |T|^q)^{1/q}. \tag{6.2b}$$

Proof. Ad (a): By the minimax formula (4.2) for singular values, we find, for each $k \in \mathbb{N}$,

$$\begin{aligned}
s_k(TS) &= \inf\{\,\|(1-P)TS\| : P = P^2 = P^*,\ \operatorname{rank} P \leq k\,\} \\
&\leq \inf\{\,\|(1-P)T\|\,\|S\| : P = P^2 = P^*,\ \operatorname{rank} P \leq k\,\} = s_k(T)\,\|S\|,
\end{aligned}$$

Summing over $k = 0, 1, \ldots, n-1$, we get $\sigma_n(TS) \leq \sigma_n(T)\,\|S\|$. Thus

$$\frac{\sigma_n(TS)}{\log n} \leq \frac{\sigma_n(T)}{\log n}\,\|S\| \quad \text{for all} \quad n \geq 2,$$

and linear interpolation gives the same relation with N replaced by and real $\lambda \geq 2$. Using the definition (4.4) of τ_λ and integrating over $\lambda \geq 3$, we get

$$\tau_\lambda(TS) \leq \tau_\lambda(T)\,\|S\| \quad \text{for all} \quad \lambda \geq 3,$$

and therefore $\operatorname{Tr}_\omega |TS| \leq (\operatorname{Tr}_\omega |T|)\|S\|$ for all ω.

Ad (b): From (6.1) and the ordinary Hölder inequality in $\mathbb{R}^n$, we get

$$\sigma_n(TS) \le \Big(\sum_{0\le k<n} s_k(T)^p\Big)^{1/p} \Big(\sum_{0\le k<n} s_k(S)^q\Big)^{1/q} = \sigma_n(|T|^p)^{1/p}\, \sigma_n(|S|^q)^{1/q}.$$

If $n \le \lambda < n+1$ with $\lambda = n+t$, then, with $a_n := \sigma_n(|T|^p)^{1/p}$ and $b_n := \sigma_n(|S|^q)^{1/q}$,

$$\begin{aligned}\sigma_\lambda(TS) &= (1-t)\sigma_n(TS) + t\sigma_{n+1}(TS)\\ &\le (1-t)a_n b_n + t a_{n+1} b_{n+1}\\ &\le \big((1-t)a_n^p + t a_{n+1}^p\big)^{1/p} \big((1-t)a_n^q + t a_{n+1}^q\big)^{1/q}\\ &= \sigma_\lambda(|T|^p)^{1/p}\, \sigma_\lambda(|S|^q)^{1/q} \quad \text{for all } \lambda \ge 2,\end{aligned}$$

where we have used the Hölder inequality in $\mathbb{R}^2$. Again we employ (4.4) and use the Hölder inequality for the integral $\frac{1}{\log\lambda}\int_3^\lambda (\cdot)\, \frac{du}{u}$. This gives

$$\tau_\lambda(TS) \le \tau_\lambda(|T|^p)^{1/p}\, \tau_\lambda(|T|^q)^{1/q}, \quad \text{for } \lambda \ge 3.$$

Thus $\tau(|TS|) \le \tau(|T|^p)^{1/p}\, \tau(|T|^q)^{1/q}$ as positive elements of the corona C*-algebra B_∞. Finally, we use the Hölder inequality for the state ω of this commutative C*-algebra, namely

$$\omega\big(\tau(|T|)^p)^{1/p}\, \tau(|S|^q)^{1/q}\big) \le \omega(\tau(|T|)^p)^{1/p}\, \omega(\tau(|S|^q)^{1/q},$$

and the result (6.2b) follows at once. □

Proposition 6.4. *Let $(\mathcal{A}, \mathcal{H}, D)$ be any spectral triple whose operator D is invertible, and let $a \in \mathcal{A}$. Then the commutator $[|D|^r, a]$ is a bounded operator for each r such that $0 < r < 1$.*

We postpone the proof of this Proposition until later. It is a crucial property of spectral triple that this bounded commutator property is *not* automatic for the case $r = 1$, that is, the commutators $[|D|, a]$ need not be bounded in general.

Theorem 6.5. *If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple such that $|D|^{-p} \in \mathcal{L}^{1+}(\mathcal{H})$ for some $p \ge 1$, then for each $a \in \mathcal{A}$ and any $T \in B(\mathcal{H})$, the following tracial property holds:*

$$\mathrm{Tr}_\omega(aT\,|D|^{-p}) = \mathrm{Tr}_\omega(Ta\,|D|^{-p}) \quad \textit{for all} \quad \omega. \tag{6.3}$$

Proof. Note that $aT\,|D|^{-p}$ and $Ta\,|D|^{-p}$ lie in $\mathcal{L}^{1+}$ since $\mathcal{L}^{1+}$ is an ideal in $B(\mathcal{H})$. Also the Hölder inequality (6.2a) gives

$$\big|\mathrm{Tr}_\omega([a,T]\,|D|^{-p})\big| = \big|\mathrm{Tr}_\omega(T\,[|D|^{-p}, a])\big| \le \|T\|\ \mathrm{Tr}_\omega\big|[|D|^{-p}, a]\big|,$$

so we must show that $\mathrm{Tr}_\omega\big|[|D|^{-p}, a]\big| = 0$ for all $a \in \mathcal{A}$. We have not supposed that $p \in \mathbb{N}$, so write $p = kr$ with $k \in \mathbb{N}$, $0 < r < 1$, and let $R := |D|^{-r}$, a positive compact operator. Then

$$[|D|^{-p}, a] = [R^k, a] = \sum_{j=1}^k R^{j-1}\,[R,a]\,R^{k-j} = -\sum_{j=1}^k R^j\,[|D|^r, a]\,R^{k-j+1}.$$

Applying Hölder's inequality to each term, we get

$$\operatorname{Tr}_\omega\bigl|R^j\,[|D|^r,a]\,R^{k-j+1}\bigr| \le \|[|D|^r,a]\|\,(\operatorname{Tr}_\omega R^{jp_j})^{1/p_j}\,(\operatorname{Tr}_\omega R^{(k-j+1)q_j})^{1/q_j},$$

where $q_j = p_j/(p_j-1)$ and the number $p_j > 1$ must be chosen so that all R^{jp_j} and all $R^{(k-j+1)q_j}$ are trace-class: for that, we need $rjp_j > p$ and $r(k-j+1)q_j > p$. This will happen if we take

$$p_j := \frac{p}{r(j-\frac12)}, \qquad q_j := \frac{p}{r(k-j+\frac12)},$$

and then $\frac{1}{p_j}+\frac{1}{q_j} = 1$, since $rk = p$. Since Tr_ω vanishes on $\mathcal{L}^1(\mathcal{H})$, we need only to check that

$$|D|^{-p} \in \mathcal{L}^{1+}(\mathcal{H}) \implies |D|^{-s} \in \mathcal{L}^1(\mathcal{H}) \quad \text{for all } s > p. \tag{6.4}$$

This is a consequence of the next lemma.

Lemma 6.6. *If $A \in \mathcal{L}^{1+}(\mathcal{H})$ and $A \ge 0$, then $A^s \in \mathcal{L}^1(\mathcal{H})$ for $s > 1$.*

Proof. We need the following result on sequence spaces. If E is a Banach space, we denote by E^* the dual Banach space of continuous linear forms on E.

Fact 6.7. *If $\mathbf{s} := \{\,(s_0,s_1,\dots) \in \mathbb{C}^{\mathbb{N}} : (s_0+\cdots+s_{n-1})/\log n$ is bounded $\}$, if $\mathbf{s}_0$ is the closure of the finite sequences in $\mathbf{s}$, and if $\mathbf{t} := \{\,(t_0,t_1,\dots) \in \mathbb{C}^{\mathbb{N}} : \sum_{k\ge0}|t_k|/(k+1) < \infty\,\}$, then $\mathbf{s}_0$, $\mathbf{s}$ and $\mathbf{t}$ are complete in the obvious norms, and under the standard duality pairing $\langle s,t\rangle := \sum_{k\ge0} s_k t_k$, there are isometric isomorphisms $\mathbf{s}_0^* \simeq \mathbf{t}$ and $\mathbf{t}^* \simeq \mathbf{s}$.*

Now let $\mathcal{K}^- := \{\,T \in \mathcal{K} : \{s_k(T)\}_{k\ge0} \in \mathbf{t}\,\}$, and let $\mathcal{L}^{1+}_0$ be the closure of the finite-rank operators in $\mathcal{L}^{1+}$. Then $(\mathcal{L}^{1+}_0)^* \simeq \mathcal{K}^-$ and $(\mathcal{K}^-)^* \simeq \mathcal{L}^{1+}$ as Banach spaces.

For $T \in \mathcal{L}^q$ with $1 < q < \infty$, the Hölder inequality for sequences gives

$$\sum_{k\ge0}\frac{s_k(T)}{k+1} \le \Bigl(\sum_{k\ge0}s_k(T)^q\Bigr)^{1/q}\Bigl(\sum_{k\ge0}\frac{1}{(k+1)^p}\Bigr)^{1/p} = \|T\|_q\,\zeta(p)^{1/p} < \infty,$$

so that $\mathcal{L}^q \subset \mathcal{K}^-$ for all $1 < q < \infty$. Since $(\mathcal{L}^q)^* \simeq \mathcal{L}^p$ with $p = q/(q-1)$, we conclude that $\mathcal{L}^{1+} \subset \mathcal{L}^p$ for all $p > 1$. (This is why we employ the notation $\mathcal{L}^{1+}$, of course.)

Now if $A \in \mathcal{L}^{1+}$ with $A \ge 0$, then $A^s \in \mathcal{L}^{p/s}(\mathcal{H})$ whenever $1 < s \le p$. In particular, when $p = s$, we see that

$$\|A^s\|_1 = \sum_{k\ge0}s_k(A^s) = \sum_{k\ge0}\lambda_k(A^s) = \sum_{k\ge0}\lambda_k(A)^s = (\|A\|_s)^s < +\infty,$$

since $A \in \mathcal{L}^{1+}$ implies $A \in \mathcal{L}^s$. ∎

This establishes (6.4) and concludes the proof of Theorem 6.5. ∎

Corollary 6.8. *If $A \ge 0$ is in $\mathcal{L}^{1+}$, and $\operatorname{Tr}^+ A > 0$, then $\operatorname{Tr}^+ A^s = 0$ for $s > 1$.*

To establish Proposition 6.4, we use the following commutator estimate, due to Helton and Howe [hh73].

Lemma 6.9. *Let D be a selfadjoint operator on $\mathcal{H}$, and let $a \in B(\mathcal{H})$ with $a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ be such that $[D, a]$ extends to a bounded operator on $\mathcal{H}$. Suppose also that $g\colon \mathbb{R} \to \mathbb{R}$ is smooth, with Fourier transform is a function $\hat{g}$ such that $t \mapsto t\,\hat{g}(t)$ is integrable on $\mathbb{R}$. Then $[g(D), a]$ extends to a bounded operator on $\mathcal{H}$, such that*

$$\|[g(D), a]\| \leq \frac{1}{2\pi}\|[D, a]\| \int_{\mathbb{R}} |t\,\hat{g}(t)|\,dt.$$

Proof. We may define

$$g(D) := \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(t)e^{itD}\,dt$$

for any smooth function g, since $\hat{g}$ has compact support. Now

$$\begin{aligned}\langle\phi \mid [e^{itD}, a]\psi\rangle &= \int_0^1 \frac{d}{ds}\langle\phi \mid e^{istD}\, a\, e^{i(1-s)tD}\psi\rangle\,ds \\ &= it\int_0^1 \frac{d}{ds}\big(\langle De^{-istD}\phi \mid ae^{i(1-s)tD}\psi\rangle - \langle\phi \mid e^{istD}aDe^{i(1-s)tD}\psi\rangle\big)\,ds\end{aligned}$$

for $\phi, \psi \in \operatorname{Dom} D$, and thus

$$\begin{aligned}&\big|\langle g(D)\phi \mid a\psi\rangle - \langle\phi \mid ag(D)\psi\rangle\big| \\ &\quad = \frac{1}{2\pi}\left|\int_{\mathbb{R}} t\hat{g}(t)\int_0^1 \frac{d}{ds}\big(\langle De^{-istD}\phi \mid ae^{i(1-s)tD}\psi\rangle - \langle\phi \mid e^{istD}aDe^{i(1-s)tD}\psi\rangle\big)\,ds\,dt\right| \\ &\quad \leq \frac{\|[D,a]\|}{2\pi}\,\|\phi\|\,\|\psi\|\int_{\mathbb{R}} |t\,\hat{g}(t)|\,dt,\end{aligned}$$

so that $[g(D), a]$ extends to a bounded operator, and the required estimate holds. ∎

Proof of Proposition 6.4. We want to apply Lemma 6.9, using $|x|^r$ instead of $g(x)$, $x \in \mathbb{R}$. But $x \mapsto |x|^r$ is not smooth at $x = 0$ (although it is homogeneous of degree r), so we modify it near $x = 0$ to get a smooth function $g(x)$ such that $g(x) = |x|^r$ for $|x| \geq \delta$, for some $\delta > 0$. Thus $g(x) = |x|^r + h(x)$, where $\operatorname{supp} h \subset [-\delta, \delta]$. We can write its derivative as a sum of two terms, $g'(x) = u(x) + h'(x)$, where $\operatorname{supp} h' \subset [-\delta, \delta]$ and u is homogeneous of negative degree $r - 1$. Taking Fourier transforms on $\mathbb{R}$, we get $it\,\hat{g}(t) = \hat{u}(t) + \hat{h}'(t)$, where $\hat{h}'(t)$ is analytic and $\hat{u}$ is homogeneous of degree $-1 - (r-1) = -r$, with $-1 < -r < 0$. Thus $t\hat{g}(t)$ is locally integrable near $t = 0$, and $t\hat{g}(t) \to 0$ rapidly for large t, since g' is smooth. We end up with an estimate

$$\|[|D|^r, a]\| \leq C_r\|[D, a]\| + \|[h(D), a]\|,$$

where $C_r := (2\pi)^{-1}\int_{\mathbb{R}} |t\hat{g}(t)|\,dt$ is finite, and $\|[h(D), a]\|$ is finite since $h(D)$ is a bounded operator. ∎

6.2 Regularity of spectral triples

The arguments of the previous section are not applicable to determine whether $[|D|, a]$ is bounded, in the case $r = 1$. This must be formulated as an assumption. In fact, we shall ask for much more: we want each element $a \in \mathcal{A}$, and each bounded operator $[D,a]$ too, to lie in the *smooth domain* of the following derivation.

Notation. We denote by δ the derivation on $B(\mathcal{H})$ given by taking the commutator with $|D|$. It is an unbounded derivation, whose domain is

$$\operatorname{Dom}\delta := \{\, T \in B(\mathcal{H}) : T(\operatorname{Dom}|D|) \subseteq \operatorname{Dom}|D|,\ [|D|,T] \text{ is bounded } \}.$$

We write $\delta(T) := [|D|,T]$ for $T \in \operatorname{Dom}\delta$.

Definition 6.10. *A spectral triple $(\mathcal{A},\mathcal{H},D)$ is called* regular*, if for each $a \in \mathcal{A}$, the operators a and $[D,a]$ lie in $\bigcap_{k\in\mathbb{N}} \operatorname{Dom}\delta^k$.*

The regularity condition does not depend on the invertibility of D (that is, the condition $\ker D = \{0\}$) which we have been assuming, to simplify certain calculations. One can always replace $|D|$ by $\langle D\rangle := (D^2+1)^{1/2}$ in the definition, since $f(D) := \langle D\rangle - |D|$ is bounded. If δ' denotes the derivation $\delta'(T) := [\langle D\rangle, T] = \delta(T) + [f(D),T]$, then clearly $\operatorname{Dom}\delta' = \operatorname{Dom}\delta$, and it is easy to show by induction that $\operatorname{Dom}\delta'^k = \operatorname{Dom}\delta^k$ for each $k \in \mathbb{N}$, so one may instead define regularity using δ'. This is the approach taken in the work of Carey et al [cprs04], who use the term "QC^∞" instead of "regular" for this class of spectral triples.

Definition 6.11. *Suppose that $(\mathcal{A},\mathcal{H},D)$ is a regular spectral triple, with D invertible. For each $s \geq 0$, consider the operator $|D|^s$ defined by functional calculus. Define also $\mathcal{H}^s := \operatorname{Dom}|D|^s$ for $s \geq 0$, with the Hilbert norm $\|\xi\|_s := \sqrt{\|\xi\|^2 + \||D|^s\xi\|^2}$. Their intersection $\mathcal{H}^\infty = \bigcap_{s\geq 0} \operatorname{Dom}|D|^s = \bigcap_{k=0}^\infty \operatorname{Dom}|D|^k$ is the smooth domain of the positive selfadjoint operator $|D|$. Its topology is defined by the seminorms $\|\cdot\|_k$, for $k \in \mathbb{N}$. Each $\mathcal{H}^s$ (and thus also $\mathcal{H}^\infty$) is complete, since the operators $|D|^s$ are closed, thus $\mathcal{H}^\infty$ is a Fréchet space.*

Since $a \in \mathcal{A}$ implies $a \in \operatorname{Dom}\delta$, we see that $a(\mathcal{H}^1) \subseteq \mathcal{H}^1$, and then we can write $a(|D|\xi) = |D|(a\xi) - [|D|,a]\,\xi$ for $\xi \in \mathcal{H}^1$. Also,

$$\begin{aligned}
\|a\xi\|_1^2 &= \|a\xi\|^2 + \||D|a\xi\|^2 \\
&= \|a\xi\|^2 + \|a|D|\xi + \delta(a)\xi\|^2 \\
&\leq \|a\xi\|^2 + 2\|\delta(a)\xi\|^2 + 2\|a|D|\xi\|^2 \\
&\leq \max\{\|a\|^2 + 2\|\delta(a)\|^2, 2\|a\|^2\}\ \|\xi\|_1^2,
\end{aligned}$$

where we have used the parallelogram law $\|\xi+\eta\|^2 + \|\xi-\eta\|^2 = 2\|\xi\|^2 + 2\|\eta\|^2$. Therefore, a extends to a bounded operator on $\mathcal{H}^1$. If $(\mathcal{A},\mathcal{H},D)$ is regular, then by induction we find that $a(\mathcal{H}^k) \subset \mathcal{H}^k$ continuously for each k, so that $a(\mathcal{H}^\infty) \subset \mathcal{H}^\infty$ continuously, too.

Definition 6.12. *If $r \in \mathbb{Z}$, let $\mathbf{Op}_D^r$ be the vector space of linear maps $T\colon \mathcal{H}^\infty \to \mathcal{H}^\infty$ for which there are constants C_k, for $k \in \mathbb{N}$, $k \geq r$, such that*

$$\|T\xi\|_{k-r} \leq C_k\,\|\xi\|_k \quad \text{for all } \xi \in \mathcal{H}^\infty.$$

Every such T extends to a bounded operator from $\mathcal{H}^k$ to $\mathcal{H}^{k-r}$, for each $k \in \mathbb{N}$. Note that $|D|^r \in \mathbf{Op}_D^r$ for each $r \in \mathbb{Z}$. If $T \in \mathbf{Op}_D^r$ and $S \in \mathbf{Op}_D^s$, then $ST \in \mathbf{Op}_D^{r+s}$.

Suppose $(\mathcal{A}, \mathcal{H}, D)$ is regular. Then $\mathcal{A} \subset \mathbf{Op}_D^0$ and $[D, \mathcal{A}] := \{\, [D, a] : a \in \mathcal{A} \,\} \subset \mathbf{Op}_D^0$, too. Moreover, if $a \in \mathcal{A}$, then

$$\begin{aligned}[D^2, a] &= [|D|^2, a] = |D|\,[|D|, a] + [|D|, a]\,|D| \\ &= |D|\delta(a) + \delta(a)|D| = 2|D|\delta(a) - [|D|, \delta(a)] \\ &= 2|D|\delta(a) - \delta^2(a),\end{aligned}$$

so that $[D^2, a] \in \mathbf{Op}_D^1$. Also $[D^2, [D, a]] \in \mathbf{Op}_D^1$ in the same way.

If b lies the subalgebra of $B(\mathcal{H})$ generated by $\mathcal{A}$ and $[D, \mathcal{A}]$, we introduce

$$\begin{aligned}L(b) &:= |D|^{-1}\,[D^2, b] = 2\delta(b) - |D|^{-1}\,\delta^2(b), \\ R(b) &:= [D^2, b]\,|D|^{-1} = 2\delta(b) + \delta^2(b)\,|D|^{-1}.\end{aligned} \tag{6.5}$$

If $b \in \bigcap_{k\geq 0} \operatorname{Dom} \delta^k$, then $L(b)$ and $R(b)$ lie in $\mathbf{Op}_D^0$. The operations L and R commute: indeed,

$$L(R(b)) = |D|^{-1}\,[D^2, [D^2, b]\,|D|^{-1}] = |D|^{-1}\,[D^2, [D^2, b]\,|D|^{-1} = R(L(b)).$$

Note also that $L^2(b) = |D|^{-2}\,[D^2, [D^2, b]]$.

Proposition 6.13. *If D is invertible, then $\bigcap_{k,l\geq 0} \operatorname{Dom}(L^k R^l) = \bigcap_{m\geq 0} \operatorname{Dom} \delta^m \subset B(\mathcal{H})$.*

Proof. We use the following identity for $|D|^{-1}$, obtained from the spectral theorem:

$$|D|^{-1} = \frac{2}{\pi}\int_0^\infty (D^2 + \mu^2)^{-1}\,d\mu, \tag{6.6}$$

in order to compute the commutators. We shall show that $\operatorname{Dom} L^2 \cap \operatorname{Dom} R \subset \operatorname{Dom} \delta$.

Indeed, if $b \in \operatorname{Dom} L^2 \cap \operatorname{Dom} R$ implies $b \in \operatorname{Dom} \delta$, then $b \in \operatorname{Dom} L^4 \cap \operatorname{Dom} L^2 R \cap \operatorname{Dom} R^2$ implies $\delta b \in \operatorname{Dom} L^2 \cap \operatorname{Dom} R$, so $b \in \operatorname{Dom} \delta^2$. By induction, $\bigcap_{k,l\geq 0} \operatorname{Dom}(L^k R^l) \subset \bigcap_{m\geq 0} \delta^m$. The converse inclusion is clear, from (6.5).

Take $b \in \operatorname{Dom} L^2 \cap \operatorname{Dom} R$, and compute $[|D|, b]$ as follows:

$$\begin{aligned}[|D|, b] &= [D^2|D|^{-1}, b] = [D^2, b]\,|D|^{-1} + D^2\,[|D|^{-1}, b] \\ &= \frac{2}{\pi}\int_0^\infty \big([D^2, b](D^2 + \mu^2)^{-1} + D^2[(D^2 + \mu^2)^{-1}, b]\big)\,d\mu \\ &= \frac{2}{\pi}\int_0^\infty \big([D^2, b](D^2 + \mu^2)^{-1} - D^2(D^2 + \mu^2)^{-1}[D^2 + \mu^2, b](D^2 + \mu^2)^{-1}\big)\,d\mu \\ &= \frac{2}{\pi}\int_0^\infty (1 - D^2(D^2 + \mu^2)^{-1})[D^2, b](D^2 + \mu^2)^{-1}\,d\mu \\ &= \frac{2}{\pi}\int_0^\infty \mu^2(D^2 + \mu^2)^{-1}[D^2, b](D^2 + \mu^2)^{-1}\,d\mu \\ &= \frac{2}{\pi}\int_0^\infty \big([D^2, b](D^2 + \mu^2)^{-2} + [(D^2 + \mu^2)^{-1}, [D^2, b]](D^2 + \mu^2)^{-1}\big)\mu^2\,d\mu \\ &= \frac{2}{\pi}\int_0^\infty \big(R(b)|D|(D^2 + \mu^2)^{-2} - (D^2 + \mu^2)^{-1}D^2L^2(b)(D^2 + \mu^2)^{-2}\big)\mu^2\,d\mu\end{aligned}$$

Now $R(b)$ and $\frac{D^2}{D^2+\mu^2}L^2(b)$ are bounded, by hypothesis. Also

$$\frac{2}{\pi}\int_0^\infty \frac{x\mu^2\,d\mu}{(x^2+\mu^2)^2} = \frac{2}{\pi}\int_0^\infty \frac{t^2\,dt}{1+t^2} = \frac{2}{\pi}\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{1}{2},$$

while

$$\int_0^\infty (D^2+\mu^2)^{-2}\mu^2\,d\mu = \int_0^1 (D^2+\mu^2)^{-2}\mu^2\,d\mu + \int_1^\infty (D^2+\mu^2)^{-2}\mu^2\,d\mu$$

is bounded by

$$\int_0^1 \|D^{-4}\|\mu^2\,d\mu + \int_0^\infty \mu^{-2}\,d\mu = \frac{1}{3}\,\|D\|^{-4} + 1.$$

Thus $[|D|, b]$ is bounded with the estimate

$$\|[|D|, b]\| \leq \frac{1}{2}\|R(b)\| + \left(\frac{1}{3}\|D\|^{-4} + 1\right)\frac{2}{\pi}\,\|L^2(b)\|.$$

Hence $b \in \operatorname{Dom}\delta$, as desired. □

Corollary 6.14. *The standard commutative example* $(C^\infty(M), L^2(M,S), D\!\!\!/)$ *is a regular spectral triple.*

Proof. We need one more fact from the theory of ΨDOs (see [t-me96], for example): over a compact manifold M, with a hermitian vector bundle E, a ΨDO of order zero is bounded as an operator on $L^2(M,E)$. Thus we need only show that, if $b = a$ or $b = [D\!\!\!/, a] = -i\,c(da)$, then $L^k R^l$ is a ΨDO of order ≤ 0, for each $k, l \in \mathbb{N}$.

For $k = l = 0$, note that $\psi \mapsto a\psi$ and $\psi \mapsto [D\!\!\!/, a]\psi = -i\,c(da)\psi$ are bounded multiplication and Clifford-action operators. Their (principal) symbols are

$$\begin{aligned} \sigma^a(x,\xi) &= a(x)\,1_{2^m}, \\ \sigma^{[D\!\!\!/,a]}(x,\xi) &= -i\sum_{j=1}^n \partial_j a(x) c^j = -i\,\{c(\xi), a(x)\} = -i\,c(da). \end{aligned}$$

For $k + l > 0$, we use $L^k R^l = |D\!\!\!/|^{-k}(\operatorname{ad} D\!\!\!/^2)^{k+l}(\cdot)|D\!\!\!/|^{-l}$. Now $D\!\!\!/^2$ is a second-order ΨDO, with $\sigma_2^{D\!\!\!/^2}(x,\xi) = g(\xi,\xi)\,1_{2^m}$, so that when P is of order d then $[D\!\!\!/^2, P]$ is of order $\leq d+1$. Hence, if $a \in C^\infty(M)$, then $(\operatorname{ad} D\!\!\!/^2)^{k+l}(a)$ is of order $\leq k+l$, and thus $L^k R^l$ is of order ≤ 0. The same is true if a is replaced by $-i\,c(da)$. Thus $L^k R^l(b)$ is bounded, if $b \in \mathcal{A}$ or $b \in [D\!\!\!/, \mathcal{A}]$. □

This example also shows why regularity is defined using the derivation $\delta = [|D|, \cdot\,]$ instead of the apparently simpler derivation $[D, \cdot\,]$. Indeed, we have just seen that for $a \in C^\infty(M)$, the operator $[|D\!\!\!/|, [D\!\!\!/, a]]$ has order zero (and therefore, it lies in $\mathbf{Op}^0_{D\!\!\!/}$. On the other hand, $[D\!\!\!/, [D\!\!\!/, a]]$ is in general a ΨDO of order 1 (and so it lies in $\mathbf{Op}^1_{D\!\!\!/}$). Indeed, the first-order terms in its symbol are

$$[\sigma^{D\!\!\!/}, \sigma^{[D\!\!\!/,a]}](x,\xi) = [c^j\xi_j, -i\,c^k\,\partial_k a(x)] = -i\,[c^j, c^k]\,\xi_j\,\partial_k a(x)$$

which need not vanish since c^j, c^k do not commute. In contrast, the principal symbol of $|D\!\!\!/|$ is a scalar matrix, which commutes with that of $[D\!\!\!/, a]$, and the order of the commutator drops to zero.

6.3 Pre-C*-algebras

If any spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the algebra $\mathcal{A}$ is a (unital) $*$-algebra of bounded operators acting on a Hilbert space $\mathcal{H}$ [or, if one wishes to regard $\mathcal{A}$ abstractly, a faithful representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$ is given]. Let A be the norm closure of $\mathcal{A}$ [or of $\pi(\mathcal{A})$] in $B(\mathcal{H})$: it is a C*-algebra in which $\mathcal{A}$ is a dense $*$-subalgebra.

A priori, the only functional calculus available for $\mathcal{A}$ is the holomorphic one:

$$f(a) := \frac{1}{2\pi i} \oint_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\, d\lambda, \tag{6.7}$$

where Γ is a contour in $\mathbb{C}$ winding (once positively) around $\operatorname{sp}(a)$, and $\operatorname{sp}(a)$ means the spectrum of a in the C*-algebra A. To ensure that $a \in \mathcal{A}$ implies $f(a) \in \mathcal{A}$, we need the following property:

If $a \in \mathcal{A}$ has an inverse $a^{-1} \in A$, then in fact a^{-1} lies in $\mathcal{A}$ (briefly: $\mathcal{A} \cap A^\times = \mathcal{A}^\times$, where $\mathcal{A}^\times$ is the group of invertible elements of A). If this condition holds, then $\frac{1}{2\pi i} \oint_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\, d\lambda$ is a limit of Riemann sums lying in $\mathcal{A}$. To ensure convergence in $\mathcal{A}$ (they do converge in A), we need only ask that $\mathcal{A}$ be complete in some topology that is finer than the C^*-norm topology.

Definition 6.15. *A* pre-C*-algebra *is a subalgebra of $\mathcal{A}$ of a C*-algebra A, which is stable under the holomorphic functional calculus of A.*

Remark 6.16. This condition appears in Blackadar's book [b-b98] under the name "local C*-algebra". However, one can wonder how such a property could be checked in practice. Consider the two conditions on a $*$-subalgebra $\mathcal{A}$ of a unital C*-algebra A:

(a) $\mathcal{A}$ is *stable under holomorphic functional calculus*; that is, $a \in \mathcal{A}$ implies $f(a) \in \mathcal{A}$, according to (6.7).

(b) $\mathcal{A}$ is *spectrally invariant* in A [s-lb92], that is,

$$a \in \mathcal{A} \text{ and } a^{-1} \in A \implies a^{-1} \in \mathcal{A}. \tag{6.8}$$

In particular, (6.8) implies $\operatorname{sp}_{\mathcal{A}}(a) = \operatorname{sp}_A(a)$, for all $a \in \mathcal{A}$.

Question: If $\mathcal{A}$ is known to have a (locally convex) vector space topology under which $\mathcal{A}$ is complete (needed for convergence of the Riemann sums defining the contour integral) and such that the inclusion $\mathcal{A} \hookrightarrow A$ is continuous, are (a) and (b) equivalent?

Ad (a) $\implies$ (b): This is clear: if $a \in \mathcal{A}$, $a^{-1} \in A$, use $f(\lambda) := 1/\lambda$ outside $\operatorname{sp}_A(a)$.

Ad (b) $\implies$ (a): To prove that the integral converges in $\mathcal{A}$, because $\mathcal{A}$ is complete, we need to show that the integrand is continuous. Note that since the inclusion $i\colon \mathcal{A} \hookrightarrow A$ is continuous, then $\mathcal{A}^\times = \{\, a \in \mathcal{A} : a^{-1} \in \mathcal{A} \,\} = \mathcal{A} \cap A^\times = i^{-1}(A^\times)$ is open in $\mathcal{A}$. But we still need to show that $a \mapsto a^{-1}\colon \mathcal{A}^\times \to \mathcal{A}^\times$ is continuous. This will follow if $\mathcal{A}$ is a *Fréchet* algebra [s-lb92].

Corollary 6.17 (Schweitzer)**.** *If $\mathcal{A}$ is a unital Fréchet algebra, and if $\|\cdot\|_A$ is continuous in the topology of $\mathcal{A}$, then Conditions (a) and (b) are equivalent.*

If $\mathcal{A}$ is a nonunital algebra, we can always adjoin a unit in the usual way, and work with $\widetilde{\mathcal{A}} := \mathbb{C} \oplus \mathcal{A}$ whose unit is $(1, 0)$, and with its C^*-completion $\widetilde{A} := \mathbb{C} \oplus A$. Since the multiplication rule in $\widetilde{\mathcal{A}}$ is $(\lambda, a)(\mu, b) := (\lambda\mu, \mu a + \lambda b + ab)$, we see that $1 + a := (1, a)$ is invertible in $\widetilde{\mathcal{A}}$, with inverse $(1, b)$, if and only if $a + b + ab = 0$.

Lemma 6.18. *If $\mathcal{A}$ is a unital, Fréchet pre-C*-algebra, then so also is $M_n(\mathcal{A}) = M_n(\mathbb{C}) \otimes \mathcal{A}$.*

Sketch proof. It is enough to show that $a \in M_n(\mathcal{A})$ is invertible for a close to the identity 1_n in the norm of $M_n(A)$. But for a close to 1_n, the procedure of Gaussian elimination gives matrix factorization $a =: ldu$, where

$$l = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \dots & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & d_n \end{pmatrix}, \qquad u = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

with $d_j \in \mathcal{A}$ such that $\|d_j - 1\|_A < 1$, for $j = 1, \dots, n$. Thus d^{-1} exists, and $a^{-1} = u^{-1} d^{-1} l^{-1} \in M_n(\mathcal{A})$.

For $n = 2$, we get explicitly

$$a = \begin{pmatrix} 1 & 0 \\ a_{21} a_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_1 1 & 0 \\ 0 & a_{22} - a_{21} a_{11}^{-1} a_{12} \end{pmatrix} \begin{pmatrix} 1 & a_{11}^{-1} a_{12} \\ 0 & 1 \end{pmatrix},$$

provided $\|1 - a_{11}\|_A < 1$. For larger n, if $\|1_n - a\|_{M_n(A)} < \delta$ for δ small enough, we can perform $(n-1)$ steps of Gaussian elimination (without any exchanges of rows or columns) and get the factorization $a = ldu$ in $M_n(\mathcal{A})$ with d invertible. □

Lemma 6.19. *The Schwartz algebra $\mathcal{S}(\mathbb{R}^n)$ is a nonunital pre-C*-algebra.*

Proof. We represent $\mathcal{S}(\mathbb{R}^n)$ by multiplication operators on $L^2(\mathbb{R}^n)$. Its C^*-completion is $C_0(\mathbb{R}^n)$. Note that $\mathbb{C}1 \oplus C_0(\mathbb{R}^n) \simeq C(\mathbb{S}^n)$. Suppose $f \in \mathcal{S}(\mathbb{R}^n)$, and that there exists $g \in C_0(\mathbb{R}^n)$ such that $(1+f)(1+g) = 1$. Then $f + g + fg = 0$, and $1 + g = 1/(1 + f)$ in $C(\mathbb{S}^n)$. Now, since f is C^∞, then in particular g is smooth on $\mathbb{R}^n$ and all derivatives $\partial^\alpha g$ are bounded. This entails that $fg \in \mathcal{S}(\mathbb{R}^n)$ also.

Finally, $g = -f - fg$ lies in $\mathcal{S}(\mathbb{R}^n)$, so that $(1 + g) = (1 + f)^{-1}$ lies in $\mathbb{C}1 \oplus \mathcal{S}(\mathbb{R}^n)$, as required. □

Example 6.20. If M is compact boundaryless smooth manifold, then $C^\infty(M)$ is a unital Fréchet pre-C*-algebra. The topology on $C^\infty(M)$ is that of "uniform convergence of all derivatives":

$$f_k \to f \text{ in } C^\infty(M) \quad \text{if and only if } \|X_1 \dots X_r f_k - X_1 \dots X_r f\|_\infty \to 0 \quad \text{as } k \to \infty,$$

for each finite set of vector fields $\{X_1, \dots, X_r\} \in \mathfrak{X}(M)$. This makes $C^\infty(M)$ a Fréchet space. If $f \in C^\infty(M)$ is invertible in $C(M)$, then $f(x) \neq 0$ for any $x \in X$, and so $1/f$ is also smooth. Thus $C^\infty(M)^\times = C^\infty(M) \cap C(M)^\times$.

We state, without proof, two important facts about Fréchet pre-C*-algebras.

Fact 6.21. *If $\mathcal{A}$ is a Fréchet pre-C*-algebra and A is its C^*-completion, then $\mathrm{K}_j(\mathcal{A}) = \mathrm{K}_j(A)$ for $j = 0, 1$. More precisely, if $i\colon \mathcal{A} \to A$ is the (continuous, dense) inclusion, then $i_*\colon \mathrm{K}_j(\mathcal{A}) \to \mathrm{K}_j(A)$ is an surjective isomorphism, for $j = 0$ or 1.*

This invariance of K-theory was proved by Bost [b-j90]. For K_0, the spectral invariance plays the main role. For K_1, one must first formulate a topological K_1-theory is a category of "good" locally convex algebras (thus whose invertible elements form an open subset and for which inversion is continuous), and it is known that Fréchet pre-C*-algebras are "good" in this sense.

Fact 6.22. *If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, we can confer on $\mathcal{A}$ the topology given by the seminorms*

$$q_k(a) := \|\delta^k(a)\|, \quad q'_k(a) := \|\delta^k([D, a])\|, \quad \textit{for each } k \in \mathbb{N}. \tag{6.9}$$

The completion $\mathcal{A}_\delta$ of $\mathcal{A}$ is then a Fréchet pre-C-algebra, and $(\mathcal{A}_\delta, \mathcal{H}, D)$ is again a regular spectral triple.*

These properties of the completed spectral triple are due to Rennie [r-a03]. We now discuss another result of Rennie, namely that such completed algebras of regular spectral triples are endowed with a C^∞ *functional calculus*.

Proposition 6.23. *If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, for which $\mathcal{A}$ is complete in the Fréchet topology determined by the seminorms (6.9), then $\mathcal{A}$ admits a C^∞-functional calculus. Namely, if $a = a^* \in \mathcal{A}$, and if $f\colon \mathbb{R} \to \mathbb{C}$ is a compactly supported smooth function whose support includes a neighbourhood of $\mathrm{sp}(a)$, then the following element $f(a)$ lies in $\mathcal{A}$:*

$$f(a) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(s) \exp(isa)\, ds. \tag{6.10}$$

Remark 6.24. One may use the continuous functional calculus in the C*-algebra A to define the one-parameter unitary group $s \mapsto \exp(isa)$, for $s \in \mathbb{R}$. Then the right hand side of (6.10) coincides with the element $f(a) \in A$ defined by the continuous functional calculus in A.

Proof. The map $\delta = \operatorname{ad}|D|\colon \mathcal{A} \to B(\mathcal{H})$ is a closed derivation [br87] since $|D|$ is a selfadjoint operator. To show that $f(a) \in \operatorname{Dom}\delta$ and that

$$\delta(f(a)) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)\, \delta(\exp(ita))\, dt, \tag{6.11}$$

we need to show that the integral on the right hand side converges. Indeed, by the same token, the formula

$$\delta(\exp(ita)) = it \int_0^1 \exp(ista)\, \delta(a)\, \exp(i(1-s)ta)\, ds$$

shows that $\exp(ita) \in \operatorname{Dom}\delta$ because

$$\left| it \int_0^1 \exp(ista)\, \delta(a)\, \exp(i(1-s)ta)\, ds \right| \leq |t| \int_0^1 \|\delta(a)\|\, ds = |t|\, \|\delta(a)\|,$$

and dominated convergence of the integral follows. Plugging this estimate into (6.11), we get

$$\frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(t)|\,\|\delta(\exp(ita))\|\,dt \le \frac{1}{2\pi}\,\|\delta(a)\|\int_{\mathbb{R}} |t\hat{f}(t)|\,dt < +\infty,$$

since $f \in C_c^\infty(\mathbb{R})$ implies $\hat{f} \in \mathcal{S}(\mathbb{R})$. Thus $f(a) \in \operatorname{Dom}\delta$, and (6.11) holds.

Now let A_m, for $m \in \mathbb{N}$, be the completion of $\mathcal{A}$ in the norm

$$a \mapsto \sum_{k=0}^{m} q_k(a) + q'_k(a) = \sum_{k=0}^{m} \|\delta^k(a)\| + \|\delta^k([D,a])\|.$$

For $m = 0$, we get

$$\|f(a)\| + \|[D, f(a)]\| \le \frac{1}{2\pi}\int_{\mathbb{R}} (|\hat{f}(t)| + \|[D,a]\|\,|t\hat{f}(t)|)\,dt$$

by replacing $|D|$ by D in the previous argument.

Therefore, $f(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(s)\,\exp(isa)\,ds$ is a limit of Riemann sums converging in the norm $q_0 + q'_0$, so that $f(a) \in \mathcal{A}_0$. Next, since δ and $(\operatorname{ad} D)$ are commuting derivations (on $\mathcal{A}$), we get $[D, f(a)] \in \operatorname{Dom}\delta$, with

$$\begin{aligned}\big\|\delta([D, f(a)])\big\| &\le \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(t)|\,\|\delta([D, \exp(ita)])\|\,dt \\ &\le \frac{1}{2\pi}\int_{\mathbb{R}} \big(|t\hat{f}(t)|\,\|\delta([D,a])\| + |t^2\hat{f}(t)|\,\|\delta(a)\|\,\|[D,a]\|\big)\,dt,\end{aligned}$$

since $t\hat{f}(t)$ and $t^2\hat{f}(t)$ lie in $\mathcal{S}(\mathbb{R})$. We conclude that δ extends to a closed derivation from A_0 to $B(\mathcal{H})$.

By an (ugly) induction on m, we find that for $k = 0, 1, \dots, m$ $f(a)$ and $[D, f(a)]$ lie in $\operatorname{Dom}\delta^k$, and that δ extends to a closed derivation from A_m to $B(\mathcal{H})$, and that $f(a) \in A_m$. By hypothesis, $\mathcal{A} = \bigcap_{m\in\mathbb{N}} A_m$, and thus $f(a) \in \mathcal{A}$. □

Before showing how this smooth functional calculus can yield useful results, we pause for a couple of technical lemmas on approximation of idempotents and projectors, in Fréchet pre-C*-algebras. The first is an adaptation of a proposition of [b-j90].

Lemma 6.25. *Let $\mathcal{A}$ be an unital Fréchet pre-C*-algebra, with C^*-norm $\|\cdot\|$. Then for each ε with $0 < \varepsilon < \frac{1}{8}$, we can find $\delta \le \varepsilon$ such that, for each $v \in \mathcal{A}$ with $\|v - v^2\| < \delta$ and $\|1 - 2v\| < 1 + \delta$, there is an idempotent $e = e^2 \in \mathcal{A}$ such that $\|e - v\| < \varepsilon$.*

Proof. Consider the holomorphic function

$$f : \{\,\lambda \in \mathbb{C} : |\lambda| < \tfrac{1}{4}\,\} \to \mathbb{C} \quad \text{defined by} \quad f(\lambda) := \tfrac{1}{2}\big(1 - \sqrt{1 + 4\lambda}\big),$$

where we choose the branch of the square root for which $\sqrt{1} = +1$. Note that $f(0) = 0$, and that $(1 - 2f(\lambda))^2 = 1 + 4\lambda$, so that

$$f(\lambda)^2 - f(\lambda) = \lambda \quad \text{for } |\lambda| < \tfrac{1}{4}.$$

If $x \in A$ with $\|x\| < \frac{1}{8}$, then $(1+4x)^{-1}$ exists since $\|1-(1+4x)\| < \frac{1}{2}$, and

$$\begin{aligned} \|x(1+4x)^{-1}\| &\le \|x\| \, \|(1+4x)^{-1}\| \\ &\le \|x\| \sum_{k=0}^{\infty} \|(-4x)^k\| \\ &\le \|x\| \sum_{k=0}^{\infty} \|4x\|^k \\ &= \frac{\|x\|}{1-4\|x\|} < \frac{1}{4}, \end{aligned}$$

since $\dfrac{t}{1-4t}$ increases from 0 to $\frac{1}{4}$ for $0 \le t \le \frac{1}{8}$.

Now let $x := v^2 - v$, (thus $\|x\| < \frac{1}{8}$), and let $y := -x(1+4x)^{-1} = (v - v^2)(1-2v)^{-2}$, for which $\|y\| < \frac{1}{4}$. Note that $\|x\| \to 0$ implies $\|y\| \to 0$, which in turn implies $\|f(y)\| \to 0$, so that for each $\varepsilon \in (0, \frac{1}{8})$, we can choose $\delta \le \varepsilon$ such that $\|1-2v\|\|f(y)\| < \varepsilon$ whenever $\|1-2v\| < 1+\delta$ and $\|v - v^2\| = \|x\| < \delta$.

Finally, let $v_t := v + (1-2v)f(ty)$ for $0 \le t \le 1$, and take $e := v_1$. Since $f(0) = 0$, we get $v_0 = v$. Our estimates show that $\|e - v\| = \|(1-2v)f(y)\| < \varepsilon$. By holomorphic functional calculus, $v \in \mathcal{A}$ implies that x, y, v_t, e all lie in $\mathcal{A}$, too. We compute

$$\begin{aligned} v_t^2 - v_t &= (v + (1-2v)f(ty))^2 - (v + (1-2v)f(ty)) \\ &= v^2 - v - (1-2v)^2 f(ty) + (1-2v)^2 f(ty)^2 \\ &= v^2 - v + (1-2v)^2 \big(f(ty)^2 - f(ty)\big) \\ &= v^2 - v + (1-2v)^2 ty = (v^2 - v)(1-t), \end{aligned}$$

and in particular $e^2 - e = 0$, as required. □

Lemma 6.25 says that in a unital Fréchet pre-C*-algebra $\mathcal{A}$, an "almost idempotent" $v \in \mathcal{A}$ that is not far from being a projector (since $\|1-2v\|$ is close to 1) can be retracted to a genuine idempotent in $\mathcal{A}$. The next Lemma says that projectors in the C^*-completion of $\mathcal{A}$ can be approximated by projectors lying in $\mathcal{A}$.

Lemma 6.26. *Let $\mathcal{A}$ be an unital Fréchet pre-C*-algebra, whose C^*-completion is A. If $\tilde{q} = \tilde{q}^2 = \tilde{q}^*$ is a projector in A, then for any $\varepsilon > 0$, we can find a projector $q = q^2 = q^* \in \mathcal{A}$ such that $\|q - \tilde{q}\| < \varepsilon$.*

Proof. For a suitable $\delta \in (0,1)$, to be chosen later, we can find $v \in \mathcal{A}$ such that $v^* = v$ and $\|v - \tilde{q}\| < \delta$, because $\mathcal{A}$ is dense in A. Now

$$\|v^2 - v\| \le \|v^2 - \tilde{q}^2 + \tilde{q} - v\| \le (\|v + \tilde{q}\| + 1)\|v - \tilde{q}\| < (3+\delta)\delta < 4\delta,$$

and

$$\|1 - 2v\| \le \|1 - 2\tilde{q}\| + 2\|\tilde{q} - v\| < 1 + 2\delta.$$

Lemma 6.25 now provides an idempotent $e = e^2 \in \mathcal{A}$ such that $\|e - v\| < \varepsilon/4$, for δ small enough (in particular, we must take $\delta < \varepsilon/4$). To replace e by a projector q, we may use Kaplansky's formula (in the C*-algebra A: see [fgv01, p. 88], for example) to define

$$q := ee^*(ee^* + (1 - e^*)(1 - e))^{-1}.$$

Indeed, $ee^* + (1 - e^*)(1 - e) = 1 + (e - e^*)(e^* - e) \geq 1$ in A, so it is invertible in A, and thus also in $\mathcal{A}$ because $\mathcal{A}$ is a pre-C*-algebra. Thus $q \in \mathcal{A}$. One checks that $q^* = q$ and $q^2 = q$. Note also that $eq = q$.

If $\mathcal{A}$ is represented faithfully on a Hilbert space $\mathcal{H}$, we can decompose $\mathcal{H}$ as $q\mathcal{H} \oplus (1 - q)\mathcal{H}$. With respect to this decomposition, we can write

$$q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & T \\ 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} R & V \\ V^* & S \end{pmatrix},$$

where $R = R^* \in \mathcal{L}(q\mathcal{H})$, $S = S^* \in \mathcal{L}((1 - q)\mathcal{H})$, and $V, T \colon (1 - q)\mathcal{H} \to q\mathcal{H}$ are bounded.

Now $\|e - v\| < \varepsilon/4$, so $\|(v - e)^*(v - e)\| < \varepsilon^2/16$; it follows that

$$\|(R - 1)^2 + VV^*\| < \frac{\varepsilon^2}{16}, \qquad \|(V - T)^*(V - T) + S^2\| < \frac{\varepsilon^2}{16}.$$

Thus $\|VV^*\| < \varepsilon^2/16$, i.e., $\|V\| < \varepsilon/4$, and likewise $\|V - T\| < \varepsilon/4$. Therefore, $\|q - e\| = \|T\| < \varepsilon/2$. Finally,

$$\|q - \tilde{q}\| \leq \|q - e\| + \|e - v\| + \|v - \tilde{q}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \delta \leq \varepsilon. \qquad \square$$

Theorem 6.27. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, in which $\mathcal{A}$ is a unital Fréchet pre-C*-algebra; and assume that $\mathcal{A}$ is* commutative. *Let $X = M(A)$ be the character space of A, a compact Hausdorff space such that $A \cong C(X)$. Then, for each finite open cover $\{U_1, \dots, U_m\}$ of X, we can choose a subordinate partition of unity $\{\phi_1, \dots, \phi_m\}$:*

$$\phi_k \in C(X), \quad 0 \leq \phi_k \leq 1, \quad \operatorname{supp} \phi_k \subset U_k, \quad \phi_1 + \dots + \phi_m = 1,$$

in such a way that $\phi_k \in \mathcal{A}$ for $k = 1, 2, \dots, m$.

Remark 6.28. By definition, $X = M(A)$ is the set of all nonzero $*$-homomorphisms $\phi\colon A \to \mathbb{C}$. Note that if $\phi \in M(A)$, then $\phi(1)^2 = \phi(1)$ implies $\phi(1) = 1$. Gelfand's theorem provided a $*$-isomorphism of unital C*-algebras from A onto $C(X)$, so that we can regard elements of A as continuous functions on X.

Now let $\phi \in M(\mathcal{A})$. Recall that $\operatorname{sp}_{\mathcal{A}}(a) = \operatorname{sp}_A(a)$ for all $a \in \mathcal{A}$, since $\mathcal{A}$ is a pre-C*-algebra. Because $(a - \lambda 1)b = 1$ in $\mathcal{A}$ implies $(\phi(a) - \lambda)\phi(b) = 1$, we see that $\lambda \notin \operatorname{sp}_A(a) \implies \lambda \neq \phi(a)$; therefore, $\phi(a) \in \operatorname{sp}_A(a)$ for all $\phi \in M(\mathcal{A})$. Thus $|\phi(a)| \leq r(a) \leq \|a\|_A$, where $r(a)$ is the spectral radius of a, so that ϕ extends to A by continuity. This means that $M(\mathcal{A}) = M(A) = X$, so that we can regard X as the character space of the pre-C*-algebra $\mathcal{A}$.

Proof of Theorem 6.27. We first choose a partition of unity $\{\tilde{\phi}_1, \dots, \tilde{\phi}_m\}$ in $C(X) = A$ subordinate to $\{U_1, \dots, U_m\}$. Let $\tilde{q} \in M_m(A)$ be the matrix whose (j, k)-entry is $\sqrt{\tilde{\phi}_j \tilde{\phi}_k}$. Then $\tilde{q} = \tilde{q}^2 = \tilde{q}^*$. Choose $\varepsilon \in (0, 1/m)$. We apply Lemma 6.26 to the pre-C*-algebra $M_m(\mathcal{A}) \subset M_m(A)$, to get $q = q^2 = q^* \in M_m(\mathcal{A})$, with $\|q - \tilde{q}\| < \varepsilon$.

Let $\psi_i := q_{ij} \in \mathcal{A}$ for $j = 1, \dots, m$. Then $\psi_1 + \cdots + \psi_m = \operatorname{tr} q$, and $\| \operatorname{tr} q - \operatorname{tr} \tilde{q} \| \leq \sum_{j=1}^m \| q_{jj} - \tilde{q}_{jj} \| \leq m\varepsilon < 1$, so that $\operatorname{tr} q = \operatorname{tr} \tilde{q}$, because $\tilde{q} \mapsto \operatorname{tr} \tilde{q}$ is an integer-valued function on X (and $\operatorname{tr} \tilde{q}$ is the rank of the vector bundle corresponding to the projector $\tilde{q}$). Thus,

$$\psi_1 + \cdots + \psi_m = \operatorname{tr} q = \operatorname{tr} \tilde{q} = \phi_1 + \cdots + \phi_m = 1.$$

Furthermore, $0 \leq \psi_k \leq 1$ since ψ_k is the (k,k)-element of $q = q^* q \in M_m(A)$, and thus $\{\psi_1, \dots, \psi_m\}$ is a partition of unity with elements in $\mathcal{A}$. We can modify $\{\psi_k\}$ to get $\{\phi_k\}$ that will be subordinated to $\{U_k\}$, as follows.

Let $g \colon \mathbb{R} \to [0,1]$ be smooth with $\operatorname{supp} g \subseteq [\varepsilon, 1+\varepsilon]$, and $g(t) > 0$ for $\varepsilon < t \leq 1$. Now $V_k := \{\, x \in X : \psi_k(x) > \varepsilon \,\} \subset U_k$, since $\| \psi_k - \phi_k \| < \varepsilon$. Let $\chi_k := g \circ \psi_k$. By the smooth functional calculus, we find that $\chi_k \in \mathcal{A}$, for $k = 1, \dots, m$, and $\sum_{j=1}^m \chi_j(x) > 0$ for all $x \in X$, since otherwise $\sum_{j=1}^m \psi_j(x) \leq m\varepsilon < 1$, impossible. Therefore, $\chi_1 + \cdots + \chi_m$ is invertible in $C(X) = A$, and thus also in $\mathcal{A}$, so we can take $\phi_k := \chi_k(\chi_1 + \cdots + \chi_m)^{-1} \in \mathcal{A}$, having $\operatorname{supp} \phi_k \subset U_k$. Now $\{\phi_1, \dots, \phi_m\}$ is the desired partition of unity. $\square$

6.4 Real spectral triples

Recall that a spin structure on an oriented compact manifold (M, ε) is represented by a pair $(\mathcal{S}, C)$, where $\mathcal{S}$ is a $\mathcal{B}$-$\mathcal{A}$-bimodule and, according to Proposition 2.18, $C \colon \mathcal{S} \to \mathcal{S}$ is an antilinear map such that $C(\psi\, a) = C(\psi)\, \bar{a}$ for $a \in \mathcal{A}$; $C(b\, \psi) = \chi(\bar{b})\, C(\psi)$ for $b \in \mathcal{B}$; and, by choosing a metric g on M, which determines a Hermitian pairing on $\mathcal{S}$, we can also require that $(C\phi | C\psi) = (\psi | \phi) \in \mathcal{A}$ for $\phi, \psi \in \mathcal{S}$. $\mathcal{S}$ may be completed to a Hilbert space $\mathcal{H} = L^2(M, S)$, with scalar product $\langle \phi \mid \psi \rangle = \int_M (\phi \mid \psi)\, \nu_g$. It is clear that C extends to a bounded antilinear operator on $\mathcal{H}$ such that $\langle C\phi \mid C\psi \rangle = \langle \psi \mid \phi \rangle$ by integration with respect to ν_g, so that (the extended version of) C is *antiunitary* on $\mathcal{H}$. Moreover, the Dirac operator is $\not{D} = -i\hat{c} \circ \nabla^S$, where by construction the spin connection ∇^S commutes with C: that is, ∇^S_X commutes with C, for each $X \in \mathfrak{X}(M)$.

The property $C(\psi a) = C(\psi)\, \bar{a}$ shows that, for each $x \in X$, $\psi(x) \mapsto C(\psi)(x)$ is an antilinear operator C_x on the fibre S_x of the spinor bundle, which is a Fock space with $\dim_{\mathbb{C}} S_x = 2^m$. Thus, to determine whether C commutes with $\not{D}$ or not, we can work with the local representation $\not{D} = -i\gamma^\alpha\, \nabla^S_{E_\alpha}$. Here $\gamma^\alpha = c(\theta^\alpha)$, for $\alpha = 1, \dots, n$, is a local section of the Clifford algebra bundle $\mathbb{C}\mathrm{l}(T^*M) \to M$, and the property $C(b\psi) = \chi(\bar{b})\, C(\psi)$ says that $C(\gamma^\alpha \psi) = -\gamma^\alpha\, C(\psi)$ whenever ψ is supported on a local chart domain.

However, replacing b by $\gamma^\alpha \in \Gamma(U, \mathbb{C}\mathrm{l}^1(T^*M))$ is only allowed when the dimension n is even. In the odd case, $\mathcal{B}$ consists of sections of the bundle $\mathbb{C}\mathrm{l}^0(T^*M)$, and we can only write relations like $C(\gamma^\alpha \gamma^\beta \psi) = \gamma^\alpha \gamma^\beta\, C(\psi)$ for $\psi \in \Gamma(U, S)$. But since C is antilinear, in the even case we get

$$C \not{D} \psi = C(-i\gamma^\alpha\, \nabla^S_{E_\alpha} \psi) = iC(\gamma^\alpha\, \nabla^S_{E_\alpha} \psi) = -i\gamma^\alpha\, C(\nabla^S_{E_\alpha} \psi) = \not{D} C \psi.$$

Thus $[\not{D}, C] = 0$ on $\mathcal{H}$, when $n = 2m$ is even.

What happens in the odd-dimensional case? Consider what happens on a single fibre S_x, which carries a selfadjoint representation of $B_x = \mathbb{C}\mathrm{l}^0(T^*_x M)$. Recall that we use the convention that $c(\omega) := c(\omega\gamma)$ to extend the action of $\mathcal{B}$ to all of $\Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$, where $\gamma = (-i)^m \theta^1 \dots \theta^{2m+1}$ is the chirality element. For $\omega = \theta^\alpha$, then gives $C\,c(\omega)\,C^{-1} = C\,c(\omega\gamma)\,C^{-1} = c(\chi(\overline{\omega\gamma})) = c(\overline{\omega\gamma})$ since $\omega\gamma$ is even for ω odd, and $\overline{\omega\gamma} = i^m\,\theta^\alpha\theta^1\dots\theta^{2m+1} = (-1)^m\omega\gamma$. We conclude that $C\,c(\omega)\,C^{-1} = (-1)^m\,c(\omega)$ for $\omega \in \mathcal{A}^1(M)$ real, and therefore $C\not{D} = (-1)^{m+1}\,\not{D}C$ by antilinearity of C. We sum up:

$$C\not{D} = \begin{cases} +\not{D}C, & \text{if } n \not\equiv 1 \mod 4, \\ -\not{D}C, & \text{if } n \equiv 1 \mod 4. \end{cases}$$

In the even case, $\mathcal{B} = \Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$ contains the operator $\Gamma = c(\gamma)$ which extends to a selfadjoint unitary operator on $\mathcal{H}$. Recall from Definition 1.18 that $c_J(\gamma)$ is the $\mathbb{Z}_2$-grading operator on the Fock space $\Lambda^\bullet W_J$, the model for S_x. If $\mathcal{H}^\pm = L^2(M, S^\pm)$ denotes the completion of $\mathcal{S}^\pm$ in the norm of $\mathcal{H}$, then $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, with Γ being the $\mathbb{Z}_2$-grading operator. Now γ is even and $\bar\gamma = (-1)^m\,\gamma$ as before, so that $C\Gamma = (-1)^m\Gamma C$ whenever $n = 2m$.

When M is a connected manifold, there is a third sign associated with C, since we know that $C^2 = \pm 1$. Once more, the sign can be found by examining the case of a single fibre S_x, so we ask whether an irreducible representation S of $\mathbb{C}\mathrm{l}(V)$ admits an *antiunitary conjugation* $C\colon S \to S$ such that $C\,c_J(v)\,C^{-1} = \pm c_J(v)$ for $v \in V$ (plus sign if $\dim V = 1 \mod 4$) and either $C^2 = +1$ or $C^2 = -1$. By periodicity of the Clifford algebras, the sign depends only on $n \bmod 8$, where $n = \dim V$.

Note that if $\{\gamma^1, \dots, \gamma^n\}$ generate $\mathrm{Cl}_{n,0}$, then $\{-i\gamma^1, \dots, -i\gamma^n\}$ generate $\mathrm{Cl}_{0,n} = \mathrm{Cl}(\mathbb{R}^n, g)$ with g negative-definite. Thus one can equally well work with $\mathrm{Cl}_{0,q}$, for $q = 0, 1, \dots, 7$. Since $\mathrm{Cl}_{p,0} \otimes_\mathbb{R} M_N(\mathbb{R}) \cong \mathrm{Cl}_{0,8-p} \otimes_\mathbb{R} M_{N'}(\mathbb{R})$ for $p = 0, 1, \dots, 7$ and suitable matrix sizes N, N', we get, from our classification (1.4) of the Clifford algebras $\mathrm{Cl}_{p,0}$:

- for $q \equiv 0, 6, 7 \bmod 8$, $\mathrm{Cl}_{0,q}$ is an algebra over $\mathbb{R}$,
- for $q \equiv 1, 5 \bmod 8$, $\mathrm{Cl}_{0,q}$ is an algebra over $\mathbb{C}$,
- for $q \equiv 2, 3, 4 \bmod 8$, $\mathrm{Cl}_{0,q}$ is an algebra over $\mathbb{H}$.

On a case-by-case basis, using this classification, one finds that $C^2 = -1$ if and only if $n = 2, 3, 4, 5 \bmod 8$.

Exercise 6.29. *Find five matrices $\varepsilon_1, \dots, \varepsilon_5 \in M_4(\mathbb{C})$, generating a representation of* Cl_{05}*, and an antiunitary operator C on $\mathbb{C}^4$ such that $C\,\varepsilon_j\,C^{-1} = -\varepsilon_j$ for $j = 1, \dots, 5$. Show that C is unique up to multiples $C \mapsto \lambda C$ with $\lambda \in \mathbb{C}$ and $|\lambda| = 1$; and that $C^2 = -1_4$.*

Summary: There are two tables of signs

$n \bmod 8$	0	2	4	6
$C^2 = \pm 1$	$+$	$-$	$-$	$+$
$C\not{D} = \pm\not{D}C$	$+$	$+$	$+$	$+$
$C\Gamma = \pm\Gamma C$	$+$	$-$	$+$	$-$

$n \bmod 8$	1	3	5	7
$C^2 = \pm 1$	$+$	$-$	$-$	$+$
$C\not{D} = \pm\not{D}C$	$-$	$+$	$-$	$+$

There is a deeper reason why only these signs can occur, and why they depend on $n \bmod 8$: the data set $(\mathcal{A}, \mathcal{H}, \not{D}, C, \Gamma)$ determines a class in the "Real"

KR-homology $\mathrm{KR}^\bullet(\mathcal{A})$, and $\mathrm{KR}^{j+8}(\mathcal{A}) \cong \mathrm{KR}^j(\mathcal{A})$ by Bott periodicity. We leave this story for Prof. Brodzki's course. (But see [fgv01, Sec. 9.5] for a pedestrian approach.)

"Real" KR-homology is a theory for algebras with involution: in the *commutative* case, we may just take $a \mapsto a^*$, and we ask that $C\,a\,C^{-1} = a^*$ i.e., that C implement the involution. This is trivial for the manifold case, since $C(\psi a) = C(\psi)\bar{a} =: a^* C(\psi)$, the a^* here being multiplication by $\bar{a}$.

In the *noncommutative* case, the operator Ca^*C^{-1} would generate a second representation of $\mathcal{A}$, in fact an antirepresentation (that is, a representation of the opposite algebra $\mathcal{A}^{\mathrm{op}}$) and we should require that this commute with the original representation of $\mathcal{A}$.

Definition 6.30. *A* real spectral triple *is a spectral triple* $(\mathcal{A}, \mathcal{H}, D)$*, together with an antiunitary operator* $J\colon \mathcal{H} \to \mathcal{H}$ *such that* $J(\operatorname{Dom} D) \subset \operatorname{Dom} D$*, and* $[a, Jb^*J^{-1}] = 0$ *for all* $a, b \in \mathcal{A}$*.*

Definition 6.31. *A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is* even *if there is a selfadjoint unitary operator* Γ *on* $\mathcal{H}$ *such that* $a\Gamma = \Gamma a$ *for all* $a \in \mathcal{A}$*,* $\Gamma(\operatorname{Dom} D) = \operatorname{Dom} D$*, and* $D\Gamma = -\Gamma D$*. If no such* $\mathbb{Z}_2$*-grading operator* Γ *is given, we say that the spectral triple is* odd*.*

We have seen that in the standard commutative example, the *even* case arises when the auxiliary algebra $\mathcal{B}$ contains a natural $\mathbb{Z}_2$-grading operator, and this happens exactly when *the manifold dimension is even*. Now, the manifold dimension is determined by the spectral growth of the Dirac operator, and this spectral version of dimension may be used for noncommutative spectral triples, too. To make this more precise, we must look more closely at spectral growth.

6.5 Summability of spectral triples

Definition 6.32. *For* $1 < p < \infty$*, there is an operator ideal* $\mathcal{L}^{p+}(\mathcal{H}) = \mathcal{L}^{p,\infty}(\mathcal{H})$*, defined as follows:*

$$\mathcal{L}^{p+}(\mathcal{H}) := \{\, T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(N^{(p-1)/p}) \quad \text{as } N \to \infty \,\},$$

with norm $\|T\|_{p+} := \sup_{N \geq 1} \sigma_N(T)/N^{(p-1)/p}$*.*

For instance, if $A \geq 0$ with $s_k(A) := \dfrac{1}{(k+1)^{1/p}}$, then $A \in \mathcal{L}^{p+}$ by the integral test:

$$\sigma_N(A) \sim \int_1^N t^{-1/p}\,dt \sim \frac{p}{p-1}\,N^{(p-1)/p}, \quad \text{as } N \to \infty.$$

Indeed, since $p > 1$, $T \in \mathcal{L}^{p+}$ implies $s_k(T) = O((k+1)^{-1/p})$. To see that, recall that $s_0(T) + \cdots + s_k(t) = \sigma_{k+1}(T)$; since $\{s_k(T)\}$ is decreasing, this implies $(k+1)s_k(T) \leq \sigma_{k+1}(T)$, and thus $s_k(T) \leq \frac{1}{k+1}\,\sigma_{k+1}(T) \leq C(k+1)^{-1/p}$ for some constant C.

Therefore, $T \in \mathcal{L}^{p+}$ implies $s_k(T^p) = O(\frac{1}{k+1})$ and then $\sigma_N(T^p) = O(\log N)$, so that $T^p \in \mathcal{L}^{1+}$, which serves to justify the notation $\mathcal{L}^{p+}$. It turns out, however, that there are, for any $p > 1$, positive operators $B \in \mathcal{L}^{1+}$ such that $B^{1/p} \notin \mathcal{L}^{p+}$, so the implication "$T \in \mathcal{L}^{p+} \implies T^p \in \mathcal{L}^{1+}$" is a one-way street. For an example, see [fgv01, Sec. 7.C].

Definition 6.33. *A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is* $\boldsymbol{p}^+$-summable *for some* p *with* $1 \leq p < \infty$ *if* $(D^2+1)^{-1/2} \in \mathcal{L}^{p+}(\mathcal{H})$*. If* D *is invertible, this is equivalent to requiring* $|D|^{-1} \in \mathcal{L}^{p+}(\mathcal{H})$*.*

Definition 6.34. *Let* $p \in [1, \infty)$*. A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *has* spectral dimension $\boldsymbol{p}$ *if it is* p^+*-summable and moreover*

$$0 < \operatorname{Tr}_\omega((D^2+1)^{-p/2}) < \infty \quad \textit{for any Dixmier trace } \operatorname{Tr}_\omega\,.$$

If D *is invertible, this is equivalent to* $0 < \operatorname{Tr}_\omega(|D|^{-p}) < \infty$ *for any* Tr_ω*.*

For positivity of all Dixmier traces, it suffices that $\liminf_{N\to\infty} \frac{1}{\log N} \sigma_N((D^2+1)^{-p/2}) > 0$. Note that, in view of Corollary 6.8, this can happen for *at most one value* of p.

Proposition 6.35. *If* $(\mathcal{A}, \mathcal{H}, D)$ *is a* p^+*-summable spectral triple, with* D *invertible, let*

$$F := D\,|D|^{-1} \tag{6.12}$$

be the phase *of the selfadjoint operator* D*. Then, for each* $a \in \mathcal{A}$*, the commutator* $[F, a]$ *lies in* $\mathcal{L}^{p+}(\mathcal{H})$*.*

Proof. First we show that $[F, a] \in \mathcal{K}(\mathcal{H})$, using the spectral formula (6.6) for $|D|^{-1}$. Indeed,

$$\begin{aligned}
[F, a] &= [D\,|D|^{-1}, a] = [D, a]\,|D|^{-1} + D\,[|D|^{-1}, a] \\
&= \frac{2}{\pi} \int_0^\infty \big([D, a]\,(D^2+\mu)^{-1} + D\,[(D^2+\mu)^{-1}, a]\big)\,d\mu \\
&= \frac{2}{\pi} \int_0^\infty \big([D, a]\,(D^2+\mu)^{-1} - D(D^2+\mu)^{-1}\,[D^2, a]\,(D^2+\mu)^{-1}\big)\,d\mu \\
&= \frac{2}{\pi} \int_0^\infty \big(\mu^2(D^2+\mu)^{-1}\,[D, a]\,(D^2+\mu)^{-1} - D(D^2+\mu)^{-1}\,[D, a]\,D(D^2+\mu)^{-1}\big)\,d\mu.
\end{aligned} \tag{6.13}$$

In the integrand, $[D, a]$ is bounded by the hypothesis that $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. Next, $(D^2+\mu)^{-1} = (D - i\mu)^{-1}(D + i\mu)^{-1} \in \mathcal{K}(\mathcal{H})$ and

$$D(D^2+\mu)^{-1} = \underbrace{D(D^2+\mu)^{-\frac{1}{2}}}_{\in B(\mathcal{H})}\;\underbrace{(D^2+\mu)^{-\frac{1}{2}}}_{\in \mathcal{K}(\mathcal{H})}$$

is also compact. Thus the integrand lies in $\mathcal{K}(\mathcal{H})$ for each μ, hence $[F, a] \in \mathcal{K}(\mathcal{H})$, that is, the integral converges in the norm of this C*-algebra.

Next to show that $[F, a] \in \mathcal{L}^{p+}(\mathcal{H})$, we may assume that $a^* = -a$, since

$$i\,[F, a] = [F, \tfrac{i}{2}(a^* + a)] - i\,[F, \tfrac{1}{2}(a^* - a)].$$

Note that this assumption implies that the bounded operators $[F, a]$ and $[D, a]$ are selfadjoint.

If we *replace* the term $[D,a]$ by its norm $\|[D,a]\|$ on the right hand side of (6.13), this integral changes into

$$\begin{aligned}
&\frac{2}{\pi}\int_0^\infty \big(\mu^2(D^2+\mu)^{-1}\,\|[D,a]\|\,(D^2+\mu)^{-1} - D(D^2+\mu)^{-1}\,\|[D,a]\|\,D(D^2+\mu)^{-1}\big)\,d\mu \\
&= \frac{2}{\pi}\,\|[D,a]\|\int_0^\infty \big((\mu^2(D^2+\mu)^{-2} + D^2(D^2+\mu)^{-2})\big)\,d\mu \\
&\le \frac{2}{\pi}\,\|[D,a]\|\int_0^\infty \big((\mu^2(D^2+\mu)^{-2} + D^2(D^2+\mu)^{-2})\big)\,d\mu \\
&= \frac{2}{\pi}\|[D,a]\|\int_0^\infty (D^2+\mu)^{-1}\,d\mu = \|[D,a]\|\,|D|^{-1},
\end{aligned}$$

where these are inequalities among selfadjoint elements of the C*-algebra $\mathcal{K}(\mathcal{H})$. Therefore, if we plug in the order relation

$$-\|[D,a]\| \le [D,a] \le \|[D,a]\|$$

among selfadjoint elements of $B(\mathcal{H})$ into the right hand side of (6.13), we obtain the operator inequalities

$$-\|[D,a]\|\,|D|^{-1} \le [F,a] \le \|[D,a]\|\,|D|^{-1}.$$

Thus the singular values of $[F,a]$ are dominated by those of $|D|^{-1}$. We now conclude that $|D|^{-1} \in \mathcal{L}^{p+}$ implies $[F,a] \in \mathcal{L}^{p+}$, for all $a \in \mathcal{A}$. □

The assumption that D is invertible in the statement of Proposition 6.35 is not essential (though the proof does depend on it, of course). With some extra work, we can modify the proof to show that $(D^2+1)^{-1/2} \in \mathcal{L}^{p+}$ implies that all $[F,a] \in \mathcal{L}^{p+}$, where F is redefined to mean $F := D\,(D^2+1)^{-1/2}$, in contrast to (6.12). This is proved in [cprs04], in full generality.

Chapter 7

Spectral triples: Examples

7.1 Geometric conditions on spectral triples

We begin by listing a set of requirements on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, whose algebra $\mathcal{A}$ is unital but not necessarily commutative, such that $(\mathcal{A}, \mathcal{H}, D)$ provides a "spin geometry" generalization of our "standard commutative example" $(C^\infty(M), L^2(M, S), D\!\!\!/)$. Again we shall assume, for convenience, that D is invertible.

Condition 1 (Spectral dimension)**.** *There is an* integer $n \in \{1, 2, \dots\}$*, called the spectral dimension of* $(\mathcal{A}, \mathcal{H}, D)$*, such that* $|D|^{-1} \in \mathcal{L}^{n+}(\mathcal{H})$*, and* $0 < \operatorname{Tr}_\omega(|D|^{-n}) < \infty$ *for any Dixmier trace* Tr_ω*.*

When n *is* even*, the spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is also even: that is, there exists a selfadjoint unitary operator* $\Gamma \in B(\mathcal{H})$ *such that* $\Gamma(\operatorname{Dom} D) = \operatorname{Dom} D$*, satisfying* $a\Gamma = \Gamma a$ *for all* $a \in \mathcal{A}$*, and* $D\Gamma = -\Gamma D$*.*

Remark 7.1. It is useful to allow the case $n = 0$ as a possible spectral dimension. There are two cases to consider:

- $\mathcal{H}$ is an infinite-dimensional Hilbert space, but the spectrum of the operator D has exponential growth, so that $N_{|D|}(\lambda) = O(\lambda^\varepsilon)$ as $\lambda \to \infty$, for any exponent $\varepsilon > 0$. This is what happens in the example by Dąbrowski and Sitarz [ds03] of a spectral triple on the standard Podleś sphere $\mathbb{S}^2_q$ with $0 < q < 1$, where the operator D has the same eigenvalue multiplicities as the Dirac operator on $\mathbb{S}^2$ (see Section 8.2), but the eigenvalue $\pm(l + \frac{1}{2})$ is replaced by $\pm(q^{-l-\frac{1}{2}} - q^{l+\frac{1}{2}})/(q^{-1} - q)$, for $l = \frac{1}{2}, \frac{3}{2}, \dots$.
- $\mathcal{H}$ is finite-dimensional, $\mathcal{A}$ is a finite-dimensional matrix algebra, and D is a hermitian matrix. In this case, we assign to $(\mathcal{A}, \mathcal{H}, D)$ the spectral dimension $n = 0$, and replace the Dixmier traces Tr_ω by the ordinary matrix trace tr.

Condition 2 (Regularity)**.** *For each* $a \in \mathcal{A}$*, the bounded operators* a *and* $[D, a]$ *lie in the smooth domain* $\bigcap_{k\geq 1} \operatorname{Dom} \delta^k$ *of the derivation* $\delta \colon T \mapsto [|D|, T]$*.*

Moreover, $\mathcal{A}$ *is complete in the topology given by the seminorms* $q_k \colon a \mapsto \|\delta^k(a)\|$ *and* $q'_k \colon a \mapsto \|\delta^k([D, a])\|$*. This ensures that* $\mathcal{A}$ *is a Fréchet pre-*C^**-algebra.*

Condition 3 (Finiteness)**.** *The subspace of smooth vectors* $\mathcal{H}^\infty := \bigcap_{k\in\mathbb{N}} \operatorname{Dom} D^k$ *is a* finitely generated projective *left* $\mathcal{A}$*-module.*

This is equivalent to saying that, for some $N \in \mathbb{N}$*, there is a projector* $p = p^2 = p^*$ *in* $M_N(\mathcal{A})$ *such that* $\mathcal{H}^\infty \cong \mathcal{A}^N p$ *as left* $\mathcal{A}$*-modules.*

Condition 4 (Real structure)**.** *There is an antiunitary operator* $J : \mathcal{H} \to \mathcal{H}$ *satisfying* $J^2 = \pm 1$*,* $JDJ^{-1} = \pm D$*, and* $J\Gamma = \pm \Gamma J$ *in the even case, where the signs depend only on* $n \bmod 8$ *(and thus are given by the table of signs for the standard commutative examples). Moreover,* $b \mapsto Jb^*J^{-1}$ *is an antirepresentation of* $\mathcal{A}$ *on* $\mathcal{H}$ *(that is, a representation of the opposite algebra* $\mathcal{A}^{\mathrm{op}}$*), which commutes with the given representation of* $\mathcal{A}$*:*

$$[a, Jb^*J^{-1}] = 0, \quad \text{for all } a, b \in \mathcal{A}.$$

Condition 5 (First order)**.** *For each* $a, b \in \mathcal{A}$*, the following relation holds:*

$$[[D, a], Jb^*J^{-1}] = 0, \quad \text{for all } a, b \in \mathcal{A}.$$

This generalizes, to the noncommutative context, the condition that D *be a first-order differential operator.*

Since

$$[[D, a], Jb^*J^{-1}] = [[D, Jb^*J^{-1}], a] + [D, \underbrace{[a, Jb^*J^{-1}]}_{=0}],$$

this is equivalent to the condition that $[a, [D, Jb^*J^{-1}]] = 0$*.*

Condition 6 (Orientation)**.** *There is a Hochschild* n*-cycle*

$$\mathbf{c} = \textstyle\sum_j (a_j^0 \otimes b_j^0) \otimes a_j^1 \otimes \cdots \otimes a_j^n \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}),$$

such that

$$\pi_D(\mathbf{c}) \equiv \textstyle\sum_j a_j^0 (Jb_j^{0*}J^{-1})\, [D, a_j^1] \dots [D, a_j^n] = \begin{cases} \Gamma, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases} \tag{7.1}$$

In many examples, including the noncommutative examples we shall meet in the next two sections, one can often take $b_j^0 = 1$, so that $\mathbf{c}$ may be replaced, for convenience, by the cycle $\sum_j a_j^0 \otimes a_j^1 \otimes \cdots \otimes a_j^n \in Z_n(\mathcal{A}, \mathcal{A})$. In the commutative case, where $\mathcal{A}^{\mathrm{op}} = \mathcal{A}$, this identification may be justified: the product map $m \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a homomorphism.

The data set $(\mathcal{A}, \mathcal{H}, D; \Gamma \text{ or } 1, J, \mathbf{c})$ satisfying these six conditions constitute a "noncommutative spin geometry". In the fundamental paper where these conditions were first laid out [c-a96], Connes added one more nondegeneracy condition (Poincaré duality in K-theory) as a requirement. We shall not go into this matter here.

To understand the orientation condition in the standard commutative example, we show that $\mathbf{c}$ arises from a *volume form* on the oriented compact manifold M. Choose a metric g on M and let ν_g be the corresponding Riemannian volume form. Furthermore, let $\{(U_j, a_j)\}$ be a finite atlas of charts on M, where $a_j \colon U_j \to \mathbb{R}^n$, and let $\{f_j\}$ be a partition of unity subordinate to the open cover $\{U_j\}$; then for $r = 1, \dots, n$, each $f_j a_j^r$ lies in $C^\infty(M)$ with

$\operatorname{supp}(f_j a_j^r) \subset U_j$. Over each U_j, let $\{\theta_j^1, \dots, \theta_j^n\}$ be a local orthonormal basis of 1-forms (with respect to the metric g). Then

$$\nu_g\big|_{U_j} = \theta_j^1 \wedge \cdots \wedge \theta_j^n = h_j \, da_j^1 \wedge \cdots \wedge da_j^n,$$

for some smooth functions $h_j \colon U_j \to \mathbb{C}$. We write $a_j^0 := (-i)^m f_j h_j \in C^\infty(M)$, where as usual, $n = 2m$ or $n = 2m+1$. With that notation, we get

$$(-i)^m \nu_g = (-i)^m \textstyle\sum_j f_j \big(\nu_g\big|_{U_j}\big) = \sum_j a_j^0 \, da_j^1 \wedge \cdots \wedge da_j^n.$$

Now we define

$$\mathbf{c} := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sum_j a_j^0 \otimes a_j^{\sigma(1)} \otimes \cdots \otimes a_j^{\sigma(n)}. \tag{7.2}$$

Exercise 7.2. *Show that the Hochschild boundary* $b\mathbf{c}$ *of the chain* (7.2) *is zero because* $\mathcal{A}$ *is commutative.*

Therefore, $\mathbf{c}$ is a Hochschild n-cycle in $Z_n(\mathcal{A}, \mathcal{A})$, for $\mathcal{A} = C^\infty(M)$. Its representative as a bounded operator on $\mathcal{H}$ is

$$\begin{aligned} \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sum_j a_j^0 \, [D\!\!\!/, a_j^{\sigma(1)}] \dots [D\!\!\!/, a_j^{\sigma(n)}] &= \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sum_j a_j^0 \, c(da_j^{\sigma(1)}) \dots c(da_j^{\sigma(n)}) \\ &= \frac{(-i)^m}{n!} \sum_j f_j \sum_{\sigma \in S_n} (-1)^\sigma c(\theta_j^{\sigma(1)}) \dots c(\theta_j^{\sigma(n)}) \\ &= \Big(\sum_j f_j \Big) (-i)^m \, c(\theta_j^1) \dots c(\theta_j^n) \\ &= c(\gamma) = \Gamma \text{ or } 1, \end{aligned}$$

since $c(\gamma) = \Gamma$ for $n = 2m$, and $c(\gamma) = 1$ for $n = 2m+1$.

This calculation shows that the elements $a_j^1, \dots, a_j^n$ occurring in the cycle c are local coordinate functions for M. An alternative approach would be to embed M in some $\mathbb{R}^N$ and take the a_j^r to be some of the cartesian coordinates of $\mathbb{R}^N$, regarded as functions on M. This is illustrated in the following example.

Example 7.3. By regarding the sphere $\mathbb{S}^2$ as embedded in $\mathbb{R}^3$,

$$\mathbb{S}^2 = \{\, (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \,\},$$

we can write down its volume form for the rotation-invariant metric g as

$$\nu = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy.$$

The corresponding Hochschild 2-cycle is

$$\mathbf{c} := -\frac{i}{2} \sum_{\text{cyclic}} (x \otimes y \otimes z - x \otimes z \otimes y),$$

summing over cyclic permutations of the letters x, y, z.

If $D\!\!\!/$ is the Dirac operator on $\mathbb{S}^2$ for this "round" metric and the unique spin structure on $\mathbb{S}^2$ compatible with its usual orientation (see Section 8.2), then

$$-\frac{i}{2} \sum_{\text{cyclic}} \big(x \, [D\!\!\!/, y] \, [D\!\!\!/, z] - x \, [D\!\!\!/, z] \, [D\!\!\!/, y] \big) = \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, which is the completion of the spinor module $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$.

Consider the following element of $M_2(\mathcal{A})$, with $\mathcal{A} = C^\infty(\mathbb{S}^2)$:

$$p := \frac{1}{2}\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}. \tag{7.3}$$

Note that $\operatorname{tr}(p - \frac{1}{2}) = 0$, where $\operatorname{tr}(a) := a_{11} + a_{22}$ means the matrix trace $\operatorname{tr}\colon M_2(\mathcal{A}) \to \mathcal{A}$.

Exercise 7.4. *Show that, if $\mathcal{A}$ is any $*$-algebra and $p \in M_2(\mathcal{A})$ is given by (7.3), then the projector relations $p = p^* = p^2$ are equivalent to the following relations among $x, y, z \in \mathcal{A}$:*

$$\begin{gathered} x^* = x, \quad y^* = y, \quad z^* = z, \\ [x,y] = [x,z] = [y,z] = 0, \\ x^2 + y^2 + z^2 = 1. \end{gathered}$$

Exercise 7.5. *Check that* $\operatorname{tr}(p\, dp \wedge dp) = -\frac{i}{2}\,\nu$.

If we replace $-\frac{i}{2}\,\nu$ by the Hochschild 2-cycle $\mathbf{c}$, the same calculation that solves the previous exercise also shows that $\pi_D(\mathbf{c}) = \Gamma$.

This computation has a deeper significance. One can show that the left $\mathcal{A}$-module $M_2(\mathcal{A})\,p$ is isomorphic to $\mathcal{E}_1$ in our classification of $\mathcal{A}$-modules of sections of line bundles over $\mathbb{S}^2$; and we have seen in Section 8.3 that $\mathcal{E}_1 \cong \Gamma(\mathbb{S}^2, L)$ where $L \to \mathbb{S}^2$ is the tautological line bundle. The first Chern class $c_1(L)$ equals (a standard multiple of) $[\nu] \in H^2_{\mathrm{dR}}(\mathbb{S}^2)$. One can trace a parallel relation between spin^c structures on $\mathbb{S}^2$ defined, via the principal $U(1)$-bundle $SU(2) \to \mathbb{S}^2$, on associated line bundles, and the Chern classes of each such line bundle. For that, we refer to [bhms07].

7.2 Isospectral deformations of commutative spectral triples

To some extent, one can recover the sphere $\mathbb{S}^2$ from spectral triple data alone. Thus, if $\mathcal{A}$ is a $*$-subalgebra of some C*-algebra containing elements x, y, z, and if the matrix

$$p = \frac{1}{2}\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix} \in M_2(\mathcal{A})$$

is a projector, i.e., $p = p^* = p^2$, then by Exercise 7.4, the elements x, y, z commute, they are selfadjoint, and they satisfy $x^2 + y^2 + z^2 = 1$. Thus, the commutative C*-algebra A generated by x, y, z is of the form $C(X)$, where $X \subseteq \mathbb{S}^2$ is a closed subset. If $\mathcal{A}$ is now a pre-C^*-subalgebra of A containing x, y, z, and is the algebra of some spectral triple $(\mathcal{A}, \mathcal{H}, D)$, then the extra condition $\pi_D(\mathbf{c}) = \Gamma$ can only hold if X is the support of the measure ν. This means that $X = \mathbb{S}^2$.

A similar argument can be tried, to obtain an "algebraic" description of $\mathbb{S}^4$. What follows is a heuristic motivation, following [cl01]. One looks for a projector $p \in M_4(\mathcal{A})$, of the form

$$p = \begin{pmatrix} 1+z & 0 & a & b \\ 0 & 1+z & -b^* & a \\ a^* & -b & 1-z & 0 \\ b^* & a^* & 0 & 1-z \end{pmatrix} = \begin{pmatrix} (1+z)1_2 & q \\ q^* & (1-z)1_2 \end{pmatrix}, \quad \text{where } q = \begin{pmatrix} a & b \\ -b^* & a \end{pmatrix}.$$

Then $p = p^*$ only if $z = z^*$, and then $p^2 = p$ implies that $-1 \leq z \leq 1$ in the C^*-completion A of $\mathcal{A}$, $\left[\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right), q\right] = 0$, and $qq^* = q^*q = \begin{pmatrix} 1 - z^2 & 0 \\ 0 & 1 - z^2 \end{pmatrix}$. From that, one finds that a, a^*, b, b^*, z commute, and $aa^* + bb^* = 1 - z^2$. Thus $A = C(X)$ with $X \subseteq \mathbb{S}^4$. Again, it can be shown that the equality $X = \mathbb{S}^4$ is reached by some extra conditions, namely,

$$\begin{aligned} \operatorname{tr}(p - \tfrac{1}{2}) &= 0, \\ \operatorname{tr}((p - \tfrac{1}{2})\, dp\, dp) &= 0 \quad \text{in } \Omega^2(\mathcal{A}), \\ \pi_D((p - \tfrac{1}{2})\, dp\, dp\, dp\, dp) &= \Gamma \quad \text{in } B(\mathcal{H}). \end{aligned}$$

However, if one takes instead $q := \begin{pmatrix} a & b \\ -\bar{\lambda} b^* & a^* \end{pmatrix}$ with $\lambda = e^{2\pi i\theta}$, then there is another, noncommutative, solution [cl01]: now A is the C*-algebra generated by a, b and $z = z^*$, where z is central, and the other relations are

$$\begin{gathered} ab = e^{-2\pi i\theta} ba, \qquad a^* b = e^{2\pi i\theta} ba^*, \\ aa^* = a^* a, \qquad bb^* = b^* b, \qquad aa^* + bb^* = 1 - z^2. \end{gathered} \tag{7.4}$$

To find a solution to these relations, where the central element z is taken to be a scalar multiple of 1, we substitute

$$\begin{aligned} a &= u \sin\psi \cos\phi \\ b &= v \sin\psi \cos\phi \\ z &= (\cos\psi)\, 1 \end{aligned}$$

with $-\pi \leq \psi \leq \pi$ and $-\pi < \phi \leq \pi$, say. In this way, the commutation relations (7.4) reduce to

$$uu^* = u^* u = 1, \qquad vv^* = v^* v = 1, \qquad vu = e^{2\pi i\theta} uv.$$

These are the relations for the unitary generators of a noncommutative 2-torus: see Section 8.4. Thus, by fixing values of ϕ, ψ with $\psi \neq \pm\pi$ and $\phi \notin \frac{\pi}{2}\mathbb{Z}$, we get a homomorphism from A to $C(\mathbb{T}^2_\theta)$, the C*-algebra of the noncommutative 2-torus with parameters $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \in M_2(A)$.

We look for a suitable algebra $\mathcal{A}$, generated by elements satisfying the above relations, by examining a *Moyal deformation* of $C^\infty(\mathbb{S}^4)$. One should first note that $\mathbb{S}^4 \subset \mathbb{R}^5 = \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ carries an obvious action of $\mathbb{T}^2$, namely,

$$(t_1, t_2) \cdot (\alpha, \beta, z) := (t_1 \alpha, t_2 \beta, z), \quad \text{for } |t_1| = |t_2| = 1,$$

which preserves the defining relation $\alpha\bar{\alpha} + \beta\bar{\beta} + z^2 = 1$ of $\mathbb{S}^4$. The action is *not free*: there are two fixed points $(0, 0, \pm 1)$, and for each t with $-1 < t < 1$ there are two circular orbits, namely $\{\, (\alpha, 0, t) : \alpha\bar{\alpha} = 1 - t^2 \,\}$ and $\{\, (\beta, 0, t) : \beta\bar{\beta} = 1 - t^2 \,\}$. The remaining orbits are copies of $\mathbb{T}^2$. The construction which follows will produce a "noncommutative space" $\mathbb{S}^4_\theta$ that can be thought of as the sphere $\mathbb{S}^4$ with each principal orbit $\mathbb{T}^2$ replaced by a noncommutative torus $\mathbb{T}^2_\theta$, while the $\mathbb{S}^1$-orbits and the two fixed points remain unchanged.

In quantum mechanics, the *Moyal product* of two functions $f, h \in \mathcal{S}(\mathbb{R}^n)$ is defined as an (oscillatory) integral of the form

$$(f \star h)(x) := (\pi\theta)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x+s)\, h(x+t)\, e^{-2is(\Theta^{-1}t)}\, ds\, dt. \tag{7.5}$$

Here $n = 2m$ is even, $\Theta = -\Theta^t \in M_n(\mathbb{R})$ is an invertible skewsymmetric matrix, and $\det\Theta = \theta^n$ with $\theta > 0$. In the next section, we shall interpret this formula in a precise manner (see Definition 7.17 below), and show that $f \star h$ lies in $\mathcal{S}(\mathbb{R}^n)$ also. Formally, at any rate, one can rewrite it as an ordinary Fourier integral:

$$(f \star h)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - \tfrac{1}{2}\Theta u)\, h(x+t)\, e^{-iut}\, du\, dt,$$

with the advantage that now Θ need not be invertible (so that n need no longer be even). It was noticed by Rieffel [r-ma93] that one can replace the translation action of $\mathbb{R}^n$ on f, h by any (strongly continuous) action α of some $\mathbb{R}^l$ on a C*-algebra A. Then, given $\Theta = -\Theta^t \in M_l(\mathbb{R})$, one can define

$$a \star b := \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \alpha_{\frac{1}{2}\Theta u}(a)\, \alpha_{-t}(b)\, e^{2\pi i ut}\, du\, dt,$$

provided that the integral makes sense. In particular, if α is *periodic action* of $\mathbb{R}^l$, i.e., $\alpha_{t+r} = \alpha_t$ for $r \in \mathbb{Z}^n$, so that α is effectively an action of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, then one can describe the Moyal deformation as follows.

Definition 7.6. *Let A be a unital C^*-algebra, and suppose that there is an action α of $\mathbb{T}^l$ on A by $*$-automorphisms, which is strongly continuous. For each $r \in \mathbb{Z}^l$, let $A_{(r)}$ be the* spectral subspace

$$A_{(r)} := \{\, a \in A : \alpha_t(a) = t^r a \text{ for all } t \in \mathbb{T}^l \,\}, \quad \text{where } t^r := t_1^{r_1} \dots t_n^{r_n} \in \mathbb{T}.$$

Let $\mathcal{A} := \{\, a \in A : t \mapsto \alpha_t(a) \text{ is smooth} \,\}$ be the "smooth subalgebra" for the action of $\mathbb{T}^l$. It can be shown that $\mathcal{A}$ is a Fréchet pre-C*-algebra, and each $a \in \mathcal{A}$ can be written as a convergent series $a = \sum_{r\in\mathbb{Z}^l} a_r$, where $a_r \in A_{(r)}$ and $\|a_r\| \to 0$ rapidly as $|r| \to \infty$.

Definition 7.7. *Fix $\Theta = -\Theta^t \in M_l(\mathbb{R})$. The* Moyal product *of two elements $a, b \in \mathcal{A}$, with $a = \sum_r a_r$ and $b = \sum_s b_s$, is defined as $a \star b := \sum_{r,s} a_r \star b_s$, where*

$$a_r \star b_s := \sigma(r,s)\, a_r b_s, \quad \text{with} \quad \sigma(r,s) := \exp\bigl\{-\pi i \textstyle\sum_{j,k=1}^{l} r_j \theta_{jk} s_k\bigr\}. \tag{7.6}$$

For actions of $\mathbb{T}^l$, Rieffel [r-ma93] showed that the integral formula and the series formula for $a \star b$ are equivalent, when a, b belong to the smooth subalgebra $\mathcal{A}$.

Definition 7.8. *Let M be a compact Riemannian manifold, carrying a continuous action of $\mathbb{T}^l$ by isometries $\{\sigma_t\}_{t\in\mathbb{T}^l}$. Then $\alpha_t(f) := f \circ \sigma_t$ is a strongly continuous action of $\mathbb{T}^l$. Given $\Theta = -\Theta^t \in M_l(\mathbb{R})$, Rieffel's construction provides a Moyal product on $C^\infty(M_\Theta) := (C^\infty(M), \star)$, whose C^*-completion in a suitable norm is $C(M_\Theta) := (C(M), \star)$. In particular, for $M = \mathbb{S}^4$ with the round metric and $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$, these are the algebras $C^\infty(\mathbb{S}^4_\theta)$ and $C(\mathbb{S}^4_\theta)$ introduced by Connes and Landi [cl01].*

To deform the spectral triple $(C^\infty(M), L^2(M,S), \not{D})$, we need a further step. Since each σ_t is an isometry of M, it defines an automorphism of the tangent bundle TM (with $T_xM \to T_{\sigma_t(x)}M$), and of the cotangent bundle T^*M (with $T^*_{\sigma_t(x)}M \to T^*_xM$), preserving the orientation and the metric on each bundle. But the group $\mathrm{SO}(T^*_xM, g_x)$ does not act directly on the fibre S_x of the spinor bundle. Instead, the action of the Clifford algebra $\mathcal{B}$ on $\mathcal{H} = L^2(M,S)$ yields a homomorphism $\mathrm{Spin}(T^*_xM, g_x) \to \mathrm{End}(S_x)$ for each $x \in M$, and we know that there is a double covering $\mathrm{Ad}_x\colon \mathrm{Spin}(T^*_xM, g_x) \to \mathrm{SO}(T^*_xM, g_x)$ by conjugation.

It turns out [cd02] that one can lift the isometric action $\alpha\colon \mathbb{T}^l \to SO(T^*M)$ to an action of another torus $\tau\colon \widetilde{\mathbb{T}}^l \to \mathrm{Aut}(S)$, where there is a covering map $\pi\colon \widetilde{\mathbb{T}}^l \to \mathbb{T}^l$ such that $\pi(\pm 1) = 1$, making the following diagram commute:

$$\begin{tikzcd} \widetilde{\mathbb{T}}^l \arrow[r, "\tau_t"] \arrow[d, "\pi"'] & \mathrm{Aut}(S) \arrow[d, "\mathrm{Ad}"] \\ \mathbb{T}^l \arrow[r, "\alpha_t"] & \mathrm{SO}(T^*M) \end{tikzcd}$$

Fact 7.9. *One can find a covering of $\mathbb{T}^l$ by a torus $\widetilde{\mathbb{T}}^l$, and a representation $\tilde{t} \mapsto \tau_{\tilde{t}}$ of $\widetilde{\mathbb{T}}^l$ on $\mathrm{Aut}(S)$ such that $\mathrm{Ad}(\tau_{\tilde{t}}) = \alpha_t$ if $\pi(\tilde{t}) = t \in \mathbb{T}^l$. For $f \in \mathcal{A} = C^\infty(M)$ and $\phi, \psi \in \mathcal{S} = \Gamma(M,S)$, this implies that*

$$\tau_{\tilde{t}}(f\psi) = \alpha_t(f)\,\tau_{\tilde{t}}\psi, \tag{7.7}$$
$$(\tau_{\tilde{t}}\phi \mid \tau_{\tilde{t}}\psi) = \alpha_t(\phi \mid \psi). \tag{7.8}$$

Integrating over M, and recalling that σ_t is an isometry, we get $\langle \tau_{\tilde{t}}\phi \mid \tau_{\tilde{t}}\psi\rangle = \langle \phi \mid \psi\rangle$, so that τ extends to a unitary representation of $\widetilde{\mathbb{T}}^l$ on $\mathcal{H} = L^2(M,S)$.

We can regard $\mathbb{T}^l$ as $\mathbb{R}^l/(\mathbb{Z}^l + \hat{\mathbb{Z}}^l)$, where $\hat{\mathbb{Z}}^l = \mathbb{Z}^l + (\frac12, \frac12, \dots, \frac12)$. With this convention, one can show that the set of commuting selfadjoint operators $P_1, \dots, P_l$ on $\mathcal{H}$ which generate the unitary representation of $\widetilde{\mathbb{T}}^l$, i.e.,

$$\tau_{\tilde{t}} =: \exp(it_1P_1 + \cdots + it_nP_n),$$

have half-integer spectra: $\mathrm{sp}(P_j) \subseteq \frac12\mathbb{Z}$.

Now define a family of unitary operators $\{\, \sigma(r,P) : r \in \mathbb{Z}^l \,\}$ by

$$\sigma(r,P) := \exp\Bigl(-\pi i \textstyle\sum_{j,k} r_j\theta_{jk}\, P_k\Bigr),$$

that is, we substitute s_k by P_k in the cocycle formula $\sigma(r,s)$ of (7.6).

Exercise 7.10. *Show that, since the action α is isometric, the unitary operator $\tau_{\tilde{t}}$ commutes with $\not{D}$ and with the charge conjugation operator C, for each $\tilde{t} \in \widetilde{\mathbb{T}}^l$.*

It follows that each $\sigma(r,P)$ commutes with $\not{D}$, too. However, the operators $\sigma(r,P)$ need *not* commute with the multiplication operators $\psi \mapsto f\psi$, for $f \in C^\infty(M)$. Indeed, (7.7) implies that $\tau_{\tilde{t}} f \tau_{-\tilde{t}} = \alpha_t(f)$ for each $\tilde{t} \in \widetilde{\mathbb{T}}^l$.

Exercise 7.11. *If $h_s \in A_{(s)}$ and $r \in \mathbb{Z}^l$, show that $\sigma(r,P)\,h_s = h_s\,\sigma(r, P+s)$.*

We are now ready to exhibit the isospectral deformation of the standard commutative example for a spin manifold M carrying an isometric action of $\mathbb{T}^l$, with respect to a fixed matrix Θ of deformation parameters. The deformation is called *isospectral* for the simple reason that the operator D of the deformed spectral triple is the *same* Dirac operator of the undeformed case, so it is no surprise that its spectrum does not change. What does change is the algebra: in fact, the underlying vector space of $\mathcal{A}$ is unchanged, but the product operation is deformed, and consequently its representation on $\mathcal{H}$ changes, too.

Theorem 7.12. *If $\mathcal{A}_\Theta = (C^\infty(M), \star)$, $\mathcal{H} = L^2(M,S)$ and $D = D\!\!\!/$, then there is a representation of $\mathcal{A}_\Theta$ by bounded operators on $\mathcal{H}$, such that $(\mathcal{A}_\Theta, \mathcal{H}, D\!\!\!/)$ is a spectral triple with the same Dirac operator as the standard commutative example $(C^\infty(M), \mathcal{H}, D\!\!\!/)$. Moreover, the charge conjugation operator C is a real structure on $(\mathcal{A}_\Theta, \mathcal{H}, D\!\!\!/)$, and the first order property holds.*

Proof. If $f \in \mathcal{A}$, write $f = \sum_{r\in\mathbb{Z}^l} f_r$ as a decomposition into spectral subspaces, where $\alpha_t(f_r) = t^r f_r$ for $t \in \mathbb{T}^l$, $r \in \mathbb{Z}^l$. Define

$$L(f) := \sum_r f_r\, \sigma(r,P) \in B(\mathcal{H}).$$

Then $f \mapsto L(f)$ is a representation of the algebra $\mathcal{A}_\Theta$:

$$\begin{aligned} L(f)L(h) &= \sum_{r,s} f_r\, \sigma(r,P)\, h_s\, \sigma(s,P) \\ &= \sum_{r,s} f_r h_s\, \sigma(r, P+s)\, \sigma(s,P) \\ &= \sum_{r,s} f_r h_s\, \sigma(r,s)\, \sigma(r+s,P) \\ &= \sum_{r,s} f_r \star h_s\, \sigma(r+s,P) = L(f\star h). \end{aligned}$$

The last equality follows because $f_r \in A_{(r)}$, $h_s \in A_{(s)}$ imply that both $f_r h_s$ and $f_r \star h_s$ lie in $A_{(r+s)}$ —these products differ only by the phase factor $\sigma(r,s)$— and therefore $(f\star h)_p = \sum_{r+s=p} f_r \star h_s$.

Since $\alpha_t(f_r^*) = \alpha_t(f_r)^* = t^{-r} f_r^*$, we see that $(f^*)_s = (f_{-s})^*$ for $s \in \mathbb{Z}^l$. Thus $L(f)^* = \sum_r f_r^*\, \sigma(-r,P) = \sum_r (f^*)_{-r}\, \sigma(-r,P) = L(f^*)$, so that L is actually a $*$-representation.

Since $D\!\!\!/$ commutes with each $\sigma(r,P)$, we get

$$[D\!\!\!/, L(f)] = \sum_r [D\!\!\!/, f_r]\, \sigma(r,P) =: L([D\!\!\!/, f]), \tag{7.9}$$

where we remark that $\tau_{\tilde t}\, [D\!\!\!/, f_r]\, \tau_{-\tilde t} = [D\!\!\!/, \tau_{\tilde t} f_r \tau_{-\tilde t}] = t^r\, [D\!\!\!/, f_r]$, so that the operators $[D\!\!\!/, f]$, for $f \in \mathcal{A}$, decompose into spectral subspaces under the action $t \mapsto \mathrm{Ad}(\tau_{\tilde t})$ by automorphisms of $B(\mathcal{H})$, which extends $t \mapsto \alpha_t$ by automorphisms of $\mathcal{A}$.

Next, since the antilinear operator C commutes with all unitaries $\sigma(r,P)$, we deduce that $CP_jC^{-1} = -P_j$ for $j = 1,2,\dots,l$. Therefore, we can define an antirepresentation of $\mathcal{A}_\Theta$ on $\mathcal{H}$ by

$$R(f) := C\, L(f)^*\, C^{-1} = \sum_r \sigma(r,P)^*\, C f_r C^{-1} = \sum_r \sigma(-r,P)\, f_r = \sum_r f_r\, \sigma(-r,P).$$

Notice that $\sigma(r,P)^* = \sigma(-r,P)$ commutes with f_r in view of Exercise 7.11 and the relation $\sigma(-r,r) = 1$.

The left and right multiplication operators commute, since

$$[L(f), R(h)] := \sum_{r,s} [\sigma(r,P)\, f_r, h_s\, \sigma(-s,P)]$$
$$= \sum_{r,s} [f_r, h_s]\, \sigma(r,s)\, \sigma(r-s,P) = 0,$$

where $[f_r, h_s] = 0$ because, with its original product. $\mathcal{A}$ is commutative. The same calculation shows also that

$$[[D\!\!\!/, L(f)], R(h)] = \big[L([D\!\!\!/, f]), R(h)\big] = \sum_{r,s} [[D\!\!\!/, f_r], h_s]\, \sigma(r,s)\, \sigma(r-s,P) = 0,$$

since $([D\!\!\!/, f])_r = [D\!\!\!/, f_r]$ for each r and $[[D\!\!\!/, f_r], h_s] = 0$ by the first-order property of the undeformed spectral triple $(C^\infty(M), \mathcal{H}, D\!\!\!/)$. □

When $\dim M$ is even, and Γ is the $\mathbb{Z}_2$-grading operator Γ on the spinor space $\mathcal{H}$, we should note that the orientation condition $\pi_{D\!\!\!/}(\mathbf{c}) = \Gamma$ says, among other things, that Γ appears in the algebra generated by the operators f and $[D\!\!\!/, f]$, for $f \in \mathcal{A}$. The representation L of $\mathcal{A}_\Theta$ extends to this algebra of operators by using (7.9) as a definition of $L([D\!\!\!/, f])$. In the formula (7.1) for $\pi_{D\!\!\!/}(\mathbf{c})$, if we replace all terms a^r_j by $L(a^r_j)$, then we obtain $L(\pi_{D\!\!\!/}(\mathbf{c})) = L(\Gamma) = \Gamma$. Thus $\mathbf{c}$ may also be regarded as a Hochschild n-cycle over $\mathcal{A}_\Theta$, and *the orientation condition* $\pi_{D\!\!\!/}(\mathbf{c}) = \Gamma$ *is unchanged by the deformation*. In odd dimensions, the same is true, with Γ replaced by 1.

In conclusion: the isospectral deformation procedure of Connes and Landi yields a family of noncommutative spectral triples that satisfy all of our stated conditions for a noncommutative spin geometry. (Moreover [cl01], Poincaré duality in K-theory is stable under deformation, too.)

7.3 The Moyal plane as a nonunital spectral triple

In order to extend the notion of spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to include the case where the algebra $\mathcal{A}$ may be nonunital, we modify Definition 4.1 as follows.

Definition 7.13. *A* nonunital spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *consists of a nonunital $*$-algebra $\mathcal{A}$, equipped with a faithful representation on a Hilbert space $\mathcal{H}$, and a selfadjoint operator D on $\mathcal{H}$ with $a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ for all $a \in \mathcal{A}$, such that*

- $[D, a]$ *extends to a bounded operator on $\mathcal{H}$, for each $a \in \mathcal{A}$;*
- $a\,(D^2+1)^{-1/2}$ *is a compact operator, for each $a \in \mathcal{A}$.*

In general, D may have continuous spectrum, so that the operator $(D^2 + 1)^{-1/2}$ will usually not be compact. But it is enough to ask that it become compact when mollified by any multiplication operator in $\mathcal{A}$. An equivalent condition is that $a(D-\lambda)^{-1}$ be compact, for all $\lambda \notin \operatorname{sp}(D)$. In the nonunital case, there is no advantage in supposing that D be invertible, so it is better to work directly with $(D^2+1)^{1/2}$ instead of $|D|$.

Remark 7.14. The simplest commutative example of a nonunital spectral triple is given by

$$\mathcal{A} = C_0^\infty(\mathbb{R}^n), \qquad \mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^m}, \qquad D\!\!\!/ = -i\,\gamma^j \,\frac{\partial}{\partial x^j},$$

describing the noncompact manifold $\mathbb{R}^n$ with trivial spinor bundle $\mathbb{R}^n \times \mathbb{C}^{2^m} \to \mathbb{R}^n$ and flat metric: as always, $n = 2m$ or $n = 2m+1$. Here $C_0^\infty(\mathbb{R}^n)$ is the space of smooth functions that vanish at infinity together with all derivatives: it is a $*$-algebra under pointwise multiplication and complex conjugation of functions. Here $\operatorname{sp}(D\!\!\!/) = \mathbb{R}$ and $(D\!\!\!/^2 + 1)^{-1/2}$ is not compact. However, it is known [s-b79] that if $f \in L^p(\mathbb{R}^n)$ with $p > n$, then $f\,(D\!\!\!/^2 + 1)^{-1/2} \in \mathcal{L}^p(\mathcal{H})$.

The simplest *noncommutative, nonunital* example is an isospectral deformation of this commutative case, where we use the same Dirac operator $D\!\!\!/ = -i\,\gamma^j\,\partial/\partial x^j$ on the same spinor space $\mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^m}$, but we change the algebra by replacing the ordinary product of functions by a Moyal product.

Before giving the details, we summarize the effect of this nonunital isospectral deformation on the conditions given in Section 7.1 to define a "noncommutative spin geometry".

- The *reality* and *first-order* conditions are unchanged: we use the same charge conjugation operator C as in the undeformed case.
- The *regularity* condition is essentially unchanged: all that is needed is to replace the derivation $\delta\colon T \mapsto [|D|, T]$ by the derivation $\delta_1\colon T \mapsto [(D\!\!\!/^2 + 1)^{1/2}, T]$, because $\operatorname{Dom} \delta_1^k = \operatorname{Dom} \delta^k$ for each $k \in \mathbb{N}$ since $(D\!\!\!/^2 + 1)^{1/2} - |D|$ is a bounded operator.
- For the *orientation condition*, the Hochschild n-cycle will not lie in $Z_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})$ but rather in $Z_p(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^{\mathrm{op}})$, where $\widetilde{\mathcal{A}}$ is a *unitization* of $\mathcal{A}$, that is, a unital $*$-algebra in which $\mathcal{A}$ is included as an essential ideal.
- For the *finiteness* condition, we ask that $\mathcal{H}^\infty = \mathcal{A}^N p$, for some projector $p = p^2 = p^*$ lying in $M_N(\widetilde{\mathcal{A}})$. Thus $\mathcal{H}^\infty$ can be regarded as the pullback, via the inclusion $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}}$, of the finitely generated projective left $\widetilde{\mathcal{A}}$-module $\widetilde{\mathcal{A}}^N p$.
- To define the integer n as the *spectral dimension*, we would like to be able to assert that $a\,(D^2 + 1)^{-1/2}$ lies in $\mathcal{L}^{n+}(\mathcal{H})$ for each $a \in \mathcal{A}$, and that $0 < \operatorname{Tr}_\omega(a\,(D^2 + 1)^{-n/2}) < \infty$ whenever a is positive and nonzero. It turns out that we can only verify this for a belonging to a certain *dense subalgebra* of $\mathcal{A}$, in the Moyal plane example: see below.

Exercise 7.15. *Check the assertion on regularity: show that* $\operatorname{Dom} \delta_1 = \operatorname{Dom} \delta$ *and that* $\operatorname{Dom} \delta_1^k = \operatorname{Dom} \delta^k$ *for each* $k \in \mathbb{N}$, *by induction on* k.

Exercise 7.16. *Show that Proposition 6.13 holds without the assumption that* D *is invertible. Namely, if* $L_1(b) := (D^2+1)^{-1/2}\,[D^2, b]$ *and* $R_1(b) := [D^2, b]\,(D^2+1)^{-1/2}$, *show that* $\bigcap_{k,l \geq 0} \operatorname{Dom}(L_1^k R_1^l) = \bigcap_{m \geq 0} \operatorname{Dom} \delta_1^m$ *by adapting the proof of Proposition 6.13.*

In what follows, we will sketch the main features of the Moyal plane spectral triple. A complete treatment can be found in Gayral *et al* [ggisv04], on which this outline is based. Our main concern here is to identify the "correct" algebra $\mathcal{A}$ and its unitization $\widetilde{\mathcal{A}}$ so that the modified spin-geometry conditions will hold.

We now recall the Moyal product over $\mathbb{R}^n$, discussed in the previous Section. It depends on a real skewsymmetric matrix $\Theta \in M_n(\mathbb{R})$ of "deformation parameters". For $n = 2$, such a matrix is of the form $\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ for some $\theta \in \mathbb{R}$; and for $n = 2m$ or $n = 2m + 1$, Θ is similar to a direct sum of m such matrices with possibly different values of θ (so Θ cannot be invertible if n is odd). For convenience, we now take n to be even, and we shall suppose that all values of θ are the same. (In applications to quantum mechanics, where the Moyal product originated [m-je49], $\theta = \hbar$ is the Planck constant.) Thus, we choose

$$\Theta := \theta\, S \in M_{2m}(\mathbb{R}), \quad \text{with } S := \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}, \quad \theta > 0. \tag{7.10}$$

Note that $\det \Theta = \theta^{2m} > 0$.

Definition 7.17. *Let $n = 2m$ be even, let $\theta > 0$, and let $f, h \in \mathcal{S}(\mathbb{R}^n)$. Their* Moyal product *$f \star_\theta h \in \mathcal{S}(\mathbb{R}^n)$ is defined as follows:*

$$\begin{aligned} (f \star_\theta h)(x) &:= (\pi\theta)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x+s)\, h(x+t)\, e^{2is(St)/\theta}\, ds\, dt \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x - \tfrac12\theta Su)\, h(x+t)\, e^{-iut}\, du\, dt \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} f(x - \tfrac12\theta Su)\, \hat{h}(u)\, e^{iux}\, du. \end{aligned} \tag{7.11}$$

Here $\hat{h}(u) := \int_{\mathbb{R}^n} h(t)\, e^{-iut}\, dt$ is the Fourier transform. Since $h \mapsto \hat{h}$ preserves the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the third integral is a twisted version of the usual convolution of f and $\hat{h}$, and one can check that this integral converges to an element of $\mathcal{S}(\mathbb{R}^n)$.

The first or second integral in (7.11) can also be regarded as defining the Moyal product $f \star_\theta h$, where f and h need not be Schwartz functions, provided that the integrals are understood in some generalized sense. Thus Rieffel [r-ma93], for instance, considers them as oscillatory integrals. Here we shall extend the Moyal product by duality, as follows. It is easy to see that $\|f \star_\theta h\|_\infty \le (\pi\theta)^{-n} \|f\|_1\, \|h\|_1$, from the first integral in (7.11). By applying similar estimates to the functions $x^\alpha\, \partial^\beta\, (f \star_\theta h)$, for $\alpha, \beta \in \mathbb{N}^n$, one can verify that the product $(f, h) \mapsto f \star_\theta h$ is a jointly continuous bilinear map from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Here are some elementary properties of the Moyal product that are easy to check formally; they can be verified rigorously by some work with oscillatory integrals: see [r-ma93].

1. The Moyal product is associative: $(f \star_\theta g) \star_\theta h = f \star_\theta (g \star_\theta h)$.
2. The Leibniz rule holds: $\partial_j(f \star_\theta h) = \partial_j f \star_\theta h + f \star_\theta \partial_j h$ for $j = 1, \dots, n$.
3. Complex conjugation is an involution: $\overline{f \star_\theta h} = \bar{h} \star_\theta \bar{f}$.

4. Integration over $\mathbb{R}^n$ is a trace for the Moyal product:

$$\int_{\mathbb{R}^n} (f \star_\theta h)(x)\,dx = \int_{\mathbb{R}^n} (h \star_\theta f)(x)\,dx = \int_{\mathbb{R}^n} f(x)\,h(x)\,dx. \tag{7.12}$$

We denote $\mathcal{S}_\theta := (\mathcal{S}(\mathbb{R}^n), \star_\theta)$. It is a Fréchet $*$-algebra, with the usual topology of $\mathcal{S}(\mathbb{R}^n)$.

The trace property gives us a (suitably normalized) bilinear pairing:

$$\langle f, h\rangle := (\pi\theta)^{-m} \int_{\mathbb{R}^n} (f \star_\theta h)(x)\,dx.$$

Together with associativity, this gives the relation

$$\langle f \star_\theta g, h\rangle = \langle f, g \star_\theta h\rangle = (\pi\theta)^{-m} \int_{\mathbb{R}^n} (f \star_\theta g \star_\theta h)(x)\,dx,$$

valid for $f, g, h \in \mathcal{S}(\mathbb{R}^n)$. Now, if $T \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, and if $f \in \mathcal{S}(\mathbb{R}^n)$, we can define $T \star_\theta f,\ f \star_\theta T \in \mathcal{S}'(\mathbb{R}^n)$ by the continuity of the Moyal product:

$$\langle T \star_\theta f, h\rangle := \langle T, f \star_\theta h\rangle, \qquad \langle f \star_\theta T, h\rangle := \langle T, h \star_\theta f\rangle.$$

In this way, $\mathcal{S}'(\mathbb{R}^n)$ becomes a bimodule over $\mathcal{S}_\theta$. Inside this bimodule, we can identify a multiplier algebra in the obvious way.

Definition 7.18. *The* Moyal algebra $\mathcal{M}_\theta = \mathcal{M}_\theta(\mathbb{R}^n)$ *is defined as the set of (left and right) multipliers for* $\mathcal{S}(\mathbb{R}^n)$ *within* $\mathcal{S}'(\mathbb{R}^n)$*:*

$$\mathcal{M}_\theta := \{\, R \in \mathcal{S}'(\mathbb{R}^n) : R \star_\theta f \in \mathcal{S}(\mathbb{R}^n),\ f \star_\theta R \in \mathcal{S}(\mathbb{R}^n) \text{ for all } f \in \mathcal{S}(\mathbb{R}^n) \,\}.$$

This is a $*$-algebra, and $\mathcal{S}'(\mathbb{R}^n)$ is an $\mathcal{M}_\theta$-bimodule, under the operations

$$\langle T \star_\theta R, f\rangle := \langle T, R \star_\theta f\rangle, \qquad \langle R \star_\theta T, f\rangle := \langle T, f \star_\theta R\rangle.$$

This Moyal algebra is very large: for instance, it contains all polynomials on $\mathbb{R}^n$. However, because it contains many unbounded elements, it cannot serve as a coordinate algebra for a spectral triple. Even so, it is a starting point for a second approach, developed in [gv88]. Consider the quadratic polynomials $H_r := \frac{1}{2}(x_r^2 + x_{m+r}^2)$ for $r = 1, \dots, m$. In the quantum-mechanical interpretation, these are Hamiltonians for a set of m independent harmonic oscillators; but for now, it is enough to know that they belong to $\mathcal{M}_\theta$. It turns out that the left and right Moyal multiplications by these H_r have a set of joint eigenfunctions $\{\, f_{kl} : k, l \in \mathbb{N}^m \,\}$ belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, with the following properties:

- The eigenvalues are half-integer multiples of θ, namely,

$$H_r \star_\theta f_{kl} = \theta(k_r + \tfrac{1}{2}) f_{kl}, \qquad f_{kl} \star_\theta H_r = \theta(l_r + \tfrac{1}{2}) f_{kl}.$$

- The eigenfunctions form a set of matrix units for the Moyal product: $f_{kl} \star_\theta f_{rs} = \delta_{lr} f_{ks}$ and $\bar{f}_{kl} = f_{lk}$ for all $k, l, r, s \in \mathbb{N}^m$.

- Any $f \in \mathcal{S}(\mathbb{R}^n)$ is given by a series $f = (2\pi\theta)^{-m/2} \sum_{kl} \alpha_{kl} f_{kl}$, converging in the topology of $\mathcal{S}(\mathbb{R}^n)$, such that $\alpha_{kl} \to 0$ rapidly.
- The subset $\{ (2\pi\theta)^{-m/2} f_{kl} : k, l \in \mathbb{N}^m \}$ of $\mathcal{S}(\mathbb{R}^n)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

For example, when $n = 2$ and $k = l \in \mathbb{N}$, f_{kk} is given by

$$f_{kk}(x_1, x_2) := 2\,(-1)^k\, e^{-(x_1^2+x_2^2)/\theta}\, L_k^0(\tfrac{2}{\theta}(x_1^2 + x_2^2)),$$

where L_k^0 is the Laguerre polynomial of order k.

Because of these properties, we can extend the Moyal product to pairs of functions in $L^2(\mathbb{R}^n)$. If $f = (2\pi\theta)^{-m/2} \sum_{k,l} \alpha_{kl} f_{kl}$ and $h = (2\pi\theta)^{-m/2} \sum_{k,l} \beta_{kl} f_{kl}$, we define

$$f \star_\theta h := (2\pi\theta)^{-m} \sum_{k,r,l} \alpha_{kr}\beta_{rl}\, f_{kl}. \tag{7.13}$$

The Schwarz inequality for sequences shows that

$$\begin{aligned}\|f \star_\theta h\|_2^2 = (2\pi\theta)^{-2m} \Bigl\| \sum_{k,r,l} \alpha_{kr}\beta_{rl}\, f_{kl} \Bigr\|_2^2 &= (2\pi\theta)^{-m} \sum_{k,l} \Bigl| \sum_r \alpha_{kr}\beta_{rl} \Bigr|^2 \\ &\leq (2\pi\theta)^{-m} \sum_{k,r} |\alpha_{kr}|^2 \sum_{r,l} |\beta_{rl}|^2 = (2\pi\theta)^{-m} \|f\|_2^2\, \|h\|_2^2. \end{aligned} \tag{7.14}$$

This calculation guarantees that the series (7.13) converges whenever $f, h \in L^2(\mathbb{R}^n)$; and that the operator $L(f)\colon h \mapsto f \star_\theta h$ extends to a bounded operator in $\mathcal{L}(L^2(\mathbb{R}^n))$ whenever $f \in L^2(\mathbb{R}^n)$. Moreover, it gives a bound on the operator norm:

$$\|L(f)\| \leq (2\pi\theta)^{-m/2}\, \|f\|_2.$$

Now the Schwartz-multiplier algebra $\mathcal{M}_\theta$ can be replaced by an L^2-multiplier algebra. By duality in sequence spaces, any $T \in \mathcal{S}'(\mathbb{R}^n)$ can be given an expansion in terms of the $\{f_{kl}\}$ basis, and in this way one can define an algebra

$$A_\theta := \{ R \in \mathcal{S}'(\mathbb{R}^n) : R \star_\theta f \in L^2(\mathbb{R}^n) \text{ for all } f \in L^2(\mathbb{R}^n) \}.$$

This is actually a C*-algebra, with operator norm $\|L(R)\| := \sup\{ \|R \star_\theta f\|_2 / \|f\|_2 : f \neq 0 \}$.

There is a unitary isomorphism $W\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^m)$ (tensor product of Hilbert spaces), such that $W\, L(f)\, W^{-1} = \sigma(f) \otimes 1$, where σ is the (irreducible) Schrödinger representation; that is to say, $f \mapsto L(f)$ is equivalent to the Schrödinger representation with infinite multiplicity. One can show that $A_\theta = W^{-1}\, \mathcal{L}(L^2(\mathbb{R}^m))\, W$, whereas the norm closure of the $*$-algebra $(\mathcal{S}(\mathbb{R}^n), \star_\theta)$ is $W^{-1}\, \mathcal{K}(L^2(\mathbb{R}^m))\, W$. For the details, consult [gv88] and [ggisv04].

The analogue of Lemma 6.19 holds, too: *$\mathcal{S}_\theta$ is a nonunital pre-C*-algebra.* As in the proof of Lemma 6.19, if $f \in \mathcal{S}_\theta$, suppose the equation $(1+f) \star_\theta (1+g) = 1$ has a solution g in the unital C*-algebra A_θ. We may also write

$$f + g + f \star_\theta g = 0 \quad \text{and} \quad f + g + g \star_\theta f = 0, \tag{7.15}$$

and we wish to show that $g \in \mathcal{S}_\theta$. Since $g = -f - f \star_\theta g$, it is enough to show that $f \star_\theta g \in \mathcal{S}_\theta$. Now, left-multiplying the second equation in (7.15) by f gives

$f \star_\theta f + f \star_\theta g + f \star_\theta g \star_\theta f = 0$, so it is enough to check that $f \star_\theta g \star_\theta f \in \mathcal{S}(\mathbb{R}^n)$ whenever $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in A_\theta$. This turns out to be true: the necessary norm estimates are given in [gv88].

However, the algebra $\mathcal{S}_\theta$ is not the best candidate for the coordinate algebra of the Moyal spectral triple. We now introduce a better algebra.

Definition 7.19. *Consider the following space of smooth functions on $\mathbb{R}^n$:*

$$\mathcal{D}_{L^2}(\mathbb{R}^n) := \{\, f \in C^\infty(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}^n \,\},$$

introduced by Laurent Schwartz in his book on distributions [s-l66]. It is a Fréchet space, under the norms $p_r(f) := \sum_{|\alpha|\leq r} \|\partial^\alpha f\|_2$, for $r \in \mathbb{N}$. The Leibniz rule for the Moyal product and the inequality (7.14) show that if $f, h \in \mathcal{D}_{L^2}(\mathbb{R}^n)$, then

$$\begin{aligned} \|\partial^\alpha(f \star_\theta h)\|_2 &\leq \sum_{0\leq\beta\leq\alpha} \|\partial^\beta f \star_\theta \partial^{\alpha-\beta} h)\|_2 \\ &\leq (2\pi\theta)^{-m/2} \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \|\partial^\beta f\|_2 \, \|\partial^{\alpha-\beta} h\|_2^2, \end{aligned}$$

so that $\mathcal{D}_{L^2}(\mathbb{R}^n)$ is actually an algebra *under the Moyal product; and that this product is continuous for the given Fréchet topology. Moreover, since complex conjugation is an isometry for each norm p_r, it is a $*$-algebra with a continuous involution. We write $\mathcal{A}_\theta := (\mathcal{D}_{L^2}(\mathbb{R}^n), \star_\theta)$ to denote this Fréchet $*$-algebra.*

It does not matter whether these derivatives $\partial^\alpha f$ are taken to be distributional derivatives only, since arguments based on Sobolev's Lemma show that if f and all its distributional derivatives are square-integrable, then f is actually a smooth function.

The algebra $\mathcal{A}_\theta$ is nonunital. Next, we introduce the preferred *unitization* of $\mathcal{A}_\theta$.

Definition 7.20. *Another space of smooth functions on $\mathbb{R}^n$ is found also in [s-l66]:*

$$\mathcal{B}(\mathbb{R}^n) := \{\, f \in C^\infty(\mathbb{R}^n) : \partial^\alpha f \text{ is bounded on } \mathbb{R}^n, \text{ for all } \alpha \in \mathbb{N}^n \,\}.$$

It is also a Fréchet space, under the norms $q_r(f) := \max_{|\alpha|\leq r} \|\partial^\alpha f\|_\infty$, for $r \in \mathbb{N}$.

We shall soon prove that $\mathcal{B}(\mathbb{R}^n)$ is also a $*$-algebra under the Moyal product; we denote it by $\widetilde{\mathcal{A}}_\theta := (\mathcal{D}_{L^2}(\mathbb{R}^n), \star_\theta)$.

It is proved in Schwartz' book that $\mathcal{D}_{L^2}(\mathbb{R}^n) \subset \mathcal{B}(\mathbb{R}^n)$, and that the inclusion is continuous for the given topologies. (This is not as obvious as it seems, because in general square-integrable functions on $\mathbb{R}^n$ need not be bounded.) Combining this with knowledge of the Moyal multiplier algebras, we end up with the following inclusions [ggisv04]:

$$\mathcal{S}_\theta \subset \mathcal{A}_\theta \subset \widetilde{\mathcal{A}}_\theta \subset A_\theta \cap \mathcal{M}_\theta.$$

The inclusion $\widetilde{\mathcal{A}}_\theta \subset A_\theta$ is a consequence of the Calderón–Vaillancourt theorem, which says that a pseudodifferential operator of order zero on $\mathbb{R}^n$, whose symbol is differentiable to a high enough order, gives a bounded operator on $L^2(\mathbb{R}^n)$; we may notice that the second integral in (7.11) says that $L(f)$ is pseudodifferential, with symbol $p(x, \xi) = f(x - \frac{1}{2}\theta S\xi)$.

Proposition 7.21. *$\mathcal{B}(\mathbb{R}^n)$ is a Fréchet $*$-algebra under the Moyal product.*

Proof. If $f, h \in \mathcal{B}(\mathbb{R}^n)$ and if $s \in \mathbb{N}$, we shall find estimates of the form

$$q_s(f \star_\theta h) \leq C_{rs}\, q_r(f)\, q_r(h) \quad \text{whenever } r \geq s + n + 2. \tag{7.16}$$

This shows that $f \star_\theta h$ lies in $\mathcal{B}(\mathbb{R}^n)$ whenever $f, h \in \mathcal{B}(\mathbb{R}^n)$, and that $(f, h) \mapsto f \star_\theta h$ is a jointly continuous bilinear operation on $\mathcal{B}(\mathbb{R}^n)$. Since complex conjugation is clearly isometric for each q_r, the involution is continuous, too.

To justify the estimates (7.16), we first notice that, for any $k \in \mathbb{N}$,

$$\begin{aligned}
&(\partial^\beta f \star_\theta \partial^\gamma h)(x) = \\
&= (\pi\theta)^{-n} \iint \frac{\partial^\beta f(x+y)}{(1+|y|^2)^k}\, \frac{\partial^\gamma h(x+z)}{(1+|z|^2)^k}\, (1+|y|^2)^k (1+|z|^2)^k\, e^{2iy(Sz)/\theta}\, dy\, dz \\
&= (\pi\theta)^{-n} \iint \frac{\partial^\beta f(x+y)}{(1+|y|^2)^k}\, \frac{\partial^\gamma h(x+z)}{(1+|z|^2)^k}\, P_k(\partial_y, \partial_z)\big[e^{2iy(Sz)/\theta}\big]\, dy\, dz \\
&= (\pi\theta)^{-n} \iint e^{2iy(Sz)/\theta}\, P_k(-\partial_y, -\partial_z)\left[\frac{\partial^\beta f(x+y)}{(1+|y|^2)^k}\, \frac{\partial^\gamma h(x+z)}{(1+|z|^2)^k}\right] dy\, dz,
\end{aligned}$$

where P_k is a certain polynomial of degree $2k$ in both y_j and z_j variables, and for the third line we integrate by parts. It is not hard to find constants such that $|\partial^\alpha((1+|x|^2)^{-k})| \leq C'_{\alpha k}(1+|x|^2)^{-k}$ for each $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. Thus, we get estimates of the form

$$\begin{aligned}
|(\partial^\beta f \star_\theta \partial^\gamma h)(x)| &\leq \sum_{|\mu|,|\nu| \leq 2k} C''_{\mu\nu} \iint \frac{|\partial^{\beta+\mu} f(x+y)|\, |\partial^{\gamma+\nu} h(x+z)|}{(1+|y|^2)^k\, (1+|z|^2)^k}\, dy\, dz \\
&\leq C'''_{kr}\, q_r(f)\, q_r(h) \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^k} \int_{\mathbb{R}^n} \frac{dz}{(1+|z|^2)^k},
\end{aligned}$$

provided $r \geq |\beta| + |\gamma| + 2k$; and we need $k > n/2$ so that the right hand side is finite. For $|\beta| + |\gamma| \leq s$, we only need to choose k so that $n < 2k \leq r - s$, and this is always possible for $r \geq s + n + 2$. $\square$

Rieffel, in [r-ma93], showed that $\widetilde{\mathcal{A}}_\theta$ is the space of smooth vectors for the action of $\mathbb{R}^n$ (by translations) on its C^*-completion; this entails that $\widetilde{\mathcal{A}}_\theta$ is a *pre-C^*-algebra*.

Now, the inclusion $\widetilde{\mathcal{A}}_\theta \subset A_\theta$ means that $\|\partial^\alpha f \star_\theta \partial^\beta h\|_2$ is finite, whenever $f \in \widetilde{\mathcal{A}}_\theta$ and $h \in \mathcal{A}_\theta$; therefore, $f \star_\theta h$ lies in $\mathcal{A}_\theta$ also. A similar argument shows that $h \star_\theta f$ lies in $\mathcal{A}_\theta$. Thus, $\mathcal{A}_\theta$ is an *ideal* in $\widetilde{\mathcal{A}}_\theta$. (In fact, it is an essential ideal; that is to say, if $f \star_\theta h = 0$ for all $h \in \mathcal{A}_\theta$, then $f = 0$; this can be seen by taking $h = f_{kl}$ for any $k, l \in \mathbb{N}^n$ and checking that f must vanish.)

Lemma 7.22. *$\mathcal{A}_\theta$ is a nonunital pre-C^*-algebra.*

Proof. Since $\mathcal{A}_\theta$ is Fréchet, we only need to show that it is spectrally invariant. In the nonunital case, this means that if $f \in \mathcal{A}_\theta$, and the equations $f + g + f \star_\theta g = f + g + g \star_\theta f = 0$ have a solution g in the C^*-completion of $\mathcal{A}_\theta$, then g lies in $\mathcal{A}_\theta$. Now since $f \in \widetilde{\mathcal{A}}_\theta$ and $\widetilde{\mathcal{A}}_\theta$ is already a pre-C*-algebra, we see that $g \in \widetilde{\mathcal{A}}_\theta$. But $\mathcal{A}_\theta$ is an ideal in $\widetilde{\mathcal{A}}_\theta$, and thus $f \star_\theta g \in \mathcal{A}_\theta$. This implies that $g = -f - f \star_\theta g$ lies in $\mathcal{A}_\theta$, too. $\square$

An important family of elements in $\widetilde{\mathcal{A}}_\theta$ that do not belong to $\mathcal{A}_\theta$ are the *plane waves*:

$$u_k(x) := e^{2\pi i k x}, \quad \text{for each } k \in \mathbb{R}^n.$$

It is immediate from the formulas (7.11) that

$$u_k \star_\theta u_l = e^{-\pi i \theta k(Sl)}\, u_{k+l}, \quad \text{for all } k, l \in \mathbb{R}^n.$$

In particular, by taking $k, l \in \mathbb{Z}^n$ to be *integral* vectors, we get an inclusion $C^\infty(\mathbb{T}^n_{\theta S}) \hookrightarrow \widetilde{\mathcal{A}}_\theta$: the smooth algebra of the noncommutative n-torus, for $\Theta = \theta S$, can be identified with a subalgebra of *periodic functions* in $\widetilde{\mathcal{A}}_\theta$.

In particular, the Hochschild n-cycle $\mathbf{c}$ representing the orientation of this noncommutative torus can also be regarded as an n-cycle over $\widetilde{\mathcal{A}}_\theta$. We can write $u_k = v_1^{k_1} \star_\theta \cdots \star_\theta v_n^{k_n}$ where $v_j = u_{e_j}$ for the standard orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. The expression for $\mathbf{c}$ is

$$\mathbf{c} = \frac{1}{n!\,(2\pi i)^n} \sum_{\sigma \in S_n} (-1)^\sigma (v_{\sigma(1)} v_{\sigma(2)} \dots v_{\sigma(n)})^{-1} \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

(When $\theta = 0$, we can write $v_j = e^{2\pi i t_j}$, and the right hand side reduces to $dt_1 \wedge \cdots \wedge dt_n$, the usual volume form for either $\mathbb{R}^n$ or the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.)

We refer to [ggisv04] for the discussion of the spectral dimension properties of the triple $(\mathcal{A}_\theta, L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^m}, D\!\!\!/)$. Briefly, the facts are these. If $\pi(f) := L(f) \otimes 1_{2^m}$ denotes the representation of $\mathcal{A}_\theta$ on the spinor space $\mathcal{H}$ by componentwise left Moyal multiplication, then one can show that, for any $f \in \mathcal{A}_\theta$, we get

$$\pi(f)\,(D\!\!\!/^2 + 1)^{-1/2} \in \mathcal{L}^p(\mathcal{H}), \quad \text{for all } p > n.$$

In particular, these operators are compact, so this triple is indeed a nonunital spectral triple. However, this is not quite enough to guarantee that

$$\pi(f)\,(D\!\!\!/^2 + 1)^{-1/2} \in \mathcal{L}^{n+}(\mathcal{H}), \tag{7.17}$$

for *every* $f \in \mathcal{A}_\theta$. Instead, what is found in [ggisv04] is that (7.17) holds for f lying in the (dense) subalgebra $\mathcal{S}_\theta$. The key lemma which makes the proof work is a "strong factorization" property of $\mathcal{S}_\theta$, proved in [gv88]: namely, that any $f \in \mathcal{S}_\theta$ can be expressed (without taking finite sums) as a product $f = g \star_\theta h$, with $g, h \in \mathcal{S}_\theta$. This factorization property fails for the full algebra $\mathcal{A}_\theta$.

Once (7.17) has been established, one can proceed to compute its Dixmier trace. It turns out that $\operatorname{Tr}_\omega(\pi(f)\,(D\!\!\!/^2 + 1)^{-1/2})$ is unchanged from its value when $\theta = 0$, namely $(2^m\,\Omega_n/n\,(2\pi)^n) \int_{\mathbb{R}^n} f(x)\,dx$. The end result is that the spectral dimension condition for nonunital spectral triples is the expected one, but that Dixmier-traceability as in (7.17) should only be required for a dense subalgebra of the original algebra.

7.4 A geometric spectral triple over $SU_q(2)$

In this section, we outline the construction of a geometric spectral triple whose algebra is the coordinate algebra of the quantum group $SU_q(2)$. As a first step, we consider the corresponding classical case, namely the commutative spectral triple for the 3-sphere $\mathbb{S}^3$, with its unique spin structure (for the usual orientation) and its rotation-invariant metric.

The spectrum of the Dirac operator for $\mathbb{S}^3$ with the round metric can be obtained explicitly. The earliest reference is probably the 1974 paper of Hitchin [h-n74]; Cahen and Gutt [cg88, 1988] studied spin structures on compact symmetric spaces, and Bär [b-c92, 1992] laid out the theory for Dirac operators on homogeneous spin manifolds; see also [t-a95]. A description of the eigenspinors was rather late in coming: Camporesi and Higuchi [ch96, 1996] treated the case of $\mathbb{S}^n$ with generalized spherical coordinates. The best treatment for $\mathbb{S}^3$ is that of Homma [h-y00, 2000], who gives both eigenvalues and eigenspinors in complete detail.

There are two keys to finding the Dirac spectrum for $\mathbb{S}^3$ (with multiplicities, of course). The first is that if $M = G/H$ is a homogeneous space of a compact Lie group G, with a G-invariant metric, and if M is also spin, so that one can find a G-invariant Dirac operator, then the spinor space $\mathcal{H} = L^2(M, S)$ can be decomposed as a direct sum $\mathcal{H} = \bigoplus_{\sigma\in\widehat{G}} \mathcal{H}_\sigma$ of finite-dimensional subspaces, where G acts on $\mathcal{H}_\sigma$ as a multiple of the irreducible representation $\sigma \in \widehat{G}$. (Some of these "isotypical components" may be trivial.) Thus $D\!\!\!/$ reduces to a direct sum of finite-dimensional hermitian matrices acting on the subspaces $\mathcal{H}_\sigma$, and one can then decompose each $\mathcal{H}_\sigma$ into eigenspaces of $D\!\!\!/$.

The second key is that $\mathbb{S}^3$ is the manifold underlying the compact Lie group $SU(2)$, so its spinor bundle is trivial (one can just translate the fibre at the identity around the group manifold), namely $S \approx \mathbb{S}^3 \times \mathbb{C}^2$, and thus its spinor space is just $\mathcal{H} = L^2(SU(2)) \otimes \mathbb{C}^2$. Therefore, the above decomposition follows at once from the Peter–Weyl decomposition of $L^2(SU(2))$, and there is no need to examine the general theory of compact group representations.

The symmetry of the sphere $\mathbb{S}^3$ is obtained from

$$\mathbb{S}^3 \approx \frac{SO(4)}{SO(3)} \approx \frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)} \approx \frac{SU(2)\times SU(2)}{SU(2)} =: \frac{G}{H}$$

where we regard $H = SU(2)$ as the *diagonal* subgroup of $G = SU(2) \times SU(2)$. The quotient map $\pi\colon G \to G/H$ is given by $\pi(p, q) := pq^{-1}$, for $p, q \in SU(2)$. We can trivialize the principal $SU(2)$-bundle $G \approx \mathbb{S}^3 \times H$ by $(p, q) \mapsto (pq^{-1}, p)$. The associated spinor bundle $S = \mathrm{Spin}(4) \times_H \mathbb{C}^2$ is trivialized by $[(p, q), \xi] \mapsto (pq^{-1}, \mathrm{id}(p)\xi)$, where id is the fundamental representation of $SU(2)$ on $\mathbb{C}^2$ — namely, the inclusion $SU(2) \hookrightarrow M_2(\mathbb{C})$.

We may recall that the irreducible representations of $SU(2)$ are given by $\{\, \pi_j : j \in \frac{1}{2}\mathbb{Z}\,\}$; for each $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ there is a unique representation space $V_j \cong \mathbb{C}^{2j+1}$. Here π_0 is the trivial representation, and $\pi_{\frac{1}{2}}$ is the fundamental representation. One can make these representations unitary by introducing a suitable hermitian scalar product on each V_j. The contragredient representation on V_j^* is equivalent to π_j on V_j since there is only one representation for each dimension: this result is a special property of the group $SU(2)$. Therefore, the Peter–Weyl decomposition is

$$L^2(SU(2)) \cong \bigoplus_{2j=0}^{\infty} \mathrm{End}\, V_j \cong \bigoplus_{2j=0}^{\infty} V_j \otimes V_j^* \cong \bigoplus_{2j=0}^{\infty} V_j \otimes V_j. \tag{7.18}$$

When we tensor this with $\mathbb{C}^2 = V_{\frac{1}{2}}$, we apply the Clebsch–Gordan isomorphism

$V_j \otimes V_{\frac{1}{2}} \cong V_{j+\frac{1}{2}} \oplus V_{j-\frac{1}{2}}$ to the *first leg only* of (7.18) to get

$$\Bigl(\bigoplus_{2j=0}^{\infty} V_j \otimes V_j\Bigr) \otimes V_{\frac{1}{2}} \simeq V_{\frac{1}{2}} \oplus \bigoplus_{2j=1}^{\infty} (V_{j+\frac{1}{2}} \otimes V_j) \oplus (V_{j-\frac{1}{2}} \otimes V_j) =: W_0^{\uparrow} \oplus \bigoplus_{2j=1}^{\infty} W_j^{\uparrow} \oplus W_j^{\downarrow}.$$

These are the building blocks of the spinor space for $SU(2)$:

$$\begin{aligned} W_j^{\uparrow} &:= V_{j+\frac{1}{2}} \otimes V_j, \quad \dim W_j^{\uparrow} = (2j+1)(2j+2), \quad \text{for } j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\ , \\ W_j^{\downarrow} &:= V_{j-\frac{1}{2}} \otimes V_j, \quad \dim W_j^{\downarrow} = 2j(2j+1), \qquad\quad \text{for } j = \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\ . \end{aligned} \tag{7.19}$$

These are in fact eigenspaces of the classical Dirac operator $D\!\!\!/$ for the "round metric" on $\mathbb{S}^3$, i.e., the Riemannian metric that is invariant under both left and right translations of the group $SU(2)$. See, for instance, [h-y00] for full details on computing the eigenvalues and eigenspaces. One finds that

- Each $W_j^{\uparrow}$ is an eigenspace of $D\!\!\!/$ with positive eigenvalue $d_j^{\uparrow} = 2j + \frac{3}{2}$;
- Each $W_j^{\downarrow}$ is an eigenspace of $D\!\!\!/$ with negative eigenvalue $d_j^{\downarrow} = -(2j + \frac{1}{2})$.

Since these eigenspaces exhaust $\mathcal{H}$, this is the full spectrum of $D\!\!\!/$. Notice that the spectrum is symmetric about 0, and that $D\!\!\!/$ has trivial kernel.

Consequently, the positive operator $|D\!\!\!/|$ has eigenvalues $(2j + \frac{3}{2})$ on $W_j^{\uparrow} \oplus W_{j+\frac{1}{2}}^{\downarrow}$, whose dimension is $2(2j+1)(2j+2)$. Now we can compute the metric dimension (which we know must be 3). Let

$$N_R := \sum_{2j=0}^{R} 2(2j+1)(2j+2) = \tfrac{2}{3}(R^3 + 6R^2 + 11R),$$

so that $\log N_R \sim 3 \log R$, as $R \to \infty$. Therefore,

$$\sigma_{N_R}(|D\!\!\!/|^{-s}) = \sum_{k=0}^{R} 2(k+1)(k+2)(k+\tfrac{3}{2})^{-s} \sim 2 \sum_{k=0}^{R} (k+\tfrac{3}{2})^{-s+2} \sim 2 \int_{\frac{3}{2}}^{R+\frac{3}{2}} t^{2-s}\, dt,$$

so the critical exponent is indeed $s = 3$ where the estimate simplifies to $2 \log R \sim \frac{2}{3} \log N_R$. Thus we find that $|D\!\!\!/|^{-3} \in \mathcal{L}^{1+}$ with $\operatorname{Tr}^+ |D\!\!\!/|^{-3} = \frac{2}{3}$.

On the other hand, we already know that

$$\operatorname{Tr}^+ |D\!\!\!/|^{-3} = \frac{1}{3(2\pi)^3} \operatorname{Wres} |D\!\!\!/|^{-3} = \frac{2(4\pi)}{3(2\pi)^3} \operatorname{Vol}(\mathbb{S}^3) = \frac{2}{3} \frac{1}{2\pi^2} \operatorname{Vol}(\mathbb{S}^3),$$

which leads to the well-known formula $\operatorname{Vol}(\mathbb{S}^3) = 2\pi^2$.

Now we turn to the " quantum group" $SU_q(2)$ and its symmetries. This is a very well known object, starting from the seminal papers of Woronowicz [w-sl87] and it can be found in textbooks, e.g., [ks97]. However, to fix notations, we summarize some details here.

Definition 7.23. *Let q be a real number with $0 < q < 1$, and let $\mathcal{A} = \mathcal{O}(SU_q(2))$ be the $*$-algebra generated by two elements a and b, subject to the following commutation rules:*

$$\begin{gathered} ba = qab, \qquad b^*a = qab^*, \qquad bb^* = b^*b, \\ a^*a + q^2 b^*b = 1, \qquad aa^* + bb^* = 1. \end{gathered} \tag{7.20}$$

This is a Hopf $*$-algebra under the coproduct

$$\Delta a := a \otimes a - q\, b \otimes b^*,$$
$$\Delta b := b \otimes a^* + a \otimes b;$$

counit $\varepsilon(a) = 1$, $\varepsilon(b) = 0$; and antipode $Sa = a^*$, $Sb = -qb$, $Sb^* = -q^{-1}b^*$, $Sa^* = a$.

A quick way to remember all these formulas is to consider the "fundamental matrix"

$$U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \in M_2(\mathcal{A}),$$

which is "formally grouplike": $\Delta(U) = U \mathbin{\dot\otimes} U$, $\varepsilon(U) = 1_2$, $S(U) = U^{-1}$ under fairly obvious entrywise extensions of these operations to matrices over $\mathcal{A}$. Note also that the commutation relations (7.20) amount to the matrix U being unitary, that is, $U^*U = UU^* = 1_2$.

When $q = 1$, the matrix U becomes the fundamental representation of $SU(2)$, the entries a and b are functions on $SU(2)$, and since all unitary irreducible representations of $SU(2)$ can be recovered from the fundamental one by a Clebsch–Gordan decomposition, the $*$-algebra $\mathcal{O}(SU(2))$ generated by a and b is linearly spanned by the "matrix elements" of all such irreducible representations. We caal it, for short, the *polynomial algebra* of the compact Lie group $SU(2)$. By analogy, $\mathcal{O}(SU_q(2))$ may be thought of as the *polynomial algebra* over a (non-existent) "compact group" $SU_q(2)$. Such bad habits become ingrained because of the Gelfand correspondence.

The symmetries of the algebra $\mathcal{A} = \mathcal{O}(SU_q(2))$ can be expressed by means of another Hopf $*$-duality, that has a separating duality with $\mathcal{A}$.

Definition 7.24. *Let $\mathcal{U} = \mathcal{U}_q(\mathfrak{su}(2))$ be the algebra generated by elements e, f, k, with k invertible, satisfying the commutation relations*

$$ek = qke, \qquad kf = qfk, \qquad k^2 - k^{-2} = (q - q^{-1})(fe - ef), \tag{7.21}$$

with the coproduct Δ is given by

$$\Delta k = k \otimes k, \qquad \Delta e = e \otimes k + k^{-1} \otimes e, \qquad \Delta f = f \otimes k + k^{-1} \otimes f,$$

and counit ϵ, antipode S and star structure * given respectively by

$$\begin{aligned} &\epsilon(k) = 1, &\quad &Sk = k^{-1}, &\quad &k^* = k,\\ &\epsilon(f) = 0, & &Sf = -qf, & &f^* = e,\\ &\epsilon(e) = 0, & &Se = -q^{-1}e, & &e^* = f. \end{aligned}$$

Morover, let ϑ be the following algebra automorphism, and coalgebra antiautomorphism, of $\mathcal{U}$:

$$\vartheta(k) := k^{-1}, \quad \vartheta(f) := -e, \quad \vartheta(e) := -f.$$

The duality between $\mathcal{U}$ and $\mathcal{A}$ is given by the bilinear pairing determined by the following matchings of algebra generators:

$$\langle k, a\rangle = q^{\frac12}, \quad \langle k, a^*\rangle = q^{-\frac12}, \quad \langle e, -qb^*\rangle = \langle f, b\rangle = 1,$$

with all other couples of generators pairing to 0. With this duality pairing, we obtain the standard left and right convolution actions of $\mathcal{U}$ on $\mathcal{A}$, given respectively by

$$\begin{aligned} h \triangleright x &:= x_{(1)} \langle h, x_{(2)} \rangle, \\ x \triangleleft h &:= \langle h, x_{(1)} \rangle\, x_{(2)}, \end{aligned} \qquad \text{for all } h \in \mathcal{U},\ x \in \mathcal{A}.$$

Since $h \mapsto S^{-1}(\vartheta(h))$ is both an algebra antiautomorphism and a coalgebra automorphism of $\mathcal{U}$, we find that

$$h \cdot x := x \triangleleft S^{-1}(\vartheta(h))$$

is a left Hopf action of $\mathcal{U}$ on $\mathcal{A}$, distinct from the action $\triangleright$ of left convolution. These two left actions commute: $h_1 \cdot (h_2 \triangleright x) = h_2 \triangleright (h_1 \cdot x)$, so that together they define a left Hopf action of $\mathcal{U} \otimes \mathcal{U}$ on $\mathcal{A}$.

When $q = 1$, this combined Hopf action of $\mathcal{U}(\mathfrak{su}(2)) \otimes \mathcal{U}(\mathfrak{su}(2))$ on $\mathcal{O}(SU(2))$ reduces to the infinitesimal action of $\mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{spin}(4)$ on $\mathbb{S}^3$, coming from the action of the group Spin(4) on $\mathbb{S}^3$.

Definition 7.25. *Let λ and ρ be two commuting representations of the Hopf algebra $\mathcal{U}$ on a finite-dimensional vector space V. A $*$-algebra representation π of $\mathcal{A}$ on V is said to be* $(\boldsymbol{\lambda}, \boldsymbol{\rho})$-equivariant *if, for all $h \in \mathcal{U}$, $x \in \mathcal{A}$, $\xi \in V$:*

$$\begin{aligned} \lambda(h)\, \pi(x)\xi &= \pi(h_{(1)} \cdot x)\, \lambda(h_{(2)})\xi, \\ \rho(h)\, \pi(x)\xi &= \pi(h_{(1)} \triangleright x)\, \rho(h_{(2)}\xi. \end{aligned} \tag{7.22}$$

(Taken together, λ, ρ and π yield a representation of the crossed product algebra $(\mathcal{U} \otimes \mathcal{U}) \ltimes \mathcal{A}$ on V.)

The irreducible representations of $\mathcal{U}_q(\mathfrak{su}(2))$ are well known, see, for instance, [ks97]. They are "deformed" versions of the Lie algebra representations of $\mathfrak{su}(2)$. We first mention the standard notation for a "q-integer",

$$[n] := \frac{q^{-n} - q^n}{q^{-1} - q}.$$

Then, for each $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, we take the vector space $V_l = \mathbb{C}^{2l+1}$, with a basis $\{\, |lm\rangle : m = -l, -l+1, \dots, l-1, l \,\}$ which we declare to be orthonormal, thereby making V_l a finite-dimensional Hilbert space. It carries the $*$-representation σ_l, defined on generators by

$$\begin{aligned} \sigma_l(k)\, |lm\rangle &= q^m\, |lm\rangle, \\ \sigma_l(f)\, |lm\rangle &= \sqrt{[l-m][l+m+1]}\, |l, m+1\rangle, \\ \sigma_l(e)\, |lm\rangle &= \sqrt{[l+m][l-m+1]}\, |l, m-1\rangle. \end{aligned}$$

To get equivariant representations, we take tensor products $V_l \otimes V_l$ and consider their *algebraic* direct sum:

$$V := \bigoplus_{2l=0}^{\infty} V_l \otimes V_l, \qquad |lmn\rangle := |lm\rangle \otimes |ln\rangle,$$

Let $\mathcal{H}_\psi$ be the Hilbert-space completion of V, for which $\{\, |lmn\rangle : l \in \frac{1}{2}\mathbb{N};\ m, n = -l, \dots, l \,\}$ is an orthonormal basis. We introduce representations λ, ρ of $\mathcal{U}_q(\mathfrak{su}(2))$ on V by declaring the respective restrictions λ_l, ρ_l to the subspaces $V_l \otimes V_l$ to be

$$\lambda_l(h) := \sigma_l(h) \otimes \mathrm{id}_{V_l}, \qquad \rho_l(h) := \mathrm{id}_{V_l} \otimes \sigma_l(h).$$

Fact 7.26. *The polynomial algebra $\mathcal{A} = \mathcal{O}(SU_q(2))$ is linearly spanned by elements t^l_{mn} with the same indices l, m, n that label this orthonormal basis, such that*

$$t^0_{00} = 1, \quad t^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} = a, \quad t^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} = b, \quad \text{and} \quad \Delta t^l_{mn} = \sum_{k=-l}^{l} t^l_{mk} \otimes t^l_{kn}.$$

The multiplication on $\mathcal{A}$ is determined by the relations

$$t^j_{rs} t^l_{mn} = \sum_{k=|j-l|}^{j+l} C_q\begin{pmatrix} j & l & k \\ r & m & r+m \end{pmatrix} C_q\begin{pmatrix} j & l & k \\ s & n & s+n \end{pmatrix} t^k_{r+m,s+n},$$

where the $C_q(-)$ factors are q-Clebsch–Gordan coefficients [ks97].

The algebra can be completed to a noncommutative C*-algebra $C(SU_q(2))$, on which there is a faithful *Haar state* ψ determined by $\psi(1) = 1$ and $\psi(t^l_{mn}) = 0$ for $l > 0$. The involution on $\mathcal{A}$ and also on $C(SU_q(2))$ satisfies

$$(t^l_{mn})^* = (-1)^{2l+m+n} q^{n-m}\, t^l_{-m,-n}. \tag{7.23}$$

The GNS representation π_ψ acts on the aforementioned Hilbert space $\mathcal{H}_\psi$, and there is an injective linear GNS map $\eta_\psi \colon C(SU_q(2)) \to \mathcal{H}_\psi$ such that $\pi_\psi(x)\eta_\psi(y) = \eta_\psi(xy)$, satisfying

$$\|\eta_\psi(t^l_{mn})\| = \psi((t^l_{mn})^*\, t^l_{mn})^{1/2} = \frac{q^{-m}}{\sqrt{[2l+1]}}.$$

Thus, concretely, the standard orthonormal basis for $\mathcal{H}_\psi$ is given by

$$|lmn\rangle := q^m \sqrt{[2l+1]}\, \eta_\psi(t^l_{mn}). \tag{7.24}$$

In particular, $|000\rangle = \eta_\psi(1)$ is a cyclic (and separating) vector for the representation π_ψ.

Before turning to the detailed form of the equivariant representations of $\mathcal{A}$, which involves much shifting of half-integer indices, we pause to introduce a small notational dodge:

$$l^\pm := l \pm \tfrac{1}{2}, \quad m^\pm := m \pm \tfrac{1}{2}, \quad n^\pm := n \pm \tfrac{1}{2}.$$

Proposition 7.27. *Any (λ, ρ)-equivariant $*$-representation π of $\mathcal{O}(SU_q(2))$ on V is given by*

$$\begin{aligned} \pi(a)\,|lmn\rangle &= A^+_{lmn}|l^+m^+n^+\rangle + A^-_{lmn}|l^-m^+n^+\rangle, \\ \pi(b)\,|lmn\rangle &= B^+_{lmn}|l^+m^+n^-\rangle + B^-_{lmn}|l^-m^+n^-\rangle, \end{aligned} \tag{7.25}$$

with constants $A^\pm_{lmn}$, $B^\pm_{lmn}$ determined up to phase factors depending only on l.

Sketch proof. We use the equivariance relations (7.22), step by step, and then the algebra relations (7.20) to pin the down how $\pi(a)$ and $\pi(b)$ act on basis vectors and then to obtain the coefficients.

(1) First take $h = k$. Then $k \cdot a = k \triangleright a = q^{\frac{1}{2}} a$ shows that

$$\begin{aligned}\lambda(k)\,\pi(a)\,|lmn\rangle &= \pi(q^{\frac{1}{2}}a)\,\lambda(k)\,|lmn\rangle = q^{m+\frac{1}{2}}\pi(a)\,|lmn\rangle,\\ \rho(k)\,\pi(a)\,|lmn\rangle &= \pi(q^{\frac{1}{2}}a)\rho(k)\,|lmn\rangle = q^{n+\frac{1}{2}}\pi(a)\,|lmn\rangle,\end{aligned}$$

and thus $\pi(a)\,|lmn\rangle$ must lie in $\operatorname{span}\{\,|l'm^+n^+\rangle : l' \in \frac{1}{2}\mathbb{N}\,\}$. Similarly, $k \cdot b = q^{\frac{1}{2}} b$, $k \triangleright b = q^{-\frac{1}{2}} b$ gives $\pi(b)\,|lmn\rangle \in \operatorname{span}\{\,|l'm^+n^-\rangle : l' \in \frac{1}{2}\mathbb{N}\,\}$.

(2) Take $h = f$, $x = a$; then $f \cdot a = 0$ implies that

$$\lambda(f)\,\pi(a)\xi = \pi(f \cdot a)\,\lambda(k)\xi + \pi(k^{-1} \cdot a)\,\lambda(f)\xi = q^{-\frac{1}{2}}\pi(a)\,\lambda(f)\xi,$$

and thus,

$$\lambda(f)^r\pi(a) = q^{-r/2}\pi(a)\,\lambda(f)^r \quad \text{for any } r \in \mathbb{N}, \tag{7.26}$$

Therefore $\lambda(f)^r\pi(a)\,|lmn\rangle \propto \pi(a)\,|l, m + r, n\rangle = 0$ for $m + r > l$; but on the other hand, $\lambda(f)^r\,|l'm^+n^+\rangle \propto |l', m + \frac{1}{2} + r, n^+\rangle \neq 0$ for $m+\frac{1}{2}+r \leq l'$. Thus, if $l' > l+\frac{1}{2}$, the basis vector $|l'm^+n^+\rangle$ cannot appear in $\pi(a)\,|lmn\rangle$. This yields $l' \leq l + \frac{1}{2}$.

A similar argument, with (f, a) replaced by (e, a^*), using $e \cdot a^* = 0$, gives $l' \geq l - \frac{1}{2}$. Also, since $l - m$ and $l' - (m + \frac{1}{2})$ are both integers, the case $l' = l$ is excluded. Thus $l' = l \pm \frac{1}{2}$ only, and we have reached the form (7.25) of $\pi(a)$.

(3) To find the coefficients, we use (7.26) again. If we apply both sides to the vector $|lmn\rangle$ for $r = 1$ and compare the results, we get a recurrence relation for the index m:

$$q^{-\frac{1}{2}} A^+_{l,m+1,n}[l + m + 1]^{\frac{1}{2}} = A^+_{lmn}[l + m + 2]^{\frac{1}{2}}.$$

Using $k \triangleright a = q^{\frac{1}{2}}a$ and $f \triangleright a = 0$, the second relation in (7.22) gives $\rho(f)\pi(a) = q^{\frac{1}{2}}\pi(a)\,\rho(f)$, that leads to a similar recurrence relation for the index n:

$$q^{-\frac{1}{2}} A^+_{lm,n+1}[l + n + 1]^{\frac{1}{2}} = A^+_{lmn}[l + n + 2]^{\frac{1}{2}}.$$

These recurrence relations are solved by

$$A^+_{lmn} = q^{\frac{1}{2}(m+n)}[l + m + 1]^{\frac{1}{2}}[l + n + 1]^{\frac{1}{2}}\ a^+_l$$

for some constants a^+_l; and similar expressions are found for $A^-_{lmn} \propto a^-_l$, $B^+_{lmn} \propto b^+_l$ and $B^-_{lmn} \propto b^-_l$.

(4) Now we use the equivariance relation

$$\rho(e)\pi(a) = \pi(e \triangleright a)\rho(k) + \pi(k^{-1} \triangleright a)\rho(e) = \pi(b)\rho(k) + q^{-\frac{1}{2}}\pi(a)\rho(e)$$

to find that $b^+_l = q^l a^+_l$ and $b^-_l = -q^{-l-1}a^-_l$. The relation

$$\lambda(e)\pi(b) = \pi(e \cdot b)\lambda(k) + \pi(k^{-1} \cdot b)\lambda(e) = q^{-1}\pi(a^*)\lambda(k) + q^{-\frac{1}{2}}\pi(b)\lambda(e)$$

then yields $(a^-_{l+\frac{1}{2}})^\star = q^{2l+\frac{3}{2}}\,a^+_l$, so we need only determine the parameters a^+_l. All equivariance relations have now been used.

(5) Next, the commutation relation $ba = qab$ implies that $\pi(b)\pi(a) = q\pi(a)\pi(b)$; comparing the matrix element $\langle l, m+1, n \mid (\cdot) \mid lmn\rangle$ for both these operators, we find the recurrence relation

$$q[2l+2]\,|a_l^+|^2 = [2l]\,|a_{l-\frac{1}{2}}^+|^2.$$

This determines all $|a_l^+|$ once $|a_0^+|$ is known. For that, note that $b_0^+ = a_0^+$ and use the relation $a^*a + q^2b^*b = 1$ to get

$$(1+q^2)\,|a_0^+|^2 = \langle 000 \mid \pi(a^*a + q^2b^*b) \mid 000\rangle = 1.$$

The phase factors $a_l^+/|a_l^+|$ remain undetermined. We are free to fix them by demanding that all a_l^+ be positive (in the quantum theory of angular momentum, this requirement is called the Condon–Shortley phase convention.) Actually, any other assignment of phases to the a_l^+ coefficients determines a unitary equivalence between π and another (λ, ρ)-equivariant representation of $\mathcal{A}$. $\square$

Having thus determined the existence, and uniqueness up to equivalence, of a (λ, ρ)-equivariant representation of $\mathcal{O}(SU_q(2))$ on V, we may realize that we have in fact given an explicit presentation of the *left regular representation* of $C(SU_q(2))$, which is just the GNS representation for the Haar state on the Hilbert space $\mathcal{H}_\psi$.

The point of this rather bare-handed approach to the regular representation is that the *spin representation* can be constructed in just the same way. Indeed, with some last-minute choice of phase factors, this will be an equivariant representation of $\mathcal{O}(SU_q(2))$ on $V \oplus V$, and also of $C(SU_q(2))$ on $\mathcal{H} = \mathcal{H}_\psi \oplus \mathcal{H}_\psi$.

The crucial extra step in constructing this spin representation is the good choice of a *spinor basis*. The spinor subspaces $W_j^\uparrow$, $W_j^\downarrow$ of (7.19) will be used, just as in the $q = 1$ case, but the isomorphisms $V_j \otimes V_{\frac{1}{2}} \simeq V_{j+\frac{1}{2}} \oplus V_{j-\frac{1}{2}}$ now depend on certain q-Clebsch–Gordan coefficients, as follows:

- for $j = \frac{1}{2}, 1, \frac{3}{2}, \dots$; $\mu = -j, \dots, j$ and $n = -j^-, \dots, j^-$, let

$$|j\mu n{\downarrow}\rangle := C_{j\mu}\,|j^-\mu^+n\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + S_{j\mu}\,|j^-\mu^-n\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle;$$

- for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$; $\mu = -j, \dots, j$ and $n = -j^+, \dots, j^+$, let

$$|j\mu n{\uparrow}\rangle := -S_{j+1,\mu}\,|j^+\mu^+n\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + C_{j+1,\mu}\,|j^+\mu^-n\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle,$$

- where in both cases the coefficients are

$$C_{j\mu} := q^{-(j+\mu)/2}\,\frac{[j-\mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}, \qquad S_{j\mu} := q^{(j-\mu)/2}\,\frac{[j+\mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}.$$

It is easy to check that $C_{j\mu}^2 + S_{j\mu}^2 = 1$.

We now modify the representations λ, ρ of $\mathcal{U}_q(\mathfrak{su}(2))$ on V to get analogous representations on $V \oplus V = V \otimes \mathbb{C}^2$. If $h \in \mathcal{U}_q(\mathfrak{su}(2))$, we set

$$\begin{aligned}\lambda'(h) &:= (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)}),\\ \rho'(h) &:= (\rho \otimes (\varepsilon \oplus \varepsilon))(\Delta h) = \rho(h) \otimes 1_2.\end{aligned}$$

Exercise 7.28. *Show that* $|j\mu n{\uparrow}\rangle$ *and* $|j\mu n{\downarrow}\rangle$ *are eigenvectors for* $\lambda'(C_q)$*, with eigenvalue* $q^{2j+1} + q^{2j-1}$*, where* C_q *is the* q-Casimir*,*

$$C_q = qk^2 + q^{-1}k^{-2} + (q - q^{-1})^2 ef,$$

which is a central element of $\mathcal{U}_q(\mathfrak{su}(2))$*. Are these also eigenvectors for* $\rho'(C_q)$*?*

Proposition 7.29. *If we write* $|j\mu n\rangle\!\rangle := \begin{pmatrix} |j\mu n{\uparrow}\rangle \\ |j\mu n{\downarrow}\rangle \end{pmatrix}$ *to denote a pair of basis elements of* V *—the lower component is* 0 *for* $j = 0$ *or* $n = \pm(j + \frac{1}{2})$*—, then the representation* $\pi' := \pi \otimes 1_2$ *on* $V \otimes \mathbb{C}^2$ *is* (λ', ρ')*-equivariant, and is given by*

$$\begin{aligned}\pi'(a)\,|j\mu n\rangle\!\rangle &= \alpha^+_{j\mu n}|j^+\mu^+n^+\rangle\!\rangle + \alpha^-_{j\mu n}|j^-\mu^+n^+\rangle\!\rangle,\\ \pi'(b)\,|j\mu n\rangle\!\rangle &= \beta^+_{j\mu n}|j^+\mu^+n^-\rangle\!\rangle + \beta^-_{j\mu n}|j^-\mu^+n^-\rangle\!\rangle,\end{aligned}$$

where $\alpha^\pm_{j\mu n}$ *and* $\beta^\pm_{j\mu n}$ *are certain triangular* 2×2 *matrices, shaped like this:*

$$\alpha^+_{j\mu n}, \beta^+_{j\mu n} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \qquad \alpha^-_{j\mu n}, \beta^-_{j\mu n} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}. \tag{7.27}$$

They are determined up to phase factors depending only on j. □

We omit the proof, which consists of running the steps of the proof of Proposition 7.27 with 2×2 matrices. The details are given in [dlsv05]. We refer to π' as the *spin representation* of the algebra $\mathcal{O}(SU_q(2))$.

In the classical case $q = 1$, the above procedures give exactly the eigenspace decomposition of the spinor space of the Dirac operator $D\!\!\!/$, with the correct multiplicities. The basis vectors $|j\mu n{\updownarrow}\rangle$, for each fixed j, can be regarded as orthonormal bases for the eigenspaces, yielding the following diagonalization of $D\!\!\!/$ when $q = 1$:

$$D\!\!\!/|j\mu n{\uparrow}\rangle_{q=1} = (2j + \tfrac{3}{2})\,|j\mu n{\uparrow}\rangle_{q=1}, \qquad D\!\!\!/|j\mu n{\downarrow}\rangle_{q=1} = -(2j + \tfrac{1}{2})\,|j\mu n{\downarrow}\rangle_{q=1}.$$

We now *define* the Dirac operator on $SU_q(2)$, for $0 < q < 1$, to be the diagonal operator given by the same formulas on the q-spinor basis:

$$D|j\mu n{\uparrow}\rangle = (2j + \tfrac{3}{2})\,|j\mu n{\uparrow}\rangle, \qquad D|j\mu n{\downarrow}\rangle = -(2j + \tfrac{1}{2})\,|j\mu n{\downarrow}\rangle. \tag{7.28}$$

In other words, D is an *isospectral deformation* of the classical Dirac operator $D\!\!\!/$, since $D = U D\!\!\!/ U^*$ by the obvious unitary map U that matches the respective spinor bases.

Fact 7.30. *The commutators* $[D, \pi'(a)]$ *and* $[D, \pi'(b)]$*, defined initially on* $V \oplus V$*, extend to bounded operators on* $\mathcal{H}$*.*

The proof of this fact is not hard, but needs the exact values of the matrices $\alpha^{\pm}_{j\mu n}$ and $\beta^{\pm}_{j\mu n}$. The boundedness is certainly true when $q = 1$, and one can check that the diagonal elements of these matrices differ from their $q \to 1$ limits by terms that are uniformly bounded as $j \to \infty$; and moreover the off-diagonal elements are small: they are $O(q^{2j})$ as $j \to \infty$.

Thus with $\mathcal{A} = \mathcal{O}(SU_q(2))$, $\mathcal{H} = \mathcal{H}_\psi \oplus \mathcal{H}_\psi$ and D given by (7.28), we have constructed a spectral triple for $SU_q(2)$. Since it is isospectral to the classical case, the metric dimension is 3. Our painstaking construction now yields an extra bonus [dlssv06].

Proposition 7.31. *The triple $(\mathcal{O}(SU_q(2)), \mathcal{H}, D)$ is a* regular *spectral triple.*

Proof. Since $|D|\,|j\mu n\rangle\rangle = \begin{pmatrix} 2j+\frac32 & 0 \\ 0 & 2j+\frac12 \end{pmatrix} |j\mu n\rangle\rangle$, we obtain

$$
\begin{aligned}
&[|D|, \pi'(a)]\,|j\mu n\rangle\rangle = \\
&= \left\{ \begin{pmatrix} 2j+\frac52 & 0 \\ 0 & 2j+\frac32 \end{pmatrix} \alpha^+_{j\mu n} - \alpha^+_{j\mu n} \begin{pmatrix} 2j+\frac32 & 0 \\ 0 & 2j+\frac12 \end{pmatrix} \right\} |j^+\mu^+n^+\rangle\rangle \\
&\qquad - \left\{ \begin{pmatrix} 2j+\frac12 & 0 \\ 0 & 2j-\frac12 \end{pmatrix} \alpha^-_{j\mu n} - \alpha^-_{j\mu n} \begin{pmatrix} 2j+\frac32 & 0 \\ 0 & 2j+\frac12 \end{pmatrix} \right\} |j^-\mu^+n^+\rangle\rangle,
\end{aligned}
$$

and the triangularity (7.27) of the matrices $\alpha^+_{j\mu n}$ and $\alpha^-_{j\mu n}$ means that the off-diagonal terms cancel exactly! This happens because of the precise form of the eigenvalues of D, since $2j^+ + \frac12 = 2j + \frac32$, and so on.

Thus $[|D|, \pi'(a)]$ is just $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ times the diagonal part of $\pi'(a)$ on each of the two-dimensional subspaces spanned by the pair $|j\mu n\rangle\rangle$; from which we obtain $\|\delta(\pi'(a))\| \leq \|\pi'(a)\|$. Similarly for $\pi'(b)$.

Next, in computing $[|D|, [D, \pi'(a)]] = [D, [|D|, \pi'(a)]]$, over each two-dimensional subspace, we must subtract two *diagonal* matrices, and the diagonal entries are preserved up to a possible change of sign. It follows that $\|\delta([D, \pi'(a)])\| \leq \|\pi'(a)\|$ also. By induction the same estimate holds for all $\|\delta^k(\pi'(a))\|$ and $\|\delta^k([D, \pi'(a)])\|$, too. Thus both $\pi'(a)$ and $[D, \pi'(a)]$ lie in the domain of δ^k for all $k \in \mathbb{N}$. Similarly for $\pi'(b)$, and indeed for the whole $*$-algebra $\pi'(\mathcal{A})$ generated by $\pi'(a)$ and $\pi'(b)$. □

We come now to the matter of the charge conjkugation J, to get a real structure on $(\mathcal{A}, \mathcal{H}, D)$. Guided by the example of noncommutative tori (in dimensions 2 and 3), we may guess that $J = J_\psi \oplus J_\psi$, where J_ψ is a *Tomita conjugation* operator on the GNS representation space $\mathcal{H}_\psi$. We pause to describe this operator.

The involutive antilinearity operator $T_\psi \colon \eta_\psi(x) \mapsto \eta_\psi(x^*)$ on $\mathcal{H}_\psi$ —usually called S_ψ in books on Tomita–Takesaki theory [t-m02] is defined, in view of (7.23) and (7.23), by

$$
T_\psi\,|lmn\rangle = (-1)^{2l+m+n} q^{m+n}\,|l, -m, -n\rangle,
$$

extended by antilinearity to all of $\mathcal{O}(SU_q(2))$. The antilinear adjoint T^*_ψ satisfies $\langle \eta \mid T^*_\psi \xi\rangle = \langle \xi \mid T_\psi \eta\rangle$ and thus $T^*_\psi\,|lmn\rangle = (-1)^{2l-m-n} q^{-m-n}\,|l, -m, -n\rangle$. The Tomita modular operator Δ_ψ thus has $\mathcal{O}(SU_q(2)) \subseteq \operatorname{Dom} \Delta_\psi$ and $\Delta_\psi\,|lmn\rangle =$

$T_\psi^* T_\psi \, |lmn\rangle = q^{2(m+n)} \, |l, -m, -n\rangle$. The defining relation $T_\psi =: J_\psi \Delta_\psi^{1/2}$ yields the action of J_ψ on basis vectors:

$$J_\psi \, |lmn\rangle = (-1)^{2l+m+n} \, |l, -m, -n\rangle.$$

Recall that $l + m$ and $l + n$ may be any nonnegative integers, so the sign may be even or odd.

Exercise 7.32. *Show that* $T_\psi \lambda(h) T_\psi^{-1} = \lambda((Sh)^*)$ *for* $h \in \mathcal{U}_q(\mathfrak{su}(2))$ *—it is enough to let* h *be a generator and to check the identities on basis vectors— and similarly with* ρ *replacing* λ*. This shows that the Tomita operator* T_ψ *implements the involution* $h \mapsto (Sh)^*$ *in the "dual" Hopf algebra.*

It turns out that the Tomita conjugation for the *spin* respresentation, namely $J_\psi \oplus J_\psi$ (since π' and $\pi \oplus \pi$ are equivalent on $\mathcal{H} = \mathcal{H}_\psi \oplus \mathcal{H}_\psi$), *does not* become diagonal in the spinor basis $\{|j\mu n\updownarrow\rangle\}$, so it does not commute with D as we might have expected. We need $JDJ^{-1} = +D$ for a real structure in dimension 3. So instead we define J directly, and we guarantee the commutation relation by ensuring that J preserves the eigenspaces of D.

Definition 7.33. *The conjugation operator* J *is the antiunitary operator on* $\mathcal{H}$ *determined by the following action on the spinor basis:*

$$\begin{aligned} J \, |j\mu n\uparrow\rangle &:= i^{2(2j+\mu+n)} \, |j, -\mu, -n, \uparrow\rangle, \\ J \, |j\mu n\downarrow\rangle &:= i^{2(2j-\mu-n)} \, |j, -\mu, -n, \downarrow\rangle. \end{aligned}$$

It is immediate that $J^2 = -1$*, since each* $2j \pm (\mu + n)$ *is the sum of an integer and a half-integer.*

This allows to define a "spin representation from the right" of $\mathcal{A}$ on $\mathcal{H}$ by setting

$$\pi'_R(x) := J \, \pi'(x) \, J^{-1}.$$

It turns out, however, that this obvious right action of $\mathcal{A}$ on $\mathcal{H}$ does not commute with the left action coming from the representation π', and the first order property is also broken. That is to say, the commutation relations $[\pi'(x), \pi'_R(y)] = 0$ and $[[D, \pi'(x)], \pi'_R(y)] = 0$ do not hold for general elements $x, y \in \mathcal{A}$.

Since these relations do hold in the limit $q \to 1$, we may conjecture that they may hold "approximately" in $\mathcal{O}(SU_q(2))$. The precise result is as follows: see [dlsv05].

Definition 7.34. *Define the operator* L_q *on* $\mathcal{H}$ *by*

$$L_q |j\mu n\updownarrow\rangle := q^j \, |j\mu n\updownarrow\rangle.$$

This is a positive trace-class operator commuting with D*. Indeed, one easily sees that* $\operatorname{Tr} L_q = 2 + \sum_{2j=1}^\infty 2(2j+1)(2j+2) q^j < \infty$*. Let* $\mathcal{K}_q$ *be the ideal of compact operators generated by* L_q *(it is not closed in any norm, since there are trace-class operators not in* $\mathcal{K}_q$*).*

Exercise 7.35. *If* $T \in \mathcal{K}_q$ *show that* $k^{1/p} s_k(T) \to 0$ *as* $k \to \infty$*, for any* $p \in \mathbb{N}$*. That is to say,* T *is an "infinitesimal of arbitrarily high order".*

Because of the exponentially fast decrease of the off-diagonal matrix elements (7.27) of $\pi'(a)$ and $\pi'(b)$, one can show that

$$[\pi'(x), \pi'_R(y)] \in \mathcal{K}_q, \quad \text{and} \quad [[D, \pi'(x)], \pi'_R(y)] \in \mathcal{K}_q \quad \text{for all } x, y \in \mathcal{A}.$$

Clearly, it is enough to check this for $x, y \in \{a, a^*, b, b^*\}$, which may be done by long but straightforward computations [dlsv05].

The conclusion is that, up to allowable infinitesimal corrections in $\mathcal{K}_q$, the reality and first-order properties of the spectral triple $(\mathcal{O}(SU_q(2)), \mathcal{H}_\psi \oplus \mathcal{H}_\psi, D)$ hold. The regularity property holds, on the nose. If desired, one can also complete $\mathcal{O}(SU_q(2))$, keeping regularity, to a Fréchet pre-C*-algebra. In this way, one reaches the noncommutative spin geometry on the quantum group $SU_q(2)$.

Chapter 8

Exercises

8.1 Examples of Dirac operators

8.1.1 The circle

Let $M := \mathbb{S}^1$, regarded as $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$; that is to say, we parametrize the circle by the half-open interval $[0, 1)$ rather than $[0, 2\pi)$, say. Then $\mathcal{A} = C^\infty(\mathbb{S}^1)$ can be identified with periodic smooth functions on $\mathbb{R}$ with period 1:

$$\mathcal{A} \cong \{\, f \in C^\infty(\mathbb{R}) : f(t+1) \equiv f(t) \,\}.$$

Since $\mathrm{Cl}(\mathbb{R}) = \mathbb{C}1 \oplus \mathbb{C}e_1$ as a $\mathbb{Z}_2$-graded algebra, we see that $\mathcal{B} = \mathcal{A}$ in this case; and since $n = 1$, $m = 0$ and $2^m = 1$, there is a "trivial" spin structure given by $\mathcal{S} := \mathcal{A}$ itself. The charge conjugation is just $C = K$, where K means *complex conjugation* of functions. With the flat metric on the circle, the Dirac operator is just

$$D\!\!\!/ := -i\,\frac{d}{dt}.$$

Exercise 8.1. *Show that its spectrum is*

$$\mathrm{sp}(D\!\!\!/) = 2\pi\mathbb{Z} = \{\, 2\pi k : k \in \mathbb{Z} \,\},$$

by first checking that the eigenfunctions $\psi_k(t) := e^{2\pi ikt}$ form an orthonormal basis for the Hilbert-space completion $\mathcal{H}$ of $\mathcal{S}$ – using Fourier series theory.

The point is that the closed span of these eigenvectors is all of $\mathcal{H}$, so that $\mathrm{sp}(D\!\!\!/)$ contains no more than the corresponding eigenvalues.

Next, consider

$$\mathcal{S}' := \{\, \phi \in C^\infty(\mathbb{R}) : \phi(t+1) \equiv -\phi(t) \,\},$$

which can be thought of as the space of smooth functions on the interval $[0, 1]$ "with antiperiodic boundary conditions".

Exercise 8.2. *Explain in detail how $\mathcal{S}'$ can be regarded as a $\mathcal{B}$-$\mathcal{A}$-bimodule, and how $C = K$ acts on it as a charge-conjugation operator. Taking $D\!\!\!/ := -i\,d/dt$ again, but now as an operator with domain $\mathcal{S}'$ on the Hilbert-space completion of $\mathcal{S}'$, show that its spectrum is now*

$$\mathrm{sp}(D\!\!\!/) = 2\pi(\mathbb{Z} + \tfrac{1}{2}) = \{\, \pi(2k+1) : k \in \mathbb{Z} \,\},$$

by checking that $\phi_k(t) := e^{\pi i(2k+1)t}$ *are a complete set of eigenfunctions.*

The circle $\mathbb{S}^1$ thus carries two inequivalent spin structures: their inequivalence is most clearly manifest in the different spectra of the Dirac operators. Notice that $0 \in \operatorname{sp}(D\!\!\!/)$ for the "untwisted" spin structure where $\mathcal{S} = \mathcal{A}$, while $0 \notin \operatorname{sp}(D\!\!\!/)$ for the "twisted" spin structure whose spinor module is $\mathcal{S}'$. There are no more spin structures to be found, since $H^1(\mathbb{S}^1, \mathbb{Z}_2) = \mathbb{Z}_2$.

8.1.2 The (flat) torus

On the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, we use the Riemannian metric coming from the usual flat metric on $\mathbb{R}^2$. Thus, if we regard $\mathcal{A} = C^\infty(\mathbb{T}^2)$ as the smooth periodic functions on $\mathbb{R}^2$ with $f(t^1, t^2) \equiv f(t^1 + 1, t^2) \equiv f(t^1, t^2 + 1)$, then (t^1, t^2) define local coordinates on $\mathbb{T}^2$, with respect to which all Christoffel symbols are zero, namely $\Gamma^k_{ij} = 0$, and thus $\nabla = d$ represents the Levi-Civita connection on 1-forms.

In this case, $n = 2$, $m = 1$ and $2^m = 2$, so we use "two-component" spinors; that is, the spinor bundles $S \to \mathbb{T}^2$ are of rank two. There is the "untwisted" one, where S is the trivial rank-two $\mathbb{C}$-vector bundle, and $\mathcal{S} \cong \mathcal{A}^2$. The Clifford algebra in this case is just $\mathcal{B} = M_2(\mathcal{A})$. Using the standard Pauli matrices:

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can write the charge conjugation operator as

$$C = -i\,\sigma^2\,K$$

where K again denotes (componentwise) complex conjugation.

Exercise 8.3. *Find three more spinor structures on* $\mathbb{T}^2$*, exhibiting each spinor module as a* $\mathcal{B}$*-*$\mathcal{A}$*-bimodule, with the appropriate action of* C*. (Use* $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$*.)*

Exercise 8.4. *Check that*

$$D\!\!\!/ = -i\,(\sigma^1\,\partial_1 + \sigma^2\,\partial_2) = \begin{pmatrix} 0 & -\partial_2 - i\partial_1 \\ \partial_2 - i\partial_1 & 0 \end{pmatrix}$$

where $\partial_1 = \partial/\partial t^1$ *and* $\partial_2 = \partial/\partial t^2$*, is indeed the Dirac operator on the untwisted spinor module* $\mathcal{S} = \mathcal{A}^2$*. Compute* $\operatorname{sp}(D\!\!\!/^2)$ *by finding a complete set of eigenvectors. Then show that*

$$\operatorname{sp}(D\!\!\!/) = \{\, \pm 2\pi\sqrt{r_1^2 + r_2^2} : (r_1, r_2) \in \mathbb{Z}\,\}$$

by finding the eigenspinors for each of these eigenvalues. What can be said of the multiplicities *of these eigenvalues? and what is the dimension of* $\ker D\!\!\!/$*?*

Notice that σ^3 does not appear in the formula for $D\!\!\!/$; its role here is to give the $\mathbb{Z}_2$-grading operator: $c(\gamma) = \sigma^3$ —regarded as a constant function with values in $M_2(\mathbb{C})$— in view of the relation $\sigma^3 = -i\,\sigma^1\sigma^2$ among Pauli matrices.

On the 3-torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$, where now $n = 3$, $m = 1$ and again $2^m = 2$, we get two-component spinors. Again we may use a flat metric and an untwisted spin structure with $\mathcal{S} = \mathcal{A}^2$. The charge conjugation is still $C = -i\,\sigma^2 K$ on $\mathcal{S}$, so that $C^2 = -1$ also in this 3-dimensional case. The Dirac operator is now

$$D\!\!\!/ = -i\,(\sigma^1\,\partial_1 + \sigma^2\,\partial_2 + \sigma^3\,\partial_3) = \begin{pmatrix} -i\,\partial_3 & -\partial_2 - i\partial_1 \\ \partial_2 - i\partial_1 & i\,\partial_3 \end{pmatrix}.$$

Exercise 8.5. *Compute* $\operatorname{sp}(D\!\!\!/^2)$ *and* $\operatorname{sp}(D\!\!\!/)$ *for this Dirac operator on* $\mathbb{T}^3$.

8.1.3 The Hodge–Dirac operator on $\mathbb{S}^2$

If M is a compact, oriented Riemannian manifold that has no spin^c structures, can one define Dirac-like operators on an $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{E}$ that is not pointwise irreducible under the action of $\mathcal{B}$? It turns out that one can do so, if $\mathcal{E}$ carries a "Clifford connection", that is, a connection $\nabla^{\mathcal{E}}$ such that

$$\nabla^{\mathcal{E}}(c(\alpha)s) = c(\nabla\alpha)\,s + c(\alpha)\,\nabla^{\mathcal{E}} s,$$

for $\alpha \in \mathcal{A}^1(M)$, $s \in \mathcal{E}$, and which is Hermitian with respect to a suitable $\mathcal{A}$-valued sesquilinear pairing on $\mathcal{E}$. For instance, we may take $\mathcal{E} = \mathcal{A}^\bullet(M)$, the full algebra of differential forms on M, which we know to be a left $\mathcal{B}$-module under the action generated by $c(\alpha) = \varepsilon(\alpha) + \iota(\alpha^\sharp)$. The Clifford connection is just the Levi-Civita connection on all forms, obtaining by extending the one on $\mathcal{A}^1(M)$ with the Leibniz rule (and setting $\nabla f := df$ on functions). The pairing $(\alpha \mid \beta) := g(\bar\alpha, \beta)$ extends to a pairing on $\mathcal{A}^\bullet(M)$; by integrating the result over M with respect to the volume form ν_g, we get a scalar product on forms, and we can then complete $\mathcal{A}^\bullet(M)$ to a Hilbert space.

If $\{E_1, \dots, E_n\}$ and $\{\theta^1, \dots, \theta^n\}$ are local orthonormal sections for $\mathcal{X}(M)$ and $\mathcal{A}^1(M)$ respectively, compatible with the given orientation, so that $c(\theta^j) = \varepsilon(\theta^j) + \iota(E_j)$ locally, then

$$\star := c(\gamma) = (-i)^m\,c(\theta^1)\,c(\theta^2)\dots c(\theta^n)$$

is globally well-defined as an $\mathcal{A}$-*linear operator* taking $\mathcal{A}^\bullet(M)$ onto itself, such that $\star^2 = 1$. This is the *Hodge star* operator, and it exchanges forms of high and low degree.

Exercise 8.6. *If* $\{1, \dots, n\} = \{i_1, \dots, i_k\} \uplus \{j_1, \dots, j_{n-k}\}$*, show that locally,*

$$\star(\theta^{i_1} \wedge \dots \wedge \theta^{i_k}) = \pm i^m\,\theta^{j_1} \wedge \dots \wedge \theta^{j_{n-k}},$$

where the sign depends on $i_1, \dots, i_k$*. Conclude that* $\star$ *maps* $\mathcal{A}^k(M)$ *onto* $\mathcal{A}^{n-k}(M)$*, for each* $k = 0, 1, \dots, n$.

(Actually, our sign conventions differ from the usual ones in differential geometry books, that do not include the factor $(-i)^m$. With the standard conventions, $\star^2 = \pm 1$ on each $\mathcal{A}^k(M)$, with a sign depending on the degree k.)

The *codifferential* δ on $\mathcal{A}^\bullet(M)$ is defined by

$$\delta := -\star d \star.$$

This operation *lowers* the form degree by 1. The *Hodge–Dirac* operator is defined to be $-i(d + \delta)$ on $\mathcal{A}^\bullet(M)$. One can show that, on the Hilbert-space completion, the operators d and $-\delta$ are adjoint to one another, so that $-i(d+\delta)$

extends to a selfadjoint operator. (With the more usual sign conventions, d and $+\delta$ are adjoint, so that the Hodge–Dirac operator is written simply $d+\delta$.)

Now we take $M = \mathbb{S}^2$, the 2-sphere of radius 1. The round (i.e., rotation-invariant) metric on $\mathbb{S}^2$ is written $g = d\theta^2 + \sin^2\theta\, d\phi^2$ in the usual spherical coordinates, which means that $\{d\theta, \sin\theta\, d\phi\}$ is a local orthonormal basis of 1-forms on $\mathbb{S}^2$. The area form is $\nu = \sin\theta\, d\theta \wedge d\phi$. The Hodge star is specified by defining it on 1 and on $d\theta$:

$$\star(1) := -i\,\nu, \qquad \star(d\theta) := i\sin\theta\, d\phi.$$

To find the eigenforms of the Hodge–Dirac operator, it is convenient to use another set of coordinates, obtained form the Cartesian relation $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ by setting $\zeta := x^1 + ix^2 = e^{i\phi}\cos\theta$, along with $x^3 = \cos\theta$; the pair (ζ, x^3) can serve as coordinates for $\mathbb{S}^2$, subject to the relation $\zeta\bar\zeta + (x^3)^2 = 1$. (The extra variable $\bar\zeta$ gives a third coordinate, extending $\mathbb{S}^2$ to $\mathbb{R}^3$.)

Exercise 8.7. *Check that in the (ζ, x^3) coordinates, the Hodge star is given by*

$$\star(\zeta) = -i\, d\zeta \wedge dx^3, \qquad \star(d\zeta) = x^3\, d\zeta - \zeta\, dx^3.$$

Exercise 8.8. *Consider the (complex) vectorfields on $\mathbb{R}^3$ given by*

$$L_+ := 2ix^3 \frac{\partial}{\partial\bar\zeta} + i\zeta\frac{\partial}{\partial x^3}, \quad L_- := 2ix^3\frac{\partial}{\partial\zeta} - i\bar\zeta\frac{\partial}{\partial x^3}, \quad L_3 := i\bar\zeta\frac{\partial}{\partial\bar\zeta} - i\zeta\frac{\partial}{\partial\zeta}.$$

Verify the commutation relations $[L_+, L_-] = -2i\,L_3$, $[L_3, L_-] = i\,L_-$ and $[L_3, L_+] = -i\,L_+$.

These commutation relations show that if $L_\pm =: L_1 \pm i\,L_2$, then L_1, L_2, L_3 generate a representation of the Lie algebra of the rotation group SO(3). One obtains representation spaces of $SO(3)$ by finding functions f_0 ("highest weight vectors") such that $L_3 f_0$ is a multiple of f_0, $L_+ f_0 = 0$, and $\{\,(L_-)^r f_0 : r \in \mathbb{N}\,\}$ spans a space of finite dimension. To get spaces of differential forms with these properties, one extends each vector field L_j to an operator on $\mathcal{A}^\bullet(\mathbb{S}^2)$, namely its *Lie derivative* $\mathcal{L}_j$, just by requiring that $\mathcal{L}_j\, d = d\,\mathcal{L}_j$. Since the Hodge star operator is unchanged by applying a rotation to an orthonormal basis of 1-forms, one can also show that $\mathcal{L}_j\star = \star\mathcal{L}_j$, so that the Hodge–Dirac operator $-i(d+\delta)$ commutes with each $\mathcal{L}_j$. This gives a method of finding subspaces of joint eigenforms for each eigenvalue of the Hodge–Dirac operator.

We introduce the following families of forms:

$$\begin{aligned} \phi_l^+ &:= i\zeta^l(1 - i\nu), \quad l = 0,1,2,3,\dots; \\ \phi_l^- &:= i\zeta^l(1 + i\nu), \quad l = 0,1,2,3,\dots; \end{aligned}$$

$$\begin{aligned} \psi_l^+ &:= \zeta^{l-1}(d\zeta + \star(d\zeta)), \quad l = 1,2,3,\dots; \\ \psi_l^- &:= \zeta^{l-1}(d\zeta - \star(d\zeta)), \quad l = 1,2,3,\dots. \end{aligned}$$

Clearly, $\star(\phi_l^\pm) = \pm\phi_l^\pm$ and $\star(\psi_l^\pm) = \pm\psi_l^\pm$. Thus ϕ_l^+ and ψ_l^+ are even, while ϕ_l^- and ψ_l^- are odd, with respecting to the $\mathbb{Z}_2$-grading on forms given by $\mathcal{A}^\bullet(\mathbb{S}^2) = \mathcal{A}^+(\mathbb{S}^2) \oplus \mathcal{A}^-(\mathbb{S}^2)$, where $\mathcal{A}^\pm(\mathbb{S}^2) := \frac{1}{2}(1 \pm \star)\,\mathcal{A}^\bullet(\mathbb{S}^2)$.

Exercise 8.9. *Show that*

$$-i(d+\delta)\phi_l^{\pm} = l\psi^{\mp}, \quad \textit{for } l = 0, 1, 2, \dots$$
$$-i(d+\delta)\psi_l^{\pm} = (l+1)\phi^{\mp}, \quad \textit{for } l = 1, 2, 3, \dots$$

and conclude that each of ϕ_l^+, ϕ_l^-, ψ_l^+ and ψ_l^- is an eigenvector for $(-i(d+\delta))^2 = -(d\delta + \delta d)$ with eigenvalue $l(l+1)$. Find corresponding eigenspinors for $-i(d+\delta)$ with eigenvalues $\pm\sqrt{l(l+1)}$.

Exercise 8.10. *Show that $L_3(\zeta^l) = -il\,\zeta^l$, $L_+\zeta^l = 0$, and that $(L_-)^k(\zeta^l)$ is a linear combination of terms $(x^3)^{k-2r}\bar{\zeta}^r\zeta^{l-k+r}$ that does not vanish for $k = 0, 1, \dots, 2l$, and that $(L_-)^{2l+1}(\zeta^l) = 0$. Check that $L_+(L_-)^k(\zeta^l)$ is a multiple of $(L_-)^{k-1}(\zeta^l)$, for $k = 1, \dots, 2l$.*

Exercise 8.11. *Show that*

$$\mathcal{L}_3\phi_l^{\pm} = -il\,\phi_l^{\pm}, \quad \mathcal{L}_+\phi_l^{\pm} = 0; \qquad \mathcal{L}_3\psi_l^{\pm} = -il\,\psi_l^{\pm}, \quad \mathcal{L}_+\psi_l^{\pm} = 0;$$

for each possible value of l. Conclude that the forms $\mathcal{L}_-^k(\phi_l^{\pm})$ and $\mathcal{L}_-^k(\psi_l^{\pm})$ vanish if and only if $k \geq 2l+1$. What can now be said about the multiplicities of the eigenvalues of $-i(d+\delta)$?

With some more works, it can be shown that all these eigenforms span a dense subspace of the Hilbert-space completion of $\mathcal{A}^\bullet(\mathbb{S}^2)$, so that these eigenvalues in fact give the full spectrum of the Hodge–Dirac operator.

8.2 The Dirac operator on the sphere $\mathbb{S}^2$

8.2.1 The spinor bundle S on $\mathbb{S}^2$

Consider the 2-dimensional sphere $\mathbb{S}^2$, with its usual orientation, $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\} \cong \mathbb{C}P^1$. The usual spherical coordinates on $\mathbb{S}^2$ are

$$p = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{S}^2.$$

The poles are $N = (0,0,1)$ and $S = (0,0,-1)$. Let $U_N = \mathbb{S}^2\backslash\{N\}$, $U_S = \mathbb{S}^2\backslash\{S\}$ be the two charts on $\mathbb{S}^2$. Consider the stereographic projections $p \mapsto z : U_N \to \mathbb{C}$, $p \mapsto \zeta : U_S \to \mathbb{C}$ given by

$$z := e^{-i\phi}\cot\frac{\theta}{2}, \qquad \zeta := e^{+i\phi}\tan\frac{\theta}{2},$$

so that $\zeta = 1/z$ on $U_N \cap U_S$. Write

$$q := 1 + z\bar{z} = \frac{2}{1-\cos\theta}, \quad \text{and} \quad q' := 1 + \zeta\bar{\zeta} = \frac{q}{z\bar{z}}.$$

The sphere $\mathbb{S}^2$ has only the "trivial" spin structure $\mathcal{S} = \Gamma(\mathbb{S}^2, S)$, where $S \to \mathbb{S}^2$ has rank two. Now $S = S^+ \oplus S^-$, where $S^{\pm} \to \mathbb{S}^2$ are complex *line bundles*, and these may be (and are) nontrivial. We argue that $S^+ \to \mathbb{S}^2$ is the "tautological" line bundle coming from $\mathbb{S}^2 \cong \mathbb{C}P^1$. We know already that

$$\mathcal{S}^\sharp \cong \mathcal{S} \iff S^* \cong S \impliedby (S^+)^* \cong S^-$$

and the converse $S^* \cong S \implies (S^+)^* \cong S^-$ will hold provided we can show that $S^\pm \to \mathbb{S}^2$ are nontrivial line bundles. (Otherwise, S^+ and S^- would each be selfdual, but we know that the only selfdual line bundle on $\mathbb{S}^2$ is the trivial one, since $H^2(\mathbb{S}^2, \mathbb{Z}) \cong \mathbb{Z}$.)

Consider now the (tautological) line bundle $L \to \mathbb{S}^2$, where

$$L_z := \{\, (\lambda z_0, \lambda z_1) \in \mathbb{C}^2 : \lambda \in \ bC \,\}, \quad \text{if } z = \frac{z_1}{z_0}, \qquad L_\infty := \{\, (0, \lambda) \in \mathbb{C}^2 : \lambda \in \mathbb{C} \,\}.$$

In other words, L_z is the complex line through the point $(1, z)$, for $z \in \mathbb{C}$. A particular *local* section of L, defined over U_N, is $\sigma_N(z) := (q^{-\frac{1}{2}}, zq^{-\frac{1}{2}})$, which is normalized so that $(\sigma_N \mid \sigma_N) = q^{-1}(1 + \bar{z}z) = 1$ on U_N: this hermitian pairing on $\Gamma(\mathbb{S}^2, L)$ comes from the standard scalar product on $\mathbb{C}^2$ —each L_z is a line in $\mathbb{C}^2$.

Let also $\sigma_S(\zeta) := (\zeta q'^{-\frac{1}{2}}, q'^{-\frac{1}{2}})$, normalized so that $(\sigma_S \mid \sigma_S) = 1$ on U_S. Now if $z \neq 0$, then

$$\sigma_S(z^{-1}) = \left(\frac{1}{z\sqrt{q'}}, \frac{1}{\sqrt{q'}}\right) = (\bar{z}/z)^{1/2} \left(\frac{1}{\sqrt{q}}, \frac{z}{\sqrt{q}}\right) = (\bar{z}/z)^{1/2}\, \sigma_N(z).$$

To avoid ambiguity, we state that $(\bar{z}/z)^{1/2}$ means $e^{-i\phi}$, and also $(z/\bar{z})^{1/2}$ will mean $e^{+i\phi}$.

A smooth section of L is given by two functions $\psi_N^+(z, \bar{z})$ and $\psi_S^+(\zeta, \bar{\zeta})$ satisfying the relation $\psi_N^+(z, \bar{z})\sigma_N(z) = \psi_S^+(\zeta, \bar{\zeta})\sigma_S(\zeta)$ on $U_N \cap U_S$. Thus we argue that

$$\psi_N^+(z, \bar{z}) = (\bar{z}/z)^{1/2} \psi_S^+(z^{-1}, \bar{z}^{-1}) \quad \text{for } z \neq 0,$$

and ψ_N^+, ψ_S^+ are regular at $z = 0$ or $\zeta = 0$ respectively. Likewise, a pair of smooth functions ψ_N^-, ψ_S^- on $\mathbb{C}$ is a section of the dual line bundle $L^* \to \mathbb{S}^2$ if and only if

$$\psi_N^-(z, \bar{z}) = (z/\bar{z})^{1/2} \psi_S^-(z^{-1}, \bar{z}^{-1}) \quad \text{for } z \neq 0.$$

We claim now that we can identify $S^+ \cong L$ and $S^- \cong L^* = L^{-1}$ —here the notation L^{-1} means that $[L^{-1}]$ is the inverse of $[L]$ in the Picard group $H^2(\mathbb{S}^2, \mathbb{Z})$ that classifies $\mathbb{C}$-line bundles— so that a *spinor* in $\mathbb{S} = \Gamma(\mathbb{S}^2, S)$ is given precisely by two pairs of smooth functions

$$\begin{pmatrix} \psi_N^+(z, \bar{z}) \\ \psi_N^-(z, \bar{z}) \end{pmatrix} \quad \text{on } U_N, \qquad \begin{pmatrix} \psi_S^+(\zeta, \bar{\zeta}) \\ \psi_S^-(\zeta, \bar{\zeta}) \end{pmatrix} \quad \text{on } U_S,$$

satisfying the above transformation rules. (The nontrivial thing is that the spinor components must both be regular at the south pole $z = 0$ and the north pole $\zeta = 0$, respectively.)

Since $\mathcal{S} \otimes_{\mathbb{A}} \mathcal{S}^* \cong \operatorname{End}_{\mathcal{A}}(\mathcal{S}) \cong \mathcal{B} \cong \mathcal{A}^\bullet(\mathbb{S}^2)$ as $\mathcal{A}$-module isomorphisms (we know that $\mathcal{B} \cong \mathcal{A}^\bullet(\mathbb{S}^2)$ as sections of *vector* bundles), it is enough to show that, as vector bundles,

$$\mathcal{A}^\bullet(\mathbb{S}^2) \cong L^0 \oplus L^2 \oplus L^{-2} \oplus L^0,$$

where $L^2 = L \otimes L$, $L^{-2} = L^* \otimes L^*$, and $L^0 = \mathbb{S}^2 \times \mathbb{C}$ is the trivial line bundle. It is clear that $\mathcal{A}^0(\mathbb{S}^2) = C^\infty(\mathbb{S}^2) = \mathcal{A} = \Gamma(\mathbb{S}^2, L^0)$; and furthermore, $\mathcal{A}^2(\mathbb{S}^2) \cong \mathcal{A} = \Gamma(\mathbb{S}^2, L^0)$ since $\Lambda^2 T^*\mathbb{S}^2$ has a nonvanishing global section, namely the volume form $\nu = \sin\theta d\theta \wedge d\phi$.

With respect to the "round" metric on $\mathbb{S}^2$, namely,

$$g := d\theta^2 + \sin^2\theta\, d\phi^2 = \frac{4}{q^2}\,(dx^1 \otimes dx^1 + dx^2 \otimes dx^2),$$

the pairs of 1-forms $\left\{\dfrac{dz}{q}, \dfrac{d\bar{z}}{q}\right\}$ and $\left\{-\dfrac{d\zeta}{q'}, -\dfrac{d\bar{\zeta}}{q'}\right\}$ are local bases for $\mathcal{A}^1(\mathbb{S}^2)$, over U_N and U_S respectively.

Exercise 8.12. *Write, for $\alpha \in \mathcal{A}^1(\mathbb{S}^2)$,*

$$\begin{aligned} \alpha &=: f_N(z,\bar{z})\,\frac{dz}{q} + g_N(z,\bar{z})\,\frac{d\bar{z}}{q} \quad \text{on } U_N, \\ &=: -f_S(\zeta,\bar{\zeta})\,\frac{d\zeta}{q'} - g_S(\zeta,\bar{\zeta})\,\frac{d\bar{\zeta}}{q'} \quad \text{on } U_S. \end{aligned}$$

Show that

$$\begin{aligned} f_N(z,\bar{z}) &= (\bar{z}/z)\, f_S(z^{-1},\bar{z}^{-1}) \\ g_N(z,\bar{z}) &= (z/\bar{z})\, g_S(z^{-1},\bar{z}^{-1}) \end{aligned}$$

on $U_N \cap U_S$, and conclude that $\mathcal{A}^1(\mathbb{S}^2) \cong \Gamma(\mathbb{S}^2, L^2 \oplus L^{-2})$.

Note that the last exercise now justifies the claim that the half-spin bundles were indeed $S^+ \oplus S^- \cong L \oplus L^*$.

8.2.2 The spin connection ∇^S over $\mathbb{S}^2$

Given any local orthonormal basis of 1-forms $\{E_1, \dots, E_n\}$, we can compute Christoffel symbols with all three indices taken from this basis, by setting $\widehat{\Gamma}^\beta_{\mu\alpha} := (E_\mu)^i \widetilde{\Gamma}^\beta_{i\alpha}$, or equivalently, by requiring that

$$\nabla_{E_\mu} E_\alpha =: \widehat{\Gamma}^\beta_{\mu\alpha}\, E_\beta$$

for $\mu, \alpha, \beta = 1, 2, \dots, n$. (This works because the first index is tensorial).

Exercise 8.13. *On U_N, take $z =: x^1 + ix^2$. Compute the ordinary Christoffel symbols Γ^k_{ij} in the (x^1, x^2) coordinates for the round metric $g = (4/q^2)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$, and then show that*

$$\widehat{\Gamma}^\beta_{\mu\alpha} = \delta_{\mu\alpha} x^\beta - \delta_{\mu\beta} x^\alpha \quad \text{for } \mu, \alpha, \beta = 1, 2.$$

This yields the local orthonormal bases $E_1 := \frac{1}{2} q\, \partial/\partial x^1$, $E_2 := \frac{1}{2} q\, \partial/\partial x^2$ for vector fields, and dually $\theta^1 = (2/q)\, dx^1$, $\theta^2 = (2/q)\, dx^2$ for 1-forms. However, since $\mathbb{S}^2 = \mathbb{C}P^1$ is a complex manifold, it is convenient to pass to "isotropic" bases, as follows. We introduce

$$\begin{aligned} E_+ &:= E_1 - iE_2 = q\frac{\partial}{\partial z}, & \theta^+ &:= \tfrac{1}{2}(\theta^1 + i\theta^2) = \frac{dz}{q}, \\ E_- &:= E_1 + iE_2 = q\frac{\partial}{\partial \bar{z}}, & \theta^- &:= \tfrac{1}{2}(\theta^1 - i\theta^2) = \frac{d\bar{z}}{q}. \end{aligned}$$

Exercise 8.14. *Verify that the Levi-Civita connection on $\mathcal{A}^1(\mathbb{S}^2)$ is given, in these isotropic local bases, by*

$$\nabla_{E_+}\left(\frac{dz}{q}\right) = \bar{z}\,\frac{dz}{q}, \qquad \nabla_{E_-}\left(\frac{dz}{q}\right) = -z\,\frac{dz}{q},$$
$$\nabla_{E_+}\left(\frac{d\bar{z}}{q}\right) = -\bar{z}\,\frac{d\bar{z}}{q}, \qquad \nabla_{E_-}\left(\frac{d\bar{z}}{q}\right) = z\,\frac{d\bar{z}}{q}.$$

The Clifford action on spinors is given (over U_N, say) by $\gamma^1 := \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^2 := \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The $\mathbb{Z}_2$-grading operator is given by

$$\chi := (-i)\,\sigma^1\sigma^2 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin connection is now specified by

$$\nabla^S_{E_\pm} := E_\pm - \tfrac{1}{4}\widehat{\Gamma}^\beta_{\pm\alpha}\,\gamma^\alpha\gamma_\beta.$$

Exercise 8.15. *Verify that, over U_N, ∇^S is determined by*

$$\nabla^S_{E_+} = q\frac{\partial}{\partial z} + \tfrac{1}{2}\bar{z}\,\chi, \qquad \nabla^S_{E_-} = q\frac{\partial}{\partial \bar{z}} - \tfrac{1}{2}z\,\chi.$$

Conclude that the Dirac operator $\not{D} = -i\sigma^1\,\nabla^S_{E_1} - i\sigma^2\,\nabla^S_{E_2}$ is given, over U_N, by

$$\not{D} = -i\begin{pmatrix} 0 & q\frac{\partial}{\partial z} - \frac{1}{2}\bar{z} \\ q\frac{\partial}{\partial \bar{z}} - \frac{1}{2}z & 0 \end{pmatrix}.$$

A similar expression is valid over U_S, by replacing $z, \bar{z}, q$ by $\zeta, \bar{\zeta}, q'$ respectively, and by changing the overall $(-i)$ factor to $(+i)$. This formal change of sign is brought about by the local coordinate transformation formulas induced by $\zeta = 1/z$. (Here is an instance of the "unique continuation property" of $\not{D}$: the local expression for the Dirac operator on any one chart determines its expressions on any overlapping chart, and then by induction, on the whole manifold.)

Exercise 8.16. *By integrating spinor pairings with the volume form $\nu = \sin\theta d\theta \wedge d\phi = 2iq^{-2}\,dz \wedge d\bar{z}$, check that $\not{D}$ is indeed symmetric as an operator on $L^2(\mathbb{S}^2, S)$ with domain $\mathcal{S}$.*

Exercise 8.17. *Show that the spinor Laplacian Δ^S is given in the isotropic basis by*

$$\Delta^s = -\tfrac{1}{2}\big(\nabla^S_{E_+}\nabla^S_{E_-} + \nabla^S_{E_-}\nabla^S_{E_+} - z\nabla^S_{E_+} - \bar{z}\nabla^S_{E_-}\big),$$

and compute directly that $\not{D}^2 = \Delta^S + \frac{1}{2}$. *This is consistent with the value $s \equiv 2$ of the scalar curvature of $\mathbb{S}^2$, taking into account how the metric g is normalized.*

8.2.3 Spinor harmonics and the Dirac operator spectrum

Newman and Penrose (1966) introduced a family of special functions on $\mathbb{S}^2$ that yield an orthonormal basis of spinors, in the same way that the conventional spherical harmonics Y_{lm} yield an orthonormal basis of L^2-functions. For functions, l and m are integers, but the spinors are labelled by "half-odd-integers" in $\mathbb{Z} + \frac{1}{2}$. When expressed in our coordinates $(z, \bar{z})$, they are given as follows.

For $l \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\} = \mathbb{N} + \frac{1}{2}$, and $m \in \{-l, -l+1, \dots, l-1, l\}$, write

$$\begin{aligned} Y^+_{lm}(z,\bar{z}) &:= C_{lm}\, q^{-l} \sum_{r-s=m-\frac{1}{2}} \binom{l-\frac{1}{2}}{r}\binom{l+\frac{1}{2}}{s} z^r(-\bar{z})^s, \\ Y^-_{lm}(z,\bar{z}) &:= C_{lm}\, q^{-l} \sum_{r-s=m+\frac{1}{2}} \binom{l+\frac{1}{2}}{r}\binom{l-\frac{1}{2}}{s} z^r(-\bar{z})^s, \end{aligned}$$

where r, s are integers with $0 \leq r \leq l \mp \frac{1}{2}$ and $0 \leq s \leq l \pm \frac{1}{2}$ respectively; and

$$C_{lm} = (-1)^{l-m}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!(l-m)!}{(l+\frac{1}{2})!(l-\frac{1}{2})!}}.$$

Exercise 8.18. *Show that $Y^\pm_{lm}$ are half-spinors in $S^\pm$, by applying the transformation laws under $z \mapsto z^{-1}$ and checking the regularity at the poles.*

Then define pairs of full spinors by

$$Y'_{lm} := \frac{1}{\sqrt{2}}\begin{pmatrix} Y^+_{lm} \\ iY^-_{lm} \end{pmatrix}, \qquad Y''_{lm} := \frac{1}{\sqrt{2}}\begin{pmatrix} -Y^+_{lm} \\ iY^-_{lm} \end{pmatrix}.$$

These turn out to be eigenspinors for the Dirac operator.

Exercise 8.19. *Verify the following eigenvalue relations:*

$$\not{D} Y'_{lm} = (l + \tfrac{1}{2})\, Y'_{lm}, \qquad \not{D} Y''_{lm} = -(l + \tfrac{1}{2})\, Y''_{lm}.$$

Goldberg *et al* (1967) showed that these half-spinors are special cases of matrix elements $\mathcal{D}^l_{nm}$ of the irreducible group representations for $SU(2)$, namely,

$$Y^\pm_{lm}(z,\bar{z}) = \sqrt{\frac{2l+1}{4\pi}}\mathcal{D}^l_{\mp\frac{1}{2},m}(-\phi,\theta,-\phi),$$

By setting $h^\pm_{lm}(\theta,\phi,\psi) := e^{\mp\frac{1}{2}(\phi+\psi)}Y^\pm_{lm}(z,\bar{z})$, we get an orthonormal set of elements of $L^2(\mathrm{SU}(2))$, such that $\int_{\mathrm{SU}(2)} |h^\pm_{lm}(g)|^2\, dg = (1/4\pi)\int_{\mathbb{S}^2} |Y^\pm_{lm}|^2\, \nu$. The Plancherel formula for SU(2) can then be used to show that these are a complete set of eigenvalues for $\not{D}$. Thus we have obtained the spectrum:

$$\mathrm{sp}(\not{D}) = \{\, \pm(l+\tfrac{1}{2}) : l \in \mathbb{N} + \tfrac{1}{2} \,\} = \{\pm 1, \pm 2, \pm 3, \dots\} = \mathbb{N} \setminus \{0\},$$

with respectively multiplicities $(2l+1)$ in each case, since the index m in $Y^\pm_{lm}$ takes $(2l+1)$ distinct values.

Postscript: Since $s \equiv 2$ and $D\!\!\!/^2 = \Delta^S + \frac{1}{2}$, we also get

$$\operatorname{sp}(\Delta^S) = \{\, (l + \tfrac{1}{2})^2 - \tfrac{1}{2} = l^2 + l - \tfrac{1}{4} : l \in \mathbb{N} + \tfrac{1}{2} \,\}$$

with multiplicities $2(2l+1)$ in each case. Note that

$$\operatorname{sp}(D\!\!\!/^2) = \{\, (l + \tfrac{1}{2})^2 = l^2 + l + \tfrac{1}{4} : l \in \mathbb{N} + \tfrac{1}{2} \,\}.$$

The operator $\underline{C}$ given by $\underline{C} := \Delta^S + \frac{1}{4} = D\!\!\!/ - \frac{1}{4}$ has spectrum

$$\operatorname{sp}(\underline{C}) = \{\, l(l+1) : l \in \mathbb{N} + \tfrac{1}{2} \,\},$$

with multiplicities $2(2l+1)$ again. This $\underline{C}$ comes from the *Casimir element* in the centre of $\mathcal{U}(\mathfrak{su}(2))$, represented on $\mathcal{H} = L^2(\mathbb{S}^2, S)$ via the rotation action of $\mathrm{SU}(2)$ on the sphere $\mathbb{S}^2$. There is a general result for compact symmetric spaces $M = G/K$ with a G-invariant spin structure, namely that $D\!\!\!/ = \underline{C}_G + \frac{1}{8}s$, or $\Delta^S = \underline{C}_G - \frac{1}{8}s$. This is a nice companion result, albeit only for homogeneous spaces, to the Schrödinger–Lichnerowicz formula. Details are given in Section 3.5 of Friedrich's book.

8.3 Spin^c Dirac operators on the 2-sphere

We know that finitely generated projective modules over the C*-algebra $A = C(\mathbb{S}^2)$ are of the form $p\,A^k$, where $p = [p_{ij}]$ is an $k \times k$ matrix with elements in A, such that p $(= p^2 = p^*)$ is an orthogonal projector, whose rank is $\operatorname{tr} p = p_{11} + \cdots + p_{kk}$. To get modules of sections of *line bundles*, we impose the condition that $\operatorname{tr} p = 1$, so that $p\,A^k$ is an A-module "of rank one". It turns out that it is enough to consider the case $k = 2$ of 2×2 matrices.

Exercise 8.20. *Check that any projector $p \in M_2(C(\mathbb{S}^2))$ is of the form*

$$p = \frac{1}{2}\begin{pmatrix} 1 + n_3 & n_1 - in_2 \\ n_1 + in_2 & 1 - n_3 \end{pmatrix},$$

where $n_1^2 + n_2^2 + n_3^2 = 1$, so that $\vec{n} = (n_1, n_2, n_3)$ is a continuous function from $\mathbb{S}^2$ to $\mathbb{S}^2$.

After stereographic projection, we can replace $\vec{n}$ by $f(z) := \dfrac{n_1 - in_2}{1 - n_3}$. where $z = e^{-i\phi}\cot\frac{\theta}{2}$ is allowed to take the value $z = \infty$ at the north pole. Then f is a continuous map from the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$ into itself. If two projectors p and q are homotopic —there is a continuous path of projectors $\{\, p_t : 0 \leq t \leq 1 \,\}$ with $p_0 = p$ and $p_1 = q$— then they give the same class $[p] = [q]$ in $K^0(\mathbb{S}^2)$; and this happens if and only if the corresponding maps $\vec{n}$, or functions $f(z)$, are homotopic.

Exercise 8.21. *Consider, for each $m = 1, 2, 3, \dots$, the maps*

$$z \mapsto f_m(z) := z^m \quad \textit{and} \quad z \mapsto f_{-m}(z) := \bar{z}^m$$

of the Riemann sphere into itself. Can you describe the corresponding maps $\vec{n}$ of $\mathbb{S}^2$ into itself? Can you show that any two of these maps are not homotopic?

Let $\mathcal{E}_{(m)} = p_m\, A^2$ and $\mathcal{E}_{(-m)} = p_{-m}\, A^2$, where

$$p_m(z) = \frac{1}{1 + z^m \bar{z}^m} \begin{pmatrix} z^m \bar{z}^m & z^m \\ \bar{z}^m & 1 \end{pmatrix}, \qquad p_{-m}(z) = \frac{1}{1 + z^m \bar{z}^m} \begin{pmatrix} z^m \bar{z}^m & \bar{z}^m \\ z^m & 1 \end{pmatrix},$$

with the obvious definition (what is it?) for $z = \infty$.

Exercise 8.22. *Show that $\mathcal{E}_{(1)}$ is isomorphic to the space of sections of the tautological line bundle $L \to \mathbb{C}P^1$ [hint: apply p_1 to any element of $\mathcal{A}^2$ and examine the result]. Show also that $\mathcal{E}_{(-1)}$ gives the space of sections of the dual line bundle $L^* \to \mathbb{C}P^1$.*

Exercise 8.23. *For $m = 2, 3, \dots$, show that $\mathcal{E}_{(m)} \cong \mathcal{E}_{(1)} \otimes_A \cdots \otimes_A \mathcal{E}_{(1)}$ (m times) by examining the components of elements of $p_m A^2$. What is the analogous result for $\mathcal{E}_{(-m)}$?*

For $m \in \mathbb{Z}$, $m \neq 0$, we redefine $\mathcal{E}_{(m)} := p_m \mathcal{A}^2$ with $\mathcal{A} = C^\infty(\mathbb{S}^2)$; so that $\mathcal{E}_{(m)}$ now denotes *smooth* sections over a nontrivial line bundle on $\mathbb{S}^2$. We can identify each element of $\mathcal{E}_{(m)}$ with a smooth function $f_N \colon U_N \to \mathbb{C}$ for which there is another smooth function $f_S \colon U_S \to \mathbb{C}$, such that

$$f_N(z) = (\bar{z}/z)^{m/2}\, f_S(z^{-1}) \quad \text{for all } \ z \neq 0. \tag{$\underline{m}$}$$

Here, as before, $(\bar{z}/z)$ means $e^{i\phi}$ in polar coordinates.

Exercise 8.24. *Writing $E_+ := q\,\partial/\partial z$ and $E_- := q\,\partial/\partial \bar{z}$ as before, where $q = 1 + z\bar{z}$, show that when the operators*

$$\nabla^{(m)}_{E_+} = q\,\frac{\partial}{\partial z} + \tfrac12 m\bar{z}, \qquad \nabla^{(m)}_{E_-} = q\,\frac{\partial}{\partial \bar{z}} - \tfrac12 m z,$$

are applied to functions f_N that satisfy ($\underline{m}$), the image also satisfies ($\underline{m}$). Thus they are components of a connection $\nabla^{(m)}$ on $\mathcal{E}_{(m)}$.

To get all the spin^c structures on $\mathbb{S}^2$, we twist the spinor module $\mathcal{S}$ for the spin structure, namely $\mathcal{S} = \mathcal{E}_{(1)} \oplus \mathcal{E}_{(-1)}$, by the rank-one module $\mathcal{E}_{(m)}$. On the tensor product $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)}$ we use the connection

$$\nabla^{S,m} := \nabla^S \otimes 1_{\mathcal{E}_{(m)}} + 1_{\mathcal{S}} \otimes \nabla^{(m)}.$$

Exercise 8.25. *Show that the Dirac operator $D\!\!\!/_m := -i\,\hat{c} \circ \nabla^{S,m}$, that acts on $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)}$, is given by*

$$D\!\!\!/_m \equiv \begin{pmatrix} 0 & D\!\!\!/_m^- \\ D\!\!\!/_m^+ & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & q\frac{\partial}{\partial z} + \frac12 (m-1)\,\bar{z} \\ q\frac{\partial}{\partial \bar{z}} - \frac12 (m+1)\, z & 0 \end{pmatrix}.$$

Check also that

$$D\!\!\!/_m^+ = -i\, q^{(m+3)/2}\, \frac{\partial}{\partial \bar{z}}\, q^{-(m+1)/2} \quad \textit{and} \quad D\!\!\!/_m^- = -i\, q^{-(m-3)/2}\, \frac{\partial}{\partial z}\, q^{(m-1)/2},$$

where these powers of q are multiplication operators on suitable spaces on functions on U_N.

Exercise 8.26. *If $m < 0$, show that any element of* $\ker D\!\!\!/_m^+$ *is of the form* $a(z)\, q^{(m+1)/2}$ *where $a(z)$ is a holomorphic polynomial of degree $< |m|$. Also, if $m \geq 0$, show that* $\ker D\!\!\!/_m^+ = 0$.

Exercise 8.27. *If $m > 0$, show that any element of* $\ker D\!\!\!/_m^-$ *is of the form* $b(\bar{z})\, q^{-(m-1)/2}$ *where $b(\bar{z})$ is an* anti*holomorphic polynomial of degree $< m$. Also, if $m \leq 0$, show that* $\ker D\!\!\!/_m^- = 0$. *Conclude that the index of $D\!\!\!/_m$ equals $-m$ in all cases.*

The *sign* of a selfadjoint operator D on a Hilbert space is given by the relation $D =: F\,|D| = F\,(D^2)^{1/2}$, where we put $F := 0$ on $\ker D$. Thus F is a bounded selfadjoint operator such that $1 - F^2$ is the orthogonal projector whose range is $\ker D$. When $\ker D$ is finite-dimensional, $1 - F^2$ has finite rank, so it is a compact operator.

An *even Fredholm module* over an algebra $\mathcal{A}$ is given by:

1. a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$;
2. a representation $a \mapsto \pi(a) = \begin{pmatrix} \pi^0(a) & 0 \\ 0 & \pi^1(a) \end{pmatrix}$ of $\mathcal{A}$ on $\mathcal{H}$ by bounded operators that commute with the $\mathbb{Z}_2$-grading;
3. a selfadjoint operator $F = \begin{pmatrix} 0 & F^- \\ F^+ & 0 \end{pmatrix}$ on $\mathcal{H}$ that anticommutes with the $\mathbb{Z}_2$-grading, such that $F^2 - 1$ and $[F, \pi(a)]$ are compact operators on $\mathcal{H}$, for each $a \in \mathcal{A}$.

We can extend the twisted Dirac operator $D\!\!\!/_m$ to a selfadjoint operator on $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, where $\mathcal{H}^0$ and $\mathcal{H}^1$ are two copies of the Hilbert space $L^2(\mathbb{S}^2, \nu)$ where $\nu = 2i\, q^{-2}\, dz\, d\bar{z}$. We define $\pi^0(a) = \pi^1(a)$ to be the usual multiplication operator of a function $a \in C^\infty(\mathbb{S}^2)$ on this L^2-space.

Exercise 8.28. *Show that $D\!\!\!/_m$, given by the above formulas on its original domain, is a symmetric operator on $\mathcal{H}$.*

Exercise 8.29. *Check that the sign F_m of the twisted Dirac operator $D\!\!\!/_m$ determines a Fredholm module over $C^\infty(\mathbb{S}^2)$.*

8.4 A spectral triple on the noncommutative torus

To define a spectral triple over a noncommutative algebra, we introduce the so-called *noncommutative torus*. In fact, there are many such tori, labelled by a dimension n and by a family of parameters θ_{ij} forming a real skewsymmetric matrix $\Theta = -\Theta^t \in M_n(\mathbb{R})$.

Fix an integer $n \in \{2, 3, 4, \dots\}$. In the algebra $A_0 := C(\mathbb{T}^n)$, one can write down Fourier-series expansions:

$$f(\phi_1, \dots, \phi_n) \longleftrightarrow \sum_{r \in \mathbb{Z}^n} c_r\, e^{2\pi i r \cdot \phi}, \qquad c_r := \int_{[0,1]^n} e^{-2\pi i r \cdot \phi}\, f(\phi)\, d^n\phi \in \mathbb{C},$$

where $r \cdot \phi := r_1\phi_1 + \cdots + r_n\phi_n$, as usual. To ensure that this series converges uniformly and represents $f(\phi)$, we retreat to the dense subalgebra $\mathcal{A}_0 := C^\infty(\mathbb{T}^n)$, in which the coefficients c_r decrease rapidly to zero as $|r| \to \infty$. On the space of multisequences $\mathbf{c} := \{c_r\}_{r\in\mathbb{Z}^n}$, we introduce the seminorms

$$p_k(\mathbf{c}) := \bigg(\sum_{r\in\mathbb{Z}^n} (1 + r\cdot r)^k |c_r|^2\bigg)^{1/2}, \quad \text{for all } k \in \mathbb{N}.$$

We say that *"$c_r \to 0$ rapidly"* if $p_k(\mathbf{c}) < \infty$ for every k. Notice that $p_{k+1}(\mathbf{c}) \geq p_k(\mathbf{c})$ for each k; these seminorms induce, on rapidly decreasing sequences, the topology of a Fréchet space, which indeed coincides with the usual Fréchet topology on $C^\infty(\mathbb{T}^n)$, i.e., the topology of uniform convergence of the functions and of all their derivatives.

We can think of A_0 as the C*-algebra generated by n commuting unitary elements, namely the functions u_j defined by $u_j(\phi_1,\dots,\phi_n) := e^{2\pi i\phi_j}$, for $j = 1,\dots,n$.

Noncommutativity appears when we choose a real skewsymmetric matrix $\Theta \in M_n(\mathbb{R})$, and introduce the (universal) C*-algebra A_Θ generated by *unitary* elements $u_1,\dots,u_n$ which no longer commute: instead, they satisfy the *commutation relations*

$$u_k\, u_j = e^{2\pi i\theta_{jk}}\, u_j\, u_k, \quad \text{for } j,k = 1,\dots,n.$$

(In quantum mechanics, these are called "Weyl's form of the canonical commutation relations".) To form polynomials with these generators, we introduce a *Weyl system* of unitary elements $\{\, u^r : r \in \mathbb{Z}^n \,\}$ in A_Θ, by defining

$$u^r := \exp\big\{\pi i \textstyle\sum_{j<k} r_j\theta_{jk}r_k\big\}\, u_1^{r_1}u_2^{r_2}\dots u_n^{r_n}.$$

Exercise 8.30. *Show that $(u^r)^* = u^{-r}$ for $r \in \mathbb{Z}^n$, and that*

$$u^r\, u^s = \sigma(r,s)\, u^{r+s}, \quad \textit{where} \quad \sigma(r,s) := \exp\big\{-\pi i \textstyle\sum_{j,k} r_j\theta_{jk}s_k\big\}.$$

Verify directly that

$$\sigma(r,s+t)\,\sigma(s,t) = \sigma(r,s)\,\sigma(r+s,t), \quad \textit{for } r,s,t \in \mathbb{Z}^n.$$

Notice that $\sigma(r,\pm r) = 1$ by skewsymmetry of Θ.

We now define $\mathcal{A}_\Theta =: C^\infty(\mathbb{T}^n_\Theta)$ to be the *dense $*$-subalgebra* of A_Θ consisting of elements of the form

$$a = \sum_{r\in\mathbb{Z}^n} a_r\, u^r$$

where $a_r \in \mathbb{C}$ for each r, and $a_r \to 0$ *rapidly.*

Exercise 8.31. *Check that this series converges in the norm of $\mathcal{A}^\Theta$, by considering the series $\sum_r (1 + r\cdot r)^{-k}$ for large enough k.*

There is an *action* of the abelian Lie group $\mathbb{T}^n$ by $*$-automorphisms on the C*-algebra A_Θ, given by

$$z\cdot u^r := z_1^{r_1}z_2^{r_2}\dots z_n^{r_n}\, u^r \quad \text{for } r \in \mathbb{Z}^n,$$

or, more simply, $z\cdot u_j = z_j\, u_j$, where $z = (z_1,\dots,z_n) \in \mathbb{T}^n$. This action is generated by a set of n commuting *derivations* $\delta_1,\dots,\delta_n$, namely,

$$\delta_j(a) := \frac{d}{dt}\bigg|_{t=0} e^{2\pi it\phi_j}\cdot a,$$

whose domain is the set of all $a \in A$ for which the map $t \mapsto e^{2\pi i t\phi_j} \cdot a$ is differentiable.

Exercise 8.32. *Show that $u_r \in \operatorname{Dom}\delta_j$, and that $\delta_j(u^r) = 2\pi i r_j\, u^r$ for all $r \in \mathbb{Z}^n$ and $j = 1, \dots, n$. Conclude that the common smooth domain $\bigcap_{m\in\mathbb{Z}^n} \operatorname{Dom}(\delta_1^{m_1} \dots \delta_n^{m_n})$ is equal to the subalgebra $\mathcal{A}_\Theta$.*

The result of the previous exercise shows that $\mathcal{A}_\Theta$ is just the "smooth subalgebra" of the C*-algebra A_Θ with respect to the action of $\mathbb{T}^n$. It is known that any such smooth subalgebra, under a continuous action of a compact Lie group on a C*-algebra, is actually a *pre-C*-algebra*.

Exercise 8.33. *Define a linear operator $E\colon A_\Theta \to A_\Theta$ by averaging over the orbits of this $\mathbb{T}^n$-action:*

$$E(a) := \int_{[0,1]^n} (e^{-2\pi i\phi_1}, \dots, e^{-2\pi i\phi_n}) \cdot a \; d\phi_1 \dots d\phi_n.$$

Check that $E(1) = 1$, that $E(a^*) = E(a)^*$, that $E(a^*a) \geq 0$ and $\|E(a)\| \leq \|a\|$ for all $a \in A_\Theta$; where "$x \geq 0$" means that x is a positive *element of A_Θ. Then show the "conditional expectation" property:*

$$E\big(E(a)\, b\, E(c)\big) = E(a)\, E(b)\, E(c) \quad \text{for all } a, b, c \in A_\Theta.$$

By considering $b = a - E(a)$, show also that $E(a^*a) \geq E(a)^*E(a)$ for $a \in A_\Theta$.

Exercise 8.34. *If $a = \sum_r a_r\, u^r \in \mathcal{A}_\Theta$, check that $E(a) = a_0\, 1$. Conclude that the range of E is the $*$-subalgebra $\mathbb{C}\, 1$, and that*

$$\tau(a)\, 1 := E(a)$$

defines a trace *on A_Θ; by continuity, it is enough to check the trace property on the dense subalgebra $\mathcal{A}_\Theta$.*

Exercise 8.35. *If instead we only consider the action of a subgroup $\mathbb{T}^k$ of $\mathbb{T}^n$, we can define a conditional expectation*

$$E_k(a) := \int_{[0,1]^k} (e^{-2\pi i\phi_1}, \dots, e^{-2\pi i\phi_k}, 1, \dots, 1) \cdot a \; d\phi_1 \dots d\phi_k.$$

In this case the range of E_k will be isomorphic to a C-algebra A_Φ where Φ is a certain real skewsymmetric matrix in $M_{n-k}(\mathbb{R})$. Compute the matrix Φ in terms of the matrix Θ. In particular, what is the range of E_k for the case $k = n-1$?*

We now define $\mathcal{H}_\tau$ to be the *completion* of A_Θ in the norm

$$\|a\|_2 := \sqrt{\tau(a^*a)}.$$

We remark that $\|a\|_2 \leq \|a\|$ for all a, so that the inclusion map $\eta_\tau\colon A_\Theta \to \mathcal{H}_\tau$ is continuous. It is convenient to write $\underline{a} := \eta_\tau(a)$ to denote the element $a \in A_\Theta$ regarded as a vector in $\mathcal{H}_\tau$. It turns out that the trace τ is faithful, so that $\mathcal{H}_\tau$ is just the Hilbert space of the "GNS representation" π_τ of A_Θ. This representation is defined —first on $\eta_\tau(A_\Theta)$, then extended by continuity— by

$$\pi_\tau(a) : \underline{b} \mapsto \underline{ab} : \mathcal{H}_\tau \to \mathcal{H}_\tau, \quad \text{for each } a \in A_\Theta.$$

Exercise 8.36. *Define an* antilinear *operator* $J_0 \colon \mathcal{H}_\tau \to \mathcal{H}_\tau$ *by setting*

$$J_0(\underline{a}) := \underline{a^*}, \quad \text{for } a \in \eta_\tau(A_\Theta).$$

Show that J_0 *is an isometry on this domain, so that it extends to all of* $\mathcal{H}_\tau$*; and show that the extended* J_0 *is an* antiunitary *operator on* $\mathcal{H}_\tau$*. For* $b \in A_\Theta$*, consider the operator*

$$\pi'_\tau(b) := J_0 \pi_\tau(b^*) J_0.$$

Check that $\pi'_\tau(b) : \underline{c} \mapsto \underline{cb}$ *for* $c \in A_\Theta$*. Conclude that* $[\pi_\tau(a), \pi'_\tau(b)] = 0$ *for all* $a, b \in A_\Theta$*.*

The analogue of the L^2-spinor space for the noncommutative torus is just the tensor product $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$, where as usual, $n = 2m$ or $n = 2m+1$ according as n is even or odd. (In the commutative case $\Theta = 0$, this means that we are using the spinor module for the untwisted spin structure on $\mathbb{T}^n$.) Recall that we can regard $\mathbb{C}^{2^m}$ as a Fock space $\Lambda^\bullet \mathbb{C}^m$, carrying an irreducible representation of the matrix algebra $B = \mathbb{C}\mathrm{l}(\mathbb{R}^n)$ if n is even, or $B = \mathbb{C}\mathrm{l}^0(\mathbb{R}^n)$ if n is odd. In the even case, there is a $\mathbb{Z}_2$-grading operator $\Gamma := 1_{\mathcal{H}_\tau} \otimes c(\gamma)$, satisfying $\Gamma^2 = 1$ and $\Gamma^* = \Gamma$.

The charge conjugation on B, that we have written $b \mapsto \chi(\bar{b})$, is implemented by an antiunitary operator on $\mathbb{C}^{2^m}$ of the form $C_0 K$, where K is complex conjugation and C_0 is a certain $2^m \times 2^m$ matrix: this means that $(C_0 K)\, b\, (C_0 K)^{-1} = \chi(\bar{b})$ as operators on $\mathbb{C}^{2^m}$.

For instance, if $n = 2$ or 3, then $C_0 = i\,\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now let $J := J_0 \otimes C_0$. This is an antiunitary operator on $\mathcal{H}$, such that $J^2 = \pm 1$ according as $C_0^2 = \pm 1$.

Exercise 8.37. *Show that* $\delta_j(a^*) = (\delta_j(a))^*$ *and that* $\tau(\delta_j(a)) = 0$ *for all* $a \in \mathcal{A}_\Theta$*. Conclude that the densely defined operator* $\underline{\delta}_j : \underline{a} \mapsto \underline{\delta_j(a)}$*, with domain* $\eta_\tau(\mathcal{A}_\Theta)$*, is* skewsymmetric *in the sense that*

$$\langle \underline{\delta}_j(\underline{a}) \mid \underline{b} \rangle = -\langle \underline{a} \mid \underline{\delta}_j(\underline{b}) \rangle, \quad \text{for all } \underline{a}, \underline{b} \in \operatorname{Dom} \underline{\delta}_j.$$

The closure of this operator, still denoted by $\underline{\delta}_j$, is then an unbounded skewadjoint operator on $\mathcal{H}$.

Let $\gamma^1, \dots, \gamma^n$ be the generators of the action of the Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ on $\mathbb{C}^{2^m}$: they are a set of $2^m \times 2^m$ matrices such that $\gamma^j \gamma^k + \gamma^k \gamma^j = 2\delta^{jk}$ for $j, k = 1, \dots, n$. The operator $C_0 K$ is determined by the relations

$$(C_0 K)\, \gamma^j\, (C_0 K)^{-1} = -\gamma^j \quad \text{for } j = 1, \dots, n.$$

We can now define the *Dirac operator* on $\mathcal{H}$ by

$$D := -i \sum_{j=1}^{n} \underline{\delta}_j \otimes \gamma^j.$$

Exercise 8.38. *Show that* $JDJ^{-1} = \pm D$ *on the domain* $\mathcal{A}_\Theta$*.*

Exercise 8.39. *If* $\{\, s_\alpha : \alpha = 1, \dots, 2^m \,\}$ *is an orthonormal basis of* $\mathbb{C}^{2^m}$*, define* $\psi_{r\alpha} := \underline{u^r} \otimes s_\alpha \in \mathcal{H}$*. Show that* $\{\, \psi_{r\alpha} : r \in \mathbb{Z}^n,\ \alpha = 1, \dots, 2^m \,\}$ *is an orthonormal basis of* $\mathcal{H}$ *that diagonalizes* D^2*, by checking that*

$$D^2 \psi_{r\alpha} = 4\pi^2 (r \cdot r)\, \psi_{r\alpha} \quad \text{for each } r, \alpha.$$

What is the spectrum (with its multiplicities) of $|D|$*? What is the spectrum of* D *itself?*

Exercise 8.40. *We can invert* D *on the orthogonal complement of the finite-dimensional space* $\ker D = \operatorname{span}\{\, \psi_{0\alpha} : \alpha = 1, \dots, 2^m \,\}$*. Show that, for each* $s > 0$*, the expression*

$$\operatorname{Tr}^+ |D|^{-s} := \lim_{N\to\infty} \frac{\sigma_N(|D|^{-s})}{\log N}$$

either exists as a finite limit, or diverges to $+\infty$*. (Show that we may use a subsequence where* $N = N_R := \#\{\, r \in \mathbb{Z}^n : r \cdot r \leq R^2 \,\}$ *for some* $R > 0$*.) Verify that the* $0 < \operatorname{Tr}^+ |D|^{-s} < +\infty$ *if and only if* $s = n$*; and compute the value of* $\operatorname{Tr}^+ |D|^{-n}$.

Exercise 8.41. *If* $a \in \mathcal{A}^\Theta$*, show that both* a *and* $[D, a]$*, considered as bounded operators on* $\mathcal{H}$*, lie in the smooth domain of the operator* $T \mapsto [|D|, T]$.

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Part IV

From Poisson to Quantum geometry

by

Nicola Ciccoli

Based on the lectures of:

- Nicola Ciccoli
  (*Dipartimento di Matematica, Universit'a di Perugia Via Vanvitelli 1, I-06123 Perugia, Italy*)
  – Chapters 1, 2, 3, 4, 5, 6, 8, 9.

With additional lectures by:

- Ludwik Dąbrowski – Chapter 7.

Chapter 1

Poisson Geometry

1.1 Poisson algebra

Definition 1.1. *A* Poisson algebra *is an associative algebra A (over a field $\mathbb{K}$) with a linear bracket $\{\cdot,\cdot\}\colon A\otimes A\to A$ such that*

1. $\{f,g\}=-\{f,g\}$ *(antisymetry),*
2. $\{f,gh\}=g\{f,h\}+\{f,g\}h$ *(Leibniz rule),*
3. $\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0$ *(Jacobi identity),*

for all $f,g,h\in A$.

Remarks 1.2.

- The first and third axiom tells us that the bracket is a Lie bracket. The second one is a compatibility relation between the associative and Lie products.
- The algebra A does not need to be commutative (and this explains the way in which the Leibniz rule is written). Non commutative Poisson algebras were studied in [x-p94] (see also [f-d95] for some algebraic theory). In what follows, however we will tacitly assume A to be commutative unless otherwise stated.
- Every associative algebra A can be made into a Poisson algebra by setting $\{f,g\}\equiv 0$.
- When A is unital we get from the assumptions

  $$\{f,g\}=\{f,g\cdot 1\}=\{f,g\}\cdot 1+g\cdot\{f,1\},$$

  so $\{f,1\}=0$ for all $f\in A$.

A natural source of examples of commutative Poisson algebras is a specific subclass of associative algebras, the so–called almost commutative algebras.

Exercise 1.3. *Let U be an almost commutative algebra, i.e. filtered associative algebra, $U^0 \subseteq U^1 \subseteq \dots$, $U^i \cdot U^j \subseteq U^{i+j}$, such that $\mathrm{gr}(U) = \bigoplus_{i=0}^{\infty} U^i/U^j$ is commutative. Let $[x] \in \mathrm{gr}(U)$ be the class of $x \in U^i$, and define*

$$\{[x],[y]\} := [xy - yx] \in \mathrm{gr}(U).$$

Prove that it is a Poisson algebra.

Definition 1.4. *A* Poisson morphism $(A, \{\}_A) \xrightarrow{\varphi} (B, \{\}_B)$ *is a morphism of associative algebras such that*

$$\varphi(\{f,g\}_A) = \{\varphi(f), \varphi(g)\}_B, \quad \textit{for all} \quad f,g \in A.$$

Exercise 1.5. *Prove that Poisson algebras together with Poisson morphisms form a category.*

Definition 1.6. *Let $(A, \{\}_A)$ be a Poisson algebra. A* Poisson subalgebra *is a subalgebra B closed with respect to $\{\}_A$.A* Poisson ideal *$I \subseteq A$ is an ideal with respect to the associative product, such that $\{f,i\}_A \in I$ for all $f \in A$, $i \in I$.*

Exercise 1.7. *For any Poisson morphism $\varphi \colon A \to B$, prove that $\ker\varphi$ is a Poisson ideal in A, $\mathrm{im}\,\varphi$ is a Poisson subalgebra in B, and there is an exact sequence of Poisson algebras*

$$0 \to \ker\varphi \to A \to \mathrm{im}\,\varphi \to 0.$$

Definition 1.8. *Let A be Poisson algebra. An element $f \in A$ is called a* Casimir *if $\{f,g\} = 0$ for all $g \in A$.*

Definition 1.9. *A linear endomorphism $X \in \mathrm{End}(A)$ is called* canonical *if it is a derivation with respect to both the associative product and the bracket, i.e. for every $f, g \in A$*

1. $X(fg) = (Xf)g + f(Xg)$
2. $X\{f,g\} = \{Xf, g\} + \{f, Xg\}$

The set of all Casimir elements in A, which is the center of the Lie algebra $(A, \{\}_A)$, will be denoted by $\mathrm{Cas}(A)$; the set of canonical endomorphisms will be denoted by $\mathrm{Can}(A)$.

Proposition 1.10. *Let A be Poisson algebra. For every $f \in A$, $X_f \colon g \mapsto \{f,g\}$ is a canonical endomorphism.*

Proof. From Leibniz rule it follows:

$$X_f(gh) = \{f, gh\} = \{f,g\}h + g\{f,h\} = (X_f g)h + g(X_f h)$$

From Jacobi identity it follows:

$$X_f(\{g,h\}) = \{f, \{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\} = \{X_f g, h\} + \{g, X_f h\}$$

□

Definition 1.11. *Canonical endomorphisms of the form X_f are called* hamiltonian *and the set of hamiltonian enodomorphism is denoted by* $\mathrm{Ham}(A)$.

Let $\mathrm{Der}(A)$ be the set of derivations of the associative algebra A. We have the following chain of inclusions.

$$\mathrm{Ham}(A) \subseteq \mathrm{Can}(A) \subseteq \mathrm{Der}(A).$$

Recall that $\mathrm{Der}(A)$ has a natural Lie algebra bracket given by the commutator of endomorphisms. $\mathrm{Can}(A)$ is a Lie subalgebra of $\mathrm{Der}(A)$. One may naturally ask whether such spaces are equal. In the trivial case we easily have $0 = \mathrm{Ham}(A) \neq \mathrm{Can}(A) = \mathrm{Der}(A)$. We will see further examples later on where all such spaces are different.

Proposition 1.12. $\mathrm{Ham}(A)$ *is a Lie ideal in* $\mathrm{Can}(A)$ *and a Lie subalgebra of* $\mathrm{Der}(A)$.

Proof. Let $X \in \mathrm{Can}(A)$, $X_f \in \mathrm{Ham}(A)$. Then

$$\begin{aligned}
[X, X_f](g) &= X(X_f(g)) - X_f(X(g)) \\
&= X\{f, g\} - \{f, X(g)\} \\
&= \{X(f), g\} + \{f, X(g)\} - \{f, X(g)\} \\
&= X_{X(f)}(g),
\end{aligned}$$

so $[X, X_f] = X_{X(f)} \in \mathrm{Ham}(A)$. To prove that $\mathrm{Ham}(A)$ is a subalgebra of $\mathrm{Der}(A)$, one computes

$$\begin{aligned}
([X_f, X_g] - X_{\{f,g\}})h &= X_f(X_g h) - X_g(X_f h) - X_{\{f,g\}}h \\
&= \{f, \{g, h\}\} - \{g, \{f, h\}\} - \{\{f, g\}, h\} \\
&= -\operatorname{Jac}(f, g, h) = 0.
\end{aligned}$$

Therefore $[X_f, X_g] = X_{\{f,g\}}$, hence $Ham(A)$ is closed under commutator. □

Exercise 1.13. *Let* $A = \mathbb{C}[X, Y]$. *After having proven that there exists a unique on* A *such that* $\{X, Y\} = X$ *try to describe the sets* $\mathrm{Ham}(A)$, $\mathrm{Can}(A)$, $\mathrm{Der}(A)$.

Proposition 1.14. *Let* $(A, \{\cdot, \cdot\}_A)$ *and* $(B, \{\cdot, \cdot\}_B)$ *be Poisson algebras. Then their tensor product* $A \otimes B$ *has a natural structure of Poisson algebra given by*

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\}_A \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}_B.$$

The maps $A \to A \otimes B$, $a \mapsto a \otimes 1$, $B \mapsto A \otimes B$, $b \mapsto 1 \otimes B$ *are Poisson morphisms and* $\{a \otimes 1, 1 \otimes b\} = 0$ *for all* $a \in A$, $b \in B$.

Definition 1.15. Poisson module *structure on a left* A*-module* M *over a Poisson algebra* A *is a linear map*

$$\{\cdot, \cdot\}_M \colon A \otimes M \to M$$

such that

1. $\{\{f, g\}_A, m\}_M = \{f, \{g, m\}_M\}_M - \{g, \{f, m\}_M\}_M$,
2. $\{fg, m\}_M = f \cdot \{g, m\}_M + g \cdot \{f, m\}_M$,
3. $\{f, g \cdot m\}_M = \{f, g\}_A \cdot m + g\{f, m\}_M$

Remark 1.16. This definition turns into a flat connection condition when M is the module of sections of a vector bundle and A is an algebra of functions on the base. Indeed, if we let

$$T\colon M \to \mathrm{Hom}_{\mathbb{K}}(A, M), \quad m \mapsto T_m := \{\cdot, m\}_M$$

then

1. $\iff T_m(\{f, g\}_A) = \{f, T_m(g)\}_M - \{g, T_m(f)\}_M$ (that is $T_m \in \mathrm{Der}((A, \{\cdot, \cdot\}_A); M))$,
2. $\iff T_{f\cdot m} = f \cdot T_m(g) + \{f, g\}_A \cdot m = f \cdot T_m(g) + X_f(g) \cdot m$,
3. $\iff T_m(fg) = fT_m(g) + gT_m(f)$ (that is $T_m \in \mathrm{Der}((A, \cdot); M))$.

One may ask whether this is a reasonable definition of Poisson module. It is, in a sense, the categorical notion of Poisson bimodule as it verifies the so-called square-zero construction which can be summarized as follows: let A be a Poisson algebra and M Poisson A-module; define a Poisson algebra structure on $A \oplus M$ using formulas

$$\begin{aligned} (f + m) \cdot (f_1 + m_1) &:= ff_1 + (f \cdot m_1 + f_1 \cdot m), \\ \{f + m, f_1 + m_1\} &:= \{f, f_1\}_A + \{f, m_1\}_M - \{f_1, m\}_M. \end{aligned}$$

Proposition 1.17. *$A \oplus M$ is a Poisson algebra if and only if M is a Poisson module. Furthermore the projection $A \oplus M \to A$ is a map of algebras, $M^2 = 0$ and M is an ideal.*

1.2 Poisson manifolds

Definition 1.18. *A* smooth Poisson manifold *M is a smooth manifold together with a on $C^\infty(M)$.*

An affine algebraic Poisson variety *M is an affine algebraic variety such that $A = \mathbb{K}[M]$ (algebra of regular functions) is a Poisson algebra over $\mathbb{K}$.*

An algebraic Poisson variety *M is an algebraic variety such that the sheaf of regular functions is a sheaf of Poisson algebras.*

In what follows we will mainly restrict ourselves to the smooth case and most of what we say can be easily adapted to the algebraic one. We will try to signal out the main point where this does not happen. This somewhat ambiguous approach is due to one fact; while the geometric theory of Poisson manifolds is usually simpler to describe then its algebraic counterpart, the quantization is better suited for regular functions (here we are thinking at the high obstacles in chracterizing the algebra of smooth functions on a manifold inside the category of commutative algebras).

Definition 1.19. *A* morphism of Poisson manifolds *is a differentiable function $\varphi\colon M \to N$ such that φ^* is a morphism of Poisson algebras, i.e.*

$$\varphi^*\{f, g\}_M = \{f, g\}_M \circ \varphi = \{f \circ \varphi, g \circ \varphi\}_N = \{\varphi^* f, \varphi^* g\}_N$$

for $f, g \in C^\infty(M)$.

Let us now translate the definitions of Casimir, hamiltonian and canonical maps to this setting. The significant remark is that the map $f \mapsto X_f$ takes values in $\mathrm{Der}(C^\infty(M)) = \mathfrak{X}^1(M)$. Thus we can write

$$\begin{aligned}
\mathrm{Cas}(M) &= \{f \in C^\infty(M) : \{f,g\} = 0 \quad \forall g \in C^\infty(M)\} = \ker(f \mapsto X_f),\\
\mathrm{Ham}(M) &= \text{Hamiltonian vector fields} = \mathrm{im}(f \mapsto X_f),\\
\mathrm{Can}(M) &= \text{canonical vector fields}\{X \in \mathfrak{X}^1(M) : X(X_f g) = X_{Xf} g + X_f(Xg)\}.
\end{aligned}$$

Hamiltonian vector fields fit into the short exact sequence

$$0 \to \mathrm{Cas}(M) \to C^\infty(M) \to \mathrm{Ham}(M) \to 0.$$

Remark 1.20. The Cartesian product of Poisson manifolds is a Poisson manifold. In fact consider

$$C^\infty(M_1 \times M_2) \supset C^\infty(M_1) \otimes C^\infty(M_2).$$

There is an easily defined Poisson structure on $C^\infty(M_1) \otimes C^\infty(M_2)$ which uniquely extends to $C^\infty(M_1 \times M_2)$ by

$$\{f(x_1,x_2), g(x_1,x_2)\} = \{f_{x_2}, g_{x_2}\}_2(x_1) + \{f_{x_1}, g_{x_1}\}_1(x_2), \quad \text{where}$$

$$\begin{aligned}
f_{x_1}&\colon x_2 \mapsto f(x_1,x_2), \quad f_{x_1} \in C^\infty(M_2),\\
f_{x_2}&\colon x_1 \mapsto f(x_1,x_2), \quad f_{x_2} \in C^\infty(M_1).
\end{aligned}$$

Examples 1.21.

1. Each manifold is a Poisson manifold with trivial bracket $\{\cdot,\cdot\}$.

2. Let (M,ω) be a symplectic manifold i.e. $\omega \in \Omega^2(M)$, $d\omega = 0$, ω non-degenerate ($\omega_x = \sum_{k<j} \omega_{ij}(x) dx^i \wedge dx^j$, where $[\omega_{ij}]$ is an antisymmetric matrix of maximal rank). Define X_f by $\omega(X_f, -) = \langle -df, -\rangle$, that is $i_{X_f}\omega = -df$, $\omega(X_f, Y) = -\langle df, Y\rangle = -Yf$. Now

   $$\{f,g\} := -\omega(X_g, X_f) = \omega(X_f, X_g) = -\{g,f\}.$$

   Indeed, $\{\cdot,\cdot\}$ is bilinear:

   $$\{f,g\} = i_{X_f} dg, \quad d(g_1+g_2) = dg_1 + dg_2$$

   $$\langle X, dg_1 + dg_2\rangle = \langle X, dg_1\rangle + \langle X, dg_2\rangle,$$

   $$X_{f+g} = X_f + X_g, \quad \omega(X_{f+g}, -) = \omega(X_f, -) + \omega(X_g, -),$$

   $\{\cdot,\cdot\}$ satisfies Leibniz identity:

   $$\begin{aligned}
   \{f, gh\} &= i_{X_f} d(gh)\\
   &= i_{X_f}(g dh + h dg)\\
   &= g i_{X_f} dh + h i_{X_f} dg\\
   &= g\{f,h\} + h\{f,g\}.
   \end{aligned}$$

$\{\cdot,\cdot\}$ satisfies Jacobi identity:

$$\begin{aligned}0 = d\omega(X_f, X_g, X_h) &= X_f\omega(X_g, X_h) - X_g\omega(X_f, X_h) + X_h\omega(X_f, X_g)\\ &\quad - \omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g) - \omega([X_g, X_h], X_f)\\ &= X_f\{g,h\} - X_g\{f,h\} + X_h\{f,g\}\\ &\quad - [X_f, X_g](h) - [X_g, X_h](f) - [X_h, X_f](g)\\ &= \{f,\{g,h\}\} - \{g,\{f,h\}\} + \{h,\{f,g\}\}\\ &\quad - [X_f, X_g](h) - [X_g, X_h](f) - [X_h, X_f](g)\\ &= \mathrm{Jac}(f,g,h) - X_f(X_g h) + X_g(X_f h) - X_g(X_h f)\\ &\quad + X_h(X_g f) - X_h(X_f g) + X_f(X_h g)\\ &= \mathrm{Jac}(f,g,h).\end{aligned}$$

Here we used

$$\begin{aligned}&d\eta(X_1,\dots,X_{k+1}) =\\ &= \sum_{i=1}^{k+1}(-1)^{i+1}X_i\cdot\eta(X_1,\dots,\widehat{X_i},\dots,X_{k+1})\\ &+ \sum_{1\le i<j\le k+1}(-1)^{i+j}\eta([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{k+1}),\end{aligned}$$

(with the usual hat notation to denote missing terms) and

$$\begin{aligned}\omega([X_f, X_g], X_h) &= (i_{X_h}\omega)([X_f, X_g])\\ &= -\langle dh, [X_f, X_g]\rangle\\ &= -[X_f, X_g](h).\end{aligned}$$

Thus every symplectic manifold is a Poisson manifold.

Consider a special case of the previous example, the standard symplectic manifold $(\mathbb{R}^{2n}, \omega = \sum dp_i \wedge dq_i)$. Let us remark that Darboux's theorem (see [cw04]) every symplectic manifold is locally symplectomorphic to this one [1]

Exercise 1.22. *Prove by applying definitions that if $M = \mathbb{R}^{2n}$, $\omega = \sum_{i=1}^n dq_i \wedge dp_i$, then $\{f,g\} = \sum_{i=1}^n -\partial_{p_i} f\partial_{q_i} g + \partial_{q_i} f\partial_{p_i} g$.*

Let $f := f(p_i, q_i)$, $\omega(X_f, Y) = -Yf$. Then for $Y = \partial_{q_i}$ and $Y = \partial_{p_i}$ we have respectively

$$\begin{aligned}-i_{\partial_{q_i}}\omega &= \omega(-,\partial_{q_i}) = -dp_i\\ -i_{\partial_{p_i}}\omega &= \omega(-,\partial_{p_i}) = -dq_i\\ X_f &= \sum_{i=1}^n a_i\partial_{q_i} + b_i\partial_{p_i}.\end{aligned}$$

[1] but that does not mean that this is unique symplectic structure on $\mathbb{R}^{2n}$! On the contrary the problem of classifying symplectic forms on, say, $\mathbb{R}^4$ is far from being trivial.

Now $\omega(X_f, \partial_{q_i}) = -b_i$, $\omega(X_f, \partial_{p_i}) = -a_i$ and

$$X_f = \sum_{i=1}^{n} -\partial_{p_i} f \partial_{q_i} + \partial_{q_i} f \partial_{p_i},$$
$$\{f, g\} = X_f(g) = \sum_{i=1}^{n} -\partial_{p_i} f \partial_{q_i} g + \partial_{q_i} f \partial_{p_i} g.$$

Exercise 1.23. *In the setting of the previous exercise derive the canonical Poisson relations.*

$$\{q_i, p_j\} = \delta_{ij},$$
$$\{q_i, q_j\} = 0,$$
$$\{p_i, p_j\} = 0.$$

Proposition 1.24. *On every Poisson manifold there is a unique bivector field* $\Pi \in \Gamma(\Lambda^2 TM)$ *such that*

$$\{f, g\} = \langle \Pi,\, df \wedge dg \rangle.$$

Proof. We need to show that $\{f, g\}(x)$ depends only on $d_x f$ and $d_x g$. Consider f fixed

$$\{f, g\}(x) = (X_f g)(x) = \langle X_f(x),\, d_x g \rangle.$$

Similarly for g fixed

$$\{f, g\}(x) = -\langle X_g(x),\, d_x f \rangle.$$

Furthermore $f \mapsto d_x f$, $C^\infty(M, \mathbb{R}) \to T_x^* M$ is surjective, therefore there exists $\Pi(x)$ bilinear, skewsymmetric on $T_x^* M$ such that

$$\{f, g\}(x) = \Pi(x)(d_x f, d_x g).$$

□

Fix on M a coordinate chart $(U; x_1, \ldots, x_n)$. Then the bivector Π is locally given by

$$\Pi_U = \sum_{i<j} \Pi_{ij} \partial_{x_i} \wedge \partial_{x_j}$$

where the coefficients Π_{ij} are functions on U explicitely given by $\Pi_{ij} = \{x_i, x_j\}$. Therefore Π is determined once you know brackets of local coordinate functions

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \partial_{x_i} f \partial_{x_j} g.$$

Let $\Pi := \sum_{i<j} \Pi_{ij} \partial_{x_i} \wedge \partial_{x_j}$ be a bivector field, where $\Pi_{ij} = \{x_i, x_j\}$. In many examples a Poisson structure on $\mathbb{R}^{2n}$ will be given simply by lifting brackets of coordinates.

Exercise 1.25. *Prove that the Jacobi identity*

$$\mathrm{Jac}(x_i, x_j, x_k) = \sum_{\text{cyclic}} \{\{x_i, x_j\}, x_k\} = 0$$

is equivalent to

$$\sum_{k=1,\ i<j<l}^{n} \frac{\partial \Pi_{ij}}{\partial x_k}\Pi_{kl} + \frac{\partial \Pi_{jl}}{\partial x_k}\Pi_{ki} + \frac{\partial \Pi_{li}}{\partial x_k}\Pi_{kj} = 0. \tag{1.1}$$

Let V be a real n-dimensional vector space. Consider coordinates $x_1, \dots, x_n$. Then $\sum_{i<j} \Pi_{ij}\partial_{x_i} \wedge \partial_{x_j}$ is *Poisson tensor* if and only if (1.1) holds.

Example 1.26. Special cases.

1. $\dim V = 1 \implies \Pi = 0$
2. $\dim V = 2 \implies \Pi = \Pi_{12}\partial_{x_1} \wedge \partial_{x_2}$. In $\mathbb{R}^2$ every bivector defines $\Pi = f(x,y)\partial_x \wedge \partial_y$, $\{x,y\} = f(x,y)$.
3. $\dim V = 3$ - Write explicitely identity 1.1. Find conditions on $f \in C^\infty(\mathbb{R}^3)$ such that $\Pi = y\partial_x \wedge \partial_y + z\partial_x \wedge \partial_z + f\partial_y \wedge \partial_z$ is a .
4. Say Π_{ij} are linear functions in $x_1, \dots, x_n$,

$$\Pi_{ij} = \sum_{k=1}^{n} c_{ij}^k x_k.$$

Therefore $\frac{\partial \Pi_{ij}}{\partial x_k} = c_{ij}^k$, identity 1.1 rewrites as:

$$\begin{aligned} 0 &= \sum_{k=1}^{n} c_{ij}^k c_{kl}^h x_h + c_{jl}^k c_{ki}^h x_h + c_{li}^k c_{kj}^h x_h \\ &= \sum_{k=1}^{n} (c_{ij}^k c_{kl}^h + c_{jl}^k c_{ki}^h + c_{li}^k c_{kj}^h) x_h \\ \Leftrightarrow &\sum_{k=1}^{n} (c_{ij}^k c_{kl}^h + c_{jl}^k c_{ki}^h + c_{li}^k c_{kj}^h) = 0, \end{aligned}$$

for all i, j, l, h. Thus c_{ij}^k are forced to be structure constants of a Lie algebra. Therefore for any given Lie algebra we have a linear Poisson structure.

Let us describe this last example in another way. Take a Lie algebra $\mathfrak{g}$, and let $V = \mathfrak{g}^*$ be the linear functionals on $\mathfrak{g}$. We want to put a on this vector space V. If one knows the bracket between elements of a basis of $\mathfrak{g}^*$, then, by linearity, he knows it on V. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$, $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$. Let $\xi_1, \dots, \xi_n$ be the dual basis of $\mathfrak{g}^*$. Say $\alpha \in \mathfrak{g}^*$, $f, g \in C^\infty(\mathfrak{g}^*)$. Then $d_\alpha f \in (\mathfrak{g}^*)^* = \mathfrak{g}$ and

$$\{f,g\}(\alpha) = \langle \alpha, [d_\alpha f, d_\alpha g] \rangle.$$

For example if $f \simeq X_i$, $g \simeq X_j$, $X_i(\xi_j) = \delta_{ij}$

$$\begin{aligned} \{X_i, X_j\}(\xi_k) &= \langle \xi_k, [X_i, X_j] \rangle \\ &= \langle \xi_k, \sum_{h=1}^{n} c_{ij}^h X_h \rangle \\ &= c_{ij}^k, \\ \{X_i, X_j\} &= c_{ij}^k X_k. \end{aligned}$$

Thus $\Pi = \sum_{k=1}^{n} c_{ij}^{k} X_i \wedge X_j$ is a linear Poisson tensor on $\mathfrak{g}^*$. In conclusion the dual of a Lie algebra has always a canonically defined Poisson tensor.

Let us now list some specific examples of Poisson manifolds. In Poisson geometry, usually, rather then being at lack of examples one has to face the opposite problem, that of being ablo to select the relevant ones. All the brackets appearing in this list appeared at some point connected to specific quantization issues, and this is the reason for our interest in them.

Example 1.27. Consider

$$M = \mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Then

$$\begin{aligned} \{\alpha, \bar{\alpha}\} &= -i|\beta|^2, \\ \{\beta, \bar{\beta}\} &= 0, \\ \{\alpha, \beta\} &= \tfrac{1}{2} i\alpha\beta \\ \{\alpha, \bar{\beta}\} &= \tfrac{1}{2} \alpha\bar{\beta} \end{aligned}$$

defines uniquely a on $\mathfrak{su}(2)$. Are you able to find Casimir functions? We will say more on this Poisson structure in the chapter on Poisson–Lie groups.

Example 1.28. Let φ be a smooth function on $\mathbb{R}^3$. Define

$$\begin{aligned} \{x, y\} &:= \partial_z \varphi \\ \{y, z\} &:= \partial_x \varphi \\ \{z, x\} &:= \partial_y \varphi. \end{aligned}$$

Then for any φ these formulas define a . In fact

$$\begin{aligned} \{x, \{y, z\}\} + \{z, \{x, y\}\} + \{y, \{z, x\}\} &= \{x, \partial_x \varphi\} + \{z, \partial_z \varphi\} + \{y, \partial_y \varphi\} \\ &= \partial_z \varphi (\partial_y \partial_x \varphi) - \partial_y \varphi (\partial_z \partial_x \varphi) \\ &\quad - \partial_x \varphi (\partial_y \partial_z \varphi) + \partial_y \varphi (\partial_x \partial_z \varphi) \\ &\quad - \partial_z \varphi (\partial_x \partial_y \varphi) + \partial_x \varphi (\partial_z \partial_y \varphi) \\ &= 0 \end{aligned}$$

Example 1.29. Let $\mathbb{S}^4 = \{(\alpha, \beta, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} : |\alpha|^2 + |\beta|^2 = t(1-t)\}$. Then

$$\Pi = \alpha\beta \partial_\alpha \wedge \partial_\beta - \alpha\beta^* \partial_\alpha \wedge \partial_{\beta^*} - \alpha^* \beta \partial_{\alpha^*} \wedge \partial_\beta + \alpha^* \beta^* \partial_{\alpha^*} \wedge \partial_{\beta^*}$$

is Poisson tensor. Can you find conditions for f to be a Casimir function? Hint: t is a Casimir function; use spherical coordinates on its level sets. This Poisson structure can be deduced to be the underlying (in a sense to be specified in the last chapters) of Connes–Landi–Matsumoto non commutative 4–spheres (see [**?**, m-o95]).

Example 1.30. The following family of Poisson structures is taken from [kp00]. Let V be a real vector space of dimension n, and U, P_j, $j = 1, \ldots, n-2$ polynomials in variables $x_1, \ldots, x_n$. Define

$$J(h_1, \ldots, h_n) := \det \left[\frac{\partial}{\partial h_i} x_j \right]$$

Prove that

$$\{f, g\} = UJ(f, g, P_1, \ldots, P_{n-2})$$

defines a . These brackets are called Jacobian s ([kp00]).

Example 1.31. On $\mathbb{R}^4$ with coordinates x_0, x_1, x_2, x_3 take real constants J_{12}, J_{23}, J_{31} and define

$$\{x_i, x_j\} := 2J_{ij}x_0x_k$$
$$\{x_0, x_i\} := -2x_jx_k, \quad \text{where } (i, j, k) = (1, 2, 3) \quad \text{or cyclic permutation.}$$

Find the conditions on J_{ij} that implies this is a . These are called Sklyanin Poisson algebras ([s-e82]). Can you find Casimir functions? Hint: two quadratic polynomials.

1.3 The sharp map

Let M be a manifold, and Π a Poisson bivector on M.

Definition 1.32. *For every Poisson manifold* (M, Π) *we define its* sharp map

$$\#_\Pi \colon T^*M \to TM$$
$$\#_{\Pi,x}(x, \alpha_x) := (x, (i_{\alpha_x}\Pi)(x)), \quad \alpha_x \in T^*_xM.$$

Remark 1.33. We have $(i_{\alpha_x}\Pi)(\beta_x) = \langle \Pi, \, \beta_x \wedge \alpha_x \rangle = \Pi_x(\alpha_x, \beta_x)$ for all $\alpha_x, \beta_x \in T^*_xM$.

Let us list its properties:

- The sharp map $\#_\Pi$ is a bundle map on M. It is also called the *anchor* of (M, Π).
- Being a bundle map it induces a map on sections (which will be denoted by the same symbol)

$$\#_\Pi \colon \Omega^1(M) \to \mathfrak{X}(M), \quad \alpha \mapsto i_\alpha\Pi$$

- When restricted to exact 1-forms the sharp map recovers hamiltonians $\#_\Pi(df) = X_f$. In fact

$$\langle \#_\Pi(df), \, dg \rangle = \Pi(df, dg) = \{f, g\} = \langle X_f, \, dg \rangle.$$

  The equality then follows remarking that a vector field is uniquely determined by its contractions with exact 1-forms (locality of vector fields).

- An easy consequence of the previous statement is that the image of the sharp map is $\operatorname{im} \#_{\Pi,x} = \operatorname{Ham}_x(M)$ - vector subspace of T_xM.
- The sharp map has the following local expression

$$\#_\Pi \left(\sum_{i=1}^n a_i dx_i \right) = \sum_{i,j=1}^n \Pi_{ij} a_i \partial_{x_j}$$

$$\#_\Pi(dx_j) = \Pi \left(\sum_{i=1}^n a_i dx_i, dx_j \right) = \sum_{i=1}^n \Pi(a_i dx_i, dx_j) = \sum_{i=1}^n a_i \Pi_{ij}.$$

  If Π_{ij} are smooth, then so is $\#_\Pi$.

The assignment of a vector subspace S_x of T_xM for every $x \in M$ is called a *(generalized) distribution*. A distribution is *differentiable* if for all $x_0 \in M$ and $v_0 \in S_{x_0}$ there exists a neighbourhood U of x_0 and a smooth vector field on U such that $X(y) \in S_y$ for all $y \in U$ and $X(x_0) = v_0$. The word generalized refers to the fact that we do not require $\dim S_xM$ to be constant in x.

Definition 1.34. *The image of the sharp map* $\operatorname{im} \#_\Pi$ *is locally generated by Hamiltonian vector fields and, thus, a differentiable distribution. It will be called the* characteristic distribution *of the Poisson manifold* (M, Π).

Exercise 1.35. *On* $(\operatorname{im} \#_\Pi)_x$ *it is possible to define a natural antisymmetric non-degenerate bilinear product. Let* $v, w \in (\operatorname{im} \#_\Pi)_x$ *and choose* α_x, β_x *such that* $v = \#_\Pi(\alpha_x)$, $w = \#_\Pi(\beta_x)$,

$$(v, w) = \langle \Pi_x,\, \alpha_x \wedge \beta_x \rangle.$$

Prove that it is well defined, and its properties.

Definition 1.36. *Let* $\rho(x) := \dim \operatorname{im} \#_{\Pi,x}$. *We call it the* rank *of the Poisson manifold (at the point* x*).*

Remark 1.37.

- The reason for the name is that in local coordinates

$$\rho(x) = \operatorname{rank}(\Pi_{ij}(x)) = \operatorname{rank}(\{X_i, X_j\}(x)).$$

- The rank is a map $\rho\colon M \to \mathbb{Z}$; from the differentiability of the distribution it follows that $\rho(x)$ is a lower semicontinuos function of x, i.e. it cannot decrease in a neghbourhood of x. Indeed, take $v_1, \dots, v_r$ to be a basis of $(\operatorname{im} \#_\Pi)_{x_0}$ and take $X_1, \dots, X_r$ to be the corresponding local vector fields, then $X_1(x), \dots, X_r(x)$ are linearly independent in a neighbourhood of x_0.

Exercise 1.38. *Show that* $\rho(x) \in 2\mathbb{Z}$ *(is always even).*

Definition 1.39. *If* $\rho(x) = k \in \mathbb{Z}$ *for all* $x \in M$ *the Poisson manifold (and also the characteristic distribution) is called* regular. *If* $x_0 \in M$ *is such that* $\rho(y) = \rho(x_0)$ *for all* y *in a neighbourhood* U_{x_0} *of* x_0*, then* x_0 *is called a* regular point *of* M. *It is called a* singular point *otherwise (i.e. if for all neighbourhoods* U *of* x_0*, there is* $y \in U$ *such that* $\rho(y) > \rho(x)$*).*

Remark 1.40.

- The set of regular points is open and dense, but not necessarily connected. That it is open is obvious from definition. Let x_0 be a singular point. Take $U \subset U_{x_0}$, there exists $y \in U$ such that $\rho(y) > \rho(x_0)$. We want to prove that y is regular. Say it is not, then there exists $y_1 \in U$ (also neighbourhood of y) such that $\rho(y_1) > \rho(y)$. If y_1 is not regular repeat. Find a seqence $\{y_n\}$ in U such that $\rho(y_{i+1}) > \rho(y_i)$. If it admits a converging subsequence we are done: the limit point will be regular. The existence of this converging subsequence follows from the fact that M is locally compact: this allows us to choose U such that $\overline{U}$ is compact.

- If in M there are singular points then $\operatorname{im} \#_\Pi$ is not a vector subbundle of TM. This is a general fact (and follows by the very definition of subbundle): the image of a bundle map is a subbundle if and only if its rank is constant.

Examples 1.41.

1. Let $M = \mathbb{R}^{2n+p}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n, y_1, \ldots, y_p)$. Let $\Pi = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i}$. Then (M, Π) is regular and $\rho(x) = 2n$. Here $\operatorname{im} \#_\Pi$ are tangent spaces to linear subspaces parallel to $y_1 = \ldots = y_p = 0$.

2. Let $\Pi = (x^2 + y^2)\partial_x \wedge \partial_y$ in $M = \mathbb{R}^2$. Then

$$\begin{aligned}(\operatorname{im} \#_\Pi)_{(0,0)} &= \{0\},\\ (\operatorname{im} \#_\Pi)_{(x,y)} &\simeq \mathbb{R}^2 \quad \text{if } (x, y) \neq (0, 0).\end{aligned}$$

   $(0, 0)$ is singular, $(x, y) \neq (0, 0)$ is regular. More generally for $f(x, y)\partial_x \wedge \partial_y$ if $\Gamma_f = \{(x, y) : f(x, y) = 0\}$ then the set of singular points is $\partial\Gamma_f$.

Proposition 1.42. *Let (M, Π) be a Poisson manifold. It is the Poisson manifold associated to a symplectic manifold if and only if it is regular, of dimension $2n$ and rank $2n$, i.e. if and only if $\#_\Pi$ is an isomorphism.*

Proof. (sketchy) M symplectic implies $\operatorname{im} \#_\Pi = TM$. In fact locally on $U \subset M$ $\omega|_U = \sum_{i=1}^n dq_i \wedge dp_i$ and the corresponding Poisson bivector is

$$\Pi|_U = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i}.$$

Therefore $\#_\Pi(dq_i) = \partial_{p_i}$, $\#_\Pi(dp_i) = -\partial_{q_i}$ and $\#_\Pi$ is isomorphism.

Let $\#_\Pi$ be an isomorphism (i.e. $\forall\ x \in M$ $\#_{\Pi,x} \colon T_x^*M \to T_xM$ is an isomorphism). Define $\flat_x \colon T_xM \to T_x^*M$, $\flat_x := \#_{\Pi,x}^{-1}$. Define $\Omega_x \in \Lambda^2 T_x^*M$ as $\Omega_x(v, w) = \Pi_x(\flat_x v, \flat_x w)$. Prove that Ω_x is a 2-form such that $\{f, g\} = \Omega(X_f, X_g)$ and therefore $\operatorname{Jac}(f, g, h) = \mathrm{d}\Omega(X_f, X_g, X_h)$. □

1.4 The symplectic foliation

Definition 1.43. *Let S be a distribution on a manifold M. An* integral *of S is a pair (N, h) of a connected differential manifold N and an immersion $h \colon N \to M$* [2] *such that*

$$T_x(h(N)) \subset S_x.$$

An integral is said to be maximal *if*

$$T_x(h(N)) = S_x.$$

An integral submanifold *of S is a connected immersed submanifold N of M such that (N, i_N) is an integral.*

Remark 1.44. Integral manifolds are immersed submanifolds but not necessarily embedded submanifolds[3]. In particular integral submanifolds are not necessarily closed.

[2] i.e. an injective map with injective tangent map $dh_x \colon T_xN \to T_{h(x)}(M)$

[3] An immersion $i \colon N \hookrightarrow M$ is an embedding if it is an immersion and a homeomorphism from N to $i(N)$ (equipped with the topology induced by M).

Definition 1.45. *A distribution is* fully integrable *if for every $x \in M$ there exists a maximal integral (N, h) of S such that $x \in h(N)$ (maximal at each point).*

Let us recall the Frobenius theorem: a constant rank differentiable distribution is fully integrable if and only if it is involutive, i.e. for all X, Y-sections of S, $[X, Y] \in S$.

For a regular Poisson manifold (M, Π) of rank $2n$, the characteristic distribution is of constant rank, and also involutive, due to $[X_f, X_g] = X_{\{f,g\}}$. Therefore it is fully integrable. Each regular Poisson manifold thus, comes equipped with a regular foliation.

Furthermore on the tangent space to the leaf $(\mathrm{im}\,\#_\Pi)_x$ it is always possible to define a natural antisymmetric nondegenerate bilinear product.

Computations similar to those of proposition 1.42, allow to prove that if (N, h) is the maximal integral containing x, then there is a symplectic 2-form ω_N on N such that $\omega_N = h^*\omega_\Pi$, where $(\omega_\Pi)_x \in \Lambda^2(\mathrm{im}\,\#_\Pi)^*_x$ is determined by the above bilinear product.
We will call this foliation the *symplectic foliation* of M. Our purpose in the following will be to understand how this generalizes to the non regular case.

Example 1.46. Let's go back to the first example in 1.41: $M = \mathbb{R}^{2n+p}$, $\Pi = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i}$. The leaves are

$$S_{c_1 \dots c_p} := \{y_1 = c_1, \dots, y_n = c_n\} \quad \text{- linear subspaces.}$$

On each leaf $\omega = \sum_{i=1}^n dq_i \wedge dp_i$.

Remark 1.47. Not every foliation is a symplectic foliation. In fact, first of all leaves need to carry a symplectic structure, a condition which already puts some topological restriction (e.g. you cannot have symplectic structure on $\mathbb{S}^{2n}$ if $n > 1$). But also a foliation into given symplectic manifolds has some more delicate obstructions depending on how the symplectic forms vary from leaf to leaf (see [b-m01] for the regular case).

Now we want to generalize this statement to non regular Poisson manifolds.

Theorem 1.48 (Weinstein's splitting theorem)**.** *Let (M, Π) be a Poisson manifold. Let $x_0 \in M$, $\rho_\Pi(x_0) = 2n$. Then there exists a coordinate neighbourhood U of x_0 with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n, y_1, \dots, y_p)$ $(\dim M = 2n + p = m)$ such that*

$$\Pi(x) = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i} + \sum_{i<j=1}^p \varphi_{ij}(x) \partial_{y_i} \wedge \partial_{y_j} \quad \forall x \in U,$$

and such that $\varphi_{ij}(x)$ depends only on coordinates $y_1, \dots, y_k$ and $\varphi_{ij}(x_0) = 0$.

Proof. The theorem is proved by induction on the semirank n, $\rho_\Pi(x_0) = 2n$. If $n = 0$ there is nothing to prove. Say the theorem holds for semirank equal to $n - 1 \neq 0$. Certainly there exists $f, g \in C^\infty(M)$ such that

$$\{f, g\}(x_0) \neq 0.$$

Let $p_1 = g$. Then $X_{p_1}(f)(x_0) \neq 0$, so $X_{p_1}(x_0)$ is a nonzero vector field. By the rectifying theorem for vector fields there exists a coordinate neighbourhood centered at x_0 such that $-X_{p_1} = \partial_{q_1}$, hence $\{q_1, p_1\} = -X_{p_1} q_1 = 1$. Remark that X_{p_1} and X_{q_1} are linearly independent (if $X_{q_1} = \lambda X_{p_1}$ then $\{q_1, p_1\} = X_{q_1}(p_1) = \lambda X_{p_1}(p_1) = -\lambda \partial_{q_1} p_1 = 0$). Furthermore $[X_{q_1}, X_{p_1}] = X_{\{q_1,p_1\}} = X_{-1} = 0$. Therefore locally around x_0 these two vector fields span a regular involutive distribution, which is integrable due to Frobenius theorem.

As a consequence there are local coordinates $(y_1, \ldots, y_m)$ centered at x_0 such that

$$\begin{aligned} X_{q_1} &= \partial_{y_1} \\ X_{p_1} &= \partial_{y_2} \\ \{q_1, y_i\} &= X_{q_1} y_i = 0 \quad \forall\, i \neq 1 \\ \{p_1, y_i\} &= X_{p_1} y_i = 0 \quad \forall\, i \neq 2. \end{aligned}$$

Lemma 1.49 (Poisson theorem)**.**

$$\begin{aligned} \{p_1, \{y_i, y_j\}\} &= 0 \quad \forall\, i, j \geq 3, \\ \{q_1, \{y_i, y_j\}\} &= 0 \quad \forall\, i, j \geq 3. \end{aligned}$$

Proof. (of lemma) Simply apply Jacobi identity. ☐

Remark 1.50. The reason for giving these equalities the dignity of a separate statement is due to the fact that historically this is the first form in which Jacobi identity was stated.

Now $(q_1, p_1, y_3, \ldots, y_m)$ is a new coordinate system for M, because $(y_1, \ldots, y_m)$ is a local coordinate system and the map $\Phi\colon (y_1, \ldots, y_m) \mapsto (q_1, p_1, y_3, \ldots, y_m)$ has Jacobian

$$\left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{$*$} \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right)$$

In these new coordinates the Poisson bivector has local expression

$$\Pi = \partial_{q_1} \wedge \partial_{p_1} + \sum_{3 \leq i < j \leq m} \Pi'_{ij}(y_3, \ldots, y_m) \partial_{y_i} \wedge \partial_{y_j}.$$

The right summand locally defines a Poisson bivector on M of rank $2(n-1)$. Applying the induction hypothesis to it proves the theorem. ☐

In the symplectic case this theorem recovers a well-known result:

Corollary 1.51 (Darboux theorem)**.** *Let (M, ω) be a symplectic manifold and $x_0 \in M$. Then there exists a coordinate neighbourhood $(U; q_1, \ldots, q_n, p_1, \ldots, p_n)$ of x_0 such that*

$$\omega|_U = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

In analogy with this last statement coordinates generated by the splitting theorem are also called *Darboux coordinates* centered at x_0.

Example 1.52. Let $\mathfrak{g}^*$ be the dual of a Lie algebra $\mathfrak{g}$ and $\Pi = \sum c_{ij}^k X_k \partial_i \wedge \partial_j$. Then these are Darboux coordinates centered at the origin in which there is no symplectic term and all functions φ_{ij} are linear.

Definition 1.53. *Let (M,Π) be a Poisson manifold. A* hamiltonian path *on M is a smooth curve $\gamma\colon [0,1] \to M$ such that each tangent vector $\dot\gamma_t$ at $\gamma(t)$ belongs to $\operatorname{im}\#_{\Pi,\gamma(t)}$ for all $t \in [0,1]$. Let $x, y \in M$. We say that x and y are in* hamiltonian relation *if there exists a piecewise Hamiltonian curve γ on M connecting x to y.*

Lemma 1.54. *Hamiltonian relation is an equivalence relation.*

Proof. This is left as an exercise. Consider the following hints. Reflexive - trivial; symmetric - change backwards the time parametrization; transitive - concatenation of Hamiltonian paths is an Hamiltonian path. □

Definition 1.55. *Equivalence classes of this relation are called* symplectic leaves *of (M,Π).*

Proposition 1.56. *Each symplectic leaf is a maximal integrable submanifold of (M,Π).*

Proof. (sketchy) Let F be a leaf, $x \in F$, $T_xF \subset \operatorname{im}\#_{\Pi,x}$ because all paths on F exiting from x are Hamiltionian paths.

Let $X \in \mathfrak{X}(M)$ such that $X(x) \in \operatorname{im}\#_{\Pi,x}$. Consider the flow of X starting at x. This is a curve $\exp_x(tX)\colon (-\varepsilon,\varepsilon) \to M$ which is an Hamiltonian path. Therefore each point of this curve is in F, so $X(x) \in T_xF$. Thus $T_xF = \operatorname{im}\#_{\Pi,x}$.

Fix Darboux coordinates centered at $x \in F$.

$$\operatorname{im}\#_{\Pi,x} = \langle \partial_{q_1}, \ldots, \partial_{q_n}, \partial_{p_1}, \ldots, \partial_{p_n} \rangle$$

F is locally given by $y_1 = \ldots = y_p = 0$ therefore F is an immersed submanifold. □

Proposition 1.57. *Let (M,Π) be a Poisson manifold. On each symplectic leaf F there is a well defined symplectic structure such that the inclusion map $i\colon F \to M$ is a Poisson map.*

Proof. Let $x \in F$ and fix Darboux coordinates centered at x in a neighbourhood U. This implies that $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are local coordinates for F around x. Define

$$\omega_x = \sum_{i=1}^{n} d_xq_i \wedge d_xp_i, \quad x \in U.$$

Check that this defines a symplectic structure on F. To verify that $i\colon F \to M$ is Poisson, it is enough to check it preserves brackets of coordinate functions. □

Examples 1.58.

1. Each symplectic manifold is a symplectic leaf of itself.
2. $(\mathbb{R}^2, f(x,y)\partial_x \wedge \partial_y)$. Each point of $\Gamma_f = \{(x,y) : f(x,y) = 0\}$ is a 0-dimensional symplectic leaf. Each connected component of $\mathbb{R}^2 \setminus \Gamma_f$ is a 2-dimensional symplectic leaf.
3. $M = \mathfrak{g}^*$. By the Leibniz rule

$$\operatorname{im}\#_{\Pi,x} = \operatorname{span}\{X_f(x) : f \quad \text{linear }\}$$

Let $f, g \in C^\infty(\mathfrak{g}^*)$ be linear i.e. $f, g \in \mathfrak{g}$. Fix $\alpha \in \mathfrak{g}^*$

$$\{f, g\}(\alpha) = \alpha([f, g]) = \langle \alpha, \mathrm{ad}_f\, g\rangle = -\langle \mathrm{ad}_f^*\, \alpha, g\rangle$$

$X_f(\alpha) = (-ad_f^*)(\alpha)$, where ad_f^* is an infinitesimal coadjoint action i.e. fundamental vector field of the adjoint action. Symplectic leaves are coadjoint orbits. Therefore each coadjoint orbit in $\mathfrak{g}^*$ carries a naturally defined symplectic form, called the Kirillov-Kostant-Souriau form (KKS).

Proposition 1.59. *Let (M, Π) be a Poisson manifold. Casimir functions are constant along the leaves (therefore leaves are contained in connected components of level sets of Casimirs).*

Proof. Let F be a leaf, f a Casimir function. We want to prove that $f|_F$ is constant. This is equivalent to $Xf = 0$ for all $X \in TF$ (vector fields tangent to F, locally). But locally $\mathfrak{X}(F) = \mathrm{im}\,\#_\Pi = \mathrm{Ham}(M)$ and

$$X_g f = \{g, f\} = 0 \quad \forall\, g \in C^\infty(M).$$

□

Proposition 1.60. *Let (M, Π) be a Poisson manifold,* $\dim M = m$*. Let $x_0 \in M$ be such that* $\mathrm{rank}_\Pi(x_0) = \rho_\Pi(x_0) = m$*. Then the symplectic leaf through x_0 is open in M.*

Proof. Around x_0 rank is constant and equal $\dim M$. Therefore $(\mathrm{im}\,\#_\Pi)_{x_0} = T_x M$ and due to lower semicontinuity the same holds for any y in an open neighbourhood U of x_0. Thus every path on M exiting x_0 is locally Hamiltonian, hence the thesis. □

Proposition 1.61. *Let (M, Π) be a Poisson manifold. Let $f_1, \dots, f_p$ be Casimir functions on M. Let*

$$\Gamma_{i,c} := \{x \in M : f_i(x) = c\}$$

If $\Gamma_{1,c} \cap \ldots \cap \Gamma_{p,c}$ has $\dim = \mathrm{rank}\,\Pi_0$ *and is smooth, then its connected components are symplectic leaves.*

Remark 1.62. It may happen that $\Gamma_{1,c} \cap \ldots \cap \Gamma_{p,c}$ may have $\dim \neq \mathrm{rank}\,\Pi_0$ and, thus, leaves are not just level sets of Casimir. As an example consider the case in which there is an open dense leaf. Casimirs, by continuity, will be forced to be constants; their level sets are the whole manifold which, nontheless may have a non trivial foliation.

Let us consider a general polynomial on $\mathbb{R}^n$, i.e. a such that $\{x_i, x_j\} = P_{ij}(x_1, \dots, x_n)$. A function $f \in C^\infty(\mathbb{R}^n)$ is a Casimir function if and only if $\{x_i, f\} = 0$ for any $i = 1, \dots, n$. This can be rewritten as $\sum_j P_{ij}\partial_{x_j} f = 0$. Therefore f has to be a smooth solution of a system of linear first order PDE's. If, as in this case, we are considering a linear Poisson structure the system has constant coefficients (which are the structural constant of the Lie algebra) and its solutions can be explicitly determined.

Example 1.63. Consider $\mathfrak{su}(2)^* \cong \mathbb{R}^3$. Its Lie-Poisson bivector is

$$\Pi(x_1, x_2, x_3) = x_1 \partial_{x_2} \wedge \partial_{x_3} + x_2 \partial_{x_3} \wedge \partial_{x_1} + x_3 \partial_{x_1} \wedge \partial_{x_2}.$$

Find the symplectic foliation.

$$\begin{aligned}\#_\Pi(dx_1) &= -x_2\partial_{x_3} + x_3\partial_{x_2},\\ \#_\Pi(dx_2) &= x_1\partial_{x_3} - x_3\partial_{x_1},\\ \#_\Pi(dx_3) &= -x_1\partial_{x_2} + x_2\partial_{x_1}.\end{aligned}$$

Thus

$$\Pi_{ij}(x_1,x_2,x_3) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

Compute $\operatorname{rank}\Pi_{ij}(x_1,x_2,x_3)$.
If $(x_1,x_2,x_3) = (0,0,0)$ then $\rho_\Pi((0,0,0)) = 0$, and if $(x_1,x_2,x_3) \neq (0,0,0)$ then $\rho_\Pi((x_1,x_2,x_3)) = 2$ (always 2×2 minor $\neq 0$). Therefore we have everywhere rank 2, except at origin, which is an isolated 0-dimensional symplectic leaf.
Remark that $x_1^2 + x_2^2 + x_3^2$ is a Casimir function

$$\begin{aligned}\{x_1,-\} &= 2x_2\{x_1,x_2\} + 2x_3\{x_1,x_3\} = 2x_2x_3 - 2x_3x_2 = 0,\\ \{x_1,-\} &= 2x_1\{x_2,x_1\} + 2x_3\{x_2,x_3\} = 2x_1x_3 - 2x_3x_1 = 0,\\ \{x_1,-\} &= 2x_1\{x_3,x_1\} + 2x_2\{x_3,x_2\} = 2x_1x_2 - 2x_2x_1 = 0.\end{aligned}$$

Thus symplectic leaves are contained in spheres $x_1^2+x_2^2+x_3^2 = r^2$ (with the singular case of a 0-dim. leaf at $r=0$) . Each leaf is a connected open 2-manifold in $\mathbb{S}^2$, so each leaf is homeomorphic to $\mathbb{S}^2$.

It is easily checked that the corresponding symplectic structure is the unique SU(2) invariant volume form on the sphere of radius r.

Exercise 1.64. *Describe the symplectic foliation of the linear Poisson structure on* $\mathfrak{sl}(2;\mathbb{R})$*:*

$$\Pi(\xi_1,\xi_2,\xi_3) = \xi_1\partial_{\xi_2}\wedge\partial_{\xi_3} - \xi_2\partial_{\xi_3}\wedge\partial_{\xi_1}\xi_3\partial_{\xi_1}\wedge\partial_{\xi_2}.$$

Exercise 1.65. *Consider the constant Poisson structure on* $\mathbb{R}^3$ *given by*

$$\Pi(x_1,x_2,x_3) = (\partial_{x_1} + \alpha_1\partial_{x_2})\wedge(\partial_{x_2} + \alpha_2\partial_{\xi_3})$$

where $\alpha_1,\alpha_2\in\mathbb{R}$*. This bivector projects onto a Poisson bivector on the torus* $\mathbb{T}^3$*. Describe the sharp map and the symplectic foliation both in space and on the torus. Check that when* $(1,\alpha_1,\alpha_2)$ *is rationally independent each leaf is two dimensional and dense on the torus.*

Example 1.66. (Natsume-Olsen Poisson sphere)

$$\mathbb{S}^2\subset\mathbb{C}\times\mathbb{R},\quad \zeta\bar\zeta + z^2 = 1$$

The following s on $\mathbb{S}^2$ were introduced in [no03]

$$\begin{aligned}\{\zeta,\, z\} &= i(1-z^2)\zeta,\\ \{\bar\zeta,\, z\} &= -i(1-z^2)\bar\zeta,\\ \{\zeta,\, \bar\zeta\} &= -2i(1-z^2)z.\end{aligned}$$

Therefore

$$\Pi = (1-z^2)[i\zeta\partial_\zeta\wedge\partial_z - i\bar\zeta\partial_{\bar\zeta}\wedge\partial_z - 2iz\partial_z\wedge\partial_{\bar\zeta}] = (1-z^2)\Pi_0.$$

where Π_0 is the standard rotation invariant symplectic (i.e. volume) form on $\mathbb{S}^2$. The sharp map $\#_\Pi$ is given by:

$$\begin{aligned} dz &\mapsto (z^2-1)i(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}), \\ dz &\mapsto (1-z^2)i(\zeta\partial_z - 2z\partial_{\bar{\zeta}}), \\ dz &\mapsto (1-z^2)i(-\bar{\zeta}\partial_z + 2z\partial_\zeta). \end{aligned}$$

so that in the obvious basis it is represented by the matrix:

$$\begin{pmatrix} 0 & (1-z^2)i\zeta & -i(1-z^2)\bar{\zeta} \\ (z^2-1)i\zeta & 0 & 2i(1-z^2)z \\ -i(z^2-1)\bar{\zeta} & 2i(z^2-1)z & 0 \end{pmatrix}$$

Such matrix has rank $= 0$ if $\zeta = \bar{\zeta} = 0$, that is if $1 - z^2 = 0$. This happens in two points

$$\begin{aligned} \zeta = \bar{\zeta} = 0, \quad z = -1 \text{ south pole}, \\ \zeta = \bar{\zeta} = 0, \quad z = 1 \text{ north pole}. \end{aligned}$$

The complement is an open symplectic manifold diffeomorphic to the punctured plane. There are no non constant Casimir functions for this manifold. Quite generally, in fact, whenever there exists a dense symplectic leaf one can apply Proposition 1.61 to deduce that Casimirs are constant on a dense subset and thus, by continuity, everywhere. Show that infinitesimal rotations around the z-axis are given by a vector field which is Poisson but not Hamiltonian.

Exercise 1.67. *Let $\mathfrak{g}$ be a real Lie algebra, and let $\xi : \wedge^2\mathfrak{g} \to \mathbb{R}$ a linear map. For any two linear functions on $\mathfrak{g}^*$, $f, g \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$ define*

$$\{f, g\}(\alpha) = \alpha([f, g]) + \xi(f, g), \qquad \forall \alpha \in \mathfrak{g}^*$$

Find conditions on ξ granting this to be a . Such brackets are called affine s.

Exercise 1.68. *Try to describe the sharp map and the corresponding symplectic foliation for the examples given at the end of the previous section.*

Chapter 2

Schouten-Nijenhuis bracket

2.1 Lie-

Let (M, Π) be a Poisson manifold.

Theorem 2.1. *There exists a unique $\mathbb{R}$-linear skewsymmetric bracket*

$$[-,-]_\Pi \colon \Omega^1 M \times \Omega^1 M \to \Omega^1 M$$

such that

1. $[df, dg] = d\{f, g\}$ *for all* $f, g \in C^\infty(M)$,
2. $[\alpha, f\beta] = f[\alpha, \beta] + (\#_\Pi(\alpha)f)\beta$ *for all* $\alpha, \beta \in \Omega^1 M,\ f \in C^\infty(M)$.

Such bracket is given by.

$$[\alpha, \beta] = L_{\#_\Pi(\alpha)}\beta - L_{\#_\Pi(\beta)}\alpha - d(\Pi(\alpha, \beta)). \tag{2.1}$$

Furthermore $[-,-]_\Pi$ *is a Lie bracket and* $\#_\Pi \colon \Omega^1 M \to \mathfrak{X}(M)$ *is a Lie algebra homomorphism:*

$$[\#_\Pi(\alpha), \#_\Pi(\beta)] = \#_\Pi([\alpha, \beta]_\Pi).$$

Before going into the proof let us explicitely describe this bracket for the canonical , i.e. $(M, \Pi) = (\mathbb{R}^{2n}, \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i})$. The bracket between exact 1–forms, due to the first property, is easily described: all brackets between exact forms are zero. Still the brackets between 1–forms can be non trivial, due to the second property which reflects the behaviour w.r. to the $C^\infty(M)$- module structure. As an example

$$\begin{aligned}
[dp_i, \sum_j a_j dp_j + b_j dq_j] &= \sum_j [dp_i, a_j dp_j] + [dp_i, b_j dq_j] \\
&= \sum_j a_j [dp_i, dp_j] + \sharp(dp_i) a_j dp_j + b_j [dp_i, dq_j] + \sharp(dp_i) b_j dq_j \\
&= -\sum_j \partial_{q_i} a_j dp_j + \partial_{q_i} b_j dq_j \,.
\end{aligned}$$

At this point all other brackets can be easily computed along the same lines. What this computation shows is that the first property fixes the value of the bracket on exact forms. Exact forms are generators for the $C^\infty(M)$– module of all forms and the second property allows to obtain formulas for all brackets from the knowledge of the one on exact ones (plus the sharp map).

Proof.

Step 1 If it exists, such $[-,-]_\Pi$ should be a local operator i.e. if $\beta_1|_U = \beta_2|_U$ in a neighbourhood U of x_0, then $[\alpha,\beta_1]_\Pi(x_0) = [\alpha,\beta_2]_\Pi(x_0)$ (the value at the point depends only on values of forms in a neighbourhood of that point) . To prove it let's take a compact neighbourhood $V_{x_0} \subset U$ of x_0 and take $f \in C^\infty(M)$ such that $f = 1$ on V_{x_0}, $f|_{M\setminus U} = 0$. Then

$$[\alpha, f\beta_1](x_0) = \underbrace{f(x_0)}_{=1}[\alpha,\beta_1]_\Pi(x_0) + (\underbrace{\#_\Pi(\alpha)f}_{=0})(x_0)\beta_1(x_0)$$

$$[\alpha, f\beta_1](x_0) = \underbrace{f(x_0)}_{=1}[\alpha,\beta_2]_\Pi(x_0).$$

Step 2 Applying twice property 2. one has the following:

Lemma 2.2. *For every pair of functions $h, f \in C^\infty(M)$*

$$[h\alpha, f\beta]_\Pi = (hf)[\alpha,\beta] + h(\#_\Pi(\alpha)f)\beta - f(\#_\Pi(\beta)h)\alpha.$$

Step 3 Locality implies that the bracket can be computed in local coordinates. After fixing a coordinate neighbourhood $(U, x_1, \ldots, x_n)$ (and a corresponding local expression for Π) take $\alpha = \sum \alpha_i dx_i$, $\beta = \sum \beta_i dx_i$, $\Pi = \sum \Pi_{ij}\partial_{x_i} \wedge \partial_{x_j}$. Then a bracket verifying the requested properties should be computed as:

$$\begin{aligned}
[\alpha,\beta]_\Pi &= \sum_{i,j}[\alpha_i dx_i, \beta_j dx_j]_\Pi \\
&= \sum_{i,j} \alpha_i\beta_j[dx_i, dx_j]_\Pi + \alpha_i(\#_\Pi(dx_i)\beta_j)dx_j - \beta_j(\#_\Pi(dx_j)\alpha_i)dx_i \\
&= \sum_{i,j}\left(\alpha_i\beta_j d\{x_i,x_j\} + \sum_k(\alpha_i\Pi_{ik}\partial_k\beta_j dx_j - \beta_j\Pi_{jk}\partial_k\alpha_i dx_i)\right) \\
&= d\left(\sum_{i,j}\Pi_{ij}\alpha_i\beta_j\right) - \sum_{i,j}\Pi_{ij}\beta_j d\alpha_i - \sum_{i,j}\Pi_{ij}\alpha_i d\beta_j \\
&= d\langle\Pi,\, \alpha\wedge\beta\rangle + i_{\#_\Pi(\alpha)}d\beta - i_{\#_\Pi(\beta)}d\alpha.
\end{aligned}$$

Since this last expression does not depend on the choice of local coordinates, we have the existence and unicity of $[-,-]_\Pi$ verifying 1. and 2.

Step4 The Jacobi identity is proved locally on a triple of 1–forms adf, bdg, cdh, by using the explicit formulae seen so forth.

Step5 Recall Cartan's magic formula:

$$L_X = di_X + i_X d$$

Using it we get that the bracket defined by (2.1) verifies:

$$\begin{aligned}[df, dg]_\Pi &= L_{\#_\Pi(df)}dg - L_{\#_\Pi(dg)}df - d\Pi\{df, dg\} \\ &= L_{X_f}dg - L_{X_g}df - d\{f,g\} \\ &= d\,\underbrace{i_{X_f}dg}_{X_f(g)} - d\,\underbrace{i_{X_g}df}_{X_g(f)} - d\{f,g\} \\ &= d\{f,g\} + d\{f,g\} - d\{f,g\} \\ &= d\{f,g\}\,,\end{aligned}$$

and furthermore

$$\begin{aligned}[\alpha, f\beta]_\Pi &= d\langle \Pi,\, \alpha\wedge f\beta\rangle + i_{\#_\Pi(\alpha)}d(f\beta) - i_{\#_\Pi(f\beta)}d\alpha \\ &= d(f\langle \Pi,\, \alpha\wedge\beta\rangle) + i_{\#_\Pi(\alpha)}d(f\beta).\end{aligned}$$

Due to unicity, then, bracket (2.1) is exactly the one we were looking for.

Step5 We want to prove that $\sharp_\Pi$ induces a Lie algebra homomorphism. This is easily checked on exact 1–forms:

$$[\sharp_\Pi(df), \sharp_\Pi(dg)] = [X_f, X_g] = X_{\{f,g\}} = \sharp_\Pi(d\{f,g\}) = \sharp_\Pi([df, dg])\,.$$

But then both maps $[,]^1_\Pi = \sharp_\Pi \circ [,]$ and $[,]^2_\Pi = [,]\circ(\sharp_\Pi\otimes\sharp_\Pi)$ are $\mathbb{R}$-bilinear, local, skewsymmetric operations satisfying

$$[\alpha, f\beta]^{1,2}_\Pi = f[\alpha,\beta]^{1,2}_\Pi + (\sharp_\Pi(\alpha)f)\sharp_\Pi(\beta)\,.$$

Considerations as before allow to prove that such a map is forcedly unique and therefore $[,]^1_\Pi = [,]^2_\Pi$, i.e. $\sharp_\Pi$ is a Lie algebra homomorphism.

□

Exercise 2.3. *Compute the bracket on* 1*–forms for linear s on* $\mathfrak{g}^*$ *(e.g. for the dual Lie algebra* $\mathfrak{su}(2)^*$*). Prove that this bracket induces the original one on* $\mathfrak{g}\simeq(\mathfrak{g}^*)^*$*.*

The existence of a bracket between 1–forms can be seen as a property of the cotangent bundle of a Poisson manifold. This bracket, furthermore, as stated in the theorem is Lie homomorphic to the natural bracket on vector fields. All of this may be summarized as a special case of the following definition.

Definition 2.4. *Let* M *be a manifold,* $E\to M$ *vector bundle. Then* E *is called a* Lie algebroid *if there exists a bilinear bracket*

$$[-,-]\colon \Gamma(E)\times\Gamma(E)\to\Gamma(E)$$

and a bundle map, called the anchor, $\rho\colon E\to TM$ $(\rho\colon\Gamma(E)\to\mathfrak{X}(M))$ *such that*

1. $(\Gamma(E), [-,-])$ *is a Lie algebra,*
2. ρ *is a Lie algebra homomorphism,*
3. $[v, fw] = f[v,w] + (\rho(v)f)w$ *for all* $v, w\in\Gamma(E)$, $f\in C^\infty(M)$.

Remarks 2.5.

- Given any Lie algebroid, the image of the anchor is always a generalized integrable distribution; its maximal integrable submanifolds are called *orbits* of the Lie algebroid.
- The tangent bundle TM to a manifold is always a Lie algebroid with the trivial anchor map $\rho = \mathrm{id}$.
- Theorem 2.1 proves that for any Poisson manifold M, its cotangent bundle T^*M is a Lie algebroid with anchor the sharp map. The orbits of this algebroid are the symplectic leaves of M

Most of the theory of Poisson manifolds can be adapted to more general Lie algebroids (this applies, for example, to the (co)homology theories we will describe later on and to the corresponding invariants). Much work on this generalization was produced around year 2000. Here we will just refer to the original papers [c-m03, elw99, h-j99] for a taste of this topic and to the book [m-k05] for an (almost) comprehensive list of the many deep interrelations between Poisson and Lie algebroid geometry.

2.2 Schouten-Nijenhuis bracket

Let M be a smooth manifold of dimension M. Let us fix the following notations

$$\begin{aligned}
\Omega^p(M) &= \Gamma(\Lambda^p T^*M) \text{ differential } p\text{-forms, } p \geq 1,\\
\mathfrak{X}^p(M) &= \Gamma(\Lambda^p TM) \text{ } p\text{-multivector fields, } p \geq 1,\\
\Omega^0(M) = \mathfrak{X}^0(M) &= C^\infty(M),\\
\Omega^\bullet(M) &= \bigoplus_p \Omega^p(M),\\
\mathfrak{X}^\bullet(M) &= \bigoplus_p \mathfrak{X}^p(M) \text{ graded vector spaces.}
\end{aligned}$$

External product gives both spaces a structure of graded, associative algebra, $\mathbb{Z}_2$-commutative (supercommutative) i.e.

$$P \wedge Q = (-1)^{\deg Q \deg P} Q \wedge P.$$

The natural duality pairing between T_xM and T^*_xM extends to a natural pairing between $\Omega^\bullet(M)$ and $\mathfrak{X}^\bullet(M)$ as follows: on 1–forms and vectors we have:

$$\langle \alpha,\, X\rangle_x := \langle \underbrace{\alpha(x)}_{\in T^*_xM},\, \underbrace{X(x)}_{\in T_xM}\rangle \quad \text{for } \alpha \in \Omega^1(M),\ X \in \mathfrak{X}^1(M);$$

on homogeneous decomposable higher degree forms and (multi)vectors we have

$$\langle \omega,\, P\rangle = \begin{cases} 0 & p \neq q, \\ \det(\langle \alpha_i,\, X_j\rangle) & p = q \quad \text{for } \omega = \alpha_1 \wedge \cdots \wedge \alpha_q,\ P = X_1 \wedge \cdots \wedge X_p, \\ & \alpha_i \in \Omega^1(M),\ X_j \in \mathfrak{X}^1(M). \end{cases} \tag{2.2}$$

Now let us remark that $\langle \omega, P\rangle(x)$ depends only on $\omega(x)$, $P(x)$. Being locally every q-form (resp. p-vector field) decomposable the formula (2.2) above defines a $C^\infty(M)$-bilinear pairing on the whole space of forms and multivectors on M.

Another operation, commonly appearing in differential geometry, that we woud like to recall here is the inner product (extended to the case of multivector fields). Given $P \in \mathfrak{X}^\bullet(M)$ and $\omega \in \Omega^\bullet(M)$ the inner product is given by:

$$\langle i_P\omega, Q\rangle = \langle \omega, P \wedge Q\rangle \quad \forall\, Q \in \mathfrak{X}^\bullet(M).$$

This operation is the left transpose of the external (wedge) product.

The $C^\infty(M)$-module of vector fields $\mathfrak{X}^1(M)$ is, as already remarked, a Lie algebra with bracket $[X,Y] = XY - YX$ of vector fields. We now want to extend this operation to the whole graded space of multivectors. Let us begin with an easy case (corresponding to multivectors at one point). Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$.

Proposition 2.6. *There is a unique bracket on $\Lambda^\bullet\mathfrak{g}$ which extends the Lie bracket on $\mathfrak{g}$ and such that if $A \in \Lambda^a\mathfrak{g}$, $B \in \Lambda^b\mathfrak{g}$, $C \in \Lambda^c\mathfrak{g}$ then:*

1. $[A,B] = -(-1)^{(a-1)(b-1)}[B,A]$;
2. $[A, B\wedge C] = [A,B]\wedge C + (-1)^{(a-1)b}B\wedge[A,C]$;
3. $$(-1)^{(a-1)(c-1)}[A,[B,C]] + (-1)^{(b-1)(c-1)}[B,[C,A]] \\ +(-1)^{(c-1)(b-1)}[C,[A,B]] = 0;$$
4. *The bracket of an element in $\Lambda^\bullet$ with an element in $\Lambda^0\mathfrak{g} = \mathbb{K}$ is 0.*

Remark 2.7. It is not correct to say that $\Lambda^\bullet\mathfrak{g}$ is a graded Lie algebra. In fact in a graded Lie algebra the 0-component should be a Lie subalgebra, therefore the 0-component should be $\mathfrak{g}$. $\Lambda^{\bullet+1}\mathfrak{g}$ is a graded Lie algebra.

Proof. Start from property 2 to prove

$$[A, B_1\wedge\cdots\wedge B_n] = \sum_{i=1}^{n}(-1)^{i(a-1)}B_1\wedge\cdots\wedge[A,B_i]\wedge\cdots\wedge B_n \quad \forall A\in\Lambda^a\mathfrak{g}, B_i\in\mathfrak{g}$$

Property 1 allows to prove a similar formula for $[B_1\wedge\cdots\wedge B_n, A]$. This can be extended by $\mathbb{K}$-linearity to sum of decomposables and shows that such a bracket exists and is unique. To verify Jacobi identity we remark that it is enough to show it holds on decomposables. □

Proposition 2.8. *Let M be a manifold. Then there exists a unique $\mathbb{R}$-bilinear bracket $[-,-]\colon \mathfrak{X}^\bullet(M)\times\mathfrak{X}^\bullet(M)\to\mathfrak{X}^\bullet(M)$ such that*

1. *$[-,-]$ is of degree -1;*
2. *For all $X\in\mathfrak{X}^1(M)$ and $Q\in\mathfrak{X}^\bullet(M)$*

   $$[X,Q] = L_XQ.$$

   In particular the bracket coincides with the usual Lie bracket of vector fields on $\mathfrak{X}^1(M)$;

3. *For all $P \in \mathfrak{X}^p(M)$ and $Q \in \mathfrak{X}^q(M)$*
$$[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P]$$

4. *For all $P \in \mathfrak{X}^p(M)$, $Q \in \mathfrak{X}^q(M)$, $R \in \mathfrak{X}^\bullet(M)$*
$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R].$$

Such bracket will be called the Schouten-Nijenhuis bracket *of multivector fields (or SN bracket for short). It will furthermore satisfy the graded Jacobi identity:*

5. *For all $P \in \mathfrak{X}^p(M)$, $Q \in \mathfrak{X}^q(M)$, $R \in \mathfrak{X}^r(M)$*
$$(-1)^{(p-1)(r-1)}[P, [Q, R]] + (-1)^{(q-1)(p-1)}[Q, [R, P]] \\ +(-1)^{(r-1)(q-1)}[R, [P, Q]] = 0.$$

Proof. Again the first step is to prove that such bracket, if it exists. has to be a local operation, i.e. for all U open in M, $[P, Q]|_U$ depends only on $P|_U$, $Q|_U$. The proof of this fact is similar to the analogous proof in theorem 2.1. Due to the graded antisymmetry required as property 3, it is enough to show that if $Q_1|_U = Q_2|_U$ then $[P, Q_1](x_0) = [P, Q_2](x_0)$ for a neighbourhood U of x_0. To prove this take a bump function $f \in C^\infty(M)$, $f = 0$ outside U, $f = 1$ in a compact neighbourhood of x_0 contained in U. Then $fQ_1 = fQ_2$ on M. Applying property 4. with $Q = f \in \mathfrak{X}^0 M$ we get
$$[P, fR] = [P, f] \wedge R - f[P, R] = (L_P f)R - f[P, R].$$
Now show that
$$[P, fQ_1](x_0) = [P, Q_1](x_0)$$
$$[P, fQ_2](x_0) = [P, Q_2](x_0)$$
The crucial property of locality allows us to work in a coordinate chart. P and Q can therefore be taken as finite sums of exterior products of vector fields. Remark that from property 4. we get
$$[X, Q_1 \wedge \cdots \wedge Q_n] = \sum_{i=1}^n (-1)^* Q_1 \wedge \cdots \wedge [X, Q_i] \wedge \cdots \wedge Q_n,$$
$$[P_1 \wedge \cdots \wedge P_n, Q_1 \wedge \cdots \wedge Q_m] =$$
$$= \sum_{i<j} (-1)^* [P_i, Q_j] \wedge P_1 \wedge \cdots \wedge \widehat{P_i} \wedge \cdots \wedge P_n \wedge Q_1 \wedge \cdots \wedge \widehat{Q_j} \wedge \cdots \wedge Q_m.$$
Now
$$[P_1 \wedge \cdots \wedge P_n, fQ_1 \wedge \cdots \wedge Q_m] =$$
$$= \underbrace{[P_1 \wedge \cdots \wedge P_n, f]}_{=(-1)^n [f, P_1 \wedge \cdots \wedge P_n]} \wedge Q_1 \wedge \cdots \wedge Q_m + (-1)^m f [P_1 \wedge \cdots \wedge P_n, Q_1 \wedge \cdots \wedge Q_m]$$
and
$$(-1)^n [f, P_1 \wedge \cdots \wedge P_n] = (-1)^n \sum_{i=1}^n L_{P_i}(f) P_1 \wedge \cdots \wedge \widehat{P_i} \wedge \cdots \wedge P_n.$$

These fixes all values and thus proves unicity. Finally one has to prove the independence of local coordinates - on the overlapping coordinate domains you have the same result. This implies existence. All other properties, in particular the graded Jacobi identity, are proved by direct (lengthy) computations. □

Definition 2.9. *A* Gerstenhaber algebra *is a triple* $(\mathfrak{A}, \wedge, [-,-])$ *such that*

1. $\mathfrak{A}$ *is a* $\mathbb{N}$*-graded vector space,* $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \ldots$*;*
2. $\wedge$ *is an associative, supercommutative multiplication of degree 0 on* $\mathfrak{A}$ *(i.e.* $\mathfrak{A}_i \wedge \mathfrak{A}_j \subset \mathfrak{A}_{i+j}$*);*
3. $[-,-]$ *is a super Lie algebra structure of degree* (-1) *on* $\mathfrak{A}$ *(i.e.* $[\mathfrak{A}_i, \mathfrak{A}_j] \subset \mathfrak{A}_{i+j-1}$*) satisfying*
$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c].$$

Examples 2.10.

- Multivector fields on a manifold M are a Gerstenhaber algebra with respect to Schouten-Nijenhuis bracket.
- Differential forms on Poisson manifold are a Gerstenhaber algebra (simply by a natural graded extension of the Lie bracket on 1–forms defined in 2.1, see also [bv88] for more on the subject).
- From any Lie algebra $\mathfrak{g}$ it is easy to construct a Gerstenhaber algebra $\Lambda^\bullet \mathfrak{g}$ following the construction of Porposition 2.6.
- Similarly from any Lie algebroid E there is a natural construction of Gerstenhaber algebra on $\Gamma(\Lambda^\bullet E)$ generalizing the costruction of the Schouten-Nijenhuis bracket (just remark that the proof of proposition 2.8 uses exactly the fact that TM is a Lie algebroid with anchor $\rho = \mathrm{id}$).
- Hochschild cohomology has a Gerstenhaber algebra structure (given by insertion) (coefficients in the given algebra). Hochschild-Kostant-Rosenberg map
$$\phi_{HKR} \colon \mathrm{HH}^\bullet_{\mathrm{cont}}(C^\infty(M)) \to \mathfrak{X}^\bullet(M)$$
fails to be a Gerstenhaber algebra morphism. This is what leads to L_∞-algebra structures and Kontsevich formality.

Let $(\mathfrak{A}, \wedge, [-,-])$ be a Gerstenhaber algebra. An operator $D \colon \mathfrak{A}^\bullet \to \mathfrak{A}^{\bullet-1}$ is said to generate the Gerstenhaber algebra if for all $a \in \mathfrak{A}^i$, $b \in \mathfrak{A}$

$$[a, b] = (-1)^i (D(a \wedge b) - Da \wedge b - (-1)^i a \wedge Db).$$

If $D^2 = 0$ we say that our Gerstenhaber algebra is *exact* or *Batalin-Vilkovisky algebra*.

We will show that the Gerstenhaber algebra of differential forms on a Poisson manifold is a Batalin-Vilkovisky algebra. Its generating operator will be called *Poisson* (or *canonical* or *Brylinski*) *differential*.

2.2.1 Schouten-Nijenhuis bracket computations

The following section is a brief description of the interesting approach to Schouten-Nijenhuis bracket that appeared recently in [dzxx]. Fix a system of local coordinates and consider two vector fields

$$X = \sum_i a_i \partial_{x_i}, \quad Y = \sum_i b_i \partial_{x_i}, \quad X, Y \in \mathfrak{X}(M), \quad x_1, \ldots, x_n \text{ coordinates}$$

$$[X, Y] = \sum_i a_i \left(\sum_j \frac{\partial b_j}{\partial x_i} \partial_{x_j} \right) - \sum_i b_i \left(\sum_j \frac{\partial a_j}{\partial x_i} \partial_{x_j} \right).$$

Let $\zeta_i = \partial_{x_i}$ and consider it as an odd formal variable

$$\zeta_i \zeta_j = -\zeta_j \zeta_i \quad (\partial_{x_i} \wedge \partial_{x_j} = -\partial_{x_j} \wedge \partial_{x_i}).$$

Then

$$X := \sum_i a_i \zeta_i, \quad Y := \sum_i b_i \zeta_i$$

$$\begin{aligned} [X, Y] &= \sum_i \left(\frac{\partial X}{\partial \zeta_i} \frac{\partial Y}{\partial x_i} - \frac{\partial Y}{\partial \zeta_i} \frac{\partial X}{\partial x_i} \right) \\ &= \left(\sum_i \partial_{\zeta_i} \wedge \partial_{x_i} \right) (X \otimes Y). \end{aligned}$$

Extend this idea to multivector fields

$$P \in \mathfrak{X}^p(M), \quad P = \sum_{i_1 < \ldots < i_p} \partial_{x_{i_1}} \wedge \ldots \wedge \partial_{x_{i_p}} = \sum_{i_1 < \ldots < i_p} P_{i_i \ldots i_p} \zeta_{i_1} \ldots \zeta_{i_p}.$$

Fix the following differentiation rule

$$\begin{aligned} &\partial_{\zeta_{i_p}} (\zeta_{i_1} \ldots \zeta_{i_p}) = \zeta_{i_1} \ldots \zeta_{i_{p-1}} \\ &\partial_{\zeta_{i_k}} (\zeta_{i_1} \ldots \zeta_{i_p}) = (-1)^{p-k} \zeta_{i_1} \ldots \widehat{\zeta_{i_k}} \ldots \zeta_{i_{p-1}}. \end{aligned}$$

Then we claim that

$$[P, Q]_{SN} = \sum_i \partial_{\zeta_i} P \, \partial_{x_i} Q - (-1)^{(p-1)(q-1)} \partial_{\zeta_i} Q \, \partial_{x_i} P.$$

2.2.2 Lichnerowicz formula

Lichnerowicz defined the Schouten-Nijenhuis bracket implicitly through the following formula.

Proposition 2.11. *For all $P \in \mathfrak{X}^p M$, $Q \in \mathfrak{X}^q M$, $\omega \in \Omega^{p+q-1} M$*

$$\langle \omega, [P, Q] \rangle = (-1)^{(p-1)(q-1)} \langle d(i_Q \omega), P \rangle - \langle d(i_P \omega), Q \rangle + (-1)^p \langle d\omega, P \wedge Q \rangle \quad (2.3)$$

With respect to our explicit construction this formula has the advantage of being well adapted and easy to use in "global type" computations.

Look at what happens, for example, when $X, Y \in \mathfrak{X}^1(M)$, $\omega \in \Omega^1 M$

$$\langle \omega,\, [X,Y]\rangle = \langle d(i_Y\omega),\, X\rangle - \langle d(i_X\omega),\, Y\rangle - \langle d\omega,\, X\wedge Y\rangle$$

which can be rewritten as

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

which gives the formula for the differential of a 1–form.

In our approach Lichnerowicz formula needs a proof. The easiest track here is to show that the bracket implicitely defined by (2.3) has the same algebraic properties as the Schouten-Nijenhuis bracket. The unicity statement of theorem 2.8 then implies the claim.

Exercise 2.12. *Let us prove property 2. for the Lichnerowicz bracket (we will denote it by* $[,]_L$*. We need to show that*

$$[X,Q]_L = L_X Q \qquad \forall X\in\mathfrak{X}^1 M,\ \forall Q\in\mathfrak{X}^q M\,.$$

Now, let $\omega\in\Omega^q M$*:*

$$\langle \omega,\, [X,Q]\rangle = \langle d(i_Q\omega),\, X\rangle - \langle d(i_X\omega),\, Q\rangle - \langle d\omega,\, X\wedge Q\rangle$$

On the other hand:

$$\begin{aligned}\langle \omega,\, L_X Q\rangle =& L_X\langle \omega,\, Q\rangle - \langle L_X\omega,\, Q\rangle\\ =&(i_X d + d i_X) i_Q\omega - \langle d i_X\omega + i_X d\omega,\, Q\rangle\\ =&\langle d(i_Q\omega),\, X\rangle + d(i_X i_Q\omega) - \langle d(i_X\omega),\, Q\rangle - \langle i_X d\omega,\, Q\rangle\end{aligned}$$

Now the second summand is zero for dimension reasons ($i_X i_Q\omega\in\mathfrak{X}^{-1}M$*) and the last summand is equal to* $-\langle d\omega,\, X\wedge Q\rangle$ *by definition of the contraction operator.*

2.2.3 Jacobi condition and Schouten-Nijenhuis bracket

The SN bracket allows to express the Jacobi condition for a Poisson bivector in short form. Let Π be a bivector on M, so that $[\Pi,\Pi]\in\mathfrak{X}^3 M$. Let ω be a 3-form on M. Then $\langle\omega,\, [\Pi,\Pi]\rangle$ is a function and, by use of (2.3)

$$\langle \omega,\, [\Pi,\Pi]\rangle = -\langle d(i_\Pi\omega),\, \Pi\rangle - \langle d(i_\Pi\omega),\, \Pi\rangle + \langle d\omega,\, \Pi\wedge\Pi\rangle.$$

Let $\omega = df\wedge dg\wedge dh$. As usual let $\{f,g\} = \langle df\wedge dg,\, \Pi\rangle$. Remark that

$$\langle i_\Pi\omega,\, X\rangle = \langle df\wedge dg\wedge dh,\, \Pi\wedge X\rangle$$

for all $X\in\mathfrak{X}^1 M$. Since $d(df\wedge dg\wedge dh) = 0$ we have

$$\langle \omega,\, [\Pi,\Pi]\rangle = -2\langle d(i_\Pi\omega),\, \Pi\rangle.$$

Lemma 2.13.

$$\langle df\wedge dg\wedge dh,\, [\Pi,\Pi]\rangle = 2\,\mathrm{Jac}(f,g,h)$$

Proof.

$$\begin{aligned}\langle df\wedge dg\wedge dh,\,[\Pi,\Pi]\rangle &= -2\langle d(i_\Pi(df\wedge dg\wedge dh)),\,\Pi\rangle = \\ &= -2\langle d(\{g,\,h\}df-\{f,\,h\}dg+\{f,\,g\}dh),\,\Pi\rangle = \\ &= -2\langle d\{g,h\}\wedge df-d\{f,\,h\}\wedge dg+d\{f,\,g\}\wedge dh,\,\Pi\rangle = \\ &= -2(\{\{g,\,h\},\,f\}-\{\{f,\,h\},\,g\}+\{\{f,\,g\},\,h\}) = \\ &= 2\,\mathrm{Jac}(f,g,h).\end{aligned}$$

□

Corollary 2.14. *A bivector $\Pi\in\mathfrak{X}^2M$ is Poisson if and only if $[\Pi,\Pi]=0$.*

The first easy consequence of the characterization $[\Pi,\Pi]=0$ is that if $\dim M=2$, then any bivector $\Pi\in\mathfrak{X}^2M$ is Poisson. Indeed $[\Pi,\Pi]\in\mathfrak{X}^3M=0$.

Let us give another application of corollary (2.14); consider a Lie algebra $\mathfrak{g}$, a manifold M and an infinitesimal action of $\mathfrak{g}$ on M i.e. a Lie algebra homomorphism$\xi\colon\mathfrak{g}\to\mathfrak{X}(M)$. Then ξ extends uniquely to a degree 0 map on higher exterior powers:

$$\begin{aligned}\wedge\xi\colon\Lambda^\bullet\mathfrak{g}&\to\mathfrak{X}^\bullet(M)\\ x_1\wedge\cdots\wedge x_n&\mapsto\xi(x_1)\wedge\cdots\wedge\xi(x_n)\,.\end{aligned}$$

Such map preserves the graded brackets:

$$\wedge\xi[\alpha,\beta]=[\wedge\xi(\alpha),\wedge\xi(\beta)]_{SN}\,.$$

This last statement is a simple consequence of the fact that both brackets are determined by their values in degree 0 and 1.

Let now G be a connected Lie group such that $\mathrm{Lie}(G)=\mathfrak{g}$ (not necessarily simply connected), with identity element e. Let us denote the translation operators by

$$\begin{aligned}l_g\colon G\to G,&\quad h\mapsto gh,\\ r_g\colon G\to G,&\quad h\mapsto hg.\end{aligned}$$

We will use the following notations for the corresponding tangent maps

$$\begin{aligned}&l_{g,*}\colon T_hG\to T_{gh}G &\qquad& r_{g,*}\colon T_hG\to T_{hg}G\\ &l^\wedge_{g,*}\colon\Lambda^\bullet T_eG\to\Lambda^\bullet T_gG && r^\wedge_{g,*}\colon\Lambda^\bullet T_eG\to\Lambda^\bullet T_gG\end{aligned}$$

Let now $\alpha\in\Lambda^\bullet\mathfrak{g}$. We will denote with α^L (resp. α^R) the left (resp. right) invariant multivector field on G whose value at $e\in G$ (identity of G) is α i.e.

$$\alpha^L(g):=l^\wedge_{g,*}\alpha,\quad(\text{resp. }\alpha^R(g):=r^\wedge_{g,*}\alpha)$$

In the same way we consider α^R to be the right invariant multivector field on G whose value at e is α.

Proposition 2.15. *For any $\gamma\in\Lambda^2\mathfrak{g}$ the following are equivalent*

1. *γ^L is a left invariant Poisson structure.*
2. *γ^R is a right invariant Poisson structure.*

3. $[\gamma, \gamma] = 0$ *(bracket in $\Lambda^\bullet \mathfrak{g}$)*

Proof. The map $l_{g,*}\colon \mathfrak{g} \to \mathfrak{X}(G)$ is an infinitesimal action of $\mathfrak{g}$ on G. Therefore it preserves brackets

$$[\gamma^L, \gamma^L]_{SN}(g) = [L^\wedge_{*,g}\gamma, L^\wedge_{*,g}\gamma](g) = L^\wedge_{*,g}[\gamma, \gamma],$$

so the left hand side is zero if and only if the right hand side is zero. But the vanishing of the left hand side $[\gamma^L, \gamma^L]_{SN} = 0$ is exactly the condition for γ^L to be Poisson, which is therefore equivalent to condition 3 in the proposition. The computation for right invariant bivectors is exactly the same. □

The condition $[\gamma, \gamma] = 0$ is called *classical Yang-Baxter equation*. The above proposition can be therefore stated as follows:

Corollary 2.16. *There is a one to one correspondence between left (resp. right) invariant Poisson structures on a Lie group G and solutions of the classical Yang-Baxter equation on* $\mathrm{Lie}(G)$.

As last application of corollary 2.14 let us discuss the problem of compatible pairs. Given two Poisson bivectors Π_1, Π_2 on M it is rather reasonable to ask under which conditions is $\Pi_1 + \Pi_2$ a Poisson bivector on M,
If this is the case we will say that they are *compatible Poisson tensors*.

Proposition 2.17. Π_1 *and* Π_2 *are compatible if and only if* $[\Pi_1, \Pi_2] = 0$.

In this case $a\Pi_1 + b\Pi_2$ is Poisson for all $a, b \in \mathbb{R}$ and $\{a\Pi_1 + b\Pi_2 : a, b \in \mathbb{R}\}$ is called a *Poisson pencil*.

Proof.

$$[\Pi_1 + \Pi_2, \Pi_1 + \Pi_2] = \underbrace{[\Pi_1, \Pi_1]}_{=0} + \underbrace{[\Pi_2, \Pi_2]}_{=0} + 2[\Pi_1, \Pi_2],$$

so $[\Pi_1 + \Pi_2, \Pi_1 + \Pi_2] = 0$ if and only if $[\Pi_1, \Pi_2] = 0$.
Furthermore

$$[a\Pi_1 + b\Pi_2, a\Pi_1 + b\Pi_2] = 2ab[\Pi_1, \Pi_2].$$

□

Poisson pencils appear quite frequently in the theory of integrable systems (under the name of bihamiltonian formalism). Such pencils also naturally arise in some specific families of Poisson homogeneous spaces of Poisson–Lie groups to be discussed in later chapters.

2.2.4 Koszul's formula

Theorem 2.18 (Koszul formula)**.** *Let $P \in \mathfrak{X}^p M$, $Q \in \mathfrak{X}^q M$. Then*

$$i_{[P,Q]} = (-1)^{(p-1)(q-1)} i_P d i_Q - i_Q d i_P + (-1)^p i_{P\wedge Q} d + (-1)^q d i_{P\wedge Q} \tag{2.4}$$

Remark 2.19. Koszul formula implies Lichnerowicz formula (2.3) after contracting with $(p+q-1)$-form.

Exercise 2.20. *(Proof of Koszul formula) Try to prove the Koszul formula alonf the following lines. 1. Use induction on* $\deg P$ *(start with* $\deg P = 0$*, i.e. P is a function). 2. Use the Leibniz rule to increase the degree of P.*

This formula can be also memorized as

$$i_{[P,Q]} = [[i_P, d], i_Q]$$

but with graded commutators on the right !

2.3 Poisson homology

Definition 2.21. Canonical *(or* Brylinski*)* operator

$$\partial_\Pi := i_\Pi d - d i_\Pi \colon \Omega^k M \to \Omega^{k+1} M$$

Proposition 2.22. *The following identities are verified*

1. $d\partial_\Pi + \partial_\Pi d = 0$.
2. $\partial_\Pi i_\Pi - i_\Pi \partial_\Pi = 0$.
3. $\partial_\Pi^2 = 0$.

Proof.

1. $d\partial_\Pi = d i_\Pi d = -\partial_\Pi d$.
2. Apply Koszul's formula to see that
$$0 = i_{[\Pi,\Pi]} = [[i_\Pi, d], i_\Pi] = [\partial_\Pi, i_\Pi].$$
3. $\partial_\Pi i_\Pi = i_\Pi \partial_\Pi$ as a consequence of Koszul formula. Thus
$$2 i_\Pi d i_\Pi = i_\Pi^2 d - d i_\Pi^2.$$
Apply d on the left
$$2 d i_\Pi d i_\Pi = d i_\Pi^2 d.$$
Apply d on the right
$$2 i_\Pi d i_\Pi d = d i_\Pi^2 d.$$
Therefore $d i_\Pi d i_\Pi = -i_\Pi d i_\Pi d$ and
$$\begin{aligned}\partial_\Pi^2 &= (i_\Pi d - d i_\Pi)(i_\Pi d - d i_\Pi)\\ &= i_\Pi d i_\Pi d - d i_\Pi i_\Pi d + d i_\Pi d i_\Pi\\ &= 2 i_\Pi d i_\Pi d - d i_\Pi i_\Pi d = 0.\end{aligned}$$

□

Definition 2.23. *The homology of the complex* $(\Omega^\bullet, \partial_\Pi)$ *is called* Poisson *(or* canonical*)* homology *and it is denoted by* $\mathrm{H}_k^\Pi(M)$.

Poisson homology was first defined by Brylinski ([b-j98], but see also [m-o95, p-g00, **?**] for additional material on the subject). The first property of the previous proposition also tells us that d and ∂_Π together form a mixed complex (mixed here refers to the fact that they have opposite degrees) and thus define a *cyclic* homology theory ([k-cxx]). The corresponding homology of the total complex will be called *cyclic Poisson homology.*

Chapter 3

Poisson maps

3.1 Poisson maps

We already mentioned that if (M_1, Π_1), (M_2, Π_2) are two Poisson manifolds then a map $\varphi\colon M_1 \to M_2$ is a Poisson map if the pull-back map $\varphi^*\colon C^\infty(M_2) \to C^\infty(M_1); \varphi^*\colon f \mapsto f \circ \varphi$ verifies

$$\varphi^*\{f,\, g\}_{M_2} = \{\varphi^* f,\, \varphi^* g\}_{M_1}$$

for all $f, g \in C^\infty(M_2)$.

Recall also from differential geometry that having a map $\varphi\colon M \to N$, you can pull-back forms, but in general you cannot push-forward vector fields. As a substitute of the notion of push-forward, there is the *relatedness*. Let us explain what we mean by this: if $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are two vector fields then they are said to be *φ-related* if

$$\varphi_{*,x}(X_x) = Y_{\varphi(x)}\,, \qquad \forall x \in M\,.$$

Here $\varphi_{*,x}\colon T_xM \to T_{\varphi(x)}N$ is the tangent map. This is indeed a relation and not a map. In fact there may be more than one vector field on N related to a given vector field on M. (Think of $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x, y) = (x, 0)$. Then saying that Y is φ-related to X says something only about values of Y along the line $(x, 0)$.) It may also happen that there are none (In the example as before if $X_{(x,0)}$ and $X_{(x,t)}$ have different projections on $\operatorname{im}\varphi_{*,x}$ for some t). Of course this relation can be easily extended to a relation on multivectors, simply by considering $\varphi^{\wedge}_{*,x}$.

If φ is a diffeomorphism, however, then none of the above mentioned behaviours arise and indeed there is a well defined map $\varphi_*\colon \mathfrak{X}(M) \to \mathfrak{X}(N)$. Remark that you can define φ-relation on multivectors

Proposition 3.1. *Let (M_1, Π_1), (M_2, Π_2) be two Poisson manifolds and let $\varphi\colon M_1 \to M_2$ be a smooth map. The following are equivalent.*

1. *φ is Poisson map.*
2. *$X_{\varphi^* f} \in \operatorname{Ham}(M_1)$ and $X_f \in \operatorname{Ham}(M_2)$ are φ-related for all $f \in C^\infty(M_2)$ i.e.*

$$\varphi_{*,x}(X_{\varphi^* f}(x)) = X_f(\varphi(x))$$

for all $x \in M$.

3. *Let* ${}^t\varphi_{*,x}\colon T^*_{\varphi(x)}M_2 \to T^*_x M_1$ *be the cotangent map. Then the sharp map intertwines the tangent and cotangent map, i.e.*

$$\#_{\Pi_2,\varphi(x)} = \varphi_{*,x} \circ \#_{\Pi_1,x} \circ^t \varphi_{*,x}$$

4. *The bivectors* Π_1, Π_2 *are* φ*-related i.e.*

$$\langle \Pi_{2,\varphi(x)},\, \alpha \wedge \beta \rangle = \langle \Pi_{1,x},\, {}^t\varphi_{*,x}\alpha \wedge^t \varphi_{*,x}\beta \rangle$$

for all $\alpha, \beta \in T^*_{\varphi(x)}M_2$ *and all* $x \in M_1$.

Proof. (3) $\Longleftrightarrow$ (4) by definitions.
(1) $\Longleftrightarrow$ (4) using

$$\{f,\, g\}(x) = \langle \Pi_x,\, d_x f \wedge d_x g \rangle$$

and the fact that for all $\alpha \in T^*_{\varphi(x)}M_2$ there exists $f \in C^\infty(M_2)$ such that $d_{\varphi(x)} f = \alpha$.
(1) $\Longleftrightarrow$ (2) using

$$\{f,\, g\} = X_f g = \langle X_f,\, dg \rangle.$$

□

Remark 3.2. From property (3) of a Poisson map we can deduce the following relation between ranks:

$$\rho_{\Pi_1}(x) \geq \rho_{\Pi_2}(\varphi(x))$$

because $(\operatorname{im} \#_{\Pi_2})_{\varphi(x)} \subseteq \varphi_{*,x}(\operatorname{im} \#_{\Pi_1,x})$. This fact has remarkable, though easy consequences.

- Let $x_0 \in M_1$ be a 0-dimensional symplectic leaf. Then its image $\varphi(x_0)$ is again a 0-dimensional symplectic leaf. Thus for example there is no Poisson map $\varphi\colon \mathfrak{g}^* \to M$ if M is symplectic and $\mathfrak{g}$ is a Lie algebra.

- Let $\varphi\colon M_1 \to M_2$ be a Poisson immersion, i.e. a Poisson map such that $\varphi_{*,x}$ is injective. Then $\operatorname{rank}_{\Pi_1}(x) = \operatorname{rank}_{\Pi_2}(x)$. This in particular holds if φ is a Poisson (local) diffeomorphism (even more so for Poisson automorphisms, of course).

- Let $\varphi\colon M_1 \to M_2$ be a Poisson map between two symplectic manifolds. Then

$$\underbrace{\rho_{\Pi_1}(x)}_{\dim M_1} \geq \underbrace{\rho_{\Pi_2}(\varphi(x))}_{\dim M_2}$$

  and φ has to be a submersion i.e. $\varphi_{*,x}$ surjective, because $\operatorname{im} \varphi_{*,x}$ is forced to be $T_{\varphi(x)}M_2$ for all $x \in M_1$. So the only Poisson maps between symplectic manifolds are submersions.

This last remark shows that being a Poisson map between symplectic manifolds is very different from being a symplectic map (which means $\varphi\colon M_1 \to M_2$, $\varphi^*\omega_2 = \omega_1$). This difference is made explicit by the following two examples.

Example 3.3. Let $\mathbb{R}^{2n}$ be considered with the standard symplectic structure and consider the map

$$\begin{aligned} i\colon \mathbb{R}^2 &\to \mathbb{R}^4, \\ (q_1, p_1) &\mapsto (q_1, p_1, 0, 0), \\ \omega_1 &= dq_1 \wedge dp_1, \\ \omega_2 &= dq_1 \wedge dp_1 + dq_2 \wedge dp_2. \end{aligned}$$

Then i is a symplectic map but it is not a Poisson map:

$$\underbrace{\{q_2,\, p_2\} \circ i}_{=1} \neq \underbrace{\{q_2 \circ i,\, p_2 \circ i\}}_{=0}.$$

Example 3.4. In the same setting as before consider the map

$$\begin{aligned} \psi\colon \mathbb{R}^4 &\to \mathbb{R}^2, \\ (q_1, p_1, q_2, p_2) &\mapsto (q_1, p_1), \\ \omega_1 &= dq_1 \wedge dp_1 + dq_2 \wedge dp_2, \\ \omega_2 &= dq_1 \wedge dp_1. \end{aligned}$$

Then ψ is a Poisson map:, e.g.

$$\underbrace{\{q_2,\, p_2\} \circ \psi}_{=1} = \underbrace{\{q_2 \circ \psi,\, p_2 \circ \psi\}}_{=1}$$

but it is not symplectic:

$$\psi^*(dq_1 \wedge dp_1) = dq_1 \wedge dp_1 \neq dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

This difference between morphisms in the Poisson and symplectic categories implies, obviously, that related concepts such as subobjects and quotients have different behaviours. We will see later an example of this issue when referring to submanifolds.

Proposition 3.5. *Let (M_i, Π_i), $i = 1, 2, 3$ be Poisson manifolds. Let $\varphi\colon M_1 \to M_2$ and $\psi\colon M_2 \to M_3$ be smooth maps.*

1. *If φ and ψ are Poisson, then $\psi \circ \varphi$ is Poisson.*
2. *If φ and $\psi \circ \varphi$ are Poisson, and φ is surjective, then ψ is Poisson.*
3. *If φ is Poisson and a diffeomorphism, then φ^{-1} is Poisson.*

Proof.

1. Obvious.
2. Take $y \in M_2$, $y = \varphi(x)$.

$$\begin{aligned} \#_{\Pi_3, \psi(y)} &= (\psi \circ \varphi)_{*,x} \circ \#_{\Pi_{1,x}} \circ^t (\psi \circ \varphi)_{*,x} \\ &= \psi_{*,y} \circ \varphi_{*,x} \circ \#_{\Pi_{1,x}} \circ^t \varphi_{*,x} \circ^t \psi_{*,y} \\ &= \psi_{*,y} \circ \#_{\Pi_{2,x}} \circ^t \psi_{*,y}. \end{aligned}$$

3. Follows immediately from (2).

□

Examples 3.6.

1. Let $\phi\colon \mathfrak{h} \to \mathfrak{g}$ be a Lie algebra morphism. Prove that $\phi^*\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map. Is the converse true?

2. Any Poisson map from M to a connected symplectic manifold S, $\varphi\colon M \to S$ is a submersion.

   Proof.
   $$\varphi(T_xM) \subseteq T_xS$$
   It is a submersion if and only if equality holds. Say there is no equality.
   $$\underbrace{\varphi_{*,x}(\Pi_M(x))}_{=\Pi_S(x)} \subseteq \varphi_{*,x}\Lambda^2 T_xM.$$
   But then choose $\xi \in T_x^*S$ such that $\xi \in \varphi_{*,x}(T_xM)^\perp$, $\xi \neq 0$. Then $\langle \xi, \Pi_S(x)\rangle = 0$ contradicting nondegeneracy of Π_S. □

3. Any Poisson map $\varphi\colon M \to N$ such that M is symplectic is called a *symplectic realization* of (N, Π_N). It can be proven that any Poisson manifold admits a surjective symplectic realization. Note that surjectivity of φ implies that functions in $C^\infty(N)$ are faithfully represented as vector fields on M by $f \mapsto X_{\varphi^* f}$.

4. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. Consider T^*G with the standard symplectic structure. Let $L\colon T^*G \to \mathfrak{g}^*$ be defined by $(g, p) \mapsto (L_g)^*p$ for $L_g^*\colon T_g^*G \to T_e^*G = \mathfrak{g}^*$. Then L is always symplectic realization.

5. Let $\mathfrak{g}$ be a Lie algebra and endow $\mathfrak{g}^*$ with its linear Poisson structure. Any Poisson map $\mu\colon M \to \mathfrak{g}^*$ is called a *moment map*. In fact the existence of such a map implies thatthe composition
   $$\mathfrak{g} \hookrightarrow C^\infty(\mathfrak{g}^*) \xrightarrow{\mu^*} C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M)$$
   is a Lie algebra homomorphism. Therefore M carries an infinitesimal $\mathfrak{g}$-action by hamiltonian vector fields; μ is the moment map for this action (see [dzxx, lm87, v-i94] for more on Hamiltonian actions).

Definition 3.7. *Let (M, Π) be a Poisson manifold. A* Poisson vector field *$X \in \mathfrak{X}(M)$ (or* infinitesimal Poisson automorphism*) is a vector field such that its flow φ induces for all $t \in \mathbb{R}$ a local Poisson morphism $\varphi_t\colon M \to M$.*

We will denote the set of Poisson vector fields on a Poisson manifold (M, π) as $\mathfrak{X}_\pi(M)$.

Proposition 3.8. *Let (M, Π) be a Poisson manifold, and $X \in \mathfrak{X}(M)$. The following are equivalent:*

1. *X is a Poisson vector field*

2. *X is a canonical derivation i.e.*

$$X\{f,g\} = \{Xf,g\} + \{f,Xg\}$$

3. *X preserves the Poisson bivector, i.e. $L_X\Pi = 0$.*

Proof. We have

$$\begin{aligned} L_X(\Pi(df,dg)) &= (L_X\Pi)(df,dg) + \Pi(L_X df,dg) + \Pi(df,L_X dg) \\ &= (L_X\Pi)(df,dg) + \Pi(dL_X f,dg) + \Pi(df,dL_X g), \end{aligned}$$

because $L_X d = dL_X$. Now rewriting this with brackets we get

$$X\{f,g\} - \{Xf,g\} - \{f,Xg\} = (L_X\Pi)(df,dg).$$

so 2) $\iff$ 3).

Let $X \in \mathfrak{X}(M)$, and let φ_t be its flow. Let $f,g \in C^\infty(M,\mathbb{R})$.

$$\begin{aligned} &\underbrace{\frac{d}{dt}\varphi^*_{-t}\{\varphi^*_t f, \varphi^*_t g\}|_{t=t_0}}_{\frac{d}{dt}\{f\circ\varphi_t, g\circ\varphi_t\}\circ\varphi_{-t}} = \\ &= -\varphi^*_{-t_0}(X\{\varphi^*_{t_0}f,\varphi^*_{t_0}g\}) + \varphi^*_{-t_0}(\{X\varphi^*_{t_0}f,\varphi^*_{t_0}g\}) + \varphi^*_{-t_0}(\{\varphi^*_{t_0}f,X\varphi^*_{t_0}g\}) \end{aligned}$$

Now 1) $\iff$ left hand side is 0, and 2) $\iff$ right hand side is 0. ☐

Remark 3.9.

1. From $L_{[X,Y]} = [L_X, L_Y]$ it immediately follows that the bracket of Poisson vector fields is a Poisson vector field.
2. Let X be a Poisson vector field. Then the rank $\rho_\Pi(x)$ is constant along the flow of X, but the flow of X need not be contained in a single leaf. This, as a flow property, is the difference between Poisson and Hamiltonian vector fields (which preserve leaves by definition).
3. Any Hamiltonian vector field is a Poisson vector field, but the opposite is false. For example on $(M,\Pi=0)$ every vector field is Poisson ($L_X 0 = 0$), but only the 0 vector field is hamiltonian (it has to stabilize every point!). For $(\mathbb{R}^{2n}, \mathrm{std})$ we have $\mathrm{Ham}(M) == \mathfrak{X}_\pi(M) = \mathfrak{X}(M)$.

This difference between Poisson and Hamiltonian vector fields has to be thought as analogous to the difference between algebra automorphisms and inner automorphisms (we will see in the last chapter some setting in which this analogy turns into a theorem). As in this last case suche difference is measured by a suitable cohomology, first introduced in [].

Definition 3.10. *Let $d_\Pi\colon \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)$, $\Pi \mapsto [\Pi, P]$. Then d_Π is called the* Lichnerowicz coboundary.

Remark 3.11. If Π is Poisson, then $d_\Pi^2 = 0$. In fact $[\Pi,[\Pi,P]] = \frac{1}{2}[[\Pi,\Pi],P]$ from the Jacobi identity of the Schouten bracket.

Definition 3.12. *The cohomology of the complex $(\mathfrak{X}^\bullet(M), d_\Pi)$ is called* Poisson (Lichnerowicz) cohomology *of (M,Π) and is denoted by $\mathrm{H}^k_\Pi(M)$.*

We have $\mathrm{H}^0_\Pi(M) = \mathrm{Cas}(M)$, $[\Pi, f] = X_f$, and $\mathrm{H}^1_\Pi(M) = \mathbf{Poiss}(M)/\,\mathrm{Ham}(M)$.

3.2 Poisson submanifolds

As we just said a map $\varphi\colon M_1 \to M_2$ between two Poisson manifolds (M_1, Π_1), (M_2, Π_2) is Poisson if and only if $\varphi^{\wedge 2}_{*,x}(\Pi_1(x)) = \Pi_2(\varphi(x))$ for all $x \in M_1$.

Recall that a submanifold of M can be described as a pair (N, i) where N is a manifold and $i\colon N \hookrightarrow M$ is an injective immersion.

Definition 3.13. *Let (M, Π_M) be a Poisson manifold. Then (N, i) is is a* Poisson submanifold *if N has a Poisson structure Π_N such that i is a Poisson map.*

Remark 3.14. If i is an immersion, then $i^{\wedge 2}_{*,x}$ is injective at every x and

$$i^{\wedge 2}_{*,x}(\Pi_N(x)) = \Pi_M(i(x))$$

uniquely determines Π_N to be Poisson diffeomorphic to the restriction of Π_M to $i(N)$. The symplectic leaves of a Poisson manifold are a natural example of Poisson submanifolds.

Proposition 3.15. *Every open subset U of (M, Π_M) is an open Poisson submanifold. A closed submanifold N of (M, Π_M) is Poisson if and only if it is a union of symplectic leaves.*

Proof. From Π being Poisson we have $\Pi|_U$ is Poisson for all open $U \subseteq M$.

Let (N, i) be a closed submanifold. The question is whether Π_N is a Poisson bivector on N. This is true if and only if Π_N is tangent to N at any of its points, which locally, around x, means exactly that the leaf through N is contained in N. Now apply the usual open-closed argument. □

What this last proposition tells us is that the property of being a Poisson submanifold is indeed a very restictive one. As an example consider the Natsume–Olsen sphere introduced in example **??**. There the only non trivial closed submanifolds are $\{N\}$, $\{S\}$ and $\{N, S\}$. Try, as an exercise, to list closed Poisson submanifolds in the examples mentioned in the first chapter.

Example 3.16. When M is a symplectic manifold the only Poisson submanifolds are open subsets (and there are no nontrivial closed Poisson submanifolds). This is in contrast with what happens for symplectic submanifolds (think again at the case $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$). To relax this rigidity the notion of Poisson–Dirac submanifold of a Poisson manifold was recently introduced (see [cf04, x-p03]).

An interesting way to construct a Poisson manifold with prescribed Poisson submanifolds is that of gluing together some symplectic structures on given symplectic leaves. The following theorem (see [v-i94], page 26, for its proof) gives a characterization for such construction. Let us remark that in general using topological constructions (like gluing, surgery, etc.) in the differential geometrical setting of Poisson manifold is, at the same time, an interesting and difficult procedure, related to what is called flexibility of the geometrical structure. A construction of suspension of Poisson structures on spheres was realized in [bct03]. For other constructions and some general consideration see [im03, cf05].

Proposition 3.17. *Let M be a differentiable manifold and let $\mathcal{F}$ be a generalized foliation on M such that every leaf $F \in \mathcal{F}$ is endowed with a symplectic* *structure ω_F. For any $f \in C^\infty(M)$ define $X_f(x) = \#_{\omega_F^{-1}}(d_x f)$. If all X_f's are differentiable vector fields then there exists a unique Poisson structure on M having symplectic (F, ω_F) as symplectic leaves.*

Example 3.18. (The following example was raised in class discussions) Let $D = D(0,1)$ be the disc and let Π_1, Π_2 be Poisson structures on D going to zero at the boundary. Then we would like to construct a (unique) Poisson structure Π on $\mathbb{S}^2$ such that

$$i_+ \colon D \to \mathbb{S}^2$$

sending D to the upper hemisphere is a Poisson submanifold (D, Π_1),

$$i_- \colon D \to \mathbb{S}^2$$

sending D to the lower hemisphere is a Poisson submanifold (D, Π_2),

$$i_0 \colon \mathbb{S}^1 \to \mathbb{S}^2$$

sending $\mathbb{S}^1$ to the equator is a Poisson submanifold $(\mathbb{S}^1, 0)$. We could of course consider also the higher dimensional analogue whith $\mathbb{S}^{2n}$ written as a gluing of two copies of the $2n$–dimensional disk along $\mathbb{S}^{2n-1}$.

There exists a uniquely determined bivector Π on $\mathbb{S}^2$ with such properties. The question is whether this bivector is Poisson and smooth. Due to dimension reasons here $[\Pi, \Pi] = 0$ is trivially verified (in a similar higher dimensional problem one could say the following: it is a local condition, it certainly holds true at any point of the lower and upper hemisphere therefrore if Π is smooth, $[\Pi, \Pi]$ is everywhere zero by continuity). So smoothness is the only real issue here.

If the Poisson structures on disks are both zero in an open neighbourhood of the boundary there is not much to be proved, everything is ok.

Let us limit ourselves to the case of rotation invariant Poisson structures on the disks. Being only interested in what happens around the boundary we can certainly use polar coordinates. Let $\Pi_1 = f(\rho)\partial_\rho \wedge \partial_\theta$ and $\Pi_2 = g(\rho)\partial_\rho \wedge \partial_\theta$. As an exercise write conditions on f and g under which such structures can be smoothly glued.

The reason for being interested in this kind of example is understanding the Poisson geometry underlying some quantum algebras recently studied (see [hms06]).

3.3 Coinduced Poisson structures

Let $\varphi \colon M_1 \to M_2$ be a surjective map. Then if we want it to be Poisson, then Π_2 is uniquely determined by Π_1.

Definition 3.19. *A surjective mapping from a Poisson manifold can be Poisson for at most one Poisson structure on M_2. If this is the case we will say that the Poisson structure on M_2 is* coinduced *via φ from that on M_1.*

Proposition 3.20. *Let (M_1, Π_1) be a Poisson manifold. If $\varphi \colon (M_1, \Pi_1) \to M_2$ is a surjective differentiable map, then M_2 has a coinduced Poisson structure if and only if*

$$\{\varphi^* f, \varphi^* g\}_{M_1}$$

is constant along the fibers of φ for all $f, g \in C^\infty(M)$.

Proof. If $\{\varphi^* f,\, \varphi^* g\}_{M_1}$ is constant then define

$$\{f,\, g\}_{M_2}(\varphi(x)) := \{\varphi^* f, \varphi^* g\}_{M_1}(x).$$

This is well defined, i.e. it does not depend on x but only on $\varphi(x)$ (and it is defined everywhere, because φ is surjective). That it is Poisson is an easy consequence of $\{-,\, -\}_{M_1}$ being Poisson (remark that here you are using again surjectivity of φ).

Conversely, let Π_2 exists. Then

$$\{\varphi^* f,\, \varphi^* g\}_{M_1}(\varphi^{-1}(y)) = \{f,\, g\}(y)$$

and the right hand side does not depend on φ, hence the left hand side is constant along fibers. □

Example 3.21. Let us apply the previous proposition to example 3.18 to give a Poisson structure on the projective plane. This has to be considered as the Poisson counterpart of the algebraic gluing-projecting procedure of [hms03]. Consider $\mathbb{S}^2 \xrightarrow{\varphi} \mathbb{R}P^2$ the projection being given by identification of antipodal points $(x, -x) \mapsto [x]$. There exists a coinduced Poisson bivector on $\mathbb{R}P^2$ if and only if for any given pair of functions on $\mathbb{R}P^2$

$$\{\varphi^* f,\, \varphi^* g\}_{D^2}(x) = \{\varphi^* f,\, \varphi^* g\}(-x).$$

So if we identify $C^\infty(\mathbb{R}P^2) \hookrightarrow C^\infty(\mathbb{S}^2)^{\mathbb{Z}_2}$, the previous equality states that maps $\widehat{f}, \widehat{g}$ belonging to $C^\infty(\mathbb{S}^2)^{\mathbb{Z}_2}$ have to satisfy

$$\langle \Pi(x),\, d_x\widehat{f} \wedge d_x\widehat{g}\rangle = \{\widehat{f},\, \widehat{g}\}(x) = \{\widehat{f},\, \widehat{g}\}(-x) = \langle \Pi(-x),\, d_x\widehat{f} \wedge d_x\widehat{g}\rangle.$$

for all $x \in \mathbb{S}^2$. Choosing functions giving you a basis of the cotangent space this implies

$$\Pi(x) = \Pi(-x).$$

In particular if Π on $\mathbb{S}^2$ is constructed by gluing this implies $\Pi_1(x) = \Pi_2(-x)$ where now $x \in D$. This does not say that for any surjective map $\varphi\colon \mathbb{S}^2 \to \mathbb{R}P^2$ you have the same condition.

Proposition 3.22. *Let (M_1, Π_1) be a Poisson manifold. Let $\varphi\colon M_1 \to M_2$ be a surjective submersion with connected fibers. If*

$$\ker \varphi_{*,x} \subseteq \#_{\Pi,x}(M_1)$$

is locally spanned by hamiltonian vector fields, then M_2 has coinduced Poisson structure.

Proof. Take $f, g \in C^\infty(M_2)$. We want to prove that $\{\varphi^* f,\, \varphi^* g\}_{M_1}$ is constant along the fibers. Because φ is a submersion fibers are submanifolds. Since $\ker \varphi_{*,x} \subseteq \#_{\Pi,x}(M_1)$ it is enough to prove that for all $\lambda \in C^\infty(M_1)$ if $\lambda \in \ker \varphi_*$ then $X_\lambda(\{\varphi^* f,\, \varphi^* g\}_{M_1}) = 0$ (because $\ker \varphi_*$ is the tangent space to the fibers). But this follows from Jacobi identity. In fact

$$X_\lambda\{\varphi^* f,\, \varphi^* g\}_{M_1} = \{\varphi^* f,\, X_\lambda(\varphi^* g)\}_{M_1} + \{X_\lambda(\varphi^* f),\, \varphi^* g\}_{M_1}.$$

But $\varphi^* f$ and $\varphi^* g$ are constant along the fibers (by definition) and therefore

$$X_\lambda(\varphi^* g) = X_\lambda(\varphi^* f) = 0,$$

from which

$$X_\lambda(\{\varphi^* f,\, \varphi^* g\}_{M_1}) = 0$$

hence thesis. □

3.4 Completeness

Let $\varphi\colon M \to N$ be a Poisson map and F a leaf in M. One could ask whether φ brings symplectic leaves of M into symplectic leaves of N. This is easily seen not to be the case. Let us take $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x, y) = x$ is Poisson with respect to the standard Poisson structure in $\mathbb{R}^2$ and zero structure in $\mathbb{R}$. But $\varphi(\mathbb{R}^2)$ is a union of leaves. From this example one could guess that in general $\phi(F)$ is a union of leaves. Even this turns out to be wrong, though for a subtler reason. Consider $U \subseteq \mathbb{R}^{2n}$ open set and $i\colon U \to \mathbb{R}^{2n}$ with the standard Poisson bivector Π on $\mathbb{R}^{2n}$ and $\Pi|_U$ on U. The image of the leaf U is not a whole leaf but just an open set in the leaf. Why is it so?

Consider now $\varphi(F)$ and take $\varphi(x) \in S$, where S is a leaf through $\varphi(x)$ in N. Take $y \in S$ and a piecewise Hamiltonian curve from y to $\varphi(x)$. We would like to lift this curve from N to M. Say the first Hamiltonian piece is the flow of X_h. Even if X_h is complete $X_{\varphi^* h}$ is not necessarily complete.

Definition 3.23. *A* complete Poisson map *is a Poisson map $\varphi\colon M \to N$ such that X_h complete implies $X_{\varphi^* h}$ complete.*

Then we immediately have

Proposition 3.24. *Let (M_1, Π_1) and (M_2, Π_2) be Poisson manifolds and $\varphi\colon M_1 \to M_2$ a complete Poisson map. Take F to be a leaf of M_1. Then $\varphi(F)$ is a union of symplectic leaves in M_2.*

Remark 3.25.

- Let M_1 be compact. Then any Poisson map $\varphi\colon M_1 \to M_2$ is complete.
- Let $\varphi\colon M_1 \to M_2$ be a proper Poisson map. Then it is complete.

Remark that also when we consider algebraic smooth Poisson varieties and algebraic maps between them, properness, in the algebraic sense, implies completeness. This is often used when dealing with algebraic Poisson groups.

Chapter 4

Poisson cohomology

Let us recall the definition of Poisson cohomology. Let (M, Π) be a Poisson manifold. Consider the cochain complex $(\mathfrak{X}^k(M), d_\Pi)$, where

$$d_\Pi \colon \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M), \quad P \mapsto [\Pi, P], \tag{4.1}$$

where $[-,-]$ is the Schouten bracket. Then $d_\Pi^2 = 0$ as a consequence of the graded Jacobi identity together with $[\Pi, \Pi] = 0$. Remark that the Poisson tensor itself always defines a 2–cocycle and, thus, a Poisson cohomology class. When $[\Pi] = 0$ the Poisson manifold is said to be *exact*. We would like now to give a different, more explicit expression for this coboundary operator.

Proposition 4.1. *In the above hypothesis, for all $P \in \mathfrak{X}^k(M)$ and for all $\alpha_i \in \Omega^k(M)$, $i = 0, \dots, k$*

$$(d_\Pi P)(\alpha_0, \dots, \alpha_k) = \sum_{i=0}^{k} (-1)^{i+1} \#_\Pi(\alpha_i) P(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_k) + \tag{4.2}$$

$$\sum_{0 \le i < j \le k} (-1)^{i+j-1} P([\alpha_i, \alpha_j], \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_k).$$

Proof. Let us first remark that the formula is true for $k = 0, 1$

$$\begin{aligned} k = 0 \qquad & (d_\Pi f)(\alpha) = \#_\Pi(\alpha) f, \\ k = 1 \qquad & (d_\Pi X)(df, dg) = X\{f, g\} + \{g, Xf\} - \{f, Xg\}. \end{aligned}$$

Let now P be a decomposable k-vector and prove (4.2) by induction on k. Due to the graded Leibniz identity for the Schouten bracket

$$d_\Pi P = [\Pi, P_1 \wedge \cdots \wedge P_k] = [\Pi, P] \wedge (P_2 \wedge \cdots \wedge P_k) + (-1)^1 P_1 \wedge [\Pi, P_2 \wedge \cdots \wedge P_k].$$

Therefore

$$
\begin{aligned}
&(d_\Pi P)(\alpha_0, \ldots, \alpha_k) = \\
&= [\Pi, P] \wedge (P_2 \wedge \cdots \wedge P_k)(\alpha_0, \ldots, \alpha_k) \\
&\quad - P_1 \wedge [\Pi, P_2 \wedge \cdots \wedge P_k](\alpha_0, \ldots, \alpha_k) \\
&= \sum_{0 \le i < j \le k} [\Pi, P_1](\alpha_i, \alpha_j) P_2 \wedge \cdots \wedge P_k(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_k) \\
&\quad - \sum_{i=0}^{k} P_1(\alpha_i) d_\Pi (P_2 \wedge \cdots \wedge P_k)(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_k).
\end{aligned}
$$

We have thus proven our claim on all decomposable k–vector fields. Due to locality of the Schouten–Nijenhuis bracket (together with the fact that locally any k-vector field is decomposable) tha claim holds true for all k–vector fields. □

Remark 4.2. From this expicit expression it would be tempting to say that the Poisson cohomology is some sort of Lie algebra cohomology, and precisely the Lie algebra cohomology of $(C^\infty(M), \{-,-\})$. This is not precise, because we do not have an identification between cochains (which are linear maps $\Lambda^k C^\infty(M) \to C^\infty(M)$) with multivectors. Multivectors are exactly those cochains which are differentiable in each argument. From this remark one can construct a homomorphism

$$j_*\colon \mathrm{H}^k_\Pi(M) \to \mathrm{H}^k_{\mathrm{Lie}}(C^\infty(M), \{-,-\}).$$

Computations of the cohomology on the right hand side are even harder than those for Poisson cohomology. This is one of the reasons why such cohomology is seldom considered.

Remark 4.3. Let f be a Casimir function for (M, Π). Let $P \in \mathfrak{X}^k(M)$. Then $d_\Pi(fP) = [\Pi, f] \wedge P + f[\Pi, P] = f[\Pi, P] = f d_\Pi P$. Hence we can define a product $f \cdot [P] = [fP]$. So there is a structure of $\mathrm{H}^0_\Pi(M) = \mathrm{Cas}(M)$-module on each $\mathrm{H}^k_\Pi(M)$.

Proposition 4.4. *The external product of multivector fields induces an associative and super commutative product in Poisson cohomology.*

$$\wedge\colon \mathrm{H}^k_\Pi(M) \times \mathrm{H}^l_\Pi(M) \to \mathrm{H}^{k+l}_\Pi(M).$$

this product will be called the Poisson product.

Proof.

$$[\Pi, P \wedge Q] = [\Pi, P] \wedge Q + (-1)^{p-1} P \wedge [\Pi, Q]$$

Therefore if $[\Pi, P] = [\Pi, Q] = 0$ also $[\Pi, P \wedge Q] = 0$, hence the product of two cocycles is a cocycle

$$[-,-]\colon Z^k_\Pi \times Z^q_\Pi \to Z^{p+q-1}_\Pi.$$

This product descends to cohomology. Define $[P] \wedge [Q] := [P \wedge Q]$. This is well defided, because

$$
\begin{aligned}
[P + [\Pi, R]] \wedge [Q] &= [(P + [\Pi, R]) \wedge Q] \\
&= [P \wedge Q] + [[\Pi, R] \wedge Q] \\
&= [P \wedge Q] + (-1)^{p-2}[P, R \wedge \underbrace{[\Pi, Q]}_{=0}] + [P, [\Pi, R \wedge Q]].
\end{aligned}
$$

The algebraic properties are a trivial consequence of analogous properties of $\wedge$. ☐

Remark 4.5. In a similar way it is easy to verify that also the Schouten bracket descends to cohomology via

$$[[P],[Q]] := [[P,Q]].$$

Here the key property is connected to the Jacobi identity for $[[\Pi,\Pi],Q]$.

Remark 4.6. H^k_Π is not functorial. In fact given a Poisson map $\varphi\colon M_1 \to M_2$ you do not have a corresponding map on chains $\varphi_*\colon \mathfrak{X}^k(M_1) \to \mathfrak{X}^k(M_2)$, where as we remarked already, only the weaker notion of φ_*-relatedness survive.

Theorem 4.7. *Let (M,Π) be a Poisson manifold. The sharp map intertwines the Poisson and de Rham cochain complexes, i.e.*

$$\#_\Pi\colon \Omega^k(M) \to \mathfrak{X}^k(M), \quad \#_\Pi \circ d = d_\Pi \circ \#_\Pi,$$

and therefore induces a homomorphism

$$\#_\Pi\colon\ \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_\Pi(M).$$

This morphism is an algebra morphism with respect to cup products. Furthermore, if M is symplectic then $\#_\Pi$ is an isomorphism.

Proof. Here $\#_\Pi$ is extended to k-forms as

$$(\#_\Pi\omega)(\alpha_1,\dots,\alpha_k) = \omega(\#_\Pi(\alpha_1),\dots,\#_\Pi(\alpha_k)).$$

Let $\omega \in \Omega^k(M)$ and $\alpha_0,\dots,\alpha_k \in \Omega^1(M)$. Then

$$\begin{aligned}
&(d_\Pi(\#_\Pi(\omega)))(\alpha_0,\dots,\alpha_k) = \\
&= \sum_{i=0}^k (-1)^{i+1}\#_\Pi(\alpha_i)(\#_\Pi(\omega))(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_k) \\
&\quad + \sum_{i<j}(-1)^{i+j-1}\#_\Pi(\omega)([\alpha_i,\alpha_j],\alpha_0,\dots,\widehat{\alpha_i},\dots,\widehat{\alpha_j},\dots,\alpha_k) \\
&= \sum_{i=0}^k (-1)^{i+1}\#_\Pi(\alpha_i)\omega(\#_\Pi(\alpha_0),\dots,\widehat{\#_\Pi(\alpha_i)},\dots,\#_\Pi(\alpha_k)) \\
&\quad + \sum_{i<j}(-1)^{i+j-1}\omega([\#_\Pi(\alpha_i),\#_\Pi(\alpha_j)],\#_\Pi(\alpha_0),\dots,\widehat{\#_\Pi(\alpha_i)}, \\
&\quad \dots,\widehat{\#_\Pi(\alpha_j)},\dots,\#_\Pi(\alpha_k)) \\
&= d\omega(\#_\Pi(\alpha_0),\dots,\#_\Pi(\alpha_k)) \\
&= \#_\Pi(d\omega).
\end{aligned}$$

The fact that the sharp map respects cup product is obvious from definitions already at the chain level

$$\#_\Pi(\omega_1 \wedge \omega_2) = \#_\Pi(\omega_1) \wedge \#_\Pi(\omega_2).$$

Lastly if M is symplectic, $\#_\Pi$ is invertible at the chain level and therefore it remains such on cohomology. ☐

Proposition 4.8. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{g}^*$ the dual vector space with the Lie-. Then*

$$\mathrm{H}^k_\Pi(\mathfrak{g}^*) \cong \mathrm{H}^k_L(\mathfrak{g}) \otimes \mathrm{Cas}(\mathfrak{g}^*),$$

where on the left H^k_L is the Lie algebra cohomology of $\mathfrak{g}$.

Remark 4.9. To complete the list of basic examples consider that if $(M, 0)$ is considered as a Poisson manifold then $\mathrm{H}^k_\Pi(M) = \mathfrak{X}^k(M)$. Therefore the Poisson cohomology has a huge variety of behaviours and is in general likely to be infinite dimensional over $\mathbb{R}$. We will see in examples that even the weaker property of being finitely generated as $H^0_\pi(M)$–modules is not always satisfied by Poisson cohomology groups.

Theorem 4.10 (Mayer–Vietoris sequence for Poisson cohomology)**.** *Let (M, Π) be a Poisson manifold. Let U and V be open subsets of M, considered as Poisson manifolds under restriction of the bivector $(U, \Pi|_U)$, $(V, \Pi|_V)$. Then there is a long exact sequence*

$$\ldots \to \mathrm{H}^{k-1}_\Pi(U \cap V) \xrightarrow{\partial} \mathrm{H}^k_\Pi(U \cup V) \to \mathrm{H}^k_\Pi(U) \oplus \mathrm{H}^k_\Pi(V) \to \mathrm{H}^k_\Pi(U \cap V) \xrightarrow{\partial} \ldots$$

Proof. Here one basically recalls how the proof of Mayer–Vietoris theorem goes on forms.

Given two open sets U and V, let $P \in \mathfrak{X}^k(U)$, $Q \in \mathfrak{X}^k(V)$, $R \in \mathfrak{X}^k(U \cup V)$. Then $R \mapsto R|_U$, $R \mapsto R|_V$ are maps to $\mathfrak{X}^k(U)$ and $\mathfrak{X}^k(V)$ respectively. Being the Shouten bracket a local operator one has $[\Pi|_U, R|_U] = [\Pi, R]|_U$ and $[\Pi|_V, R|_V] = [\Pi, R]|_V$. Therefore restriction induces a map on chains, trivially injective. Now start from P and Q. We want a vector field on $U \cap V$. We can of course consider $P - Q|_{U\cap V}$. If $[\Pi|_U, P] = 0 = [\Pi|_V, Q]$ then $[\Pi, P - Q|_{U\cap V}] = 0$. So again we have a cochain map. This map is surjective. Indeed, given a vector field S on $U \cap V$ we may extend, by the usual trick of smoothing function, to P on U and Q on V such that. $P - Q = S$ on $U \cap V$. Therefore we have a short exact sequence of cochain complexes. This induces as usual a long exact sequence in cohomology. Given $S \in \mathfrak{X}^k(U \cap V)$, $[\Pi, S] = 0$ consider (P, Q) as before such that $P - Q = S$. Being $[\Pi, (P, Q)] = ([\Pi|_U, P], [\Pi|_V, Q])$ we have $[\Pi|_U, P] - [\Pi|_V, Q] = [\Pi|_{U\cap V}, P - Q] = 0$. Therefore there exists $T \in \mathfrak{X}(U \cup V)$ such that $P = T|_U$, $Q = T|_V$. Define $\partial[S] := [T]$. The usual arguments, based on the snake lemma, prove the theorem. □

4.1 Modular class

Let (M, Π) be a Poisson manifold. Let us assume, for simplicity that M is orientable. Let Ω be a volume form on Ω. Consider for any $f \in C^\infty(M)$, $L_{X_f}\Omega \in \Omega^n M$. There exists a function $\phi_\Omega(f)$ such that $L_{X_f}\Omega = \phi_\Omega(f)\Omega$.

Fact 4.11. *ϕ_Ω is a vector field.*

Proof. We have to prove $\phi_\Omega(fg) = \phi_\Omega(f)g + f\phi_\Omega(g)$. But $X_{fg} = gX_f + fX_g$ (from $\{fg, h\} = g\{f, h\} + f\{g, h\}$). Therefore

$$\begin{aligned} L_{X_{fg}}\Omega &= L_{gX_f + fX_g}\Omega \\ &= gL_{X_f}\Omega + X_f(g)\Omega + fL_{X_g}\Omega + X_g(f)\Omega \\ &= g\phi_\Omega(f)\Omega + f\phi_\Omega(g)\Omega \end{aligned}$$

hence thesis. □

Definition 4.12. *ϕ_Ω is called the* modular vector field *of* (M, Π) *with respect to* Ω.

Fact 4.13. *The modular vector field is an infinitesimal Poisson field.*

Proof. We have seen that this is equivalent to $\phi_\Omega \in \mathrm{Der}(C^\infty(M), \{-, -\})$. Now

$$\begin{aligned} L_{X_{\{f,\, g\}}} &= L_{[X_f, X_g]}\Omega \\ &= [L_{X_f}, L_{X_g}]\Omega \\ &= L_{X_f}(\phi_\Omega(g)\Omega) - L_{X_g}(\phi_\Omega(f)\Omega) \\ &= \phi_\Omega(g)L_{X_f}\Omega + \{f,\, \phi_\Omega(g)\}\Omega - \phi_\Omega(f)L_{X_g}\Omega - \{g,\, \phi_\Omega(f)\}\Omega \\ &= \phi_\Omega(g)\phi_\Omega(f)\Omega + \{f,\, \phi_\Omega(g)\}\Omega + \phi_\Omega(f)\phi_\Omega(g)\Omega + \{\phi_\Omega(f),\, g\}\Omega, \end{aligned}$$

so

$$\phi_\Omega(\{f,\, g\}) = \{\phi_\Omega(f),\, g\} + \{f,\, \phi_\Omega(g)\}.$$

□

Fact 4.14. $L_{\phi_\Omega}\Omega = 0$.

Proof.

$$L_{\phi_\Omega}\Omega = di_{\phi_\Omega}\Omega + i_{\phi_\Omega}d\Omega = di_{\phi_\Omega}\Omega = d(di_\Pi\Omega),$$

because $di_\Pi\Omega = i_{\phi_\Omega}\Omega$. □

Take another volume form $\Omega' = a\Omega$. Then

$$L_{X_f}\Omega' = \phi_{\Omega'}(f)\Omega' = \phi_{\Omega'}(f)a\Omega$$

$$L_{X_f}(a\Omega) = aL_{X_f}\Omega + X_f(a)\Omega = a\phi_\Omega(f) + X_f(a)$$

Furtermore

$$a\phi_{\Omega'}(f) = a\phi_\Omega(f) + X_f(a),$$

$$\phi_{\Omega'}(f) = \phi_\Omega(f) + \frac{1}{a}X_f(a),$$

and

$$\frac{1}{a}X_f(a) = \frac{1}{a}\{f,\, a\} = \{f,\, \log|a|\} = -\{\log|a|,\, f\}$$

Hence the modular vector fields with respect to different volume forms differ for a hamiltonian vector field.

$$\phi_{\Omega'} = \phi_\Omega + X_{-\log|a|}.$$

Definition 4.15. *The vector field ϕ_Ω defines a class $[\phi_\Omega] \in \mathrm{H}^1_\Pi(M)$. This class is independent of Ω, and is called the* (Poisson) modular class.

Definition 4.16. *Let (M, Π) be a Poisson manifold such that $[\phi_\Omega] = 0$. Then (M, Π) is called* unimodular.

Exercise 4.17. *On $(\mathbb{R}^2, f(x, y)dx \wedge dy)$ compute the modular class.*

Examples 4.18.

1. Let (M, ω) be a compact symplectic manifold, $\Pi = \omega^{-1}$. Then the modular class is 0. In fact the volume form $\frac{\omega^n}{n!}$ is invariant under all Hamiltonian vector fields.

2. Let $(M, \Pi) = (\mathfrak{g}^*, \Pi_{\text{lin}})$. Then M is unimodular if and only if $\mathfrak{g}$ is unimodular as Lie algebra, i.e. $\operatorname{tr}(\operatorname{ad}_X) = 0$ for all $X \in \mathfrak{g}$.

3. Let (M, Π) be a regular Poisson structure. It can be proved that there exists an injective map
$$\mathrm{H}^1(M) \hookrightarrow \mathrm{H}^1_\Pi(M)$$
sending Reeb class $[Reeb]$ to $[\phi_\Omega]$. The Reeb class is an obstruction to the existence of a volume form of the usual bundle invariant for vector fields tangent to leaves [ab03].

Let (M, Π) be compact unimodular Poisson manifold. Then there exists Ω such that $L_{\phi_\Omega}\Omega = 0$. Then

$$\begin{aligned}
\int_M \{f, g\}\Omega &= \int_M (L_{x_f} g)\Omega \\
&= \int_M L_{X_f}(g\Omega) - \int_M g L_{X_f}\Omega \\
&= \underbrace{\int_M d(i_{X_f} g\Omega)}_{=0 \text{ by Stokes theorem}} + \underbrace{i_{X_f} d(g\Omega)}_{=0} - \int_M g L_{X_f}\Omega \\
&= -\int_M g\phi_\Omega(f)\Omega.
\end{aligned}$$

This is called also *infinitesimal KMS condition*. Being (M, Π) unimodular, we can choose a volume form Ω such that $\phi_\Omega \equiv 0$, so $\int_M \{f, g\}\Omega = 0$, i.e.

$$\int_M \Omega \colon C^\infty(M) \to \mathbb{R}$$

is a *Poisson trace*.

4.2 Computation for Poisson cohomology

Let us consider the quadratic Poisson structure on $\mathbb{R}^2$

$$\Pi_0(x, y) = (x^2 + y^2)\partial_x \wedge \partial_y.$$

We want to prove the following

Proposition 4.19 (Ginzburg)**.** *The Poisson cohomology of* $(\mathbb{R}^2, \Pi_0)$ *is given by*

$$\begin{aligned}
&\mathrm{H}^0_{\Pi_0}(\mathbb{R}^2) = \mathbb{R}, \\
&\mathrm{H}^1_{\Pi_0}(\mathbb{R}^2) = \mathbb{R}\langle x\partial_x + y\partial_y, y\partial_x - x\partial_y\rangle, \\
&\mathrm{H}^2_{\Pi_0}(\mathbb{R}^2) = \mathbb{R}\langle \partial_x \wedge \partial_y, \Pi_0\rangle.
\end{aligned}$$

Proof. To make computations easier let us identify $\mathbb{R}^2 \simeq \mathbb{C}$, $z = x+iy$, $\bar{z} = z-iy$, $\partial_z = \partial_z - i\partial_y$, $\partial_{\bar{z}} = \partial_x + i\partial_y$, $\Pi_0 = -2iz\bar{z}\partial_z \wedge \partial_{\bar{z}}$. We will omit the factor $(-2i)$ from now on.

Let us start by considering vector fields having coefficients which are formal power series in z and $\bar{z}$ (real coefficients): $\mathfrak{X}^k_f(\mathbb{R}^2)$. Let us denote with V_n the space of homogeneous polynomials in z and $\bar{z}$ of degree n, $V_n = \langle z^n, z^{n-1}\bar{z}, \ldots, \bar{z}^n\rangle$, $\dim V_n = n+1$.

$$\begin{aligned}
\mathfrak{X}^0_f(\mathbb{R}^2) &= \text{ formal power series in } z \text{ and } \bar{z} \ = \prod_{i=1}^{\infty} V_i, \\
\mathfrak{X}^1_f(\mathbb{R}^2) &= \prod_{i=1}^{\infty} V_i\partial_z \oplus \prod_{i=1}^{\infty} V_i\partial_{\bar{z}} = \prod_{i=1}^{\infty}(V_i\partial_z \oplus V_i\partial_{\bar{z}}), \\
\mathfrak{X}^2_f(\mathbb{R}^2) &= \prod_{i=1}^{\infty} V_i\partial_z \wedge \partial_{\bar{z}}.
\end{aligned}$$

Now we compute $[\Pi, f]$ on $f \in C^\infty(\mathbb{R}^2)$:

$$[\Pi, f] = z\bar{z}((\partial_z(f))\partial_{\bar{z}} - (\partial_{\bar{z}}(f))\partial_z), \tag{4.3}$$

and $[\Pi, X]$ with $X \in \mathfrak{X}^1(\mathbb{R}^2)$, $X = f(z,\bar{z})\partial_z + g(z,\bar{z})\partial_{\bar{z}}$:

$$[\Pi, X] = z\bar{z}(\partial_z(f) + \partial_{\bar{z}}(g)) - \bar{z}f - zg. \tag{4.4}$$

From this formulae it is evident that

$$d_\Pi \colon V_{i-1} \to V_i\partial_z \oplus V_i\partial_{\bar{z}}$$
$$d_\Pi \colon V_i\partial_z \oplus V_i\partial_{\bar{z}} \to V_{i+1}\partial_z \wedge \partial_{\bar{z}}.$$

Therefore the complex on formal vector fields splits into a direc sum complexes

$$0 \to V_{i-1} \to V_i^{\oplus 2} \to V_{i+1} \to 0$$

If we denote with $\varphi_i := d_\pi|_{V_i} \colon V_i \to V_{i+1}^{\oplus 2}$, and $\psi_i := d_\Pi|_{V_i^{\oplus 2}} \colon V_i^{\oplus 2} \to V_{i+1}$, this means that

$$\begin{aligned}
\mathrm{H}^0_\Pi &= \bigoplus_{i\in\mathbb{N}} \ker\varphi_i \\
\mathrm{H}^1_\Pi &= \bigoplus_{i\in\mathbb{N}} \ker\psi_i/\operatorname{im}\varphi_i \\
\mathrm{H}^2_\Pi &= \bigoplus_{i\in\mathbb{N}} \operatorname{im}\psi_i
\end{aligned}$$

Let us first consider the case $i \geq 2$.

$$0 \to V_{i-1} \to V_i^{\oplus 2} \to V_{i+1} \to 0$$

The cohomology contributions of these complexes are

$$\ker\varphi_{i-1} \hookrightarrow \mathrm{H}^0_\Pi, \quad \ker\psi_i/\operatorname{im}\varphi_{i-1} \hookrightarrow \mathrm{H}^1_\Pi, \quad V_{i+1}/\operatorname{im}\psi_i \hookrightarrow \mathrm{H}^2_\Pi\,.$$

Observe that $\dim V_{i-1} = i$, $\dim V_i^{\oplus 2} = 2(i+1)$, $\dim V_{i+1} = i+2$. Now

1. φ_{i-1} is injective. In fact from 4.3
$$\varphi_{m+l}(z^m \bar{z}^l) = m z^m \bar{z}^{l+1} \partial_{\bar{z}} - l z^{m+1} \bar{z}^l \partial_z.$$
Therefore $\ker \varphi_{i-1} = \{0\}$, $\dim \operatorname{im} \varphi_{i-1} = i$.
2. ψ_i is surjective. In fact from 4.4
$$\psi_{m+l}(z^m \bar{z}^l \partial_z) = (m-1) z^m \bar{z}^{l+1} \partial_z \wedge \partial_{\bar{z}},$$
$$\psi_{m+l}(z^m \bar{z}^l \partial_{\bar{z}}) = (l-1) z^{m+1} \bar{z}^l \partial_z \wedge \partial_{\bar{z}}.$$
Therefore $V_{i+1} / \operatorname{im} \psi_i = 0 \hookrightarrow \mathrm{H}^2_\Pi$.
3. lastly $\dim \ker \psi_i = \dim V_i^{\oplus 2} - \dim \operatorname{im} \psi_i = 2(i+2) - (i+2) = i+2$ which implies $\ker \psi_i / \operatorname{im} \varphi_{i-1} = 0 \hookrightarrow \mathrm{H}^1_\Pi$.

Thus no contributions to cohomology comes from $i \geq 2$. Let us look what happens when $i = 0, 1$.

$$\begin{tikzcd}
V_0 \arrow[dr, "\varphi_0"'] & V_0 \oplus V_0 \arrow[dr, "\psi_0"'] & V_0 \\
& V_1 \oplus V_1 \arrow[dr, "\psi_1"'] & V_1 \\
& & V_2
\end{tikzcd}$$

Again an easy and explicit computation shows that

$$\psi_0 \colon \begin{cases} \partial_z \mapsto \bar{z} \partial_z \wedge \partial_{\bar{z}} \\ \partial_{\bar{z}} \mapsto z \partial_z \wedge \partial_{\bar{z}}, \end{cases}$$
$$\varphi_0 = 0,$$
$$\psi_1 \colon \begin{cases} z\partial_z \mapsto 0 \\ \bar{z}\partial_z \mapsto -\bar{z}^2 \partial_z \wedge \partial_{\bar{z}} \\ z\partial_{\bar{z}} \mapsto -z^2 \partial_z \wedge \partial_{\bar{z}} \\ \bar{z}\partial_{\bar{z}} \mapsto 0. \end{cases}$$

Now
$$\ker \varphi_0 = V_0 \simeq \mathbb{R} \hookrightarrow \mathrm{H}^0_\Pi$$
$$\ker \psi_0 \oplus \ker \psi_1 / \operatorname{im} \varphi_0 = \ker \psi_1 \hookrightarrow \mathrm{H}^1_\Pi, \quad \ker \psi_1 = \langle z\partial_z, \bar{z}\partial_{\bar{z}} \rangle$$
$$V_0 \oplus V_1 / \operatorname{im} \psi_0 \oplus V_2 / \operatorname{im} \psi_1 = V_0 \oplus \langle z\bar{z} \partial_z \wedge \partial_{\bar{z}} \rangle \hookrightarrow \mathrm{H}^2_\Pi\,.$$

Moving to real coordinates

$$\begin{aligned} V_0 &= \langle \partial_x \wedge \partial_y \rangle \\ \langle z\bar{z} \partial_z \wedge \partial_{\bar{z}} \rangle &= \langle \Pi \rangle \\ z\partial_z &= x\partial_x + y\partial_y \\ \bar{z}\partial_{\bar{z}} &= y\partial_x - x\partial_y. \end{aligned}$$

Now we need to prove that only this formal vector fields contribute to the Poisson smooth cohomology. Define flat functions to be those $f \in C^\infty(\mathbb{R}^2)$ such that all their derivatives at the origin are 0 and $f(0) = 0$. Similarly flat multivector fields are those with flat coefficients. Then we have a short exact sequence of complexes

$$0 \to \mathfrak{X}^\bullet_{flat}(\mathbb{R}^2) \to \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet_{formal}(M) \to 0$$

Exactness is a consequence of Borel's theorem.
If we prove that $\mathrm{H}^*_{flat,\Pi_0}(\mathbb{R}^2) = 0$ we are done.

Now consider $\#_\Pi \colon \Omega^*_{flat} \to \mathfrak{X}^*_{flat}$ (here as usual by $\#_\Pi$ we denote the extension to all $\Omega^* M$; as we have seen $\#_\Pi(f) = X_f$). We claim that $\#_\Pi$ is an isomorphism. Let us prove it on 1-forms

$$\#_\Pi(fdx + gdy) = (x^2 + y^2)(f\partial_y - g\partial_x).$$

Now tha point is that if f is flat, then $(x^2 + y^2)f$ is also flat, but also the other way around, i.e.

$$(x^2 + y^2)f = \bar{f}$$

has always a flat solution in f, i.e.

$$\frac{\bar{f}}{x^2 + y^2}$$

is a well defined flat function.

The key points are that Π has polynomial coefficients and has isolated singular points (this can be weakened). □

Example 4.20. Let SU(2) have Poisson structure we already mentioned. The adjoint action of SU(2) on $\mathfrak{su}(2) \simeq \mathbb{R}^3$ is then action by rotations. The isotropy subgroup of $(1, 0, 0)$ is $\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} : |\alpha| = 1 \right\}$. The orbit of $(1, 0, 0)$ is $\mathbb{S}^2$. The map

$$\phi\colon \mathrm{SU}(2) \to \mathrm{SU}(2)/\,\mathrm{U}(1) \simeq \mathbb{S}^2$$

is given by the formula

$$\phi \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = (\underbrace{|\alpha|^2 - |\beta|^2}_{x_1}, \underbrace{-i(\alpha\beta - \bar{\alpha}\bar{\beta})}_{x_2}, \underbrace{-(\alpha\beta + \bar{\alpha}\bar{\beta})}_{x_3})$$

Check that indeed $x_1^2 + x_2^2 + x_3^2 = 1$.

We claim that ϕ coinduces a Poisson structure on $\mathbb{S}^2$. Using the explicit expression for p one can explicitly compute

$$\begin{aligned}
\{x_1, x_2\} &= (1 - x_1)x_3 \\
\{x_2, x_3\} &= (1 - x_1)x_1 \\
\{x_3, x_1\} &= (1 - x_1)x_2 \\
\Pi_0 &= (1 - x_1)[x_3\partial_{x_1} \wedge \partial_{x_2} + x_1\partial_{x_2} \wedge \partial_{x_3} + x_2\partial_{x_3} \wedge \partial_{x_1}] \\
&= (1 - x_1)\Pi,
\end{aligned}$$

Symplectic foliation consists of two 0-leaves - the north and south pole, and complement which is a 2-leaf.

Example 4.21. Take the stereographic projection from the south pole, i.e.

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{S}^2 \setminus \{N\} \\ (x, y) &\mapsto \left(\frac{x_2}{1 + x_1}, \frac{x_3}{1 + x_1}\right) \end{aligned}$$

Then Π_0 on $\mathbb{R}^2$ becomes $(x^2 + y^2)\partial_x \wedge \partial_y$.

Of course if you take the stereographic projection from the north pole you get

$$\mathbb{R}^2 \to \mathbb{S}^2 \setminus \{S\}$$

with the symplectic structure (it can be proved that it is the standard one).

We wanto to use the Mayer–Vietoris exact sequence to compute the Poisson cohomology of $(\mathbb{S}^2, \Pi_0)$

$$\begin{array}{ccc} U = \mathbb{S}^2 \setminus \{N\} & V = \mathbb{S}^2 \setminus \{N\} & U \cap V = \mathbb{S}^2 \setminus \{N, S\} \\ (x^2 + y^2)\partial_x \wedge \partial_y & \text{symplectic} & \text{symplectic} \\ \mathrm{H}^0_\Pi(U) = \mathbb{R} & \mathrm{H}^0_\Pi(V) = \mathbb{R} & \mathrm{H}^0_\Pi(U \cap V) = \mathbb{R} \\ \mathrm{H}^1_\Pi(U) = \mathbb{R}^2 & \mathrm{H}^1_\Pi(V) = 0 & \mathrm{H}^1_\Pi(U \cap V) = \mathbb{R} \\ \mathrm{H}^2_\Pi(U) = \mathbb{R}^2 & \mathrm{H}^2_\Pi(V) = 0 & \mathrm{H}^2_\Pi(U \cap V) = 0 \end{array}$$

The sequence is

$$\begin{gathered} 0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to \\ \to \mathrm{H}^1_\Pi(\mathbb{S}^2) \to \mathbb{R}^2 \oplus 0 \to \mathbb{R} \to \\ \to \mathrm{H}^2_\Pi(\mathbb{S}^2) \to \mathbb{R}^2 \oplus 0 \to 0 \end{gathered}$$

The first row is exact (a Casimir function is constant on each of U and V).

$$0 \to \mathrm{H}^1_\Pi(\mathbb{S}^2) \to \underbrace{\mathbb{R}^2 \xrightarrow{\lambda} \mathbb{R}}_{x\partial_x + y\partial_y \mapsto y\partial_x - x\partial_y} \xrightarrow{\mu} \mathrm{H}^2_\Pi(\mathbb{S}^2) \to \mathbb{R}^2 \to 0$$

Because λ is surjecitve $\dim \ker \lambda = 0$ and $\dim \operatorname{im} \mu \leq 1$, so $\mu = 0$ and $\mathrm{H}^2_\Pi(\mathbb{S}^2) \simeq \mathbb{R}^2$. If you know that $\mathrm{H}^1_\Pi(\mathbb{S}^2)$ is nontrivial then $\mathrm{H}^1_\Pi(\mathbb{S}^2) \simeq \mathbb{R}$.

Chapter 5

Poisson homology

Recall that
$$\partial_\Pi = i_\Pi d - d i_\Pi \colon \Omega^k M \to \Omega^{k-1} M.$$
We showed that $\partial_\Pi^2 = 0$ and defined Poisson homology as the homology of the complex $(\Omega^\bullet, \partial_\Pi)$.

Proposition 5.1. *The Poisson homology is explicitely computed (in local coordinates) by*
$$\partial_\Pi(f_0 df_1 \dots df_k) = \sum_{1\le i\le k} (-1)^{i+1}\{f_0,\, f_i\} df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_k$$
$$+ \sum_{1\le i<j\le k} (-1)^{i+j} f_0 d\{f_i,\, f_j\} df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge \widehat{df_j} \wedge \dots \wedge df_k$$

Proof.
$$\begin{aligned}
&\partial_\Pi(f_0 df_1 \dots df_k) = \\
&= i_\Pi(df_0 \wedge df_1 \dots df_k) \\
&\quad - d[\sum_{1\le i<j\le k} (-1)^{i+j} f_0 d\{f_i,\, f_j\} df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge \widehat{df_j} \wedge \dots \wedge df_k] \\
&= \sum_{1\le i<j\le k} (-1)^{i+j+1}\{f_i,\, f_j\} df_0 \wedge df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge \widehat{df_j} \wedge \dots \wedge df_k \\
&+ \quad \sum_{1\le i<j\le k} (-1)^{i+j}\{f_i,\, f_j\} df_0 \wedge df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge \widehat{df_j} \wedge \dots \wedge df_k \\
&+ \quad \sum_{1\le i<j\le k} (-1)^{i+j} f_0 d\{f_i,\, f_j\} df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge \widehat{df_j} \wedge \dots \wedge df_k.
\end{aligned}$$
□

Remark 5.2. One could use this formulas a definition for ∂_Π. This is correct, but requires also checking that the formula does not depend on local choices and this is quite difficult.

Note that $\partial_\pi(f_0 df_1) = \{f_0,\, f_1\}$ and therefore The 0–th Poisson homology is just given by: $C^\infty(M)/\{C^\infty(M),\, C^\infty(M)\}$. Thus it can be considered as the dual space to Poisson traces. This apparently easy definition does not mean that, even in very explicit examples, such invariant can be easily computed.

Theorem 5.3 (Brylinski)**.** *If M is symplectic manifold then*

$$\mathrm{H}_k^{\Pi}(M) \simeq \mathrm{H}_{DR}^{m-k}(M;\mathbb{R}) \simeq \mathrm{H}_{\Pi}^{m-k}(M).$$

Proof. (sketch) Given ω symplectic form, take the volume form $\frac{\omega^m}{m!} = \Omega_0$. You can use it to define a "Hodge-like" *-operator

$$*\colon \Omega^k(M) \to \Omega^{2m-k}(M),$$

implicitly as

$$(\beta \wedge *\alpha) = \underbrace{\Pi^{\wedge k}(\alpha,\beta)}_{\in C^\infty(M)}\Omega.$$

This operator verifies the following:

1. $** = \mathrm{id}$,
2. $\beta \wedge (*\alpha) = (-1)^k \alpha \wedge (*\beta)$,
3. on $\Omega^k(M)$, $\partial_\Pi = (-1)^{k+1} * d *$.

Therefore it intertwines d with * and therefore induces an isomorphism in homology. □

Remark 5.4. This map is similar to Poincare duality. In fact one could recover the same result through the existing duality between Poisson homology and cohomology.

Poisson homology is functorial. Given a Poisson map $\varphi\colon M_1 \to M_2$ there is a map $\varphi^*\colon \mathrm{H}_k^{\Pi}(M_2,\Pi_2) \to \mathrm{H}_k^{\Pi}(M_1,\Pi_1)$. In particular for any leaf S of M

$$\begin{tikzcd}
\mathrm{H}_k^{\Pi}(S) \arrow[r, "\varphi^*"] \arrow[d, "\simeq"'] & \mathrm{H}_k^{\Pi}(M) \\
\mathrm{H}_{\Pi}^{n-k}(S) \arrow[d, "\simeq"'] & \\
\mathrm{H}_{\mathrm{DR}}^{n-k}(S) \arrow[uur, "\varphi*"'] &
\end{tikzcd}$$

Again deciding whether this map is injective or surjective is a difficult problem.

In the canonical double (mixed) complex you have $d\partial_\Pi + \partial_\Pi d = 0$

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
\Omega^2(M) \arrow[d, "\partial_\Pi"] & \Omega^1(M) \arrow[l, "d"'] \arrow[d, "\partial_\Pi"] & \Omega^0(M) \arrow[l, "d"'] \\
\Omega^1(M) \arrow[d, "\partial_\Pi"] & \Omega^0(M) \arrow[l, "d"'] & \\
\Omega^0(M) & &
\end{tikzcd}$$

Starting from this you can define cyclic (negative, periodic) Poisson homology and a long exact sequence of Connes-type.

Example 5.5. Consider on $\mathbb{R}^3$ the

$$\{x_2, x_3\} = 2px_2x_3 - qx_1^2 = g_1,$$
$$\{x_1, x_3\} = 2px_1x_3 - qx_2^2 = g_2,$$
$$\{x_1, x_2\} = 2px_1x_2 - qx_3^2 = g_3.$$

Check that

$$\phi = \frac{q}{3}(x_1^3 + x_2^3 + x_3^3) - 2px_1x_2x_3$$

is a Casimir element. Can you prove that there are no other functionally independent Casimirs?

Let

$$\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

Verify that

$$\nabla\phi = (g_1, g_2, g_3),$$

and that

$$\nabla \times (g_1, g_2, g_3) = 0.$$

(here we are denoting $\nabla\times$ to be the curl as in usal vector calculus). Then, again by direct computation you can verify that

$$\partial_\Pi(x_1dx_1 + x_2dx_2 + x_3dx_3) = \nabla(x_1, x_2, x_3) \cdot \nabla\phi,$$

$$\partial_\Pi(x_1dx_2\wedge dx_3 + x_2dx_3\wedge dx_1 + x_3dx_1\wedge dx_2) = \nabla(x_1, x_2, x_3)d\phi - d[(x_1, x_2, x_3)\cdot\nabla\phi],$$

$$\partial_\Pi(f dx_1 \wedge dx_2 \wedge dx_3) = -df \wedge d\phi.$$

These formulas are basically all one needs to thoroughly compute in an explicit manner the Poisson homology groups, as explained in [v-m94].

The result of computation of Poisson homology is that $\mathrm{H}_*^\Pi(\mathbb{R}^3)$ is a free $\mathbb{R}[\phi]$-module of rank 8, 8, 1, 1. $\mathrm{H}_2^\Pi(\mathbb{R}^3)$ is generated by $x_1dx_2dx_3$, $x_2dx_3dx_1$, $x_3dx_1dx_2$. $\mathrm{H}_3^\Pi(\mathbb{R}^3)$ is generated by $dx_1dx_2dx_3$.

5.1 Poisson homology and modular class

Say Ω is a volume form on M.

$$\partial_\Pi\Omega = i_\Pi d\Omega - di_\Pi\Omega = -i_{\phi_\Omega}\Omega.$$

If M is unimodular Poisson then there exists $\Omega \in \Omega^n(M)$ such that $\phi_\Omega = 0$, so $\partial_\Pi\Omega = 0$ and thus $[\Omega] \neq 0 \in \mathrm{H}_n^\Pi(M)$.

For this reason in quantization you can regard Connes axiom of having "quantum" homological dimension equal classical dimension as a condition of unimodularity of the underlying Poisson manifold.

Let us now consider the Poisson structure of example (5.5). We want to compute its modular form starting from the standard volume form $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. This means we want, for any $f \in C^\infty(M)$

$$L_{X_f}\Omega = \phi(f)\Omega$$

Then, explicitely

$$\begin{aligned} L_{X_{x_1}}\Omega &= di_{g_3\partial_2 - g_2\partial_3}\Omega \\ &= d\left(g_3 i_{\partial_2}\Omega - g_2 i_{\partial_3}\Omega\right) \\ &= d\left(-g_3 dx_1 \wedge dx_3 - g_2 dx_1 \wedge dx_2\right) \\ &= \left(\partial_2 g_3 - \partial_3 g_2\right)\Omega \end{aligned}$$

And similar computations show that

$$\phi_\Omega(x_i) = \det \begin{pmatrix} \partial_j & \partial_k \\ g_j & g_k \end{pmatrix}$$

with $(i,j,k) = (1,2,3)$ or cyclic permutations. Therefore $\phi_\Omega = \nabla \times g$ (up to now we've never used the explicit form of g). Lastly, as remarked, g is defined in such a way that $\nabla \times g = 0$ and therefore such Poisson structure is unimodular. It is worth remarking that van den Bergh in its paper was commenting that this condition is exactly what makes computations of Poisson homology accesible through explicit formulas (unimodularity was at that time not recognized as an easily accesible, though very relevant, invariant of Poisson manifolds).

Chapter 6

Coisotropic submanifolds

Let (M, Π) be a Poisson manifold, C a submanifold of M and N^*C its conormal bundle defined as:

$$N^*C = \{\alpha \in T^*M : \langle \alpha,\, v \rangle = 0 \quad \forall\, v \in TC\}.$$

Definition 6.1. *C is called* coisotropic sumbanifold *of M if*

$$\#_\Pi(N^*C) \subseteq TC$$

Remark 6.2. On symplectic manifolds, for a submanifold N of M you consider TN and

$$TN^{\perp\omega} = \{w \in TM : \omega(v, w) = 0 \quad \forall\, v \in TN\}.$$

Then you have

$$\begin{aligned} TN &\subseteq TN^{\perp\omega} && \textit{isotropic},\\ TN &= TN^{\perp\omega} && \textit{Lagrangian},\\ TN &\supseteq TN^{\perp\omega} && \textit{coisotropic}. \end{aligned}$$

Exercise 6.3. *Prove that if (M, Π) is the Poisson manifold associated to a symplectic manifold then a submanifold verifies*

$$\#_\Pi(N^*C) \subseteq TC \quad \textit{iff } (TC)^{\perp\omega} \subseteq TC.$$

Proposition 6.4. *The following are equivalent*

1. *C is coisotropic in (M, Π).*
2. *For all $f, g \in C^\infty(M)$ such that $f|_C, g|_C = 0$, $\{f,\, g\}|_C = 0$.*
3. *For all $f \in C^\infty(M)$ such that $f|_C = 0$, $X_f|_C$ is tangent to C.*

Proof. The point here is that if $I = \{f \in C^\infty(M) | f|_C = 0\}$ then

$$\{d_x f : f \in I\} = N^*_x C,$$

$$\langle d_x f,\, v_x \rangle = v(f)(x).$$

The fact that we get all conormal vectors as differentials of functions in I follows from local equalities for C of the form $x^1 = \ldots = x^p = 0$ in a coordinate neighbourhood $(U; x^1, \ldots, x^n)$ $(p \leq n)$ adapted to C.
Then we have easily (3) $\implies$ (1) $\implies$ (2) $\implies$ (3). ∎

Remark 6.5.

- If C is Poisson submanifold then I is a Poisson ideal.
- If C is coisotropic then I is a Poisson subalgebra.

Exercise 6.6. *Let $\mathfrak{h}$ be a Lie subalgebra in $\mathfrak{g}$. Prove that $\mathfrak{h}^{\perp}$ is a coisotropic submanifold in $\mathfrak{g}^*$.*

Theorem 6.7. *$\varphi\colon (M_1, \Pi_1) \to (M_2, \Pi_2)$ is a Poisson map if and only if*

$$\Gamma_\varphi := \{(x, \varphi(x)) : x \in M_1\}$$

is a coisotropic submanifold of $M_1 \times \overline{M_2}$.

The notation:

$$M_1 \times \overline{M_2} = (M_1 \times M_2, \Pi_1 \oplus (-\Pi_2))$$

with product Poisson structure.

Proof. We have

$$T_{(x,\varphi(x))}\Gamma_\varphi = \{(v, \varphi_{*,x}v) : v \in T_x M_1\}$$
$$N^*\Gamma_\varphi = \{(-\varphi^*\lambda, \lambda) : \lambda \in T^*_{\varphi(x)}M_2\}$$

Then

$$\#_\Pi(N^*\Gamma_\varphi) \subseteq T\Gamma_\varphi$$

is equivalent to

$$\varphi_*(\#_{\Pi_1}(-\varphi^*\lambda)) = -\#_{\Pi_2}\lambda, \quad \forall\, \lambda \in T^*_{\varphi(x)}M_2,$$

which is one of the conditions equivalent to being Poisson. ◻

Definition 6.8. *Let C be a coisotropic submanifold of (M, Π) and let $I := \{f \in C^\infty(M) : f|_C = 0\}$. Define*

$$N(I) := \{g \in C^\infty(M) : \{g,\, I\} \subseteq I\}.$$

Proposition 6.9. *$N(I)$ is a Poisson subalgebra of $C^\infty(M)$, I is a Poisson ideal of $N(I)$ and therefore $N(I)/I$ is a Poisson algebra.*

Proof. From the Jacobi identity we get the first part:

$$\{\{g_1,\, g_2\},\, f\} = -\{\{g_2,\, f\},\, g_1\} + \{\{g_1,\, f\},\, g_2\}$$

Furthermore

$$\begin{aligned} N(I)/I &= C^\infty(\underline{C}) \\ &= \{f \in C^\infty(C) : Xf = 0 \quad \forall\, X \in \Gamma(\#_\Pi N^*C)\} \subseteq \text{ Poisson manifold} \end{aligned}$$

◻

Proposition 6.10. *A submanifold C is coisotropic if and only if $f|_C = 0$ and $g|_C = 0$ implies $\{f,\, g\}|_C = 0$.*

Remark 6.11. Is it true that C is a coisotropic submanifold of M if and only if $C \cap F$ is a coisotropic submanifold of any leaf F of M? To show this is not true take for example $\mathbb{R}^3$, $\Pi = \partial_x \wedge \partial_y$. Symplectic foliation is given by planes parallel to $\{z = 0\}$, $F_h = \{z = h\}$. The standard embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, $x^2 + y^2 + z^2 = 1$ gives a coisotropic submanifold. This can be checked directly proving that functions which are zero on $\mathbb{S}^2$ form a Poisson subalgebra or through the following:.

Exercise 6.12. *Show that every codimension 1 locally closed submanifold is coisotropic.*

Now let us look at intersections:

$$\mathbb{S}^2 \cap F_h = \begin{cases} \emptyset & h \notin [-1,1] \\ * & h \in \{-1,1\} - \text{ is never coisotropic in the leaf to which it belongs} \\ \mathbb{S}^1 & h \in (-1,1) \end{cases}$$

Therefore a submanifold maybe coisotropic without its intersections being coisotropic in the leaves. From this example it is also quite evident the reason for it: the submanifold and the leaves may intersect not transversally. In fact adding suitable transversality conditions it is possible to relate coisotropy to coisotropy in the leaves (see for example [v-i94]).

6.1 Poisson–Morita equivalence

Take (M, Π) to be Poisson, (S, ω) symplectic,

$$\begin{aligned} \#_\Pi &: T^*M \to TM \\ \#_{\omega^{-1}} &: T^*S \to TS \\ \flat_\omega &: TS \to T^*S \end{aligned}$$

Say we have $\rho \colon S \to M$ surjective submersion,

$$\rho_{*,p} \colon T_pS \to T_{\rho(p)}M.$$

For $p \in S$, $x = \rho(p)$, $\rho^{-1}(x)$ is a closed submanifold. We have

$$\begin{aligned} \ker(\rho_{*,p}) &= \{v \in T_pS : \rho_{*,p}(v) = 0\} = T_p\rho^{-1}(x), \\ N_p^*\rho^{-1}(x) &= \{\alpha \in T_p^*S : \langle \alpha, v\rangle = 0 \quad \forall\, v \in T_p\rho^{-1}(x)\} \\ &= \{\alpha \in T_p^*S : \alpha = d_pf,\ f \in \rho^*(C^\infty(M))\}, \\ (\ker(\rho_{*,p}))^{\perp\omega} &= \{w \in T_pS : \omega(v, w) = 0 \quad \forall\, v \in T_p\rho^{-1}(x)\} \\ &= \{w \in T_pS : \langle \flat_\omega(w), v\rangle = 0 \quad \forall\, v \in T_p\rho^{-1}(x)\} \\ &= \{w \in T_pS : \flat_\omega(w) \in N_p^*\rho^{-1}(x)\} \\ &= \#_{\omega^{-1}}(N_p^*\rho^{-1}(x)) \\ &= \{X^\omega_{\rho^*(f)} : f \in \rho^*(C^\infty(M))\}. \end{aligned}$$

Definition 6.13. *Two Poisson manifolds (M_1, Π_1) and (M_2, Π_2) form a* dual pair *if there exists a symplectic manifold (S, ω) and two Poisson submersions (i.e. symplectic realizations)*

$$\begin{tikzcd} & S \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

such that the fibers are symplectic orthogonal, i.e. for any $p \in S$, $\rho_1(p) = x$, $\rho_2(p) = y$

$$T_p\rho_1^{-1}(x) = (T_p\rho_2^{-1}(y))^{\perp\omega}.$$

The pair is called full *if ρ_1, ρ_2 are surjective.*

Remark 6.14. We have seen that in some sense a symplectic realization of M_1 is a notion like "one sided module over M_1". The dual pair is thus a notion of bimodule.

Our task now is to unravel this definition.

Proposition 6.15. *Let (S, ω) with $\rho_i \colon (S, \omega) \to (M, \Pi_i)$, $i = 1, 2$ be a full dual pair. Then*

$$\{\rho_1^*(f), \rho_2^*(g)\}_S = 0 \quad \forall\, f \in C^\infty(M_1),\ g \in C^\infty(M_2). \tag{6.1}$$

Condition 6.1 is equivalent to symplectic orthogonality of tangent spaces if fibers are connected.

Proof.

$$\begin{aligned} \ker((\rho_1)_{*,p}) = T_p\rho_1^{-1}(x) = (T_p\rho_2^{-1}(y))^{\perp\omega} = (\ker((\rho_2))_{*,p})^{\perp\omega} \\ = \{X^\omega_{\rho_2^*(g)}(p) : g \in \rho_2^*(C^\infty(M_2))\}. \end{aligned}$$

Take $f \in C^\infty(M_1)$

$$\{\rho_1^*(f), \rho_2^*(g)\}(p) = -X^\omega_{\rho_2^*(g)}(\rho_1^*(f))(p) = 0$$

because $-X^\omega_{\rho_2^*(g)} \in T\rho^{-1}(x)$ and $\rho_1^*(f)$ is constant along $\rho^{-1}(x)$.
The argument can be reversed provided fibers are connected. ☐

Example 6.16. Let S be a symplectic manifold, $J \colon S \to \mathfrak{g}^*$ constant rank Poisson map. (Moment map, Hamiltonian action of G on S). Assume that J is a surjective submersion and that G-action on S is regular, S/G is a manifold. Then there exists a coinduced Poisson structure on S/G and

$$\begin{tikzcd} & S \arrow[dl, "J"'] \arrow[dr, "p"] & \\ \mathfrak{g}^* & & S/G \end{tikzcd}$$

form a full dual pair.

If regularity is missing one can ask if $p^*(C^\infty(S/G))$ is the Poisson commutant of $J^*(C^\infty(\mathfrak{g}^*))$ (admissible functions),

$$\{C^\infty(S)^G,\ J^*(C^\infty(\mathfrak{g}^*))\} = 0.$$

One can also ask such questions only after restriction to an open subset U of $\mathfrak{g}^*$.

Proposition 6.17. *Let (M_i, Π_i), $i = 1, 2$, be Poisson manifolds. Let (S, ω) be a symplectic manifold. Let $\rho_i \colon S \to M_i$, $i = 1, 2$, form a full dual pair with connected fibers. Then there is a 1-1 correspondence between symplectic leaves of M_1 and M_2, inducing homeomorphism on leaf spaces.*

Proof. The basic idea is the following. Take a leaf F_1 in M_1. Consider $\rho_2(\rho_1^{-1}(F_1))$, which is a leaf in M_2. The correspondence $\Phi \colon F_1 \mapsto \rho_2(\rho_1^{-1}(F_1))$ is bijective and Φ is a homeomorphism.

The details are as follows. Fix $x \in M_1$ and let F_1 be a leaf through x

$$T_x F_1 = \operatorname{im} \#_{\Pi_1, x}.$$

Consider $\rho_1^{-1}(F_1)$ and take $p \in \rho_1^{-1}(x)$. Prove that

$$(\rho_2)_*(T_p \rho^{-1}(F_1)) = \operatorname{im} \#_{\Pi_2, \rho_2(p)}.$$

Indeed

$$\begin{aligned} T_{\rho_1(p)} F_1 &= \{X_f^{\Pi_1}(\rho_1(p)) : f \in C^\infty(M_1)\} \\ &= \{(\rho_1)_{*,p} X_f^\omega(p) : f \in \rho_1^*(C^\infty(M_2))\} \\ &= (\rho_1)_{*,p}(\#_{\omega^{-1}} N^* \ker((\rho_1)_{*,p})). \end{aligned}$$

Take $\mathcal{D}_1$, $\mathcal{D}_2$ be the distributions spanned by Hamiltonian vector fields of pullbacks.

$$\begin{aligned} \mathcal{D}_1 &= \{X^\omega_{\rho_1^* f} : f \in \rho_1^*(C^\infty(M_1))\} \\ \mathcal{D}_2 &= \{X^\omega_{\rho_2^* f} : f \in \rho_2^*(C^\infty(M_2))\} \end{aligned}$$

Take as usual $p \in S$, $\rho_1(p) = x$, $\rho_2(p) = y$.

$\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable disributions. In fact we know also the maximal integral submanifolds - fibers. Surjectivity grants that each point belongs to a fiber. Connectedness grants that the fibers are submanifolds.

Consider the distribution $\mathcal{D}_1 + \mathcal{D}_2$. We claim that it is also integrable. Let F_1 be a symplectic leaf in M_1. Show that $\rho_1^{-1}(F_1)$ is a connected integral sumanifold of $\mathcal{D}_1 + \mathcal{D}_2$.

$$\begin{aligned} T_p(\rho^{-1}(F_1)) &= (\ker \rho_{1,*})_p + \{v \in T_p S : (\rho_{1,*})_p v \in T_{\rho_1(p)} F_1\} \\ &= (\ker \rho_{2,*})_p^{\perp\omega} + \{X^\omega_{\rho_1^* f}(p) : f \in C^\infty(M_1)\} \\ &= \{X^\omega_{\rho_2^* g}(p) : g \in C^\infty(M_2)\} + \{X^\omega_{\rho_1^* f}(p) : f \in C^\infty(M_1)\} \\ &= \mathcal{D}_1 + \mathcal{D}_2. \end{aligned}$$

Now also $\rho_2^{-1}(F_2)$, F_2 symplectic leaf of M_2 are integral submanifolds of $\mathcal{D}_2 + \mathcal{D}_2$.

Let $\mathcal{L}$ be the set of integral submanifolds of $\mathcal{D}_1 + \mathcal{D}_2$. Then $\mathcal{L}$ is in one to one correspondence with the set of leaves of M_1 and the set of leaves of M_2. The bijection on the sets of leaves is given by

$$F_1 \mapsto \rho_2(\rho_1^{-1}(F_1))$$

It remains to show that it is homeomorphism of topological spaces. $\square$

Lemma 6.18. *Let $f\colon S^{(n)} \to M^{(m)}$ be a submersion, $n \geq m$, F is $(m-k)$-dimensional submanifold of M, $x \in S$, $f(x) = y \in F$. Then we can find local coordinates*

$$\begin{aligned} &(U, \phi) \text{ around } x \text{ in } S, \\ &(V, \psi) \text{ around } y \text{ in } M, \end{aligned}$$

such that for all $w \in f^{-1}(V) \cap U$

$$\psi_i(f(w)) = \phi_i(w), \quad i = 1, \ldots, m$$

and

$$f(w) \in F \cap V \Longleftrightarrow \psi_1 = \ldots = \psi_k = 0.$$

Therefore $f^{-1}(F)$ is an $(n-k)$-dimensional submanifold of S given by $\phi_1 = \ldots = \phi_k = 0$.

Definition 6.19 ([x-p91])**.** *Two Poisson manifolds are called* Poisson–Morita equivalent *if there exists a full dual pair (S, ω), ρ_1, ρ_2 between (M_1, Π_1) and $(M_2, -\Pi_2)$ such that*

1. *ρ_1, ρ_2 are complete,*
2. *fibers of ρ_1, ρ_2 are connected, simply connected.*

$$\begin{tikzcd} & S \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

Remark 6.20. Despite its name Poisson–Morita equivalence is not an equivalence relation as it fails to be reflexive. In such cases it is natural to single out the subclass of objects on which a relation indeed defines an equivalence:

Definition 6.21. *Poisson manifolds Poisson–Morita equivalent to themeselves are called* integrable*.*

Reason for the name is that the associated Lie algebroid can be integrated to a Lie grupoid.

Proposition 6.22. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. They are Poisson–Morita equivalent if and only if they have isomorphic fundamental groups.*

In particular any connected and simply connected symplectic manifold is Poisson–Morita equivalent to a point.

Proof. Let

$$\begin{tikzcd} & S \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

be the Poisson–Morita equivalence.

Look at the long exact sequence in homotopy

$$\begin{tikzcd} 0 = \pi_1(\text{fiber}_1) \arrow[r] & \pi_1(S) \arrow[r] \arrow[dr] & \pi_1(M_1) \arrow[r] & \pi_0(\text{fiber}_1) = 0 \\ 0 = \pi_1(\text{fiber}_2) \arrow[ur] & & \pi_1(M_2) \arrow[r] & \pi_0(\text{fiber}_2) \end{tikzcd}$$

Conversely: say $\pi_1(M_1) \simeq \pi_1(M_2) \simeq G$. Let $\widetilde{M_j}$ be the universal cover of M_j, $j = 1, 2$. Both are principal G-bundles over M_j. The product $\widetilde{M_1} \times \overline{\widetilde{M_2}}$ has symplectic structure given by $(\omega_1, -\omega_2)$.

$$\begin{tikzcd} & \widetilde{M_1} \times \overline{\widetilde{M_2}} \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

□

Example 6.23. Let (S, ω) be a connected, simply connected symplectic manifold and let M be a connected manifold with the zero Poisson structure. Then M is Poisson–Morita equivalent to S.

$$\begin{tikzcd} & S \times T^*M \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S & & M \end{tikzcd}$$

where p_1 denotes the projection of $S \times T^*M$ on its first component, while p_2 is the projection on the second component composed with the cotangent bundle projection.

Proposition 6.24 (Lu-Ginzburg)**.** *Poisson–Morita equivalent manifolds have isomorphic first Poisson cohomology* $\mathrm{H}^1_\Pi(-)$*, but can have non-isomorphic* $\mathrm{H}^k_\Pi(-)$*.*

Remark 6.25. With some more work one can prove that the induced map between set of leaves is in fact a heomeomorphism of topological spaces.

Remark 6.26. The first Poisson cohomology and modular class are Poisson–Morita invariants.

6.2 Dirac structures

Definition 6.27. *Let M be a smooth manifold. A* Dirac structure *on M is a subbundle $L \subset TM \oplus T^*M$ which gives pointwise linear Dirac structures in $T_xM \oplus T_x^*M$ and such that its sections are closed under the* Courant bracket

$$[(X, \alpha), (Y, \beta)] = ([X, Y], L_X\beta - L_Y\alpha + \tfrac{1}{2}d(\alpha(Y) - \beta(X))) \qquad (6.2)$$

Remark 6.28. The Courant bracket is not a Lie bracket. However it turns out to be a Lie bracket on sections of a Dirac bundle.

Proposition 6.29. *Let* $\Pi \in \Gamma(\Lambda^2 TM)$ *be a bivector on* M*. Then* graph(Π) *defines a subbundle of* $TM \oplus T^*M$ *which is pointwiese a linear Dirac structure;* $\Gamma(\Pi)$ *is a Dirac structure if and only if* Π *is Poisson.*

Remark 6.30. Not every Dirac structure comes from a Poisson bivector.

Proof. For any $\Pi \in \Gamma(\Lambda^2 TM)$ define

$$\Gamma_\Pi = \{(\#_\Pi(\alpha), \alpha) : \alpha \in \Omega^1(M)\}$$

This is pointwise Dirac.

$$\begin{aligned}[(\#_\Pi(\alpha), \alpha), (\#_\Pi(\beta), \beta)] = ([&\#_\Pi(\alpha), \#_\Pi(\beta)], \\ &L_{\#_\Pi(\alpha)}\beta - L_{\#_\Pi(\beta)}\alpha + \tfrac{1}{2} d(\alpha(\#_\Pi(\beta)) - \beta(\#_\Pi(\alpha)))) \\ = ([&\#_\Pi(\alpha), \#_\Pi(\beta)], [\alpha, \beta]_\Pi)\end{aligned}$$

Now the point is that $\#_\Pi$ is a Lie algebra map if and only if $[\Pi, \Pi]_{\mathrm{SN}} = 0$. □

Proposition 6.31. *Let* B *be a skewsymmetric bilinear form on* V*,* $B \in \Lambda^2 V^*$*. Then for any linear Dirac structure* L

$$\mathcal{C}_B(L) := \{(v, \mu + Bv) : (v, \mu) \in L\}$$

is a linear Dirac structure.

Proof. Dimension is obviously unchanged. Therefore it suffices to show isotropy

$$\begin{aligned}((v, \mu + Bv), (w, \eta + Bw)) &= \tfrac{1}{2}((\mu + Bv)(w) + (\eta + Bw)(v)) \\ &= \tfrac{1}{2}(\mu(w) + \eta(v)) + \tfrac{1}{2}(\underbrace{B(v, w) + B(w, v)}_{=0}) \\ &= ((v, \mu), (w, \eta)) = 0.\end{aligned}$$

□

Proposition 6.32. *Let* $\Pi \in \Lambda^2 V$ *and let* Γ_Π *be the linear Dirac structure corresponding to the graph of* Π*. Let* $B \in \Lambda^2 V^*$*. Then there exists* $\Pi' \in \Lambda^2 V$ *such that*

$$\mathcal{C}_B(\Gamma_\Pi) = \Gamma_{\Pi'} \iff (\mathrm{id} + \flat_B \circ \#_\Pi) \textit{ is invertible.}$$

Here

$$\flat_B \colon V \to V^*,\ \flat_B(v) = B(v, -),$$
$$\#_\Pi \colon V^* \to V,\ \#_\Pi(\xi) = \Pi(\xi, -)$$

with the identifcation $V \simeq V^{**}$*.*

Proof.

$$\mathcal{C}_B(\Gamma_\Pi) = \Gamma_{\Pi'} \iff \mathcal{C}_B(\Gamma_\Pi) \cap V = \{0\}$$

Now $\mathrm{id}\flat_B \circ \#_\Pi \colon V^* \to V^*$ is invertible if and only if it is injective, therefore

$$\alpha + B(\#_\Pi(\alpha)) = 0,\ \alpha \neq 0 \iff \mathrm{id} + \flat_B \circ \#_\Pi \text{ is injective.}$$

□

Proposition 6.33. *Let L be a Dirac structure on M and let $B \in \Omega^2(M)$. Then*

$$\mathcal{C}_B(L) \text{ is Dirac } \iff dB = 0$$

Proof. As we have already seen $\mathcal{C}_B(L)$ is pointwise a linear Dirac structure. We have to show what happens if we require in $\mathcal{C}_B(L)$ closeness with respect to the Courant bracket.

$$\begin{aligned}
[(X, \eta + B(X)), (Y, \xi + B(Y))] &= ([X, Y], L_X(\mu + B(Y)) - L_Y(\omega + B(X)) \\
&\quad + \tfrac{1}{2} d((\omega + B(X))(Y) - (\mu + B(Y))(X))) \\
&= ([X, Y], L_X\mu - L_Y\omega + \tfrac{1}{2} d(\omega(Y) - \mu(X)) \\
&\quad + L_X B(Y) - L_Y B(X) + d(B(X, Y)))
\end{aligned}$$

Lemma 6.34.

$$L_X B(Y) - L_Y B(X) + d(B(X, Y)) = (dB)(X, Y) - B([X, Y])$$

Proof.

$$L_X = di_X + i_X d, \;\; L_Y = di_Y + i_Y d$$

$$L_X B(Y) = d(B(Y, X)) + i_X(dB(Y))$$

$$L_Y B(X) = d(B(X, Y)) + i_Y(dB(X))$$

$$(i_X d(B(Y)))(Z) = \langle d(B(Y)),\, X \wedge Z\rangle = ZB(X, Y) - XB(Y, Z) - B(Y, [X, Z])$$

Use formula for

$$\begin{aligned}
(dB)(X, Y, Z) = XB(Y, Z) &- YB(X, Z) + ZB(X, Y) \\
&- B([X, Y], Z) + B([X, Z], Y) - B([Y, Z], X).
\end{aligned}$$

□

□

Definition 6.35. *Two Poisson bivectors Π_1, Π_2 on the manifold M are said to be* gauge equivalent *if there exists a closed 2-form B such that*

$$\mathcal{C}_B(\Gamma_{\Pi_1}) = \Gamma_{\Pi_2}$$

(i.e. if the corresponding Dirac structures are equivalent)

Two Poisson manifolds (M_1, Π_1) and (M_2, Π_2) are said to be gauge equivalent up to diffeomorphism *if there exists a Poisson diffeomorphism*

$$\varphi\colon (M_1, \Pi_1) \to (M_2, \Pi_0)$$

such that Π_0 and Π_2 are gauge equivalent.

Remark 6.36. Two symplectic structures on a given manifold are gauge equivalent. Two symplectic manifolds are gauge equivalent up to diffeomorphism if and only if they are symplectomorphic.

Chapter 7

Poisson Lie groups

7.1 Poisson Lie groups

Recall the two presentations of a Poisson manifold:

1. $\{-,-\}\colon C^\infty(M)\otimes C^\infty(M)\to C^\infty(M)$ such that
   - $\{-,-\}$ is a Lie bracket (antisymmetric + Jacobi identity)
   - $\{f,\, gh\} = \{f,\, g\}h + g\{f,\, h\}$ (Leibniz rule)
2. (M,Π), $\Pi\in\Gamma(\Lambda^2 TM)$ such that $[\Pi,\Pi]=0$

connected by the equality

$$\{f,\, g\}(x) = \langle \Pi(x),\, d_x f\otimes d_x g\rangle$$

Recall that a smooth map $\phi\colon M\to N$ between Poisson manifolds is a map that preserves s

$$\{f_1,\, f_2\}_M\circ\phi = \{f_1\circ\phi,\, f_2\circ\phi\}_N$$

or equivalently

$$\phi_{*,x}^{\otimes 2}\Pi_M(x) = \Pi_N(\phi(x))$$

Recall also that if M, N are Poisson manifolds the structure of product Poisson manifold on $M\times N$ is the one given by

$$\{f_1,\, f_2\}_{M\times N}(x,y) = \{f_1(-,y),\, f_2(-,y)\}_M(x) + \{f_1(x,-),\, f_2(x,-)\}_N(y)$$

or equivalently

$$\Pi_{M\times N} = \Pi_M\oplus\Pi_N\in\Gamma(\Lambda^2 T(M\times N)) = \Gamma(\Lambda^2 TM\oplus\Lambda^2 TN)$$

Proposition 7.1. *Let G be a Lie group, Π Poisson tensor on G. Then the following are equivalent:*

1. *The product $m\colon G\times G\to G$ is a Poisson map*
2. $$\Pi(g_1g_2) = L_{g_1,*}\Pi(g_2) + R_{g_2,*}\Pi(g_1),$$

   where

   $$L_g\colon G\to G,\quad h\mapsto gh,\qquad R_g\colon G\to G,\quad h\mapsto hg$$

   and $L_{g,*}$, $R_{g,*}$ are derivatives.

Proof. Let $m\colon G \times G \to G$ be a Poisson map, that is

$$\{f_1,\, f_2\}(m(g_1, g_2)) = \{f_1 \circ m,\, f_2 \circ m\}_{G\times G}(g_1, g_2)$$

i.e.

$$\{f_1,\, f_2\}(g_1 g_2) = \{f_1 \circ L_{g_1},\, f_2 \circ L_{g_1}\}(g_2) + \{f_1 \circ R_{g_2},\, f_2 \circ R_{g_2}\}(g_1)$$

or equivalently

$$\langle \Pi(g_1 g_2),\, d_{g_1 g_2} f_1 \otimes d_{g_1 g_2} f_2\rangle =$$
$$\langle \Pi(g_2),\, d_{g_2}(f_1 \circ L_{g_1}) \otimes d_{g_2}(f_2 \circ L_{g_1})\rangle + \langle \Pi(g_1),\, d_{g_1}(f_1 \circ R_{g_2}) \otimes d_{g_1}(f_2 \circ R_{g_2})\rangle$$

Now use

$$d_g(f \circ L_h) = L_{h,*} d_g f, \quad d_g(f \circ R_h) = R_{h,*} d_g f$$

to obtain

$$\begin{aligned}
\langle \Pi(g_1 g_2),\, d_{g_1 g_2} f_1 \otimes \Pi_{g_1 g_2} f_2\rangle &= \langle \Pi(g_2),\, L^{\otimes 2}_{g_1,*}(d_{g_2} f_1 \otimes d_{g_2} f_2)\rangle \\
&\quad + \langle \Pi(g_1),\, R^{\otimes 2}_{g_2,*}(d_{g_1} f_1 \otimes d_{g_1} f_2)\rangle \\
&= \langle L^{\otimes 2}_{g_1,*}\Pi(g_2),\, d_{g_2} f_1 \otimes d_{g_2} f_2\rangle \\
&\quad + \langle R^{\otimes 2}_{g_2,*}\Pi(g_1),\, d_{g_1} f_1 \otimes d_{g_1} f_2\rangle
\end{aligned}$$

hence the thesis. □

Definition 7.2. *When one of the conditions of proposition (7.1) is verified* (G, Π) *is called a* Poisson Lie group.

Remarks 7.3.

- For a Poisson Lie group (G, Π) we have $\Pi(e) = 0$. In fact $\Pi(ee) = 2\Pi(e)$.
- $$0 = \Pi(e) = \Pi(gg^{-1}) = L_{g,*}\Pi(g^{-1}) + R_{g^{-1},*}\Pi(g)$$

  so

  $$\Pi(g^{-1}) = -\operatorname{Ad}_{g^{-1},*}\Pi(g)$$

  This means that the inverse $g \mapsto g^{-1}$ is not a Poisson map, but anti-Poisson.
- Another equivalent condition is

  $$L_X L_Y \Pi = 0, \quad \forall\, X \text{ right invariant, and } Y \text{ left invariant}$$

  and additionally $\Pi(e) = 0$.

  This obviously suggests what if $\Pi(e) \neq 0$? We have

  $$\Pi(g_1 g_2) = L_{g_1}\Pi(g_2) + R_{g_2}\Pi(g_1) + L_{g_1} R_{g_2}\Pi(e)$$

  what is called an *affine Poisson structure on G*.

Let us move on to the infinitesimal description of the Poisson Lie groups. Consider

$$\eta\colon G \to \Lambda^2\mathfrak{g}$$

given by right translating the Poisson tensor

$$\eta(g) = R_{g^{-1},*}\Pi(g)$$

(obviously $\eta(e) = 0$). Now

$$\begin{aligned}
\eta(g_1g_2) &= R_{(g_1g_2)^{-1},*}\Pi(g_1g_2) \\
&= R_{g_1^{-1},*}R_{g_2^{-1},*}(L_{g_1,*}\Pi(g_2) + R_{g_2,*}\Pi(g_1)) \\
&= R_{g_1^{-1},*}\Pi(g_1) + \mathrm{Ad}_{g_1} R_{g_2^{-1},*}\Pi(g_2) \\
&= \eta(g_1) + \mathrm{Ad}_{g_1}\eta(g_2)
\end{aligned}$$

i.e. Π multiplicative $\implies$ η is a cocycle of G with values in $\Lambda^2\mathfrak{g}$.

Define now

$$\delta\colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$$

to be its derivative at e, i.e.

$$\delta(X) := \frac{d}{dt}\eta(e^{tX})|_{t=0}.$$

What are the properties of δ coming from the fact that Π is Poisson and multiplicative?

Proposition 7.4.

1. Π *multiplicative* $\implies$

$$\delta([X,Y]) = \mathrm{ad}_X\,\delta(Y) - \mathrm{ad}_Y\,\delta(X).$$

2. Π *Poisson* $\implies$ ${}^t\delta\colon \Lambda^2\mathfrak{g}^* \to \mathfrak{g}$ *satisfies Jacobi identity.*

Proof.

1.

$$\begin{aligned}
\eta(e^{tX}e^{tY}) &= \eta(e^{tX}) + \mathrm{Ad}_{e^{tX}}\,\eta(e^{tY}) \\
\eta(e^{tY}e^{tX}) &= \eta(e^{tY}) + \mathrm{Ad}_{e^{tY}}\,\eta(e^{tX}) \\
\eta(e^{tX}e^{tY}) - \eta(e^{tY}e^{tX}) &= \eta(e^{tX}) - \eta(e^{tY}) + \mathrm{Ad}_{e^{tX}}\,\eta(e^{tY}) - \mathrm{Ad}_{e^{tY}}\,\eta(e^{tX})
\end{aligned}$$

2.

Lemma 7.5. *Let $\xi_1, \xi_2 \in \mathfrak{g}^*$. Choose $f_1, f_2 \in C^\infty(G)$ such that $d_e f_i = \xi_i$, $i = 1, 2$. Then*

$${}^t\delta(\times_1, \xi_2) = d_e\{f_1,\, f_2\}$$

Proof.

$$\begin{aligned}
\{f_1,\, f_2\}(g) &= \langle \Pi(g),\, d_g f_1 \otimes d_g f_2\rangle \\
&= \langle \eta(g),\, R_{g,*}^{\otimes 2}(d_g f_1 \otimes d_g f_2)\rangle
\end{aligned}$$

Take $g = e^{tX}$ and the derivative at $t = 0$.

$$\underbrace{\frac{d}{dt}\{f_1,\, f_2\}(e^{tX})|_{t=0}|_{t=0}}_{\langle X,\, d_e\{f_1, f_2\}\rangle} = \underbrace{\frac{d}{dt}\langle \eta(g^{tX}),\, R^{\otimes 2}_{e^{tX},*}(d_{e^{tX}} f_1 \otimes d_{e^{tX}} f_2)\rangle|_{t=0}}_{\langle \frac{d}{dt}\eta e^{tX}|_{t=0},\, d_e f_1 \otimes d_e f_2\rangle}$$

$$= \langle \delta(X),\, d_e f_1 \otimes d_e f_2\rangle = \langle X,\, {}^t\delta(d_e f_1, d_e f_2)\rangle$$

Thus the claim. Remark that this proves indirectly independence of the right hand side from choices. ☐

Now the statement follows easily from

$$\mathrm{Jac}_{{}^t\delta}(\xi_1, \xi_2, \xi_3) = d_e\, \mathrm{Jac}_{\{-,-\}}(f_1, f_2, f_3)$$

☐

7.2 Lie bialgebras

Definition 7.6. *A* Lie bialgebra *is a pair* $(\mathfrak{g}, \delta)$ *where* $\mathfrak{g}$ *is a Lie algebra and* $\delta\colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$ *is such that*

1. ${}^t\delta$ *satisfies Jacobi identity (coJacobi:* $\mathrm{cyclic}((\delta \otimes \mathrm{id})\delta(X)) = 0$*)*
2. $\delta([X, Y]) = \mathrm{ad}_X\, \delta(Y) - \mathrm{ad}_Y\, \delta(X)$

We have just proven that the tangent space of a Poisson Lie group has a Lie bialgebra structure. To what extend is the converse true?

Example 7.7. $\mathfrak{g}$ abelian Lie algebra. Thus any $\delta\colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$ such that ${}^t\delta$ satisfies Jacobi identity gives a Lie bialgebra. Choose a nontrivial one. Therefore $\mathfrak{g}^*$ is a non trivial Lie algebra, which implies that $\mathfrak{g}$ itself has a non trivial Poisson linear structure ($\mathfrak{g} \cong \mathfrak{g}^{**}$).

Take $\Gamma \in \mathfrak{g}$ a lattice under which the Poisson structure is not invariant. Take a Lie group $H = \mathfrak{g}/G$. Then $\mathrm{Lie}(H) = \mathfrak{g}$ is a Lie bialgebra which does not integrate to a Poisson Lie group structure.

The point here is

Lemma 7.8. *Given a 1-cocycle* $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ *there is a unique 1-cocycle* $\eta\colon G \to \mathfrak{g} \wedge \mathfrak{g}$*, where* G *is the connected simply connected Lie group integrating* $\mathfrak{g}$ *(i.e.* $\mathrm{Lie}(G) = \mathfrak{g}$*).*

Basically this is all you need to prove

Theorem 7.9 (Drinfel'd)**.** *The correspondence* $G \mapsto \mathfrak{g}$ *gives you a 1:1 correspondence between Lie bialgebras and Poisson Lie groups.*

Given any Poisson Lie group (G, Π) consider its Lie bialgebra $(\mathfrak{g}, \delta)$. Then $(\mathfrak{g}^*, {}^t[-,-])$ is a Lie bialgebra. Therefore it integrates to a unique connected, simply connected Poisson Lie group G^* called the *dual Poisson Lie group* of G.

Lie bialgebras form a category. Morphisms are those homomorphisms which respect δ

$$\begin{tikzcd} \mathfrak{g} \wedge \mathfrak{g} \arrow[r, "\chi\otimes\chi"] & \mathfrak{g}' \wedge \mathfrak{g}' \\ \mathfrak{g} \arrow[u, "\delta"] \arrow[r, "\chi"'] & \mathfrak{g}' \arrow[u, "\delta'"'] \end{tikzcd}$$

Proposition 7.10. *Given a Lie bialgebra $(\mathfrak{g}, \delta)$, the vector space $\mathfrak{g}^*$ has a canonical Lie bialgebra structure. The cobracket δ' being dual to bracket $[-,-]$ in $\mathfrak{g}$, and the bracket $[-,-]'$ in $\mathfrak{g}^*$ being dual to δ.*

Definition 7.11. *$\mathfrak{g}^*$ is called dual bialgebra of $\mathfrak{g}$.*

Examples 7.12.

1. Any Lie algebra with $\delta = 0$.
2. Dual of previous, $\mathfrak{g}^*$ as vector space, $[-,-] = 0$, $\delta' = [-,-]^*_{\mathfrak{g}}$.
3. $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $X^+, X^-, H \in \mathfrak{sl}(2,\mathbb{C})$

$$[H, X^{\pm}] = \pm 2X^{\pm}, \quad [X^+, X^-] = H$$

$$\delta(X^{\pm}) = X^{\pm} \wedge H, \quad \delta(H) = 0$$

$$\text{cyclic}(\delta \otimes \text{id}) \circ \delta(X^{\pm}) = \text{cyclic}(\delta \otimes \text{id})(\underbrace{X^{\pm} \wedge H}_{X^{\pm}\otimes H - H \otimes X^{\pm}})$$

$$= \text{cyclic}(X^{\pm} \otimes H - H \otimes X^{\pm}) \otimes H = 0$$

(co-Jacobi identity). Now check the 1-cocycle condition

$$\delta([a,b]) = a \cdot \delta(b) - b \cdot \delta(a)$$

We have

$$\delta([H, X^{\pm}]) \stackrel{?}{=} H \cdot (X^{\pm} \wedge H) - X^{\pm} \cdot \delta(H)$$

$$LHS = \pm 2\delta(X^{\pm}) = \pm 2X^{\pm} \wedge H$$

$$RHS = [H, X^{\pm}] \wedge H + X^{\pm} \wedge [H, H] = \pm 2X^{\pm} \wedge H$$

Similarly

$$\delta([X^+, X^-]) \stackrel{?}{=} X^+ \cdot \delta(X^-) - X^- \cdot \delta(X^+)$$

$$LHS = \delta(H) = 0$$

$$RHS = X^+ \cdot X^- \wedge H - X^- \cdot X^+ \wedge H$$

$$= [X^+, X^-] \wedge H + X^- \wedge [X^+, H] - [X^-, X^+] \wedge H - X^+ \wedge [X^-, H]$$

$$= H \wedge H - 2X^- \wedge X^+ - 2X^+ \wedge X^- = 0$$

4. Let $\mathfrak{g}$ be a $\mathbb{C}$ simple Lie algebra with fixed bilinear, nondegenerate, symmetric form $(-,-)$ on $\mathfrak{g}$ (and on $\mathfrak{g}^*$). choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ ($n = \dim \mathfrak{h}$ is the rank of $\mathfrak{g}$). Choose a simple roots $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$. This gives a decomposition
$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$
where $\mathfrak{n}_\pm$ are nilpotent and $\mathfrak{h}$ abelian. Let $X_i^\pm$, H_i be the corresponding Chevalley generators and $A = [a_{ij}]$ the Cartan matrix
$$a_{ij} = \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$
Recall
$$[H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, \quad [X_i^+, X_i^-] = \delta_{ij} H_j$$
The following cobracket
$$\delta(H_i) = 0, \quad \delta(X_i^\pm) = d_i X^\pm \wedge H_i,$$
where d_i symmetrize $[a_{ij}]$, i.e. $d_i a_{ij} = a_{ij} d_j$, gives the structure of a Lie bialgeba.

Definition 7.13. *This example is called a* standard Lie bialgebra *structure on* $\mathfrak{g}$.

Remark. There exist other structures, and all standard structures are equivalent up to conjugation.

7.3 Manin triples

Definition 7.14. *Let $\mathfrak{g}$ be a Lie algebra with a non degenerate invariant symmetric bilinear form. Let $\mathfrak{g}_+$ and $\mathfrak{g}_-$ be Lie subalgebras such that*
$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$
as vector spaces, and such that $\mathfrak{g}_+, \mathfrak{g}_-$ are maximal isotropic subspaces of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is called a Manin triple.

Using the form we can identify
$$\mathfrak{g}_- \cong (\mathfrak{g}_+)^*, \quad \mathfrak{g}_+ \cong (\mathfrak{g}_-)^*$$
In particular $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$.

Theorem 7.15.

1. *Suppose $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. Let*
$$[-,-] \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \quad [-,-]_+ = [-,-]|_{\mathfrak{g}_+ \otimes \mathfrak{g}_+}, \quad [-,-]_- = [-,-]|_{\mathfrak{g}_- \otimes \mathfrak{g}_-}$$
Put
$$\delta_+ ={}^t [-,-]_- \colon (\mathfrak{g}_-^*) = \mathfrak{g}_+ \to \Lambda^2 \mathfrak{g}_+, \quad \delta_- ={}^t [-,-]_+ \colon (\mathfrak{g}_+^*) = \mathfrak{g}_- \to \Lambda^2 \mathfrak{g}_-$$
Then $(\mathfrak{g}_+, \delta_+)$ and $(\mathfrak{g}_-, \delta_-)$ are Lie bialgebras.

2. Let $(\mathfrak{g}, \delta)$ *be a Lie bialgebra. Define on* $\mathfrak{g} \oplus \mathfrak{g}^*$

$$\langle X + \xi,\, Y + \eta\rangle := \xi(Y) + \eta(X)$$

$$[X + \xi, Y + \eta] = \langle [X, Y] + \mathrm{ad}^*_X \xi - \mathrm{ad}^*_Y \eta,\, [\xi, \eta] + \mathrm{ad}^*_\xi X - \mathrm{ad}^*_\eta Y\rangle$$

Then $\mathfrak{g} \oplus \mathfrak{g}^*$ *with this form and bracket is a Manin triple.*

Proof. Let us rewrite the cocycle condition in a Lie bialgebra

$$\langle \delta([X, Y]),\, \xi \otimes \eta\rangle = \langle [X, Y],\, [\xi, \eta]\rangle$$

Indeed,

$$\begin{aligned}
\langle \delta([X, Y]),\, \xi \otimes \eta\rangle &= \langle \mathrm{ad}_X \delta(Y) - \mathrm{ad}_Y \delta(X),\, \xi \otimes \eta\rangle \\
&= -\langle \delta(Y),\, \mathrm{ad}^*_X(\xi \otimes \eta)\rangle + \langle \delta(X),\, \mathrm{ad}^*_Y(\xi \otimes \eta)\rangle \\
&= -\langle \delta(Y),\, \mathrm{ad}^*_X(\xi) \otimes \eta + \xi \otimes \mathrm{ad}^*_X \eta\rangle \\
&\quad + \langle \delta(X),\, \mathrm{ad}^*_Y(\xi) \otimes \eta + \xi \otimes \mathrm{ad}^*_Y \eta\rangle \\
&= \langle Y,\, [\mathrm{ad}^*_X \xi, \eta] + [\xi, \mathrm{ad}^*_X \eta]\rangle - \langle X,\, [\mathrm{ad}^*_Y \xi, \eta] + [\xi, \mathrm{ad}^*_Y \eta]\rangle \\
&= \langle \mathrm{ad}^*_\eta Y,\, \mathrm{ad}^*_X \xi\rangle - \langle \mathrm{ad}^*_\xi Y,\, \mathrm{ad}^*_X \eta\rangle \\
&\quad - \langle \mathrm{ad}^*_\eta X,\, \mathrm{ad}^*_Y \xi\rangle + \langle \mathrm{ad}^*_\xi X,\, \mathrm{ad}^*_Y \eta\rangle
\end{aligned}$$

Invariance of bilinear form is equivalent to

$$[\xi, X] = \mathrm{ad}^*_\xi X - \mathrm{ad}^*_X \xi$$

$$\langle [\xi, X],\, \eta\rangle = (\xi, [X, \eta]) = -\langle \mathrm{ad}^*_X \xi,\, \eta\rangle, \quad \forall\, \eta$$

$$\langle [\xi, X],\, Y\rangle = \langle \xi,\, [X, Y]\rangle = -\langle \mathrm{ad}^*_X \xi,\, Y\rangle, \quad \forall\, Y$$

Therefore

$$\begin{aligned}
\langle \delta([X, Y]),\, \xi \otimes \eta\rangle &= \langle [X, Y],\, [\xi, \eta])\rangle \\
&= -\langle X,\, [Y, [\xi, \eta]]\rangle \\
&= -\langle X,\, [\eta, [Y, \xi]] + [\xi, [\eta, Y]]\rangle \qquad \text{(from Jacobi identity)} \\
&= -\langle X,\, [\eta, \mathrm{ad}^*_Y \xi - \mathrm{ad}^*_\xi Y] + [\xi, \mathrm{ad}^*_\eta Y - \mathrm{ad}^*_Y \eta]\rangle \\
&= \langle \mathrm{ad}^*_\eta Y,\, \mathrm{ad}^*_X \xi\rangle - \langle \mathrm{ad}^*_\xi Y,\, \mathrm{ad}^*_X \eta\rangle \\
&\quad - \langle \mathrm{ad}^*_\eta X,\, \mathrm{ad}^*_Y \xi\rangle + \langle \mathrm{ad}^*_\xi X,\, \mathrm{ad}^*_Y \eta\rangle
\end{aligned}$$

and this is formula obtained before. This proves (1). □

Proposition 7.16. *Let* $(\mathfrak{g}, \delta)$ *be a Lie bialgebra,* $(D\mathfrak{g}, [-,-])$ *Lie algebra in* $\mathfrak{g} \oplus \mathfrak{g}^*$*. Then*

$$\delta \colon D\mathfrak{g} \to \Lambda^2 D\mathfrak{g}$$

given by

$$\delta(X + \xi) = \delta(X) +^t [-,-](\xi)$$

is a Lie cobracket.

Example 7.17. $\mathfrak{g}$ complex simple Lie bialgebra, $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ diagonal embedding. Fix $\mathfrak{h}$ Cartan subalgebra and choice of positive roots.

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

$$\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$$

$$S := \{(x, y) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- : x|_\mathfrak{h} = -y|_\mathfrak{h}\}$$

Let on $\mathfrak{g} \oplus \mathfrak{g}$

$$\langle (a, b),\, (c, d) \rangle = \langle a,\, c\rangle - \langle b,\, d\rangle$$

Then $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, S)$ is a Manin triple.

Example 7.18. With the notation as before $(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-)$ is a Manin triple.

Chapter 8

Poisson actions

Recall some notations. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, $L_g, R_g \colon G \to G$ left and right translations with derivatives $L_{g,*} \colon T_hG \to T_{gh}G$, $R_{g,*} \colon T_hG \to T_{hg}G$.

Let (G, Π) be a Poisson Lie group, i.e.

$$\Pi(g_1 \cdot g_2) = L_{g_1,*}\Pi(g_2) + R_{g_2,*}\Pi(g_1)$$

and let $\eta \colon G \to \Lambda^2 T_eG = \Lambda^2\mathfrak{g}$ be

$$\eta(g) = R_{g^{-1},*}\Pi(g).$$

Then η is a 1-cocycle of G with respect to adjoint action on $\Lambda^2\mathfrak{g}$, i.e.

$$\eta(g_1g_2) = \eta(g_1) + \mathrm{Ad}_{g_1}\eta(g_2)$$

Let $\delta \colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$,

$$\delta(X) = \frac{d}{dt}\eta(e^{tX})\big|_{t=0}$$

Then $(\mathfrak{g}, \delta)$ is a Lie bialgebra

$$\begin{aligned} (\mathfrak{g}, [-,-]) &\quad \text{is Lie} \\ (\mathfrak{g}^*, {}^t\delta) &\quad \text{is Lie} \end{aligned}$$

satisfying compatibility

$$\delta([X,Y]) = \mathrm{ad}_X\,\delta(Y) - \mathrm{ad}_Y\,\delta(X).$$

The Lie algebra $\mathfrak{g}^*$ integrates to a (unique) connected (simply connected) Poisson Lie group G^*. Furthermore on $\mathfrak{g} \oplus \mathfrak{g}^*$ we have the following Lie bracket

$$[X + \xi, Y + \eta] = ([X,Y] + \mathrm{ad}^*_X\,\eta - \mathrm{ad}^*_Y\,\xi, [\xi,\eta] + \mathrm{ad}^*_\xi\,Y - \mathrm{ad}^*_\eta\,X)$$

and Lie cobracket

$$\delta_D(X + \xi) = \delta(X) + \delta^*(\xi)$$

This makes $\mathfrak{g}\oplus\mathfrak{g}^*$ a Lie bialgebra, which is called *Drinfeld double of a Lie algebra* $\mathfrak{g}$. It integrates to (a unique sonnected, simply connected) Poisson Lie group DG called *Drinfeld double of a Lie group G*.

8.1 Poisson actions

Definition 8.1. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. Let (M, Π) be a Poisson manifold, together with an infinitesimal action, i.e. a Lie algebra morphism*

$$\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$$

Then ρ is called an infinitesimal Poisson action *if*

$$L_{\rho(X)}\Pi = \rho^{\wedge 2}(\delta(X)), \quad \forall\, X \in \mathfrak{g} \tag{8.1}$$

Remark 8.2.

1. Π is not invariant under an infinitesimal Poisson action. If the infinitesimal action is effective it is invariant if and only if $\delta = 0$.
2. To be precise this could be considered an infinitesimal left Poisson action. An infinitesimal right Poisson action is then a Lie algebra antihomomorphism such that (8.1) is verified.

Let now $\phi\colon G \times M \to M$ be a Lie group action. Let us fix the following notations

$$\phi(g, x) = g \cdot x$$
$$\forall\, g \in G, \quad \phi_g\colon M \to M, \quad x \mapsto g \cdot x$$
$$\forall\, x \in M, \quad \phi_x\colon G \to M, \quad g \mapsto g \cdot x$$

Remark that

$$\phi_{g\cdot x} = \phi_x R_g, \quad \phi_g \phi_x = \phi_x L_g.$$

For $f \in C^\infty(M)$ let $\theta_f\colon M \to \mathfrak{g}^*$ be defined by

$$\theta_f(x) = d_g f(g \cdot x)\big|_{g=e}$$

If we have only the infinitesimal action we can define equivalently

$$\langle \theta_f, Y\rangle = \rho(Y) f$$

Theorem 8.3 (Semonov-Tian-Shanskii)**.** *Let (G, Π_G) be a connected, simply connected, Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta)$. Let (M, Π_M) be a Poisson manifold. Let $\phi\colon G \times M \to M$ be a Lie group action with infinitesimal map $\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$. then following are equivalent*

1. *ϕ is a Poisson map with respect to the product Poisson structure;*
2. *for all $x \in M$, $g \in G$*
$$\Pi_M(g \cdot x) = \phi_{g,*}\Pi_M(x) + \phi_{x,*}\Pi_G(g);$$
3. *for all $X \in \mathfrak{g}$ and $f, g \in C^\infty(M)$*
$$\rho(X)\{f, g\} - \{\rho(X)f, g\} - \{f, \rho(X)g\} = \langle [\theta_f, \theta_g], X\rangle;$$
4. *ρ is an infinitesimal Poisson action, i.e. for all $X \in \mathfrak{g}$*
$$L_{\rho(X)}\Pi_M = \rho^{\wedge 2}(\delta(X)).$$

Proof. (1) $\iff$ (2) by definition of product Poisson structure ϕ is Poisson if and only if for all $f_1, f_2 \in C^\infty(M)$, $g \in G$, $x \in M$

$$\{f_1 \circ \phi,\, f_2 \circ \phi\}_{G\times M}(g,x) = \{f_1,\, f_2\}_M(g \cdot x)$$

But the left hand side equals

$$\{f_1 \circ \phi_x,\, f_2 \circ \phi_x\}_G(g) + \{f_1 \circ \phi_g,\, f_2 \circ \phi_g\}_M(x)$$
$$= \langle \Pi_G(g),\, \phi_x^* d_{g\cdot x} f_1 \wedge \phi_c^* d_{g\cdot x} f_2\rangle + \langle \Pi_M(x),\, \phi_g^* d_{g\cdot x} f_1 \wedge \phi_g^* d_{g\cdot x} f_2\rangle$$
$$= \langle \phi_{x,*}\Pi_G(g),\, d_{g\cdot x} f_1 \wedge d_{g\cdot x} f_2\rangle + \langle \phi_{g,*}\Pi_M(x),\, d_{g\cdot x} f_1 \wedge d_{g\cdot x} f_2\rangle$$

and the right hand side is

$$\langle \Pi_M(g\cdot x),\, d_{g\cdot x} f_1 \wedge d_{g\cdot x} f_2\rangle$$

Hence (1) $\iff$ (2).
(3) $\iff$ (4)

$$L_{\rho(X)}\Pi_M = \rho^{\wedge 2}(\delta(X)) = \langle \delta(X),\, \theta_f \wedge \theta_g\rangle$$
$$\iff \langle L_{\rho(X)}\Pi_M,\, df \wedge dg\rangle = \langle \rho^{\wedge 2}(X),\, df \wedge dg\rangle$$
$$\iff$$
$$L_{\rho(X)}\langle \Pi_M,\, df \wedge dg\rangle = \langle \Pi_M,\, (L_{\rho(X)df}) \wedge dg\rangle - \langle \Pi_M,\, df \wedge L_{\rho(X)} dg\rangle$$
$$= \langle \delta(X),\, \theta_f \wedge \theta_g\rangle$$
$$\iff \rho(X)\{f,\, g\} - \{\rho(X)f,\, g\} - \{f,\, \rho(X)g\} = \langle X,\, [\theta_f, \theta_g]\rangle$$

because

$$\langle \delta(X),\, \theta_f \wedge \theta_g\rangle = \langle X,\, [\theta_f, \theta_g]\rangle.$$

(2) $\implies$ (4) by applying $\phi_{g^{-1},*}$ to both sides of (2) we have

$$\phi_{g^{-1},*}\Pi_M(g\cdot x) = \Pi_M(x) + \phi_{g^{-1},*}\phi_{x,*}\Pi_G(g)$$
$$\phi_{g^{-1},*}\Pi_M(g\cdot x) = \Pi_M(x) + \phi_{x,*}L_{g^{-1},*}\Pi_G(g)$$

Now let $g = e^{tX}$, $X \in \mathfrak{g}$ and differentiate with respect to t at $t = 0$

$$\frac{d}{dt}\phi_{e^{-tX},*}\Pi_M(e^{tX}x)\big|_{t=0} = L_{\rho(X)}\Pi_M$$

$$\frac{d}{dt}\phi_{x,*}L_{e^{-tX},*}\Pi_G(e^{tX})\big|_{t=0} = \phi_{x,*}\frac{d}{dt}L_{e^{-tX},*}\Pi_G(e^{tX})\big|_{t=0}$$

(4) $\implies$ (2) Prove that

$$\phi_{e^{-tX},*}\Pi_M(e^{tX}\cdot x) = \Pi_M(x) + \phi_{e^{-tX},*}\phi_{x,*}\Pi_G(e^{tX})$$

Then prove that derivatives $\frac{d}{dt}$ at $t = 0$ are equal.

$$\begin{aligned}
\phi_{e^{-tX},*}\Pi_M(e^{tX}\cdot x) &= \phi_{e^{-tX},*}(L_{\rho(X)}\Pi_M)(e^{tX}x)\\
&= \phi_{e^{-tX},*}(\rho^{\wedge 2}(\delta(X)))(e^{tX}x)\\
&= \phi_{e^{-tX},*}\phi_{e^{tX}x,*}[(L_X\Pi_G)(e)]\\
&= \phi_{e^{-tX},*}\phi_{x,*}R_{e^{tX},*}[(L_X\Pi_G)(e)]\\
&= \phi_{x,*}L_{e^{-tX},*}R_{e^{tX},*}[(L_X\Pi_G)(e)]\\
&= \phi_{x,*}\,\mathrm{Ad}_{e^{-tX}}[(L_X\Pi_G)(e)].
\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{d}{dt}(\Pi_M(x)+\phi_{x,*}L_{e^{-tX},*}\Pi_G(e^{tX})) &= \phi_{x,*}\frac{d}{dt}L_{e^{-tX},*}\Pi_G(e^{tX})\\ &= \phi_{x,*}\operatorname{Ad}_{e^{-tX}}(L_X\Pi_G(e)).\end{aligned}$$

Therefore the two sides coincide for any $x \in M$ in an open neighbourhood of $e \in G$. Being G connected any open neighbourhood generates it and the theorem is proven. $\square$

Remark 8.4.

1. ϕ does not preserve Π_M unless $\Pi_G = 0$. Neither ϕ_g nor ϕ_x are in general Poisson maps.

2. The multiplicity condition (2) is often referred to as Π_M being covariant with respect to Π_G.

3. $m\colon G\times G \to G$ is a left Poisson action on the Poisson Lie group itself. As a special case of the previous statement neither left nor right translations are Poisson maps.

4. Another way of stating the infinitesimal Poisson action condition is

$$d_\Pi(\rho(X)) = \rho^{\wedge 2}(\delta(X))$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\rho"] \arrow[d, "\delta"'] & \mathfrak{X}(M) \arrow[d, "d_\Pi"] \\ \Lambda^2\mathfrak{g} \arrow[r, "\rho^{\wedge 2}"'] & \mathfrak{X}^2(M) \end{tikzcd}$$

   Thus ϕ looks like some sort of intertwining operator between differentials. In fact δ can be extended to a degree 1 derivation of $\Lambda^\bullet\mathfrak{g}$, simply by letting

$$\delta(X_1\wedge\cdots\wedge X_n) = \sum_{i=1}^n (-1)^i X_1\wedge\cdots\wedge\delta(X_i)\wedge\cdots\wedge X_n$$

   The coJacobi condition on δ implies $\delta^2 = 0$. This turns $\Lambda^\bullet\mathfrak{g}$ into a differential Gerstenhaber algebra $(\Lambda^\bullet\mathfrak{g}, \wedge, [-,-])$. The infinitesimal action condition shows that $\rho\colon \mathfrak{g}\to\mathfrak{X}(M)$ with its natural extension $\rho^\wedge\colon \Lambda^\bullet\mathfrak{g}\to \mathfrak{X}^\bullet(M)$ provides a morphism of differential Gerstenhaber algebras.

5. Right hand side of (4) does not depend on Π_M. Consider Π_M and Π'_M such that (G,Π_G) acts in a Poisson way on both. Then $L_{\rho(X)}(\Pi_M - \Pi'_M) = 0$. Thus $\Pi_M - \Pi'_M$ is an invariant bivector (not necessarily Poisson).

We can give a slightly different look on conditions (3)-(4).

$$\theta\colon \Omega^1(M)\to C^\infty(M;\mathfrak{g}^*) \in \mathfrak{X}(M)\otimes C^\infty(M;\mathfrak{g}^*)$$

Recall the Poisson coboundary introduced in (4.1).

$$d_\Pi\colon \mathfrak{X}^p(M)\to\mathfrak{X}^{p+1}(M),\quad d_\Pi(P) = [\Pi, P]$$

$$(d_\Pi X)(df, dg) = (L_X\Pi)(df, dg) = X\{f,\, g\} - \{Xf,\, g\} - \{f,\, Xg\}$$

Therefore LHS of (3) can be rewritten as

$$(d_\Pi\theta)(df, dg)$$

and phrased with suitable conventions as

$$(d_\Pi\theta - \tfrac{1}{2}[\theta,\theta])(df, dg) = 0$$

i.e. θ satisfies a Maurer-Cartan type of equation.

Proposition 8.5. *Let $(\mathfrak{g},\delta)$ be a Lie bialgebra with an infinitesimal Poisson action $\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$ on the Poisson manifold (M,Π_M). Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$,*

$$C^\infty(M)^{\mathfrak{h}} = \{f \in C^\infty(M) : \rho(X)f = 0 \quad \forall\, X \in \mathfrak{h}\},$$
$$\mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^* : \langle \xi,\, x\rangle = 0 \quad \forall\, X \in \mathfrak{h}\}.$$

Then

1. *If $\mathfrak{h}^\perp$ is a Lie subalgebra then $C^\infty(M)^{\mathfrak{h}}$ is a Poisson subalgebra.*
2. *If $C^\infty(M)^h$ is a Poisson subalgebra and $\{\theta_f : f \in C^\infty(M)^{\mathfrak{h}}\}$ span $\mathfrak{h}^\perp$, then $\mathfrak{h}^\perp$ is a Lie subalgebra.*

Proof. Let $f, g \in C^\infty(M)^{\mathfrak{h}}$. This means that for any $X \in \mathfrak{h}$, $\rho(X)f = 0 = \rho(X)g$. Using condition (3)

$$\rho(X)\{f,\, g\} = \{\rho(X)f,\, g\} + \{f,\, \rho(X)g\} + \langle[\theta_f, \theta_g],\, X\rangle = 0$$

is equivalent to $[\theta_f, \theta_g] \in \mathfrak{h}^\perp$. Now simply remark that for the case of an infinitesimal action θ_f is defined via

$$\langle \theta_f,\, Y\rangle = \rho(Y)f \quad \forall\, Y \in \mathfrak{g}$$

Therefore f is invariant if and only if $\theta_f \in \mathfrak{h}^\perp$ and $\mathfrak{h}^\perp$ is generated by such elements. Thus the statement. □

Corollary 8.6. *If $\mathfrak{h} \setminus M$ is a smooth manifold then it posesses a Poisson structure and $p\colon M \to \mathfrak{h} \setminus M$ is a Poisson map.*

Corollary 8.7. *If we have a global action and a closed connected subgroup H such that $\mathfrak{h}^\perp$ is a Lie subalgebra then the same holds true for $H \setminus M$.*

8.2 Poisson homogeneous spaces

Definition 8.8. *A* Poisson homogeneous space *is a Poisson manifold (M, Π_M) together with a transitive Poisson action of a Poisson Lie group.*

Remark 8.9. The covariance condition is

$$\Pi_M(g \cdot x) = \phi_{g,*}\Pi_M(x) + \phi_{x,*}\Pi_G(g)$$

When H is homogeneous for a given $x \in M$ this formula allows to compute Π_M at all points from $\Pi_M(x)$, i.e. Π_M is uniquely determined by its value at one fixed point.

Homogeneous G-spaces are of the form G/H for a closed Lie subgroup H. We will show how, and why, such description does not work any more at the Poisson level. First we need to describe properties of subgroups of Poisson Lie group.

Definition 8.10. *A Lie subgroup H of a Poisson Lie group G is called a* Poisson Lie subgroup *if it is a Poisson submanifold. It is called a* coisotropic subgroup *if it is a coisotropic submanifold.*

Remark 8.11. Let $H \leq G$ be a Poisson (coisotropic) Lie subgroup and $g \in G$. Then $\mathrm{Ad}_g(H) = gHg^{-1}$ may be Poisson, coisotropic or none of the above.

Proposition 8.12. *Let H be a connected Lie subgroup of a Poisson Lie group (G, Π_G).*

1. *H is a Poisson Lie subgroup if and only if $\mathfrak{h}^\perp$ is an ideal in $\mathfrak{g}^*$.*
2. *H is coisotropic if and only if $\mathfrak{h}^\perp$ is a Lie subalgebra.*

Proof. H is a Poisson submanifold if and only if

$$I_H = \{f \in C^\infty(G) : f|_H = 0\}$$

is a Poisson ideal. Being $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ spanned by covectors $d_e f$, $f \in I_H$, I_H is a Poisson ideal implies $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is an ideal. The converse is true due to connectedness.

The second statement is proved analogously, but now we request that I_H is a Poisson subalgebra. $\mathfrak{h}^\perp$ is still spanned by $d_e f$, $f \in I_H$, therefore the thesis. $\square$

Poisson homogeneous spaces $\phi \colon G \times M \to M$ contain a number of special cases.

1. Invariant Poisson structures ($\Pi_G = 0$)
2. Affine Poisson structures ($M = G$)
3. Non symplectic covariant (i.e. $\Pi_G \neq 0$) Poisson structures, which include
   (a) "Highly singular" covariant Poisson structures ($\exists\, x_0\ \Pi_M(x_0) = 0$)
   (b) Quotients by coisotropic subgroups
   (c) Quotients by Poisson Lie subgroups

   Furthermore $(a) = (b) \supset (c)$.

Some relevant examples of Poisson Lie groups:

- (G, Π_G) any Poisson Lie group. Drinfeld double DG has a natural Poisson Lie structure. $G, G^* \hookrightarrow DG$ (if it can be embedded) is a Poisson Lie subgroup.
- G complex semisimple Lie group. K compact real form with standard Poisson structure. Then $DK = G$. Furthermore, as the standard Poisson structure is defined via simple roots any Dynkin diagram embedding

sorresponds to a Poisson Lie group. In particular to each node there corresponds a distinct Poisson embedding

$$\mathrm{SU}(2) \subset \mathrm{SU}(n)$$

Remark though that SL_2 triples not corresponding to simple roots are not Poisson Lie subgroups. For example

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

are Poisson Lie subgroups, but

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}$$

is not.

Exercise 8.13. *Classify Poisson Lie subgroups of* $\mathrm{SU}(2)$.

Hint: Compute the dual Lie bialgebra. Classify ideals in this 3-dimensional Lie algebra, distinguishing between 2-dimensional ideals and 1-dimensional ideals. Check which of them is the $\perp$ of a Lie algebra, and you have that the only pair $(\mathfrak{h}, \mathfrak{h}^\perp)$ such that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{su}(2)$ and $\mathfrak{h}^\perp$ is a Lie ideal in $\mathfrak{su}(2)^*$ is when $\mathfrak{h} = \langle H \rangle$, H being the Cartan diagonal element. Therefore the only connected Poisson-Lie subgroup is $\mathbb{S}^1$ diagonally embedded in $\mathrm{SU}(2)$ and the disconnected ones are its discrete subgroups.

Exercise 8.14. *Classify Poisson Lie subgroups of* $\mathrm{SL}(n, \mathbb{C})$ *with respect to the standard structure.*

It requires some work. A good start is to look at the first pages of [s-j03].

Coisotropy condition is much weaker. For example let $H \leq G$ be a Lie subgroup of codimension 1. Then H is coisotropic. In fact $\dim \mathfrak{h}^\perp = 1$ and therefore $\mathfrak{h}^\perp$ is a Lie algebra, $[X, X] = 0$.

Let M be a Poisson homogeneous space. Fix $x \in M$

$$T_x M \simeq \mathfrak{g}/\mathfrak{h}_x, \quad \mathfrak{h}_x \text{ - stabilizer of } x$$

Proposition 8.15. *For any* $v \in \Lambda^2 \mathfrak{g}/\mathfrak{h}_x$

$$L_x := \{X + \xi : X \in \mathfrak{g},\ \xi \in \mathfrak{h}_x^\perp,\ (\xi \otimes \mathrm{id})(v) = X + \mathfrak{h}_x\}$$

is a Lagrangian subspace of the double.

Proof.

$$\langle X + \xi, Y + \eta \rangle = (\xi \otimes \eta + \eta \otimes \xi)(v) = 0$$

so L_x is isotropic. Surjectivity follows from surjectivity of $X + \xi \to X$, which implies maximality. ∎

Theorem 8.16. *For any* $x \in M$ *let* L_x *be the Lagrangian subspace in* $D\mathfrak{g}$. *Then:*

1. *L_x is a Lie subalgebra in $D\mathfrak{g}$*

2. *$L_{gx} = gL_x$ where gL_x is given by the adjoint action of G in $D\mathfrak{g}$*

3. *There is a bijection between Poisson G-homogeneous structures on M and G-equivariant maps from M to the set of Lagrangian subalgebras such that if $x \in M$ then $L_x \cap \mathfrak{g} = \mathfrak{h}_x$.*

Remark 8.17. Let $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ be a Drinfeld double, $G \times D\mathfrak{g} \to D\mathfrak{g}$ adjoint action

$$\mathrm{Ad}_g(X + \xi) = \mathrm{Ad}_g X + \mathrm{Ad}^*_{g^{-1}} \xi \, i_{R_{g^{-1},*}\Pi(g)} + \mathrm{Ad}^*_{g^{-1}} \xi$$

$\mathcal{L}(D\mathfrak{g})$ is an algebraic variety; the set of Lagrangian subalgebras of the double. The adjoint action of G passes to an action on this variety

$$G \times \mathcal{L}(D\mathfrak{g}) \to \mathcal{L}(D\mathfrak{g})$$

Then theorem (8.16) says that on $\mathcal{L}(D\mathfrak{g})$ orbits are "models" for Poisson homogeneous spaces ([el01]).

Proposition 8.18. *Let M be a Poisson homogeneous space of (G, Π_G). For $x_0 \in M$ the following are equivalent:*

1. $\Pi_M(x_0) = 0$

2. *$\phi_{x_0} \colon G \to M$ is a Poisson map*

3. *H_{x_0} (stabilizer $= \{g \in G : gx_0 = x_0\}$) is coisotropic; $M \simeq G/H_{x_0}$*

Proof. (1) $\implies$ (2) Take the same x_0

$$\Pi_M(gx_0) = \underbrace{\phi_{g,*}\Pi_M(x_0)}_{=0} + \phi_{x_0,*}\Pi_G(g)$$

Therefore ϕ_{x_0} is Poisson.

(2) $\implies$ (1) Let ϕ_{x_0} be a Poisson map

$$\Pi_M(x_0) = \Pi_M(ex_0) = \phi_{x_0,*}\Pi_G(e) = 0$$

(2) $\iff$ (3) We have already proven (3) $\implies$ (2). Furthermore we know that $\phi_{x_0} \colon G \to M$ is Poisson if and only if $\{\phi^*_{x_0} f_1, \phi^*_{x_0} f_2\}_G$ is constant along the fibers of ϕ_{x_0} (proposition (3.20)).

Lemma 8.19. *$\{\phi^*_{x_0} f_1, \phi^*_{x_0} f_2\}_G$ is constant along all fibers if and only if*

$$\{\phi^*_{x_0} f_1, \phi^*_{x_0} f_2\}_G\big|_{\phi^{-1}_{x_0}(x_0) = H_{x_0}} = 0$$

Proof. Let $\{\phi^*_{x_0} f_1, \phi^*_{x_0} f_2\}_G$ be constant along all fibers. Then it is constant when restricted to H_{x_0}. But $e \in H_{x_0}$ and

$$\{\phi^*_{x_0} f_1, \phi^*_{x_0} f_2\}_G(e) = 0$$

due to $\Pi_G(e) = 0$. Therefore it is 0 on all H_{x_0}.

Let $\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G\big|_{H_{x_0}} = 0$. Take $g, g' \in G$ on the same fiber of ϕ_{x_0}. Then there exists $h \in H_{x_0}$ such that $g' = gh$. Now

$$\begin{aligned}
\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G(g') &= \langle \Pi_G(gh),\, d_{gh}(\phi_{x_0}^* f_1) \otimes d_{gh}(\phi_{x_0}^* f_2)\rangle \\
&= \langle L_{g,*}\Pi_G(h) + R_{h,*}\Pi_G(g),\, d_{gh}(\phi_{x_0}^* f_1) \otimes d_{gh}(\phi_{x_0}^* f_2)\rangle \\
&= \{L_g^*\phi_{x_0}^* f_1,\, L_g^*\phi_{x_0}^* f_2\}(h) + \{R_h^*\phi_{x_0}^* f_1,\, R_h^*\phi_{x_0}^* f_2\}(g) \\
&= \underbrace{\{\phi_{x_0}^*(\phi_g \circ f_1),\, \phi_{x_0}^*(\phi_g \circ f_2)\}(h)}_{=0 \text{ by hypothesis}} \\
&\quad + \{\phi_{hx_0}^*(\phi_g \circ f_1),\, \phi_{hx_0}^*(\phi_g \circ f_2)\}(g) \\
&= \{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}(g)
\end{aligned}$$

because $h \in H_{x_0} \implies \phi_{hx_0} = \phi_{x_0}$. □

Now we want to show

$$\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G\big|_{H_{x_0}} = 0 \Longleftrightarrow H_{x_0} \text{ is coisotropic.}$$

$\phi_{x_0}^* f_1$ is constant along H_{x_0}, so

$$(\phi_{x_0}^* f_1)(e) = c + f', \quad f' \in I_{H_{x_0}} = \{f \in C^\infty(G) : f|_{H_{x_0}=0}\}$$

$$\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G = \{f_1' + c,\, f_2' + c\}_G = \{f_1',\, f_2'\}_G$$

Remember that

$$\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G(h) = \langle \Pi_G(h),\, d_h\phi_{x_0}^* f_1 \otimes d_h\phi_{x_0}^* f_2\rangle$$

We want to prove that $\operatorname{im} \#_\Pi \subseteq N^*H$. The point is that we can restrict to a neighbourhood of identity (due to multiplicativity and connectedness). There choose $h = e^{tH}$. It is enough to show that

$$\frac{d}{dt}\langle \Pi_G(h),\, d_h\phi_{x_0}^* f_1 \otimes d_h\phi_{x_0}^* f_2\rangle|_{t=0} = 0$$

because we know that it is 0 at e. But this equals

$$\langle \delta(H),\, d_e\phi_{x_0}^* f_1 \otimes d_e\phi_{x_0}^* f_2\rangle$$

and $d_e\phi_{x_0}^* f_1 \in \mathfrak{h}^\perp$, and in fact generates it. It is 0 if and only if $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$, that is precisely when H_{x_0} is coisotropic. □

Proposition 8.20. *Let G be a Poisson Lie group. Let K be a Poisson Lie subgrouop and let H' be a coisotropic subgroup. Then $H = K \cap H'$ is coisotropic in K and*

$$i\colon K/K \cap H' \to G/H'$$

is a Poisson embedding.

Proof.

$$I_K := \{f \in C^\infty(G) : f|_K = 0\}$$

I_K is a Lie ideal with respect to $\{-,-\}$ and $I_{H'}$ is a Lie subalgebra with respect to $\{-,-\}$.

$$I_{H' \cap K} = I_{H'} + I_K$$

$$f = f_1 + f_2$$

Take $f' \in I_K$

$$\{f',\, f_1 + f_2\} = \underbrace{\{f',\, f_1\}}_{?} + \underbrace{\{f',\, f_2\}}_{\in I_K}$$

so $I_{H' \cap K}$ is not in general a Lie ideal.

Take $l_1, l_2 \in I_{H' \cap K}$

$$\{l_1 + l_2,\, f_1 + f_2\} = \underbrace{\{l_1,\, f_1\}}_{\in I_{H'}} + \underbrace{\{l_1,\, f_2\}}_{\in I_K} + \underbrace{\{l_2,\, f_1\}}_{\in I_K} + \underbrace{\{l_2,\, f_2\}}_{\in I_K} \in I_{H'} + I_K$$

Therefore $I_{H' \cap K}$ is a Lie subalgebra with respect to $\{-,\, -\}$. The second statement follows from the fact that in this diagram everything is Poisson

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & G \arrow[d] \\
K/H' \cap K \arrow[r] & G/H'
\end{tikzcd}$$

□

The following example is carried out in all details in [cs06].

Example 8.21. Take $\mathrm{SU}(n)$ with standard Poisson Lie structure

$$\begin{aligned}
\delta(H_i) &= 0 \\
\delta(E_i) &= H_i \wedge E_i \\
\delta(F_i) &= H_i \wedge F_i
\end{aligned}$$

E_i, F_i simple roots, $i = 1, \ldots, n$.

Then $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1))$, $(a, A) \mapsto \begin{pmatrix} A & 0 \\ O & a \end{pmatrix}$ is a Poisson Lie subgroup of $\mathrm{SU}(n)$. For every $k \in \{1, \ldots, n\}$, $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ gives a Poisson Lie subgroup $K_k := \mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n-k)) \hookrightarrow \mathrm{SU}(n)$. In particular $\mathrm{SU}(n)/K_{n-1} = \mathbb{C}P^{n-1}$ with covariant Poisson structure.

Now take $H' = K_{n-1}$, $K = K_k$, $k = 1, \ldots, n-2$

$$H' \cap K \cong \mathrm{U}(k) \times \mathrm{U}(n-k-1)$$

$$H'/H' \cap K \simeq \mathbb{C}P^{n-k-1}$$

with the same Poisson structure.

Therefore we get

$$* \hookrightarrow \mathbb{C}P^1 \hookrightarrow \ldots \hookrightarrow \mathbb{C}P^{n-2} \hookrightarrow \mathbb{C}P^{n-1}$$

This gives all symplectic foliation of $\mathbb{C}P^{n-1}$. Now change things a little bit. There exists a family $\sigma_c \in \mathrm{SU}(n)$ such that $\mathrm{Ad}_{\sigma_c} H'$ is coisotropic. We want to study

$$\mathbb{C}P_c^{n-1} \simeq \mathrm{SU}(n)/\,\mathrm{Ad}_{\sigma_c} H'$$

Now $H' \cap K_k$ changes.

Proposition 8.22. *The embedding*

$$H'/H' \cap K_k \hookrightarrow \mathbb{C}P_c^{n-1}$$

is an embedding of

$$\mathbb{S}^{2k-1} \times \mathbb{S}^{2(n-k)-1} \hookrightarrow \mathbb{C}P_c^{n-1}$$

In particular, when $k = 1$*, this gives*

$$\mathbb{S}^{2n-3} \hookrightarrow \mathbb{C}P_c^{n-1}$$

where odd spheres have the standard Poisson structure.

Recall that we have defined a Lie bracket on $\Omega^1(M)$ (where M is Poisson)

$$[\alpha, \beta] = L_{\#_\Pi(\alpha)}\beta - L_{\#_\Pi(\beta)}\alpha - d(\Pi(\alpha, \beta)) \tag{8.2}$$

What happens to this bracket when $M = G$ is a Poisson-Lie group?

Theorem 8.23 (Dazord-Karasev-Weinstein)**.** *The left (resp. right) invariant 1-forms on a Poisson Lie group* (G, Π_G) *form a Lie subalgebra with respect to (8.2). Furthermore this induces a Lie bracket on* $\mathfrak{g}^*$ *isomorphic to* ${}^t\delta$*.*

Proof. Let α, β be left invariant 1-forms. Let $X \in \mathfrak{X}^1(G)$ be a left invariant vector field. We will prove that $[\alpha, \beta]$ is left invariant by proving that $\langle X, [\alpha, \beta]\rangle$ is constant for any such X.

$$\langle X, [\alpha, \beta]\rangle = \langle X, L_{\#_\Pi(\alpha)}\beta - L_{\#_\Pi(\beta)}\alpha\rangle - \langle X, d(\Pi(\alpha, \beta))\rangle$$

Let's look at the second summand

$$\begin{aligned}\langle X, d(\Pi(\alpha, \beta))\rangle &= L_X(\Pi(\alpha, \beta))\\ &= (L_X\Pi)(\alpha, \beta) + \Pi(L_X\alpha, \beta) + \Pi(\alpha, L_X\beta)\\ &= (L_X\Pi)(\alpha, \beta) - \langle \#_\Pi, L_X\alpha\rangle + \langle \#_\Pi(\alpha), L_X\beta\rangle\end{aligned}$$

Now consider first summand

$$\begin{aligned}\langle X, L_{\#_\Pi(\alpha)}\beta\rangle &= \underbrace{L_{\#_\Pi(\alpha)}\langle X, \beta\rangle}_{=0 \text{ because } \langle X, \beta\rangle \text{ is constant}} - \langle [\#_\Pi(\alpha), X], \beta\rangle\\ &= -\langle [\#_\Pi(\alpha), X], \beta\rangle\\ &= \langle L_X(\#_\Pi(\alpha)), \beta\rangle\\ &= \langle i_{L_X\Pi}\alpha + \#_\Pi(\alpha), \beta\rangle\end{aligned}$$

because $\#_\Pi(\alpha) = i_\Pi(\alpha)$, $[L_X, i_\Pi] = i_{L_X\Pi}$.
Therefore

$$\langle X, L_{\#_\Pi(\alpha)}\beta\rangle = (L_X\Pi)(\alpha, \beta) - \langle \#_\Pi(\beta), L_X\alpha\rangle$$

Similarly

$$\langle X, L_{\#_\Pi(\alpha)}\beta\rangle = -(L_X\Pi)(\alpha, \beta) - \langle \#_\Pi(\alpha), L_X\beta\rangle$$

Now substitute

$$L_X\Pi(\alpha, \beta) - \langle \#_\Pi(\beta), L_X\alpha\rangle + L_X\Pi(\alpha, \beta) + \langle \#_\Pi(\alpha), L_X\beta\rangle$$

$$-L_X\Pi(\alpha, \beta) + \langle \#_\Pi(\beta), L_X\alpha\rangle - \langle \#_\Pi(\alpha), L_X\beta\rangle = (L_X\Pi)(\alpha, \beta)$$

Lemma 8.24. *If Π is a Poisson Lie bracket on G then for any X left invariant vector field $L_X\Pi$ is left invariant.*

Proof. If X is left invariant on G its flow are right translations

$$\begin{aligned}(L_X\Pi)(g) &= \frac{d}{dt}R_{e^{-tX},*}\Pi(ge^{tX})|_{t=0}\\ &= \frac{d}{dt}(R_{e^{-tX},*}L_{g,*}\Pi(e^{tX}) + R_{e^{-tX},*}R_{e^{tX},*}\Pi(g))|_{t=0}\\ &= L_{g,*}\frac{d}{dt}R_{e^{tX},*}\Pi(e^{tX})|_{t=0}\\ &= L_{g,*}(L_X\Pi(e))\end{aligned}$$

□

This proves that the bracket of left invariant 1-forms is left invariant because $\langle X, [\alpha,\beta]\rangle = (L_X\Pi)(\alpha,\beta) = \langle L_X\Pi(e), \alpha_e \wedge \beta_e\rangle$ so $L_X\Pi$ is a left invariant 2-vector field.

Now the statement follows from

$$^t\delta(d_e f, d_e g) = d_e\{f, g\}$$

which gives the same as (8.2) computed at e.

$$\begin{aligned}[][df, dg] &= L_{\#_\Pi(df)}dg - L_{\#_\Pi(dg)}df - d(\Pi(df, dg))\\ &= L_{X_f}dg - L_{X_g}df - d\{f, g\}\\ &= d\langle X_f, dg\rangle - d\langle X_g, df\rangle - d\{f, g\}\\ &= d\{f, g\} + d\{f, g\} - d\{f, g\}\\ &= d\{f, g\}\end{aligned}$$

Left invariant 1-forms evaluated at e give you all of $\mathfrak{g}^*$ and therefore you can say

$$^t\delta(\xi_1, \xi_2) = [df_1, df_2](e), \text{ where } \xi_1 = d_e f_1,\ \xi_2 = d_e f_2.$$

□

Exercise 8.25. *Consider the standard Poisson Lie group structure on* SU(2). *Then* $\mathfrak{su}(2)$ *has a basis*

$$E_1 = \tfrac{1}{2}\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}, \quad E_2 = \tfrac{1}{2}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \quad E_3 = \tfrac{1}{2}\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}$$

Now $\delta\colon \mathfrak{su}(2) \to \mathfrak{su}(2) \wedge \mathfrak{su}(2)$

$$\begin{aligned}\delta(E_1) &= 0\\ \delta(E_2) &= E_1 \wedge E_2\\ \delta(E_3) &= E_1 \wedge E_3\end{aligned}$$

Prove that this defines a Lie bialgebra, that is δ is a 1-cocycle

$$\delta([X, Y]) = \mathrm{ad}_X\,\delta(Y) - \mathrm{ad}_Y\,\delta(X), \quad \mathrm{ad}_X = \mathrm{ad}_X \otimes 1 - 1 \otimes \mathrm{ad}_X$$

and
$$^t\delta\colon \mathfrak{su}(2)^* \wedge \mathfrak{su}(2)^* \to \mathfrak{su}(2)^*$$
satisfies Jacobi identity. Check it is enough to verify the cocycle conditions for $(X, Y) = (E_1, E_2)$, (E_1, E_3), (E_2, E_3).

Prove that for $^t\delta = [-, -]$
$$[e_1, e_2] = e_2$$
$$[e_1, e_3] = e_3$$
$$[e_2, e_3] = 0$$
define a Lie algebra structure.

Use the Killing form
$$\langle A,\, B\rangle = \mathrm{im}(\mathrm{Tr}(AB))$$
to identify $\mathfrak{su}(2)^*$ *with*
$$\left\{ \begin{pmatrix} x & a+ib \\ 0 & -x \end{pmatrix} : x, a, b \in \mathbb{R} \right\}$$
Therefore the connected simply connected dual group
$$\mathrm{SB}(2) = \left\{ \begin{pmatrix} x & z \\ 0 & x^{-1} \end{pmatrix} : x \in \mathbb{R}_{>0},\ z \in \mathbb{C} \right\} \cong \mathbb{R} \ltimes \mathbb{C}$$
Now let us describe all $(\mathrm{SU}(2), \Pi_G)$ *Poisson homogeneous space structures on* $\mathbb{S}^2$. *Let* Π_1, Π_2 *be Poisson homogeneous bivectors on* $\mathbb{S}^2$. *Then*

1. $\Pi_1 - \Pi_2$ *is* $\mathrm{SU}(2)$*-invariant (general)*
2. $\Pi_1 - \Pi_2$ *is Poisson (because of dimension 2)*
3. *On* $\mathbb{S}^2$ *there is a "unique" invariant symplectic form* ω_0 *corresponding to bivector* Π_0.
$$\Pi_1 - \Pi_2 = f\Pi_0$$
but being $\Pi_1 - \Pi_2$ $\mathrm{SU}(2)$*-invariant,* $f =$ *constant,* $\Pi_1 - \Pi_2 = C\Pi_0$. *Therefore we have a Poisson pencil of invariant Poisson structures on* $\mathbb{S}^2$
$$c\Pi_0 + \Pi_1$$
Choose as Π_1 *the quotient with respect to the Poisson Lie subgroup of diagonal matrices. It is explicitely given by*
$$\{x_1,\, x_2\} = (1 - x_1)x_3$$
$$\{x_2,\, x_3\} = (1 - x_1)x_1$$
$$\{x_3,\, x_1\} = (1 - x_1)x_2 = (1 - x_1)\Pi_0$$
Now
$$c\Pi_0 + (1 - x_1)\Pi_0 = (\lambda - x_1)\Pi_0, \quad \lambda \in \mathbb{R}$$
Prove that $\lambda \mapsto -\lambda$ *is a Poisson isomorphism.*
In the corresponding symplectic foliation 0-dimensional leaves are given by
$$\{x_1^2 + x_2^2 + x_3^2 = 1\} \cap \{x_1 = \lambda\}$$
There are 3 cases

$\lambda > 1$ *no 0-dimensional leaves, $(\lambda - x_1)\Pi_0$ is symplectic*

$\lambda = 1$ *(corresponding to Π_1 the quotient by Poisson Lie group) $\{N\}$ is a 0-dimensional leaf, and $\mathbb{S}^2 \setminus \{N\}$ is a 2-dimensional leaf*

$0 \leq \lambda < 1$ *(corresponding to Π_1 the quotient by Poisson Lie subgroup) $\mathbb{S}^1$-family of 0-dimensional leaves, two distinct 2-dimensional leaves.*

If $\lambda > 1$ they all have different symplectic volume, thus they are not symplectomorphic. If $0 \leq \lambda < 1$ they are not unimodular. The modular class is $x_2\delta_{x_3} - x_3\delta_{x_2}$. Using 2-Poisson cohomology it is possible to show that $\mathbb{S}^2_\lambda \ncong \mathbb{S}^2_{\lambda'}$ for $\lambda \neq \lambda'$ in $[0,1]$.

8.3 Dressing actions

Take $\xi \in \mathfrak{g}^*$, and denote by ξ^L the associated left invariant 1-form and by ξ^R the associated right invariant 1-form, i.e.

$$\xi^L(g) = L^*_{g^{-1}}\xi \in T^*_g G; \qquad \xi^R(g) = R^*_{g^{-1}}\xi \in T^*_g G\,.$$

Definition 8.26. *Define $\lambda, \rho \colon \mathfrak{g}^* \to \mathfrak{X}(G)$*

$$\lambda(\xi) := \#_\Pi(\xi^L)$$

$$\rho(\xi) := -\#_\Pi(\xi^R)$$

Lemma 8.27. *λ is a Lie algebra morphism, ρ is a Lie algebra antimorphism.*

Proof.

$$\begin{aligned}\lambda([\xi_1,\xi_2]) &= \#_\Pi([\xi_1,\xi_2]^L)\\ &= \#_\Pi([\xi_1^L,\xi_2^L])\\ &= [\#_\Pi(\xi_1^L),\#_\Pi(\xi_2^L)]\end{aligned}$$

$$\begin{aligned}\rho([\xi_1,\xi_2]) &= \#_\Pi([\xi_1,\xi_2]^R)\\ &= \#_\Pi([\xi_1^R,\xi_2^R])\\ &= -[\#_\Pi(\xi_1^R),\#_\Pi(\xi_2^R)]\end{aligned}$$

□

Therefore λ defines an infinitesimal left action of $\mathfrak{g}^*$ on G and ρ defines an infinitesimal right action of $\mathfrak{g}^*$ on G. These are called *infinitesimal dressing actions.*

Exercise 8.28. *Prove that the inversion map $S \colon g \mapsto g^{-1}$ intertwines left and right infinitesimal dressing actions, i. e. $S_* \circ \lambda = \rho$.*

Definition 8.29. *If the dressing action can be integrated to a global action of G^* on G, the Poisson Lie group G is said to be* complete.

We recall that the notion of Poisson-Lie group is self dual, therefore the above defines also the left and right infinitesimal dressing actions of $\mathfrak{g}$ on the dual Poisson-Lie group G^*.

Proposition 8.30. *Locally symplectic leaves of G coincide with the orbits of the left (or right) dressing action. If the Poisson Lie group is complete then the symplectic leaves coincide with such orbits.*

Proof. By definition left dressing vector fields are hamiltonian vector fields. They are tangent to leaves. Therefore locally orbits are contained in leaves.

On the other hand values of the left dressing vector fields at any $\mathfrak{g} \in G$ span the tangent space to the leaf through g. Therefore orbits and leaves coincide locally. If the action is global consider the whole orbit $\mathcal{O} \in S$ and $T_p\mathcal{O} = \mathrm{im}_{\#_{\Pi,p}} = T_pS$ for all p. Thus $\mathcal{O}$ is a Poisson submanifold of S and therefore $\mathcal{O} = S$. $\square$

Dressing action is the most powerful tool for computing the symplectic foliation of Poisson Lie group.

Proposition 8.31. *Taking the derivative at e of left (resp. right) infinitesimal dressing action you get (resp. minus) the coadjoint action of $\mathfrak{g}^*$ on $\mathfrak{g}$.*

Theorem 8.32 (Semonov-Tian-Shansky,[])**.** *Left and right dressing actions are Poisson actions.*

How can one integrate the dressing action? Recall the Drinfeld double $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$. Then locally (around $e \in DG$)

$$DG|_U = GG^*|_U$$

For any $d \in U$ denote with d_G its component in G, and with d_{G^*} its component in G^*, such that $d = d_G d_{G^*}$.

Proposition 8.33. *The local action given by this splitting*

$$\mathfrak{g}^* \cdot \mathfrak{g} := (\mathfrak{g}^*\mathfrak{g})_G$$

is a local left action of G^* on G integrating the infinitesimal dressing action λ.

The proof relies on a characterisation of the dressing action we could not give.
Whenever $DG = GG^*$ holds globally you have the *global dressing action.*

Example 8.34. Standard Poisson Lie structure on K compact. $DK = G$ complex semisimple in which K compact real form. $DG = KAN_+$ Iwasawa decomposition is a global splitting of the double. Therefore symplectic leaves on K are orbits of an AN_+ action.

Take $(G, \Pi = 0)$. Then $G^* \simeq \mathfrak{g}^*$ abelian Lie group with Lie-. The dressing action of G on G^* is given by

$$\lambda \colon \mathfrak{g} \to \mathfrak{X}(G^*)$$
$$X \mapsto \#_{\mathrm{LP}}(X^L)$$

where X^L is identified with an invariant 1-form on G^* (remark that $T_eG^* = \mathfrak{g}^*$, $T_e^*G^* = \mathfrak{g}^{**} = \mathfrak{g}$).

$$\langle \underbrace{\#_{\mathrm{LP}}(X^L)}_{\in\Omega^1_{\mathrm{inv}}(G^*)}, \underbrace{Y^L}_{\in\mathfrak{X}_{\mathrm{inv}}(G^*)} \rangle(\xi) = \{X^L, Y^L\}(\xi) = \langle \xi, [X,Y] \rangle = \langle \mathrm{ad}^*_X \xi, Y \rangle$$

Therefore $\#_{\mathrm{LP}}(X^L)$ as vector field is the same as $-\operatorname{ad}^*_X$. Thus locally it is given by coadjoint action of G on $\mathfrak{g}^*$. But this action is global. We recover the result that symplectic leaves for the Lie-Poisson structure are orbits of the coadjoint action.

How to integrate the dressing action? Recall that the Drinfeld double is a Lie bialgebra on $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ which integrates to a Poisson Lie group DG. Then locally around $e \in DG$ we have

$$DG|_U = GG^*|_U$$

Let $d \in U \subseteq DG$

$$d := d_G \cdot d_{G^*}$$

with obvious notation.

Proposition 8.35. *The local action given by the splitting*

$$DG|_U = GG^*|_U$$

as

$$\mathfrak{g}^* \cdot \mathfrak{g} := (\mathfrak{g}^* \mathfrak{g})_G$$

integrates the infinitesimal dressing action.

Remark 8.36. When you have a global splitting of the double, you have a global dressing action.

Examples 8.37.

1. K compact with standard Poisson Lie structure. Then $G = KAN_+$ (Iwasawa decomposition) is the double.
2. $(G, \Pi = 0)$, $(G^* = \mathfrak{g}^*, \Pi_{\mathrm{PL}})$. Then the dressing action of G on G^* is the coadjoint action.

Theorem 8.38. *Let $g \in G$ (around e). The leaf through g locally is the image of the double coset $G^* g G^*$ under the natural projection*

$$DG \to DG/G^* \cong G$$

If the dressing action is global they are exactly those.

Chapter 9

Quantization

9.1 Introduction

The purpose will be here to give a definition of quantization and estabilish a vocabulary given us the link between two languages: Poisson geometry and noncommutative algebras. Something like

classical	semiclassical	quantum
manifold	Poisson manifold	noncommutative algebra
group	Poisson Lie group	noncommutative Hopf algebra
point	0-leaf	character

Of course to state all this correctly we need to be very precise on the setting in which we will work. Apart from some preliminaries we will content ourselves to deal with the group case where, for a number of reasons and still with a high degree of attention on details, such dictionary behaves particularly well (i.e. is a functor).

Let us start with a general definition of quantization. On the formal level that will first require from us some definitions. We will work over the field $k = \mathbb{C}$. Basically all what follows work on any field of characteristic 0 and a not so trivial part still holds in characteristic p.

Let us denote with $\mathbb{C}[[\hbar]]$ the ring of formal power series in an indeterminant $\hbar$ with coefficients in $\mathbb{C}$. The algebraic structure here is obvious:

$$\sum_{n\geq 0} a_n\hbar^n + \sum_{n\geq 0} b_n\hbar^n = \sum_{n\geq 0}\sum_{n\geq 0}(a_n+b_n)\hbar^n$$

$$\left(\sum_{n\geq 0} a_n\hbar^n\right)\cdot\left(\sum_{n\geq 0} b_n\hbar^n\right) = \sum_{n\geq 0}\left(\sum_{p+q=n} a_p b_q\right)\hbar^n$$

This is a ring with unit 1. Invertible elements are exactly those power series with $a_0 \neq 0$ (check this as an exercise).

Let now M be a $\mathbb{C}[[\hbar]]$-module. For every $x \in M$ define

$$\kappa(x) := \max\{k : x \in \hbar^k M\}$$

Define for every $x, y \in M$

$$d(x,y) := 2^{-k(x-y)}$$

Lemma 9.1. *d is a pseudo metric on M.*

This metric induces a topology on M which is called the *$\hbar$-adic topology.* A $\mathbb{C}[[\hbar]]$-module is called *torsion free* if the multiplication by $\hbar$ is an injective map.

Proposition 9.2. *Let M be a topological $\mathbb{C}[[\hbar]]$-module. Then there exists a $\mathbb{C}$-vector space M_0 such that $M \cong_{\mathbb{C}} M_0[[\hbar]]$ if and only if M is Hausdorff, complete, $\hbar$-torsion free.*

Proof. If $M \cong M_0[[\hbar]]$ then one simply applies definitions.

In the opposite direction let M be Hausdorff, complete, torsion free. Let $M_0 := M/\hbar M$. Take $\pi\colon M \to M_0$. Choose a section $\sigma\colon M_0 \to M$ and define

$$\widetilde{\sigma}\colon M_0[[\hbar]] \to M$$

$$\sum_{n\geq 0} \hbar^n m_n \mapsto \sum_{n\geq 0} \hbar^n \sigma(m_n)$$

This $\widetilde{\sigma}$ is well defined on formal power series because of completeness. In fact

$$\sum_{n=0}^{N} \hbar^n \sigma(m_n)$$

is a Cauchy sequence in N, therefore we have a well defined

$$\lim_{N\to\infty} \sum_{n=0}^{N} \hbar^n \sigma(m_n)$$

This $\widetilde{\sigma}$ is injective as a consequence of $\hbar$-torsion freeness. In fact

$$\sum_{n\geq 0} \hbar^n \sigma(m_n) = 0 \implies \pi\left(\sum_{n=0}^{N} \hbar^n \sigma(m_n)\right) = 0$$

$$\implies m_0 = 0 \implies \hbar \sum_{n=1}^{N} \hbar^{n-1} \sigma(m_n) = 0$$

Now divide by $\hbar$ and repeat the argument.

$\widetilde{\sigma}$ is injective because of Hausdorffness. □

A module M over $\mathbb{C}[[\hbar]]$ of this form is called a *topologically free module.*

Take A to be a topologically free $\mathbb{C}[[\hbar]]$-algebra (completed tensor product). Then being $\hbar A$ an ideal $A/\hbar A$ is an algebra over $\mathbb{C}$.

Definition 9.3. *A* quantization *of an algebra A_0 is a topological free $\mathbb{C}[[\hbar]]$-algebra A such that $A/\hbar A$ is commutative.*

Proposition 9.4. *Let A be a quantization of A_0. Then A_0 is a Poisson algebra.*

Proof. Take $a, b \in A_0$, $\bar{a}, \bar{b} \in A$ respective lifts (i.e. $a = \bar{a} \mod \hbar$, $b = \bar{b} \mod \hbar$). Remark that $[\bar{a}, \bar{b}] \in \hbar A$ from the commutativity of A_0. Define

$$\{a,\, b\} := \frac{[\bar{a}, \bar{b}]}{\hbar} \mod \hbar$$

This is well defined

$$\frac{[\bar{a}+\hbar u, \bar{b}+\hbar v]}{\hbar} = \frac{[\bar{a},\bar{b}]}{\hbar} + \frac{\hbar[u,\bar{b}]+\hbar[\bar{a},v]}{\hbar} + \frac{\hbar^2[u,v]}{\hbar} = [\bar{a},\bar{b}] \mod \hbar$$

□

In fact, when you have a Lie group, then you have two algebraic objects to describe with: $F[G]$ and $U(\mathfrak{g})$. What is their relation?

$U(\mathfrak{g})$ is a Hopf algebra (cocommutative). The "right" choice of $F[G]$ is a Hopf algebra:

- G affine algebraic group and $\mathbb{C}[G]$ algebra of regular functions (sheaf of Hopf algebras wnen you do not have affine)
- K compact group and $R[K]$ algebra of representative functions (matrix elements of irreducible representations)
- G Lie group and $\mathbb{C}_f[G]$ algebra of formal functions

If you consider everything as real objects you have a Hopf-*-algebras. $(H, m, \Delta, \varepsilon, S)$ is a Hopf-*-algebra if $*\colon A \to A$ is an involution, i.e.

$$\begin{gathered}(ab)^* = b^*a^* \\ (\lambda a)^* = \bar{\lambda}a^* \\ \text{and} \\ \Delta(a^*) = (\Delta a)^* \\ (a \otimes b)^* = a^* \otimes b^*\end{gathered}$$

(this implies $(*\circ S)^2 = \mathrm{id}$). Then $U(\mathfrak{g})$ and $F[G]$ can be seen as Hopf-*-algebras.

9.2 Duality

Take $X \in U(\mathfrak{g})$. Then it defines a left invariant differential operator on G. Take $f \in F[G]$

$$\begin{gathered}(Xf)(e) = \langle X,\, f\rangle \\ \langle \Delta X,\, f_1 \otimes f_2\rangle = \langle X,\, f_1 f_2\rangle\end{gathered}$$

It gives you a nondegenerate pairing of Hopf-*-algebras. In general it is a map

$$\langle -, -\rangle\colon A \otimes B \to \mathbb{C}$$

such that

$$\begin{aligned}\langle a,\, b\rangle = 0\ \forall\, a \in A \implies b = 0 \\ \langle a,\, b\rangle = 0\ \forall\, b \in B \implies a = 0 \\ \langle 1,\, b\rangle &= \varepsilon(b) \\ \langle a,\, 1\rangle &= \varepsilon(a) \\ \langle a_1 a_2,\, b\rangle &= \langle a_1 \otimes a_2,\, \Delta b\rangle \\ \langle \Delta a,\, b_1 \otimes b_2\rangle &= \langle a,\, b_1 b_2\rangle \\ \langle S(a),\, b\rangle &= \langle a,\, S(b)\rangle \\ \langle a^*,\, b\rangle &= \overline{\langle a,\, S(b)^*\rangle}\end{aligned}$$

So you have a pair of Hopf-*-algebras in nondegenerate duality. More structure when (G, Π) is a Poisson-Lie group, $F[G]$ is a Poisson algebra such that multiplication $m\colon G \times G \to G$ satisfies

$$\{f_1 \circ m,\, f_2 \circ m\}_{G\times G} = \{f_1,\, f_2\}_G \circ m$$

Definition 9.5. Poisson Hopf algebra *is defned by condition*

$$\{\Delta f_1,\, \Delta f_2\}_{G\times G} = \Delta\{f_1,\, f_2\}_G$$

From our point of view it will be better to start with the infinitesimal description, i.e. universal enveloping algebra level. Let us first see what happens at the universal enveloping algebra of a Lie bialgebra.

Definition 9.6. *A* coPoisson Hopf algebra *is a pair $(U, \widehat{\delta})$, where U is a Hopf algebra and the linear map $\widehat{\delta}\colon U \to U \otimes U$ is such that*

$$\widehat{\delta}(ab) = (\Delta a)\widehat{\delta}(b) + \widehat{\delta}(a)(\Delta b)$$

and the dual map $\delta^*\colon U^* \otimes U^* \to U^*$ is a .

Proposition 9.7. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and $U = U(\mathfrak{g})$ its universal enveloping algebra. Then there exists unique $\widehat{\delta}\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ such that*

$$\widehat{\delta}|_{\mathfrak{g}} = \delta$$

In particular $U(\mathfrak{g})$ has a canonical coPoisson Hopf algebra structure.

Proof. The formula

$$\widehat{\delta}(ab) = (\Delta a)\widehat{\delta}(b) + \widehat{\delta}(a)(\Delta b)$$

plus $\widehat{\delta}|_{\mathfrak{g}} = \delta$ defines $\widehat{\delta}$ uniquely on all of $U(\mathfrak{g})$ once you have checked

$$\delta([a, b]) = [\Delta a, \delta(b)] + [\delta(a), \Delta b], \quad \forall\, a, b \in \mathfrak{g}$$

which is equivalent to the 1-cocycle condition

$$[\Delta, a] = \mathrm{ad}_a \text{ on } \mathfrak{g} \otimes \mathfrak{g}.$$

$\square$

Definition 9.8. *A topologically free Hopf algebra H over $\mathbb{C}[[\hbar]]$ is a* quantized universal enveloping algebra *if*

$$H/\hbar H \cong U(\mathfrak{g})$$

for some Lie algebra $\mathfrak{g}$.

Proposition 9.9. *Let H be a quantized universal enveloping algebra. Then $\mathfrak{g}$ has a Lie bialgebra structure defined by*

$$\delta(X) = \frac{\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X}}{\hbar} \mod \hbar$$

where $\overline{X}$ is any lifting of $X \in \mathfrak{g}$ to H.

Proof. $\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X} \in \hbar H$ because $U(\mathfrak{g})$ is cocommutative and therefore

$$\frac{\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X}}{\hbar} \in H$$

$\delta(X)$ as defined does not depend on the choice of $\overline{X}$

$$\frac{\Delta(\overline{X} + \hbar u) - \Delta^{\mathrm{op}}(\overline{X} + \hbar v)}{\hbar} = \frac{\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X}}{\hbar} + \alpha, \quad \alpha \in \hbar H$$

Modulo $\hbar$ one obtains

$$\frac{\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X}}{\hbar} \mod \hbar$$

$\delta(X)$ is skewsymmetric (clear) and belongs to $\mathfrak{g} \otimes \mathfrak{g}$. $\delta(X) \in \mathfrak{g} \otimes \mathfrak{g}$ if and only if its two components are primitive elements.

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\delta(X) &= \left[\frac{1}{\hbar}(\Delta \otimes \mathrm{id})(\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X})\right] \mod \hbar \\
&= \left[\frac{1}{\hbar}(\mathrm{id} \otimes \Delta - \mathrm{id} \otimes \Delta^{\mathrm{op}})\Delta\overline{X} + \sigma_{23}(\Delta \otimes \mathrm{id} - \Delta^{\mathrm{op}} \otimes \mathrm{id})\Delta\overline{X}\right] \\
&= (\mathrm{id} \otimes \delta)\Delta X + \sigma_{23}(\delta \otimes \mathrm{id})\Delta X
\end{aligned}$$

CoJacobi identity for δ follows from coassociativity. Cocycle condition follows from Δ being an algebra morphism. □

So for us a *quantum group* will be the following set of data. A pair $F_\hbar[G]$ (quantum functions algebra), $U_\hbar(\mathfrak{g})$ (quantum universal enveloping algebra) of topological Hopf algebras over $\mathbb{C}[[\hbar]]$ together with a nondegenerate Hopf pairing

$$\langle -, - \rangle \colon F_\hbar[G] \times U_\hbar(\mathfrak{g}) \to \mathbb{C}[[\hbar]]$$

The pairing gives you $U_\hbar(\mathfrak{g})$ as "dual" of $F_\hbar[G]$ and vice-versa. You can start with one of the two legs and construct the other. On the way you have some choices. Many technical problems containing remarkable details.

This "pairing" contains the $(X^L f)(e)$ kind of pairing, i.e. the interpretation of $U(\mathfrak{g})$ as differentiable distributions supported at e. But it contains something completely different.

9.3 Local, global, special quantizations

The discussion in the preceeding section was about local quantization. Their main advantage is that they are well suited to capture relations between the classical, semiclassical, and quantum properties (we will see some examples of these relations in more details later). However they miss part of the relevant information, or at least of the full geometry. For example local quantization does not allow to specialize the deformation parameter to complex values $\neq 0$. Being $(\hbar)$ the only maximal ideal in the local ring $\mathbb{C}[[\hbar]]$ they can describe only the limit $\hbar \to a$. But we know of some relevant parts of the theory of quantum groups staying out of this range. This is the case, for example, of the theory of quantum groups at roots of unity, which links quantum groups to 3-manifold invariants and Lie algebras in characteristic p.

Let us denote with $\mathbb{C}(q)$ the field of rational functions in the variable q.

Definition 9.10. *Let A_q be a $\mathbb{C}(q)$- Hopf algebra. An* integer form *(resp.* rational form*) of A_q is a $\mathbb{Z}[q,q^{-1}]$-Hopf subalgebra (resp. $\mathbb{Q}[q,q^{-1}]$) $\mathcal{A}$ of A_q such that*

$$\mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) = A_q$$

(resp. $\mathcal{A} \otimes_{\mathbb{Q}[q,q^{-1}]} \mathbb{C}(q) = A_q$)

Definition 9.11. *Given a $\mathbb{C}(q)$- Hopf algebra A_q together with an integer form $\mathcal{A}$ a* specialization *of A_q to the complex number λ is*

$$A_\lambda := \mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$$

where the tensor product is taken with respect to $\varphi\colon \mathbb{Z}[q,q^{-1}] \to \mathbb{C}$, $\varphi(q) = \lambda$.

In this way starting from a $\mathbb{C}(q)$- Hopf algebra we obtain a $\mathbb{C}$- Hopf algebra.

Example 9.12. Let $\mathfrak{g}$ be a finitely dimensional complex simple Lie algebra. Then $U_q(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra with generators $X_i^{\pm}$, $K_i^{\pm 1}$, $1 \le i \le n$ and relations

$$\begin{aligned}
K_iK_j &= K_jK_i \\
K_iK_i^{-1} &= K_i^{-1}K_i = 1 \\
K_iX_j^{+}K_i^{-1} &= q_i^{a_{ij}}X_j^{+} \\
K_iX_j^{-}K_i^{-1} &= q_i^{-a_{ij}}X_j^{-} \\
[X_i^{+}, x_j^{-}] &= \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \\
\sum_{r=0}^{1-a_ij} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_q (X_i^{\pm})^{1-a_{ij}} X_j^{\pm} (X_i^{\pm})^r &= 0, \text{ if } i \ne j
\end{aligned}$$

together with the Hopf algebra structure

$$\begin{aligned}
\Delta_q K_i^{\pm 1} &= K_i^{\pm} \\
\Delta_q X_i^{+} &= X_i^{+} \otimes K_i + 1 \otimes X_i^{+} \\
\Delta_q X_i^{-} &= X_i^{-} \otimes K_i + K_i^{-1} \otimes X_i^{+}
\end{aligned}$$

$$\begin{aligned}
S_q(K_i) &= K_i^{-1} \\
S_q(X_i^{+}) &= -X_i^{+}K_i^{-1} \\
S_q(X_i^{-}) &= -K_iX_i^{-}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_q(K_i) &= 1 \\
\varepsilon_q(X_i^{\pm}) &= 0
\end{aligned}$$

where $[a_{ij}]$ is the Cartan matrix of $\mathfrak{g}$, $q_i = q^{d_i}$, and d_i are positive integers such that $[d_ia_{ij}]$ is symmetric,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

$$(q;q)_n = (1-q) \cdot \ldots \cdot (1-q^n)$$

are the q-binomial coefficients.

Remark 9.13.

- If we have a relation $xy = qyx$, then there is a following formula using q-binomial coefficients
$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$$
- It is not true that $U_q(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} \mathbb{C}(q) = U(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}(q)$. For example in $U(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}(q)$ you do not have that many invertibles.
- Let $q = e^h$, $K_i = e^{d_i h H_i}$. This defines a local quantization $U_h(\mathfrak{g})$ of the standard bialgebra structure on $\mathfrak{g}$. To be precise you have, after modding out relations, take closure in the h-adic topology.
- Examples of ambiquities in choices of integer form. You can declare
$$\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} = \widehat{H}_i \text{ to belong to } A_q$$
or
$$\frac{K_i^2 - 1}{q_i^2 - q_i} \text{ to belong to } A_q$$
Choose between $K_i^{\pm 1} X_i^{\pm}$ or $X_i^{\pm}$. Connected to choice of a lattice in between weight and root lattice, which is equivalent to choice in between different groups with the same Lie algebra.

Definition 9.14. *Let $F_q(\mathrm{GL}_n(\mathbb{C}))$ be the $\mathbb{C}(q)$-algebra generated by $t_{ij}, \det_q^{-1}$, $1 \leq i, j \leq n$ with relations*
$$\begin{aligned}
t_{ki} t_{kj} &= q t_{kj} t_{ki}, \quad i < j \\
t_{ik} t_{jk} &= q t_{jk} t_{ik}, \quad i < j \\
t_{il} t_{kj} &= t_{kj} t_{il}, \quad l < k,\ j < l \\
t_{ij} t_{kk} - t_{kl} t_{ij} &= (q - q^{-1}) t_{il} t_{kj}, \quad l < k,\ j < l \\
\det{}_q &= \sum_{\sigma \in \Sigma_n} (-1)^{l(\sigma)} t_{1\sigma(1)} \dots t_{n\sigma(n)}
\end{aligned}$$
together with the Hopf algebra structure
$$\begin{aligned}
\Delta t_{ij} &= \sum_{k=1}^{n} t_{ik} \otimes t_{kj} \\
\varepsilon(t_{ij}) &= \delta_{ij} \\
S(t_{ij}) &= (-q)^{i-j} \xi_{i_c}^{j_c} \det{}_q^{-1}
\end{aligned}$$
where
$$\xi_{i_c}^{j_c} = \sum_{\substack{\sigma \in \Sigma_n \\ i_1, \dots, i_{n-1} \in [1, \dots, n] \setminus \{i_c\} \\ \sigma(i_1), \dots, \sigma(i_{n-1}) \in [1, \dots, n] \setminus \{j_c\}}} (-q)^{l(\sigma)} t_{i_1 \sigma(1)} t_{j\sigma(1)} \dots t_{i_{n-1}\sigma(n-1)} t_{j\sigma(n-1)}$$

Here apparently there is no need to use the machinery of $\mathbb{C}(q)$-algebras and integer forms to specialize the parameter to complex values. This is why often in this context one does not mention integer forms. Still they are relevant in the duality between $F_g[G]$ and $U_q(\mathfrak{g})$.

Definition 9.15. *Let G be an affine algebraic complex Poisson group. A* global quantized function algebra on G *is a $\mathbb{C}(q)$- Hopf algebra A_q together with an integer form $\mathcal{A}$ such that $A_{q=1} \cong F[G]$ as Hopf algebras.*

Another good aspect of global quantization is that it provides you with genuine (non topological) Hopf algebras.

9.4 Real structures

The usual approach to real structures is to consider $\mathbb{C}$-Hopf algebras endowed with a *-structure.

Definition 9.16. *A Hopf-*-algebra is a Hopf algebra over $\mathbb{C}$ endowed with the unital, involutive, antimultiplicative morphism $*\colon A \to A$ such that Δ and ε are *-homomorphisms.*

One can then prove that $* \circ S = S^{-1} \circ *$.

Proposition 9.17. *Let G be a complex algebraic group with Lie algebra $\mathfrak{g}$. Then there is a 1-1 correspondence between*

1. *real forms of G*
2. *Hopf-* structures on $U(\mathfrak{g})$*
3. *Hopf-* structures on $F[G]$*

Definition 9.18. *A* real quantum group *is a global quantized function algebra with a compatible *-structure.*

Example 9.19. Consider the example of $F_q[\mathrm{GL}_n(\mathbb{C})]$. Fix on it the *-structure given by

$$t_{ij}^* = S(t_{ji})$$

This gives you what is called the unitary $F_q[\mathrm{U}(n)]$ (compact form of $F_q[\mathrm{GL}_n(\mathbb{C})]$).

Example 9.20. Let $0 < q < 1$. Consider the *-algebra generated by α, γ ($= t_{11,t_{22}}$) subject to relations

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha \\ \alpha\gamma^* &= q\gamma^*\alpha \\ \gamma^*\gamma &= \gamma\gamma^* \\ \alpha\alpha^* + q^2\gamma\gamma^* &= 1 \\ \alpha^*\alpha + \gamma^*\gamma &= 1 \end{aligned}$$

together with the Hopf-algebra structure

$$\Delta\alpha = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$$
$$\Delta\gamma = \gamma\alpha + \alpha^*\gamma$$
$$\varepsilon(\alpha) = 1$$
$$\varepsilon(\gamma) = 0$$
$$S(\alpha) = \alpha^*$$
$$S(\gamma) = -q\gamma$$

This is called the (standard) quantum $\mathrm{SU}_q(2)$; $F_q[\mathrm{SU}(2)]$.

Example 9.21. Let $0 < q < 1$. Consider the *-algebra generated by v, n subject to relations

$$vv^{-1} = v^{-1}v = 1$$
$$vn = qnv$$
$$nn^* = qn^*n$$
$$vn^* = qn^*v$$

together with the Hopf-algebra structure

$$\Delta v = v \otimes v$$
$$\Delta n = v^* \otimes n + n \otimes 1$$
$$\varepsilon(v) = 1$$
$$\varepsilon(n) = 0$$
$$S(v) = v^{-1}$$
$$S(n) = -qn$$
$$S(n^*) = -q^{-1}n^*$$

This is called the (standard) quantum $E_q(2)$; $F_q[E(2)]$.

Example 9.22. Consider now the *-algebra generated by v, n subject to relations

$$vv^{-1} = v^{-1}v = q$$
$$vn - nv = q(1-v)^2$$
$$[n, n^*] = in$$

The Hopf algebra structure as before. This is called the non standard quantum $E_q(2)$.

9.5 Dictionary

In the following we would like to set up a whole dictionary

classical	semiclassical	quantum
algebraic group	Poisson algebraic group	quantum group
compact group	Poisson compact group	compact quantum group
Lie algebra	Lie bialgebra	quantum universal enveloping agebra
	Poisson dual	quantum duality principle
	Poisson double	quantum double construction
point	0-leaf	character

It is known in examples that quantum groups have few characters (classical points). Why is it so?

Proposition 9.23. *Let $A_\hbar$ be a local quantization of $A_0 = (F[M], \Pi)$. There is an injective map between set of characters of $A_\hbar$ (i.e. maps $\varepsilon\colon A_\hbar \to \mathbb{C}[[\hbar]]$ such that $\varepsilon([A_\hbar, A_\hbar]) = 0$) and 0-leaves of the Poisson bivector Π.*

Proof. Let ε be the character of $A_\hbar$. Then ε defines a character of A_0. Thus there exists $x_0 \in M$ such that $\varepsilon(f) = f(x_0)$ for all $f \in A_0$.

Now

$$\begin{aligned} &\varepsilon([a,b]) = 0 \quad \forall\, a, b \in A_\hbar \\ \Longrightarrow\ &\varepsilon(\{f_1,\, f_2\}) = 0 \quad \forall\, f_1, f_2 \in A_0 \\ \Longrightarrow\ &\underbrace{\{f_1,\, f_2\}(x_0)}_{\langle \Pi(x_0),\, d_{x_0} f_1 \wedge d_{x_0} f_2 \rangle} = 0 \end{aligned}$$

Thus if A_0 is an algebra of functions on a smooth manifold such that $d_{x_0} f$ generate $\Omega^1_{x_0}(M)$ we have $\Pi(x_0) = 0$. ☐

Example 9.24. $F_q[\mathrm{SU}(2)]$ (here *-characters - looking for real points)

$$\underbrace{\varepsilon(\alpha\gamma)}_{\varepsilon(\alpha)\varepsilon(\gamma)} = \underbrace{\varepsilon(q\gamma\alpha)}_{q\varepsilon(\gamma)\varepsilon(\alpha)} \Longrightarrow (1-q)\varepsilon(\alpha)\varepsilon(\gamma) = 0$$

Thus $\varepsilon(\alpha) = 0$ or $\varepsilon(\gamma) = 0$ and so on. We end up with

$$\varepsilon(\alpha) = t, \quad \varepsilon(\alpha^*) = t^{-1}$$

This is just an issue of a more general situation. In principle you would like to have a correspondence between primitive ideals of $F_\hbar[G]$ and symplectic foliation of (G, Π). For example if we take $U_q(\mathfrak{g}) = F_q[\mathfrak{g}^*]$ then by orbit method we obtain a homeomorphism between primitive ideals in $U_q(\mathfrak{g})$ and coadjoint orbits of G on $\mathfrak{g}^*$. It would be nice to have a "quantum orbit method". In fact it works for compact quantum groups.

9.6 Quantum subgroups

Let H be a closed or algebraic subgroup of G.

$$I_H = \{f \in F[G] : f|_H = 0\}$$

is a Hopf ideal and

$$F[G]/I_H \cong F[H]$$

as Hopf algebras. To put it another way

$$H \text{ subgroup of } G \quad \Longleftrightarrow \quad F[G] \to F[H] \quad \text{Hopf algebra epimorphism}$$

Alternatively thinking at the infinitesimal level

$$\mathfrak{h} \text{ subalgebra of } \mathfrak{g} \quad \Longleftrightarrow \quad U(\mathfrak{h}) \to U(\mathfrak{g}) \quad \text{Hopf algebra monomorphism}$$

It is therefore natural to say

Definition 9.25. *A* quantum subgroup *of a (global, local, special) quantized algebra of functions is a topological Hopf algebra epimorphism*

$$F_q[G] \to F_q[H]$$

Therefore quantum subgroups correspond to Hopf ideals in $F_q[G]$.

9.7 Quantum homogeneous spaces

Let B be a unital *-algebra and let A be a Hopf-*-algebra.

Definition 9.26. *A *-algebra homomorphism $\delta\colon B \to b \otimes A$ is a* right coaction *if*

$$(\mathrm{id} \otimes \Delta) \circ \delta = (\delta \otimes \mathrm{id}) \circ \delta$$
$$(\mathrm{id} \otimes \varepsilon) \circ \delta = \mathrm{id}$$

B is called A-right quantum space.

Which right coactions correspond to homogeneous actions? Here we mean $A = F[G]$, $B = F[X]$, δ dual of action $\phi\colon G \times X \to X$.

Definition 9.27. *Two right quantum spaces (B, δ), (B', δ') are* equivalent *if and only if there exists $\Phi\colon B \to B'$ *-algebra isomorphism such that*

$$\delta' \circ \Phi = (\Phi \otimes \mathrm{id})\delta \tag{9.1}$$

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B' \arrow[d] \\ B \otimes A \arrow[r] & B' \otimes A \end{tikzcd}$$

Modifying the following definition replacing the identity in (9.1) by a *-algebra morphism $\Psi\colon A \to A'$

$$\delta' \circ \Phi = (\Phi \otimes \Psi) \circ \delta$$

gives the definition of *equivariant map* of quantum spaces on different Hopf algebras.

Proposition 9.28. *Let (B, δ) be A-right quantum space. There is a 1:1 correspondence between *-algebra homomorphisms $\widetilde{\varepsilon}\colon B \to \mathbb{C}$ and *-algebra homomorphisms $i\colon B \to A$ such that*

$$\Delta \circ i = (i \otimes \mathrm{id}) \circ \delta$$

The correspondence is given by

$$i_{\widetilde{\varepsilon}} = (\widetilde{\varepsilon} \otimes \mathrm{id}) \circ \delta$$
$$\widetilde{\varepsilon} = \varepsilon \circ i$$

Proof. Say $\widetilde{\varepsilon}\colon B \to \mathbb{C}$ is given. Define

$$i_{\widetilde{\varepsilon}} := (\widetilde{\varepsilon} \otimes \mathrm{id}) \circ \delta$$

We have

$$\begin{aligned}
\Delta \circ i_{\widetilde{\varepsilon}} &= \Delta \circ (\widetilde{\varepsilon} \otimes \mathrm{id}) \circ \delta \\
&= (\widetilde{\varepsilon} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \delta \qquad (\Delta \text{ is } \mathbb{C}\text{-linear}) \\
&= (\widetilde{\varepsilon} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id}) \circ \delta \qquad (\delta \text{ is a coaction}) \\
&= (i_{\widetilde{\varepsilon}} \otimes \mathrm{id}) \circ \delta
\end{aligned}$$

hence $i_{\widetilde{\varepsilon}}$ verifies the required identity. Furthermore we then have

$$\varepsilon \circ i_{\widetilde{\varepsilon}} = (\widetilde{\varepsilon} \otimes \varepsilon) \circ \delta = (\widetilde{\varepsilon} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varepsilon) \circ \delta = \widetilde{\varepsilon}$$

Say $i\colon B \to A$ is given. Let $\widetilde{\varepsilon} = \varepsilon \circ i$. Then

$$(\widetilde{\varepsilon} \otimes \mathrm{id}) \circ \delta = ((\varepsilon \circ i) \otimes \mathrm{id}) \circ \delta = (\varepsilon \otimes \mathrm{id}) \circ (i \otimes \mathrm{id}) \circ \delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta \circ i = i$$

□

Thus for any A-right quantum space (B, δ) such that B has a character there exists an equivariant map between (B, δ) and a subalgebra $(i_\varepsilon(B), \Delta|_{i_\varepsilon(B)})$ of A.

What is $i_\varepsilon(B)$ in usual language? Take a G-space X. Fix $x_0 \in X$. Then consider

$$F[X] \to F[G], \quad f \mapsto \widetilde{f_{x_0}}$$

where $\widetilde{f_{x_0}}(g) := f(gx_0)$. When X is a classical homogeneous space we have that this map is injective.

Definition 9.29. *An* embeddable *quantum homogeneous space is an A-right quantum space (B, δ) with a *-homomorphism $\widetilde{\varepsilon}\colon B \to \mathbb{C}$ such that $i_{\widetilde{\varepsilon}}$ is injective.*

Identifying (B, δ) with $(i_{\widetilde{\varepsilon}}(B), \Delta|_{i_{\widetilde{\varepsilon}}(B)})$ we can equivalently declare an embeddable quantum homogeneous space to be a *-subalgebra and right coideal of $F_q[G]$.

Remark 9.30. This is not the most general fdefinition of quantum homogeneous space. In fact it requires B to have a character, which is in noncommutative algebras something not so trivial.

Let us understand this from the point of view of semiclassical limit. Everything above can be rephrased on $\mathbb{C}[\hbar]$-Hopf-*-algebras. Now we have

Having one
character

$$\updownarrow$$

Being embeddable
i.e $F_{\hbar[X]} \hookrightarrow F_\hbar[G]$

But we have seen already this at the semiclassical level

$$
\begin{tikzcd}
\text{SEMICLASSICAL} & \text{QUANTUM} \\
\begin{array}{l}\text{Having a 0-dimensional}\\ \text{leaf } \Pi_X(x_0)=0\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{l}\text{Having one}\\ \text{character}\end{array} \arrow[d, leftrightarrow] \\
\begin{array}{l}\text{Having Poisson}\\ \text{surjective map}\\ G\to X\\ \text{i.e. having a coinduced}\\ \text{Poisson structure}\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{l}\text{Being embeddable}\\ \text{i.e } F_{\hbar[X]}\hookrightarrow F_\hbar[G]\end{array} \arrow[d, leftrightarrow] \\
\begin{array}{l}\text{Being the quotient}\\ \text{by a coisotropic subgroup}\end{array} \arrow[r, leftrightarrow] & \begin{array}{l}\text{THEN WE WANT}\\ \text{TO FILL THIS !}\end{array}
\end{tikzcd}
$$

Before going into this we want to understatnd the relation betwwen quantum subgroups and embeddable quantum homogeneous spaces.

Proposition 9.31. *Let $F_q[G] = A$ be a quantum group and let $F_q[H]$ be a quantum subgroup with defining ideal I_H, i.e.*

$$F_q[H] = F_q[G]/I_H, \quad p_H\colon F_q[G] \to F_q[H]$$

If our quantum group is real require also $I_H^* = I_H$. Define

$$B_H := \{b \in A : (p_H \otimes \mathrm{id})\Delta b = 1 \otimes b\} = B^{\mathrm{co}I_H}$$

*Then B_H is a *-subalgebra and right coideal of A. Furthermore B_H is S^2-invariant and $p_H(b) = \varepsilon(b)1$ for all $b \in B$.*

Proof. Remark that

$$y \in B_H \otimes A \quad \Longleftrightarrow \quad (p_H \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})y = 1 \otimes y$$

Take $b \in B_H$. We want to show that $\Delta b \in B_H \otimes A$.

$$\begin{aligned}(p_H \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta b &= (p_H \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \Delta b \\ &= (id \otimes \Delta) \circ \underbrace{(p_H \otimes \mathrm{id})\Delta b}_{1\otimes b} \\ &= 1 \otimes \Delta b\end{aligned}$$

Now we will prove that B_H is S^2-invariant. In fact S^2 is a Hopf algebra automorphism

$$\begin{aligned}(p_H \otimes \mathrm{id}) \circ \Delta \circ S^2(b) &= (p_H \otimes \mathrm{id}) \circ (S^2 \otimes S^2) \circ \Delta b \\ &= ((p_H \circ S^2) \otimes S^2) \circ \Delta b \\ &= (S^2 \otimes S^2) \circ (1 \otimes b) \\ &= 1 \otimes S^2 b\end{aligned}$$

Lastly, apply $\mathrm{id} \otimes \varepsilon$ to $(p_H \otimes \mathrm{id}) \circ \Delta b = 1 \otimes b$ to prove $p_H(b) = \varepsilon(b)1$. $\square$

We would like to check whether all quantum homogeneous spaces are of this form. We have a necessary condition, S^2-invariance. Is it always verified?

Example 9.32. Consider on the standard $F_q[E(2)]$

$$z = \lambda v + n, \quad \bar{z} = \bar{\lambda} v^* + n^*, \quad \lambda \in \mathbb{C},\ |\lambda| = 1$$

$B =$ *-subalgebra generated by $z, \bar{z}$. Then B coincides with polynomials in z and $\bar{z}$.

$$z\bar{z} = q^2 \bar{z} z + (1 - q^2)$$
$$z^* = \bar{z}$$

Furthermore B is a coideal, $\Delta B \subset A \otimes B$.

$$\Delta z = v \otimes z + n \otimes 1$$

$$\Delta \bar{z} = v^* \otimes \bar{z} + n^* \otimes 1$$

But B is not S^2-invariant unless $\lambda = 0$.

$$S(z) = \lambda S(v) + S(n) = \lambda v^* - q^{-1} n$$

$$S^2(z) = \lambda v - q^{-2} n$$

Thus $z, S^2(z) \in B$, so $n \in B$, which is not true if $\lambda \neq 0$.

Example 9.33. Similarly consider $F_q[\mathrm{SU}(2)]$. Take

$$K := s(\gamma\alpha + \alpha^*\gamma^*) + (1 - s^2)\gamma^*\gamma$$

$$L := s(\alpha^2 - q\gamma^{*2}) + (1 - s^2)\alpha\gamma^*$$

One can check that:

1. The *-subalgebra generated by K and L is isomorphic to the universal *-algebra on these two generators and relations
$$\begin{aligned} K &= K^* \\ LK &= q^2 KL \\ LL^* + K^2 &= (1 - s^2)K + s^2 \\ LL^* + qK^2 &= (1 - s^2)q^2 K + s^2, \qquad s \in [0, 1] \end{aligned}$$
2. This *-subalgebra is always a right coideal and therefore is an embeddable quantum homogeneous space
3. This *-subalgebra is a quotient by a quantum subgroup if and only if $s = 1$.

We are looking for a quantum analogue of a coisotropic subgroup.

9.8 Coisotropic creed

When $A_\hbar$ is a quantization of (M, Π) then one-sided ideals in $A_\hbar$ should correspond to coisotropic submanifolds. The motivation for this comes from characterization

$$\begin{tikzcd}
\text{Poisson submanifold } N \arrow[d] & \text{Coisotropic submanifold } N \arrow[d, leftrightarrow] \\
\begin{array}{l} I_N \text{ is an ideal} \\ \text{and a Poisson ideal} \end{array} \arrow[d, leftrightarrow] & \begin{array}{l} I_N \text{ is an ideal} \\ \text{and a Poisson subalgebra} \end{array} \arrow[d, leftrightarrow] \\
\begin{array}{l} I_{q,N} \text{ is an ideal} \\ \text{(two-sided ideal)} \end{array} & \begin{array}{l} \text{SOMETHING WEAKER} \\ \text{BUT STRONGER THAN} \\ \text{BEING A SUBALGEBRA} \end{array}
\end{tikzcd}$$

Proposition 9.34. *Let $A_\hbar$ is a quantization of M. Take I to be a right ideal in M. Then $I_0 = I/\hbar I$ is an ideal in A_0 and a Poisson subalgebra in M.*

Proof. Let $i \in I$, $f \in A_\hbar$

$$f * i = fi + \hbar\{f, i\} + \ldots \in I$$

$$[f * i]_{\hbar I} = fi \in I/\hbar I$$

To be precise, take $f \in A_0$, $i \in I_0$. Take any lift $\bar{f} \in A_\hbar$, $\bar{i} \in I$.

$$\bar{f} = f + O(\hbar), \quad \bar{i} = i + O(\hbar)$$

Then

$$\begin{aligned} \bar{f} * \bar{i} = fi + O(\hbar) \quad &\Longrightarrow \quad [\bar{f} * \bar{i}] = f_0 i_0 \in A_0 I_0 \\ \bar{i} * \bar{f} = fi + O(\hbar) \quad &\Longrightarrow \quad [\bar{i} * \bar{f}] = i_0 f_0 \in A_0 I_0 \end{aligned}$$

But now

$$\bar{f} * \bar{i} - \bar{i} * \bar{f} \in \hbar A_\hbar \notin I$$

so we cannot define $\{f, i\} \in I_0$. Still what we have is the following. Let $i, j \in I_0$. Take $\bar{i}, \bar{j} \in I$ lifting i, j.

$$\bar{i} * \bar{j},\ \bar{j} * \bar{i} \in I \quad \Longrightarrow \quad [\bar{i}, \bar{j}] \in \hbar I$$

$$\Longrightarrow \{i, j\} \in I_0$$

so I_0 is a Poisson subalgebra. $\square$

We will stick to this creed and declare the following

Definition 9.35. *Let A be (*)- Hopf algebra. A* right (real) coisotropic quantum subgroup *C is a coalgebra and A-right module C such that there exists surjective linear map $p\colon A \to C$, which is a morphism of coalgebras and right A-module (endowed with an involution σ such that $p \circ (* \circ S) = \sigma \circ p$).*

Proposition 9.36. *C is a right (real) coisotropic quantum subgroup if and only if there exists $I_C \subseteq A$, which is a $((* \circ S)$-invariant) two sided coideal and right ideal such that*

$$p \colon A \to A/I_C \cong C$$

Remark 9.37. All Poisson subgroups can be quantized in a context of functorial quantization, but it is not known in such context whether all coisotropic subgroups can be quantized.

Proposition 9.38.

1. *Let C be a coisotropic quantum subgroup of A with defining ideal I. Then*

$$B_C := \{a \in A : (p \otimes \mathrm{id}) \circ \Delta b = p_{I_C}(1) \otimes B\}$$

is an embeddable quantum homogeneous space of A.

2. *Let B be an embeddable quantum homogeneous space. Then*

$$I_B := \{(b - \varepsilon(b)1) : b \in B\}$$

is a right ideal and two sided coideal of A.

Is this

$$\begin{matrix}\text{coisotropic quantum} \\ \text{subgroups}\end{matrix} \longleftrightarrow \begin{matrix}\text{embeddable quantum} \\ \text{homogeneous spaces}\end{matrix}$$

a bijective correspondence? Is it true that quotient by quantum subgroups are characterized by S^2-invariance? Almost.

Let B be a right coideal subalgebra. Take

$$AB^+ := B \cap \ker \varepsilon = \{b - \varepsilon(b)1 : b \in B\}$$

In general $B \subseteq A^{\mathrm{co}A/AB^+}$ but not necessarily equal. If the antipode is bijective and we restrict to left faithfully flat right coideal subalgebras and left faithfully coflat coisotropic quantum subgroups, then in that case S^2-invariance corresponds to quotient by a coisotropic quantum subgroup.

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Part V

Cyclic Homology Theory

by

Jean-Louis Loday

Mariusz Wodzicki

Based on the lectures of:

- Jean-Louis Loday
  (*Institut de Recherche Mathématique Avancée, CNRS et Université de Strasbourg, 7 rue R. Descartes, 67084 Strasbourg Cedex, France*)
  – Chapters 1, 2, 5, 6, 7.

- Mariusz Wodzicki
  (*Department of Mathematics, 970 Evans Hall 3840, University of California, Berkeley, USA*)
  – Chapters 8, 9, 10.

With additional lectures by:

- Piotr M. Hajac – Chapter 3.

- Ulrich Krähmer – Chapter 4.

Chapter 1

Cyclic category

1.1 Circle and disk as a cell complexes

The circle in its simplest decomposition has one 0-cell (a point) and one 1-cell (an interval).

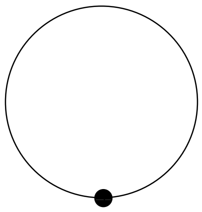
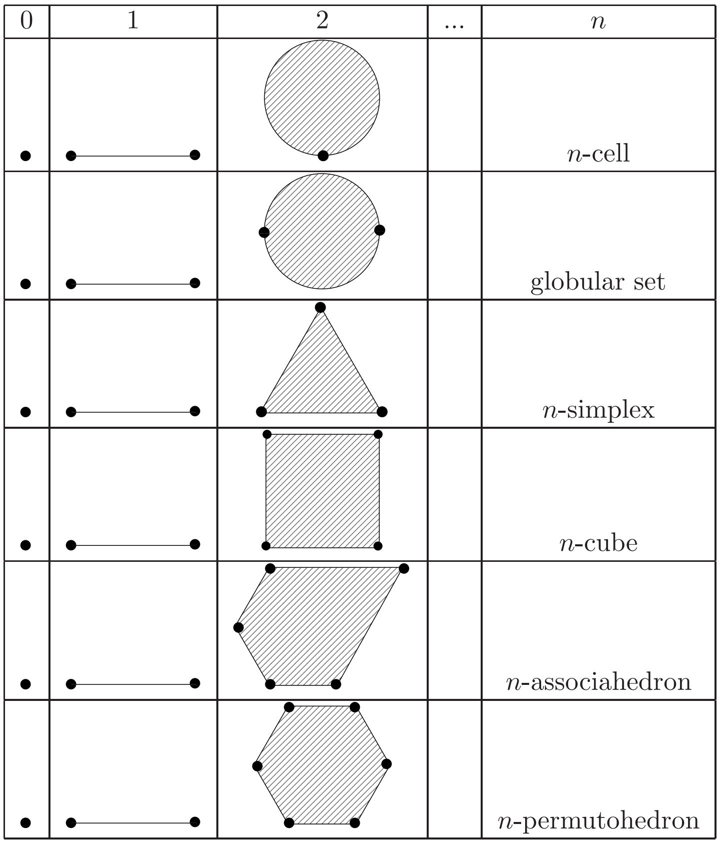
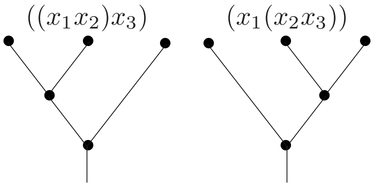


Figure 1.1: Circle

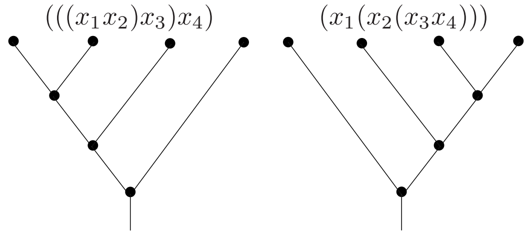
This is the only way to form a circle from an interval. If we try to decompose a disk of higher dimension, then we have choices. In the table below we give a few examples of decomposition of an n-cell.



The construction of an n-associahedron can be given by the use of *Stasheff complex*. Its vertices are defined to be all ways of putting parentheses to a word of length $(n+1)$. They are in bijection with the set of planar binary rooted trees as we can see on the example of words of length 3 and 4.



There is a partial order on trees in which the first tree on the picture is before the second one. This can be generalized for the trees with more leaves, and is called the Tamari order.



We can associate a tree to each vertex of a 2-associahedron and order them using the ordering on trees.

The realization of the Stasheff polytope as a subspace in $\mathbb{R}^n$ is homeomorphic to a ball. To each planar binary tree t we associate a point $M(t) = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ as follows. The i-th coordinate is the product of the number of leaves to the left of i-th vertex times the number of leaves to the right.

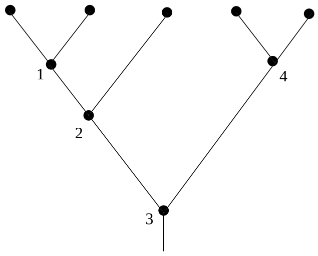


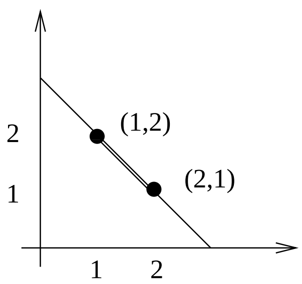
Figure 1.2: Tree t

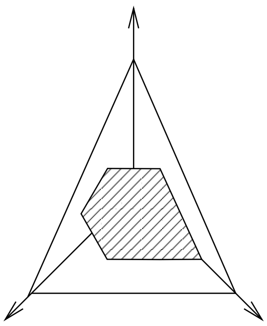
$$M(t) = (1 \cdot 1, 2 \cdot 1, 3 \cdot 2, 1 \cdot 1) = (1, 2, 6, 1) \in \mathbb{R}^4$$

The Stasheff polytope $\mathcal{K}^{n-1}$ of dimension $n-1$ is the convex hull of the points $M(t)$ for all planar binary tree with $(n+1)$ leaves. The sum of coordinates is

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$$

so the Stasheff polytope lies in the hyperplane given by this equation. The examples of Stasheff polytopes $\mathcal{K}^1$ and $\mathcal{K}^2$ are in the following pictures.





The Stasheff polytope $\mathcal{K}^3$ has 14 vertices and 9 faces. The faces are three squares and six pentagons (2-associahedrons). In general, the Stasheff polytope $\mathcal{K}^n$ has faces of the form $\mathcal{K}^p \times \mathcal{K}^q$, where $p+q=n$.

What about the permutohedron? Take an element σ in the symmetric group S_n. Associate to it the point $M(\sigma) = (\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$. Then we have permutohedron $\mathcal{P}^{n-1}$ as a convex hull of all points $M(\sigma)$ for all permutations. Of course $\sum_{i=1}^n \sigma(i) = \frac{n(n+1)}{2}$, so it lies in the hyperplane given by the equation $\sum_{i=1}^n x_i = \frac{n(n+1)}{2}$.

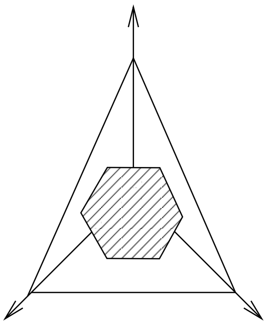
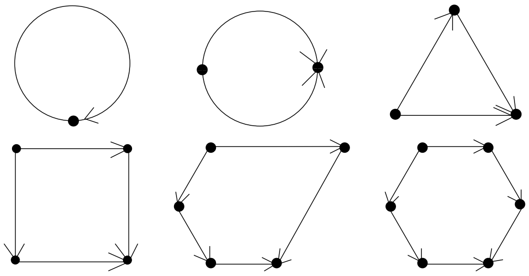


Figure 1.3: 2-permutohedron

In general $\mathcal{P}^n$ has faces of the form $\mathcal{P}^p \times \mathcal{P}^q$, where $p+q=n-1$.

Observe that we have an order on the vertices of our complexes.



On the set of vertices of the n-simplex the order comes from the order on natural numbers, because the vertices are numbered from 0 to n.

On the set of vertices of the n-associahedron the order is called the Tamari order.

On the n-permutohedron the order comes from the weak Bruhat order on the symmetric group S_n.

1.2 Simplicial sets

Definition 1.1. *The* n-simplex *is the subspace* $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_i x_i = 1, 0 \leq x_i \leq 1\}$.

Denote by i the vertex on the x_i-axis. On the set of vertices of an n-simplex we have an ordering coming from the order on the set $[n] = \{0, \ldots, n\}$.

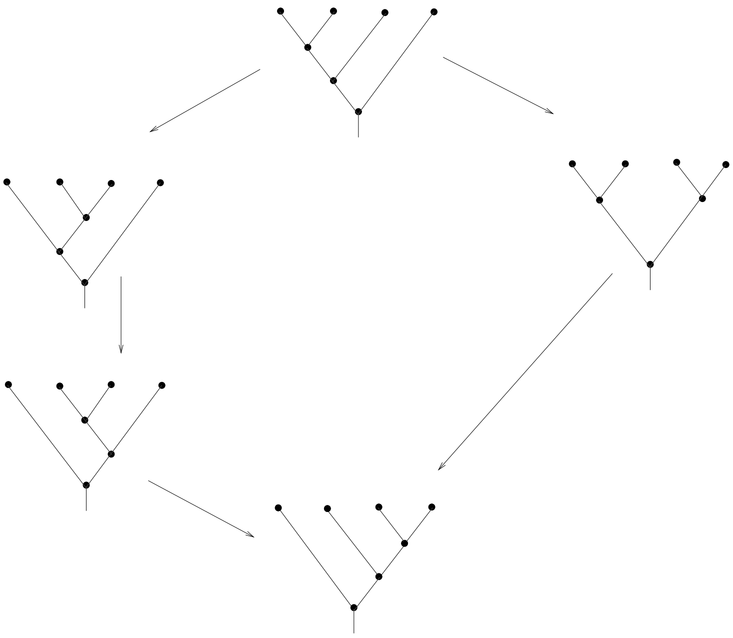
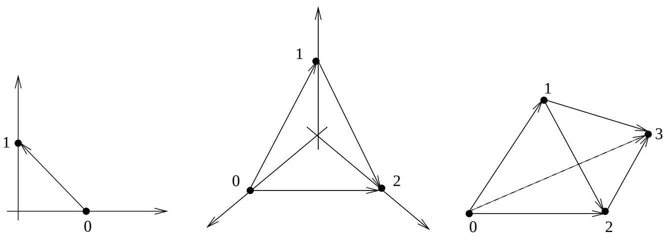


Figure 1.4: Tamari order on trees



Definition 1.2. *Define two kinds of order preserving maps on simplices*

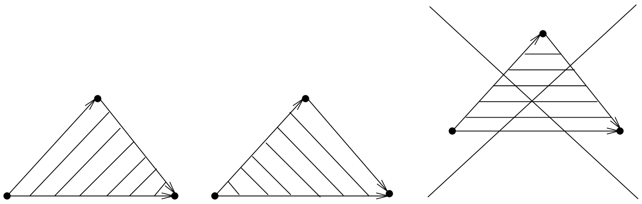
- Face maps $\delta_i\colon \Delta^{n-1} \to \Delta^n$, $i = 0, \dots, n$, *whose image is the face not containing* i *as image:*

$$\delta_i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n).$$

- Degeneracy maps $\sigma_j\colon \Delta^{n+1} \to \Delta^n$, $j = 0, \dots, n$ *which squeezes the* j*-th face:*

$$\sigma_j(x_0, \dots, x_{n+1}) = (x_0, \dots, x_{j-1}, x_j + x_{j+1}, x_{j+2}, \dots, x_{n+1}).$$

Degeneracy map which does not preserve the ordering on vertices is not allowed. For example if $n = 2$ we have two allowed degeneracies s_0, s_1



The face and degeneracy maps satisfy the following identities

$$\begin{aligned}
\delta_j\delta_i &= \delta_i\delta_{j-1}, \quad i<j\\
\sigma_j\sigma_i &= \sigma_i\sigma_{j+1}, \quad i\le j\\
\sigma_j\delta_i &= \begin{cases}\delta_i\sigma_{j-1} & i<j\\ \mathrm{id} & i=j,\ i=j+1\\ \delta_{i-1}\sigma_j & i>j+1\end{cases}
\end{aligned}$$

Definition 1.3. *A* simplicial set *is a collection of sets* $\{K_n\}_{n\ge 0}$ *with a collection of maps*

$$\begin{aligned}
d_i\colon K_n &\to K_{n-1}, \quad i=0,\dots,n\\
s_j\colon K_n &\to K_{n+1}, \quad j=0,\dots,n
\end{aligned}$$

satisfying "the dual relations"

$$\begin{aligned}
d_id_j &= d_{j-1}d_i, \quad i<j\\
s_is_j &= s_{j+1}s_i, \quad i\le j\\
d_js_i &= \begin{cases}s_{j-1}d_i & i<j\\ \mathrm{id} & i=j,\ i=j+1\\ s_jd_{i-1} & i>j+1\end{cases}
\end{aligned}$$

A simplicial morphism $\varphi_\bullet\colon K_\bullet\to K'_\bullet$ *is a collection of maps* $\varphi_n\colon K_n\to K'_n$ *which commute with face and degeneracy maps*

$$\begin{tikzcd}
K_n \arrow[r, "\varphi_n"] \arrow[d, "d_i^K"'] & K'_n \arrow[d, "d_i^{K'}"] \\
K_{n-1} \arrow[r, "\varphi_{n-1}"'] & K'_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
K_{n+1} \arrow[r, "\varphi_{n+1}"] & K'_{n+1} \\
K_n \arrow[u, "s_j^K"'] \arrow[r, "\varphi_n"'] & K'_n \arrow[u, "s_j^{K'}"]
\end{tikzcd}$$

Now suppose we have a simplicial set $K_\bullet$. For all $x\in K_n$ we take a simplex Δ^n and we will build a topological space out of these data.

The *geometric realization* of a simplicial set is the following topological space

$$|X_\bullet| := \coprod_{n\ge 0} X_n\times\Delta^n/\sim,$$

where the equivalence relation $\sim$ is defined as follows. We identify $(x,\delta_it)\in X_n\times\Delta^n$ with $(d_ix,t)\in X_{n-1}\times\Delta^{n-1}$ for any $x\in X_n$, $t\in\Delta^{n-1}$ and $(x,\sigma_jt)\in$

$X_n \times \Delta^n$ with $(s_j x, t) \in X_{n+1} \times \Delta^{n+1}$ for any $x \in X_{n-1}$ and $t \in \Delta^{n+1}$. The topology on $|X_\bullet|$ is the quotient topology.

There exists a *simplicial category* Δ, whose objects are finite ordered sets $[n] = \{0, \dots, n\}$, and morphism $\mathrm{Mor}([n], [m])$ are nondecreasing set maps.

The category Δ can be described by generators and relations. As generators we take face and degeneracy maps

$$\begin{aligned} \delta_i &\colon [n-1] \to [n] \\ \sigma_j &\colon [n+1] \to [n] \end{aligned}$$

and relations are as before

$$\begin{aligned} \delta_j \delta_i &= \delta_i \delta_{j-1}, \quad i < j \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1}, \quad i \leq j \\ \sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & i < j \\ \mathrm{id} & i = j,\ i = j+1 \\ \delta_{i-1} \sigma_j & i > j+1 \end{cases} \end{aligned}$$

A simplicial set is a functor $X \colon \Delta^{op} \to \mathbf{Sets}$.

Example 1.4. Take $X_n = \{*\}$ for all $n \geq 0$, d_i, s_j- the identity. Then $|\{*\}| = *$.

Example 1.5. Take a monoid M (or a group). Define $M_\bullet$ as follows.

$$M_n := M \underbrace{\times \dots \times}_{n \text{ times}} M = M^n$$

$$d_i(m_1, \dots, m_n) = \begin{cases} (m_2, \dots, m_n) & i = 0 \\ (m_1, \dots, m_i m_{i+1}, \dots, m_n) & 0 < i < n \\ (m_1, \dots, m_{n-1}) & i = n \end{cases}$$

$$s_j(m_1, \dots, m_n) = (m_1, \dots, m_j, 1, m_{j+1}, \dots, m_n)$$

Example 1.6. Let $\mathcal{C}$ be a small category. The *nerve* of $\mathcal{C}$ is the following simplicial set

$$\mathcal{C}_n := \{C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n\}$$

$$\begin{aligned} d_i(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n) &= \text{ forget about } C_i \\ &= (C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \to \\ &\quad \to C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \to \dots \xrightarrow{f_n} C_n) \\ s_j(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n) &= \text{ insert } \mathrm{id}_{C_j} \\ &= (C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \to \\ &\quad \to C_{j-1} \xrightarrow{f_j} C_j \xrightarrow{\mathrm{id}} C_j \xrightarrow{f_{j+1}} C_{j+1} \to \dots \xrightarrow{f_n} C_n) \end{aligned}$$

The axioms of a category are exactly the conditions for $\mathcal{C}_\bullet$ to be a simplicial set.

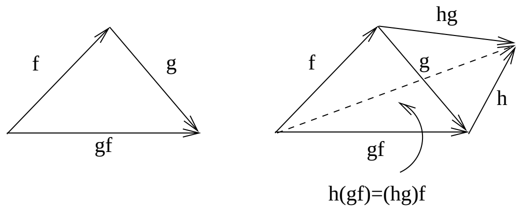


Figure 1.5: Associativity relation

To each category we associate its

$$\mathrm{B}\,\mathcal{C} := |\mathcal{C}_\bullet|$$

The $\mathrm{B}\,G$ of a group G is obtained from the realization of simplicial set in example (1.5). If G is discrete, then we can prove the following

$$\begin{aligned}\pi_1(\mathrm{B}\,G) &= G\\ \pi_n(\mathrm{B}\,G) &= 0, \quad n > 1.\end{aligned}$$

If all X_n are topological spaces, and the face and degeneracy maps are continuous, then we call $X_\bullet$ a *simplicial space*. Then the geometric realization is defined as before, but we keep track of the topology of X_n in the construction.

$$\left|X_\bullet\right| := \coprod_{n\geq 0} X_n \times \Delta^n / \sim,$$

$$\begin{aligned}(x, \delta_i t) &\sim (d_i x, t)\\ (x, \sigma_j t) &\sim (s_j x, t)\end{aligned}$$

1.3 Fibrations

A *locally trivial fibration* is a surjective map of topological spaces $f\colon E \to B$ such that for every $b \in B$ there exists an neighbourhood U_b of b in B such that $f^{-1}(U_b) \cong U_b \times F$, where F is a fiber.

Example 1.7. The Möbius band is a fibration over S^1. It is not a trivial fibration because it is not a product.

There is a fibration

$$G \to \mathrm{EG} \to \mathrm{BG}$$

where EG is a contractible space. For example if $G = \mathbb{Z}$, then this fibration is homotopy equivalent to

$$\mathbb{Z} \to \mathbb{R} \to S^1$$

But $\mathrm{B}\,\mathbb{Z}$ is not a space with one 0-cell and one 1-cell. The 0-cells are in bijection with $\mathbb{Z}$, and the 1-cells are in bijection with pairs of distinct integers.

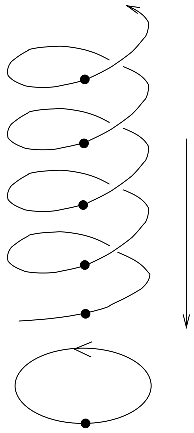


Figure 1.6: $\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1$

Example 1.8. The *Hopf fibration* is a map $f\colon S^3 \to S^2$ with fiber S^1 which can be described as follows.

$$\begin{aligned} S^3 &:= \{(z,z') : |z|^2 + |z'|^2 = 1\} \subset \mathbb{C} \times \mathbb{C}, \\ S^2 &:= \{(t,z) : t^2 + |z|^2 = 1\} \subset \mathbb{R} \times \mathbb{C}, \\ f(z,z') &= (|z|^2 - |z'|^2, 2zz') \in \mathbb{R} \times \mathbb{C}. \end{aligned}$$

The restriction of f to the north (resp. south) hemisphere is a trivial fibration.

Another description of the sphere S^3 is given by gluing two solid tori $S^1 \times D^2$ and $D^2 \times S^1$ along the boundary $S^1 \times S^1$.

If X and Y are pointed spaces, then we can perform the *join construction* $X * Y$:

$$X * Y := X \times I \times Y/\sim,$$

$$(x, 0, *) \sim (x', 0, *),$$

$$(*, 1, y) \sim (*, 1, y).$$

For example $S^1 * S^1 = S^3$.

Exercise 1.9. *Show that* $\Delta^p * \Delta^q \cong \Delta^{p+q+1}$.

1.4 Cyclic category

We know that $\mathrm{B}\,\mathbb{Z}$ is homotopy equivalent to S^1. Consider a question: what is the simplicial set $C_\bullet$ whose geometric realization is the circle with the cell structure consisting of one 0-cell and one 1-cell (not up to homotopy)?

The 0-cell $* \in C_0$ generate only one element, still denoted by $*$ in each C_n.

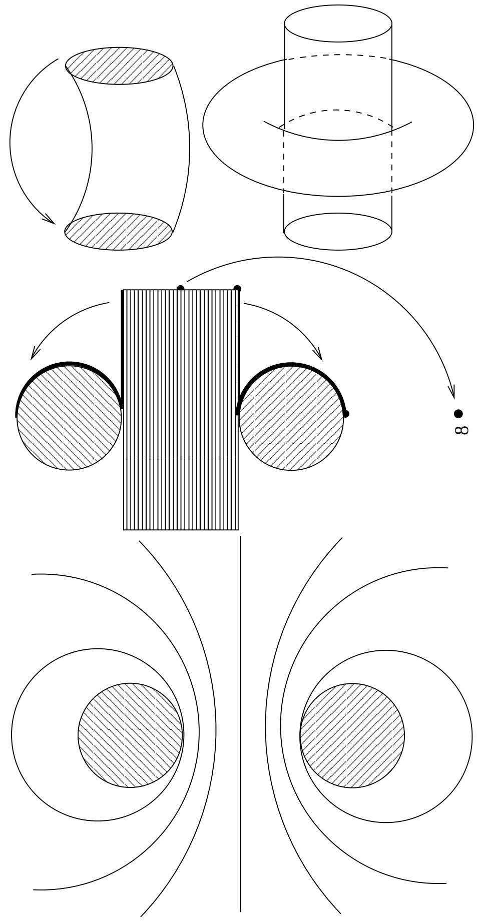


Figure 1.7: $S^3 = S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1$

Suppose we an add additional element τ to C_1. Then we get

$$
\begin{aligned}
&C_0 = \{*\} \\
&C_1 = \{*, \tau\} \\
&C_1 = \{*, s_0\tau, s_1\tau\} \\
&C_3 = \{*, s_1s_0\tau, s_2s_0\tau, s_2s_1\tau\} \\
&\dots \quad \dots \\
&C_n = \{*, \dots, s_{n-1}\dots\widehat{s_i}, \dots s_0\tau, \dots\}
\end{aligned}
$$

The faces are obvious to find. In particular $d_0(\tau) = * = d_1(\tau)$. Then $|C_\bullet|$ is a circle with its simplest cell structure. We can identify

$$C_n = \{*, \dots, s_{n-1}\dots\widehat{s_i}, \dots s_0\tau, \dots\}$$

with the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z} =: C_n$ by sending $*$ to 0, and $s_{n-1}\dots\widehat{s_i}, \dots s_0\tau$ to $i+1$. Denote the generator of C_n by t_n.

There exists a *cyclic category* ΔC whose objects are finite ordered sets $[n] = \{0, \dots, n\}$, and morphism $\mathrm{Mor}([n],[m])$ are generated by δ_i, σ_j as in the simplicial category, and an additional morphism $\tau_n \colon [n] \to [n]$ for all $n \geq 0$ satisfying the relations

$$\begin{aligned}
\tau_n^{n+1} &= \mathrm{id}_{[n]}, \\
\tau_n \delta_i &= \delta_{i-1}\tau_{n-1}, \quad 1 \leq i \leq n, \\
\tau_n \delta_0 &= \delta_n, \\
\tau_n \sigma_j &= \sigma_{j-1}\tau_{n+1}, \quad 1 \leq j \leq n, \\
\tau_n \sigma_0 &= \sigma_n \tau_{n+1}^2.
\end{aligned}$$

If in this presentation we omit the relation $\tau_n^{n+1} = \mathrm{id}_{[n]}$, then we get a different category, denoted $\Delta\mathbb{Z}$.

Definition 1.10. *A cyclic set is a functor $\Delta C^{op} \to$ **Sets**.*

Proposition 1.11. *$C_\bullet$ is a cyclic set.*

Proposition 1.12.

$$\mathrm{Aut}_\Delta([n]) = \{1\}$$
$$\mathrm{Aut}_{\Delta C}([n]) = C_n = \mathbb{Z}/(n+1)\mathbb{Z}$$

Every morphism of ΔC can be written uniquely as $\phi \circ g$, where $\phi \in \mathrm{Mor}_\Delta([n],[m])$, $g \in C_n = \mathrm{Mor}_{\Delta C}([n],[n])$. As sets

$$\mathrm{Hom}_{\Delta C}([n],[m]) \cong \mathrm{Hom}_\Delta([n],[m]) \times C_n$$

The composition of two morphisms $(g \circ \phi)$ and $(h \circ \psi)$ is in ΔC, so there exist $\phi^*(h) \in C_n$ and $h_*(\phi) \in \mathrm{Mor}_\Delta([n],[m])$ such that the following diagram commutes.

$$\begin{tikzcd}
{[n]} \arrow[r, "g"] & {[n]} \arrow[r, "\phi"] \arrow[dr, "\phi^*(h)"] & {[m]} \arrow[r, "h"] & {[m]} \arrow[r, "\psi"] & {[r]} \\
& & {[n]} \arrow[ur, "h_*(\phi)"'] & &
\end{tikzcd}$$

Analogously, suppose we have two subgroups $A, B \subseteq G$ such that every element of G can be written uniquely as $g = ab$, $a \in A$, $b \in B$. In this situation

$$gg' = aba'b' = a\,\underbrace{b^*(a')}_{\in A}\,\underbrace{a'_*(b)}_{\in B}\,b'.$$

The relations satisfied by ϕ^* and h_* are exactly the same as the relations satisfied by $b^* \colon A \to A$ and $a_* \colon B \to B$.

Remark 1.13. There is a way of constructing a category ΔS along the same lines, such that $\mathrm{Aut}_{\Delta S}([n]) = S_{n+1}$ - the symmetric group. Every morphism of ΔS can be written uniquely as $\phi \circ g$, where $\phi \in \mathrm{Mor}_\Delta([n],[m])$, $g \in S_n = \mathrm{Mor}_{\Delta S}([n],[n])$. As sets

$$\mathrm{Hom}_{\Delta C}([n],[m]) \cong \mathrm{Hom}_\Delta([n],[m]) \times S_n.$$

Example of such subgroups are $S_{n-1} \subset S_n$ and C_n generated by the cycle $(1\ 2\ \dots\ n)$. We can replace the cyclic group C_n by the symmetric group S_{n+1}

and construct a category ΔS. It means that for any $\phi \in \mathrm{Mor}_\Delta([m],[n])$ and $\sigma \in S_n$ there exist $\phi^*(\sigma) \in S_{m+1}$ and $\sigma_*(\phi) \in \mathrm{Mor}_\Delta([m],[n])$ such that the following diagram commutes:

$$\begin{tikzcd}
{[m]} \arrow[r, "\phi"] \arrow[d, "\phi^*(g)\in S_{n+1}"'] & {[n]} \arrow[d, "\sigma\in S_{n+1}"] \\
{[n]} \arrow[r, "\sigma_*(\phi)"'] & {[n]}
\end{tikzcd}$$

Denote by ΔB the braided category, defined along the same lines using braid

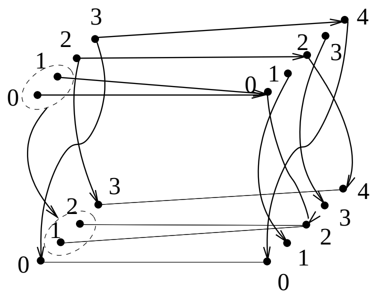


Figure 1.8: Morphisms in ΔS

groups, which contains ΔS as a subcategory. Let $H_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n = \mathbb{Z}/2\mathbb{Z} \int S_n$ and denote corresponding hyperdihedral category by ΔH. Furthermore we have a dihedral category ΔD. We can arrange them in a diagram of inclusions

$$\begin{tikzcd}
 & \Delta C \arrow[r] \arrow[d] & \Delta S \arrow[r] & \Delta B \\
\Delta\mathbb{Z}/2 \arrow[r] & \Delta D \arrow[r] & \Delta H &
\end{tikzcd}$$

There is an exact sequence of groups

$$0 \to \mathbb{Z} \xrightarrow{\cdot(n+1)} \mathbb{Z} \to \mathbb{Z}/(n+1)\mathbb{Z} \to 0$$

If we treat $\mathbb{Z}$ as category, then we have the following diagram of functors

$$\Delta \times \mathbb{Z} \to \Delta\mathbb{Z} \to \Delta C$$

We can ask what kind of structure on the geometric realization of the underlying simplicial set $X_\bullet$, that is $|X_\bullet|$, does the cyclic structure give? The answer is a structure of S^1-space. An open question is can we discretize analogously $S^3 = \mathrm{SU}(2)$?

1.5 Noncommutative sets

Let **Fin** denote the skeleton category of the category of finite sets. This means that the objects in **Fin** are the sets $[n] = \{0, 1 \ldots, n\}$ and the morphisms are arbitrary functions. Let F' denote a category with the same objects, but whose morphisms satisfy $f(0) = 0$. Then there is a following diagram of categories:

$$\begin{tikzcd} \Delta^{op} \arrow[r] \arrow[d] & \Delta S'^{op} \arrow[r] \arrow[d] & \mathbf{Fin}' \arrow[d] \\ \Delta C = \Delta C^{op} \arrow[r] & \Delta S \arrow[r] & \mathbf{Fin} \end{tikzcd}$$

For a set $[n]$ we have

$$\mathrm{Aut}_{\Delta S}([n]) = S_{n+1},$$
$$\mathrm{Aut}_{\Delta S'}([n]) = S_n.$$

The top row of this diagram will correspond to Hochschild homology, and the bottom row to cyclic homology, which we will define in the next chapter.

If A is an algebra, then $[n] \mapsto A^{\otimes(n+1)}$ is a well defined functor $\Delta S \to \mathbf{Mod}$.

$$A^{\otimes 2} \rightrightarrows A,\ a \otimes b \mapsto ab,\ a \otimes b \mapsto ba.$$

The two maps $d_1, d_0 \colon [1] \to [0]$ become the same in **Fin**. If A is commutative, then $[n] \to A^{\otimes(n+1)}$ factors through **Fin**.

Thus ΔS can be viewed as a category of noncommutative sets. It has the following description

$$\mathrm{Ob}(\Delta S) = \{[n]\}$$

$\mathrm{Mor}_{\Delta S}([n],[m]) =$ set maps preserving the order on fibers $f^{-1}(i)$ for any $i \in [m]$.

1.6 Adjoint functors

Suppose we have two categories $\mathcal{A}$ and $\mathcal{B}$ and a pair of functors $F \colon \mathcal{A} \to \mathcal{B}$, $G \colon \mathcal{B} \to \mathcal{A}$. We say that F is *right adjoint* to G and G is *left adjoint* to F if there is an isomorphism of sets

$$\mathrm{Hom}_{\mathcal{A}}(G(B), A) \cong \mathrm{Hom}_{\mathcal{B}}(B, F(A))$$

for every $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and the isomorphism is functorial in A and B.

Example 1.14. Let $\mathcal{A}, \mathcal{B} = \mathbf{Sets}$. Take a set X and define

$$G(B) = B \times X, \qquad F(A) = \mathrm{Hom}_{\mathbf{Sets}}(X, A)$$

Then

$$\mathrm{Hom}(B \times X, A) \cong \mathrm{Hom}(B, \mathrm{Hom}(X, A))$$
$$\varphi \colon B \times X \to A \mapsto (B \to \mathrm{Hom}(X, A))$$

Many examples follow the pattern in (1.14), but with additional structure.

Example 1.15. Let $\mathcal{A}, \mathcal{B} = \mathbf{Vect}$, V vector space over a field k. Define

$$G(B) = B \otimes_k V, \qquad F(A) = \mathrm{Hom}_k(V, A)$$

Then

$$\mathrm{Hom}_k(B \otimes_k V, A) = \mathrm{Hom}_k(B, \mathrm{Hom}_k(V, A))$$

Example 1.16. Let R be a ring, $\mathcal{A}$ be the category of left R-modules, and $\mathcal{B}$ the category or right R-modules. Take a left R-module V and define

$$G(B) = B \otimes_R V, \qquad F(A) = \mathrm{Hom}_R(V, A)$$

$$\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R V, A) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathrm{Hom}_R(V, A))$$

Example 1.17. Define the loop space and the suspension of a topological space X with base point as follows.

$$\Omega X = \{f \colon S^1 \to X : f(*) = *\}$$

$$SX = S^1 \wedge X / S^1 \vee X$$

Then

$$\mathrm{Hom}_{\mathbf{Top}_*}(SX, Y) \cong \mathrm{Hom}_{\mathbf{Top}_*}(X, \Omega Y)$$

where $\mathbf{Top}_*$ is the category of topological spaces with base point.

1.7 Generic example of a simplicial set

Let X be a topological space. Define

$$\mathcal{S}_n(X) := \{f \colon \Delta^n \to X, \text{ continuous}\}$$

We claim that $\mathcal{S}_\bullet(X)$ is a simplicial set with the following face and degeneracy maps:

$$\begin{aligned} d_i \colon \mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X), &\quad d_i(f) := f \circ \delta_i \\ s_j \colon \mathcal{S}_n(X) \to \mathcal{S}_{n+1}(X), &\quad s_j(f) := f \circ \sigma_j \end{aligned}$$

It is called the *singular functor*. It goes from the category of topological spaces to the category of simplicial sets.

$$\mathcal{S}_\bullet(-) \colon \mathbf{Top} \to \mathbf{SSets}$$

Recall the functor of geometric realization of a simplicial set,

$$K_\bullet \mapsto |K_\bullet|, \quad |-| \colon \mathbf{SSets} \to \mathbf{Top}$$

Proposition 1.18. *The functors $\mathcal{S}_\bullet(-)$ and $|-|$ are adjoint, that is*

$$\mathrm{Hom}_{\mathbf{Top}}(|K_\bullet|, X) \cong \mathrm{Hom}_{\mathbf{SSets}}(K_\bullet, \mathcal{S}_\bullet(X)).$$

In the example (1.16) R-modules can be replaced by functors. Left modules correspond to covariant functors, and right modules correspond to contravariant functors. Then the geometric realization functor can be seen as a tensor product over the simplicial category

$$|K_\bullet| = K_\bullet \otimes_\Delta \Delta^\bullet$$

In an analogous way we can present the singular functor as

$$\mathcal{S}_\bullet(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^\bullet, X)$$

Hence we can derive adjointness

$$\mathrm{Hom}_{\mathbf{Top}}(K_\bullet \otimes_\Delta \Delta^\bullet, X) \cong \mathrm{Hom}_\Delta(K_\bullet, \mathrm{Hom}_{\mathbf{Top}}(\Delta^\bullet, X))$$

Now the question arises: how to compare X and $|\mathcal{S}_\bullet(X)|$? Take the identity

$$\mathrm{id} \in \mathrm{Hom}_{\mathbf{SSets}}(\mathcal{S}_\bullet(X), \mathcal{S}_\bullet(X))$$

which goes to a map

$$\varepsilon\colon |\mathcal{S}_\bullet(X)| \to X$$

which is called a *unit*. Also $\mathrm{id} \in \mathrm{Hom}_{\mathbf{Top}}(|K_\bullet|, |K_\bullet|)$ goes to a map

$$\eta\colon K_\bullet \to \mathcal{S}_\bullet(|K_\bullet|)$$

which is called a *counit*. If X is a CW-complex, then the map ε is a homotopy equivalence.

Now we will prove the following theorem.

Theorem 1.19. *If $X_\bullet$ is a cyclic set, then the geometric realization $|X_\bullet|$ is an S^1-space.*

Before the proof, we will give some necessary propositions.

Lemma 1.20. *The functor $\Delta \to$ **Top** given by $[n] \mapsto \Delta^n$ is in fact a functor on ΔC (it is a* cocyclic space*).*

Proof. It is enough to define the image of τ_n

$$\tau_n \mapsto \{\Delta^n \to \Delta^n\}$$

$$\text{vertex } i \mapsto \text{vertex } i-1$$

$$\text{vertex } 0 \mapsto \text{vertex } n$$

□

Let $C_\bullet$ be the cyclic set, whose geometric realization is the circle. A naive way to define an S^1-action would be to use

$$C_\bullet \times X_\bullet \to X_\bullet$$

$$(g, x) \mapsto g_*(x)$$

But it does not work, since it gives a trivial action of S^1 for $X_\bullet = C_\bullet$.

There is a forgetful functor from the category of cyclic sets to the category of simplicial sets

$$G\colon \mathbf{CSets} \to \mathbf{SSets}.$$

We will define its left adjoint

$$F\colon \mathbf{SSets} \to \mathbf{CSets}.$$

If $Y_\bullet$ is a simplicial set, then put

$$F(Y_\bullet)_n := C_n \times Y_n, \quad C_n = \mathbb{Z}/(n+1)\mathbb{Z}$$

If f is a morphism in Δ^{op}, then we define

$$f_*(g, y) := (f_*(g), (g^*(f))_*(y))$$

$$\begin{tikzcd}
[n] \arrow[r, "f"] \arrow[d, "f_*(g)"'] & [m] \arrow[d, "g"] \\
[n] \arrow[r, "g^*(f)"'] & [m]
\end{tikzcd}$$

If h is a morphism in C_m, then we define

$$h^*(g, y) := (h(g), y)$$

Proposition 1.21. *The set $F(Y_\bullet)$ equipped with the Simplicial structure given by f_* and the cyclic structure given by h^* is a cyclic set.*

Proposition 1.22. *If $X_\bullet$, $Y_\bullet$ are simplicial sets, and if $|X_\bullet| \times |Y_\bullet|$ is a CW-complex, then the map*

$$|X_\bullet \times Y_\bullet| \to |X_\bullet| \times |Y_\bullet|$$

is a homeomorphism.

Proposition 1.23. *If $X_\bullet$ is a cyclic set, then we have a homeomorphism*

$$|F(X_\bullet)| \cong |C_\bullet| \times |X_\bullet| = S^1 \times |X_\bullet|$$

Observe that the composite

$$|F(X_\bullet)| \to |C_\bullet| \times |X_\bullet| \xrightarrow{\cong} |C_\bullet \times X_\bullet|$$

is not the geometric realization of a simplicial map.

Proof. It is induced by the two projections

$$|F(X_\bullet)| \xrightarrow{p_1 \times p_2} |C_\bullet| \times |X_\bullet|$$

The map p_1 is induced by $(g, y) \mapsto g$, and p_2 is induced by $(g, y) \mapsto y$.

Next we define

$$C_n \times X_n \times \Delta^n \to X_n \times \Delta^n$$

$$(g, y, t) \mapsto (y, g^*(t))$$

and show that it is compatible with the equivalence relation. It induces a cyclic map called the *evaluation*

$$F(X_\bullet) \xrightarrow{ev} X_\bullet$$

which gives a map

$$|F(X_\bullet)| \xrightarrow{|ev|} |X_\bullet|$$

□

Proof. (of theorem (1.19)) Define a map

$$S^1 \times |X_\bullet| \xrightarrow{\cong} |C_\bullet| \times |X_\bullet| \xrightarrow{(p_1,p_2)^{-1}} |F(X_\bullet)| \xrightarrow{ev} |X_\bullet|$$

If we want it to be an S^1-action on $|X_\bullet|$, then the following diagram has to commute

$$\begin{tikzcd} S^1 \times S^1 \times |X_\bullet| \arrow[r] \arrow[d] & S^1 \times |X_\bullet| \arrow[d] \\ S^1 \times |X_\bullet| \arrow[r] & |X_\bullet| \end{tikzcd}$$

Let $X_\bullet = C_\bullet$. We will show that, the action $S^1 \times S^1 \to S^1$ is the classical multiplication of units in $\mathbb{C}$.

$$\begin{aligned} F(C_\bullet)_0 &= (*,*), \quad 1 \in C_0, \\ F(C_\bullet)_1 &= \underbrace{(*,t_1),\ (t_1,*),\ (t_1,t_1),}_{\text{nondegenerate simplices}}\ (*,*), \\ F(C_\bullet)_2 &= (t_2,t_2),\ (t_2^2,t_2^2),\ \text{all other simplices are degenerate.} \end{aligned}$$

The higher rank simplices are degenerate.

We will examine the evaluation map

$$S^1 \times S^1 = |F(C_\bullet)| \to |C_\bullet| = S^1$$

Take $(u,v) \in |F(C_\bullet)|$. Then

$$(u,v) \in \begin{cases} \{(t_2^2,t_2^2) \times \Delta^2\} & \text{if } u+v \le 1 \\ \{(t_2,t_2) \times \Delta^2\} & \text{if } u+v \ge 1 \end{cases}$$

The formulas

$$\begin{aligned} d_0(t_2,t_2) &= (*,t_1) \\ d_2(t_2^2,t_2^2) &= (t_1,*) \end{aligned}$$

show that the 0-th face of the triangle (t_2,t_2) has to be identified with the 2-nd face of the triangle (t_2^2,t_2^2).

$$F(C_\bullet) \xrightarrow{ev} C_\bullet, \quad (t_2,t_2) \mapsto t_2$$

$$\begin{aligned} ev(t_2^2,t_2^2) &= t_2^4 = t_2 = s_1(t_1), \text{ because } t_2^3 = 1 \\ ev(t_2,t_2) &= t_2^2 = s_0(t_1) \end{aligned}$$

$$\begin{tikzcd} & F(C_\bullet) \arrow[dl, "p_1 \times p_2"'] \arrow[dr, "|ev|"] & \\ |C_\bullet| \times |C_\bullet| \arrow[rr] & & |C_\bullet| \end{tikzcd}$$

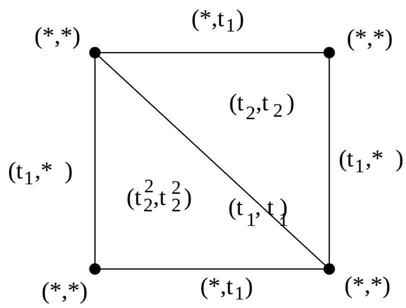


Figure 1.9: 0,1, and 2-faces

$$S^1 \times |C_\bullet| \to |C_\bullet|$$

$$\begin{aligned} C_0 &= \{1\} \\ C_1 &= \{1, t_1\} \\ C_2 &= \{1, t_2, t_2^2\} \end{aligned}$$

Degenerate simplices will be identified with the interval. There are two ways to do that.

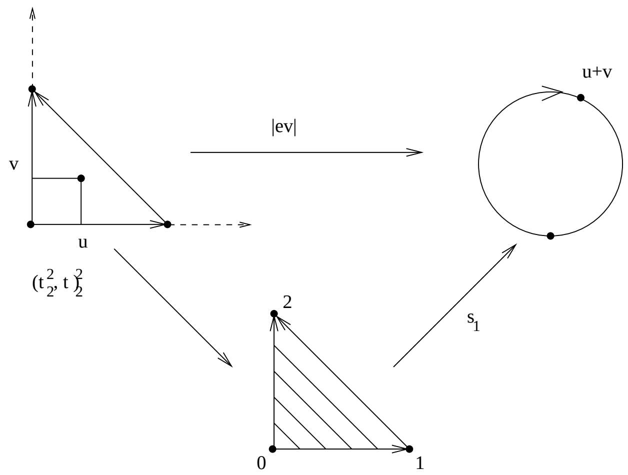


Figure 1.10:

$$\begin{array}{ccc} \bigcup_{n\geq 0} F(C_\bullet) \times \Delta^n & \longrightarrow & \bigcup_{n\geq 0} C_n \times \Delta^n \\ \downarrow & & \downarrow \\ |F(C_\bullet)| & \longrightarrow & |C_\bullet| \end{array}$$

$$\begin{aligned} ev\colon (t_2, t_2) \times \Delta^2 &\mapsto (s_0 t_1, \Delta^2) \\ ev\colon (t_2^2, t_2^2) \times \Delta^2 &\mapsto (s_1 t_1, \Delta^2) \end{aligned}$$

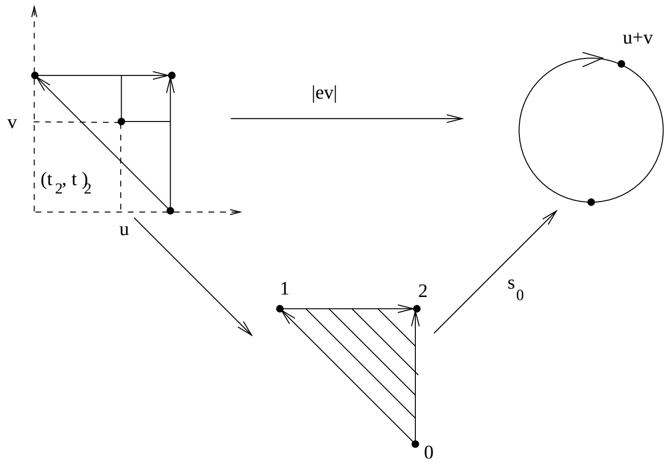


Figure 1.11:

Therefore the map $|ev|\colon S^1 \times S^1 \to S^1$ is the multiplication of complex units (under the exponential map $\exp(2\pi i -)\colon \mathbb{R}/\mathbb{Z} \to \mathrm{SO}(2)$).

At the end we get a commutative diagram:

$$\begin{tikzcd}
S^1 \times S^1 \times |X_\bullet| \arrow[rrrr] \arrow[dddd] & & & & S^1 \times |X_\bullet| \arrow[dddd] \\
& & F(C_\bullet) \times |X_\bullet| \arrow[ull] \arrow[urr] & & \\
& |C_\bullet| \times F(X_\bullet) \arrow[uul] \arrow[ddl] & |F(F(X_\bullet))| \arrow[l] \arrow[r] \arrow[u] \arrow[d] & |F(X_\bullet)| \arrow[uur] \arrow[ddr] & \\
& & |F(X_\bullet| \arrow[dll] \arrow[drr] & & \\
S^1 \times |X_\bullet| \arrow[rrrr] & & & & |X_\bullet|
\end{tikzcd}$$

As a consequence $|X_\bullet|$ is an S^1-space. ☐

1.8 Simplicial modules

Definition 1.24. *A* simplicial module *is a functor*

$$\Delta^{op} \to \mathbf{Mod}_k, \quad [n] \mapsto M_n$$

There is a chain complex associated to any simplicial module

$$M_\bullet : \quad \ldots \to M_n \xrightarrow{b_n} M_{n-1} \xrightarrow{b_{n-1}} M_{n-1} \to \ldots$$

where $b = b_n = \sum_{i=0}^{n} (-1)^i d_i$. We have $b^2 = 0$ as an immediate consequence of $d_i d_j = d_{j-1} d_i$, $i < j$, for example:

$$\ldots M_2 \xrightarrow{d_0 - d_1 + d_2} M_1 \xrightarrow{d_0 - d_1} M_0$$

$$(d_0 - d_1)(d_0 - d_1 + d_2) = \underbrace{d_0 d_0 - d_0 d_1}_{0} + \underbrace{d_0 d_2 - d_1 d_0}_{0} + \underbrace{d_1 d_1 - d_1 d_2}_{0} = 0$$

We define the homology of a simplicial module as

$$\mathrm{H}_n(M_\bullet) := \ker(b_n)/\operatorname{im}(b_{n-1})$$

It is well defined for pre simplicial module, that is using only face maps.

Lemma 1.25. *The submodule M'_n of M_n spanned by the degeneracy elements gives a subcomplex $M'_\bullet$ of $M_\bullet$.*

Proof. This is a consequence of the relations between s_j, d_i. □

Define the *normalized* complex $\overline{M}_\bullet$ as a quotient

$$0 \to M'_\bullet \to M_\bullet \to \overline{M}_\bullet \to 0$$

Theorem 1.26. *The quotient map $M_\bullet \to \overline{M}_\bullet$ is a quasi-isomorphism, i.e. it induces an isomorphism in homology.*

Proof. From the long exact sequence in homology

$$\ldots \to \mathrm{H}_n(M'_\bullet) \to \mathrm{H}_n(M_\bullet) \to \mathrm{H}_n(\overline{M}_\bullet) \xrightarrow{\delta} \mathrm{H}_{n-1}(M'_\bullet) \to \ldots$$

it is enough to prove that $\mathrm{H}_n(M'_\bullet) = 0$.

If one wants to prove that some complex $C_\bullet$ is acyclic, then it is enough to construct a homotopy from id to 0 (contraction), that it a sequence of maps $h_n \colon C_n \to C_{n+1}$ such that $h_{n-1}d_{n-1} + d_n h_n = \mathrm{id}$. Unfortunately it is hard to find a contracting homotopy for $M'_\bullet$ to prove that it is acyclic. But one can define a filtration on $M'_\bullet$

$$F_k \hookrightarrow F_{k+1} \to G_k$$

with F_k spanned by the first k degeneracies, and quotient $G_\bullet$ for which we can construct a contracting homotopy. Then we can proceed by induction. □

Let A be a k-algebra and M an A-bimodule. There is a simplicial module

$$C_\bullet(A, M) := M \otimes A^{\otimes n}$$

$$\begin{aligned}
d_i(a_0, a_1, \ldots, a_n) &= (a_0, \ldots, a_i a_{i+1}, \ldots, a_n), \quad i = 0, \ldots, n-1 \\
d_n(a_0, a_1, \ldots, a_n) &= (a_n a_0, \ldots, a_{n-1}) \\
s_j(a_0, a_1, \ldots, a_n) &= (a_0, \ldots, a_j, 1, a_{j+1}, \ldots, a_n)
\end{aligned}$$

Define

$$b := \sum_{i=0}^{n} (-1)^i d_i$$

Then $(C_\bullet(A, M), b)$ is called the *Hochschild chain complex*, and its homology $\mathrm{H}_*(A; M)$ the *Hochschild homology* of A with coefficients in M. If $M = A$, then we denote

$$\mathrm{H}_*(A; A) =: \mathrm{HH}_*(A)$$

Suppose that A is augmented and let $\overline{A}$ be its augmentation ideal, that is $A = \overline{A} \oplus k1$. Define the *reduced Hochschild complex* as

$$\overline{C}_n(A, M) := M \otimes \overline{A}^{\otimes n}$$

If $M = A = \bar{A} \oplus k1$, then $C_\bullet(A, A)$ has extra degeneracy

$$s_{-1}(a_0, \dots, a_n) = (1, a_0, \dots, a_n).$$

We have

$$\begin{aligned} d_0(1, a_1, \dots, a_n) &= (a_1, \dots, a_n), \\ d_n(1, a_1, \dots, a_n) &= (a_n, \dots, a_1). \end{aligned}$$

Define also two maps on $\overline{A}^{\otimes n}$

$$\begin{aligned} t(a_1, \dots, a_n) &:= (-1)^n (a_n, a_1, \dots, a_{n-1}), \\ b' &:= \sum_{i=0}^{n-1} (-1)^i d_i, \quad (b = b' + (-1)^n d_n). \end{aligned}$$

1.9 Bicomplexes

Assume we have an array of k-modules

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ C_{02} \arrow[d, "d^v"'] & C_{12} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{22} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\ C_{01} \arrow[d, "d^v"'] & C_{11} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{21} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\ C_{00} & C_{10} \arrow[l, "d^h"'] & C_{20} \arrow[l, "d^h"'] & \cdots \arrow[l] \end{tikzcd}$$

We call it a *bicomplex* of k-modules if the maps d^v and d^h, called vertical and horizontal differential, satisfy

$$\begin{aligned} d^v \circ d^v &= 0, \\ d^h \circ d^h &= 0, \\ d^h \circ d^v + d^v \circ d^h &= 0. \end{aligned}$$

For a bicomplex $C_{\bullet\bullet}$ we define a *total complex* as

$$\mathrm{Tot}(C_{\bullet\bullet})_n := \bigoplus_{p+q=n} C_{pq}, \quad d := d^h + d^v$$

After taking homology with respect to the vertical differential we obtain a complex

$$\ldots \leftarrow \mathrm{H}^v_{(p-1),\bullet} \leftarrow \mathrm{H}^v_{p,\bullet} \leftarrow \mathrm{H}^v_{(p+1),\bullet} \leftarrow \ldots$$

with the differential induced on homology by horizontal differential in the bicomplex. Now we can take homology of this complex and obtain

$$E^2_{pq} := \mathrm{H}^h_q(\mathrm{H}^v_{p,\bullet}).$$

There is a decomposition of the reduced Hochschild complex

$$\overline{C}_n(A,A) = A \otimes \overline{A}^{\otimes n} = (\overline{A} \oplus k1) \otimes \overline{A}^{\otimes n} = \overline{A}^{\otimes(n+1)} \oplus \overline{A}^{\otimes n}$$

and a map

$$\begin{pmatrix} b & 1-t \\ 0 & -b' \end{pmatrix} : \overline{A}^{\otimes(n+1)} \oplus \overline{A}^{\otimes n} \to \overline{A}^{\otimes n} \oplus \overline{A}^{\otimes(n-1)}$$

which fits in the diagram

$$\begin{tikzcd}
\overline{C}_n(A,A) \arrow[r, "\cong"] \arrow[d, "b"'] & \overline{A}^{\otimes(n+1)} \oplus \overline{A}^{\otimes n} \arrow[d, "\begin{pmatrix} b & 1-t \\ 0 & -b' \end{pmatrix}"] \\
\overline{C}_{n-1}(A,A) \arrow[r, "\cong"] & \overline{A}^{\otimes n} \oplus \overline{A}^{\otimes(n-1)}
\end{tikzcd}$$

This complex can be thought of as the total complex of a bicomplex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\overline{A}^{\otimes 3} \arrow[d, "b"'] & \overline{A}^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] \\
\overline{A}^{\otimes 2} \arrow[d, "b"'] & \overline{A}^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] \\
\overline{A} & \overline{A} \arrow[l, "1-t"']
\end{tikzcd}$$

Here we see the beginning of the complex computing the homology of the cyclic group with coefficients in a module. This will lead to the cyclic bicomplex.

1.10 Spectral sequences

Having computed $E^2_{pq} = \mathrm{H}^h_q(\mathrm{H}^v_{p,\bullet})$ of a bicomplex $C_{\bullet\bullet}$ it seems that we have used all data, that is vertical and horizontal differentials in the bicomplex. However, there is a piece of information which we can extract in addition to E^2_{pq}. We can define a homomorphism

$$d^2 \colon E^2_{pq} \to E^2_{p-2,q+1}$$

as follows.

$$\begin{tikzcd} C_{p-2,q+1} & C_{p-1,q+1} \arrow[l, "d_h"'] \arrow[d, "d^v"'] & \\ & C_{p-1,q} & C_{pq} \arrow[l, "d^h"'] \end{tikzcd}$$

Starting with a horizontal cycle $x \in \mathrm{Z}_p(C_{\bullet,q})$ we want to define an element in $C_{p-2,q-1}$ which represents an element in horizontal cycles of vertical homology complex, that is in $\mathrm{Z}^h_p(\mathrm{H}^v_q(C_{\bullet\bullet}))$. Our cycle x gives a class $[x] \in \mathrm{H}^v_q(C_{\bullet\bullet})$. Using the induced map

$$d^h_*\colon\ \mathrm{H}^v_q(C_{p,\bullet}) \to \mathrm{H}^v_q(C_{p-1,\bullet})$$

we have $d^h_*([x]) = 0 = [d^h(x)]$. Saying that the homology class is zero means that the cycle is in fact a boundary. Therefore there exists an element $y \in C_{p-1,q+1}$ such that $d^v(y) = d^h(x)$. Now we define our cycle as $d^h(y) \in C_{p-2,q+1}$.

$$\begin{tikzcd} d^h(y) & y \arrow[l, "d_h"'] \arrow[d, "d^v"'] & \\ & d^v(y) = d^h(x) & x \arrow[l, "d^h"'] \end{tikzcd}$$

We claim that this element defines an element in $E^2_{p-2,q+1}$ which does not depend on the choice of y nor on the choice of the representative of $[x]$. Thus we have defined

$$d^2\colon E^2_{pq} \to E^2_{p-2,q+1}, \quad [x] \mapsto [d^h(y)].$$

Furthermore $d^2 \circ d^2 = 0$, so now we can take homology to obtain E^3_{pq} and

$$d^3\colon E^2_{pq} \to E^3_{p-3,q+2}.$$

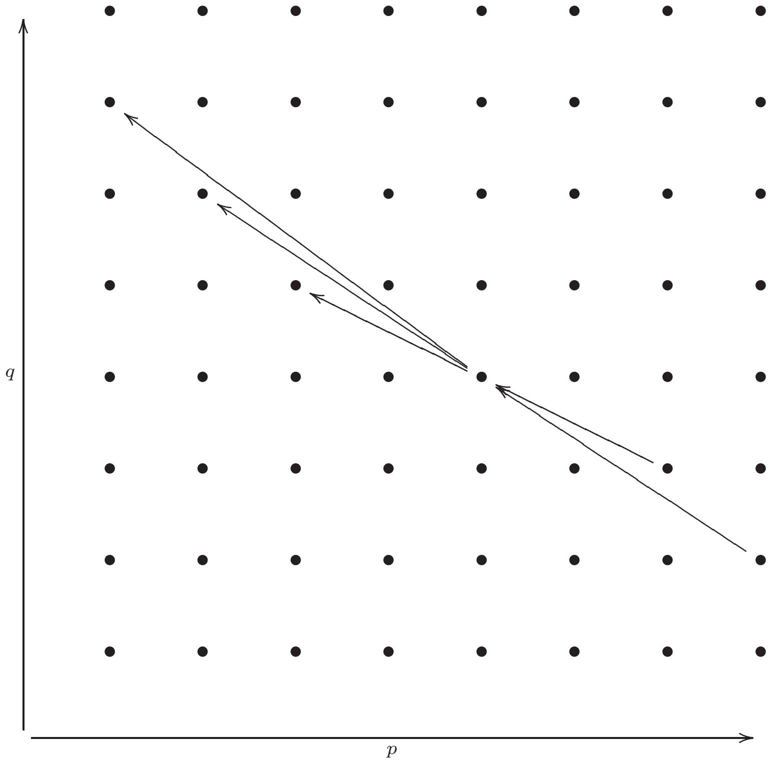
This procedure can be continued and as a result we get a sequence of arrays E^r_{pq} for any $r \geq 2$ and maps

$$d^r\colon E^r_{pq} \to E^r_{p-r,q+r-1}$$

such that E^r_{pq} is the homology of the complex (E^{r-1}, d^{r-1}) at the place (p, q). Furthermore there are subspaces B^r_{pq}, Z^r_{pq} of C_{pq}

$$B^2_{pq} \subseteq B^3_{pq} \subseteq \ldots \subseteq B^\infty_{pq} \subseteq Z^\infty_{pq} \subseteq \ldots \subseteq Z^2_{pq} \subseteq Z^2_{pq} \subseteq C_{pq}$$

such that $E^r_{pq} = Z^r_{pq}/B^r_{pq}$.



When both differentials (leaving and entering) for E^r_{pq} are zero, this component does not change furthermore and we have $E^r_{pq} = E^{r+1}_{pq} = \ldots$. We denote this stable component by E^∞_{pq}.

There is a filtration on the total complex

$$\mathrm{F}_p \operatorname{Tot} C_{\bullet\bullet} := \operatorname{Tot} \bigoplus_{k \le p} C_{k\bullet},$$

$$0 \subseteq F_0 \subseteq F_1 \subseteq \ldots \subseteq F_{p-1} \subseteq F_p \subseteq \ldots \subseteq \operatorname{Tot} C_{\bullet\bullet}.$$

This filtration induces a filtration on $\mathrm{H}_*(\operatorname{Tot} C_{\bullet\bullet})$

$$\mathrm{F}_p := \mathrm{F}_p \,\mathrm{H}_*(\operatorname{Tot} C_{\bullet\bullet}) := \operatorname{im}(\mathrm{H}_*(\mathrm{F}_p \operatorname{Tot} C_{\bullet\bullet}) \to \mathrm{H}_*(\operatorname{Tot} C_{\bullet\bullet})).$$

Denote the quotient

$$F_p/F_{p-1} =: \operatorname{gr}_p(\mathrm{H}_{p+q}(\operatorname{Tot} C_{\bullet\bullet})).$$

All data defined above, that is $\{E^r_{pq}, d^r\}_{p,q,r}$ and a filtration $\{F_p\}_p$ define a *spectral sequence* of a bicomplex $C_{\bullet\bullet}$. We say that the spectral sequence abuts to $\mathrm{H}_*(\operatorname{Tot} C_{\bullet\bullet})$, which means that there is an isomorphism

$$E^\infty_{pq} \cong \operatorname{gr}_p(\mathrm{H}_{p+q}(\operatorname{Tot} C_{\bullet\bullet}))$$

We write

$$E^2_{pq} = \mathrm{H}^h_p(\mathrm{H}^v_q(C_{\bullet\bullet})) \implies \mathrm{H}_{p+q}(\operatorname{Tot} C_{\bullet\bullet}).$$

which is to read as: there is a spectral sequence starting at E^2_{pq} and converging to $\mathrm{H}_{p+q}(\operatorname{Tot} C_{\bullet\bullet})$

Example 1.27. The typical theorem using spectral sequences in algebraic topology looks as follows

Theorem 1.28. *Let $F \to E \to B$ be a fibration of connected spaces, with B simply connected. Then there is a spectral sequence*

$$E^2_{pq} = \mathrm{H}_p(B; \mathrm{H}_q(F)) \implies \mathrm{H}_{p+q}(E).$$

The implicit data in this theorem are $E^3_{pq}, E^4_{pq}, \ldots$, the filtration F_p on $\mathrm{H}_*(E)$. The sign " $\implies$ " means that there is an isomorphism

$$E^\infty_{pq} \cong \mathrm{gr}_p(\mathrm{H}_{p+q}(E)).$$

In many cases we do not need to look at E^r_{pq} for $r \geq 3$ and at the filtration. That is why these data are often omitted in the theorems.

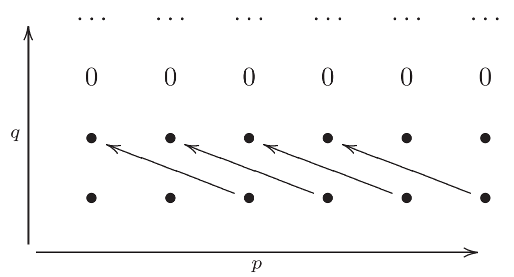
Example 1.29. Let X be an S^1-space, $\mathrm{E}\,S^1$ the contractible space of paths on S^1. Consider the Borel space $\mathrm{E}\,S^1 \times_{S^1} X$ and the S^1-fibration

$$S^1 \hookrightarrow \mathrm{E}\,S^1 \times_{S^1} X \to X.$$

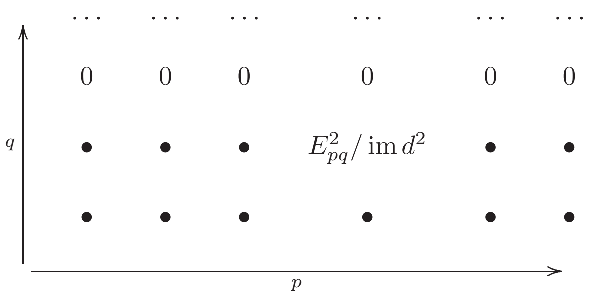
The homology of a fiber is

$$\begin{aligned}
&\mathrm{H}_0(S^1) = \mathbb{Z}, \\
&\mathrm{H}_1(S^1) = \mathbb{Z}, \\
&\mathrm{H}_q(S^1) = 0, \quad q \geq 2.
\end{aligned}$$

E^2 :



$E^3 = \ldots = E^\infty$:



For any S^1-fibration $S^1 \hookrightarrow E \xrightarrow{f} B$ of pointed spaces we obtain a Gysin sequence

$$\ldots \to \mathrm{H}_n(E) \xrightarrow{f_*} \mathrm{H}_n(B) \xrightarrow{d^2} \mathrm{H}_{n-2}(B) \to \mathrm{H}_{n-1}(E) \to \ldots.$$

Recall that for the bicomplex we took the vertical homology and then horizontal homology. We could have done it the other way. Any bicomplex gives a rise to two spectral sequences

$$\begin{aligned} E'^2_{pq} &= \mathrm{H}^h_p(\mathrm{H}^v_q(C_{\bullet\bullet})) \implies \mathrm{H}_{p+q}(\mathrm{Tot}(C_{\bullet\bullet})), \\ E''^2_{pq} &= \mathrm{H}^v_p(\mathrm{H}^h_q(C_{\bullet\bullet})) \implies \mathrm{H}_{p+q}(\mathrm{Tot}(C_{\bullet\bullet})). \end{aligned}$$

But remark that the filtrations are different on $\mathrm{Tot}(C_{\bullet\bullet})$.

Chapter 2

Cyclic homology

2.1 The cyclic bicomplex

Let $C_\bullet$ be the cyclic module with

$$\begin{aligned} d_i &: C_n \to C_{n-1}, \\ t_n &: C_n \to C_n. \end{aligned}$$

Consider the following two-column bicomplex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
C_2 \arrow[d, "b"'] & C_2 \arrow[l, "1-t"'] \arrow[d, "-b'"] \\
C_1 \arrow[d, "b"'] & C_1 \arrow[l, "1-t"'] \arrow[d, "-b'"] \\
C_0 & C_0 \arrow[l, "1-t"']
\end{tikzcd}$$

One checks that it has anticommuting squares, so it is indeed a bicomplex. It can be extended to the right using the map $N := 1 + t + \ldots t^n : C_n \to C_n$.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
C_2 \arrow[d, "b"'] & C_2 \arrow[l, "1-t"'] \arrow[d, "-b'"] & C_2 \arrow[l, "N"'] \arrow[d, "b"'] & C_2 \arrow[l, "1-t"'] \arrow[d, "-b'"] & \cdots \arrow[l, "N"'] \\
C_1 \arrow[d, "b"'] & C_1 \arrow[l, "1-t"'] \arrow[d, "-b'"] & C_1 \arrow[l, "N"'] \arrow[d, "b"'] & C_1 \arrow[l, "1-t"'] \arrow[d, "-b'"] & \cdots \arrow[l, "N"'] \\
C_0 & C_0 \arrow[l, "1-t"'] & C_0 \arrow[l, "N"'] & C_0 \arrow[l, "1-t"'] & \cdots \arrow[l, "N"']
\end{tikzcd}$$

For example if $C_n = A \otimes A^{\otimes n}$ we have a cyclic bicomplex $C_{\bullet\bullet}(A)$ with t being the cyclic operator, and $N = 1 + t + \ldots t^n$.

Definition 2.1. *The* cyclic homology *of a cyclic module $C_\bullet$ is defined as*

$$\mathrm{HC}_n(C_\bullet) := \mathrm{H}_n(\mathrm{Tot}(C_{\bullet\bullet})).$$

When $C_n = A \otimes A^{\otimes n}$ then the cyclic homology of an algebra A is denoted by $\mathrm{HC}_n(A)$.

Proposition 2.2. *The complex $(C_\bullet, b')$ is acyclic.*

Proof. Use the extra degeneracy

$$(a_0, \ldots, a_n) \mapsto (1, a_0, \ldots, a_n)$$

to construct a homotopy of the identity and the zero map. □

Whenever we have a sequence of complexes

$$K'_\bullet \rightarrowtail K_\bullet \twoheadrightarrow K''_\bullet$$

and we know that $K'_\bullet$ is acyclic, then the complexes $K_\bullet$ and $K''_\bullet$ are quasi-isomorphic. This allows us to quotient out the acyclic subcomplexes of a given complex when computing homology. But $(C_\bullet, -b')$ is not a subcomplex. We will get rid of one column at a time using

Lemma 2.3 (Killing contractible complexes)**.** *Suppose we have o complex*

$$\ldots \to A_n \oplus A'_n \xrightarrow{d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} A_{n-1} \oplus A'_{n-1} \to \ldots$$

and $(A'_\bullet, \delta)$ has a homotopy h between id *and 0. Then the following inclusion is a quasi-isomorphism*

$$(A_\bullet, \alpha - \beta h \gamma) \xrightarrow{(\mathrm{id}, -h\gamma)} (A_\bullet \oplus A'_\bullet, d).$$

The cokernel of $(\mathrm{id}, -h\gamma)$ is $(A'_\bullet, \delta)$. Applied infinitely many times to the cyclic bicomplex we end up with the total complex of the bicomplex $B_\bullet C_\bullet$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
C_2 \arrow[d, "b"'] & C_2 \arrow[d, "b"'] \arrow[ul, "B"'] & C_2 \arrow[d, "b"'] \arrow[ul, "B"'] & \cdots \\
C_1 \arrow[d, "b"'] & C_1 \arrow[d, "b"'] \arrow[ul, "B"'] & C_1 \arrow[d, "b"'] \arrow[ul, "B"'] & \cdots \arrow[ul] \\
C_0 & C_0 \arrow[ul, "B"'] & C_0 \arrow[ul, "B"'] & \cdots \arrow[ul]
\end{tikzcd}$$

This is the normalized version of a bicomplex $C_{\bullet\bullet}$ used to define cyclic homology. Because of the quasi-isomorphism in the lemma (2.3) we have

$$\mathrm{H}_*(C_\bullet) = \mathrm{H}_*(\mathrm{Tot}(B_\bullet C_\bullet)).$$

We can rearrange the bicomplex $B_\bullet C_\bullet$ to obtain

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
C_2 \arrow[d, "b"'] & C_1 \arrow[l, "B"'] \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & \\
C_1 \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & & \\
C_0 & & &
\end{tikzcd}$$

It is indeed a bicomplex, that is we have the identities

$$b^2 = 0, \quad B^2 = 0, \quad bB + Bb = 0.$$

The morphism B on the normalized complex $B_\bullet C_\bullet(A)$ is given explicitly by

$$B = (1-t)sN \colon A \otimes \overline{A}^{\otimes n} \to A \otimes \overline{A}^{\otimes(n+1)},$$

$$(a_0, \dots, a_n) \mapsto \sum_{i=0}^{n} (-1)^{in}(1, a_i, \dots, a_n, a_0, \dots, a_{n-1}).$$

In the non-normalized complex there are more terms, but they are trivial in the normalized complex.

Theorem 2.4. *For a cyclic module $C_\bullet$ there exits a* periodicity exact sequence

$$\dots \to \mathrm{H}_n(C_\bullet) \xrightarrow{I} \mathrm{HC}_n(C_\bullet) \xrightarrow{S} \mathrm{HC}_{n-2}(C_\bullet) \xrightarrow{B} \mathrm{H}_{n-1}(C_\bullet) \to \dots, \tag{2.1}$$

where the map I is induced by the inclusion of the simplicial complex for $C_\bullet$ into the bicomplex $C_{\bullet\bullet}$.

If $C_n = A^{\otimes n}$ the sequence takes the form

$$\dots \to \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \to \dots. \tag{2.2}$$

Proof. It follows from the bicomplex $(B_\bullet C_\bullet, b, B)$ and the sequence of complexes

$$C_\bullet \rightarrowtail \mathrm{Tot}(B_\bullet C_\bullet) \twoheadrightarrow \mathrm{Tot}\, B_\bullet C_\bullet[-2].$$

Prove that the boundary map is given by B. Find an explicit formula for S. $\square$

2.2 Characteristic 0 case

Recall the computation of the homology of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Let M be a module over $\mathbb{Z}/n\mathbb{Z}$, that is a module over the group ring $k[\mathbb{Z}/n\mathbb{Z}]$ for some ring k. to compute $\mathrm{H}_i(\mathbb{Z}/n\mathbb{Z}; M)$ one uses a complex

$$M \xleftarrow{1-t} M \xleftarrow{N} M \xleftarrow{1-t} M \xleftarrow{N} \dots$$

When the ring k is a field of characteristic 0, there is a homotopy from id to 0,

$$M \xrightarrow{h} M \xrightarrow{h'} M \xrightarrow{h} M \xrightarrow{h'} \dots,$$

$$\begin{aligned} h &:= -\frac{1}{n}\sum_{i=1}^{n-1} it^i, \\ h' &:= \frac{1}{n}\mathrm{id}, \\ h(1-t) + Nh' &= t^n = \mathrm{id}. \end{aligned}$$

It proves that

$$\begin{aligned} \mathrm{H}_0(\mathbb{Z}/n\mathbb{Z}; M) &= M/1-t, \\ \mathrm{H}_n(\mathbb{Z}/n\mathbb{Z}; M) &= 0, \quad n \geq 1. \end{aligned}$$

Now instead of considering all bicomplex $C_{\bullet\bullet}$ we can take the reduced complex $C^\lambda_\bullet$ which is defined as a cokernel of the map $(1-t)$ between first and zeroth column of $C_{\bullet\bullet}$

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \\ C_3/(1-t) \arrow[d, "b"] & 0 \\ C_2/(1-t) \arrow[d, "b"] & 0 \\ C_1/(1-t) \arrow[d, "b"] & 0 \\ C_0/(1-t) & 0 \end{tikzcd}$$

If $C_n = A^{\otimes(n+1)}$, then $C^\lambda_n(A) = A^{\otimes(n+1)}/(1-t)$ and we denote

$$\mathrm{H}^\lambda_n(A) := \mathrm{H}_n(C^\lambda_\bullet)$$

As a corollary we have that if $k \supset \mathbb{Q}$, then $\mathrm{H}^\lambda(A) \cong \mathrm{HC}_n(A)$ and there exists an exact sequence

$$\dots \to \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{H}^\lambda_n(A) \xrightarrow{S} \mathrm{H}^\lambda_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \to \dots.$$

In the case of characteristic not equal 0 the maps are still well defined, but the sequence is not exact.

2.3 Computations

Let $A = k$, the ground ring. Then

$$\mathrm{HH}_0(k) = k,$$
$$\mathrm{HH}_n(k) = 0, \quad n \geq 1.$$

The periodicity exact sequence (2.2) implies that

$$\mathrm{HC}_{2n}(k) = k,$$
$$\mathrm{HC}_{2n+1}(k) = 0,$$

so also

$$\mathrm{H}^{\lambda}_{2n}(k) = k,$$
$$\mathrm{H}^{\lambda}_{2n+1}(k) = 0.$$

Let $A = T(V)$ be the tensor algebra over V, that is

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}, \quad (v_1, \dots, v_n)(v_{n+1}, \dots, v_{n+m}) = (v_1, \dots, v_{n+m}) \in V^{\otimes(n+m)}$$

Then

$$\begin{aligned}
\mathrm{HH}_0(T(V)) &= \bigoplus_{m\geq 0} V^{\otimes m}/(1-\tau) = \bigoplus_{m\geq 0} (V^{\otimes m})_{\mathbb{Z}/m\mathbb{Z}}, \\
\mathrm{HH}_1(T(V)) &= \bigoplus_{m\geq 0} (V^{\otimes m})^{\mathbb{Z}/m\mathbb{Z}}, \\
\mathrm{HH}_1(T(V)) &= 0,
\end{aligned}$$

where τ is the cyclic operator without sign.

$$\mathrm{HC}_n(T(V)) = \mathrm{HC}_n(k) \oplus \underbrace{\bigoplus_{m>0} \mathrm{H}_n(\mathbb{Z}/m\mathbb{Z}; V^{\otimes m})}_{\text{This is zero in the characteristic 0 case.}}$$

Consider now the matrix algebras $M_n(A)$ for a unital associative algebra A over a field k. There are isomorphisms

$$\mathrm{HH}_*(M_r(A)) \cong \mathrm{HH}_*(A),$$
$$\mathrm{HC}_*(M_r(A)) \cong \mathrm{HC}_*(A).$$

The map $A \to M_r(A)$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

In the opposite way we have the trace map $\mathrm{Tr}\colon M_r(A) \to A$

$$\alpha = [\alpha_{ij}] \mapsto \sum_i \alpha_{ii}.$$

There is also a trace map $\mathrm{Tr}\colon M_r(A)^{\otimes(n+1)} \to A^{\otimes(n+1)}$

$$\mathrm{Tr}(\alpha^0, \dots, \alpha^n) := \sum_{(i_0,\dots,i_n)} \alpha^0_{i_0 i_1} \otimes \alpha^1_{i_1 i_2} \otimes \dots \otimes \alpha^n_{i_n i_0}$$

We claim that this map commutes with the faces and with the cyclic operator.

Let k be a field and A a commutative k-algebra. Define the space of 1-forms on A, denoted by $\Omega^1_{A/k} = \Omega^1_A$, as an A-module generated by elements da for every $a \in A$ satisfying the following relations

$$d(\lambda a + \mu b) = \lambda da + \mu db \text{ (linearity)},$$
$$d(ab) = adb + bda \text{ (Leibniz rule)}.$$

Define the space of n-forms as an n-th exterior power of Ω^1_A

$$\Omega^n_A := \Lambda^n_A \Omega^1_A.$$

Elements of Ω^n_A can be written as $a_0 da_1 \dots da_n$, $a_i \in A$, $i = 0, \dots, n$, with the relation

$$dada' = -da'da.$$

Define a differential of an n-form as

$$d(a_0 da_1 \dots da_n) := 1 da_0 da_1 \dots da_n.$$

$$d\colon \Omega^n_A \to \Omega^{n+1}_A, \quad d \circ d = 0.$$

Now $\Omega^\bullet_A$ is a cochain complex and its homology is called deRham cohomology of an algebra A

$$\mathrm{H}_{\mathrm{dR}}(A) := \mathrm{H}_n(\Omega^\bullet_A, d).$$

If A is commutative, M an A-module, then

$$\mathrm{H}_1(A; M) = M \otimes_A \Omega^1_A.$$

There is a map

$$\begin{aligned} \pi\colon C_n(A) = A^{\otimes(n+1)} &\to \Omega^n_A \\ (a_0, \dots, a_n) &\mapsto a_0 da_1 \dots da_n \end{aligned} \tag{2.3}$$

There is a map also in the opposite way

$$\Omega^n_A \xrightarrow{\varepsilon_n} \mathrm{HH}_n(A)$$

$$\varepsilon_n(a_0 da_1 \dots da_n) := \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)(a_0, a_{\sigma(1)}, \dots, a_{\sigma(n)}). \tag{2.4}$$

Passing to Hochschild homology it gives a well defined map $\Omega^n_A \to \mathrm{HH}_n(A)$. In charecteristic 0 case the composition of the maps in (2.4) and (2.3) gives an isomorphism

$$\Omega^n_A \to \mathrm{HH}_n(A) \to \Omega^n_A.$$

Proposition 2.5. *The following diagram is commutative*

$$\begin{tikzcd} \Omega^n_A \arrow[r, "\varepsilon_n"] \arrow[d, "d"'] & \mathrm{HH}_n(A) \arrow[d, "B"] \\ \Omega^{n+1}_A \arrow[r, "\varepsilon_{n+1}"] & \mathrm{HH}_{n+1}(A) \end{tikzcd}$$

Proof. There is a bijection of sets $S_{n+1} \cong S_n \times \mathbb{Z}/(n+1)\mathbb{Z}$. First one proves the commutativity of the following diagram

$$\begin{tikzcd}
A \otimes \Lambda^n A \arrow[r, "\varepsilon_n"] \arrow[d] & C_n(A) \arrow[d] \\
A \otimes \Lambda^{n+1} A \arrow[r, "\varepsilon_{n+1}"] & C_{n+1}(A)
\end{tikzcd}$$

and then passes to the quotient. □

Now we can form a map of bicomplexes

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[r, "\pi_*"] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
C_2 \arrow[d, "b"'] & C_1 \arrow[l, "B"'] \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & & \Omega^2 \arrow[d, "0"'] & \Omega^1 \arrow[l, "d"'] \arrow[d, "0"'] & \Omega^0 \arrow[l, "d"'] & \\
C_1 \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & & & \Omega^1 \arrow[d, "0"'] & \Omega^0 \arrow[l, "d"'] & & \\
C_0 & & & & \Omega^0 & & &
\end{tikzcd}$$

Definition 2.6. *A commutative algebra A is* formally smooth *if for any commutative algebra R and two sided ideal $R \supset I$ such that $I^2 = 0$ and a map $A \to R/I$, there is a lifting $\phi\colon A \to R$.*

$$\begin{tikzcd}
 & R \arrow[d] \\
A \arrow[ur, "\varphi"] \arrow[r] & R/I
\end{tikzcd}$$

Theorem 2.7 (Hochschild-Kostant-Rosenberg)**.** *If A is formally smooth, then*

$$\varepsilon_*\colon M \otimes_A \Omega_A^n \to \mathrm{H}_*(A; M)$$

is an isomorphism.

As a corollary we have that for a formally smooth algebra A over characteristic 0 field k

$$\mathrm{HC}_n(A) \cong \Omega_A^n / d\Omega_A^{n-1} \oplus {\mathrm{H}_{\mathrm{dR}}}^{n-2}(A) \oplus {\mathrm{H}_{\mathrm{dR}}}^{n-4}(A) \oplus \ldots \oplus {\mathrm{H}_{\mathrm{dR}}}^{0}(A) \text{ or } {\mathrm{H}_{\mathrm{dR}}}^{1}(A).$$

2.4 Periodic and negative cyclic homology

Recall the cyclic bicomplex

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
C_2 \arrow[d, "b"'] & C_1 \arrow[l, "B"'] \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & \\
C_1 \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & & \\
C_0 & & &
\end{tikzcd}
$$

which after passing to total complex gives a complex computing cyclic homology of an algebra. There is an obvious way to extend this bicomplex to the left using the same differentials

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
\cdots & C_3 \arrow[l] \arrow[d, "b"'] & C_2 \arrow[l, "B"'] \arrow[d, "b"'] & C_1 \arrow[l, "B"'] \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & \\
\cdots & C_2 \arrow[l] \arrow[d, "b"'] & C_1 \arrow[l, "B"'] \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & & \\
\cdots & C_1 \arrow[l] & C_0 \arrow[l, "B"'] & & &
\end{tikzcd}
$$

Furthermore we can repeat each row going down continuing the same pattern.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
\cdots & C_3 \arrow[l] \arrow[d, "b"'] & C_2 \arrow[l, "B"'] \arrow[d, "b"'] & C_1 \arrow[l, "B"'] \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & \\
\cdots & C_2 \arrow[l] \arrow[d, "b"'] & C_1 \arrow[l, "B"'] \arrow[d, "b"'] & C_0 \arrow[l, "B"'] & & \\
\cdots & C_1 \arrow[l] \arrow[d] & C_0 \arrow[l, "B"'] & & & \\
\cdots & C_0 \arrow[l] & & & & \\
\cdots & & & & &
\end{tikzcd}
\tag{2.5}
$$

This is called a *periodic bicomplex.* If the columns of the cyclic bicomplex we started with were indexed by natural numbers starting from 0, then in the periodic bicomplex (2.5) we have columns indexed by integers.

To work with the total complex of the periodic bicomplex one should use the product instead of the sum. Otherwise one would get zero in the homology.

Definition 2.8. *The cohomology of the total complex of bicomplex (2.5) is called the* periodic cyclic homology*. If $C_n = A^{\otimes(n+1)}$, then we denote this homology by $\mathrm{HP}_*(A)$ or $\mathrm{HC}_*^{per}(A)$.*

The cohomology of the total complex consisting of columns with nonpositive indices is called negative cyclic homology*. If $C_n = A^{\otimes(n+1)}$, then we denote this homology by $\mathrm{HN}_*(A)$ or $\mathrm{HC}_*^{-}(A)$.*

2.5 Harrison homology

Recall that when A is a commutative algebra over characteristic 0 field k, then

$$\mathrm{HH}_*(A) \xrightarrow{\cong} \Omega_A^*.$$

In general there is a decomposition into direct sum

$$\begin{aligned} \mathrm{HH}_n(A) &= \underbrace{\square \oplus \ldots \oplus \square}_{n \text{ terms}} \oplus \Omega_A^n \\ &\ldots \\ \mathrm{HH}_2(A) &= \square \oplus \Omega_A^2 \\ \mathrm{HH}_1(A) &= \square \end{aligned}$$

When one considers the first summands in each gradation then what one obtains is called *Harrison homology* of the commutative algebra A. When M is an A-bimodule, then $C_n(A, M) = M \otimes_A A^{\otimes n}$ gives a complex computing Hochschild homology of an algebra A with coefficients in M. The complex for Harrison homology can be obtained by taking a quotient by the shuffles in $C_n(A, M)$.

2.6 Derived functors

The Hochschild homology of an algebra A over a field k with coefficients in an A-bimodule M can be interpreted as a derived functor.

Proposition 2.9. *There is an isomorphism*

$$\mathrm{H}_n(A; M) \cong \mathrm{Tor}_n^{A^e}(M, A),$$

where $A^e = A \otimes A^{op}$ (so M is a right A^e-module).

The definition of the derived functor $\mathrm{Tor}_n^{A^e}$ goes as follows. Having an exact sequence of right A^e-modules

$$0 \to M' \to M \to M'' \to 0$$

we tensor it with A over A^e to get a sequence which is exact on the right

$$M' \otimes_{A^e} A \to M \otimes_{A^e} A \to M'' \otimes_{A^e} A \to 0,$$

but the map $M' \otimes_{A^e} A \to M \otimes_{A^e} A$ can have a nontrivial kernel. There exists a group $\mathrm{Tor}_1^{A^e}(M'', A)$ which maps onto it. Next we can define in an analogous way $\mathrm{Tor}_1^{A^e}(M, A)$ and $\mathrm{Tor}_1^{A^e}(M', A)$ which fit into an exact sequence

$$\mathrm{Tor}_1^{A^e}(M', A) \to \mathrm{Tor}_1^{A^e}(M, A) \to \mathrm{Tor}_1^{A^e}(M'', A) \to$$

$$\to M' \otimes_{A^e} A \to M \otimes_{A^e} A \to M'' \otimes_{A^e} A \to 0.$$

There exists a general construction using a resolution of A by free left A^e-modules, $C_\bullet \twoheadrightarrow A \to 0$,

$$\begin{tikzcd} \cdots \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r] & C_0 \arrow[d] \\ & & & A \arrow[d] \\ & & & 0 \end{tikzcd}$$

Then we define

$$\mathrm{Tor}_n^{A^e}(M, A) := \mathrm{H}_n(M \otimes_{A^e} C_\bullet).$$

As a resolution we can take $C_n := A^e \otimes A^{\otimes n}$ and obtain the isomorphism $\mathrm{H}_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A)$.

Recall that the simplicial module $C_\bullet$ is a functor $\Delta^{op} \to \mathbf{Mod}$, for example $[n] \mapsto M \otimes_{A^e} A^n$. The homology of $C_\bullet$ with respect to $b = \sum_i (-1)^i d_i$ can be written as a derived functor

$$\mathrm{H}_n(C_\bullet) \cong \mathrm{Tor}_n^{\Delta^{op}}(k, C_\bullet),$$

where $C_\bullet$ is a left module over Δ^{op}, and k is a right module over Δ^{op}, that is a functor $\Delta \to \mathbf{Mod}$, $[n] \mapsto k$. The resolution for k can be given by

$$\begin{tikzcd} \cdots \arrow[r] & k[\mathrm{Hom}_\Delta([n], -)] \arrow[r] & \cdots \arrow[r] & k[\mathrm{Hom}_\Delta([1], -)] \arrow[r] & k[\mathrm{Hom}_\Delta([0], -)] \arrow[d] \\ & & & & k \end{tikzcd}$$

In general for a category $\mathcal{C}$ we have the following correspondence

Category $\mathcal{C}$	Algebra A
Functor $F\colon \mathcal{C} \to \mathbf{Mod}$	Left A-module M
Functor $G\colon \mathcal{C}^{op} \to \mathbf{Mod}$	Right A-module N
Tensor product over a category $G \otimes_{\mathcal{C}} F$	Tensor product over algebra $N \otimes_A M$

The tensor product over a category is defined as

$$G \otimes_{\mathcal{C}} F := \bigoplus_{C \in \mathrm{Ob}(\mathcal{C})} G(C) \otimes F(C) / \sim,$$

where the equivalence relation $\sim$ is given by

$$y \otimes f_*(x) \sim f^*(y) \otimes x, \quad C \xrightarrow{f} D, \quad x \in F(C),\ y \in G(D),$$

$$F(C) \xrightarrow{f_*} F(D), \quad G(C) \xleftarrow{f^*} G(D).$$

Using cyclic category ΔC we can present cyclic homology of a cyclic module $C_\bullet$ as a derived functor.

Proposition 2.10. *There is an isomorphism*

$$\mathrm{HC}_n(C_\bullet) \cong \mathrm{Tor}_n^{\Delta C^{op}}(k, C_\bullet).$$

We can write $\mathrm{Tor}_0^{\mathcal{C}}(G,F)$ simply as the tensor product $G \otimes_{\mathcal{C}} F$. To define higher derived functors $\mathrm{Tor}_n^{\mathcal{C}}(G,F)$ we need a notion of a free module over a category. Let $\mathcal{C}^{triv}$ be the category with the same objects as $\mathcal{C}$, but with only the identity morphisms. For a functor $F\colon \mathcal{C} \to \mathbf{Mod}$ there is a corresponding forgetful functor forget(F)$\colon \mathcal{C}^{triv} \to \mathbf{Mod}$. Suppose we have an adjoint pair

$$\mathrm{Funct}(\mathcal{C}, \mathbf{Mod}) \underset{\text{left adjoint}}{\overset{\text{forgetful}}{\rightleftarrows}} \mathrm{Funct}(\mathcal{C}^{triv}, \mathbf{Mod})$$

Then we say that a functor $F\colon \mathcal{C} \to \mathbf{Mod}$ is free if it is an image of this left adjoint functor to a forgetful functor. For example

$$A - \mathbf{Mod} \to k - \mathbf{Mod}$$

has a left adjoint

$$k^n \mapsto A^n.$$

Chapter 3

Cyclic duality and Hopf–cyclic homology

3.1 Cyclic duality

Definition 3.1. *A* cyclic (cocyclic) object *is a contravariant (covariant) functor from the cyclic category ΔC to an abelian category $\mathcal{A}$.*

Cyclic objects are used to define homology

$$\begin{tikzcd} \cdots & \mathcal{A}_{n-2} & \mathcal{A}_{n-1} \arrow[l, "d_i"] \arrow[r, shift left, "s_i"] \arrow[loop below, "t_{n-1}"] & \mathcal{A}_n \arrow[l, shift left, "d_j"] \arrow[r, shift left, "s_j"] \arrow[loop below, "t_n"] & \mathcal{A}_{n+1} \arrow[l, shift left, "d_k"] \arrow[r, "s_k"] \arrow[loop below, "t_{n+1}"] & \mathcal{A}_{n+2} & \cdots \end{tikzcd}$$

where $0 \leq i \leq n-1$, $0 \leq j \leq n$, $0 \leq k \leq n+1$, $n \in N$, and the faces d and degeneracies s satisfy the ΔC^{op} relations.

Cocyclic objects are used to define cohomology

$$\begin{tikzcd} \cdots & \mathcal{A}^{n-2} \arrow[r, "\delta_i"] & \mathcal{A}^{n-1} \arrow[r, shift right, "\delta_j"'] \arrow[loop below, "\tau_{n-1}"] & \mathcal{A}^n \arrow[l, shift right, "\sigma_i"'] \arrow[r, shift right, "\delta_k"'] \arrow[loop below, "\tau_n"] & \mathcal{A}^{n+1} \arrow[l, shift right, "\sigma_j"'] \arrow[loop below, "\tau_{n+1}"] & \mathcal{A}^{n+2} \arrow[l, "\sigma_k"'] & \cdots \end{tikzcd}$$

where $0 \leq i \leq n-1$, $0 \leq j \leq n$, $0 \leq k \leq n+1$, $n \in N$, and the faces δ and degeneracies σ satisfy the ΔC relations.

Having a cyclic object, we can construct a cocyclic one, and vice versa.

$$\begin{tikzcd} \Delta C^{op} \arrow[r, shift left, "\hat{}"] & \Delta C \arrow[l, shift left, "\check{}"] \end{tikzcd}$$

$$\begin{aligned} \hat{\mathcal{A}}^n &:= \mathcal{A}_n & \qquad \check{\mathcal{A}}_n &:= \mathcal{A}^n \\ \hat{\delta}_0 &:= t_n s_{n-1} & \check{d}_0 &:= \sigma_{n-1}\tau_n \\ \hat{\delta}_{j+1} &:= s_j,\ 0 \leq j \leq n-1 & \check{d}_{j+1} &:= \sigma_j,\ 0 \leq j \leq n-1 \\ \hat{\sigma}_j &:= d_j & \check{s}_j &:= \delta_j \\ \hat{\tau}_n &:= t_n^{-1} & \check{t}_n &:= \tau_n^{-1} \end{aligned}$$

The cyclic category ΔC is isomorphic with its opposite ΔC^{op}. The compositions $\hat{\ } \circ \check{\ }$ and $\check{\ } \circ \hat{\ }$ are inner automorphisms implemented by τ_* and t_* respectively.

3.2 Cyclic homology of algebra extensions

Let B be a subalgebra of A and M an A-bimodule. Define

$$\mathcal{B}_n := B \otimes_{B \otimes B^{op}} (M \otimes_B \underbrace{A \otimes_B \ldots \otimes_B A}_{n \text{ times}}).$$

Then

$$\mathcal{B}_0 = B \otimes_{B \otimes B^{op}} M = M/[B, M]$$

and B_n can be written in a circle.

The simplicial structure is given by

$$\begin{aligned}
d_0(m, a_1, \ldots, a_n) &:= (ma_1, a_2, \ldots, a_n) \\
d_j(m, a_1, \ldots, a_n) &:= (m, a_1, \ldots, a_j a_{j+1}, \ldots, a_n),\ 1 \leq j \leq n-1 \\
d_n(m, a_1, \ldots, a_n) &:= (a_n m, a_1, \ldots, a_{n-1}) \\
s_j(m, a_1, \ldots, a_n) &:= (m, a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n).
\end{aligned}$$

Lemma 3.2. *The collection $\{B_n\}_{n \in \mathbb{N}}$ is a simplicial module. If $M = A$, then adjoining the morphisms $t_n \colon B_n \to B_n$*

$$t_n(a_0, a_1, \ldots, a_n) := (a_n, a_0, \ldots, a_{n-1})$$

makes $\{B_n\}_{n \in N}$ a cyclic module.

3.3 Hopf–Galois extensions

Let H be a Hopf algebra with comultiplication $\Delta \colon H \to H \otimes H$, $\Delta(h) := h^{(1)} \otimes h^{(2)}$. Let M be a right H-comodule with coaction $\Delta_R \colon M \to M \otimes H$, $\Delta_R(m) := m^{(0)} \otimes m^{(1)}$.

Let A be a right H-comodule algebra via $\Delta_R \colon A \to A \otimes H$ (G-space). Denote

$$B := \{a \in A \mid \Delta_R(a) = a \otimes 1\}$$

(functions on quotient).

Definition 3.3. *The extension of algebras $B \subset A$ is called* Hopf–Galois extension *if*

$$g \colon A \otimes_B A \to A \otimes H$$
$$g(a \otimes_B a') := (a \otimes 1)\Delta_R(a') = a(a')^{(0)} \otimes (a')^{(1)}$$

(canonical map) is an isomorphism.

Since g is a B-module map, we can extend it to B_n

$$\tilde{g}(a_0, \ldots, a_{n-1}, a_n) := (a_0, \ldots, a_{n-1} a_n^{(0)}) \otimes a_n^{(1)}.$$

Using $\tilde{g}$ we have

$$B \otimes_{B \otimes B^{op}} (\underbrace{A \otimes_B \ldots \otimes_B A}_{n \text{ times}}) \xrightarrow{\tilde{g}} B \otimes_{B \otimes B^{op}} (\underbrace{A \otimes_B \ldots \otimes_B A}_{n-1 \text{ times}}) \otimes H$$

After n iterations we land in $B \otimes_{B \otimes B^{op}} A \otimes H^{\otimes n} = A/[A, B] \otimes H^{\otimes n}$. The key idea is to transport the cyclic structure via $\tilde{g}_*$.

3.4 Hopf– cyclic homology with coefficients

Definition 3.4. *Let M be a left H-module and right H-comodule. It is called* anti-Yetter-Drinfeld module *if*

$$\Delta_R(hm) = h^{(2)}m^{(0)} \otimes h^{(3)}m^{(1)}S(h^{(1)}) \in M \otimes H.$$

M is stable *if $m^{(1)}m^{(0)} = m$.*

Denote $\mathcal{H}_n := (H^{\otimes(n+1)}) \otimes_H M$ with diagonal action on $H^{\otimes(n+1)}$.

Theorem 3.5 (Jara-Stefan, Hajac-Khalkhali-Rangipour-Sommerhaus)**.** *The following formulas define a cyclic module structure on $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$.*

$$\begin{aligned}
d_j(h_0 \otimes \cdots \otimes h_n) \otimes_H m &:= (h_0 \otimes \cdots \otimes \varepsilon(h_j) \otimes \cdots \otimes h_n) \otimes_H m \\
s_j(h_0 \otimes \cdots \otimes h_n) \otimes_H m &:= (h_0 \otimes \cdots \otimes \Delta h_j \otimes \cdots \otimes h_n) \otimes_H m \\
t_n(h_0 \otimes \cdots \otimes h_n) \otimes_H m &:= (h_n m^{(1)} \otimes h_0 \otimes \cdots \otimes h_{n-1}) \otimes_H m^{(0)}.
\end{aligned}$$

It is well defined if and only if M is a stable anti-Yetter-Drinfeld module.

Let $B \subset A$ be a Hopf–Galois extension for a Hopf algebra H. The map $g\colon A \otimes H \to A \otimes_B A$ allows to define the translation map

$$T\colon H \to A \otimes_B A, \quad T(h) := g^{-1}(1 \otimes h) = h^{[1]} \otimes_B h^{[2]}.$$

Lemma 3.6 (Mijaschita-Ulbrich, Jara-Stefan, Hajac-Khalkhali-Rangipour-Sommerhaus)**.** *The formula*

$$H \otimes A/[B,A] \xrightarrow{\triangleright} A/[B,A], \quad h \triangleright \bar{a} \mapsto \overline{h^{[2]}ah^{[1]}}$$

defines a left action. Moreover this action satisfies the stable anti-Yetter-Drinfeld module compatibility condition for the induced coaction on $A/[B,A]$.

Example 3.7. If $A = H$ then $h \triangleright k = h^{(2)}kS(h^{(1)})$.

Theorem 3.8 (Jara-Stefan)**.** *The cyclic modules $\{B \otimes_{B\otimes B^{op}} A^{\otimes_B n}\}_{n\in\mathbb{N}}$ and $\{H^{\otimes(n+1)} \otimes_H A/[B,A]\}_{n\in\mathbb{N}}$ are isomorphic.*

Theorem 3.9. *Let A be a left H-module algebra with respect to $H \otimes A \to A$, $h(ab) = h^{(1)}(a)h^{(2)}(b)$, $h(1) = \varepsilon(h)$. Let $M \otimes A^{\otimes(n+1)}$ be a right H-module via $(m \otimes \widetilde{a})h := mh^{(1)} \otimes S(h^{(2)})\widetilde{a}$ and k be a right H-module via ε. Then $\{\mathrm{Hom}_H(M \otimes A^{\otimes(n+1)}, k)\}_{n\in\mathbb{N}}$ is a cocyclic module with the cocyclic structure given by*

$$\begin{aligned}
(\delta_j f)(m \otimes a_0 \otimes \cdots \otimes a_n) &:= f(m \otimes a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n) \\
(\delta_n f)(m \otimes a_0 \otimes \cdots \otimes a_n) &:= f(m^{(0)} \otimes (S^{-1}(m^{-1})a_n)a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \\
(\sigma_j f)(m \otimes a_0 \otimes \cdots \otimes a_n) &:= f(m \otimes a_0 \otimes \cdots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_n) \\
(\tau_n f)(m \otimes a_0 \otimes \cdots \otimes a_n) &:= f(m^{(0)} \otimes S^{-1}(m^{-1})a_n \otimes a_0 \otimes \cdots \otimes a_n).
\end{aligned}$$

Special cases:

1. $H = k = M$ - the standard cyclic homology.
2. $H = k[\sigma, \sigma^{-1}]$, $M = {}^{\sigma}k_{\varepsilon}$ - the twisted cyclic homology.

Chapter 4

Twisted homology and Koszul duality

4.1 Hochschild homology of the Quantum plane

Let $k = \mathbb{C}$, and A be a quadratic algebra with the relation $xy = qyx$, where $q \in k^\times$ not a root of unity . The elements $e_{ij} = x^i y^j$, $i, j \geq 0$ form a vector space basis ($\Rightarrow$ PBW algebra). Consider the following automorphism

$$\sigma(x) = \lambda x,\ \sigma(y) = \mu y, \quad \lambda, \mu \in k^\times$$

We will compute twisted cyclic homology $\mathrm{HC}^\sigma_\bullet(A)$.

The general strategy is to compute the simplicial theory underlying the paracyclic object as

$$\mathrm{H}_\bullet(A, M) = \mathrm{Tor}^{A^e}_\bullet(M, A)$$

(in our case $M = {}_\sigma A$) using a nice resolution of A as A^e-module or other techniques.

There is a morphisms to the simplicial object underlying the cyclic object (in our case this simplicial object will be denoted by $\mathrm{HH}^\sigma_\bullet(A)$). We will try to compute this using the spectral sequence arising from the (b, B)-bicomplex to obtain as its limit the cyclic theory (here $\mathrm{HC}^\sigma_\bullet(A)$).

First we look for free resolution, i.e., exact complex

$$\ldots \to (A^e)^{b_1} \to (A^e)^{b_0} \to A \to 0$$

of A^e-modules. It should be as small as possible. Thus $b_0 = 1$, and augmentation is a multiplication.

Lemma 4.1. *If A generated by x_i, then* $\ker \mu$ *is generated as A^e-module by* $1 \otimes x_i - x_i \otimes 1$.

Proof. If $\sum a_i b_i = 0$ then

$$\begin{aligned}\sum a_i \otimes b_i &= \sum a_i \otimes b_i - a_i b_i \otimes 1\\ &= \sum (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1),\end{aligned}$$

hence it is generated by elements of the form $1 \otimes f - f \otimes 1$. But $f \mapsto (1 \otimes f - f \otimes 1)$ satisfies the Leibniz rule

$$1 \otimes fg - fg \otimes 1 = (f \otimes 1)(1 \otimes g - g \otimes 1) + (1 \otimes g)(1 \otimes f - f \otimes 1).$$

The claim follows. □

Thus here $b_0 = 1, b_1 = 2$,

$$\ldots \to (A^e)^2 \to A^e \to A \to 0$$

Now we will determine the kernel of the first map

$$\begin{aligned}(a \otimes b, c \otimes d) \mapsto &(a \otimes b)(1 \otimes x - x \otimes 1) + (c \otimes d)(1 \otimes y - y \otimes 1) \\ &= a \otimes xb - ax \otimes b + c \otimes yd - cy \otimes d.\end{aligned}$$

Thus

$$(e_{ij} \otimes e_{kl}, 0) \mapsto e_{ij} \otimes e_{k+1l} - q^{-j} e_{i+1j} \otimes e_{kl},$$

and

$$(0, e_{ij} \otimes e_{kl}) \mapsto q^{-k} e_{ij} \otimes e_{kl+1} - e_{ij+1} \otimes e_{kl}.$$

Playing a bit with grading arguments gives that the kernel is generated as A^e-module by a single element

$$\omega := (1 \otimes y - qy \otimes 1, -q \otimes x + x \otimes 1).$$

Hence we can take $b_2 = 1$. Since A is a domain (again grading argument), the new map

$$\begin{gathered} A^e \to (A^e)^2, \\ (a \otimes b) \mapsto (a \otimes b)\omega \end{gathered}$$

is injective, so we have a free resolution

$$0 \to A^e \to (A^e)^2 \to A^e \to 0$$

of A as left A^e-module.

Now we tensor this resolution with ${}_\sigma A$ and take homology. As vector space our complex is

$$0 \to A \to A^2 \to A \to 0.$$

The two morphisms are

$$f \mapsto (\sigma(y)f - qfy, -q\sigma(x)f + fx),$$

and

$$(f, g) \mapsto \sigma(x)f - fx + \sigma(y)g - gy.$$

Thus on bases:

$$\begin{aligned} e_{ij} &\mapsto ((\mu q^{-i} - q)e_{ij+1}, (-q\lambda + q^{-j})e_{i+1j}) \\ (e_{ij}, 0) &\mapsto (\lambda - q^{-j})e_{i+1j} \\ (0, e_{kl}) &\mapsto (\mu q^{-k} - 1)e_{kl+1} \end{aligned}$$

The generators of HH_2 are

$$e_{ij}, \quad \lambda = q^{-j-1}, \mu = q^{i+1},$$

and the generators of HH_0 are

$$\begin{aligned} &e_{00}, \\ &e_{0,l+1} \text{ for } \mu = 1, \\ &e_{i+1,0} \text{ for } \lambda = 1, \\ &e_{i+1,j+1} \text{ for } \lambda = q^{-j-1}, \mu = q^{i+1}. \end{aligned}$$

The image contains $e_{i+1,j}$ except when $\lambda = q^{-j}$ and $e_{k,l+1}$ except when $\mu = q^k$.

For the computation of HH_1 we write generators of the kernel

$$\begin{aligned} (e_{ij,0}) &\text{ for } \lambda = q^{-j} \\ (0, e_{kl}) &\text{ for } \mu = q^k \\ ((1 - \mu q^{-i-1})e_{i,j+1}, (\lambda - q^{-j-1})e_{i+1,j}) \end{aligned}$$

The latter are always trivial in homology. Furthermore

$$(e_{ij+1}, 0), \quad \lambda = q^{-j-1}$$

is trivial except when also $\mu = q^{i+1}$. Finally

$$(e_{i0}, 0), \quad \lambda = 1$$

are always nontrivial. Similarly,

$$(0, e_{k+1l}) \quad \mu = q^{k+1}$$

is trivial except when $\lambda = q^{-l-1}$ and always nontrivial is

$$(0, e_{0l}), \quad \mu = 1.$$

From now on for simplicity

$$\lambda = q^{-1} \quad \mu = q.$$

Then we can write the generators of Hochschild homology

$$\begin{aligned} &\mathrm{HH}_2 : 1 \\ &\mathrm{HH}_1 : (y, 0), (0, x) \\ &\mathrm{HH}_0 : 1, xy \end{aligned}$$

In the original Hochschild homology

$$\mathrm{HH}_2 \ : \ 1 \otimes x \otimes y - \alpha \otimes y \otimes x$$

is a boundary only when $\lambda = q^{-1}, \mu = \alpha = q$. In degree one

$$\mathrm{HH}_1 \ : \ x \otimes y, y \otimes x.$$

Now we will compute $\mathrm{HC}^{\sigma}_{*}(A)$.

The (b, B)-bicomplex is not a bicomplex, since

$$b \circ B + B \circ b = \mathrm{id} - T.$$

But the columns form a complex. It computes the Hochschild homology $\mathrm{HH}_*(A, {}_\sigma A)$ with coefficients in the bimodule ${}_\sigma A = A$.

Define $\mathrm{C}_n^0 = \ker(\mathrm{id} - T)$. If $\mathrm{C}_n = \mathrm{C}_n^0 \oplus \mathrm{C}_n^1$, then we have

$$\mathrm{HH}_*(A, {}_\sigma A) \cong \mathrm{HH}_*^\sigma(A).$$

Since $[b, \mathrm{id} - T] = 0$, $\mathrm{C}_* = \mathrm{C}_*^0 \oplus \mathrm{C}_*^1$ as complexes, we have

$$\begin{aligned} \mathrm{HH}_*^\sigma(A) &= \mathrm{H}_*(\mathrm{C}_*^0, b), \\ \mathrm{H}_*(A, {}_\sigma A) &= \mathrm{HH}_*^\sigma(A) \oplus \mathrm{H}_*(\mathrm{C}_*^1, b). \end{aligned}$$

But $(\mathrm{id} - T)|_{\mathrm{C}_*^1}$ is a bijection, and on C_*^1 we have

$$b \circ (\mathrm{id} - T)^{-1} \circ B + (\mathrm{id} - T)^{-1} \circ B \circ b = \mathrm{id}.$$

Hence $(\mathrm{id} - T)^{-1} \circ B$ is a contracting homotopy. This applies for example when σ is diagonalizable.

4.2 Cyclic homology of the Quantum plane

The B-map in normalised form:

$$\begin{aligned} B \colon f &\mapsto 1 \otimes f \\ f \otimes g &\mapsto 1 \otimes f \otimes g - 1 \otimes \sigma(g) \otimes f \\ f \otimes g \otimes h &\mapsto 1 \otimes f \otimes g \otimes h + 1 \otimes \sigma(g) \otimes \sigma(h) \otimes f + 1 \otimes \sigma(h) \otimes f \otimes g. \end{aligned}$$

On our generators:

$$\begin{aligned} 1 &\mapsto 1 \otimes 1, \\ xy &\mapsto 1 \otimes xy \\ x \otimes y &\mapsto 1 \otimes x \otimes y - q \otimes y \otimes x \\ y \otimes x &\mapsto 1 \otimes y \otimes x - q^{-1} \otimes x \otimes y \end{aligned}$$

We have (consider $b(1 \otimes x \otimes y)$)

$$[1 \otimes xy] = [x \otimes y] + q[y \otimes x]$$

On the page 2 of the spectral sequence there is nothing in degree 2, the generator of HH_2 is in $\operatorname{im} B$. The kernel of B_1 is spanned by $\omega := [x \otimes y] + q[y \otimes x]$ which is in the image of B_0. The kernel of B_0 is spanned by $[1]$.

Here the spectral sequence stabilises. So periodically:

$$\mathrm{HP}_{even} = k[1], \quad \mathrm{HP}_{odd} = 0.$$

and not periodically we have to correct

$$\mathrm{HC}_0 = k[1] \oplus k \cdot [xy].$$

That is, the quantum plane has the same cyclic theory as the classical one.

4.3 On Koszul duality

Let $A = A(V, I)$ be quadratic, $A^! = A(V^*, I^\perp)$ its Koszul dual, x_i, x^i dual bases in V, V^* The original Koszul complex

$$K = (A^!)^* \otimes_k A$$

has differential given by multiplication from the right by $e := x^i \otimes x_i$. Here $A^!$ acts on the dual space from the right. Why $d^2 = 0$? We have

$$\mathrm{Hom}_k(V^{\otimes 2}, V^{\otimes 2}) \simeq V^* \otimes V \otimes V^* \otimes V \simeq (A_1^! \otimes A_1)^{\otimes 2}$$

and

$$\begin{aligned} A_2^! \otimes_k A_2 &\simeq (V^* \otimes V^*/I^\perp) \otimes (V \otimes V/I) \\ &\simeq I^* \otimes (V \otimes V/I) \\ &\simeq \mathrm{Hom}_k(I, V \otimes V/I) \end{aligned}$$

Then μ just become the canonical map

$$\mathrm{Hom}_k(V^{\otimes 2}, V^{\otimes 2}) \to \mathrm{Hom}_k(I, V \otimes V/I).$$

Under the identification $e \otimes e$ becomes id. Hence $\mu(e \otimes e)$ is zero.

The bimodule complex is

$$A \otimes (A^!)^* \otimes A$$

built as the total complex of a bicomplex:

$$\begin{aligned} b_L &: r \otimes f \otimes s \mapsto r x_i \otimes f x^i \otimes s \\ b_R &: r \otimes f \otimes s \mapsto r \otimes x^i f \otimes x_i s \end{aligned}$$

These commute and square to zero. Spectral sequence argument: One complex acyclic if and only if the other is. One is a resolution of k, and the second of A.

Chapter 5

Relation with K-theory

We will define invariant of rings, called algebraic K-theory and denoted by $\mathrm{K}_*(A)$ for a ring A. Next we will describe its relation with cyclic homology by defining a map

$$\mathrm{K}_*(A) \to \mathrm{HC}_*(A).$$

5.1 K-theory

First we will define *K-theory* of a ring A in gradation 0, that is $\mathrm{K}_0(A)$. We say that a finitely generated module over A is *free* if it is isomorphic to the product A^n for some n. A finitely generated A-module P is *projective* if it is a direct summand in a free A-module, that is there exists an A-module Q such that $P \oplus Q \cong A^n$ for some n. Such projective module P corresponds to an idempotent in the matrix algebra $M_n(A)$. The set of isomorphism classes of finitely generated projective modules over A is a monoid with respect to direct sum of classes defined by

$$[P] + [Q] =: [P \oplus Q].$$

There is a universal abelian group for this monoid (called the Grothendieck group), and we take it as the definition of the K-theory of A, denoted by $\mathrm{K}_0(A)$.

Let A be a commutative algebra over k. Suppose we want to construct a map

$$\mathrm{ch}\colon\ \mathrm{K}_0(A) \to {\mathrm{H}_{\mathrm{dR}}}^2(A).$$

First consider an example of a map from a tori $S^1 \times S^1$ to a sphere S^2 given by contracting the boundary of a square with opposite edges identified. This map has degree 1 and induces an isomorphism

$$\mathrm{H}^2(S^2) \xrightarrow{\deg(f)} \mathrm{H}^2(S^1 \times S^1).$$

If we want to find an algebraic map of corresponding coordinate rings

$$S_a^2 := \mathbb{C}[X,Y,Z]/(X^2+Y^2+Z^2-1) \to \mathbb{C}[U,U^{-1},V,V^{-1}] =: S^1 \times S_a^1$$

then we will not succeed, because any algebraic map $S^1 \times S^1 \to S^2$ is homotopic to the constant map (Loday). The situation is very different now than it is in

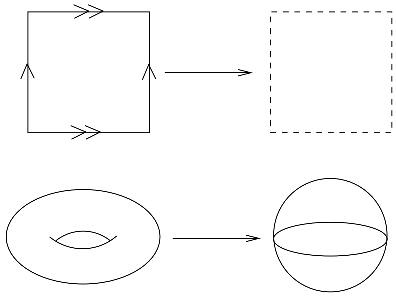


Figure 5.1: $f\colon S^1 \times S^1 \to S^2$

case of maps $S^3 \to S^2$. Indeed, assume we have the map

$$f^*\colon S_a^2 \to S_a^1 \times S_a^1.$$

Then it induces a map on K-theory

$$\mathrm{K}_0(S_a^2) \to \mathrm{K}_0(S_a^1 \times S_a^1),$$

and we would have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] \arrow[d] & \widetilde{\mathrm{K}}_0(S_a^2) \arrow[r] \arrow[d] & \widetilde{\mathrm{K}_0}(S_a^1 \times S_a^1) \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
\mathbb{C} \arrow[r, equal] & {\mathrm{H}_{\mathrm{dR}}}^2(S_a^2) \arrow[r, "\deg(f)"] & {\mathrm{H}_{\mathrm{dR}}}^2(S_a^1 \times S_a^1) \arrow[r, equal] & \mathbb{C}
\end{tikzcd}$$

which gives a contradiction, because a generator of $\mathbb{Z} = \widetilde{\mathrm{K}}_0(S_a^2)$ goes to the generator of $\mathbb{C} = {\mathrm{H}_{\mathrm{dR}}}^2(S_a^2)$.

Define a projector p and idempotent e in $M_2(S_a^2)$ by the formulas

$$p := \begin{pmatrix} X & Y + iZ \\ Y - iZ & -X \end{pmatrix}, \quad p^2 = 1, \quad e := \frac{p+1}{2}, \quad e^2 = e.$$

Fact 5.1. *The class of the image of* e*, denoted* $[\operatorname{im} e]$*, generates* $\mathrm{K}_0(S_a^2)$*.*

Fact 5.2 (Grothendieck)**.** *For any noetherian ring* A *there is an isomorphism*

$$\widetilde{\mathrm{K}}_0(A[X, X^{-1}]) \cong \mathrm{K}_0(A).$$

5.2 Trace map

There is a trace map defined as

$$\mathrm{Tr}\colon M_r(A) \to A, \quad [a_{ij}]_{i,j=1}^r \mapsto \sum_{i=1}^r a_{ii}.$$

We can extend it to a map

$$\mathrm{Tr}\colon M_r(A)^{\otimes(n+1)} \to A^{\otimes(n+1)},$$

$$[a_{i_0 j_0}] \otimes \ldots \otimes [a_{i_n j_n}] \mapsto \sum_{k_0, k_1, \ldots, k_n} a_{k_0 k_1} \otimes a_{k_1 k_2} \otimes \ldots \otimes a_{k_n k_0}$$

for any $r \geq 1$, $n \geq 0$. It induces a maps on Hochschild, cyclic, periodic cyclic and negative cyclic homology.

$$\mathrm{HH}_n(M_r(A)) \to \mathrm{HH}_n(A), \quad \mathrm{HC}_n(M_r(A)) \to \mathrm{HC}_n(A), \text{ etc.}$$

Let us take an idempotent $e^2 = e$ in $M_r(A)$. Under the map b in Hochschild complex for $M_r(A)$ we have

$$e^{\otimes(n+1)} \mapsto \begin{cases} 0 & n \text{ even} \\ e^{\otimes n} & n \text{ odd} \end{cases}$$

In $C_n^\lambda(M_r(A))$ we have $e^{\otimes(n+1)} = (-1)^n e^{\otimes(n+1)}$. If n is odd, then $[e^{\otimes(n+1)}] = 0$. If $n = 2m$ is even, then $b[e^{\otimes(n+1)}] = 0$, so $[e^{\otimes(n+1)}]$ is a cycle, and we can define a map $[e] \mapsto [\mathrm{Tr}(e^{\otimes(n+1)})]$,

$$\mathrm{K}_0(A) \to \mathrm{H}_{2m}^\lambda(M(A)) \xrightarrow{\mathrm{Tr}} \mathrm{H}_{2m}^\lambda(A),$$

$$M(A) = \bigcup_r M_r(A), \quad M_r(A) \hookrightarrow M_{r+1}(A), \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}.$$

We have to show that the element $[\mathrm{Tr}(e^{\otimes(n+1)})] \in \mathrm{H}_{2m}^\lambda(A)$ depends only on the isomorphism class.

Lemma 5.3. *An interior automorphism (conjugation) induces an identity for Hochschild, cyclic, periodic cyclic, negative cyclic homology.*

We have constructed a functorial map $\mathrm{K}_0(A) \to \mathrm{H}_{2m}^\lambda(A)$. Now we ask if we can construct a map $\mathrm{K}_0(A) \to \mathrm{HC}_{2m}(A)$?

Recall the cyclic bicomplex $C_{\bullet\bullet}(A)$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
C_2 \arrow[d, "b"'] & C_2 \arrow[l, "1-t"'] \arrow[d, "-b'"] & C_2 \arrow[l, "N"'] \arrow[d, "b"'] & C_2 \arrow[l, "1-t"'] \arrow[d, "-b'"] & \cdots \arrow[l, "N"'] \\
C_1 \arrow[d, "b"'] & C_1 \arrow[l, "1-t"'] \arrow[d, "-b'"] & C_1 \arrow[l, "N"'] \arrow[d, "b"'] & C_1 \arrow[l, "1-t"'] \arrow[d, "-b'"] & \cdots \arrow[l, "N"'] \\
C_0 & C_0 \arrow[l, "1-t"'] & C_0 \arrow[l, "N"'] & C_0 \arrow[l, "1-t"'] & \cdots \arrow[l, "N"']
\end{tikzcd}$$

Define

$$y_i := (-1)^i \frac{(2i)!}{i!} \mathrm{Tr}(e^{\otimes(2i+1)}),$$
$$z_i := (-1)^{i-1} \frac{(2i)!}{2(i!)} \mathrm{Tr}(e^{\otimes(2i)}).$$

Proposition 5.4. *The element* $\mathrm{ch}([e]) := (y_m, z_m, y_{m-1}, z_{m-1}, \ldots, y_0, z_0) \in (\mathrm{Tot}(C_{\bullet\bullet}(A)))_n$, $n = 2m+1$ *is a cycle. Furthermore the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "\mathrm{ch}"] \arrow[dr, "\mathrm{ch}"] \arrow[ddr] \arrow[dddr, "\mathrm{Tr}"'] & \mathrm{HC}_{2m}(A) \arrow[d, "S"] \\
& \mathrm{HC}_{2m-2}(A) \arrow[d] \\
& \cdots \arrow[d] \\
& \mathrm{HC}_0(A)
\end{tikzcd}$$

For the $B_\bullet C_\bullet$ we have to use $\mathrm{ch}([e]) := (y_n, y_{n-1}, \ldots, y_0) \in (\mathrm{Tot}(B_\bullet C_\bullet(A)))_n$. We can define a map

$$\mathrm{ch}\colon\ \mathrm{K}_0(A) \to {\mathrm{H}_{\mathrm{dR}}}^{ev}(A), \quad \mathrm{ch}([e]) := \mathrm{Tr}(edede \ldots de).$$

5.3 Algebraic K-theory

Let A be a ring with unit. Define a discrete group $\mathrm{GL}(A)$ as a direct limit of the groups $\mathrm{GL}_r(A)$ with respect to the maps

$$\mathrm{GL}_r(A) \hookrightarrow \mathrm{GL}_{r+1}(A), \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

There is a $\mathrm{B\,GL}(A)$ with

$$\pi_1(\mathrm{B\,GL}(A)) = \mathrm{GL}(A),$$
$$\pi_n(\mathrm{B\,GL}(A)) = 0, \quad n \neq 1.$$

We can apply the Quillen's plus construction to obtain a space $\mathrm{B\,GL}(A)^+$ with the following three properties

1. the fundamental group is an abelianization of $\mathrm{GL}(A)$,
$$\pi_1(\mathrm{B\,GL}(A)^+) = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)],$$
2. there is an isomorphism on homology $\mathrm{H}_i(\mathrm{B\,GL}(A)) \cong \mathrm{H}_i(\mathrm{B\,GL}(A)^+)$,
3. there is an H-space structure on $\mathrm{B\,GL}(A)^+$.

Thus $\mathrm{H}_*(\mathrm{B\,GL}(A)^+)$ is commutative, cocommutative (and connected) Hopf algebra.

Definition 5.5. *Higher K-theory groups of A are defined as*

$$\mathrm{K}_n(A) := \pi_n(\mathrm{B\,GL}(A)^+), \quad n \geq 1.$$

Prior to this definition there were defined K_1, K_2, K_3. We will describe these earlier definitions.

The K_1 group of a ring A was defined as an abelianization of $\mathrm{GL}(A)$,

$$\mathrm{K}_1(A) = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)].$$

For example if $A = F$ is a field, then $\mathrm{K}_1(F) = F^\times$, the group of invertible elements in F. The determinant map $\det\colon \mathrm{GL}(F) \to F^\times$ can be generalized to noncommutative rings by the map $\mathrm{GL}(A) \to \mathrm{K}_1(A)$.

Denote by $\mathrm{E}(A)$ the group generated by elementary matrices e_{ij}^a, where each e_{ij}^a is an identity matrix plus the matrix with only one nonzero entry equal to a in i-th row and j-th column. Then

$$[\mathrm{GL}(A), \mathrm{GL}(A)] = \mathrm{E}(A).$$

The elementary matrices e_{ij}^a satisfy the following relations

$$\begin{cases} e_{ij}^a e_{ij}^b & = e_{ij}^{a+b}, \\ e_{ij}^a e_{kl}^b & = e_{kl}^b e_{ij}^a, \text{ for } j \neq k,\ i \neq l, \\ e_{ij}^a e_{jk}^b & = e_{jk}^b e_{ik}^{ab} e_{ij}^a. \end{cases} \tag{5.1}$$

The group $\mathrm{E}(A)$ can be presented using generators e_{ij}^a which satisfy the relations (5.1) above plus some relations which depend on A. Define the Steinberg group $\mathrm{St}(A)$ of A as the group with the set of generators $\{x_{ij}^a\}$ with the relations (5.1). There is an epimorphism $\mathrm{St}(A) \twoheadrightarrow \mathrm{E}(A)$ and we define $\mathrm{K}_2(A)$ as the kernel of this map. Then $\mathrm{K}_2(A)$ is abelian, and the sequence

$$\mathrm{K}_2(A) \rightarrowtail \mathrm{St}(A) \twoheadrightarrow \mathrm{E}(A)$$

is a central extension.

Theorem 5.6 (Kervaire)**.** *The group* $\mathrm{E}(A)$ *is perfect, that is*

$$\mathrm{H}_1(E(A)) = 0,$$

and

$$\mathrm{H}_2(E(A)) \cong \mathrm{K}_2(A).$$

Proof. The proof relies on the spectral sequence of the fibration

$$\mathrm{B}\,\mathrm{K}_2(A) \to \mathrm{B}\,\mathrm{St}(A) \to \mathrm{B}\,\mathrm{E}(A)$$

On the second table we have

$$E_{pq}^2 = \mathrm{H}_p(\mathrm{B}\,\mathrm{E}(A); \mathrm{H}_q(\mathrm{B}\,\mathrm{K}_2(A)))$$

and the sequence converges to $\mathrm{H}_{p+q}(\mathrm{B}\,\mathrm{St}(A))$. We have

$$\mathrm{H}_p(\mathrm{B}\,\mathrm{E}(A); \mathrm{H}_q(\mathrm{B}\,\mathrm{K}_2(A))) \cong \mathrm{H}_p(\mathrm{E}(A); \mathrm{H}_q(\mathrm{K}_2(A))) \cong \mathrm{H}_p(\mathrm{E}(A)) \otimes \mathrm{H}_q(\mathrm{K}_2(A))$$

The second table looks like follows.

$$
\begin{tikzcd}
\cdots & 0 & \cdots & \cdots & \cdots \\
\Lambda^2 K_2(A) & 0 & \cdots & \cdots & \cdots \\
K_2(A) & 0 & \mathrm{H}_2(\mathrm{E}(A) \otimes K_2(A) \arrow[ull] & \mathrm{H}_3(\mathrm{E}(A) \otimes K_2(A) & \cdots \\
\mathbb{Z} & 0 & \mathrm{H}_2(\mathrm{E}(A)) \arrow[ull] & \mathrm{H}_3(\mathrm{E}(A)) \arrow[ull] & \cdots
\end{tikzcd}
$$

One needs to prove that $\mathrm{H}_2(\mathrm{St}(A)) = 0$, and that E^∞_{pq} looks like

$$
\begin{array}{ccccc}
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} & 0 & 0 & \mathrm{H}_3(\mathrm{St}(A)) & \cdots
\end{array}
$$

□

Theorem 5.7 (Gersten)**.** *There is an isomorphism*

$$\mathrm{H}_3(\mathrm{St}(A)) \cong \mathrm{K}_3(A).$$

Proof. One has to prove that there is a fibration

$$
\begin{tikzcd}
\mathrm{B}\, K_2(A) \arrow[r] \arrow[d, equal] & \mathrm{B}\,\mathrm{St}(A)^+ \arrow[r] & \mathrm{B}\,\mathrm{E}(A)^+ \\
\mathrm{B}\, K_2(A)^+ & &
\end{tikzcd}
$$

and then use a spectral sequence. □

Summarizing earlier results we have

$$
\begin{aligned}
\mathrm{H}_1(\mathrm{GL}(A)) &= \mathrm{K}_1(A), \\
\mathrm{H}_2(\mathrm{E}(A)) &= \mathrm{K}_2(A), \\
\mathrm{H}_3(\mathrm{St}(A)) &= \mathrm{K}_3(A).
\end{aligned}
$$

Let us look once more at the relations for Steinberg group (5.1). We can label the edges of a Stasheff polytope of dimension 2 as follows

$$\begin{tikzcd}
 & \bullet \arrow[dl, "e^b_{jk}"'] \arrow[dr, "e^a_{ij}"] & \\
\bullet \arrow[d, "e^{ab}_{ik}"'] & & \bullet \arrow[ddl, "e^b_{jk}"] \\
\bullet \arrow[dr, "e^a_{ij}"'] & & \\
 & \bullet &
\end{tikzcd}$$

to encode the relation $e^a_{ij}e^b_{jk} = e^b_{jk}e^{ab}_{ik}e^a_{ij}$. There is a way to put labels on the Stasheff polytope of dimension 3 in the coherent way. It can be generalized to higher dimension.

Proposition 5.8 (Kapranov-Saito)**.** *The space* $\mathrm{B}\,\mathrm{GL}(A)^+$ *is an* H*-space and there is an isomorphism*

$$\mathrm{Prim}\,\mathrm{H}_*(\mathrm{B}\,\mathrm{GL}(A)^+;\mathbb{Q}) = \pi_*(\mathrm{B}\,GL(A)^+)\otimes\mathbb{Q}$$

where the primitive elements are the set

$$\{x\in \mathrm{H}_*(\mathrm{B}\,\mathrm{GL}(A)^+;\mathbb{Q}) : \Delta(x) = x\otimes 1 + 1\otimes x\}.$$

Chapter 6

Homology of Lie algebras of matrices

Theorem 6.1 (Loday-Quillen, Tsygan). *Let k be a characteristic 0 field, A an associative unital k-algebra. Then*

$$\mathrm{H}_*(\mathfrak{gl}(A)) \cong \Lambda(\mathrm{HC}_{*-1}(A)). \tag{6.1}$$

On the left hand side of isomorphism (6.1) we have matrices of any size, while on the right hand side there are no matrices, and the computations are easier.

There are generalizations of the theorem (6.1) for Lie algebras $\mathfrak{so}(A)$ and $\mathfrak{sp}(A)$. Also, if instead of algebra we take an operad, then on the right hand side cyclic homology is replaced by some graph homology (Kontsevich).

6.1 Leibniz algebras

Definition 6.2. *A* Leibniz (right) algebra *over k is an algebra A with bracket*

$$[-,-]\colon A \otimes A \to A,$$

such that $[-,z]$ is a derivation for each $z \in A$, that is

$$[[x,y],z] = [[x,z],y] + [x,[y,z]].$$

Definition 6.3. *A* Lie algebra *is a Leibniz algebra such that*

$$[x,y] = -[y,x].$$

Under this symmetry property, the Leibniz relation is equivalent to Jacobi relation.

There is a chain complex associated to a Leibniz algebra $\mathfrak{g}$:

$$\mathrm{CL}_*(\mathfrak{g})\colon \ \ldots \to \mathfrak{g}^{\otimes n} \xrightarrow{d} \mathfrak{g}^{\otimes(n-1)} \xrightarrow{d} \ldots \xrightarrow{d} \mathfrak{g} \xrightarrow{0} k$$

$$\mathrm{CL}_n(\mathfrak{g}) = \mathfrak{g}^{\otimes(n-1)}, \quad d\colon \mathrm{CL}_{n+1}(\mathfrak{g}) \to \mathrm{CL}_n(\mathfrak{g})$$

$$d(x_1,\ldots,x_n) := \sum_{1\le i<j\le n} (-1)^j (x_1,\ldots,[x_i,x_j],x_{i+1},\ldots,\widehat{x_j},\ldots,x_n).$$

Lemma 6.4. *The map d is a differential, that is $d^2 = 0$.*

Proof. We will check only the composition

$$\begin{array}{ccccc} \mathfrak{g}^{\otimes 3} & \xrightarrow{\quad d \quad} & \mathfrak{g}^{\otimes 2} & \xrightarrow{\quad d \quad} & \mathfrak{g} \\ & & x \otimes y & \longmapsto & [x,y] \end{array}$$

$$x \otimes y \otimes z \mapsto -[x,z] \otimes y - x \otimes [y,z] + [x,y] \otimes z$$

In this case $d^2 = 0$ is equivalent to Leibniz relation. The general case is analogous. □

Definition 6.5. *The* Leibniz homology *of the Leibniz algebra $\mathfrak{g}$ is*

$$\mathrm{HL}_*(\mathfrak{g}) := \mathrm{H}_*(\mathrm{CL}_*(\mathfrak{g}), d).$$

When $\mathfrak{g}$ is a Lie algebra, then one can pass to the quotient by the action of symmetric group (with signature)

$$\mathfrak{g}^{\otimes n} \twoheadrightarrow \Lambda^n \mathfrak{g}.$$

Then d also passes to the quotient and one obtains a Chevalley-Eilenberg chain complex of $\mathfrak{g}$:

$$C_*(\mathfrak{g}) : \ldots \to \Lambda^n \mathfrak{g} \xrightarrow{d} \Lambda^{n-1}\mathfrak{g} \xrightarrow{d} \ldots \xrightarrow{d} \mathfrak{g} \xrightarrow{0} k$$

$$C_n(\mathfrak{g}) = \Lambda^{n-1}\mathfrak{g}, \quad d\colon C_{n+1}(\mathfrak{g}) \to C_n(\mathfrak{g})$$

$$d(x_1, \ldots, x_n) := \sum_{1 \le i < j \le n} (-1)^{i+j-1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n.$$

For a Lie algebra $\mathfrak{g}$ we define a *Lie algebra homology*:

$$\mathrm{H}_*(\mathfrak{g}) := \mathrm{H}_*(C_*(\mathfrak{g}), d).$$

There is a map $\mathrm{HL}_*(\mathfrak{g}) \to \mathrm{H}_*(\mathfrak{g})$, which is not an isomorphism in general. For example if $\mathfrak{g}$ is abelian, then the boundary in the Leibniz and Chevalley-Eilenberg complex is 0, so we have

$$\begin{aligned} \mathrm{HL}_n(\mathfrak{g}) &= \mathfrak{g}^{\otimes n} \\ \mathrm{H}_n(\mathfrak{g}) &= \Lambda^n \mathfrak{g} \end{aligned}$$

Also if $\mathfrak{g}$ is a simple Lie algebra, then $\mathrm{HL}_n(\mathfrak{g}) = 0$, for $n \ge 1$, but $\mathrm{H}_n(\mathfrak{g})$ does not have to be 0 for $n \neq 1$.

Let $\mathfrak{g}$ be a Lie algebra, and $g \in \mathfrak{g}$. Then g acts on $\mathfrak{g}^{\otimes n}$

$$[g_1 \otimes \cdots \otimes g_n, g] = \sum_{i=1}^{n} g_1 \otimes \cdots \otimes [g_i, g] \otimes \cdots \otimes g_n.$$

Proposition 6.6. *This action is compatible with the boundary map d and it is zero on $\mathrm{H}_*(\mathfrak{g})$.*

Proof. The first part is easy. For the second part we construct for $y \in \mathfrak{g}$ a map

$$\sigma(y)\colon \Lambda^n\mathfrak{g} \to \Lambda^{n+1}\mathfrak{g}$$

$$\alpha \mapsto (-1)^n \alpha \wedge y.$$

Then $\sigma(y)$ is a homotopy from conjugation to zero map, that is

$$d\sigma(y) + \sigma(y)d = [-, y].$$

□

Proposition 6.7. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ be a reductive sub- Lie algebra of $\mathfrak{g}$. Then the surjective map*

$$(\Lambda^n\mathfrak{g}) \twoheadrightarrow (\Lambda^n\mathfrak{g})_{\mathfrak{h}}$$

induces an isomorphism on homology

$$\mathrm{H}_*(\mathfrak{g}) \cong \mathrm{H}_*((\Lambda^n\mathfrak{g})_{\mathfrak{h}}, d).$$

6.2 Computation of Lie algebra homology $\mathrm{H}_*(\mathfrak{gl}(A))$

Let k be a field, A an associative unital algebra over k. Denote by $M_r(A)$ the algebra of $r \times r$ matrices with coefficients in A, and by $\mathfrak{gl}_r(A)$ the same space, but with its Lie algebra structure given by $[\alpha, \beta] = \alpha\beta - \beta\alpha$. There is an inclusion of Lie algebras $\mathfrak{gl}_r(A) \hookrightarrow \mathfrak{gl}_{r+1}(A)$,

$$\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix},$$

but it does not preserve the identity in $M_r(A)$. With respect to these inclusions we define $\mathfrak{gl}(A) := \bigcup_r \mathfrak{gl}_r(A)$, the Lie algebra of matrices over A. Our aim is to compute $\mathrm{H}_*(\mathfrak{gl}(A)) = \varinjlim_r \mathrm{H}_*(\mathfrak{gl}_r(A))$, but unfortunately we cannot compute $\mathrm{H}_*(\mathfrak{gl}_r(A))$.

Proof. (of the theorem (6.1)) The strategy of the proof of theorem (6.1) can be summarized in the following four steps:

1. Koszul trick.
2. Coinvariant theory.
3. Hopf-Borel (type) theorem.
4. Computation of primitive part.

The idea is to prove that the composition of the following maps is a quasi-isomorphism

$$\begin{array}{c}
(\mathfrak{gl}(A)^{\otimes n})_{S_n} \\
\big\downarrow \\
((\mathfrak{gl}(A)^{\otimes n})_{S_n})_{\mathfrak{sl}(k)} \\
\big\| \\
((\mathfrak{gl}(k)^{\otimes n} \otimes A^{\otimes n})_{\mathfrak{sl}(k)})_{S_n} \\
\big\| \\
((\mathfrak{gl}(k)^{\otimes n})_{\mathfrak{sl}(k)} \otimes A^{\otimes n})_{S_n} \\
\big\| \\
(k[S_n] \otimes A^{\otimes n})_{S_n} \\
\big\downarrow \\
\Lambda((k[U_n] \otimes A^{\otimes n})_{S_n}) \\
\big\downarrow \\
\Lambda(A^{\otimes n}/(1-t))
\end{array}$$

where U_n denotes the set of permutations with only one cycle, and $A^{\otimes *}/1-t$ is the Connes complex computing cyclic homology.

1. The algebra $\mathfrak{sl}_r(k)$ is reductive, $(\mathfrak{gl}_r(A)^{\otimes n})_{S_n}$ is an $\mathfrak{sl}_r(k)$-module, and we can consider the projection on the component corresponding to the trivial representation

$$K \rightarrowtail (\mathfrak{gl}_r(A)^{\otimes n})_{S_n} \twoheadrightarrow ((\mathfrak{gl}_r(A)^{\otimes n})_{S_n})_{\mathfrak{sl}_r(k)}.$$

   The kernel K has trivial homology, so the projection is a quasi-isomorphism.

2. There is an isomorphism $\mathfrak{gl}_r(A) \cong \mathfrak{gl}_r(k) \otimes A$ which can be proved by decomposing a matrix with entries in A into the elementary matrices E_{ij}^a, having one nonzero entry a in the place (i,j), that is

$$\sum_{i,j=1}^{r} E_{ij}^{a_{ij}} = \sum_{i,j=1}^{r} E_{ij}^1 \otimes a_{ij}.$$

   From this we derive

$$(\mathfrak{gl}(A)^{\otimes n})_{\mathfrak{sl}_r(k)} = (\mathfrak{gl}_r(k) \otimes A^{\otimes n})_{\mathfrak{sl}_r(k)} = (\mathfrak{gl}_r(k)^{\otimes n})_{\mathfrak{sl}_r(k)} \otimes A^{\otimes n}.$$

   Now we use

   Theorem 6.8. *When k is a characteristic 0 field there is an isomorphism of S_n-modules*

$$(\mathfrak{gl}_r(k)^{\otimes n})_{\mathfrak{sl}_r(k)} \cong k[S_n].$$

Proof. The isomorphism is given by

$$\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \sum_{\sigma \in S_n} \underbrace{T(\sigma)(\alpha)}_{\in k} \sigma,$$

where for $sg \in S_n$ which is decomposed into cycles $(i_1 \dots i_k)(j_1 \dots j_l)(\dots)\dots$ we define a map $T(\sigma)\colon (\mathfrak{gl}_r)^{\otimes n} \to k$ by

$$T(\sigma)(\alpha) = \mathrm{Tr}(\alpha_{i_1} \dots \alpha_{i_k})\,\mathrm{Tr}(\alpha_{j_1} \dots \alpha_{j_l})\,\mathrm{Tr}(\dots)\dots,$$

which is a product of finite number of elements in k. From the trace property $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ we know that $T(\sigma)$ is well defined.

Observe that

$$E^1_{i_1 i_2} \otimes E^1_{i_2 i_3} \otimes \cdots \otimes E^1_{i_n i_1} \mapsto (12\dots n).$$

The action of S_n on $k[S_n] \otimes A^{\otimes n}$ is conjugation in $k[S_n]$, place permutation on $A^{\otimes n}$ and multiplication by sign. □

3. The diagonal map $\mathfrak{g} \xrightarrow{\Delta} \mathfrak{g} \times \mathfrak{g}$ induces a graded cocommutative coproduct on homology $\mathrm{H}_*(\mathfrak{g}) \to \mathrm{H}_*(\mathfrak{g} \times \mathfrak{g}) = \mathrm{H}_* \otimes \mathrm{H}_*(\mathfrak{g})$. For $\mathfrak{g} = \mathfrak{gl}(A)$ there is a map
$$\mathfrak{gl}(A) \times \mathfrak{gl}(A) \xrightarrow{\oplus} \mathfrak{gl}(A),$$
which we can schematically describe as
$$\left(\begin{pmatrix} * & * & * & \dots \\ * & * & * & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \begin{pmatrix} \star & \star & \star & \dots \\ \star & \star & \star & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \right) \mapsto$$
$$\begin{pmatrix} * & 0 & * & 0 & * & 0 & \dots \\ 0 & \star & 0 & \star & 0 & \star & \dots \\ * & 0 & * & 0 & * & 0 & \dots \\ 0 & \star & 0 & \star & 0 & \star & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$
It induces a graded cocommutative product.
$$\mathrm{H}_*(\mathfrak{gl}(A)) \otimes \mathrm{H}_*(\mathfrak{gl}(A)) \xrightarrow{\mu = \oplus_*} \mathrm{H}_*(\mathfrak{gl}(A))$$

Theorem 6.9. $\mathrm{H}_*(\mathfrak{gl}(A))$ *is a commutative cocommutative bialgebra.*

For any coalgebra $\mathcal{H}$ there is a filtration $F_*\mathcal{H}$ such that , $F_0 = k$ and

$$F_r := \{x \in \mathcal{H} \mid \overline{\Delta}(x) \in F_{r-1} \otimes F_{r-1}\},$$

where $\overline{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$.

Definition 6.10. *We say that a coalgebra $\mathcal{H}$ is* conilpotent *if $\mathcal{H} = \sum_{r \geq 0} F_r\mathcal{H}$.*

Now we recall the Hopf-Borel theorem:

Theorem 6.11. *If k is a characteristic 0 field, and $\mathcal{H}$ is a conilpotent graded commutative cocommutative bialgebra, then*

$$\mathcal{H} \cong \Lambda(\mathrm{Prim}(\mathcal{H})).$$

The main point now is that $C_*(\mathfrak{gl}(A))$ is a graded commutative cocommutative bialgebra.

Proposition 6.12. *$\bigoplus_n (k[S_n] \otimes A^{\otimes n})_{S_n}$ is a graded commutative cocommutative bialgebra.*

4. The last step in the proof of theorem (6.1) is determining the primitive part of $(k[S_n] \otimes A^{\otimes n})_{S_n}$. Let U_n denote the permutations with only one cycle. Then

   Proposition 6.13.

   $$\mathrm{Prim}((k[S_n] \otimes A^{\otimes n})_{S_n}) = (k[U_n] \otimes A^{\otimes n})_{S_n}.$$

   Proof. Assume that σ can be decomposed into more than one cycle, $\sigma = (i_1 \dots i_k)(j_1 \dots j_l)$. Then the coproduct gives

   $$\Delta((i_1 \dots i_k)(j_1 \dots j_l)) = \sigma \otimes 1 + 1 \otimes \sigma + (i_1 \dots i_k) \otimes (j_1 \dots j_l) \pm (j_1 \dots j_l)(i_1 \dots i_k).$$

   We see that σ is primitive if and only if σ has only one cycle. $\square$

   Now
   $$\mathrm{H}_*(\mathfrak{gl}(A)) = \Lambda(\mathrm{H}_*((k[U_*] \otimes A^{\otimes *})_{S_*})).$$
   The symmetric group S_n is acting by conjugation in $k[S_n]$ and $k[U_n]$. As an S_n-representation
   $$k[U_n] = \mathrm{Ind}_{C_n}^{S_n} k$$
   and the dimension of $k[U_n]$ is $(n-1)!$. Furthermore

   $$(\mathrm{Ind}_{C_n}^{S_n} k \otimes A^{\otimes n})_{S_n} \cong (A^{\otimes n})_{C_n} = A^{\otimes n}/(1-t) = C_n^\lambda(A)$$

   and for $a_1 \otimes \dots \otimes a_n \in C_n^\lambda(A)$ we have by tracing all the steps in the proof

   $$\begin{tikzcd}
   a_1 \otimes \dots \otimes a_n \in C_n^\lambda(A) \arrow[r, "b"] & b(a_1 \otimes \dots \otimes a_n) \arrow[d, equal] \\
   (12 \dots n) \otimes (a_1, \dots, a_n) \in k[S_n] \otimes A^{\otimes n} \arrow[u, mapsto] \arrow[r] & \begin{array}{c} a_1a_2 \otimes a_3 \otimes \dots \otimes a_n \\ -a_1 \otimes a_2a_3 \otimes \dots \otimes a_n \\ + \dots + \\ +(-1)^n a_na_1 \otimes a_2 \otimes \dots \otimes a_{n-1} \end{array} \\
   E_{12}^{a_1} \otimes E_{23}^{a_2} \otimes \dots \otimes E_{n1}^{a_n} \arrow[u, mapsto] \arrow[r, "d"] & \begin{array}{c} E_{12}^{a_1a_2} \otimes E_{34}^{a_3} \otimes \dots \otimes E_{n1}^{a_n} \\ +E_{12}^{a_1} \otimes E_{23}^{a_2a_3} \otimes \dots \otimes E_{n1}^{a_n} \\ + \dots + \\ +E_{12}^{a_1} \otimes E_{23}^{a_2} \otimes \dots \otimes E_{n1}^{a_na_1} \end{array} \arrow[u, mapsto]
   \end{tikzcd}$$

   We proved that $(k[U_n] \otimes A^{\otimes n})_{S_n}$ is the Connes complex, and thus

   $$\Lambda(\mathrm{HC}_{*-1}(A)) \cong \mathrm{H}_*(\mathfrak{gl}(A)).$$

□

Example 6.14. If $A = k$ we know that

$$\begin{cases} \mathrm{HC}_{2n}(k) & = k \\ \mathrm{HC}_{2n-1}(k) & = 0 \end{cases}$$

From this we can derive

$$\begin{aligned} \mathrm{H}_*(\mathfrak{gl}(k)) &= \Lambda(x_1, x_3, \ldots, x_{2n+1}, \ldots), \\ \mathrm{H}_*(\mathfrak{gl}_n(k)) &= \Lambda(x_1, x_3, \ldots, x_{2n-1}), \\ \mathrm{H}_*(\mathfrak{sl}_2(k)) &= \Lambda(x_3). \end{aligned}$$

6.3 Computation of Leibniz homology $\mathrm{HL}_*(\mathfrak{gl}(A))$

Our aim now is to compute $\mathrm{HL}_*(\mathfrak{gl}(A))$. Recall the steps in the proof of theorem (6.1).

$$\begin{array}{c} \mathfrak{gl}(A)^{\otimes n} \\ \Big\downarrow \text{quasi-isomorphism} \\ (\mathfrak{gl}(A)^{\otimes n})_{\mathfrak{sl}(k)} \\ \Big\| \\ (\mathfrak{gl}(k)^{\otimes n} \otimes A^{\otimes n})_{\mathfrak{sl}(k)} \\ \Big\| \\ (\mathfrak{gl}(k)^{\otimes n})_{\mathfrak{sl}(k)} \otimes A^{\otimes n} \\ \Big\| \\ (k[S_n] \otimes A^{\otimes n})_{S_n} \end{array}$$

We have to modify the third step, because Leibniz homology is not a Hopf algebra. If $\mathfrak{g}$ is a Leibniz algebra, then $\mathrm{HL}_*(\mathfrak{g})$ is a graded Zinbiel coalgebra which definition we give below.

Definition 6.15. *A* Zinbiel algebra A *is an algebra such that its multiplication satisfies the following identity*

$$(x \cdot y) \cdot z = x \cdot (y \cdot z + z \cdot y).$$

Lemma 6.16. *If we define a new product $xy := x \cdot y + y \cdot x$, then it will be associative.*

Zinbiel algebras play the same role to the commutative algebras as the associative algebras to Lie algebras.

Definition 6.17. *A* graded Zinbiel algebra A *is an algebra such that its multiplication satisfies the following identity*

$$(x \cdot y) \cdot z = x \cdot (y \cdot z + (-1)^{|y||z|} z \cdot y).$$

Definition 6.18. *A* Zinbiel coalgebra C *is a coalgebra such that its comultiplication* $\Delta\colon C \to C \otimes C$ *satisfies the following identity*

$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta + (\mathrm{Id} \otimes \tau\Delta)\Delta,$$

where $\tau\colon C \otimes C \to C \otimes C$ *is given by*

$$\tau(x \otimes y) = \begin{cases} y \otimes x & \text{in the non graded case,} \\ (-1)^{|y||x|} y \otimes x & \text{in the graded case.} \end{cases}$$

Proposition 6.19. *The Leibniz homology* $\mathrm{HL}_*(\mathfrak{gl}(A))$ *is graded Zinbiel as coalgebra and associative as algebra.*

In short we say that $\mathrm{HL}_*(\mathfrak{gl}(A))$ is a graded $Zinb^c$-As-bialgebra. It means that the Zinbiel coalgebra coproduct and the associative algebra product satisfy some compatibility relation. If one compares the product with the symmetric coproduct, then one obtatins the Hopf formula.

There is a following structure theorem for $Zinb^c$-As-bialgebras.

Theorem 6.20. *If a* $Zinb^c$-As*-bialgebra* $\mathcal{H}$ *is conilpotent, then it is free and cofree over its primitive part.*

Corollary 6.21.

$$\mathrm{HL}_*(\mathfrak{gl}(A)) \cong T(\mathrm{Prim}(\bigoplus_{n\geq 0} k[S_n] \otimes A^{\otimes n})) = T(\bigoplus_{n\geq 0} k[U_n] \otimes A^{\otimes n}).$$

Our aim now is to compute $\mathrm{H}_*(\bigoplus_{n\geq 0} k[U_n] \otimes A^{\otimes n})$.

Theorem 6.22 (Cuvier)**.** *There is a quasi-isomorphism of complexes*

$$\begin{tikzcd} \cdots \arrow[r] & k[U_n] \otimes A^{\otimes n} \arrow[r, "d"] \arrow[d] & k[U_{n-1}] \otimes A^{\otimes (n-1)} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & A^{\otimes n} \arrow[r, "b"] & A^{\otimes (n-1)} \arrow[r] & \cdots \end{tikzcd}$$

The map is given by

$$g(12\dots n)g^{-1} \otimes \alpha \mapsto g(\alpha),$$

where $g \in S_n$, $g(1) = 1$ is chosen in such way that $g(12\dots n)g^{-1}$ is the cycle which we want to send to $A^{\otimes n}$. The map in the opposite direction is

$$\alpha \mapsto (12\dots n) \otimes \alpha.$$

and the one composition is identity on A^* and the second one is homotopic to the identity.

Corollary 6.23.

$$\begin{tikzcd} \mathrm{HL}_*(\mathfrak{gl}(A)) \arrow[r, "\cong"] \arrow[d] & T(\mathrm{HH}_{*-1}(A)) \arrow[d] \\ \mathrm{H}_*(\mathfrak{gl}(A)) \arrow[r, "\cong"] & \Lambda(\mathrm{HC}_{*-1}(A)). \end{tikzcd}$$

Now we can make a digression on some algebraic topology theorems. Suppose there is a fibration $F \to E \to B$ of H-spaces. There is a spectral sequence

$$E^2_{pq} = \mathrm{H}_p(B, \mathrm{H}_q(F)) \implies \mathrm{H}_{p+q}(E).$$

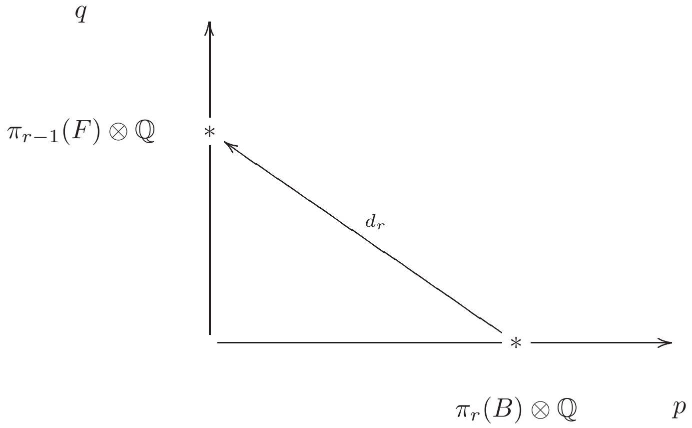
Furthermore for any H-space X

$$\mathrm{Prim}(\mathrm{H}_*(X; \mathbb{Q})) = \pi_{*(X)} \otimes \mathbb{Q},$$

so

$$\mathrm{H}_*(X; \mathbb{Q}) = \Lambda(\pi_*(X) \otimes \mathbb{Q}).$$

If we take the primitive parts on second term of this spectral sequence, then it can be proved that the only nonzero terms will be on the row $q = 0$ and column $p = 0$, and it will be isomorphic to rational homotopy of the basis and the fiber.



From this spectral sequence we obtain the long exact sequence of homotopy groups.

The motivation for computing $\mathrm{H}_*(\mathfrak{gl}_r(A))$ for fixed r comes from Macdonald conjecture, which is some identity with sum on the left hand side and product on the right. To prove it, it is sufficient to compute $\mathrm{H}_n(\mathfrak{gl}_r(k[t]/t^k))$. On one side there will be an Euler-Poincaré characteristic of the complex, and on the other the Euler-Poincaré characteristic of the homology, which are equal.

Theorem 6.24. *If k is a characteristic 0 field, and A is an associative unital algebra, then*

$$\mathrm{H}_n(\mathfrak{gl}_n(A)) \cong \mathrm{H}_n(\mathfrak{gl}_{n+1}(A)) \cong \ldots \cong \mathrm{H}_n(\mathfrak{gl}(A)).$$

Furthermore for commutative A the following sequence is exact

$$\mathrm{H}_n(\mathfrak{gl}_{n-1}(A)) \to \mathrm{H}_n(\mathfrak{gl}_n(A)) \twoheadrightarrow \Omega^{n-1}_A / d\Omega^{n-2}_A.$$

Theorem 6.25. *If k is a characteristic 0 field, and A is an associative unital algebra, then*

$$\mathrm{H}_n(\mathrm{GL}_n(F)) \cong \mathrm{H}_n(\mathrm{GL}_{n+1}(F)) \cong \ldots \cong \mathrm{H}_n(\mathrm{GL}(A)),$$

and the following sequence is exact

$$\mathrm{H}_n(\mathrm{GL}_{n-1}(F)) \to \mathrm{H}_n(\mathrm{GL}_n(F)) \to \mathrm{K}^M_n(F) \otimes \mathbb{Q}.$$

where K^M is the Milnor's K-theory.

Theorem 6.26. *If k is a characteristic 0 field, and A is an associative unital algebra, then*

$$\mathrm{HL}_n(\mathfrak{gl}_n(A)) \cong \mathrm{HL}_n(\mathfrak{gl}_{n+1}(A)) \cong \ldots \cong \mathrm{HL}_n(\mathfrak{gl}(A)).$$

Furthermore for commutative A there is an exact sequence

$$\mathrm{HL}_n(\mathfrak{gl}_{n-1}(A)) \to \mathrm{HL}_n(\mathfrak{gl}_n(A)) \twoheadrightarrow \Omega_A^{n-1}.$$

Chapter 7

Algebraic operads

7.1 Schur functors and operads

Definition 7.1. *An* algebraic operad *is a functor* $\mathcal{P}\colon \mathbf{Vect} \to \mathbf{Vect}$*, together with a natural transformation of functors* $\iota\colon \mathrm{Id} \to \mathcal{P}$*,* $\gamma\colon \mathcal{P}\circ\mathcal{P} \to \mathcal{P}$*. They are supposed to satisfy the following relations*

- γ *is associative,*

$$\begin{tikzcd}
\mathcal{P}\circ\mathcal{P}\circ\mathcal{P} \arrow[r, "\mathrm{id}\circ\gamma"] \arrow[d, "\gamma\circ\mathrm{id}"'] & \mathcal{P}\circ\mathcal{P} \arrow[d, "\gamma"] \\
\mathcal{P}\circ\mathcal{P} \arrow[r, "\gamma"'] & \mathcal{P}
\end{tikzcd}$$

- ι *is a unit for* γ*.*

If X is a set, then the structure is just inclusion $\{*\} \to X$ and if $X \times X \to X$ is an operation, then we have the notion of *set operad* $\mathcal{P}\colon \mathbf{Sets} \to \mathbf{Sets}$. In analagous way we can define topological operad, chain complex operad etc.

In the sequel, we suppose that $\mathcal{P}$ is Schur functor, which definition we give below.

Definition 7.2. *A* Schur functor *is defined from an* S*-module* $\mathcal{P}$*, which is a collection of right* S_n*-modules, and*

$$\mathcal{P}(V) := \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}.$$

We can as well write

$$\mathcal{P}(V) := \bigoplus_{n\geq 0} (\mathcal{P}(n) \otimes V^{\otimes n})_{S_n}.$$

In characteristic 0 we can take the invariants as well as coinvariants.

In these notes we restrict the study of algebraic operads to Schur functors.

The natural transformation γ gives us for each vector space V a linear map

$$\gamma\colon \mathcal{P}(\mathcal{P}(V)) = \mathcal{P}\left(\bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}\right) \to \mathcal{P}(V).$$

and so for each $n \geq 0$, a map

$$\gamma_{i_1 \dots i_n} \colon \mathcal{P}(n) \otimes \mathcal{P}(i_1) \otimes \dots \otimes \mathcal{P}(i_n) \to \mathcal{P}(i_1 + \dots + i_n), \quad n \geq 0.$$

Starting with $\gamma_{i_1 \dots i_n}$, in order to reconstruct the operad we need to assume that it is compatible with the action of the symmetric groups

$$S_n \times (S_{i_1} \times \dots \times S_{i_n}) \to S_{i_1 + \dots + i_n},$$

and that it satisfies a certain associativity property, namely the associativity of $\gamma \colon \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$.

Definition 7.3. *An* algebra over the operad $\mathcal{P}$ *(or* $\mathcal{P}$-algebra*) is a vector space* A *equipped with a linear map* $\gamma_A \colon \mathcal{P}(A) \to A$ *such that the following diagrams commute*

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(A)) \arrow[r, "\mathcal{P}(\gamma_A)"] \arrow[d, "\gamma(A)"'] & \mathcal{P}(A) \arrow[d, "\gamma_A"] \\
\mathcal{P}(A) \arrow[r, "\gamma_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\iota(A)"] \arrow[dr, "="'] & \mathcal{P}(A) \arrow[d, "\gamma_A"] \\
& A
\end{tikzcd}$$

For an algebra over the operad $\mathcal{P}$ and each $n \geq 0$ there is a map $\gamma_n \colon \mathcal{P}(n) \otimes_{S_n} A^{\otimes n} \to A$ and we write

$$(\mu; a_1, \dots, a_n) \mapsto \gamma(\mu \otimes (a_1, \dots, a_n)) =: \mu(a_1, \dots, a_n).$$

We call $\mathcal{P}(n)$ the space of n-ary operations.

Let V be a vector space, and $\mathcal{P}$ an operad. Suppose that we have a type of algebras (for example associative, Leibniz, Lie). We name it $\mathcal{P}$-algebras, where $\mathcal{P}$ denotes the given type. Then we define

Definition 7.4. *The* $\mathcal{P}$*-algebra* A_0 *is* free *over* V *if for any map* $V \to A$ *to a* $\mathcal{P}$*-algebra* A *there is a unique map of* $\mathcal{P}$*-algebras* $A_0 \to A$ *such that the following diagram commutes*

$$\begin{tikzcd}
V \arrow[r] \arrow[dr] & A_0 \arrow[d, dashed] \\
& A
\end{tikzcd}$$

Let $V = kx_1 \oplus \dots \oplus kx_n$ be an n-dimensional vector space over k, and $\mathcal{P}(V)$ denote the free algebra of a given type over V. The multilinear part of $\mathcal{P}(V)$ of degree n (linear in each variable) is a subspace which we denote by $\mathcal{P}(n)$ and it inherits an S_n-action. Thus it allows us to construct an operad $\mathcal{P}$ as a Schur functor. If $k \supseteq \mathbb{Q}$ then

$$\mathcal{P}(V) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}.$$

7.2 Free operads

The notion of free $\mathcal{P}$- algebra over a vector space V gives rise to a functor from the category **Vect** of vector spaces to the category of $\mathcal{P}$-algebras, which is a left adjoint to the forgetful functor

$$\mathrm{Hom}_{\mathcal{P}-\mathbf{alg}}(\mathcal{P}(V), A) \cong \mathrm{Hom}_{\mathbf{Vect}}(V, A).$$

There is a forgetful functor from the category **Op** of operads to the category of Schur functors. It has a left adjoint $\mathcal{F}$ giving rise to free operads.

Example 7.5. An S-module is given by the family $M(n)$ of S_n-modules. Suppose $M(n) = 0$ except $M(2) = k[S_2]\mu$. What is $\mathcal{F}(M)$? We have $\mathrm{id} \in \mathcal{F}(M)(1)$, $\mu \in \mathcal{F}(M)(2)$ and the following two operations in $\mathcal{F}(M)(3)$

$$\mu \circ (\mu, \mathrm{id}), \quad \mu \circ (\mathrm{id}, \mu).$$

Thus

$$\mathcal{F}(M)(3) = k[S_3]\mu \circ (\mu, \mathrm{id}) \oplus k[S_3]\mu \circ (\mathrm{id}, \mu).$$

Proposition 7.6. *The free operad $\mathcal{F}(M)$, where M is binary and free over S_2, has $\mathcal{F}(M)(n) = k[Y_{n-1}] \otimes k[S_n]$, where Y_{n-1} is the set of planar binary trees with n leaves.*

Exercise 7.7. *What is the free operad on N, where $N(n) = 0$ except $N(2) = k$ - the trivial representation?*

7.3 Operadic ideals

Definition 7.8. *For a given operad $\mathcal{P}$ and a family of operations in $\mathcal{P}$, the* operadic ideal *generated by this family is a sub-Schur functor I (that is $I(n) \subseteq \mathcal{P}(n)$) linearly generated by all the compositions where at least one of the operations is in the family.*

In another words whenever one of the operations is in I, then the image by γ is also in I.

$$\gamma \colon \mathcal{P}(n) \otimes (\mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_n)) \to \mathcal{P}(i_1 + \ldots + i_n)$$

Proposition 7.9. *The quotient $\mathcal{P}/I$ defined as $(\mathcal{P}/I)(n) := \mathcal{P}(n)/I(n)$ is an operad. It is called the* quotient operad*.*

A type of algebras consists of generating operations and relations (multilinear). Let $\mathcal{F}(M)$ be the free operad over an S-module M such that $M(n)$ is defined by the n-ary operations. If we take the ideal I generated by the relators in $\mathcal{F}(M)(-)$, then the we have an operad associated to a given type of algebras represented as a quotient operad $\mathcal{P} = \mathcal{F}(M)/I$. The space $\mathcal{P}(V)$ is exactly the free algebra over V for the given type.

7.4 Examples

Example 7.10. Associative algebras over k with binary associative operation $\mu\colon A \otimes A \to A$, $\mu(xy) =: xy$. The corresponding operad **As** has

$$\mathbf{As}(n) = k[S_n],$$

$$\mathbf{As}(n) \otimes_{S_n} V^{\otimes n} = V^{\otimes n},$$

$$\gamma\colon \mathbf{As}(n) \otimes \mathbf{As}(i_1) \otimes \cdots \otimes \mathbf{As}(i_n) \to \mathbf{As}(i_1 + \ldots + i_n),$$

$$k[S_n] \otimes k[S_{i_1}] \otimes \cdots \otimes k[S_{i_n}] \to k[S_{i_1+\ldots i_n}]$$

$$(\sigma; \omega_1, \ldots, \omega_n) \mapsto \sigma(\omega_1, \ldots, \omega_n) = (\omega_{\sigma(1)} \times \cdots \times \omega_{\sigma(n)}).$$

Example 7.11. Commutative algebras over k with binary commutative operation $\mu\colon A \otimes A \to A$, $\mu(xy) =: xy$. The corresponding operad **Com** has

$$\mathbf{Com}(n) = k,$$

$$\mathbf{Com}(n) \otimes_{S_n} V^{\otimes n} = S^n V,$$

$$\gamma\colon \mathbf{Com}(n) \otimes \mathbf{Com}(i_1) \otimes \cdots \otimes \mathbf{Com}(i_n) \to \mathbf{Com}(i_1 + \ldots + i_n),$$

$$\gamma\colon k^{\otimes(n+1)} \xrightarrow{\cong} k$$

The general construction of the operad associated to algebras of given type uses the following data:

- generating operations μ_n with symmetries which define a right S_n-module $M(n)$, $n \geq 0$
- multilinear relations in $M(n)$, $n \geq 0$.

From the generating operations we can construct a free operad $\mathcal{F}(M)$, and then quotient by the ideal I generated by the relations which gives us $\mathcal{P} = \mathcal{F}(M)/I$.

For example if we have one binary operation μ and one relator $\mu \circ (\mu \otimes \mathrm{id}) - \mu \circ (\mathrm{id} \otimes \mu)$, then we can construct an operad **As** for associative algebras.

7.5 Koszul duality of algebras

Definition 7.12. *A* quadratic data *is a pair* (V, R)*, where* V *is a vector space and* $R \subset V^{\otimes 2}$*.*

Definition 7.13. *A* quadratic algebra *associated with quadratic data* (V, R) *is a quotient algebra of a tensor algebra* $A(V, R) := T(V)/R$*.*

The algebra $A := A(V, R)$ has universal property

$$\begin{tikzcd}
R \arrow[r, tail] \arrow[rr, bend left, "0"] & T(V) \arrow[r, two heads] \arrow[dr] & T(V)/R \arrow[d, dashed] \\
& & A'
\end{tikzcd}$$

Let $T^c(V)$ be the tensor module with the deconcatenation operation $\Delta\colon T^c(V) \to T^c(V) \otimes T^c(V)$,

$$\Delta(v_1, \dots, v_n) = \sum_{i=1}^{n} v_1 \dots v_i \otimes v_{i+1} \dots v_n.$$

Definition 7.14. *A* quadratic coalgebra *associated with quadratic data* (V, R) *is a subcoalgebra of a cotensor algebra* $C := C(V, R)$ *having the following universal property*

$$\begin{tikzcd}
C(V,R) \arrow[r, tail] \arrow[rr, bend left, "0"] & T^c(V) \arrow[r, two heads] & V^{\otimes 2}/R \\
C' \arrow[u, dashed] \arrow[ur] & &
\end{tikzcd}$$

We can write explicitly

$$A = k \oplus V \oplus V^{\otimes 2}/R \oplus V^{\otimes 3}/(V \otimes R + R \otimes V) \oplus \dots \oplus V^{\otimes n}/(\sum_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}) \oplus \dots$$

$$C = k \oplus V \oplus R \oplus (V \otimes R \cap R \otimes V) \oplus \dots \oplus \bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \oplus \dots$$

$$\alpha : \quad \begin{tikzcd}
C \arrow[rr] \arrow[dr] & & A \\
& V \arrow[ur] &
\end{tikzcd}$$

We can define a map $d_\alpha\colon C \otimes A \to C \otimes A$ by the composition

$$\begin{tikzcd}
C \otimes A \arrow[r, "\Delta \otimes \mathrm{id}"] \arrow[rrr, bend right, "d_\alpha"'] & C \otimes C \otimes A \arrow[r, "\mathrm{id} \otimes \alpha \otimes \mathrm{id}"] & C \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \cdot"] & C \otimes A
\end{tikzcd}$$

Lemma 7.15. *The map* d_α *is a differential, that is* $d_\alpha \circ d_\alpha = 0$.

Proof. This immediately follows from $\alpha * \alpha = 0$ and $d_\alpha \circ d_\alpha = d_{\alpha * \alpha}$. □

Definition 7.16. *A* Koszul complex *of the quadratic algebra* $A(V, R)$ *is the complex* $(C \otimes A, d_\alpha)$.

Definition 7.17. *A quadratic algebra* $A(V, R)$ *is said to be* Koszul algebra *if the Koszul complex is acyclic.*

Definition 7.18. *The* Koszul dual algebra *to an algebra* $A = A(V, R)$ *is defined as*

$$A^! := C^* := C(V, R)^*.$$

We have $A^! = T(V^*)/(R^\perp)$, where $R^\perp$ is defined as the kernel

$$R^\perp \rightarrowtail V^{*\otimes 2} \to R^*.$$

If $\dim V < \infty$, then $V^{**} = V$ and there is an epimorphism $V^{*\otimes 2} \twoheadrightarrow R^*$.

7.6 Bar and cobar constructions

Recall that for associative algebras there is so called bar construction

$$B\colon \mathbf{As}-\mathbf{alg}\to \mathbf{DGA}-\mathbf{coalg},$$

and for coalgebras there is a dual cobar construction

$$\Omega\colon \mathbf{As}-\mathbf{coalg}\to \mathbf{DGA}-\mathbf{alg}.$$

Theorem 7.19. *Let (V,R) be a quadratic data. Then the following are equivalent*

1. *$A(V,R)$ is Koszul.*
2. *$C\rightarrowtail B(A)$ is a quasi-isomorphism.*
3. *$\Omega(C)\twoheadrightarrow A$ is a quasi-isomorphism.*

The last two conditions mean that

2. $C\cong \mathrm{H}^0(B(A))$, $\mathrm{H}^n(B(A))=0$ for $n\neq 0$.
3. $A\cong \mathrm{H}_0(\Omega(C))$, $\mathrm{H}_n(\Omega(C))=0$ for $n\neq 0$.

Analogous constructions we can perform for quadratic operads. Starting from generating operations E and relators $R\subset \mathcal{F}(E)(3)$ we can construct an operad $\mathcal{P}(E,R)$ and a cooperad $\mathcal{C}(E,R)$. The cooperads are constructed on the same pattern but using comonoids instead of monoids, that is they are Schur functors with the comonoid structure with comultiplication $\gamma\colon \mathcal{P}\to\mathcal{P}\circ\mathcal{P}$ and counit $\eta\colon \mathcal{P}\to \mathrm{Id}$.

There are bar and cobar constructions

$$B\colon \mathbf{Op}\to \mathbf{DGA}-\mathbf{coOp}$$
$$\Omega\colon \mathbf{coOp}\to \mathbf{DGA}-\mathbf{Op}$$

Along the same lines we can construct a Koszul complex as $(\mathcal{C}\circ\mathcal{P},d)$, and if it is acyclic then $\mathcal{P}$ is called a Koszul operad.

Define dual cooperad $\mathcal{P}^!:=\mathcal{C}^\vee:=\mathcal{C}^*\otimes \mathrm{sgn}$, where sgn is the signature representation of S_n. For any $\mathcal{P}$-algebra A we define a chain complex $\mathcal{P}^{!\vee}(A):=C_*^{\mathcal{P}}(A)$. The Koszul complex $(\mathcal{P}^{!\vee}\circ\mathcal{P}(V))$ is a particular case of this construction.

Definition 7.20. *The homology $\mathrm{H}_*^{\mathcal{P}}(A):=\mathrm{H}_*(C_*^{\mathcal{P}}(A),d)$ is called* operadic homology *of $\mathcal{P}$-algebra A.*

Proposition 7.21. *The Koszul complex is acyclic if and only if $\mathrm{H}_n^{\mathcal{P}}(\mathcal{P}(V))=0$ for $n>1$, and $\mathrm{H}_1^{\mathcal{P}}(\mathcal{P}(V))=V$.*

Example 7.22. If $\mathcal{P}=\mathbf{Lie}$, $\mathbf{Lie}(n)=\mathrm{Ind}_{C_n}^{S_n}k[U_n]$ then $\mathcal{P}^!=\mathbf{Com}$, $\mathbf{Com}(n)=k$ for $n\geq 0$. If $\mathfrak{g}$ is a Lie algebra, then $C_n^{\mathbf{Lie}}=\Lambda^n\mathfrak{g}$ is the Chevalley-Eilenberg complex.

Example 7.23. If $\mathcal{P}=\mathbf{Leib}$, $\mathbf{Leib}(n)=k[S_n]$ then $\mathcal{P}^!=\mathbf{Zinb}$, $\mathbf{Zinb}(n)=k[S_n]$ for $n\geq 0$. If A is a Leibniz algebra, then $C_n^{\mathbf{Leib}}=\mathfrak{g}^{\otimes n}$ is the Leibniz complex.

7.7 Bialgebras and props

Recall that the operad is a Schur functor $\mathcal{P}\colon \mathbf{Vect} \to \mathbf{Vect}$ together with natural transformations $\iota\colon \mathrm{Id} \to \mathcal{P}$, $\gamma\colon \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ defining a monoid structure. An algebra of given type $\mathcal{P}$ gives rise to an operad by the construction of free algebra $\mathcal{P}(V)$ over V. The free algebra is an adjoint functor to the forgetful functor $\mathbf{Alg} \to \mathbf{Vect}$. For bialgebras however, the left adjoint functor to the forgetful functor does not exist. Recall that for an operad $\mathcal{P}$ we have a family of S_n-modules $\mathcal{P}(n)$ such that $\mathcal{P}(V) = \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$. The module $\mathcal{P}(n)$ is called a space of n-ary operations. For bialgebras one replaces these modules by $\mathcal{P}(n,m)$ for $n,m \geq 0$, because we can do operations and cooperations.

Definition 7.24. *A* symmetric monoidal category $\mathbb{S}$ *is a category with distinguished object* $0 \in \mathrm{Ob}(\mathcal{S})$ *and an associative product* $\square\colon \mathbb{S} \times \mathbb{S} \to \mathbb{S}$. *We say that the category is* strict *if the associativity relation is equality, not only an isomorphism.*

Definition 7.25. *A* prop *is a strict symmetric monoidal category* $\mathbb{S}$ *such that*

$$\mathrm{Ob}(\mathbb{S}) = \{\underline{0}, \underline{1}, \ldots, \underline{n}, \ldots\} \cong \mathbb{Z}$$

$$\underline{n} \square \underline{m} := \underline{n+m},$$

and the morphisms $\mathrm{Mor}_{\mathcal{S}}(\underline{n}, \underline{m})$ *is a vector space over* k.

Definition 7.26. *An* algebra over the prop $\mathcal{P}$ *(*gebra*) is a functor between symmetric monoidal categories*

$$(\mathcal{P}, \square) \to (\mathbf{Vect}, \otimes)$$

$$\underline{1} \mapsto A, \quad \underline{n} \mapsto A^{\otimes n}.$$

Example 7.27. Consider the skeleton category of the cateogory of finite sets, denoted by **Fin**. Objects are $\underline{n} = \{1, 2, \ldots, n\}$ and the morphisms are the set-theoretic maps.

The category of gebras over **Fin** is the category of unital commutative algebras.

$$\begin{tikzcd}
\underline{3} \arrow[d, shift right] \arrow[d, shift left] & \{1,2,3\} \arrow[d, shift right] \arrow[d, shift left] & A^{\otimes 3} \arrow[d, shift right, "\mathrm{id}\otimes\mu"'] \arrow[d, shift left, "\mu\otimes\mathrm{id}"] \\
\underline{2} \arrow[d] & \{1,2\} \arrow[d] & A^{\otimes 2} \arrow[d, "\mu"] \\
\underline{1} & \{1\} & A \\
\underline{0} \arrow[u] & \emptyset \arrow[u] & k \arrow[u]
\end{tikzcd}$$

What if we would like to have a prop corresponding to unital associative algebras? Then the answer is the category of noncommutative sets ΔS. Its skeleton category **NFin** has the same objects $\{\underline{n}\}$ as **Fin**, but the morphism $f\colon \underline{n} \to \underline{m}$ is a set map together with a total order on each fiber $f^{-1}(i)$. For example we have one map $\underline{2} = \{1,2\} \to \{1\} = \underline{1}$, but two morphisms $\{1 < 2\} \to \{1\}$ and $\{2 < 1\} \to \{1\}$, which correspond to the two maps $A^{\otimes 2} \to A$ given by $a \otimes b \mapsto ab$ and $a \otimes b \mapsto ba$.

For the Hochschild complex $C_n(A,M) = M \otimes A^{\otimes n}$ there are idempotents $e_n^{(i)}$ which commute with the Hochschild boundary b

$$\begin{tikzcd} C_n(A,M) \arrow[r, equal] \arrow[d, "b"'] & M \otimes A^{\otimes n} \arrow[loop above, "e_n^{(i)}"] \\ C_{n-1}(A,M) \arrow[r, equal] & M \otimes A^{\otimes (n-1)} \arrow[loop below, "e_n^{(i)}"] \end{tikzcd}$$

$$be_n^{(i)} = e_{n-1}^{(i)} b.$$

When $A = M$ and there is a B-map we also have

$$Be_n^{(i)} = e_{n-1}^{(i)} B.$$

All these formulas live in $k\mathbf{Fin}$, where $\mathrm{Mor}_{k\mathbf{Fin}}(\underline{n}, \underline{m}) = k[\mathrm{Mor}_{\mathbf{Fin}}(\underline{n}, \underline{m})]$. If A is commutative, then we have

$$b\colon \underline{n} \to \underline{n-1}, \quad b = \sum_{i=1}^{n} d_i$$

$$e_n^{(i)}\colon \underline{n} \to \underline{n}, \qquad \begin{tikzcd} \underline{n} \arrow[r, "e_n^{(i)}"] \arrow[d, "b"'] & \underline{n} \arrow[d, "b"] \\ \underline{n-1} \arrow[r, "e_n^{(i)}"] & \underline{n-1} \end{tikzcd}$$

in $k[\mathrm{Mor}_{\mathbf{Fin}}(\underline{n}, \underline{m})]$. There is a functor $\mathcal{L}\colon k\mathbf{Fin} \to M \otimes A^{\otimes n}$ such that

$$\begin{tikzcd} \underline{n} \arrow[r, mapsto] \arrow[d, "f"'] & M \otimes A^{\otimes n} \arrow[d, "f_*"] \\ \underline{m} \arrow[r, mapsto] & M \otimes A^{\otimes m} \end{tikzcd}$$

$$f_*(m, a_1, \dots, a_n) = (m, b_1, \dots, b_m), \quad b_i = \prod_{j,\, f(j)=i} a_j.$$

When A is not commutative we take $\mathbf{NFin}$ and get Loday's functor.

Any operad $\mathcal{P}$ gives rise to a prop. One defines

$$\mathcal{P}(n,1) := \mathcal{P}(n)$$

$$\mathcal{P}(n,m) := \dots$$

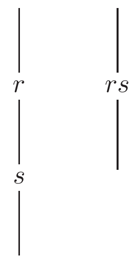
We can say that operads and free operads correspond to abstract rooted trees with data what happens when we contract an edge. If one considers planar rooted trees, then what one gets is so called non-symmetric operad (with no action of symmetric group). If any leaf could be a root, that is we consider abstract trees, then what we get is so called cyclic operads (there is a cyclic action of $\mathbb{Z}/(n+1)\mathbb{Z}$) on $\mathcal{P}(n)$.

Let $\tau = \tau_n \in \mathbb{Z}/(n+1)\mathbb{Z}$ denote the generator. For every n-ary operation $\mu \in \mathcal{P}(n)$ we have $\tau(\mu) \in \mathcal{P}(n)$.

Definition 7.28. *A* cyclic operad *is an operad such that $\mathcal{P}(n)$ has a $\mathbb{Z}/(n+1)\mathbb{Z}$-action. This action together with the S_n-action makes it an S_{n+1}-module.*

There is a relation between cyclic action and composition.

Example 7.29. Let $\mathcal{P}(1) = R$ an associative algebra, and $\mathcal{P}(n) = 0$ for $n \geq 0$.



If there is a cyclic action $r \mapsto \bar{r}$, $\bar{\bar{r}} = r$ on R, then $\overline{rs} = \bar{s}\bar{r}$. The cyclic operad here correspond to cyclic algebra with involution.

Fact 7.30. As*,* **Lie***,* **Com***,* **Poiss** *are cyclic operad, but* **Leib** *is not.*

Let $\mathcal{P}$ be a cyclic operad. Then we can construct three homology theories HA_*, HB_*, HC_* (Getzler-Kapranov) which fit into an exact sequence

$$\ldots \to \mathrm{HA}_n(A) \to \mathrm{HB}_n(A) \to \mathrm{HC}_n(A) \to \mathrm{HA}_{n-1}(A) \to \ldots,$$

where A is a $\mathcal{P}$-algebra, and $\mathrm{HB}_n(A) = \mathrm{H}_n^{\mathcal{P}}(A)$ is an operadic homology of A. If $\mathcal{P} = \mathbf{As}$, then HC_n is cyclic homology, $\mathrm{HA}_n = \mathrm{HC}_{n-1}$, $\mathrm{HB}_n = \mathrm{HH}_n$. If $\mathcal{P} = \mathbf{Lie}$, then $\mathrm{HB}_n = \mathrm{H}_n^{\mathrm{Lie}}(\mathfrak{g})$, $\mathrm{HA}_n = \mathrm{H}^{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra and a $\mathfrak{g}$-module via the adjoint representation.

$$\begin{tikzcd}
\mathfrak{g} \otimes \Lambda^n \mathfrak{g} \arrow[r] \arrow[d, "d^{\mathrm{CE}}"'] & \Lambda^{n+1} \mathfrak{g} \arrow[d, "d^{\mathrm{CE}}"'] \\
\mathfrak{g} \otimes \Lambda^{n-1} \mathfrak{g} \arrow[r] & \Lambda^n \mathfrak{g}
\end{tikzcd}$$

$$\mathrm{H}_n^{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}) \qquad \mathrm{H}_{n+1}^{\mathrm{Lie}}(\mathfrak{g})$$

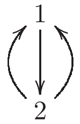
7.8 Graph complex

In the degree n of graph complex the space of n-chains is generated by connected graphs without loops with n edges and the valence of each vertex is ≥ 3. The differential

$$d(\gamma) = \sum_{e \text{ edges}} \pm \gamma/e, \quad d\colon C_n \to C_{n-1}$$

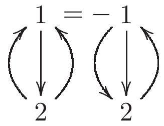
(when there is a loop we send the graph to 0).

To describe it precisely we assume that the graphs are oriented and the set of vertices is labelled by $1, \dots, K$.

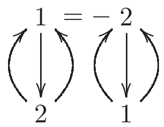


The equivalence relation:

1. changing the orientation of one edge, $\sim_1$



2. permutation of indices, $\sim_2$



The differential in the complex $\widetilde{C_n}$ is given by

$$d(\widetilde{\gamma}) = \sum_{i \xrightarrow{e} j} (-1)^j \widetilde{\gamma}/e$$

and is compatible with the equivalence relation $\sim$. It is also compatible with the first part $\sim_1$ of the equivalence relation. The complex $\widetilde{C_n}$ quotient by $\sim$ gives a graph complex C_n. We denote

$$\mathcal{G}_n := \widetilde{C_n}/\sim_1$$
$$(\mathcal{G}_s)_n := \widetilde{C_n}/\sim_1, \sim_2$$

The complex $(\mathcal{G}_s)_*$ is called the *Kontsevich graph complex*.

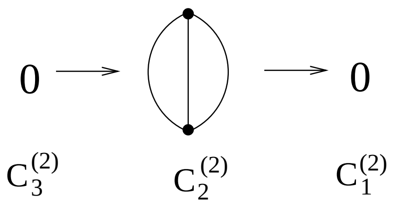


Figure 7.1: $\mathrm{H}_2^{(2)} = \mathbb{Q}$

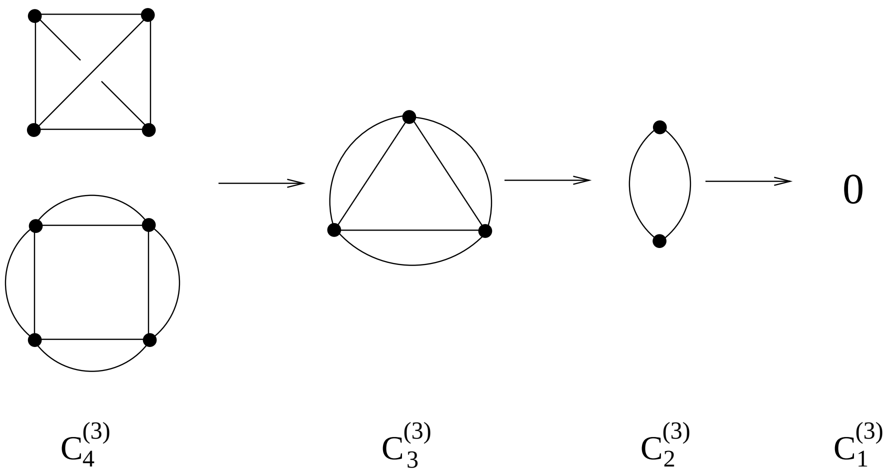


Figure 7.2: $\mathrm{H}_4^{(3)} = \mathbb{Q}$

7.9 Symplectic Lie algebra of the commutative operad

We consider the polynomial algebra in $p_1, \ldots, p_n, q_1, \ldots, q_m$, that is $k[p_1, \ldots, q_m] = S(V)$. The , defined as

$$\{F, G\} := \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}$$

is a Lie bracket, and so, $S(V)$ is a Lie algebra. We denote it by $\mathfrak{sp}_{2m}(\mathbf{Com})$.

Theorem 7.31 (Kontsevich)**.** *If $\mathcal{G}_*$ is the graph complex, then*

$$\mathrm{H}_*(\mathfrak{sp}(\mathbf{Com})) \cong \Lambda(\mathrm{H}_*(\mathcal{G}_*)).$$

The proof follows the same pattern as before:

1. Koszul trick.
2. Coinvariant theory.
3. Hopf-Borel type theorem, primitives.
4. Making the primitive complex smaller by dividing out acyclic complexes.

The key point in step 2 is the following result.

Theorem 7.32. *Let A_r^- be a linear subspace of degree r in $k[\varphi_{ij}]_{\substack{1\le i\le r\\ 1\le i\le r}}/\sim$, $\varphi_{ij} \sim \varphi_{ji}$. Then*

$$(V^{\otimes 2m})_{\mathfrak{sp}(V)} \cong A_r^-.$$

Kontsevich's idea is to compute

$$\bigoplus_{k_1,\ldots,k_n} (A_r^-)_{S_{k_1}\times\cdots\times S_{k_n}}$$

which appears to be a vector space generated by graphs divided by the relation $\sim_1$.

In step 3 one proves that the subcomplex of primitives is connected.

In step 4 we get rid of loops and graphs which have vertices of valence 2.

An analogue of this theorem for Leibniz homology is due to E. Burgunder.

Chapter 8

The algebra of classical symbols

8.1 Local definition of the algebra of symbols

Let X be a C^∞-manifold (not necessarily compact), and E a vector bundle on X. Consider a coordinate patch

$$f_U \colon U \to X, \quad U \subset \mathbb{R}^n.$$

The cotangent bundle $T^*X \to X$ pulls back to U

$$\begin{tikzcd} T_0^*U \arrow[r] \arrow[d, "\pi"'] & T^*U \arrow[r] \arrow[d] & T^*X \arrow[d] \\ U \arrow[r, equal] & U \arrow[r, "f_U"] & X \end{tikzcd}$$

The bundle T_0^*U is defined as $T^*U \setminus U$. There is an isomorphism

$$\begin{tikzcd} T_0^*U \arrow[r, "\cong"] \arrow[d, "\pi"'] & U \times \mathbb{R}_0^n \subset \mathbb{R}^n \times \mathbb{R}_0^n \\ U & \end{tikzcd}$$

Using it we can denote the coordinates on T_0^*U by (u, ξ), where $u = (u_1, \ldots, u_n) \in \mathbb{R}_0^n$, and $\xi \in (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

To each open set U we associate a section $a^U := \sum_{j=0}^\infty a_j^U$, where each a_j^U is a section of the bundle $\mathrm{End}(\pi^* f_U^* E)$, where

$$\begin{tikzcd} \pi^* f_U^* E \arrow[r] \arrow[d] & f_U^\bullet E \arrow[r] \arrow[d] & E \arrow[d] \\ T_0^*U \arrow[r, "\pi"] & U \arrow[r, "f_U"] & X \end{tikzcd}$$

More precisely by a_{m-j}^U we denote the homogeneous part of degree $m-j$

$$a_{m-j}^U \in C^\infty(T_0^*U, \mathrm{End}(\pi^* f_U^* E))(m-j).$$

There is a natural action of $\mathbb{R}_+^*$ on T_0^*X given by $t \cdot (u, \xi) := (u, t\xi)$. The infinitesimal action is provided by the Euler field

$$\Xi = \sum_{i=1}^{n} \xi_i \partial_{\xi_i}.$$

The homogenity condition for a_{m-j}^U is given by $a_{m-j}^U(u, t\xi) = t^{m-j} a_{m-j}^U(u, \xi)$.

The section a^U belongs to the product

$$\prod_{j=0}^{\infty} C^\infty(T_0^*U, \mathrm{End}(\pi^* f_U^* E))(m-j)$$

which has a natural structure of Frechet space. With the norm

$$|\xi| := \sqrt{\xi_1^2 + \ldots \xi_n^2}$$

we can write

$$|\xi|^{j-m} a_{m-j}^U \in C^\infty(T_0^*U, \mathrm{End}(\pi^* f_U^* E))(0) \cong C^\infty(S^*U, \mathrm{End}(\pi^* f_U^* E)),$$

where S^*U is the cosphere bundle $T_0^*U/\mathbb{R}_+^* \xrightarrow{\pi} U$. The cotangent bundle $T^*X \to X$ is canonically oriented and S^*X is canonically oriented (even though we do not have the orientation on X). Now S^*U is a canonically oriented $(2n-1)$-manifold and $S^*U \cong U \times S^{n-1}$.

The sections a^U are given locally, so we need a compatibility condition. We need a composition law such that it will depend on all jets, not only on 1-jets as usual composition.

$$a^U \circ_u b^U \colon \sum_\alpha \delta_\xi^\alpha a^U D_u^{[\alpha]} b^U$$

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_i \in \mathbb{N}$$

$$D_{u_i} := \frac{1}{i}\partial_{u_i}, \quad D_u^{[\alpha]} = \frac{1}{\alpha!} D_u^\alpha = \frac{1}{\alpha! i^{|\alpha|}} \partial_u^\alpha.$$

If a^U is of order m, b^U of order m' using the notation for classical symbols

$$\mathrm{CS}_U^m(U, E) := \prod_{j=0}^{\infty} (T_0^*U, \mathrm{End}(\pi^* f_U^* E))(m-j)$$

we can write

$$\circ_u \colon \mathrm{CS}_U^m(U, E) \times \mathrm{CS}_U^{m'}(U, E) \to \mathrm{CS}_U^{m+m'}(U, E), \quad m, m' \in \mathbb{C}.$$

Now suppose we have two open sets $U, V \in \mathbb{R}^n$ such that the images of charts $f_U \colon U \to X$, $f_V \colon V \to X$ have nonempty intersection $f(U) \cap f(V)$. Denote

$$U' := f_U^{-1}(f(U) \cap f(V)), \quad V' := f_V^{-1}(f(U) \cap f(V)),$$

$$f_{UV} := f_U^{-1} \circ f_V \colon V' \to U'.$$

For a smooth map $f \colon X \to Y$ there are induced maps

$$Tf \colon TC \to TY, \quad (Tf)_x \colon (TX)_x \to (TY)_{f(x)},$$
$$Tf^* \colon T^*X \to T^*Y, \quad (Tf)_x^* \colon (T^*X)_x \leftarrow (T^*Y)_{f(x)}.$$

Assume that Tf is invertible

$$((Tf)^*_x)^{-1}\colon (TX)^*_x \to (TY)^*_{f(x)}$$

Define a maps

$$\begin{aligned} X \times TX \to Y \times TY, &\quad (x,v) \mapsto (f(x),(Tf)_x(v)),\\ X \times T^*X \to Y \times T^*Y, &\quad (x,\xi) \mapsto (f(x),((Tf)^*)^{-1}_x(\xi)). \end{aligned}$$

Now comes the question, to what extend a^V and $(T^*f)^*a^U$ agree? We have

$$\begin{aligned} a^V &= (T^*f)^*(a^U + \text{(arbitrary high order correction terms)})\\ &= (T^*f_{UV})^*(\sum_\alpha \psi_\alpha \partial_\xi^\alpha a^U), \end{aligned}$$

where

$$\psi_\alpha(u,\xi) = D_z^{[\alpha]} e^{i\langle j_u^{>1}(z),\,(Tf_{UV})^*_v(\xi)\rangle}\big|_{z=u,\ v=(f_V^{-1}\circ f_U)(u)},$$

$$j_u^{>1}(z) = f_V^{-1}\circ f_U - j_U^1(f_V^{-1}\circ f_U),$$

so $j_u^{>1}$ vanishes up to second order at point $u \in U$. The $\psi_\alpha(u,\xi)$ are scalar valued functions on coordinate charts. They do not depend on symbols, only on manifold.

In the whole notes we will be using a projective tensor product of topological vector spaces desribed in the appendix (**??**).

The product

$$\mathrm{CS}^m(X,E) \times \mathrm{CS}^{m'}(X,E) \to \mathrm{CS}^{m+m'}(X,E)$$

of Frechet spaces is associative. Define the algebra of symbols as

$$\mathrm{CS}(X,E)\colon = \bigcup_{m\in\mathbb{Z}} \mathrm{CS}^m(X,E).$$

Let $a := \{a^U\}_{f_U\colon U\to X}$. The topology on $\mathrm{CS}(X,E)$ is defined as follows. We say that the net $\{a_\lambda\}$ converges to a symbol a if for any $m \in \mathbb{C}$ there exists λ_0 such that $a_\lambda - a \in \mathrm{CS}^m(X,E)$ for all $\lambda \geq \lambda_0$.

The subalgebra $\mathrm{CS}^0(X,E)$ is a Frechet algebra, and $\mathrm{CS}^{-j}(X,E)$, $j \in \mathbb{Z}_+$ is a two sided ideal in $\mathrm{CS}^0(X,E)$.

Remark 8.1. The multiplication

$$\mathrm{CS}^m(X,E) \otimes \mathrm{CS}(X,E) \to \mathrm{CS}(X,E)$$

is not continuous in both arguments.

8.2 Classical pseudodifferentials operators

Let $A\colon C_c^\infty(X,E) \to C^\infty(X,E)$ be a pseudo differential operator. For a chart $f_U\colon U \to X$ there is an operator

$$f_U^\# A\colon C_c^\infty(U, f_U^*E) \to C^\infty(U, f_U^*E)$$

We can define it for $\varphi \in C_c^\infty(U, f_U^* E)$ as follows. First take $(\varphi \circ f_U^{-1})|_{f(\operatorname{supp}\varphi)}$, and then extend by 0, apply A and pullback, as in the following diagram

$$\begin{tikzcd} C_c^\infty(X,E) \arrow[r, "A"] & C^\infty(X,E) \arrow[d, "f_U^*"] \\ C_c^\infty(U, f_U^* E) \arrow[u, "(f_U)_!"] \arrow[r, "f_U^\# A"'] & C^\infty(U, f_U^* E) \end{tikzcd}$$

Explicitly

$$(f_U^\# A)\varphi(u) = \int_{\mathbb{R}^n_\xi} \int_U e^{i\langle u-u',\, \xi\rangle} \beta(u,u',\xi)\varphi(u')du' \bar{d}\xi + (T\varphi)(u),$$

where $\beta \in C^\infty(U \times T^*U, \operatorname{End}(\pi^* f_U^* E))$ is called an amplitude,

$$\beta(u,u',\xi) \sim \sum_{j=0}^\infty \beta_{m-j}(u,u',\xi),$$

$$\beta_{m-j}(u,u',t\xi) = t^{m-j}\beta(u,u',\xi),$$

T is a smoothing operator

$$(T\phi)(u) = \int_U K(u,u')\varphi(u')|du'|,$$

and

$$|du| = |du_1 \wedge \cdots \wedge du_n|, \quad \bar{d}\xi = \frac{1}{(2\pi)^n}|d\xi_1 \wedge \cdots \wedge d\xi_n|.$$

By $\mathrm{CL}^m(X,E)$ we denote the space of classical pseudo differential operators, and by $\mathrm{CL}^m_{prop}(X,E)$ the subset of operators which take functions with compact support into functions with compact support. For $A \in \mathrm{CL}^m(X,E)$ there is a decomposition $A = A_{prop} + S$ into a proper part A_{prop} and non proper smoothing part S. Define a Frechet space of arbitrary low order operators by

$$L^{-\infty}(X,E) := \bigcap_{m\in\mathbb{Z}} \mathrm{CL}^m(X,E).$$

There is an isomorphism

$$\mathrm{CL}^m(X,E)/L^{-\infty}(X,E) \xrightarrow{\cong} \mathrm{CS}^m(X,E).$$

Classical symbols have a product

$$\mathrm{CL}^m_{prop}(X,E) \times \mathrm{CL}^{m'}_{prop}(X,E) \to \mathrm{CL}^{m+m'-1}_{prop}(X,E), \quad m,m' \in \mathbb{C}.$$

We define the algebra of classical symbols as

$$\mathrm{CL}(X,E) := \bigcup_{m\in\mathbb{Z}} \mathrm{CL}^m(X,E).$$

The space of smoothing operators $\mathcal{L}^\infty(X,E)$ is defined as a kernel

$$\mathcal{L}^\infty(X,E) \rightarrowtail \mathrm{CL}(X,E) \twoheadrightarrow \mathrm{CS}(X,E)$$

and if X is closed it is isomorphic (non canonically) to the space of rapidly decaying matrices

$$L^{-\infty} = \{(a_{ij})_{i,j=1\dots,\infty} \mid |a_{ij}|(i+j)^N \to 0, \text{ as } i+j \to \infty\}.$$

This is the noncommutative orientation class of a closed manifold and index theorem is the way to state that. Index measures to what extend this sequence is not split.

The map

$$\mathrm{CL}(X,E)/L^{-\infty}(X,E) \to \mathrm{CS}(X,E)$$

is defined as follows. For a classical pseudo differential operator

$$A\colon C_c^\infty(X,E) \to C^\infty(X,E)$$

we take the amplitude

$$\beta^U(u,u',\xi) \sim \sum_{j=0}^{\infty} \beta^U_{m-j}(u,u',\xi)$$

and then define $a^U \in \mathrm{CS}(X,E)$ by

$$a^U := \left(e^{\sum_{i=1}^n \partial_{\xi_i} D_{u_i} \beta^U}\right)\big|_{u=u'}.$$

8.3 Statement of results

The main goal is to compute the Hochschild and cyclic homology of the algebra of symbols $\mathrm{CS}(X)$. Let $T_0^*X = T^*X \setminus X$ and Y^c be the $\mathbb{C}^*$-bundle over the cosphere bundle S^*X defined as

$$\begin{array}{c} Y^c := T_0^*X \times_{\mathbb{R}_+} \mathbb{C}^* \\ \big\downarrow{\scriptstyle \mathbb{C}^*} \\ S^*X \end{array}$$

Theorem 8.2. *There is a canonical isomorphism*

$$\mathrm{HH}_q(\mathrm{CS}(X)) \cong \mathrm{H}_{\mathrm{dR}}^{2n-q}(Y^c).$$

Regarding cyclic homology, consider on $\mathrm{HC}_q^{\mathrm{cont}}(\mathrm{CS}(X))$ the filtration by the kernels of the iterated S-map:

$$\{0\} = \mathcal{S}_{q0} \subset \mathcal{S}_{q1} \subset \ldots \subset \mathcal{S}_{qt} = \mathrm{HC}_q(\mathrm{CS}(X)),$$

where $t = \left[\frac{q}{2}\right]$ and $\mathcal{S}_{qr} := \ker S_*^{1+r} \cap \mathrm{HC}_q(\mathrm{CS}(X))$.

Theorem 8.3. *The canonical map*

$$I\colon\ \mathrm{HH}_*(\mathrm{CS}(X)) \to \mathrm{HC}_*(\mathrm{CS}(X))$$

is injective. In particular

$$\mathrm{HC}_{qr}(\mathrm{CS}(X)) = \mathrm{gr}_r^S \mathrm{HC}_q(\mathrm{CS}(X)) := \mathcal{S}_{qr}/\mathcal{S}_{q,r-1}$$

is canonically isomorphic with

$$\mathrm{H}_{\mathrm{dR}}^{2n-q+2r}(Y^c), \quad r = 0, 1, \ldots.$$

8.4 Derivations of the de Rham algebra

Let $\mathcal{O}$ be a commutative k-algebra with unit, and k any commutative ring of coefficients. We define

$$\Omega^*_{\mathcal{O}/k} := \Lambda^*_{\mathcal{O}}\Omega^1_{\mathcal{O}/k},$$

where $\Omega^1_{\mathcal{O}/k}$ can be defined in a three ways:

- Serre's picture
$$\Omega^1_{\mathcal{O}/k} := I_\Delta/I_\Delta^2,$$
where $I_\Delta := \ker(\mathcal{O}^{\otimes 2} \to \mathcal{O})$.
- Hochschild picture
$$\Omega^1_{\mathcal{O}/k} := \mathcal{O}^{\otimes 2}/b\mathcal{O}^{\otimes 3}.$$
- Leibniz picture
$$\Omega^1_{\mathcal{O}/k} := \frac{\mathcal{O}\langle df \mid f \in \mathcal{O}\rangle}{\mathcal{O}\langle d(f+g) - df - dg,\ dc = 0\ (c \in k),\ d(fg) - fdg - gdf\rangle}.$$

The differential $d\colon \mathcal{O} \to \Omega^1_{\mathcal{O}/k}$ is defined in those three pictures as follows

- $f \mapsto d_\Delta f \mod I_\Delta^2 = (1 \otimes f - f \otimes 1) \mod I_\Delta^2$ (Serre's picture),
- $f \mapsto d_\Delta f \mod b\mathcal{O}^{\otimes 3} = (1 \otimes f - f \otimes 1) \mod b\mathcal{O}^{\otimes 3}$ (Hochschild picture),
- $f \mapsto df$ (Leibniz picture).

The derivation $d_\Delta\colon \mathcal{O} \to I_\Delta \subset \mathcal{O} \otimes \mathcal{O}$ is universal in the sense that if we have an $\mathcal{O}$-bimodule M and a derivation $\delta\colon \mathcal{O} \to M$, then there exists a unique $\mathcal{O}$-bimodule map $\widetilde{\delta}$ such that the following diagram commutes

$$\begin{tikzcd} & M \\ \mathcal{O} \arrow[ur, "\delta"] \arrow[r, "d"] \arrow[dr, "d_\Delta"'] & I_\Delta/I_\Delta^2 \arrow[u, dashed, "\widetilde{\delta}"'] \\ & I_\Delta \arrow[u] \end{tikzcd}$$

Let $\mathrm{Der}^m(\Omega^*) = \mathrm{Der}^m_k(\Omega^*)$ denote the algebra of degree m derivations, and

$$\mathrm{Der}^*(\Omega^*) := \bigoplus_{m \in \mathbb{Z}} \mathrm{Der}^m(\Omega^*).$$

If η is of degree p and ζ of degree q, then for $\delta \in \mathrm{Der}^m(\Omega^*)$ we have

$$\delta(\eta \wedge \zeta) = \delta(\eta) \wedge \zeta + (-1)^{pm}\eta \wedge \delta(\zeta).$$

$$\delta\colon \Omega^p \to \Omega^{p+m}.$$

Furthermore $\mathrm{Der}^*(\Omega^*)$ is a super Lie algebra, that is the commutators satisfy the super Jacobi identity

$$0 = [[a,b],c] + (-1)^{|a|(|b|+|c|)}[[b,c],a] + (-1)^{|c|(|a|+|b|)}[[c,a],b].$$

Denote $\delta_p := \delta|_{\Omega_p}$.

Proposition 8.4. *The set* $\mathrm{Der}^m(\Omega^*)$ *is naturally identified with the set of pairs* (δ_0, δ_1)*, where*

$$\delta_0 \colon \mathcal{O} \to \Omega^m$$

is a k*-linear derivation of* $\mathcal{O}$ *with values in* Ω^m*,*

$$\delta^1 \colon \Omega^1 \to \Omega^{m+1}$$

is a k*-linear map such that*

$$\delta_1(f\alpha) = \delta_0(f) \wedge \alpha + f\delta_1(\alpha).$$

and

$$\delta_1(\alpha)\alpha - (-1)^{m+1}\alpha\delta_1(\alpha) = 0,$$

that is the super commutator $[\delta_1(\alpha), \alpha] = 0$*.*

Any derivation of degree m is uniquelly determined by δ_0 and δ_1. Thus $\mathrm{Der}^m(\Omega^*) = 0$ for $m < -1$.

For $\delta_0 = 0$ we have

$$\delta(f\alpha_1 \wedge \cdots \wedge \alpha_p) = \sum_{i=1}^{p} (-1)^{m(i-1)} f\alpha_1 \wedge \cdots \wedge \delta_1(\alpha_i) \wedge \cdots \wedge \alpha_p.$$

Similarly for any $\phi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^{m+1})$ there exists a corresponding derivation

$$\delta_\phi(f\alpha_1 \wedge \cdots \wedge \alpha_p) := \sum_{i=1}^{p} (-1)^{m(i-1)} f\alpha_1 \wedge \cdots \wedge \phi(\alpha_i) \wedge \cdots \wedge \alpha_p.$$

Example 8.5. (The de Rham derivation) Let $d_0 = d \colon \mathcal{O} \to \Omega^1$. Now we will give a construction of $d_1 \colon \Omega^1 \to \Omega^2$. Consider a k-linear pairing

$$\mathcal{O} \times \mathcal{O} \to \Omega^2, \quad (f, g) \mapsto df \wedge dg$$

$$\begin{tikzcd} \mathcal{O} \times \mathcal{O} \arrow[r] \arrow[d] & \Omega^2 \\ \mathcal{O} \otimes_k \mathcal{O} \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O} \times \mathcal{O} \arrow[r] \arrow[d] & \Omega^2 \\ (\mathcal{O} \otimes_k \mathcal{O})/I_\Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

Now we can take a restriction to $I_\Delta/I_\Delta^2 \subset (\mathcal{O} \otimes_k \mathcal{O})/I_\Delta^2$. Recall that I_Δ consists of sums of terms of the form

$$\begin{aligned} f_0 d_\Delta f_1 &= f_0(1 \otimes f_1 - f_1 \otimes 1) \\ &= f_0 \otimes f_1 - f_0 f_1 \otimes 1. \end{aligned}$$

Similarly I_Δ^2 consists of sums of terms of the form

$$\begin{aligned} f_0 d_\Delta f_1 d_\Delta f_2 &= f_0(1 \otimes f_1 - f_1 \otimes 1)(1 \otimes f_2 - f_2 \otimes 1) \\ &= f_0(1 \otimes f_1 f_2 + f_1 f_2 \otimes 1 - f_1 \otimes f_2 - f_2 \otimes f_1) \\ &= f_0 \otimes f_1 f_2 + f_0 f_1 f_2 \otimes 1 - f_0 f_1 \otimes f_2 - f_0 f_2 \otimes f_1) \end{aligned}$$

The last expression maps to

$$
\begin{aligned}
&df_0 \wedge d(f_1 f_2) + d(f_0 f_1 f_2) \wedge d1 - d(f_0 f_1) \wedge df_2 - d(f_0 f_2) \wedge df_1 \\
&= df_0 \wedge ((df_1)f_2 + f_1 df_2) - ((df_0)f_1 + f_0 df_1) \wedge df_2 - ((df_0)f_2 + f_0 df_2) \wedge df_1 \\
&= f_2 df_0 \wedge df_1 + f_1 df_0 \wedge df_2 - f_1 df_0 \wedge df_2 - f_0 df_1 \wedge df_2 - f_2 df_0 \wedge df_1 - f_0 df_2 \wedge df_1 \\
&= -f_0 df_1 \wedge df_2 - f_0 df_2 \wedge df_1 \\
&= 0.
\end{aligned}
$$

Proposition 8.6. *Any derivation $\delta \in \mathrm{Der}_k^m(\Omega^*)$ can be uniquelly expressed as*

$$[\delta_\phi, d] + \delta_\psi$$

for $\phi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^m)$, $\psi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^{m+1})$.

Example 8.7. ($\mathcal{O}$-linear derivation) For $m = -1$ $\mathrm{Der}_k^{-1}(\Omega^*) = \mathrm{Der}_{\mathcal{O}}^{-1}(\Omega^*)$ and by restriction to Ω^1

$$\mathrm{Der}_{\mathcal{O}}^{-1}(\Omega^*) = \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^*).$$

If $\mathcal{O} = \mathcal{O}(X)$, then $\mathrm{Der}_k \mathcal{O} = \mathcal{T}X$.

$$
\begin{tikzcd}
\Omega^1 \arrow[r] & \Omega^* \\
\mathcal{O} \arrow[u, "d"] \arrow[ur] &
\end{tikzcd}
$$

Suppose that $\delta, \delta' \in \mathrm{Der}_k^m(\Omega^*)$ are such that

$$\delta_0 = \delta|_{\mathcal{O}} = \delta'|_{\mathcal{O}} = \delta'_0.$$

Then

$$\delta - \delta' \in \mathrm{Der}_{\mathcal{O}}^m(\Omega^*) \quad \mathcal{O} - \text{linear}.$$

Suppose that we have a derivation $D \in \mathrm{Der}_k^1(\Omega^*)$. Then for any $\phi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^m)$ there is a $\delta_\phi \in \mathrm{Der}_{\mathcal{O}}^{m-1}(\Omega^*)$ and

$$[\delta_\phi, D] \in \mathrm{Der}^m(\Omega^*)$$

$$[\delta_\phi, D]_0 = \delta_\phi D = \phi \circ D = d(\text{the de Rham derivation})$$

If there exists $d_1 \colon \Omega^1 \to \Omega^2$, k-linear and satisfying

$$d_1(f\alpha) = df \wedge \alpha + f d\alpha,$$

then there exists a derivation $d \in \mathrm{Der}_k^1(\Omega^*)$.

There is a natural identification between $\mathcal{O}$-modules

$$
\begin{tikzcd}
\mathrm{Der}(\mathcal{O}, \Omega^m) \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^m) \arrow[d, leftrightarrow] \\
& \mathrm{Der}_{\mathcal{O}}^{m-1}(\Omega^*)
\end{tikzcd}
$$

Let $\eta = \phi \circ d$ which on Ω^0 is $= [\delta_\phi, d]$. Then $\iota_\eta = \delta_\phi$ is the interior product with derivation η. If $m = -1$ this is the classical product of differential forms with a given vector field. Define a Lie derivative with respect to η

$$\mathcal{L}_\eta := [\delta_\phi, d] = [\iota_\eta, d].$$

Then
$$[\mathcal{L}_\eta, d] = [[\iota_{\eta,d}], d] = (-1)^{m-1} d\iota_\eta d - (-1)^m d\iota_\eta d = 0.$$
Any derivation δ is of the form $\delta = \mathcal{L}_\eta + \iota_\zeta$ where $\zeta = \psi \circ d$ for some $\psi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^{m+1})$. Consider a special $\phi\colon \Omega^1 \to \Omega^m$
$$\phi(\alpha) = \varphi \wedge \alpha$$
for some $\varphi \in \Omega^{m-1}$. Then
$$\begin{aligned}[\delta_\phi, d](\omega) &= \varphi \wedge d\omega - (-1)^{m-1} p d\varphi \wedge \omega \\ &= \varphi \wedge d\omega - (-1)^{m-1} d\varphi \wedge \deg.\end{aligned}$$
A degree map deg is a derivation $\deg = \delta_{\mathrm{id}}$, $\mathrm{id}\colon \Omega^1 \to \Omega^1$, $[\delta_{\mathrm{id}}, d] = d$.

Remark 8.8. To prove identities like $\delta = \delta'$, where δ, δ' are $\mathcal{O}$-linear derivations on Ω^*, it is enough to prove it on $d\mathcal{O} \subset \Omega^1$. For example, for vector fields there is an identity
$$[\mathcal{L}_\eta, \iota_\zeta] = [\iota_\eta, \mathcal{L}_\zeta] = \iota_{[\eta,\zeta]}.$$
The expressions are $\mathcal{O}$-linear, so we can check the equalities by evaluating on df, $f \in \mathcal{O}$.

For $\omega \in \Omega^p$ we have the formula
$$[\delta_{\varphi\wedge -}, d]^2(\omega) = \begin{cases} 0 & m = 1 \\ \frac{1-m}{2} d(\varphi \wedge \varphi) \wedge d\omega & \text{if } m \text{ is odd } \neq 1 \\ (m+p) p d\varphi \wedge d\varphi \wedge \omega & \text{if } m \text{ is even.} \end{cases}$$
For example if $m = 1$ φ is the contact 1-form on $\mathbb{A}^1$, that is $\sum_{i=1}^n \xi_i dx_i$.
$$\omega = \mathcal{L}_\Xi \omega = d\iota_\Xi \omega.$$
In case $m = 0$, for any function $f \in \mathcal{O}$ let $f \cdot -$ denote the multiplication by the function f
$$[\delta_{f\cdot -}, d] = fd - df \wedge \deg, \quad [\delta_{1\cdot -}, d] = d_{\mathrm{dR}}.$$
Let $\eta_1, \dots, \eta_p \in \mathrm{Der}_k(\mathcal{O})$ (vector fields if $\mathcal{O} = \mathcal{O}(X)$). Then there is a formula
$$\begin{aligned}[d, \iota_{\eta_1} \dots \iota_{\eta_p}] = &\sum_{1 \le i \le p} (-1)^{i-1} \iota_{\eta_1} \dots \widehat{\iota_{\eta_i}} \dots \iota_{\eta_p} \mathcal{L}_{\eta_i} + \\ &+ \sum_{1 \le i < j \le p} (-1)^{i+j-1} \iota_{[\eta_i, \eta_j]} \iota_{\eta_1} \dots \widehat{\iota_{\eta_i}} \dots \widehat{\iota_{\eta_j}} \dots \iota_{\eta_p}.\end{aligned} \tag{8.1}$$
where $\deg \iota_{\eta_i} = -1$ for all $i = 1, \dots, p$. Similarly
$$\begin{aligned}[\iota_{\eta_p} \dots \iota_{\eta_1}, d] = &\sum_{1 \le i \le p} (-1)^{i-1} \mathcal{L}_{\eta_i} \iota_{\eta_p} \dots \widehat{\iota_{\eta_i}} \dots \iota_{\eta_1} + \\ &+ \sum_{1 \le i < j \le p} (-1)^{i+j} \iota_{\eta_p} \dots \widehat{\iota_{\eta_j}} \dots \widehat{\iota_{\eta_i}} \dots \iota_{\eta_1} \iota_{[\eta_i, \eta_j]}.\end{aligned} \tag{8.2}$$
This is in analogy to the Cartan formula for $\omega \in \Omega^{p-1}$
$$\begin{aligned}(d\omega)(\eta_1, \dots, \eta_p) = &\sum_{1 \le i \le p} (-1)^{i-1} \mathcal{L}_{\eta_i} \omega(\eta_1, \dots, \widehat{\eta}_i, \dots, \eta_p) + \\ &+ \sum_{1 \le i < j \le j} (-1)^{i+j} \omega([\eta_i, \eta_j], \eta_1, \dots, \widehat{\eta}_i, \dots, \widehat{\eta}_j, \dots, \eta_p).\end{aligned} \tag{8.3}$$

8.5 Koszul-Chevalley complex

Let $\mathfrak{m}$ be a $\mathfrak{g}$-module, where $\mathfrak{g}$ is a Lie k-algebra This means that $[,]\colon \mathfrak{g}\otimes_k \mathfrak{g} \to \mathfrak{g}$ satisfies Jacobi identity, each $g \in \mathfrak{g}$ acts as an endomorphism of a k-module $\mathfrak{m}$, and the map

$$\mathfrak{g} \to \mathfrak{gl}_k(\mathfrak{m}) = \mathrm{Lie}(\mathrm{End}_k(\mathfrak{m})), \quad g \mapsto \rho_g \text{ - action of } g \text{ on } \mathfrak{m}$$

is a homomorphism of Lie-k-algebras. We have

$$\rho_{[g_1,g_2]} = [\rho_{g_1}, \rho_{g_2}]$$

and $\mathfrak{gl}_k(\mathfrak{m})$ has the right $\mathfrak{g}$-module structure

$$\widetilde{\rho}_g(m) := mg,$$

$$mg_1g_2 - mg_2g_1 = (\widetilde{\rho}_{g_2}\widetilde{\rho}_{g_1} - \widetilde{\rho}_{g_1}\widetilde{\rho}_{g_2})(m) = [\widetilde{\rho}_{g_2}, \widetilde{\rho}_{g_1}] = m[g_2, g_1].$$

This shows that $\widetilde{\rho}\mathfrak{g} \to \mathfrak{gl}(m)$ is an antihomomorphism of Lie algebras (it corresponds to the fact that the inverse $G \to G$, $g \mapsto g^{-1}$ corresponds to $g \mapsto -g$ on $\mathfrak{g}$).

Definition 8.9. Koszul-Chevalley complex *of a Lie k-algebra $\mathfrak{g}$ with coefficients in $\mathfrak{m}$*

$$C_*(\mathfrak{g}, \mathfrak{m}) := \mathfrak{m} \otimes \Lambda_k^* \mathfrak{g}, \quad \partial\colon C_p(\mathfrak{g}, \mathfrak{m}) \to C_{p+1}(\mathfrak{g}, \mathfrak{m}),$$

where

$$\partial(m \otimes g_1 \wedge \dots \wedge g_p) := \sum_{1\le i\le p} (-1)^{i-1} g_i m \otimes g_1 \wedge \dots \wedge \widehat{g_i} \wedge \dots \wedge g_p +$$

$$+ \sum_{1\le i<j\le p} (-1)^{i+j-1} m \otimes [g_i, g_j] \wedge g_1 \wedge \dots \wedge \widehat{g_i} \wedge \dots \wedge \widehat{g_j} \wedge \dots \wedge g_p.$$

$$C^*(\mathfrak{g}, \mathfrak{m}) :== \mathrm{Alt}^*(\mathfrak{g} \times \dots \times \mathfrak{g}, \mathfrak{m}), \quad \delta\colon C_{p-1}(\mathfrak{g}, \mathfrak{m}) \to C_p(\mathfrak{g}, \mathfrak{m}),$$

where for $\gamma \in \mathrm{Alt}^{p-1}(\mathfrak{g} \times \dots \times \mathfrak{g}, \mathfrak{m})$ we define $\delta(\gamma) \in \mathrm{Alt}^p(\mathfrak{g} \times \dots \times \mathfrak{g}, \mathfrak{m})$ by

$$\delta(\gamma)(g_1, \dots, g_p) := \sum_{1\le i\le p} (-1)^{i-1} g_i \gamma(g_1, \dots, \widehat{g_i}, \dots, g_p) +$$

$$+ \sum_{1\le i<j\le p} (-1)^{i+j-1} \gamma([g_i, g_j], g_1, \dots, \widehat{g_i}, \dots, \widehat{g_j}, \dots, g_p).$$

In the next definition we use a relative Tor and Ext groups, which are the derived functorsin the sense of relative homological algebra ([**?**], [**?**]).

Definition 8.10. Lie algebra homology and cohomology *with coefficients in a $\mathfrak{g}$-module $\mathfrak{m}$*

$$\mathrm{H}_*(\mathfrak{g}; \mathfrak{m}) := \mathrm{H}(C_*(\mathfrak{g}, \mathfrak{m}), \partial) \cong \mathrm{Tor}_*^{(\mathcal{U}(\mathfrak{g}),k)}(k, \mathfrak{m}),$$

$$\mathrm{H}_*(\mathfrak{g}; \mathfrak{m}) := \mathrm{H}(C^*(\mathfrak{g}, \mathfrak{m}), \delta) \cong \mathrm{Ext}^*_{(\mathcal{U}(\mathfrak{g}),k)}(k, \mathfrak{m}).$$

8.6 A relation between Hochschild and Lie algebra homology

Consider the following situation: A is an associative k-algebra with unit, M an A-bimodule. Let $\mathrm{Lie}(A) = A$ as a k-module with commutator bracket $[a, b] := ab - ba$. Let $a \in A$ act on $m \in M$ by $m \mapsto am - ma$. Consider $d_\Delta \colon A \to A \otimes A^{op}$, $a \mapsto 1 \otimes a^{op} - a \otimes 1$,

$$\begin{aligned}[d_\Delta a, d_\Delta b] &= -\underbrace{[1 \otimes a^{op}, b \otimes 1]}_{=0} - \underbrace{[a \otimes 1, 1 \otimes b^{op}]}_{=0} + [1 \otimes a^{op}, 1 \otimes b^{op}] + [a \otimes 1, b \otimes 1] \\ &\quad \text{(because } A \otimes 1 \text{ and } 1 \otimes A^{op} \text{ commute in } A) \\ &= 1 \otimes [a^{op}, b^{op}] + [a, b] \otimes 1 \\ &= 1 \otimes [b, a]^{op} - [b, a] \otimes 1 \\ &= -d_\Delta[a, b].\end{aligned}$$

Universal derivation is an antihomomorphism, so

$$-d_\Delta \colon \mathrm{Lie}(A) \to \mathrm{Lie}(A \otimes A^{op})$$

is a homomorphism of Lie algebras.

In what follows we will use many arguments based on spectral sequences, and the necessary basics of the theory is presented in appendix (**??**).

Let $R = \mathcal{U}(\mathrm{Lie}(A))$, $S = A \otimes A^{op}$. Any bimodule N can be viewed as a left $A \otimes A^{op}$-module. The base change spectral sequence takes the form

$$E^2_{pq} = \mathrm{Tor}_p^{A \otimes A^{op}}(\mathrm{Tor}_q^{\mathcal{U}(\mathrm{Lie}(A))}(k, A \otimes A^{op}), N)$$

$$a \cdot (b \otimes c^{op}) = ab \otimes c^{op} - b \otimes a^{op} c^{op} = ab \otimes c^{op} - b \otimes (ca)^{op}$$

Assume that $\mathcal{U}(\mathrm{Lie}(A))$ is flat over k. Then

$$\mathrm{Tor}_p^{A \otimes A^{op}}(\mathrm{Tor}_q^{\mathcal{U}(\mathrm{Lie}(A))}(k, A \otimes A^{op}), N) \cong \mathrm{Tor}_p^{A \otimes A^{op}}(\mathrm{H}_q(\mathrm{Lie}(A); A \otimes A^{op}), N).$$

In our base change spectral sequence we get an edge homomorphism

$$\mathrm{H}_p(\mathrm{Lie}(A); N) \to \mathrm{Tor}_p^{A \otimes A^{op}}(\mathrm{H}_0(\mathrm{Lie}(A); A \otimes A^{op}), N).$$

In general if $\mathfrak{g}$ is a Lie algebra, and M a $\mathfrak{g}$-module, then $\mathrm{H}_0(\mathfrak{g}; M) = M_{\mathfrak{g}}$ - the coinvariants of the $\mathfrak{g}$-action. Thus we have a map from Lie algebra homology to Hochschild homology

$$\mathrm{H}_p(\mathrm{Lie}(A); N) \to \mathrm{Tor}_p^{A \otimes A^{op}}(\underbrace{\mathrm{H}_0(\mathrm{Lie}(A); A \otimes A^{op})}_{A}, N) = \mathrm{Tor}_p^{A \otimes A^{op}}(A, N) = \mathrm{H}_p(A; N).$$

When k is of characteristic 0, that map, up to a sign, is induced by inclusion

$$\eta \colon C_*(\mathrm{Lie}(A); N) \to C_*(A; N)$$

$$n \otimes a_1 \wedge \cdots \wedge a_p \mapsto \sum_{l_1, \ldots, l_p} (-1)^{\overline{l_1 \ldots l_p}} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_p},$$

where on the right hand side we have a sum over all permutations of the set $\{1, \ldots, p\}$, and $\overline{l_1 \ldots l_p}$ denotes the sign of a permutation.

Proposition 8.11. *The map η is a map of complexes, that is*

$$b\eta = -\eta\partial,$$

where b is the Hochschild boundary, and ∂ the boundary of the Koszul-Chevalley complex.

Proof. On the left hand side we have:

$$
\begin{aligned}
b\eta(n \otimes a_1 \wedge \cdots \wedge a_p) &= \sum_{l_1,\ldots,l_p} (-1)^{\overline{l_1\ldots l_p}} n a_{l_1} \otimes \cdots \otimes a_{l_p} \\
&\quad + \sum_{1\le m\le p-1} \sum_{l_1,\ldots,l_p} (-1)^{\overline{l_1\ldots l_p}+m} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p} \\
&\quad + \sum_{l_1,\ldots,l_p} (-1)^{\overline{l_1\ldots l_p}+p} a_{l_p} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_{p-1}} \\
&= \sum_{1\le i\le p} (-1)^{i-1} \sum_{\substack{l_1,\ldots,l_p \\ l_1=i}} (-1)^{\overline{l_2\ldots l_p}} n a_i \otimes a_{l_2} \otimes \cdots \otimes a_{l_p} \\
&\qquad\qquad (\text{because } \overline{i l_2 \ldots l_p} = \overline{l_2 \ldots l_p} \cdot (-1)^{i-1}) \\
&\quad - \sum_{1\le i\le p} (-1)^{i-1} \sum_{\substack{l_1,\ldots,l_p \\ l_p=i}} (-1)^{\overline{l_1\ldots l_{p-1} i}} a_i n \otimes a_{l_1} \otimes \cdots \otimes a_{l_{p-1}} \\
&\qquad\qquad (\text{because } \overline{l_1 \ldots l_{p-1} i} = \overline{l_1 \ldots l_{p-1}} \cdot (-1)^{p-i}) \\
&\quad + \sum_{1\le m\le p-1} \sum_{1\le i<j\le p} \sum_{\substack{l_1\ldots l_p \\ l_m=i,\ l_{m+1}=j}} (-1)^{\overline{l_1\ldots l_p}+m} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p} \\
&\quad + \sum_{1\le m\le p-1} \sum_{1\le j<i\le p} \sum_{\substack{l_1\ldots l_p \\ l_m=j,\ l_{m+1}=i}} (-1)^{\overline{l_1\ldots l_p}+m} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p} \\
&= \sum_{1\le i\le p} (-1)^{i} [a_i, n] \otimes a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_p \\
&\qquad\qquad (\text{because } \overline{l_1 \ldots l_p} \cdot (-1)^m = \overline{l_1 \ldots l_{m-1} l_{m+2} \ldots l_p} \cdot (-1)^{(i-1)+(j-1)}) \\
&\quad + \sum_{1\le m\le p-1} \sum_{1\le i<j\le p} (-1)^{(i-1)+(j-1)} \sum_{\substack{l_1\ldots l_p \\ l_m=i,\ l_{m+1}=j}} (-1)^{\overline{l_1\ldots l_{m-1} l_{m+2}\ldots l_p}} \\
&\qquad\qquad n \otimes a_{l_1} \otimes \cdots \otimes \underbrace{a_{l_m} a_{l_{m+1}}}_{a_i a_j} \otimes \cdots \otimes a_{l_p} \\
&\quad + \sum_{1\le m\le p-1} \sum_{1\le j<i\le p} (-1)^{(i-1)+(j-1)} \sum_{\substack{l_1\ldots l_p \\ l_m=j,\ l_{m+1}=i}} (-1)^{\overline{l_1\ldots l_{m-1} l_{m+2}\ldots l_p}} \\
&\qquad\qquad n \otimes a_{l_1} \otimes \cdots \otimes \underbrace{a_{l_m} a_{l_{m+1}}}_{a_j a_i} \otimes \cdots \otimes a_{l_p} \\
&= \eta\Bigg(\sum_{1\le i\le p} (-1)^{i} [a_i, n] \otimes a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_p \\
&\quad + \sum_{1\le i<j\le p} (-1)^{i+j+1} [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge \widehat{a_j} \wedge \cdots \wedge a_p \Bigg) \\
&= -\eta\partial(n \otimes a_1 \wedge \cdots \wedge a_p).
\end{aligned}
$$

□

8.7 Poisson trace

Consider the Lie algebra of derivations $\operatorname{Der}\mathcal{O} = \operatorname{Der}_k \mathcal{O}$. The algebra $\mathcal{O}$ is always a $\operatorname{Der}\mathcal{O}$-module via the natural representation. Let $\varphi \in \Omega^p_{\mathcal{O}/k}$. Then it defines an alternating $\mathcal{O}$-p-linear map

$$\underbrace{\operatorname{Der}\mathcal{O} \times \cdots \times \operatorname{Der}\mathcal{O}}_{p} \to \mathcal{O}$$

$$(\eta_1, \dots, \eta_p) \mapsto \varphi(\eta_1, \dots, \eta_p) := \iota_{\eta_p} \dots \iota_{\eta_1} \varphi \in \Omega^0 = \mathcal{O}.$$

There is an $\mathcal{O}$-linear map,

$$\Omega^* \to \operatorname{Alt}^*_{\mathcal{O}}(\operatorname{Der}\mathcal{O}, \mathcal{O}) \hookrightarrow \operatorname{Alt}^*_k(\operatorname{Der}\mathcal{O}, \mathcal{O})$$

which transforms the de Rham differential d into δ

$$d\varphi \mapsto \delta(\iota_{\eta_p} \dots \iota_{\eta_1} \varphi).$$

(Cartan's picture of de Rham complex).

Let $\Omega^{vol} = \Omega^n$, where n is such that $\Omega^n \neq 0$, $d\colon \Omega^n \to \Omega^{n+1}$ identically 0. Then

$$C_*(\operatorname{Der}\mathcal{O}; \Omega^{vol}) = \Omega^{vol} \otimes_k \Lambda^*_k \operatorname{Der}_k \mathcal{O} \twoheadrightarrow \Omega^{vol} \otimes_{\mathcal{O}} \Lambda^*_{\mathcal{O}} \operatorname{Der}_k \mathcal{O}$$

where the last epimorphism is $\mathcal{O}$-linearization and is an isomorphism if $\mathcal{O}$ is smooth algebra of $\dim n$.

Fact 8.12. *The kernel of $\mathcal{O}$-linearization is a subcomplex of* $C_*(\operatorname{Der}\mathcal{O}; \Omega^{vol})$.

For $\nu \in \Omega^{vol} = \Omega^n$

$$\nu \otimes \eta_1 \wedge \cdots \wedge \eta_p \mapsto \iota_{\eta_1} \dots \iota_{\eta_p} \nu \in \Omega^{n-p} =: \Omega_p.$$

The composition

$$C_*(\operatorname{Der}\mathcal{O}; \Omega^{vol}) \to \Omega^{vol} \otimes_{\mathcal{O}} \Lambda^*_{\mathcal{O}} \operatorname{Der}_k \mathcal{O}$$

is the map of complexes. It suffices to apply the formula for $[d, \iota_{\eta_1} \dots \iota_{\eta_p}]$ only to n-forms.

$$(C_*(\operatorname{Der}_k \mathcal{O}, \Omega^{vol}), \partial) \twoheadrightarrow (\Omega_*, d)$$

(Spencer's picture of de Rham complex).

Now we fix the volume form ν, and denote

$$\operatorname{Der}_k \mathcal{O}_\nu := \{\text{derivations annihilating } \nu\}.$$

There is an $\mathcal{O}$-module morphism

$$\mathcal{O} \to \Omega^{vol}, \quad f \mapsto f\nu,$$

$$C_*(\operatorname{Der}_k \mathcal{O}_\nu, \mathcal{O}) \to C_*(\operatorname{Der}\mathcal{O}, \Omega^{vol}) \to \Omega_*$$

("Divergentless vector fields").

Suppose that $\mathcal{O} = \mathcal{O}(X)$, where X is a symplectic manifold of dimension $2n$, $\omega \in \Omega^2$ is closed and nondegenerate.

$$\omega\colon \operatorname{Der}\mathcal{O} \to \Omega^1, \quad \eta \mapsto \iota_\eta \omega$$

is injective. Furthermore $\omega^n \in \Omega^{vol}$ and we can take $\nu = \omega^n$.

Define $\operatorname{Ham}(X,\omega) \subset \operatorname{Der}_k \mathcal{O}_{\omega^n}$ as

$$\operatorname{Ham}(X,\omega) := \{\eta \in \operatorname{Der}\mathcal{O}_{\omega^n} \mid \mathcal{L}_\eta \omega = 0\}.$$

Define $\mathbf{Poiss}(X,\omega)$ as an algebra $\mathcal{O}$ with the

$$\{f,\, g\} := \mathcal{L}_{H_f} g = \omega(H_f, H_g) = \iota_{H_g}\iota_{H_f}\omega,$$

where H_f is the vector field characterized by

$$\iota_{H_f}\omega = -df.$$

There is a homomorphism of Lie algebras

$$\mathbf{Poiss}(X,\omega) \to \operatorname{Ham}(X,\omega),$$

and an $\mathcal{O}$-linear map of complexes

$$C_*(\mathbf{Poiss}(X,\omega), \operatorname{ad}) \to C_*(\operatorname{Ham}(X,\omega), \omega^n).$$

$$f_0 \otimes f_1 \wedge \cdots \wedge f_p \mapsto f_0\omega^n \otimes f_1 \wedge \cdots \wedge f_p.$$

There is also a map

$$C_*(\operatorname{Ham}(X,\omega), \omega^n) \to \Omega_*$$

$$f_0\omega^n \otimes f_1 \wedge \cdots \wedge f_p \mapsto f_0 \iota_{H_{f_1}} \ldots \iota_{H_{f_p}} \omega^n.$$

We have

$$\mathcal{L}_{H_f} = [d, \iota_{H_f}]\omega = 0.$$

Proposition 8.13. *For any $f, g \in \mathcal{O}$*

$$H_{f,g} = [H_f, H_g].$$

Proof. It is sufficient to prove the corresponding identity for contractions

$$\iota_{[H_f,H_g]} = \iota_{H_{\{f,\,g\}}}.$$

We have

$$\begin{aligned}
\iota_{[H_f,H_g]}\omega &= [\mathcal{L}_{H_f}, \iota_{H_g}] \\
&= \mathcal{L}_H(\iota_{H_g}\omega) - \iota_{H_g}\underbrace{\mathcal{L}_{H_f}\omega}_{0} \\
&= -\mathcal{L}_{H_f}(dg) \\
&= -d(\mathcal{L}_{H_f} g) \\
&= -d\{f,\, g\} \\
&= -\iota_{H_{\{f,\,g\}}}.
\end{aligned}$$

∎

There is a well defined map, called a *Poisson trace*

$$\mathrm{ptr}_*\colon (C_*(\mathbf{Poiss}(X,\omega);\mathrm{ad}),\partial) \twoheadrightarrow (\Omega_*, d).$$

Let Y be a symplectic manifold, $\dim Y = 2n$, with a symplectic 2-form ω. Then we have a canonical morphism of chain complexes

$$\mathrm{ptr}\colon C_*(\mathbf{Poiss}(Y,\omega);\mathrm{ad}) \twoheadrightarrow \Omega_*(Y),$$

where $\Omega_q(Y) = \Omega^{\dim Y - q}(Y)$, given by

$$f_0 \otimes f_1 \wedge \cdots \wedge f_q \mapsto f_0 \iota_{H_{f_1}} \ldots \iota_{H_{f_q}} \omega^n.$$

An important special case is when Y is a symplectic cone, i.e. Y is acted upon by $\mathbb{R}_+^*$. Let Ξ be the corresponding Euler field (the image of $t\frac{d}{dt}$). We have $t^*\omega = t\omega$ or equivalently

$$\mathcal{L}_\Xi \omega = \omega.$$

8.7.1 Graded Poisson trace

We consider the graded algebra of functions on Y

$$\mathcal{O}_* := \bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m),$$

where

$$\mathcal{O}(m) := \{f \in \mathcal{O} \mid \mathcal{L}_\Xi f = mf\}.$$

Then the $\{\cdot,\cdot\}$ agrees with the grading in the following way

$$\{\mathcal{O}(l), \mathcal{O}(m)\} \subseteq \mathcal{O}(l+m-1).$$

Let

$$\mathcal{P}_l := \mathcal{O}(l+1), \quad \mathcal{P}_* := \bigoplus_{l\in\mathbb{Z}} \mathcal{P}_l$$

be the graded Lie algebra when equipped with the $\{\cdot,\cdot\}$. The map $f \mapsto H_f$ is a homomorphism of Lie algebras $\mathcal{O} \to \mathcal{P} = \mathbf{Poiss}(Y,\omega)$, and furthermore

$$\mathcal{L}_\Xi H_f = (\deg(f) - 1)H_f.$$

To check this identity one computes

$$\iota_{[\Xi,H_f]}\omega = [\mathcal{L}_\Xi, \iota_{H_f}]\omega = -\deg(f)df + df = (1-\deg(f))df = (\deg(f)-1)H_f$$

because $\iota_{H_f}\omega = -df$. Thus there is a *graded Poisson trace*

$$\mathrm{ptr}_*\colon C_*(\mathcal{P}_*, \mathrm{ad}) \to \Omega_{**}(Y)$$

$$\mathrm{ptr}_*\colon \bigoplus_{k\in\mathbb{Z}} C_*^{(k)}(\mathcal{P}_*, \mathrm{ad}) \to \Omega_{*,k+n}(Y),$$

where

$$C_*^{(k)}(\mathcal{P}_*, \mathrm{ad}) = (\mathcal{P}_* \otimes \Lambda^q \mathcal{P}_*)(k+q)$$

and ∂ preserves k. Explicitely we have

$$\begin{aligned}\mathcal{L}_\Xi(f_0\iota_{H_{f_1}}\ldots\iota_{H_{f_q}}\omega^n) &= (l_0 + (l_1-1) + \ldots + (l_q-1) + m) f_0 \iota_{H_{f_1}} \ldots \iota_{H_{f_q}}\omega^n \\ &= ((l_0 + \ldots + l_q) + n - q) f_0 \iota_{H_{f_1}} \ldots \iota_{H_{f_q}}\omega^n, \\ &(\mathcal{P}_* \otimes \Lambda^q \mathcal{P}_*)(l) \to \Omega_q(l-q)\end{aligned}$$

8.8 Hochschild homology

Let $C_*(\mathrm{CS}(X))$ be the completed Hochschild complex of $\mathrm{CS}(X)$. Define

$$C_*^{(m)} := C_*(\mathrm{CS}(X))/F_{m-1}C_*(\mathrm{CS}(X)),$$

where $F_{m-1}C_*(\mathrm{CS}(X))$ is the filtration induced by order. Then

$$C_j = \lim_{m\to-\infty} C_j^{(m)}, \quad j \in \mathbb{N}$$

The complexes $C_*^{(m)}$ inherit filtration from C_*

$$\{0\} = F_{m-1}C_*^{(m)} \subset F_m C_*^{(m)} \subset \dots$$

where

$$F_p C_*^{(m)} := \begin{cases} F_p C_*(\mathrm{CS}(X))/F_{m-1}C_*(\mathrm{CS}(X)) & \text{for } p \geq m-1, \\ 0 & \text{for } p \leq m-1. \end{cases} \tag{8.4}$$

We have

$$C_j^{(m)} = \lim_{p\to\infty} F_{pj}^{(m)}, \quad m \in \mathbb{Z},\ j \in \mathbb{N}.$$

Let $\mathrm{HH}_*^{(m)}$ denote the homology of $C_*^{(m)}$ and HH_* the homology of C_*. Our first objective will be to find $\mathrm{HH}_*^{(m)}$.

There is a *Milnor short exact sequence*

$$0 \to \lim{}^1 \mathrm{H}_{q+1}(C_*^{(m)}) \to \mathrm{HH}_q(\mathrm{CS}(X)) \to \lim \mathrm{H}_q(C_*^{(m)}) \to 0.$$

If the system $\{\mathrm{H}_{q-1}(C_*^{(m)})\}_{m\to-\infty}$ satisfies the Mittag-Leffler condition, then $\lim^1$ vanishes.

Suppose $\{V_\lambda\}$ is an inverse system of sets (k-modules). It satisfies Mittag-Leffler condition if for all λ the system of subsets $(\mathrm{im}(V_\mu \to V_\lambda))$ for $\mu > \lambda$ stabilizes. The inverse system $\{V_\lambda\}$ can be treated as a sheaf $\widetilde{V}$ over the indexing set Λ with partial order topology. Then

$$\lim{}^p\{V_\lambda\} = \mathrm{H}^p(\Lambda, \widetilde{V}),$$

and in particular $\lim\{V_\lambda\} = \Gamma(\Lambda, \widetilde{V})$.

Theorem 8.14 (Emmanouil)**.** *For $\Lambda = \omega$ - the first infinite ordinal, the inverse system of vector spaces $\{V_\lambda\}$ is Mittag-Leffler if and only if one of the following conditions is satisfied*

$$\lim{}^1\{V_\lambda \otimes_k W\} = 0, \text{ for all vector spaces } W \text{ over } k, \tag{8.5}$$

$$\lim{}^1\{V_\lambda \otimes_k W\} = 0, \text{ for some infinite dimensional vector space } W \text{ over } k. \tag{8.6}$$

Recall that $T_0^*X = T^*X \setminus X$ and Y^c is the $\mathbb{C}^*$-bundle over the cosphere bundle S^*X defined as

$$\begin{tikzcd} Y^c := T_0^*X \times_{\mathbb{R}_+} \mathbb{C}^* \arrow[d, "\mathbb{C}^*"] \\ S^*X \end{tikzcd}$$

Consider the eigenspace of the action of the Euler field $\Xi = \sum_{i=1}^n \xi_i \partial \xi_i$ on T_0^*X

$$\Omega^*(T_0^*X)(m) \subset \Omega^*_{C^\infty}(T_0^*X)$$

$$t^*\eta = t^m \eta$$

Then

$$\Omega^{**}(T_0^*X) := \bigoplus_{m\in\mathbb{Z}} \Omega^*(T_0^*X)(m)$$

is a bigraded algebra whose cohomology is naturally isomorphic with $\mathrm{H}^*(Y^c)$. We denote it by $\mathrm{H}^*_{\mathrm{dR}}(Y^c)$.

There is a spectral sequence ${}'E^{(m),r}_{**}$ converging to $\mathrm{HH}^{(m)}_*$ which is associated with the filtration (8.4) of $C^{(m)}_*$. Its complete description is provided in the following proposition.

Proposition 8.15. *Assume $m \leq -\dim X = -n$. Then*

a) the second term of a spectral sequence ${}'E^{(m),r}_{**}$ which is associated with the filtration on $C^{(m)}_\bullet$ which is induced by the order filtration as in (8.4) is given by

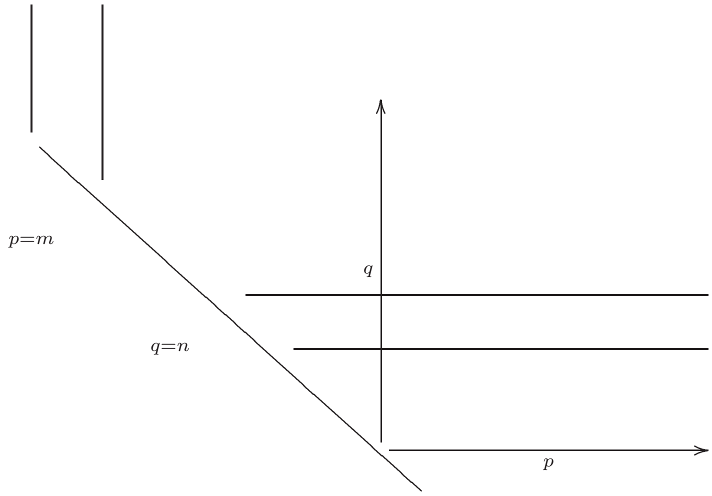
$$
{}'E^{(m),2}_{pq} \cong \begin{cases} \mathrm{H}^{n-p}_{\mathrm{dR}}(Y^c) & q = n \\ \Omega^{2n-m-q}(n-q)/d\Omega^{2n-1-m-q}(n-q) & p = m \\ 0 & \textit{otherwise} \end{cases}
$$

b) the spectral sequence ${}'E^{(m),r}_{**}$ degenerates at ${}'E^2$

c) the identification in a) are compatible with the spectral sequence morphisms induced by the canonical spectral sequence projections

$$C^{(l)}_* \twoheadrightarrow C^{(m)}_*$$

for $l \leq m$.



Corollary 8.16. *The inverse system of the homology groups* $\{\mathrm{HH}_p^{(m)}\}_{m\in\mathbb{Z}_{<-n}}$ *satisfies Mittag-Leffler condition, in fact*

$$\mathrm{HH}_p^{(l_1,m)} = \mathrm{HH}_p^{(l_2,m)}$$

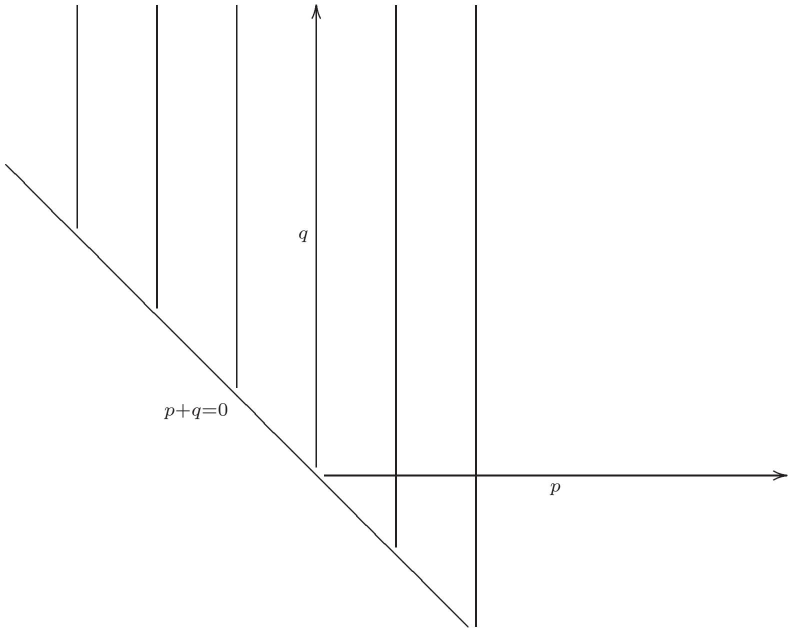
for any $l_1 \leq l_2 \leq m < -n$, *where* $\mathrm{HH}_p^{(l,m)} := \mathrm{im}(\mathrm{HH}_p^{(l)} \to \mathrm{HH}_p^{(m)})$.

Proof. From the proposition (8.15) we obtain a commutative diagrams whose rows are exact.

$$\begin{tikzcd}
\frac{\Omega^{2n-p}(n+m-p)}{d\Omega^{2n-1-p}(n+m-p)} \arrow[r, tail] & \mathrm{H}_p^{(m)} \arrow[r, two heads] & \mathrm{H}_{\mathrm{dR}}^{2n-p}(Y^c) \\
\frac{\Omega^{2n-p}(n+l-p)}{d\Omega^{2n-1-p}(n+l-p)} \arrow[r, tail] \arrow[u, "0"] & \mathrm{H}_p^{(l)} \arrow[r, two heads] \arrow[u] & \mathrm{H}_{\mathrm{dR}}^{2n-p}(Y^c) \arrow[u, equal]
\end{tikzcd}$$

□

Consider a spectral sequence with ${}'E^0_{p*}$ being the p-th component of the graded complex $\mathrm{gr}^F(\mathrm{CS}(X))$.



Taking homology with respect to the differential $d^0_{p*}\colon {}'E^0_{p*} \to' E^0_{p,*-1}$ we obtain

$$'E^1_{pq} = \mathrm{HH}_{p+q}(\mathcal{O}_*(X))(p),$$

calculated in terms of differential forms.

If $\mathcal{O}$ is a smooth algebra, there is a map of complexes

$$(C_*, b) \to (\Omega^*, 0)$$

$$f_0 \otimes \cdots \otimes f_q \to f_0 df_1 \wedge \cdots \wedge df_q.$$

But instead of this map we take

$$f_0 \otimes \cdots \otimes f_q \to \frac{(-1)^q}{q!} f_0 \iota_{H_{f_1}} \ldots \iota_{H_{f_q}} \omega^n.$$

We can compose the two maps

$$\begin{tikzcd}
(C_*(\mathrm{Lie}(\mathrm{CS}(X))), \partial) \arrow[r] \arrow[rr, bend right] & (C_*(\mathrm{CS}(X)), b) \arrow[r] & (\Omega_{**}, d).
\end{tikzcd}$$

The first map

$$\eta\colon a_0 \otimes a_1 \wedge \cdots \wedge a_q \mapsto \sum_{l_1,\ldots,l_q} (-1)^{\overline{l_1 \ldots l_q}} a_0 \otimes a_{l_1} \otimes \cdots \otimes a_{l_q},$$

is a map of complexes, while the second one is a map of complexes only if $d = 0$. But the composition is still a map of complexes.

We identified $'E^{(m),1}_{pq}$ with $\Omega^{2n-p-q}_{\mathcal{O}}(n-q)$ for $p \geq m$ and d^1 with d_{dR}.

To demonstrate that the spectral sequence degenerates at ${}'E_2$ one has to show that the only possibly nontrivial differentials

$$d_{pn}^{(m),p-m} : {}'E_{pn}^{(m),p-m} \to' E_{m,n+p-m-1}^{(m),p-m}$$

all vanish. This is a consequence of the commutativity of the diagram

$$\begin{tikzcd}
{}'E_{pn}^{(m),p-m} \arrow[r, "d_{pn}^{(m),p-m}"] & {}'E_{m,n+p-m-1}^{(m),p-m} \\
{}'E_{p,n}^{(l),p-m} \arrow[u, "\cong"] \arrow[r, "d_{pn}^{(l),p-m}"] & {}'E_{m,n+p-m-1}^{(l),p-m} \arrow[u]
\end{tikzcd}$$

for $l < m$.

Now $\mathrm{H}_* = \mathrm{HH}_*(\mathrm{CS}(X))$ is the homology of the projective limit $\lim C_*^{(m)}$. The projective system $C^{(m)}$ satisfies Mittag-Leffler condition. The same holds for the projective systems of homology groups $\{\mathrm{HH}_*^{(m)}\}_{m\in\mathbb{Z}_{<-n}}$ by corollary (8.16). Hence

$$\mathrm{HH}_j = \lim_m \mathrm{HH}_j^{(m)} \cong \mathrm{H}_{\mathrm{dR}}^{2n-j}(Y^c),$$

and we proved the theorem.

Theorem 8.17. *There is a canonical isomorphism*

$$\mathrm{HH}_q(\mathrm{CS}(X)) \cong \mathrm{H}_{\mathrm{dR}}^{2n-q}(Y^c).$$

8.9 Cyclic homology

We will use the Connes double complex $\mathcal{B}_{**}(\mathrm{CS}(X))$. The maps I, B, S which involve Hochschild and cyclic homology HH_*, HC_* are induced by morphism of filtered chain complexes.

$$C_*(\mathrm{CS}(X)) \rightarrowtail \mathrm{Tot}(\mathcal{B}_{**}(\mathrm{CS}(X))) \twoheadrightarrow \mathrm{Tot}(\mathcal{B}_{**}(\mathrm{CS}(X)))[2]$$

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"'] & {} \arrow[d, "b"'] \\
\mathrm{CS}(X)^{\otimes 3} \arrow[d, "b"'] & \mathrm{CS}(X)^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"'] & \mathrm{CS}(X) \arrow[l, "B"'] \\
\mathrm{CS}(X)^{\otimes 2} \arrow[d, "b"'] & \mathrm{CS}(X) \arrow[l, "B"'] & \\
\mathrm{CS}(X) & &
\end{tikzcd}$$

The first column is a Hochschild complex $C_*(\mathrm{CS}(X))$. The rest is the same complex but shifted diagonally by 1, so the total complex is shifted by 2.

Let us put

$$\mathcal{B}_{**}^{(m)} := \mathcal{B}_{**}/F_{m-1}\mathcal{B}_{**},$$

where $F_p\mathcal{B}_{kl} := F_pC_{l-k}$. Much as we did before we consider the projective system of quotient complexes

$$\operatorname{Tot}\mathcal{B}_{**}^{(m)} = \operatorname{Tot}\mathcal{B}_{**}/F_{m-1}\mathcal{B}_{**}, \quad m \to -\infty.$$

Then we have

$$\mathcal{B}_{kl}^{(m)} = \lim_{p\to\infty} F_{pkl}^{(m)}, \quad m \in \mathbb{Z},\ k,l \geq 0$$

and

$$\mathcal{B}_{kl} = \lim_{m\to-\infty} \mathcal{B}_{kl}^{(m)}, \quad k,l \geq 0,$$

where

$$F_{pkl}^{(m)} := F_p\mathcal{B}_{kl}/F_{m-1}\mathcal{B}_{kl}.$$

Let $\mathrm{HC}_*^{(m)}$ denote the homology of $\operatorname{Tot}\mathcal{B}_{**}^{(m)}$, and HC_{**} the homology of $\operatorname{Tot}\mathcal{B}_{**}$.

Proposition 8.18. *Assume that $m \leq 0$ and $q \geq 2n+1$. Then there exist isomorphisms*

$$\mathrm{HC}_q^{(m)} \cong \begin{cases} \mathrm{H}_{\mathrm{dR}}^{ev}(Y^c) & q \text{ even} \\ \mathrm{H}_{\mathrm{dR}}^{odd}(Y^c) & q \text{ odd} \end{cases}$$

compatible with the canonical maps $\mathrm{HC}_q^{(m')} \to \mathrm{HC}_q^{(m)}$ for $m' \leq m$.

In particular, the systems $\{\mathrm{HC}^{(m)}\}_{m\in\mathbb{Z}_{\leq 0}}$ satisfy for $q \geq 2n+1$ the Mittag-Leffler condition. This gives us a corollary.

Corollary 8.19. *There are, for $q \geq 2n+1$, natural isomorphisms*

$$\mathrm{HC}_q \cong \lim_{m\to-\infty} \mathrm{HC}_q^{(m)} \cong \begin{cases} \mathrm{H}_{\mathrm{dR}}^{ev}(Y^c) & q \text{ even} \\ \mathrm{H}_{\mathrm{dR}}^{odd}(Y^c) & q \text{ odd} \end{cases}$$

This corollary together with a theorem (8.17) imply the following theorem for cyclic homology of an algebra of symbols if $\dim \mathrm{H}_{\mathrm{dR}}^*(Y^c) < \infty$.

Theorem 8.20. *The canonical map*

$$I\colon\ \mathrm{HH}_*(\mathrm{CS}(X)) \to \mathrm{HC}_*(\mathrm{CS}(X))$$

is injective. In particular

$$\mathrm{HC}_{qr}(\mathrm{CS}(X)) = \mathrm{gr}_r^S\,\mathrm{HC}_q(\mathrm{CS}(X)) := \mathcal{S}_{qr}/\mathcal{S}_{q,r-1}, \quad \mathcal{S}_{qr} = \ker S_*^{1+r} \cap \mathrm{HC}_q(\mathrm{CS}(X))$$

is canonically isomorphic with

$$\mathrm{H}_{\mathrm{dR}}^{2n-q+2r}(Y^c), \quad r = 0, 1, \ldots.$$

With some more work we can prove the theorem without assumption of finite dimension of $\mathrm{H}_{\mathrm{dR}}^*(Y^c)$. Then one represents X as a union $\bigcup_{j\in\mathbb{N}} X_j$, where each X_j is compact (with smooth or empty boundary) and $X_j \subset \operatorname{Int} X_{j+1}$. Then the restriction maps $\mathrm{CS}(X) \to \mathrm{CS}(X_j)$ induce homomorphisms

$$\theta\colon\ \mathrm{HH}_*(\mathrm{CS}(X)) \to \widehat{\mathrm{HH}}_* := \lim_j \mathrm{HH}_*(\mathrm{CS}(X)), \tag{8.7}$$

$$\eta\colon\ \mathrm{HC}_*(\mathrm{CS}(X)) \to \widehat{\mathrm{HC}}_* := \lim_j \mathrm{HC}_*(\mathrm{CS}(X)). \tag{8.8}$$

For each q there is a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_q(\mathrm{CS}(X)) \arrow[r, "\theta_q"] \arrow[d, "\cong"'] & \widehat{\mathrm{HH}}_q \arrow[d, "\cong"] \\
\mathrm{H}^{2n-q}_{\mathrm{dR}} \arrow[r] & \lim_j \mathrm{H}^{2n-q}_{\mathrm{dR}}(Y_j^c)
\end{tikzcd}$$

Notice that also the lower arrow is an isomorphism, since

$$\Omega^*_{\mathcal{O}} = \lim_j \Omega^*_{\mathcal{O}_j},$$

where $\mathcal{O}_j$ denotes the corresponding graded algebra of functions on Y_j^c. Since both projective systems $\{\Omega_j^*\}$ and $\{\mathrm{H}^*_{\mathrm{dR}}(Y_j^c)\}$ satisfy Mittag-Leffler condition, we have that θ in (8.7) is an isomorphism.

The naturality of the Connes exact sequence gives us the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & \widehat{\mathrm{HH}}_q \arrow[r, "\widehat{I}"] & \widehat{\mathrm{HC}}_q \arrow[r, "\widehat{S}"] & \widehat{\mathrm{HC}}_{q-2} \arrow[r, "0"] & \widehat{\mathrm{HH}}_{q-1} \arrow[r, "\widehat{I}"] & \cdots \\
\cdots \arrow[r, "B"] & \mathrm{HH}_q \arrow[r, "I"] \arrow[u, "\theta_q", "\cong"'] & \mathrm{HC}_q \arrow[r, "S"] \arrow[u, "\eta_q"] & \mathrm{HC}_{q-2} \arrow[r, "B"] \arrow[u, "\eta_{q-2}"] & \mathrm{HH}_{q-1} \arrow[r, "I"] \arrow[u, "\theta_{q-1}", "\cong"'] & \cdots
\end{tikzcd}$$

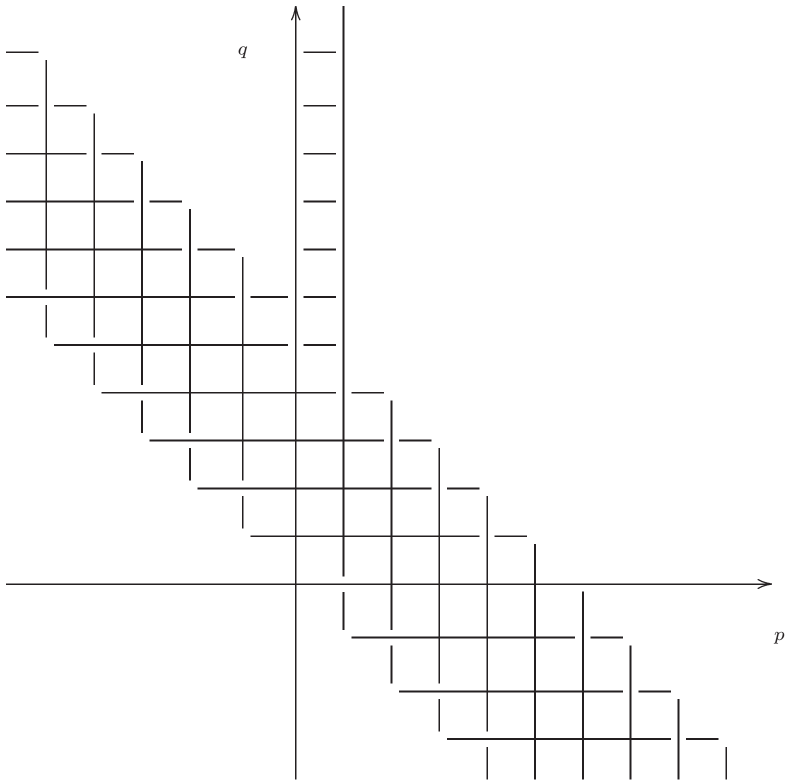
with a priori only the lower sequence being exact. The exactness of the upper sequence follows from

$$\overset{1}{\lim}\, \mathrm{HH}_q(\mathrm{CS}(X_j)) = 0, \text{ for all } q \in \mathbb{N},$$

which is a consequence of the finite-dimensionality of the groups $\mathrm{HH}_q(\mathrm{CS}(X_j)) = \mathrm{H}_{\mathrm{dR}}(Y_j^c)$. Thus the "five lemma" and an easy inductive argument prove that η is an isomorphism and $B = 0$.

Now it remains to prove the proposition (8.18). The filtration $\{F^{(m)}_{p**} \mid p = m, m+1, \ldots\}$ on $\mathcal{B}^{(m)}_{**}$ induces a filtration on $\operatorname{Tot}\mathcal{B}^{(m)}_{**}$. Denote by $E^{(m),r}_{pq}$ the associated spectral sequence which converges to $\mathrm{HC}^{(m)}_*$.

This spectral sequence is a priori located in the region $\{(p,q) \mid p \geq m,\ p+q \geq 0\}$. We shall see that $E^{(m),r}_{pq}$ for $r \geq 1$ vanishes in fact outside the region shown below



i.e. $E_{pq}^{(m),r} = 0$ also if $p + q \geq 2n$ and $p \neq 0$.

Indeed, $E_{pq}^{(m),1}$ is equal, for $p \geq m$, to

$$\mathrm{H}_{p+q}(\mathrm{Tot}\,\mathcal{B}_{**}(\mathcal{O})(p)) = \mathrm{HC}_{p+q}(\mathcal{O})(p).$$

Actually, the first spectral sequence of the double complex $\mathcal{B}_{**}(\mathcal{O})(p)$ degenerates at E^2 yielding thus that

$$E_{pq}^{(m),1} \cong \Omega_{\mathcal{O}}^{p+q}(p)/d\Omega_{\mathcal{O}}^{p+q-1}(p), \quad p \geq m,\ p \neq 0,$$

and

$$E_{0q}^{(m),1} \cong \mathrm{H}_{\mathrm{dR}}^{\tilde{q}}(Y^c), \quad q \geq 2n,$$

where $\tilde{q}$ is the parity of q and $\mathrm{H}_{\mathrm{dR}}^{*} = \mathrm{H}_{\mathrm{dR}}^{(0)}(Y^c) \oplus \mathrm{H}_{\mathrm{dR}}^{(1)}(Y^c)$. This implies the required location of non-vanishing $E_{pq}^{(m),r}$ and as a corollary gives

$$\mathrm{HC}_q^{(m)} \cong E_{0q}^{(m),1} \cong \mathrm{H}_{\mathrm{dR}}^{(\tilde{q})}(Y^c)$$

for $q \geq 2n + 1$. The isomorphisms are also compatible with the canonical mappings $\mathrm{HC}_q^{(m')} \to \mathrm{HC}_q^{(m)}$.

8.9.1 Further analysis of spectral sequence

We will use the notation ${}'E^{(m),r}_{pq}$ for the earlier spectral sequence converging to Hochschild homology $\mathrm{HH}^{(m)}$.

First, let us consider the morphism of spectral sequences induced by S

$$\begin{tikzcd} {}'E^{(m),r}_{pq} \arrow[d, "S^{(m),r}_{pq}"] \\ {}'E^{(m),r}_{p,q-2} \end{tikzcd}$$

For r=1 we have

$$E^{(m),1}_{pq} = \begin{cases} \mathrm{HC}_{p+q}(\mathcal{O})(p), \quad \mathcal{O} = \mathrm{gr}(\mathrm{CS}(X)) = \bigoplus_{p\in\mathbb{Z}} \mathcal{O}(p) & p \geq m \\ 0 & p < m \end{cases}$$

Then

$$\begin{tikzcd} E^{(m),1}_{pq} \arrow[d, "S^{(m),1}_{pq}"] \\ E^{(m),1}_{p,q-2} \end{tikzcd}$$

is the corresponding component of the S-map on cyclic homology of graded algebra $\mathcal{O}$.

If $p = 0$

$$\mathrm{HC}_{p+q}(\mathcal{O}) = \overline{\Omega}^q \oplus \mathrm{H}^{q-2}_{\mathrm{dR}} \oplus \mathrm{H}^{q-4}_{\mathrm{dR}} \oplus \ldots,$$

where

$$\Omega^* := \Omega^*_{\mathcal{O}}, \quad \mathrm{H}^*_{\mathrm{dR}} := \mathrm{H}^*(\Omega^*).$$

$$\overline{\Omega}^k(p) := \Omega^k(p)/d\Omega^{k-1}(p)$$

For $p \neq 0$

$$\mathrm{HC}_{p+q}(\mathcal{O})(p) = \begin{cases} \overline{\Omega}^{p+q}(p) & p \geq m \\ 0 & p < m \end{cases}$$

$p=-2$ $\quad p=-1$ $\quad p=0$ $\quad p=1$ $\quad p=2$

$$\begin{tikzcd} \overline{\Omega}^{q-2}(-2) \arrow[d, "0"'] & \overline{\Omega}^{q-1}(-1) \arrow[l, "d^1"'] \arrow[d, "0"'] & \overline{\Omega}^{q} \oplus \mathrm{H}^{q-2}_{\mathrm{dR}} \oplus \mathrm{H}^{q-4}_{\mathrm{dR}} \oplus \ldots \arrow[l, "d^1"'] \arrow[d] & \overline{\Omega}^{q+1}(1) \arrow[l, "d^1"'] \arrow[d] & \overline{\Omega}^{q+2}(2) \arrow[l, "d^1"'] \arrow[d] \\ \overline{\Omega}^{q-2}(-4) & \overline{\Omega}^{q-1}(-3) \arrow[l, "d^1"'] & \overline{\Omega}^{q-2} \oplus \mathrm{H}^{q-4}_{\mathrm{dR}} \oplus \mathrm{H}^{q-4}_{\mathrm{dR}} \oplus \ldots \arrow[l, "d^1"'] & \overline{\Omega}^{q-1}(1) \arrow[l, "d^1"'] & \overline{\Omega}^{q}(2) \arrow[l, "d^1"'] \end{tikzcd}$$

where for $p = 0$ we have

$$\begin{tikzcd} \overline{\Omega}^q \arrow[d] & \oplus & \mathrm{H}^{q-2}_{\mathrm{dR}} \arrow[d] & \oplus & \mathrm{H}^{q-4}_{\mathrm{dR}} \arrow[d, equal] & \oplus & \cdots \\ 0 & \oplus & \overline{\Omega}^{q-2} & \oplus & \mathrm{H}^{q-4}_{\mathrm{dR}} & \oplus & \cdots \end{tikzcd}$$

Denote

$$\overline{E}^{(m),1}_{pq} := \begin{cases} \overline{\Omega}^{p+q}(p) & p \geq 0 \\ 0 & p < 0 \end{cases}$$

Corollary 8.21. *There is an isomorphism of chain complexes*

$$(E^{(m),1}_{*,q}, d^1_{*,q}) \cong (\overline{E}^{(m),1}_{*,q} \oplus (\mathrm{H}^{q-2}_{\mathrm{dR}} \oplus \mathrm{H}^{q-4}_{\mathrm{dR}} \oplus \ldots)[0], d)$$

and there is an exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[d] \\
(\mathrm{H}^{q-1}_{\mathrm{dR}}[0], 0) \arrow[d] \\
(\overline{E}^{(m),1}_{*,q-1}, d^1) \arrow[d, "B"] \\
({}'E^{(m),1}_{*,q}, d^1) \arrow[d, two heads] \\
(\overline{E}^{(m),1}_{*,q}, d^1) \arrow[d] \\
0
\end{tikzcd}$$

Consider the second spectral sequence of the double complex but arranged according to conventions of Cartan-Eilenberg's book. Denote it by ${}_q\mathcal{E}^r_{**}$, although it depends also on m.

The ${}_q\mathcal{E}^2_{**}$ looks as follows.

$$\begin{tikzcd}
 & & \mathrm{H}^{q-2}_{\mathrm{dR}} & & & \\
\overline{E}^{(m),2}_{-2,q-1} \arrow[ddr, two heads] & \overline{E}^{(m),2}_{-1,q-1} \arrow[ddr, two heads] & \overline{E}^{(m),2}_{0,q-1} \arrow[ddr, two heads] & \overline{E}^{(m),2}_{1,q-1} \arrow[ddr, two heads] & \overline{E}^{(m),2}_{2,q-1} \arrow[ddr, two heads] & \overline{E}^{(m),2}_{3,q-1} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\overline{E}^{(m),2}_{-2,q-1} & \overline{E}^{(m),2}_{-1,q-1} & \overline{E}^{(m),2}_{0,q-1} & \overline{E}^{(m),2}_{1,q-1} & \overline{E}^{(m),2}_{2,q-1} & \overline{E}^{(m),2}_{3,q-1}
\end{tikzcd}$$

There is an isomorphism

$$\overline{E}^{(m),2}_{pq} \xrightarrow{\cong} \overline{E}^{(m),2}_{p+1,q+1}$$

except $(p, q) = (0, q)$, $(1, q-1)$, $(1, q)$, $(2, q)$.

The term $\overline{E}^{(m),2}_{pq}$ appears twice, in ${}_q\mathcal{E}^r_{**}$ and ${}_{q+1}\mathcal{E}^r_{**}$.

There are two cases:

$q < n$ then for $l = \left[\frac{q}{2}\right] + 1$

$$\overline{E}^{(m),2}_{0} \xleftarrow{\cong} \overline{E}^{(m),2}_{-1,q-1} \xleftarrow{\cong} \overline{E}^{(m),2}_{-2,q-2} \xrightarrow{\cong} \ldots \xrightarrow{\cong} \overline{E}^{(m),2}_{-l,q-l} \subseteq \mathrm{HC}_{q-2l}(\mathcal{O})(-l) = 0$$

because $q-2l<0$.

The $\mathcal{E}^1$-term is the same as the $\mathcal{E}^2$-term:

$$\begin{tikzcd}
 & & & \mathrm{H}^{q-1}_{\mathrm{dR}} \arrow[dddrr] & & \\
0 & 0 & 0 & \overline{E}^{(m),2}_{0,q-1}=0 & \overline{E}^{(m),2}_{1,q-1}=0 & \overline{E}^{(m),2}_{2,q-1} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \overline{E}^{(m),2}_{1,q} & \overline{E}^{(m),2}_{2,q}
\end{tikzcd}$$

In $\mathcal{E}^3$ there are only two terms and the spectral sequence collapses at $\mathcal{E}^4$.

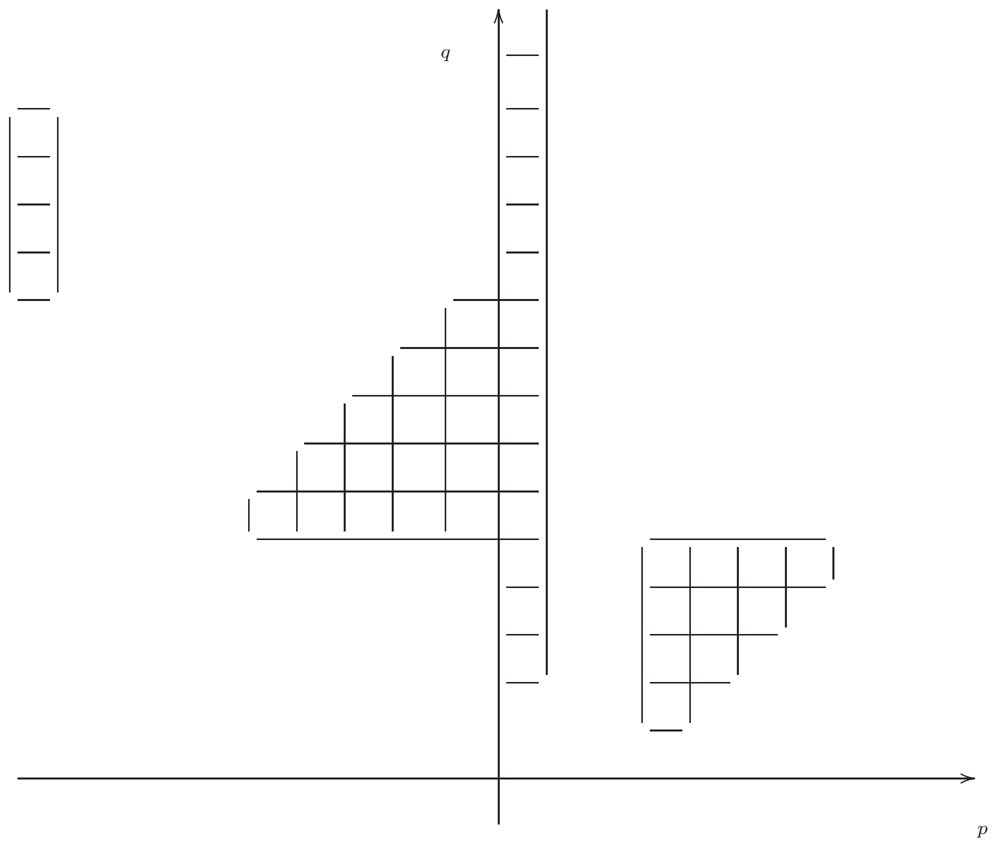
$$\begin{tikzcd}
 & & & \mathrm{H}^{q-1}_{\mathrm{dR}} \arrow[dddrr] & & & \\
0 & 0 & 0 & 0 & 0 & \mathrm{H}^{q-2}_{\mathrm{dR}} & \mathrm{H}^{q-3}_{\mathrm{dR}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{H}^{q-1}_{\mathrm{dR}} & \mathrm{H}^{q-2}_{\mathrm{dR}}
\end{tikzcd}$$

$q-1\geq n$ then for $l=n-\left[\frac{q}{2}\right]$

$$\overline{E}^{(m),2}_{2,q-1}\xrightarrow{\cong}\overline{E}^{(m),2}_{3,q}\xrightarrow{\cong}\overline{E}^{(m),2}_{4,q+1}\xrightarrow{\cong}\dots\xrightarrow{\cong}\overline{E}^{(m),2}_{2+l,q+l-1}\cong\overline{\Omega}^{2l+q-1}(2+l)=0$$

because $2l+q-1>2n$.

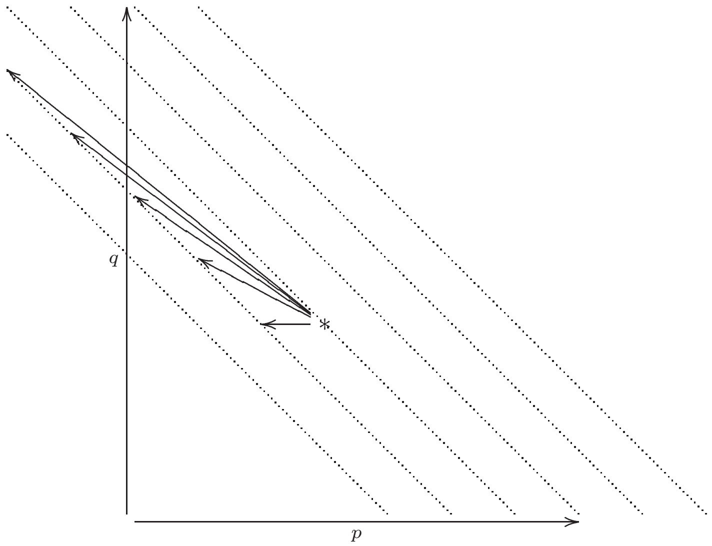
$$\begin{array}{ccccccc}
 & & & \mathrm{H}^{n-1}_{\mathrm{dR}} & & & \\
0 & 0 & 0 & 0 & 0 & \overline{E}^{(m),2}_{2,n-1} & \overline{E}^{(m),2}_{3,n-1} \\
\mathrm{H}^{n+3}_{\mathrm{dR}} & \mathrm{H}^{n+2}_{\mathrm{dR}} & \mathrm{H}^{n+1}_{\mathrm{dR}} & \mathrm{H}^{n}_{\mathrm{dR}} & \mathrm{H}^{n-1}_{\mathrm{dR}} & \mathrm{H}^{n-2}_{\mathrm{dR}} & \mathrm{H}^{n-3}_{\mathrm{dR}} \\
0 & \overline{E}^{(m),2}_{-2,n} & \overline{E}^{(m),2}_{-1,n} & \overline{E}^{(m),2}_{0,n} & 0 & 0 & 0
\end{array}$$


q
p

						$\cdots$	0	0	0	$H^{2n+2} \oplus H^{2n+1}$	0	$\cdots$								
						$\cdots$	0	0	0	$H^{2n+1} \oplus H^{2n}$	0	$\cdots$								
						$\cdots$	0	0	0	$H^{2n} \oplus H^{2n-1}$	0	$\cdots$								
						$\cdots$	0	0	H^{2n}	$H^{2n-1} \oplus H^{2n-3}$	0	$\cdots$								
						$\cdots$	0	H^{2n}	H^{2n-1}	$H^{2n-2} \oplus H^{2n-4}$	0	$\cdots$								
						$\cdots$	H^{2n}	H^{2n-1}	H^{2n-2}	$H^{2n-3} \oplus H^{2n-5}$	0	$\cdots$								
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$n+2$	0	0	0	H^{2n}	H^{2n-1}	$\cdots$	H^{n+5}	H^{n+4}	H^{n+3}	$H^{n+2} \oplus H^{n}$	0	0	0	0	0	$\cdots$	0	0	0	0
$n+1$	0	0	H^{2n}	H^{2n-1}	H^{2n-2}	$\cdots$	H^{n+3}	H^{n+3}	H^{n+2}	$H^{n+1} \oplus H^{n-1}$	0	0	0	0	0	$\cdots$	0	0	0	0
n	0	H^{2n}	H^{2n-1}	H^{2n-2}	H^{2n-3}	$\cdots$	H^{n+3}	H^{n+2}	H^{n+1}	$H^{n} \oplus H^{n-2}$	0	0	0	0	0	$\cdots$	0	0	0	0
$n-1$	0	0	0	0	0	$\cdots$	0	0	0	$H^{n-3} \oplus H^{n-5}$	0	H^{n-2}	H^{n-3}	H^{n-4}	H^{n-5}	$\cdots$	H^{3}	H^{2}	H^{1}	H^{0}
$n-2$	0	0	0	0	0	$\cdots$	0	0	0	$H^{n-4} \oplus H^{n-6}$	0	H^{n-3}	H^{n-4}	H^{n-5}	H^{n-6}	$\cdots$	H^{2}	H^{1}	H^{0}	0
	0	0	0	0	0	$\cdots$	0	0	0	$H^{n-5} \oplus H^{n-7}$	0	H^{n-4}	H^{n-5}	H^{n-6}	H^{n-7}	$\cdots$	H^{1}	H^{0}	0	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
						$\cdots$	0	0	0	$H^{2} \oplus H^{0}$	0	H^{3}	H^{2}	H^{1}	H^{0}	$\cdots$				
						$\cdots$	0	0	0	H^{1}	0	H^{2}	H^{1}	H^{0}	0	$\cdots$				
						$\cdots$	0	0	0	H^{0}	0	H^{1}	H^{0}	0	0	$\cdots$				
						$\cdots$	0	0	0	0	0	H^{0}	0	0	0	$\cdots$				
						$\cdots$	0	0	0	0	0	0	0	0	0	$\cdots$				
$\cdots$	$-n-1$	$-n$		$\cdots$		-4	-3	-2	-1	0	1	2	3	4		$\cdots$		$n-1$	n	$\cdots$

8.9.2 Higher differentials

For $r = 1, 2, \ldots$ the differentials in the spectral sequence are as follows



Let E^r_{pq} be a spectral sequence such that each E^r_{pq} (for $r > r_0$) is a finite dimensional vector space. Let R be a region in the (p,q)-plane which contains finitely many boxes. Then

$$\sum_{(p,q)\in R} \dim E^r_{pq} \geq \sum_{(p,q)\in R} \dim E^{r+1}_{pq} \geq \ldots \geq \sum_{(p,q)\in R} \dim E^{\infty}_{pq}.$$

The equality holds if and only if there is no nontrivial differential originating or leaving R, that is the equality

$$\sum_{(p,q)\in R} \dim E^{r'}_{pq} = \sum_{(p,q)\in R} \dim E^{\infty}_{pq}$$

is another way of saying that the spectral sequence in region R degenerates at $E^{r'}$.

In our spectral sequence

$$E^{(m),2}_{pq} \implies \mathrm{H}_{p+q}(\operatorname{Tot}\mathcal{B}_{**}(\mathrm{CS}(X))/F_{m-1}\operatorname{Tot}\mathcal{B}_{**}(\mathrm{CS}(X)))$$

We claim that the only nonvanishing differentials d^r_{pq} for $r \geq 2$ are

$$d^p_{pq} \colon E^{(m),p}_{pq} \to E^{(m),2}_{0,p+q-1}$$

which inject $E^{(m),2}_{pq} = E^{(m),2}_{pq} \cong \mathrm{H}^{q-2}_{\mathrm{dR}}$ into $E^{(m),p}_{0,p+q-1}$.

We can define two regions R, R' as follows.

[PICTURE]

Then

$$\sum_{(p,q)\in R} \dim E^{r'}_{pq} = \sum_{(p,q)\in R} \dim E^{\infty}_{pq}.$$

Suppose that ther is no nontrivial differential originating from R' or nontrivial differential hitting R and originating outside. Then

$$\sum_{(p,q)\in R'} \dim E^r_{pq} - \sum_{(p,q)\in R} \dim E^r_{pq} \geq \sum_{(p,q)\in R'} \dim E^{r+1}_{pq} - \sum_{(p,q)\in R} \dim E^{r+1}_{pq}$$

Equality holds if and only if all d^r inside R are zero, and then for all $r > r_0$ for some r_0

$$\sum_{(p,q)\in R'} \dim E^r_{pq} - \sum_{(p,q)\in R} \dim E^r_{pq} = \sum_{(p,q)\in R'} \dim E^\infty_{pq} - \sum_{(p,q)\in R} \dim E^\infty_{pq}.$$

We can write

$$\sum_{0\leq q\leq n} \dim E^{(m),2}_{0q} - \sum_{0\leq q\leq n} \dim E^{(m),\infty}_{0q} = \sum_{p>0} \dim E^{(m),2}_{pq}.$$

For $r \geq 2$ let us introduce the following statements:

$(A)_r$ The natural maps
$$E^{(m),r}_{pq} \to E^{(m),r}_{pq}\langle Y^{n-1}\rangle$$
are isomorphisms for $p > 0$, r fixed.

$(B)_r$ The differentials
$$d^r_{pq}\colon E^{(m),r}_{rq} \to E^{(m),r}_{0,q+r-1}$$
are injective.

$(C)_r$ The differentials
$$d^r_{pq}\colon E^{(m),r}_{pq} \to E^{(m),r}_{p-r,q+r-1}$$
are zero for $p \geq r$.

We prove them by induction on r, simultaneously

$$\begin{tikzcd}[column sep=small] & (B)_2 & & & (B)_3 \\ (A)_2 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (B)_2 \wedge (C)_2 \arrow[r, Rightarrow] & (A)_3 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & (C_2) & & & (C)_3 \end{tikzcd}$$

and so on. Furthermore let us introduce two more sequences of statements:

$(D)_r$ For $p > m$
$$d^r_{pq} = \lim_j d^r_{pq,j}.$$

$(E)_r$ For $p > m$
$$E^{(m),r}_{pq} = \lim_j E^{(m),r}_{pq}\langle Y_j\rangle.$$

These are also proved by induction on r in the following way. The $(E)_r$ implies $(D)_r$ and $(E)_r$ and $(D)_r$ together with the condition that $\{E_{pq}^{(m),r}\langle Y^j\rangle\}$, $\{E_{pq}^{(m),r+1}\langle Y^j\rangle\}$ satisfy Mittag-Leffler condition, imply $(E)_{r+1}$.

The $(A)_2$ statement follows from the following remark. Suppose $\mathrm{H}^k_{\mathrm{dR}}(Y^c) = 0$ for $k > n$ and that $\dim \mathrm{H}^*_{dR}(Y^c) < \infty$. Then

$$\sum_{j=0}^{2n-2} \dim E_{0j}^{(m),2} - \sum_{p>0,q} \dim E_{pq}^{(m),2} = \sum_{j=0}^{2n-2} \dim \mathrm{HC}_j(\mathrm{CS}_Y).$$

The maps

$$\mathrm{H}^j_{\mathrm{dR}}(Y^c) \to \mathrm{H}^j_{\mathrm{dR}}((Y^k)^c)$$

are isomorphisms for $j < k$, monomorphism for $j = k$, zero for $j > k+1$.

Chapter 9

Appendix: Topological tensor products

Let $(E, \{p_\alpha\}_{\alpha\in A})$, $(F, \{q_\beta\}_{\beta\in B})$ be vector spaces with the sytems of seminorms $\{p_\alpha\}_{\alpha\in A}$, $\{q_\beta\}_{\beta\in B}$ respectively. Define a system of seminorms on $E \otimes F$ by

$$(p_\alpha \otimes q_\beta)(\tau) := \inf \sum_{\imath\in I} p_\alpha(e_i) q_\beta(f_i), \tag{9.1}$$

where infimum is taken over all representations $\tau = \sum_{i\in I} e_i \otimes f_i$, in which I is a finite set.

Definition 9.1. *A locally convex space $E \otimes F$ with topology induced by the system of seminorms $\{p_\alpha \otimes q_\beta\}_{(\alpha,\beta)\in A\times B}$ is calles a* projective tensor product *and denoted by $E \otimes_\pi F$. Its completion is denoted by $E\widehat{\otimes}_\pi F$.*

A bilinear map

$$\phi\colon E \times F \to E\widehat{\otimes}_\pi F, \quad (e, f) \mapsto e \otimes f,$$

is continuous in both variables and has the following universal property.

Fact 9.2. *For every bilinear jointly continuous mapping $f\colon E \times F \to W$ into locally convex space W there exists unique continuous linear map $L_\phi\colon E\widehat{\otimes}_\pi F \to W$ such that following diagram commutes.*

$$\begin{tikzcd}
E \times F \arrow[rr, "f"] \arrow[dr, "\phi"'] & & W \\
& E\widehat{\otimes}_\pi F \arrow[ur, dashed, "L_\phi"'] &
\end{tikzcd}$$

Remark 9.3. There are also different tensor products on topological vector spaces, like injective and inductive tensor products, but we will not describe them here.

Suppose that $E' = \bigcup_{m\in\mathbb{Z}} E'_m$, where

$$\ldots \subseteq E'_{m-1} \subseteq E'_m \subseteq \ldots$$

is a $\mathbb{Z}$-filtration of E' by locally convex closed vector subspaces of E', and analogously for the space E''. Then define

$$E'\widetilde{\otimes}E'' := \lim_{(l_1,l_2)\in\mathbb{Z}\times\mathbb{Z}} E'_{l_1}\widehat{\otimes}_\pi E''_{l_2}.$$

If for any m there is a continuous projections $E'_m \to E'_{m-1}$, $E''_m \to E''_{m-1}$, then the space $E'_{l_1}\widehat{\otimes}_\pi E''_{l_2}$ is a closed subspace in $E'_{m_1}\widehat{\otimes}_\pi E''_{m_2}$ for any $m_1 \geq l_1$, $m_2 \geq l_2$.

Define a $\mathbb{Z}$-filtration on $E'\widetilde{\otimes}E''$

$$(E'\widetilde{\otimes}E'')_m := \bigcup_{\substack{(l_1,l_2)\in\mathbb{Z}\times\mathbb{Z}\\ l_1+l_2\leq m}} E'_{l_1}\widehat{\otimes}_\pi E''_{l_2}.$$

In similar way we define $E^{(1)}\widetilde{\otimes}\dots\widetilde{\otimes}E^{(p)}$ with $\mathbb{Z}$-filtration

$$(E^{(1)}\widetilde{\otimes}\dots\widetilde{\otimes}E^{(p)})_m := \bigcup_{\substack{(l_1,\dots,l_p)\in\mathbb{Z}^p\\ l_1+\dots+l_p\leq m}} E^{(1)}_{l_1}\widehat{\otimes}_\pi\dots\widehat{\otimes}_\pi E^{(p)}_{l_p}.$$

Chapter 10

Appendix: Spectral sequences

10.1 Spectral sequence of a filtered complex

Let $(C_\bullet, F, \partial)$ be a filtered chain complex, that is

$$\ldots \subseteq F_p C_\bullet \subseteq F_{p+1} C_\bullet \subseteq \ldots \subseteq C_\bullet.$$

We say that the filtration is

1. *separable* if $\bigcap_p F_p C_n = \{0\}$,
2. *complete* if $C_n \xrightarrow{\cong} \varprojlim_p C_n / F_p C_n$,
3. *cocomplete* if $\bigcup_p F_p C_n \xrightarrow{\cong} C_n$,

for all $n \in \mathbb{Z}$.

We define $E^0_{**} := \operatorname{gr}^F_* C_\bullet$ (the associated graded complex), where $E^0_{pq} := F_p C_{p+q} / F_{p-1} C_{p+q}$, and d^0_{**} is the boundary operator induced by ∂,

$$d^0_{pq} \colon E^0_{pq} \to E^0_{p,q-1}.$$

Thus (E^0_{**}, d^0_{**}) is the direct sum of complexes

$$(E^0_{**}, d^0_{**}) = \bigoplus_{p \in \mathbb{Z}} (E^0_{p*}, d^0_{p*}).$$

Next we define

$$\begin{aligned} E^1_{pq} &:= \mathrm{H}_q(E^0_{p*}, d^0_{p*}) \\ &= \frac{\{c \in F_p C_{p+q} \mid \partial c \in F_{p-1} C_{p+q-1}\}}{\{c \in F_p C_{p+q} \mid c = \partial b \text{ for some } b \in F_p C_{p+q+1}\}} \quad \operatorname{mod} F_{p-1} C_{p+q} \\ &=: \frac{Z^1_{pq} + F_{p-1} C_{p+q}}{B^1_{pq} + F_{p-1} C_{p+q}}. \end{aligned}$$

On E^1_{pq} the boundary operator ∂ from a complex

$$\cdots \qquad C_{p+q-1} \xleftarrow{\partial} C_{p+q} \xleftarrow{\partial} C_{p+q+1} \qquad \cdots$$

induces a boundary operator

$$d^1_{pq} \colon E^1_{pq} \to E^1_{p-1,q}.$$

Let us define, for $r = 1, 2, \dots$

$$\begin{aligned} E^r_{pq} &= \frac{\{c \in F_pC_{p+q} \mid \partial c \in F_{p-r}C_{p+q-1}\}}{\{c \in F_pC_{p+q} \mid c = \partial b \text{ for some } b \in F_{p+r-1}C_{p+q+1}\}} \quad \operatorname{mod} F_{p-1}C_{p+q} \\ &=: \frac{Z^r_{pq} + F_{p-1}C_{p+q}}{B^r_{pq} + F_{p-1}C_{p+q}} \cong \frac{Z^r_{pq}}{B^r_{pq} + Z^r_{pq} \cap F_{p-1}C_{p+q}} = \frac{Z^r_{pq}}{B^r_{pq} + Z^{r-1}_{p-1,q+1}}. \end{aligned}$$

Now E^r_{**} equipped with the boundary operator induced by ∂ becomes a direct sum of complexes

$$\dots \leftarrow E^r_{p-r,q+r-1} \xleftarrow{d^r_{pq}} E^r_{pq} \xleftarrow{d^r_{p+r,q-r+1}} E^r_{p+r,q-r+1} \leftarrow \dots,$$

which we can denote by $(E^r_{p+*r,q-*(r-1)}, d^r_{p+*r,q-*(r-1)})$. Now E^{r+1}_{pq} is canonically isomorphic to the homology of the complex

$$(E^r_{p+*r,q-*(r-1)}, d^r_{p+*r,q-*(r-1)})$$

at the E^r_{pq}.

One can arrange terms E^r_{pq} in a table:

$$\begin{tikzcd}
\cdots & E^r_{p-r,q+r-1} & E^r_{p-r,q+r} & E^r_{p-r,q+r+1} & \cdots \\
\cdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & E^r_{p-1,q} & E^r_{p-1,q+1} & E^r_{p-1,q+2} & \cdots \\
\cdots & E^r_{p,q-1} & E^r_{p,q} \arrow[uuul, "d^r_{pq}"'] & E^r_{p,q+1} \arrow[uuul, "d^r_{pq}"'] & \cdots \\
\cdots & E^r_{p+1,q-2} & E^r_{p+1,q-1} & E^r_{p+1,q}C_{p+q+1} & \cdots \\
\cdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & E^r_{p+r,q-r-1} & E^r_{p+r,q-r} \arrow[uuul, "d^r_{pq}"'] & E^r_{p+r,q-r+1} \arrow[uuul, "d^r_{pq}"'] & \cdots \\
\cdots & \vdots & \vdots & \vdots & \cdots
\end{tikzcd}$$

For each (p,q) we defined a system of subobjects of F_pC_{p+q}:

$$\{0\} = B^0_{pq} \subseteq B^1_{pq} \subseteq \ldots \subseteq B^r_{pq} \subseteq \ldots$$
$$\subseteq \bigcup_r B^r_{pq} =: B^\infty_{pq} \subseteq Z^\infty_{pq} := \bigcap_r Z^r_{pq} \subseteq$$
$$\ldots \subseteq Z^r_{pq} \subseteq \ldots \subseteq Z^1_{pq} \subseteq Z^0_{pq} = F_pC_{p+q},$$

such that

$$E^r_{pq} = Z^r_{pq}/B^r_{pq} \mod F_{p-1}C_{p+q}.$$

Morphism $\varphi\colon (C_\bullet, F, \partial) \to (C'_\bullet, F', \partial')$ of filtered complexes induces a morphism

$$E^r_{**}(\varphi)\colon E^r_{**} \to E'^r_{**}, \quad r \geq 0,$$

of corresponding spectral sequences.

Theorem 10.1 (Eilenberg-Moore)**.** *If $E^r_{**}(\varphi)$ is an isomorphism for some r and both filtrations are complete and cocomplete, then φ is a quasi-isomorphism.*

We say that the spectral sequence E^r_{**} *converges* to a filtered module M if

$$E^\infty_{pq} \cong F_pM_{p+q}/F_{p-1}M_{p+q}, \quad p,q \in \mathbb{Z}.$$

Then we write $E^r_{pq} \implies M_{p+q}$.

If the filtration is locally bounded from below (i.e., $F_pC_n = \{0\}$ for $p \ll 0$) and cocomplete, then E^r_{**} converges to $\mathrm{H}_*(C_\bullet, \partial)$. The homology of complex $(C_\bullet, \partial)$ is equipped with canonical filtration

$$F_p \mathrm{H}_*(C_\bullet, \partial) := \operatorname{im}(\mathrm{H}_*(F_pC_\bullet, \partial) \to \mathrm{H}_*(C_\bullet, \partial)).$$

We say that the spectral sequence E^r_{**} *degenerates* (or *collapses*) at E^s if $E^s_{**} \cong E^\infty_{**}$.

Consider the r-th term E^r of the spectral sequence.

There is a sequence of maps

$$E^r_{pq} \twoheadrightarrow E^{r+1}_{pq} \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{pq} \to \mathrm{H}_{p+q}(C),$$

and similarly

$$\mathrm{H}_{p'+q'}(C) \to E^\infty_{p'q'} \rightarrowtail \cdots \rightarrowtail E^{r+1}_{p'q'} \rightarrowtail E^r_{p'q'}.$$

These maps are called the *edge homomorphisms.* For the first quadrant spectral sequence they correspond to the maps from leftmost column $p = 0$

$$E^r_{0q} \to \mathrm{H}_q(C),$$

and to the bottom row, $q = 0$,

$$\mathrm{H}_p(C) \to E^r_{p0}.$$

10.2 Examples

Example 10.2. Two spectral sequences associated with the double complex $(C_{**}, \partial', \partial'')$.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d, "\partial''"'] & C_{p,q+1} \arrow[l, "\partial'"'] \arrow[d, "\partial''"'] & C_{p+1,q+1} \arrow[l, "\partial'"'] \arrow[d, "\partial''"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "\partial''"'] & C_{pq} \arrow[l, "\partial'"'] \arrow[d, "\partial''"'] & C_{p+1,q} \arrow[l, "\partial'"'] \arrow[d, "\partial''"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "\partial'"'] \arrow[d] & C_{p+1,q-1} \arrow[l, "\partial'"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Here

$$\partial'^2 = \partial''^2 = 0, \quad [\partial', \partial''] = \partial'\partial'' + \partial''\partial' = 0,$$

and the total complex is defined by

$$(\operatorname{Tot} C)_n := \prod_{p=-\infty}^{-1} C_{p,n-p} \oplus \bigoplus C_{p,n-p}, \quad \partial := \partial' + \partial''.$$

$$
\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \\
\cdots & * & * & * & \cdots & * & * & \cdots \\
\cdots & C_{0,n} & * & * & \cdots & * & * & \cdots \\
\cdots & * & C_{1,n-1} & * & \cdots & * & * & \cdots \\
\cdots & * & * & C_{2,n-2} & \cdots & * & * & \cdots \\
\cdots & * & * & * & \cdots & C_{n-1,1} & * & \cdots \\
\cdots & * & * & * & \cdots & * & C_{n0} & \cdots \\
 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots &
\end{array}
$$

There are two filtrations on $\operatorname{Tot} C$:

filtration by columns

$$F'_p(\operatorname{Tot} C)_n := \prod_{r \le p} C_{r,n-r}$$

$$
\begin{array}{cccccccccccc}
 & \vdots & \vdots & \vdots & \cdots & \vdots & p & \vdots & \cdots & \vdots & \vdots & \\
\cdots & * & * & * & \cdots & * & \vert & * & \cdots & * & * & \cdots \\
\cdots & C_{0,n} & * & * & \cdots & * & \vert & * & \cdots & * & * & \cdots \\
\cdots & * & C_{1,n-1} & * & \cdots & * & \vert & * & \cdots & * & * & \cdots \\
\cdots & * & * & C_{2,n-2} & \cdots & * & \vert & * & \cdots & * & * & \cdots \\
\cdots & * & * & * & \cdots & C_{p,n-p} & \vert & * & \cdots & * & * & \cdots \\
\cdots & * & * & * & \cdots & * & \vert & C_{p+1,n-p-1} & \cdots & * & * & \cdots \\
\cdots & * & * & * & \cdots & * & \vert & * & \cdots & C_{n-1,1} & * & \cdots \\
\cdots & * & * & * & \cdots & * & & * & \cdots & * & C_{n0} & \cdots \\
 & \vdots & \vdots & \vdots & \cdots & \vdots & p & \vdots & \cdots & \vdots & \vdots &
\end{array}
$$

filtration by rows

$$F''_p(\operatorname{Tot} C)_n := \bigoplus_{p\le s} C_{n-s,s}$$

$$
\begin{array}{ccccccccccc}
 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \\
\cdots & * & * & * & \cdots & * & * & \cdots & * & * & \cdots \\
\cdots & C_{0,n} & * & * & \cdots & * & * & \cdots & * & * & \cdots \\
\cdots & * & C_{1,n-1} & * & \cdots & * & * & \cdots & * & * & \cdots \\
\cdots & * & * & C_{2,n-2} & \cdots & * & * & \cdots & * & * & \cdots \\
\cdots & * & * & * & \cdots & C_{n-p,p} & * & \cdots & * & * & \cdots \\
p & \multicolumn{9}{c}{\hrulefill} & p \\
\cdots & * & * & * & \cdots & * & C_{n-p-1,p+1} & \cdots & * & * & \cdots \\
\cdots & * & * & * & \cdots & * & * & \cdots & C_{n-1,1} & * & \cdots \\
\cdots & * & * & * & \cdots & * & * & \cdots & * & C_{n0} & \cdots \\
 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots &
\end{array}
$$

Both filtrations are cocomplete. The filtration by columns is also complete for each n. The filtration by rows is complete if and only if, $C_{p,n-p} = 0$ for $p \ll 0$.

There are two spectral sequences associated to double complex $(C_{**}, \partial', \partial'')$.

1. First spectral sequence associated to the filtration by columns

$$E'^1_{pq} = \mathrm{H}_q(C_{p*}, \partial'').$$

It converges to $\mathrm{H}_{p+q}(C_{**}) := \mathrm{H}_{p+q}(\operatorname{Tot}(C_{**}))$ if $C_{p,n-p} = 0$ for $p \ll 0$ $(n \in \mathbb{Z})$.

2. Second spectral sequence associated to the filtration by rows

$$E''^1_{pq} = \mathrm{H}_q(C_{*p}, \partial').$$

It converges to $\mathrm{H}_{p+q}(C_{**})$ if $C_{p,n-p} = 0$ for $p \ll 0$ and $p \gg 0$ $(n \in \mathbb{Z})$.

Example 10.3. Double complex $\mathcal{B}(A)_{**}$ (Connes double complex). Let A be the associative algebra with unit.

$$\mathcal{B}(A)_{pq} := \begin{cases} A^{\otimes(q-p+1)} & \text{if } q \geq p \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"'] & A \arrow[l, "B"'] & \\
A^{\otimes 2} \arrow[d, "b"'] & A \arrow[l, "B"'] & & \\
A & & &
\end{tikzcd}$$

Here b is the Hochschild boundary operator and B is defined as

$$B := (1-t)sN,$$

where

$$\begin{aligned}
s(a_0 \otimes \cdots \otimes a_n) &:= 1 \otimes a_0 \otimes \cdots \otimes a_n \\
t(a_0 \otimes \cdots \otimes a_n) &:= (-1)^n \otimes a_0 \otimes \cdots \otimes a_{n-1} \\
N(a_0 \otimes \cdots \otimes a_n) &:= (\mathrm{id} + t + \ldots + t^n)(a_0 \otimes \cdots \otimes a_n)
\end{aligned}$$

Example 10.4. Double complex $\mathcal{D}(A)_{**}$. Here A is commutative k-algebra with unit.

$$\mathcal{D}(A)_{pq} := \begin{cases} \Omega^{q-p}_{A/k} & \text{if } q \geq p \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
\Omega^2_{A/k} \arrow[d, "0"'] & \Omega^1_{A/k} \arrow[l, "d"'] \arrow[d, "0"'] & A \arrow[l, "d"'] & \\
\Omega^1_{A/k} \arrow[d, "0"'] & A \arrow[l, "d"'] & & \\
A & & &
\end{tikzcd}$$

If $A \xrightarrow{\cong} A \otimes_{\mathbb{Z}} \mathbb{Q}$ (i.e. the additive group $(A,+)$ is uniquely divisible), then the formula

$$\mu(a_0 \otimes \cdots \otimes a_n) := \frac{1}{n!} a_0 da_0 \wedge \cdots \wedge da_n$$

induces a morphism of double complexes $\mu \colon \mathcal{B}(A)_{**} \to \mathcal{D}(A)_{**}$.

On the level of spectral sequences associated with the filtration by columns we obtain surjective maps

$$E^1(pq)(\mu)\colon A^{\otimes(q-p+1)} \twoheadrightarrow \Omega^{q-p}_{A/k}.$$

These maps are isomorphisms if A is a function algebra on the smooth algebraic variety over a perfect field (i.e. of characteristic 0 or such that $k^p = k$ if $\mathrm{char}(k) = p$), or iductive limit of such (for example $A = \mathbb{C}$ as $\mathbb{Q}$-algebra).

The first spectral sequence of a double complex

$$(\mathcal{D}(A)_{**}, 0, d) = \bigoplus_{q\geq 0}(\Omega^q_{A/k} \xleftarrow{d} \ldots \xleftarrow{d} A)$$

degenerates at the term E^2:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
\Omega^2_{A/k}/d\Omega^1_{A/k} \arrow[d, "0"'] & \mathrm{H}^1_{\mathrm{dR}}(A) \arrow[l, "d"'] \arrow[d, "0"'] & \mathrm{H}^0_{\mathrm{dR}}(A) \arrow[l, "d"'] & \\
\Omega^1_{A/k}/dA \arrow[d, "0"'] & \mathrm{H}^0_{\mathrm{dR}}(A) \arrow[l, "d"'] & & \\
A & & &
\end{tikzcd}$$

Thus the first spectral sequence of the double complex $(\mathcal{B}(A)_{**}, b, B)$ also degenerates at the term E^2, and we get an isomorphism

$$\mathrm{HC}_n(A) := \mathrm{H}_n(\mathcal{B}(A)_{**}) = \Omega^n_{A/k}/d\Omega^{n-1}_{A/k} \oplus \mathrm{H}^{n-2}_{\mathrm{dR}}(A) \oplus \mathrm{H}^{n-4}_{\mathrm{dR}}(A) \oplus \ldots.$$

Example 10.5. Let P_* be a projective resolution of a right R-module M, and Q_* a projective resolution of a left R-module N. Consider the double complex $P_* \otimes_R Q_*$. Then

$$E'^2_{pq} = \begin{cases} \mathrm{H}_p(P_* \otimes_R N) & q = 0, \\ 0 & q \neq 0 \end{cases}$$

$$E''^2_{pq} = \begin{cases} \mathrm{H}_p(M \otimes_R Q_*) & q = 0, \\ 0 & q \neq 0 \end{cases}$$

Both spectral sequences converge to $\mathrm{H}_{p+q}(P_* \otimes_R Q_*) =: \mathrm{Tor}^R_{p+q}(M, N)$, so we get an important canonical isomorphisms

$$\mathrm{H}_p(P_* \otimes_R N) \cong \mathrm{Tor}^R_p(M, N) \cong \mathrm{H}_p(M \otimes_R Q_*).$$

They express the fact that the bifunctor $\otimes_R\colon \mathbf{Mod}\text{-}R \times R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is balanced.

Example 10.6. Two hyperhomology spectral sequences. A Cartan-Eilenberg resolution of a complex (C_*, ∂) is a double complex $(P_{**}, \partial', \partial'')$ with augmentation $\eta\colon P_{*0} \to C_*$ satisfying the following conditions:

1. for all p, q the modules P_{pq}, $\operatorname{im}\partial'_{pq}$, $\ker\partial'_{pq}$, $\mathrm{H}_p(P_{*q}, \partial')$ are projective,
2. the augmented complexes

$$\begin{tikzcd}
P_{p*}, \arrow[d, "\eta"'] & \operatorname{im}\partial'_{p*}, \arrow[d, "\eta"'] & \ker\partial'_{p*}, \arrow[d, "\eta"'] & \mathrm{H}_p(P_{*q}, \partial') \arrow[d, "\eta"'] \\
C_p & \operatorname{im}\partial_p & \ker\partial_p & \mathrm{H}_p(C_*, \partial)
\end{tikzcd}$$

are projective resolutions of modules C_p, $\operatorname{im}\partial_p$, $\ker\partial_p$, $\mathrm{H}_p(C_*, \partial)$.

$$\begin{tikzcd}
\cdots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \cdots \\
\cdots & P_{p-1,q} \arrow[l] \arrow[d, "\partial''_q"'] & P_{p,q} \arrow[l, "\partial'_p"'] \arrow[d, "\partial''_q"'] & P_{p+1,q} \arrow[l, "\partial'_{p+1}"'] \arrow[d, "\partial''_q"'] & \cdots \arrow[l] \\
\cdots & P_{p-1,q-1} \arrow[l] \arrow[d] & P_{p,q-1} \arrow[l, "\partial'_p"'] \arrow[d] & P_{p+1,q-1} \arrow[l, "\partial'_{p+1}"'] \arrow[d] & \cdots \arrow[l] \\
\cdots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \cdots \\
\cdots & P_{p-1,1} \arrow[l] \arrow[d, "\partial''_1"'] & P_{p,1} \arrow[l, "\partial'_p"'] \arrow[d, "\partial''_1"'] & P_{p+1,1} \arrow[l, "\partial'_{p+1}"'] \arrow[d, "\partial''_1"'] & \cdots \arrow[l] \\
\cdots & P_{p-1,0} \arrow[l] \arrow[d, "\eta"'] & P_{p,0} \arrow[l, "\partial'_p"'] \arrow[d, "\eta"'] & P_{p+1,0} \arrow[l, "\partial'_{p+1}"'] \arrow[d, "\eta"'] & \cdots \arrow[l] \\
\cdots & C_{p-1} \arrow[l] & C_p \arrow[l, "\partial_p"'] & C_{p+1} \arrow[l, "\partial_{p+1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

Such resolution can be obtained from the arbitrary projective resolutions of $\mathrm{H}_p(C_*, \partial)$ and $\operatorname{im}\partial_{p-1}$ by gluing.

$$\begin{tikzcd}
P^H_{p*} \arrow[d] & P^Z_{p*} \arrow[l, two heads, dashed] \arrow[d, dashed] & P^B_{p-1,*} \arrow[l, tail, dashed] \arrow[d] & P^B_{p*} \arrow[d] & P_{p*} \arrow[l, two heads, dashed] \arrow[d, dashed] & P^Z_{p*} \arrow[l, tail, dashed] \arrow[d] \\
\mathrm{H}_p(C_p, \partial) & \ker\partial_p \arrow[l, two heads] & \operatorname{im}\partial_{p-1} \arrow[l, tail] & \operatorname{im}\partial_p & C_p \arrow[l, two heads] & \ker\partial_p \arrow[l, tail]
\end{tikzcd}$$

For an additive functor F the hyperhomology spectral sequences are the first and second spectral sequences of a double complex $(F(P_{**}), F(\partial'), F(\partial''))$

$$E'^1_{pq} = (L_qF)(C_p),$$
$$E''^2_{pq} = F(P^H_{pq}),$$

and

$$E'^2_{pq} = \mathrm{H}_p((L_qF)(C_*)),$$
$$E''^2_{pq} = (L_pF)(\mathrm{H}_q(C_*)).$$

Both spectral sequences converge to

$$\mathbb{L}_{p+q}F(C_*) := \mathrm{H}_{p+q}(F(P_{**})).$$

if C_* is bounded from below, that is $C_n = 0$ for $n \ll 0$.

Assume that $C_n = 0$ for $n < 0$, C_* is F-acyclic, that is $(L_0F)(C_n) \xrightarrow{\cong} C_n$, $(L_pF)(C_n) = 0$ for $p > 0$, and that

$$\mathrm{H}_n(C_*) = \begin{cases} M & n = 0, \\ 0 & n > 0. \end{cases}$$

Such complex is called an F-acyclic resolution of the module M. In that case

$$E'^2_{pq} \cong \begin{cases} \mathrm{H}_p(F(C_*)) & q = 0, \\ 0 & q \neq 0, \end{cases}$$

$$E''^2_{pq} \cong \begin{cases} L_pF(M) & p = 0, \\ 0 & p \neq 0. \end{cases}$$

Thus we obtain an isomorphism

$$\mathrm{H}_p(F(C_*)) \cong (L_pF)(M).$$

We established a very important fact: to compute $(L_pF)(M)$ it suffices to use an arbitrary F-acyclic resolution of M.

Example 10.7. A flat module is an F-acyclic module for $F = (-) \otimes_R N$, where N is an arbitrary left R-module. For $R = \mathbb{Z}$ flat modules are precisely torsion free abelian groups. Thus

$$0 \leftarrow \mathbb{Q}/\mathbb{Z} \leftarrow \mathbb{Q} \leftarrow \mathbb{Z} \leftarrow 0$$

is a flat resolution of group $\mathbb{Q}/\mathbb{Z}$ (which is an injective cogenerator of a category of abelian groups **Ab**). From this we obtain

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A) = \ker(A \to A \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{Torsion}(A).$$

Example 10.8. Consider two composable additive functors

$$\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C},$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories. Let M be an object in $\mathcal{A}$, P_* its projective resolution. In the hyperhomology spectral sequence we put $C_* = G(P_*)$. Then if G sends projective objects into F-acyclic objects

$$E'^2_{pq} = \mathrm{H}_p((L_qF)(G(P_*))) \cong \begin{cases} \mathrm{H}_p((F \circ G)(P_*)) = (L_p(F \circ G))(M) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

$$E''^2_{pq} = (L_pF \circ L_qG)(M)$$

In this case we obtain that

$$E''^2_{pq} = (L_pF \circ L_qG)(M) \implies (L_{p+q}(F \circ G))(M).$$

$$E'^2_{pq} = E^\infty_{pq} = \quad \begin{array}{c|cccc} & \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & \cdots & 0 \\ q & 0 & 0 & \cdots & 0 \\ & (L_0(F \circ G))(M) & (L_1(F \circ G))(M) & \cdots & (L_p(F \circ G))(M) \\ \hline & & p & & \end{array}$$

$$E''^2_{pq} = \quad \begin{array}{c|cccc} & \cdots & \cdots & \cdots & \cdots \\ & (L_0F \circ L_qG)(M) & (L_1F \circ L_qG)(M) & \cdots & (L_pF \circ L_qG)(M) \\ & \cdots & \cdots & \cdots & \cdots \\ q & (L_0F \circ L_1G)(M) & (L_1F \circ L_1G)(M) & \cdots & (L_pF \circ L_1G)(M) \\ & (L_0F \circ L_0G)(M) & (L_1F \circ L_0G)(M) & \cdots & (L_pF \circ L_0G)(M) \\ \hline & & p & & \end{array}$$

This spectral sequence is called the *spectral sequence of a composition of functors*.

Example 10.9. Let $\varphi\colon R \to S$ be a homomorphism of unital rings, M a right R-module, N a left S-module. Consider a composition

$$\mathbf{Mod}\text{-}R \xrightarrow{(-)\otimes_R S} \mathbf{Mod}\text{-}S \xrightarrow{(-)\otimes_R N} \mathbf{Ab}$$

The spectral sequence of a composition of these two functors (G sends projective R-modules into projective S-modules) in looks as follows:

$$E^2_{pq} = \operatorname{Tor}^S_p(\operatorname{Tor}^R_q(M,S),N) \implies \operatorname{Tor}^R_{p+q}(M,N)$$

and it is called a base change spectral sequence.

Suppose that $R \to S$ is a homomorphism of k-algebras, M_R, ${}_SN$ are respectively right R-module and left S-module. Their tensor product $M \otimes RN$ gives rise to a sequence of derived functors $\operatorname{Tor}^R_*(M,N)$.

Suppose that $P_* \twoheadrightarrow M$ is a projective R-module resolution of M, and $Q_* \twoheadrightarrow N$ a projective S-module resolution for N.

$$M \otimes_R N \leftarrow P_* \otimes_R Q_* \cong (P_* \otimes_R S) \otimes_S Q_*$$

Suppose $F(\cdot,\cdot)$ is a functor covariant in both arguments.

$$\begin{tikzcd} L_qF(\cdot,\cdot) \arrow[r, equal] & L_q^{\{1,2\}}F(\cdot,\cdot) \arrow[dl] \arrow[dr] & \\ L_q^{\{1\}}F(\cdot,\cdot) \arrow[dr] & & L_q^{\{2\}}F(\cdot,\cdot) \arrow[dl] \\ & L_q^{\emptyset}F(\cdot,\cdot) \arrow[r, equal] & \begin{cases} F & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} \end{tikzcd}$$

We say that it is *left balanced* if there are isomorphisms $L_q^{\{1\}} \cong L_q^{\{1,2\}} \cong L_q^{\{2\}}$.

$$\begin{tikzcd}
R^q F(\cdot,\cdot) \arrow[r, equal] & R^q_{\{1,2\}} F(\cdot,\cdot) & \\
R^q_{\{1\}} F(\cdot,\cdot) \arrow[ur] & & R^q_{\{2\}} F(\cdot,\cdot) \arrow[ul] \\
& R^q_{\emptyset} F(\cdot,\cdot) \arrow[ul] \arrow[ur] \arrow[r, equal] & \begin{cases} F & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}
\end{tikzcd}$$

We say that it is *right balanced* if there are isomorphisms $R^q_{\{1\}} \cong R^q_{\{1,2\}} \cong R^q_{\{2\}}$.

There is an isomorphism

$$P_* \otimes_R N \xleftarrow{\cong} P_* \otimes_R Q_* \cong (P_* \otimes_R S) \otimes_S Q_*$$

$$\mathrm{Tor}_q^R(M, S \otimes_S Q_*) \xrightarrow{\cong} \mathrm{Tor}_q^R(M, S) \otimes_S Q_*.$$

Taking homology we get

$$\mathrm{H}_p(\mathrm{Tor}_q^R(M, S) \otimes Q_*) \cong \mathrm{Tor}_p^S(\mathrm{Tor}_q^R(M, S), N),$$

and a base change spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^S(\mathrm{Tor}_q^R(M, S), N) \implies \mathrm{Tor}^R_{p+q}(M, N).$$

The boundary maps (transgressions) of this spectral sequences are as follows:

$$E^2_{0n} = \mathrm{Tor}_n^R(M, S) \otimes_S N \to \mathrm{Tor}_n^R(M, N)$$
$$\mathrm{Tor}_n^R(M, N) \to E^2_{n0} = \mathrm{Tor}_n^S(M \otimes S, N)$$

Example 10.10. For an unital k-algebra A let $\mathrm{Lie}(A)$ denote the associated Lie algebra with bracket $[a, a'] = aa' - a'a$. The universal derivation

$$d_\Delta \colon A \to A \otimes_k A^{op}, \quad d_\Delta(a) = 1 \otimes a^{op} - a \otimes 1$$

is a homomorphism of Lie algebras $\mathrm{Lie}(A) \to \mathrm{Lie}(A \otimes_k A^{op})$, so it induces a homomorphism of associative algebras $R := U(\mathrm{Lie}(A)) \to A \otimes_k A^{op} =: S$. Let $M = k$ (trivial representation of a Lie algebra $\mathrm{Lie}(A)$). The base change spectral sequence has the form

$$E^2_{pq} = \mathrm{Tor}_p^{A \otimes_k A^{op}}(\mathrm{Tor}_q^{U(\mathrm{Lie}(A))}(k, A \otimes_k A^{op}), N) \implies \mathrm{Tor}_{p+q}^{U(\mathrm{Lie}(A))}(k, N),$$

that is if A is flat over k then

$$E^2_{pq} = \mathrm{Tor}_p^{A \otimes_k A^{op}}(\mathrm{H}_q^{\mathrm{Lie}}(A; A \otimes_k A^{op}), N) \implies \mathrm{H}^{\mathrm{Lie}}_{p+q}(k, N).$$

Because $k \otimes_{U(\mathrm{Lie}(A))} (A \otimes A^{op}) \cong A$ as a right $A \otimes A^{op}$-module, we have that the second boundary map gives a canonical homomorphism

$$\mathrm{H}_n^{\mathrm{Lie}}(A; N) \to \mathrm{H}_n(A; N) \cong E^2_{n0}.$$

There is a homomorphism of standard chain complexes

$$(C_*(\mathrm{Lie}(A);N),\partial) \to (C_*(A,N),b)$$

where

$$\partial(n\otimes a_1\wedge\cdots\wedge a_n) := \sum_{i=1}^{n}(-1)^i \underbrace{(a_i n - n a_i)}_{-(d_\Delta a)n} \otimes a_1\wedge\cdots\wedge\widehat{a_i}\wedge\cdots\wedge a_n$$

$$+\sum_{1\le i<j\le n}(-1)^{i+j} n\otimes[a_i,a_j]\wedge a_1\wedge\cdots\wedge\widehat{a_i}\wedge\cdots\wedge\widehat{a_j}\wedge\cdots\wedge a_n$$

In the special case $N = A$ we obtain canonical homomorphism

$$\mathrm{H}_n^{\mathrm{Lie}}(A;\mathrm{ad}) \to \mathrm{HH}_n(A)$$

Example 10.11. hyper-Tor spectral sequences and Künneth spectral sequence. For a right R-module M and a complex of left modules C_* we define

$$\mathbf{Tor}_n^R(M,C_*) := \mathrm{H}_n(P_*\otimes_R C_*)$$

where $P_*\to M$ is a projective resolution of M. Then the first and second spectral sequence of a bicomplex $P_*\otimes_R C_*$ are as follows:

$$\begin{aligned} E'^1_{pq} &= P_p\otimes_R \mathrm{H}_q(C)\\ E'^2_{pq} &= \mathrm{Tor}_p^R(M,\mathrm{H}_q(C)) \implies \mathbf{Tor}_{p+q}^R(M,C_*)\end{aligned}$$

and

$$\begin{aligned} E''^1_{pq} &= \mathrm{Tor}_q^R(M,C_p)\\ E''^2_{pq} &= \mathrm{H}_p(\mathrm{Tor}_q^R(M,C_*)) \cong \begin{cases}\mathrm{H}_p(M\otimes_R C_*) & q=0\\ 0 & q\neq 0\end{cases}\end{aligned}$$

where the isomorphism for E^2_{pq} holds if the complexes $\mathrm{Tor}_q^R(M,C_*)$ are acyclic for $q>0$, for example if C_n are flat. Then we obtain a Künneth spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^R(M,\mathrm{H}_q(C)) \implies \mathrm{H}_{p+q}(M\otimes_R C_*)$$

if $C_n = 0$ for $n\ll 0$.

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Part VI

Equivariant KK-theory

by

Paul F. Baum

Jacek Brodzki

Based on the lectures of:

- Paul F. Baum
  (*Mathematics Department, McAllister Building The Pennsylvania State University, University Park, PA 16802, USA*)
  – Chapter 6.

- Jacek Brodzki
  (*School of Mathematics, University of Southampton Southampton SO17 1BJ, UK*)
  – Chapters 1, 2, 3, 4, 5, 6.2, **??**.

With additional lectures by:

- Piotr M. Sołtan – Sections 2.10, 2.11, 2.12, 2.13.

- Christian Voigt – Introduction VI.

Introduction to KK-theory

KK-theory was defined and developed by Kasparov in the 1980s. Since then it has played a fundamental rôle in the theory of operator algebras and its applications. In this lecture we explain some of the origins of Kasparov's theory, sketch its definition and basic properties, and indicate some applications. A large part of the material presented here will be discussed in much more detail in later lectures. Therefore we will skip almost all proofs and only give some references to the literature.

Motivation and background

Topological K-theory

The origins of KK-theory go back to topological K-theory in the sense of Atiyah and Hirzebruch which was introduced around 1960.
For a compact topological space X the topological K-theory group $\mathrm{K}^0(X)$ is the Grothendieck group of the semigroup of isomorphism classes of complex vector bundles over X. This definition can be extended to locally compact spaces by setting

$$\mathrm{K}^0(X) = \mathrm{coker}(\iota^* : \mathrm{K}^0(+) \to \mathrm{K}^0(X^+))$$

where X^+ denotes the one-point compactification of X, $+$ is a one-point space and $\iota : + \to X^+$ the inclusion of the base point. Using the n-fold suspension $\mathbb{R}^n \times X$ one defines $\mathrm{K}^{-n}(X) := \mathrm{K}^0(\mathbb{R}^n \times X)$.
With these definitions at hand we can already formulate the most fundemantal result in topological K-theory: The Bott periodicity theorem asserts that there is a natural isomorphism $\mathrm{K}^{-n-2}(X) \cong \mathrm{K}^{-n}(X)$ for all n. As a consequence we see that there are only two K-groups to consider, namely K^0 and K^1.

K-theory for Banach algebras

The K-group $\mathrm{K}_0(A)$ of a unital complex Banach algebra A is defined as the Grothendieck group of the semigroup of isomorphism classes of finitely generated projective modules over A. For nonunital Banach algebras one sets

$$\mathrm{K}_0(A) = \ker(\pi_* : \mathrm{K}_0(A^+) \to \mathrm{K}_0(\mathbb{C}))$$

where A^+ is the unitarization of A and $\pi : A^+ \to \mathbb{C}$ denotes the augmentation homomorphism.

The Serre-Swan theorem states that the category of vector bundles over a compact space X is equivalent to the category of finitely generated projective modules over the algebra of continuous functions $C(X)$. It follows that $\mathrm{K}^0(X)$ can be identified with $\mathrm{K}_0(C_0(X))$ for all locally compact spaces X.
The higher topological K-groups of a Banach algebra A are defined by $\mathrm{K}_n(A) = \mathrm{K}_0(C_0(\mathbb{R}^n, A))$. Here $C_0(\mathbb{R}^n, A)$ is the algebra of continuous functions $\mathbb{R}^n \to A$ vanishing at infinity. We remark that the definition of $\mathrm{K}_0(A)$ uses only the algebraic structure, and not the topology of the Banach algebra A. In contrast, the definition of the higher topological K-theory groups $\mathrm{K}_n(A)$ relies on the Banach algebra structure of A.
The Bott periodicity theorem carries over to the setting of Banach algebras: There is a natural isomorphism $\mathrm{K}_{n+2}(A) \cong \mathrm{K}_n(A)$ for all n.

The index theorem

One main motivation for the study of topological K-theory comes from index theory. Let D be an elliptic pseudodifferential operator on a closed oriented manifold M, a classical example is given by the Dirac operator on a spin manifold. The Atiyah-Singer index theorem allows to calculate the index

$$\mathrm{Index}(D) = \dim\ker(D) - \dim\mathrm{coker}(D)$$

of D in terms of topological data.
More precisely, the symbol of D gives a class $[\sigma(D)] \in \mathrm{K}^0(T^*M)$. Here T^*M denotes the cotangent bundle of M. Atiyah and Singer defined two maps

$$a\text{-}\mathrm{Index},\ t\text{-}\mathrm{Index}\colon\ \mathrm{K}^0(T^*M) \to \mathbb{Z},$$

called the analytical and topological index, respectively. These maps are made in such a way that $a\text{-}\mathrm{Index}([\sigma(D)]) = \mathrm{Index}(D)$, and $t\text{-}\mathrm{Index}([\sigma(D)])$ is defined topologically. The Atiyah-Singer index theorem states that

$$a\text{-}\mathrm{Index} = t\text{-}\mathrm{Index}\,.$$

Using the Chern character from K-theory to cohomology, this result leads to an explicit expression for the index involving characteristic classes. For instance, in the case of the Dirac operator D on a spin manifold M the corresponding formula reads

$$\mathrm{Index}(D) = \int_M \hat{A}(M)$$

where $\hat{A}(M)$ is the $\hat{A}$-genus of M.

K-homology

Index theory is a natural starting point for the definition of K-homology, the homology theory dual to K-theory. The existence of such a dual homology theory follows from abstract homotopy theory, but homotopy theory does not provide a useful description for the cycles of K-homology.
Atiyah proposed an operator theoretic approach to K-homology based on "abstract elliptic operators" [a-mf68]. The definition of an abstract elliptic operator encodes the main properties of elliptic pseudodifferential operators on closed

manifolds.
Let X be compact topological space. An abstract elliptic operator over X is a triple (ϕ_0, ϕ_1, T), where $\phi_i \colon C(X) \to B(\mathcal{H}_i)$ are $*$-representations on Hilbert spaces $\mathcal{H}_i$ and $T \in B(\mathcal{H}_0, \mathcal{H}_1)$ is a Fredholm operator such that $\phi_1(f)T - T\phi_0(f)$ is a compact operator for all $f \in C(X)$.
Let us write $\mathrm{Ell}(X)$ for the set of all such triples. There is a binary operation on $\mathrm{Ell}(X)$ given by direct sum. Atiyah defined a map $\mathrm{Ell}(X) \to \mathrm{K}_0(X)$ and showed that it is surjective provided X is a finite CW-complex.
The remaining problem was to describe explicitely the equivalence relation $\sim$ such that $\mathrm{Ell}(X)/\sim$ is isomorphic to $\mathrm{K}_0(X)$. Eventually this problem was solved by Kasparov via KK-theory.

Brown-Douglas-Fillmore theory

Another approach to K-homology which precedes KK-theory is the extension theory of Brown, Douglas and Fillmore [bdf-77]. This theory was motivated by questions in operator theory in the first place.
Let $\mathcal{H}$ be a Hilbert space and consider the exact sequence

$$0 \to \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}) \xrightarrow{\pi} \mathcal{Q}(\mathcal{H}) \to 0,$$

where $\mathcal{K}(\mathcal{H})$ is the ideal of compact operators on $\mathcal{H}$, and $\mathcal{Q}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra. An operator $T \in B(\mathcal{H})$ is called essentially normal (selfadjoint) if $\pi(T)$ is normal (selfadjoint). The essential spectrum of T is the spectrum of $\pi(T)$.
The Weyl-von Neumann theorem states that if T is essentially selfadjoint, then $T = S + K$, where S is selfadjoint and K compact. Moreover, one has $T = URU^* + K$ where U is unitary and K compact if and only if T and R have the same essential spectrum.
Brown, Douglas and Fillmore addressed the following two natural questions. If T is essentially normal, then

- under what conditions can one write $T = N + K$, where N is normal and K compact?
- under what conditions on R can one write $T = URU^* + K$, where U is unitary and K compact?

This led them to study extensions of C*-algebras. We say that E is and extension of A by B if there exists an exact sequence

$$0 \to B \to E \to A \to 0$$

of C*-algebras, that is, B is an ideal in E and A is isomorphic to the quotient of E by B. If A is separable and nuclear and B is σ-unital, then there is an abelian group $\mathrm{Ext}(A, B)$ constructed out of equivalence classes of extensions of A by $B \otimes \mathcal{K}$.
If T is an essentially normal operator on $\mathcal{H}$ and $X \subset \mathbb{C}$ its essential spectrum, then one has an extension

$$0 \to \mathcal{K}(\mathcal{H}) \to C^*(T, 1, \mathcal{K}(\mathcal{H})) \to C(X) \to 0.$$

The crucial point is that a computation of $\mathrm{Ext}(C(X), \mathbb{C})$ answers the questions stated above. We refer to chapter 16 in [b-b98] for a precise statement of the results and more information.

Definition of KK-theory

Hilbert modules

As a first ingredient in Kasparov theory we need Hilbert modules [l-e95]. If B is a C*-algebra, then a Hilbert B-module is a right B-module $\mathcal{E}$ with a positive definite sesquilinear form $\langle -, -\rangle\colon \mathcal{E}\times\mathcal{E}\to B$ such that

$$\begin{aligned}\langle \xi,\, \eta\cdot b\rangle &= \langle \xi,\, \eta\rangle\cdot b,\\ \langle \xi,\, \eta\rangle^* &= \langle \eta,\, \xi\rangle,\\ \langle \xi,\, \xi\rangle &\geq 0\\ \langle \xi,\, \xi\rangle &= 0 \text{ iff } \xi = 0\end{aligned}$$

for all $\xi, \eta\in\mathcal{E}$, $b\in B$ and $\mathcal{E}$ is complete in the norm $\|\xi\| = \sqrt{\|\langle \xi,\, \xi\rangle\|}$.
Let us consider some examples of Hilbert modules.

a) In the case $B = \mathbb{C}$ a Hilbert B-module is the same thing as a Hilbert space.

b) If $B = C_0(X)$ for a locally compact space X then Hilbert B-modules can be identified with continuous fields of Hilbert spaces over X.

c) Every C*-algebra B is a Hilbert B-module over itself with the bracket $\langle b,\, c\rangle = b^* c$.

d) If $(\mathcal{E}_i)_{i\in I}$ is a family of Hilbert B-modules, then the completed direct sum $\bigoplus_{i\in I}\mathcal{E}_i$ is a Hilbert B-module. For a C*-algebra B the Hilbert B-module $\mathcal{H}_B = \bigoplus_{i=1}^\infty B$ is a standard module in a certain sense. More precisely, the Kasparov stabilization theorem states that if $\mathcal{E}_B$ is any countably generated Hilbert B-module, then $\mathcal{E}_B\oplus\mathcal{H}_B = \mathcal{H}_B$.

Let $\mathcal{E}, \mathcal{F}$ be Hilbert B-modules. Denote by $\mathcal{L}(\mathcal{E},\mathcal{F})$ the set of all maps $T\colon \mathcal{E}\to\mathcal{F}$ such that there exists $T^*\colon \mathcal{F}\to\mathcal{E}$ satisfying $\langle T\xi,\, \eta\rangle = \langle \xi,\, T^*\eta\rangle$ for all $\xi\in\mathcal{E}$, $\eta\in\mathcal{F}$. Such maps are automatically B-linear and bounded, and they are simply referred to as bounded operators.
A bounded operator $T\in\mathcal{L}(\mathcal{E},\mathcal{F})$ is called finite rank operator if it is a finite sum of rank-one operators $|\eta\rangle\langle\xi|$ given by $|\eta\rangle\langle\xi|(\lambda) = \eta\langle\xi,\, \lambda\rangle$ for $\xi\in\mathcal{E}$, $\eta\in\mathcal{F}$. The space $\mathcal{K}(\mathcal{E},\mathcal{F})$ of compact operators is defined as the closed linear span of the space of finite rank operators. In the case $\mathcal{E}=\mathcal{F}$ we have that $\mathcal{L}(\mathcal{E},\mathcal{E}) = \mathcal{L}(\mathcal{E})$ is a C*-algebra, and $\mathcal{K}(\mathcal{E},\mathcal{E}) = \mathcal{K}(\mathcal{E})\subset\mathcal{L}(\mathcal{E})$ is an ideal.

Kasparov modules

Let A and B be separable C*-algebras. A Kasparov A-B-module is a triple $(\mathcal{E},\phi,F)$, where $\mathcal{E}$ is countably generated graded Hilbert B-module $\mathcal{E} = \mathcal{E}^+\oplus\mathcal{E}^-$, $\phi\colon A\to\mathcal{L}(\mathcal{E})$ is a $*$-homomorphism of degree 0, that is,

$$\phi(a) = \begin{pmatrix}\phi_+(a) & 0\\ 0 & \phi_-(a)\end{pmatrix},$$

with $*$-homomorphisms $\phi_\pm : A\to\mathcal{L}(\mathcal{E}^\pm)$, and $F\in\mathcal{L}(\mathcal{E})$ is an operator of degree one,

$$F = \begin{pmatrix}0 & P\\ Q & 0\end{pmatrix}$$

such that
$$[\phi(a), F], \qquad \phi(a)(F - F^*), \qquad \phi(a)(F^2 - \mathrm{Id})$$
are compact for all $a \in A$.
Let us consider some examples of Kasparov modules.

a) If $\phi\colon A \to B$ is a *-homomorphism, then $(B \oplus 0, \phi, 0)$ is a Kasparov A-B-module.

b) Let M be a closed manifold, and let $P\colon \Gamma(E^+) \to \Gamma(E^-)$ be an elliptic pseudodifferential operator of order zero between vector bundles $E^\pm$ over M. Moreover let Q be a parametrix for P. If we set $\mathcal{H} = L^2(E^+) \oplus L^2(E^-)$ and let $\phi\colon C(M) \to B(\mathcal{H})$ be the $*$-homomorphism given by pointwise multiplication of functions with sections, then
$$\left(\mathcal{H}, \phi, \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}\right),$$
is a Kasparov $C(M)$-$\mathbb{C}$-module.

c) Finally, let A and B be Morita–Rieffel equivalent C^*-algebras, and let ${}_A\mathcal{E}_B$ be an imprimitivity bimodule. If we write ϕ for the left action of A then $({}_A\mathcal{E}_B, \phi, 0)$ is a Kasparov A-B-module. For instance, the Hilbert space $l^2(\mathbb{N})$ with the canonical action of the compact operators yields an element in $\mathrm{KK}(\mathcal{K}, \mathbb{C})$.

KK-theory

There is an obvious notion of isomorphism of Kasparov modules. More generally, a homotopy between Kasparov A-B-modules $\mathcal{E}_0$, $\mathcal{E}_1$ is a Kasparov A-$B \otimes C[0,1]$-module $(\mathcal{E}, \phi, F)$ such that
$$(\mathcal{E}_i, \phi_i, F_i) \cong (\mathcal{E} \otimes_{\mathrm{ev}_i} B, \phi \otimes \mathrm{Id}, F \otimes \mathrm{Id}),$$
for $i = 0, 1$. Here $\mathrm{ev}_i\colon B \otimes C[0,1] \to B$ is the evaluation at i, and $\mathcal{E} \otimes_{\mathrm{ev}_i} B$ is the so-called inner tensor product of $\mathcal{E}$ and B with respect to ev_i.
Let A and B be separable C*-algebras and denote by $\mathbb{E}(A, B)$ the set of isomorphism classes of Kasparov A-B-modules. There is a binary operation on $\mathbb{E}(A, B)$ given by direct sum.
By definition, the KK-group $\mathrm{KK}(A, B)$ is the set of equivalence classes in $\mathbb{E}(A, B)$ with respect to homotopy. The set $\mathrm{KK}(A, B)$ is an abelian group with addition induced by direct sum. The zero element in this group is the class of $0 = (0, 0, 0)$.

Some properties of KK-theory

Let us state some fundamental properties of KK-theory [b-b98].

- $\mathrm{KK}(A, B)$ defines a bifunctor on the category of separable C*-algebras, covariant in B and contravariant in A.
- There exists an associative, natural product
$$\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \to \mathrm{KK}(A, C)$$

for all A, B, C. This product is called the Kasparov product, and it is by far the most important feature of Kasparov theory. Using the Kasparov product we can view KK as a category with separable C*-algebras as objects and morphism sets $\mathrm{Mor}_{\mathrm{KK}}(A, B) = \mathrm{KK}(A, B)$.

- There are several equivalent ways to define higher Kasparov groups, one possible definition is $\mathrm{KK}_n(A, B) = \mathrm{KK}(A, C_0(\mathbb{R}^n) \otimes B)$. As for ordinary K-theory there are Bott periodicity isomorphisms $\mathrm{KK}_{n+2}(A, B) \cong \mathrm{KK}_n(A, B)$, natural in A and B.
- Topological K-theory and K-homology are contained in KK-theory as a special case. In fact, one has $\mathrm{KK}_*(\mathbb{C}, A) = \mathrm{K}_*(A)$, $\mathrm{KK}_*(A, \mathbb{C}) = \mathrm{K}^*(A)$ for every separable C*-algebra A. If $A = C(X)$ where X is a finite CW-complex we obtain in this way the K-theory and K-homology of X, respectively.
- Let
$$0 \to K \to E \to Q \to 0$$
be an extension of C*-algebras with a completely positive, contractive splitting of the quotient map. Then there are exact sequences
$$\begin{tikzcd}
\mathrm{KK}_0(A,K) \arrow[r] & \mathrm{KK}_0(A,E) \arrow[r] & \mathrm{KK}_0(A,Q) \arrow[d] \\
\mathrm{KK}_1(A,Q) \arrow[u] & \mathrm{KK}_1(A,E) \arrow[l] & \mathrm{KK}_1(A,K) \arrow[l]
\end{tikzcd}$$
and
$$\begin{tikzcd}
\mathrm{KK}_0(Q,A) \arrow[r] & \mathrm{KK}_0(E,A) \arrow[r] & \mathrm{KK}_0(K,A) \arrow[d] \\
\mathrm{KK}_1(K,A) \arrow[u] & \mathrm{KK}_1(E,A) \arrow[l] & \mathrm{KK}_1(Q,A) \arrow[l]
\end{tikzcd}$$
The boundary maps in these sequences are determined by an element in $\mathrm{KK}_1(Q, K)$ naturally associated to the extension.
If Q is nuclear, then every extension $0 \to K \to E \to Q \to 0$ has a completely positive splitting.
- If A is nuclear there is a natural isomorphism $\mathrm{KK}_1(A, B) \cong \mathrm{Ext}(A, B)$.

Further developments

Universal coefficient theorem

For computations it is important that the groups $\mathrm{KK}(A, B)$ are determined by the K-groups $\mathrm{K}_*(A)$, $\mathrm{K}_*(B)$ in many cases. More precisely, the universal coefficient theorem of Rosenberg-Schochet [rs87] states that there is a short exact sequence of graded abelian groups

$$0 \to \mathrm{Ext}_*(\mathrm{K}_{*+1}(A), \mathrm{K}_*(B)) \to \mathrm{KK}_*(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to 0$$

if A is isomorphic in KK to a commutative C*-algebra.
For instance, using this in the case $A = C(X)$ where $X \subset \mathbb{C}$ is compact and $B = \mathbb{C}$ one can reprove the results of Brown-Douglas-Fillmore.

The Kasparov index theorem

Many results in index theory can be formulated and proved elegantly using KK-theory. As an example let us consider the following version of the index theorem due to Kasparov.
Let M be a closed manifold. The cotangent bundle T^*M is an almost complex manifold in a natural way. In particular, there is the Dolbeault operator $D = \bar{\partial} + \bar{\partial}^*$ which gives a class $[\bar{\partial}_M]$ in $\mathrm{KK}(C_0(T^*M), \mathbb{C})$. Now if P is an elliptic pseudodifferential operator $P\colon \Gamma(E^+) \to \Gamma(E^-)$ on M, then, as we have seen above, P defines a class $[P] \in \mathrm{KK}(C(M), \mathbb{C})$. Its symbol yields a class $[\sigma(P)] \in \mathrm{KK}(\mathbb{C}, C_0(T^*M)) = \mathrm{K}^0(T^*M)$. In fact, one may define a bivariant symbol class $[[\sigma(P)]] \in \mathrm{KK}(C(M), C_0(T^*M))$ such that $[\sigma(P)] = 1 \cdot [[\sigma(P)]]$.
The Kasparov index theorem states that

$$[P] = [[\sigma(P)]] \cdot [\bar{\partial}_M].$$

This implies the index theorem of Atiyah-Singer since

$$a\text{-}\operatorname{Index}([\sigma(P)]) = 1 \cdot [P] = 1 \cdot [[\sigma(P)]] \cdot [\bar{\partial}_M] = [\sigma(P)] \cdot [\bar{\partial}_M] = t\text{-}\operatorname{Index}([\sigma(P)]).$$

The universal property of KK

It is remarkable that KK-theory can be characterized abstractly by a universal property.
A functor F from the category of C*-algebras to an additive category $\mathcal{C}$ is called

- homotopy invariant if $F(f_0) = F(f_1)$ for f_0, f_1 homotopic *-homomorphisms,
- stable if $F(A \otimes \mathcal{K}(\mathcal{H})) \cong F(A)$ (naturally),
- split exact if for every split extension

  $$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & E \arrow[r, "\pi"'] & Q \arrow[r] \arrow[l, bend right, "\sigma"'] & 0, \end{tikzcd}$$

  where $\sigma\colon Q \to E$ is a $*$-homomorphism such that $\pi\sigma = \mathrm{id}$, there is a split exact sequence

  $$\begin{tikzcd} 0 \arrow[r] & F(K) \arrow[r] & F(E) \arrow[r] & F(Q) \arrow[r] \arrow[l, bend right, "F(\sigma)"'] & 0. \end{tikzcd}$$

A theorem due to Higson and Cuntz [h-n87] states that the obvious functor from the category C^*-**Alg** of separable C*-algebras to the category KK is the universal split exact stable homotopy functor. That is, whenever $f\colon C^*\text{-}\mathbf{Alg} \to \mathcal{C}$ is a split exact stable homotopy invariant functor, then there exists a unique additive functor $F\colon \mathrm{KK} \to \mathcal{C}$ such that the diagram

$$\begin{tikzcd} C^*\text{-}\mathbf{Alg} \arrow[r] \arrow[dr, "f"'] & \mathrm{KK} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$$

commutes.

There are many important topics that we do not have time to touch upon, in particular the equivariant versions of KK-theory and its applications to the Novikov conjecture [k-g88]. The Novikov conjecture was one of the principal motivations for the invention of KK-theory.
In a completely different direction, KK-theory plays a prominent rôle in the classification of purely infinite simple C*-algebras due to Kirchberg and Philipps. This classification is one of the deepest achievements in C*-algebra theory up to now.

Chapter 1

C*-algebras

1.1 Definitions

Definition 1.1. *A* Banach algebra *(complex) is an algebra A which is a Banach space with norm satisfying the inequality*

$$\|ab\| \leq \|a\|\|b\|, \text{ for all } a, b \in A.$$

Assume that we have an involution on Banach algebra, $*\colon A \to A$ that is for all $a, b \in A$, $\lambda, \mu \in \mathbb{C}$

$$\begin{aligned} a^{**} &= a, \\ (\lambda a + \mu b)^* &= \bar{\lambda} a^* + \bar{\mu} b^*, \\ (ab)^* &= b^* a^*. \end{aligned}$$

Definition 1.2. *A* C*-algebra *is a Banach algebra A with involution $*\colon A \to A$ which satisfies the C*-identity*

$$\|a^* a\| = \|a\|^2$$

for all $a \in A$.

We say that A is unital if there exists $1 \in A$ such that $a \cdot 1 = 1 \cdot a = a$. The involution $*$ is an isometry

$$\|a\|^2 = \|a^* a\| \leq \|a^*\|\|a\|, \quad \|a\| \leq \|a^*\|.$$

The C*-identity forces a strong connection between algebra and analysis. A **-morphism* is an algebra homomorphism $\varphi\colon A \to B$ such that $\varphi(a^*) = (\varphi(a))^*$ for all $a \in A$.

Theorem 1.3. *Let A, B be a C*-algebras (unital or not). If $\phi\colon A \to B$ is $*$-homomorphism then*

1. *for all $a \in A$ we have $\|\phi(a)\| \leq \|a\|$, i.e. ϕ is continuous with norm $\|\phi\| \leq 1$.*
2. *$\phi(A)$ is closed in B, in particular $\phi(A)$ is a subalgebra of B and the induced homomorphism $A/\ker\phi \to \phi(A)$ is an isometry. An injective C*-homomorphism is an isometry.*

1.2 Examples

Example 1.4. Let X be a locally compact Hausdorff space, and $C_0(X)$ the algebra of functions vanishing at infinity. Then with respect to conjugation and norm $\|f\| = \sup_{x\in X} |f(x)|$, the algebra $C_0(X)$ is a C*-algebra.

Example 1.5. The matrix algebra $M_n(\mathbb{C})$ is a C*-algebra. Furthermore

Theorem 1.6. *Every finite dimensional C*-algebra A is of the form* $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$.

More generally direct limits of finite dimensional C*-algebras are called *AF algebras*.

Example 1.7. Let $B(\mathcal{H})$ be tha algebra of bounded operators on Hilbert space. It is not separable unless it is finite dimensional. If $\dim \mathcal{H} = n$, then $B(\mathcal{H}) = M_n(\mathbb{C})$. If $\dim \mathcal{H} = \infty$, then there is a closed ideal of compact operators $\mathcal{K}(\mathcal{H}) \subset B(\mathcal{H})$ which takes over the role of matrices. There is an extension

$$0 \to \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to 0,$$

where the quotient algebra $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is denoted $\mathcal{Q}(\mathcal{H})$, and is called the *Calkin algebra*.

Theorem 1.8. *[Gelfand, Naimark] Every C*-algebra A admits a faithful representation on* $\mathcal{H}$ *i.e. there is an injective C*-homomorphism* $\phi\colon A \to B(\mathcal{H})$ *for some* $\mathcal{H}$*. Then* ϕ *is an isometry, so A can be identified with a C*-subalgebra of* $B(\mathcal{H})$.

Example 1.9. Let G be a discrete group (for simplicity). Its group ring $\mathbb{C}[G]$ is the ring of finitely supported functions $f\colon G \to \mathbb{C}$, $f = \sum_{g\in G} f_g\delta_g$, $f_g \in \mathbb{C}$, $\delta_g(s) = 1$ if $s = g$ and 0 otherwise. The multiplication is given by convolution

$$(f * g)(s) := \sum_{\alpha,\beta=s} f(\alpha)g(\beta) = \sum_{t\in G} f(st^{-1})g(t).$$

We have $\delta_s * \delta_t = \delta_{st}$. We will assume that G is countable and then $\{\delta_s\}_{s\in G}$ will provide a basis for $l^2(G)$. For fixed g the action of $\delta_g * -$ on $l^2(G)$ produces a permutation of $\{\delta_s\}_{s\in G}$ and so an operator $U_g\colon l^2(G) \to l^2(G)$,

$$(U_g\xi)(t) = (\delta_g * \xi)(t) = \xi(g^{-1}t)$$

The operator U_g is unitary $U_g^{-1} = U_{g^{-1}} = U_g^*$. Indeed

$$\begin{aligned}
\langle U_g\xi,\, \eta\rangle &= \sum_{t\in G}(U_g\xi)(t)\overline{\eta(t)} \\
&= \sum_{t\in G}\xi(g^{-1}t)\overline{\eta(t)} \\
&= \sum_{t'\in G}\xi(t')\overline{\eta(gt')} \\
&= \langle \xi,\, U_{g^{-1}}\eta\rangle
\end{aligned}$$

$$\|U_g\xi\|^2 = \sum_{t\in G} |\xi(g^{-1}t)|^2 = \sum_{t'\in G} |\xi(t')|^2 = \|\xi\|^2.$$

The left regular representation $\lambda\colon \mathbb{C}[G] \to \mathcal{L}(l^2(G))$

$$\lambda(f) = \sum_{g\in G} f_g U_g$$

$$\|\lambda(f)\| \le \sum_{g\in G} |f_g| = \|f\|_1$$

extends to $\lambda\colon l^1(G) \to \mathcal{L}(l^2(G))$.

Definition 1.10. *The* reduced group algebra $C^*_r(G)$ *of* G *is the norm closure* $\overline{\lambda(\mathbb{C}[G])} = \lambda(\overline{l^1(G)})$.

If G is abelian, then $C^*_r(G) = C_0(\widehat{G})$, where $\widehat{G}$ is the *Pontryagin dual*, $\widehat{G} = \mathrm{Hom}(G, \mathrm{U}(1))$.

There is a canonical trace on $\mathbb{C}[G]$

$$\tau\colon \sum_{g\in G} f_g\delta_g \mapsto f_e \in \mathbb{C}.$$

Proposition 1.11. *If* $\phi\colon G_1 \to G_2$ *is an injective group homomorphism, then there is an induced map* $\phi\colon C^*_r(G_1) \to C^*_r(G_2)$.

Let Π_U be the direct sum of all irreducible representations of G (up to unitary equivalence). The algebra $C^*(G)$ is defined as a closure of $\Pi_U(\mathbb{C}[G])$. Equivalently, if $\|f\| = \sup\{\|\pi(f)\| \mid f \in l^1(G)\}$, where the supremum is taken over all *-representations of $l^1(G)$, then $C^*(G)$ is the completion of $l^1(G)$ in this norm. Our λ extends to a C*-algebra homomorphism $\lambda\colon C^*(G) \to C^*_r(G)$. The following theorem holds for all locally compact groups.

Theorem 1.12. *The homomorphism* $\lambda\colon C^*(G) \to C^*_r(G)$ *is an isomorphism if and only if* G *is amenable.*

Proposition 1.13. *If* $\phi\colon G_1 \to G_2$ *is a group homomorphism, then there is an induced map* $\phi\colon C^*(G_1) \to C^*(G_2)$.

If X is a compact Hausdorff space, then $f \in C(X)$ is a projection if and only if $\bar{f} = f$, $f^2 = f$. It follows that $f(x) = 0$ or 1 for all $x \in X$. Denote $S_i := \{x \in X \mid f(x) = i\}$ for $i = 0, 1$. Then $S_0 \cap S_1 = \emptyset$, $S_0 \cup S_1 = X$. If F is continuous, integer valued, then δ_0, δ_1 are open and closed. So if f is a nontrivial projection, then X must be disconnected.

Conjecture 1 (Idempotent conjecture). *If* G *is discrete, torsion free, then* $\mathbb{C}[G]$ *has no nontrivial idempotents.*

Conjecture 2 (Strong idempotent conjecture, Kadison-Kaplansky conjecture). *If* G *is discrete, torsion free, then* $C^*_r(G)$ *has no nontrivial idempotents.*

Both conjectures follow from the Baum-Connes conjecture.

Example 1.14. If a locally compact group G acts on locally compact Hausdorff space X, then there is a crossed product algebra $C_0(X) \rtimes G$. When G acts freely, properly on X, then $C_0(X) \rtimes G$ is Morita equivalent to $C_0(X/G)$. Remark that X/G is not a Hausdorff space in general.

Example 1.15. We will define a *Toeplitz algebra* as $\mathcal{T} := C^*(v)$, where $v^*v = 1$ (isometry), $vv^* \neq 1$ (not unitary). There is an isomorphism $C^*(v) \cong C^*(S)$, where $S\colon l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is the shift operator

$$S(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots), \quad S^*(x_1, x_2, \ldots) := (x_2, x_3, \ldots).$$

Theorem 1.16 (Coburn). *The algebra $C^*(S)$ contains the compact operators $\mathcal{K}$ as an ideal and there is an extension*

$$0 \to \mathcal{K} \to C^*(S) \to C(S^1) \to 0,$$

where S^1 is the circle.

We can give another description using Hardy space $H^2 \subset L^2(S^1)$

$$H^2 = \operatorname{span}\{z^n \mid n \geq 0\} \text{ (closed span)}.$$

Let $P\colon L^2(S^1) \to H^2$ be the orthogonal projection. For each $f \in C(S^1)$ define an operator $T_f\colon H^2 \to H^2$, $T_f(g) = P(fg)$ for $g \in H^2$. The operator T_z, where z is the identity funcion in $C(S^1)$, acts as a shift operator on H^2, so $C^*(T_z) \cong \mathcal{T} \cong C^*(S)$.

For $f \in C(S^1)$ let M_f be the operator of pointwise multiplication by f.

Exercise 1.17. $\|M_f\| = \|f\|$.

Consider the action of $[P, M_z]$ on the basis $\{z^n \mid n \in \mathbb{Z}\}$ of $L^2(S^1)$.

$$\begin{aligned} PM_z\colon z^n &\mapsto z^{n+1}, \quad n \geq -1 \\ M_zP\colon z^n &\mapsto z^{n+1}, \quad n \geq 0. \end{aligned}$$

Both operators are zero outside this range. It follows that $[P, M_z]$ is of rank one, and $[P, M_{z^n}]$ is of rank n on $L^2(S^1)$. If p is a polynomial in z, then $[P, M_p]$ is of finite rank.

For $f \in C(S^1)$ there exist a sequence of Laurent polynomials $p_n \to f$ such that

$$\|M_{p_n} - M_f\| = \|M_{p_n - f}\| = \|p_n - f\| \to 0, \text{ and so } M_{p_n} \to M_f.$$

From this we have that $[P, M_{p_n}] \to [P, M_f]$, so $[P, M_f]$ is compact.

For $f, g \in C(S^1)$

$$\begin{aligned} T_fT_g &= PM_fPM_g \\ &= P(PM_f - [P, M_f])M_g \\ &= PM_fM_g - P[P, M_f]M_g \\ &= T_{fg} + K, \end{aligned}$$

where K is compact operator. Denote

$$B := \{T_f + K \mid f \in C(S^1),\ K \in \mathcal{K}\}.$$

Theorem 1.18 (Coburn). *There is an isomorphism $B \cong C^*(T_z) \cong \mathcal{T}$.*

The map $f \mapsto \pi(T_f) \in Q$, where $\pi B(\mathcal{H}) \to Q$ is a projection on Calkin algebra, gives an isomorphism $C(S^1) \cong C^*(T_z)/\mathcal{K}$. Furthermore

$$\pi(T_f)\pi(T_g) = \pi(T_f T_g) = \pi(T_{fg} + K) = \pi(T_{fg}).$$

Consider the Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0.$$

We may ask whether there are other extensions

$$0 \to \mathcal{K} \to \mathcal{E} \to C(S^1) \to 0$$

not equivalent to the Toeplitz extension. The example is $\mathcal{E} = \mathcal{C}$, where

$$\mathcal{C} := \{M_f + K \mid f \in C(S^1),\ K \in \mathcal{K}\}.$$

There is no *-isomorphism $\mathcal{T} \to \mathcal{C}$. Now we can ask about the classification of such extensions. The answer was given by Brown, Douglas and Filmore, who introduced Ext-groups, which have relation with K-homology.

Example 1.19. More general construction than the Toeplitz algebra are the *Cuntz algebras* $\mathcal{O}_n$. These are generated by $S_1, \ldots, S_n$ such that $S_i^* S_i = 1$ (isometries), $\sum_{i=1}^n S_i S_i^* = 1$. The algebras O_n are unique up to isomorphism, simple, purely infinite for $n \geq 2$. There exist an extension $\mathcal{E}_n$

$$0 \to \mathcal{K} \to \mathcal{E}_n \to O_n \to 0.$$

We recall that:

Definition 1.20. *A projection $p \in A$ is* infinite *if p is equivalent to a proper subprojection of itself. Otherwise it is called* finite.

A simple C^*-algebra is purely infinite *if and only if the closure of xAx contains an infinite projection for every positive $x \in A$.*

Example 1.21. Noncommutative Riemann surfaces. Let Γ_g be a fundamental group of compact oriented Riemann surface Σ_g of genus $g \geq 1$.

$$\Gamma_g = \{u_j v_j \mid j = 1, \ldots, g,\ \prod_{j=1}^{g} [u_j, v_j] = 1\},$$

$$\mathrm{B}\,\Gamma_g = \Sigma_g,\ \mathrm{H}^2(\Gamma_g; \mathrm{U}(1)) = \mathbb{R}/\mathbb{Z}.$$

For all $\theta \in [0,1)$ there is a cocycle $\delta_\gamma * \delta_\mu = \sigma_\theta(\gamma, \mu)\delta_{\gamma\mu}$. By completion in operator norm we get $C_r^*(\Gamma_g, \sigma_\theta)$.

We can give an alternative description by unitaries u_j, v_j such that $\prod_{j=1}^g [u_j, v_j] = e^{2\pi i \theta}$. Noncommutative torus is a special case for $g = 1$.

1.3 Gelfand transform

Let A be a unital C*-algebra. For an element $a \in A$ we define its *spectrum* as

$$\mathrm{sp}_A(a) := \{\lambda \in \mathbb{C} \mid \lambda 1 - a \text{ is not invertible}\},$$

and the *resolvent* as

$$\rho_A(a) := \mathbb{C} \setminus \mathrm{sp}_A(a).$$

The *spectral radius* of an element is

$$r(a) := \sup\{|\lambda| \mid \lambda \in \mathrm{sp}_A(a)\}, \quad r(a) \leq \|a\|.$$

Proposition 1.22. *1. If A is a Banach algebra, then for every $a \in A$*

$$\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = r(a).$$

2. If A is a C^*-algebra, and $a \in A$ is a normal element ($a^*a = aa^*$), then $r(a) = \|a\|$.

3. If A is a C^*-algebra, then for every $a \in A$

$$\|a\|^2 = r(a^*a).$$

Let B be a C*-algebra, $a \in B$. Consider C*-algebra $C^*(a)$ generated by a (when B is unital we assume $1 \in C^*(a)$). The algebra $C^*(a)$ is commutative if and only if a is normal. Define

$$\widehat{A} := \{\phi\colon A \to \mathbb{C} \mid \phi \text{ is a homomomorphism}, \|\phi\| \leq 1\}.$$

Definition 1.23. *Let A be a commutative C^*-algebra.The* Gelfand transform *is the homomorphism*

$$A \to C_0(\widehat{A}), \quad a \mapsto \widehat{a},$$
$$\widehat{a}(\phi) := \phi(a).$$

Theorem 1.24 (Gelfand)**.** *If A is commutative, then the Gelfand transform is an isometric *-isomorphism form A to $C_0(\widehat{A})$.*

Corollary 1.25. *If a is normal element of a C^*-algebra A, then the Gelfand transform gives an isometric *-isomorphism $C^*(a) \to C(\mathrm{sp}(a))$.*

Definition 1.26. *If a is a normal element in a unital C^*-algebra A and $f \in C(\mathrm{sp}(a))$, then the inverse of Gelfand transform $f \mapsto f(a) \in C^*(a)$ is called the functional calculus for a.*

Example 1.27. Let A be a C*-algebra, $u \in A$ unitary element. Then $\mathrm{sp}(u) \subset S^1$. Assume $\mathrm{sp}(u) \subsetneq S^1$. Take a branch of logarithm defined on subset of S^1 containing $\mathrm{sp}(u)$. Use functional calculuc to define a family of unitary groups $u_t := \exp(t \log u)$, $t \in [0, 1]$. This family constitutes a continuous path which connects u to the identity through unitaries.

There is also a *holomorphic functional calculus.* Let A be a unital Banach algebra, $a \in A$. Assume that f is a holomorphic sunction in on an open set containing $\mathrm{sp}(a)$. Choose a piecewise linear closed curve C in that set, but not intersecting $\mathrm{sp}(a)$. Then

$$f(a) := \frac{1}{2\pi i} \int_C f(z)(z-a)^{-1} dz$$

defines an element of A. If $H(a)$ is the set of holomorphic functions of this type, then this gives an algebra homomorphism $H(a) \to A$ - holomorphic functional calculus.

If A is a subalgebra of a Banach algebra B, and $\widetilde{A}$, $\widetilde{B}$ are unitizations, then we say that A is stable under holomorphic functional calculus if and only if for any $a \in A$, and f which is holomorphic in an open set containing $\mathrm{sp}_{\widetilde{B}}(a)$, we have $f(a) \in \widetilde{a}$.

Proposition 1.28. *Let A be a C^*-algebra. Then for any $x \in A$ the following are equivalent*

1. $x = x^*$, $\mathrm{sp}(x) \subset \mathbb{R}_+$,
2. *there exists $y \in A$ such that $x = y^* y$,*
3. *there exists $y \in A$ such that $y = y^*$, $y^2 = x$.*

If x satisfies any of thers, then we say that it is positive *and write $x \geq 0$.*

If $x \geq 0$ and $-x \geq 0$ then $x = 0$. Positivity induces a partial order on elements of A. We say that $x \leq y$ if and only if $y - x \geq 0$. Positive elements form a cone $A_+ \subset A$. For projections p, q we have $p \leq q$ if and only if $pq = p$.

Now we will define tensor products of C*-algebras. Let A, B be C*-algebras and $A \odot B$ be the algebraic tensor product (as vector spaces). The vector space $A \odot B$ is a *-algebra

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb', \quad (a \otimes b)^* = a^* \otimes b^*.$$

C*-algebra norm on $A \odot B$ is a cross norm $\| - \|_\alpha$, $\|a \otimes b\|_\alpha = \|a\|\|b\|$, and satisfies

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha, \quad \|x^* x\|_\alpha = \|x\|_\alpha^2.$$

A completion of $A \odot B$ with respect to such norm is a C*-algebra $A \otimes_\alpha B$. Let $\pi \colon A \to B(\mathcal{H})$, $\sigma \colon B \to \mathcal{L}(\mathcal{H}')$ be faithful representations. The algebraic tensor product gives a representation

$$\pi \odot \sigma \colon A \odot B \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}'),$$

$$((\pi \odot \sigma)(a \otimes b))(\xi \otimes \eta) = \pi(a)\xi \otimes \sigma(b)\eta.$$

Define a *minimal norm* $\|x\|_{min} := \|(\pi \odot \sigma)(x)\|_{\mathcal{L}(\mathcal{H} \otimes \mathcal{H}')}$. The theorem of Takesaki states that this definition does not depend on π, σ.

Definition 1.29. *A C^*-algebra A is* nuclear *if and only if for any C^*-algebra B there is a unique C^*-norm on $A \odot B$.*

A is exact *if and only if the functor $B \mapsto A \otimes_{min} B$ is exact (i.e. sends exact sequences of C^*-algebras to exact sequences).*

Theorem 1.30 (Kirchberg-Wassermann)**.** *A discrete group G is exact if and only if $C^*_r(G)$ is exact.*

Nuclear algebras are exact. For a free group on two generators F_2 the reduced group algebra $C^*_r(F_2)$ is exact but not nuclear. The full C*-subalgebra $C^*(F_2)$ of the nonabelian free group on two generators is not exact.

Proposition 1.31. *The reduced group algebra $C^*_r(G)$ is nuclear if and only if G is amenable.*

Maximal tensor product $\otimes_{max}$ has the following universal property. There is a natural bijection between non degenerate C*-homomorphisms

$$A_1 \otimes_{max} A_2 \to B(\mathcal{H}),$$

and pairs of commuting non degenerate C*-homomorphisms

$$A_1 \to B(\mathcal{H}), \quad A_2 \to B(\mathcal{H}).$$

One can also replace $B(\mathcal{H})$ be the multiplier algebra $\mathcal{M}(D)$ for any C*-algebra D.

There is a canonical C*-algebra homomorphism

$$A_1 \otimes_{max} A_2 \to A_1 \otimes_{min} A_2$$

for any C*-algebras A_1, A_2. We can give a second definition

Definition 1.32. *A C*-algebra A_1 is nuclear if this map is an isomorphism for any C*-algebra A_2.*

One can say that A_1 is K-nuclear if this map induces an isomorphism on K-theory for any C*-algebra A_2.

Chapter 2

K-theory

2.1 Definitions

Definition 2.1. *If A is a unital C*-algebra, then $p \in A$ is a* projection *if and only if $p^* = p$, $p^2 = p$.*

Definition 2.2. *Let $p, q \in A$ be a projections. We say that they are*

1. Murray-von Neumann equivalent*, $p \sim_v q$, if there exist $v \in A$ such that $p = v^*v$, $q = vv^*$.*
2. unitarily equivalent*, $p \sim_u q$, if there exist a unitary $u \in A$ such that $upu^* = q$.*
3. homotopic*, $p \sim_h q$, if there exist a continuous map $\gamma\colon [0,1] \to A$ such that $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma(t)$ is a projection for all $t \in [0,1]$.*

In a general C*-algebra there are implications

$$p \sim_h q \implies p \sim_u q \implies p \sim_v q.$$

Let $M_\infty(A) = \bigcup_{n \geq 1} M_n(A)$. Then these three notions of equivalence coincide in $M_\infty(A)$.

Denote by $P(A)$ the set of projections in $M_\infty(A)$. We have the following structure:

- Semigroup, for $p \in M_n(A)$, $q \in M_n(A)$
$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{n+m}(A).$$
- A projection $p \in M_n(A)$ is equivalent to $q \in M_m(A)$, $n \leq m$, if and only if $p \oplus 0_{m-n} \sim q$ in $M_m(A)$.
- Projections p and q are stably isomorphic if and only if $p \oplus r \sim q \oplus r$ for some projection $r \in P(A)$.
- The set of stable equivalence classes of projections in $P(A)$ with the addition induced from $P(A)$ is denoted by $[P(A)]$.

- Two pairs $([p_1],[p_2])$ and $([q_1],[q_2])$ are equivalent if and only if
$$[p_1]\oplus[q_2]=[p_2]\oplus[q_1].$$

Definition 2.3. *The set of equivalence classes of pairs* $([p_1],[p_2])$ *with componentwise addition is an abelian group denoted by* $\mathrm{K}_0(A)$.

Example 2.4. If $A=\mathbb{C}$, then two projections in $M_n(\mathbb{C})$ are homotopic if and only if they have the same rank. It follows that $\mathrm{K}_0(\mathbb{C})=\mathbb{Z}$.

Example 2.5. If $\mathcal{H}$ is a separable Hilbert space, and $A=B(\mathcal{H})$ is the algebra of bounded operators on $\mathcal{H}$, then two projections $p,q\in B(\mathcal{H})$ are equivalent in the sense of Murray- von Neumann if and only if there exists a unitary isomorphism from the range of p to the range of q. The set of projections in $B(\mathcal{H})$ can be indexed by the dimension of the range (including 0 and ∞). Thus any two projections of infinite range are equivalent. If $p\in B(\mathcal{H})$ is any projection, then $p\oplus 1\sim 0\oplus 1$, $[p]+[1]=[0]+[1]$ in $\mathrm{K}_0(A)$, so $[p]=[0]=0$ in $\mathrm{K}_0(A)$, and $\mathrm{K}_0(B(\mathcal{H}))=0$.

Proposition 2.6.

1. K_0 *is a covariant functor. If* $\phi\colon A\to B$ *is a homomorphism of C*-algebras, then there is an induced map* $\phi_*\colon \mathrm{K}_0(A)\to\mathrm{K}_0(B)$.
2. *If* $\phi_0,\phi_1\colon A\to B$ *are homotopic homomorphisms then* $\phi_{0*}=\phi_{1*}\colon \mathrm{K}_0(A)\to\mathrm{K}_0(B)$.
3. *If* A *is a unital C*-algebra and* $A_1\subset A_2\subset A_3\subset\ldots$ *is an increasing sequence of unital C*-algebras whose union is dense in* A *then* $\varinjlim \mathrm{K}_0(A_n)=\mathrm{K}_0(A)$.

For any nonunital C*-algebra J there exists an unique (up to isomorphism) unital C*-algebra $\widetilde{J}$ which contains J as an ideal of codimension 1.
$$0\to J\to\widetilde{J}\to\mathbb{C}\to 0.$$
Define $\mathrm{K}_0(J):=\ker(\mathrm{K}_0(\widetilde{J})\to\mathrm{K}_0(\mathbb{C}))$. When J is unital, then $\mathrm{K}_0(\widetilde{J})=\mathrm{K}_0(\mathbb{C})\oplus\mathrm{K}_0(J)$.

2.2 Unitizations and multiplier algebras

There are at least two ways to adjoin a unit to a C*-algebra A.

1. Represent A on a Hilbert space $\mathcal{H}$. The image of A in $B(\mathcal{H}))$ may not contain 1, even if A is unital, as the following example shows
$$\mathbb{C}\to M_2(\mathbb{C}),\quad \mu\mapsto\begin{pmatrix}\mu & 0\\ 0 & 0\end{pmatrix}.$$
Let $\widetilde{A}$ be the C*-subalgebra of $B(\mathcal{H})$ generated by A and 1. It contains 1 as an ideal of codimension 1.
2. Use the left multiplication to represent A on the Banach space A. Regard $\widetilde{A}$ as generated by A and 1.

Is there a reasonable maximal unitization?

Definition 2.7. *A is an* essential ideal *in a C*-algebra B if and only if for all $b \in B$ if $bA = \{0\}$ then $b = 0$.*

There is a unique (up to isomorphism) unital C*-algebra which contains A as an essential ideal and is maximal in the sense that it contains any other algebra with this property. This is the *multiplier algebra* $M(A)$.

We will give an interpretation of the two, minimal and maximal, unitizations, in the case of commutative C*-algebras. Let $A = C_0(X)$, and B a unital commutative C*-algebra, $B = C(Y)$ for a compact space Y. Then the inclusion

$$A = C_0(X) \hookrightarrow C(Y) = B$$

corresponds to inclusion of X as an open subset in Y, and is given by extension by 0. Then A is essential in B if and only if X is dense in Y, that is Y is a compactification of X. The minimal choice of compactification is the one-point compactification X^+. Then $B = \widetilde{A}$. The maximal choice is the Stone Čech compactification βX. Then $M(C_0(X)) = C(\beta X)$.

2.3 Stabilization

Stabilization map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

is an example of a nonunital C*-algebra morphism $A \to M_n(A)$ even when A is unital.

Proposition 2.8. *The stabilization map induces an isomorphism in K-theory for all n.*

Proof. For all k there is an isomorphism $M_k(M_n(A)) \cong M_{kn}(A)$, so any matrix in $M_k(M_n(A))$ can be regarded as a projection in $M_{kn}(A)$ which provides the two-sided inverse to the stabilization map. □

Example 2.9. Take $M_2(\mathbb{C}) \subset M_4(\mathbb{C}) \subset M_8(\mathbb{C}) \subset \ldots$. The direct limit $\bigcup_{n\geq 1} M_{2^n}(\mathbb{C})$ is dense in $\mathcal{K}$, so

$$\varinjlim \mathrm{K}_0(M_{2^n}(\mathbb{C})) = \mathrm{K}_0(\mathcal{K}) \implies \mathrm{K}_0(\mathcal{K}) = \mathbb{Z}.$$

By applying similar argument to $M_n(A)$ we get the following stability property.

Proposition 2.10. *For any C*-algebra A and the algebra of compact operators $\mathcal{K}$ there is an isomorphism*

$$\mathrm{K}_0(A) = \mathrm{K}_0(A \otimes \mathcal{K}).$$

2.4 Higher K-theory

Let A be a unital C*-algebra. Define the *cone* of A as a C*-algebra

$$CA := \{f\colon [0,1] \to A \mid f \text{ is continuous, } f(0) = 0\}.$$

This is a contractible algebra, and a map $\phi_s\colon CA \to CA$ given by

$$\phi_s(f)(t) = f(ts), \quad s \in [0,1]$$

gives a homotopy between $\mathrm{id}\colon A \to A$ ($s = 1$) and $0\colon A \to 0$ ($s = 0$).

Define the *suspension* of A as a C*-algebra

$$SA := \{f \in CA \mid f(1) = 0\}.$$

There is a suspension extension

$$0 \to SA \to CA \to A \to 0.$$

Definition 2.11. *The* higher K-theory *groups are defined by*

$$\begin{aligned} \mathrm{K}_1(A) &:= \mathrm{K}_0(SA) = \mathrm{K}_0(C_0(\mathbb{R}) \otimes A) \\ \mathrm{K}_p(A) &:= \mathrm{K}_0(S^pA) = \mathrm{K}_0(C_0(\mathbb{R}^p) \otimes A) \end{aligned}$$

2.5 Excision and relative K-theory

Let J be an ideal in a C*-algebra A,

$$0 \to J \to A \to A/J \to 0.$$

Then the induced sequence of K_0-groups

$$\mathrm{K}_0(J) \to \mathrm{K}_0(A) \to \mathrm{K}_0(A/J)$$

is exact in the middle (half-exactness). If the sequence is split-exact, then K_0 is additive, $\mathrm{K}_0(A) = \mathrm{K}_0(J) \oplus \mathrm{K}_0(A/J)$.

Definition 2.12. *A* relative cycle *is a triple (p, q, x), where p, q are projections in $M_n(A)$ for some n, and $x \in M_n(A)$ is such that $\pi(x) \in M_n(A/J)$ for $\pi\colon A \to A/J$ is a partial isometry implementing the Murray-von Neumann equivalence between $\pi(p)$ and $\pi(q)$.*

Such a triple is nondegenerate *if and only if x provides the Murray-von Neumann equivalence between p and q.*

Definition 2.13. Relative K-theory *group $\mathrm{K}_0(A, A/J)$ is the abelian group with one generator $[p, q, x]$ for each relative cycle modulo homotopy equivalence and degeneracy.*

If J is an ideal in a unita algebra A, then $\widetilde{J}$ may be regarded as a subalgebra of A. The *excision map* is a homomorphism

$$\mathrm{K}_0(J) = \mathrm{K}_0(\widetilde{J}, \mathbb{C}) \to \mathrm{K}_0(A, A/J).$$

Theorem 2.14. *The excision map $\mathrm{K}_0(J) \to \mathrm{K}_0(A, A/J)$ is an isomorphism.*

Example 2.15. Let D be the open unit disc in $\mathbb{R}^2$, $A = C(\overline{D})$. Let $J = C_0(D)$ - continuous functions on $\overline{D}$ which vanish on ∂D. Then $A/J = C(\partial D)$.

The inclusion $\overline{D} \hookrightarrow \mathbb{C}$ can be regarded as an element of A. The triple $(1, 1, \bar{z})$ defines a relative K-cycle in $\mathrm{K}_0(C(\overline{D}), C(\partial D))$. By excision this gives an element of $\mathrm{K}_0(C_0(D))$. Since $D \cong \mathbb{R}^2$ we have an element $b \in \mathrm{K}_0(C_0(\mathbb{R}^2))$. This is the *Bott generator*. Under the isomorhism $\mathrm{K}_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$, the Bott generator b is mapped to $1 \in \mathbb{Z}$.

Definition 2.16. *The* mapping cone *of a surjective morphism* $\pi\colon A \twoheadrightarrow B$ *of* C^**-algebras is the* C^**-algebra*

$$C(A, B) := \{(a, f) \mid a \in A,\ f\colon [0, 1] \to B \text{ is continuous },\ f(0) = 0,\ f(1) = \pi(a)\}.$$

If $\pi = \mathrm{id}\colon A \to A$ then $C(A, A) = CA$. This construction is useful in the following situation. If J is an ideal in A, $\pi\colon A \to A/J$, we get $C(A, A/J)$. There is a map $C(A, A/J) \to A$, $(a, f) \mapsto a$. An element (a, f) is in the kernel of this map if and only if $a = 0$ and $f(1) = 0$. Since $f(0) = 0$ by definition, this means that $f \in S(A/J)$. Thus we have the following exact sequence

$$0 \to S(A/J) \to C(A, A/J) \to A \to 0,$$

where the first map is given by $f \mapsto (0, f)$.

There is also a homomorphism $J \to C(A, A/J)$ given by $a \mapsto (a, 0)$.

Proposition 2.17. *Excision map* $\mathrm{K}_0(J) \to \mathrm{K}_0(C(A, A/J))$ *is an isomorphism.*

By applying K_0 to the above exact sequence we get

$$0 \to \mathrm{K}_0(S(A/J)) \to \mathrm{K}_0(C(A, A/J)) \to \mathrm{K}_0(A) \to 0.$$

Using the definition of K_1 and the isomorphism in proposition we can write a sequence

$$\mathrm{K}_1(A/J) \to \mathrm{K}_0(J) \to \mathrm{K}_0(A) \to \mathrm{K}_0(A/J),$$

which is exact at $\mathrm{K}_0(J)$ and $\mathrm{K}_0(A)$. By iterating this we obtain

Proposition 2.18. *Let* $0 \to J \to A \to A/J \to 0$ *be a short exact sequence of* C^**-algebras. Then there is a natural exact sequence of abelian groups.*

$$\ldots \to \mathrm{K}_{n+1}(A/J) \to \mathrm{K}_n(J) \to \mathrm{K}_n(A) \to \mathrm{K}_n(A/J) \to \mathrm{K}_{n-1}(J) \to \ldots \to \mathrm{K}_0(A/J).$$

Example 2.19. Consider a Hilbert space $\mathcal{H}$ and an exact sequence

$$0 \to \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) \to 0,$$

where $\mathcal{Q}(\mathcal{H})$ is the Calkin algebra. Take $T \in B(\mathcal{H})$ such that $T^*T - 1 \in \mathcal{K}(\mathcal{H})$ and $TT^* - 1 \in \mathcal{K}(\mathcal{H})$ (T is *essenitally unitary*). Then $(1, 1, T)$ is a relative K-cycle for $(B(\mathcal{H}), \mathcal{Q}(\mathcal{H}))$,

$$\pi(T)^*\pi(T) = 1, \quad \pi(T)\pi(T)^* = 1.$$

By excision and computation of $\mathrm{K}_0(\mathcal{K}(\mathcal{H}))$ we have

$$\mathrm{K}_0(B(\mathcal{H}), \mathcal{Q}(\mathcal{H})) = \mathrm{K}_0(\mathcal{K}(\mathcal{H})) = \mathbb{Z}, \quad [T] \mapsto m \in \mathbb{Z}.$$

Let p be an orthogonal projection onto $\ker T$, and q an orthogonal projection onto $\ker T^*$. Then

$$(1,1,T) = (p,q,0) + (1-p, 1-q, T(1-p)).$$

The second cycle is degenerated because T restricts to an invertible operator from $\operatorname{im}(1-p)$ to $\operatorname{im}(1-q)$. The cycle $(p,q,0) \in \mathrm{K}_0(\widehat{\mathcal{K}}, \mathbb{C})$ corresponds to

$$\dim \operatorname{im} p - \dim \operatorname{im} q = \operatorname{Index}(T).$$

To summarise, the relative K-theory leads to half-exactness of K-theory and the cone construction provides the connecting homomorphism ∂ in the and long exact sequence in K-theory. Bott periodicity provides a six term exact sequence

$$\begin{tikzcd}
\mathrm{K}_0(J) \arrow[r] & \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(A/J) \arrow[d] \\
\mathrm{K}_1(A/J) \arrow[u] & \mathrm{K}_1(A) \arrow[l] & \mathrm{K}_1(J) \arrow[l]
\end{tikzcd}$$

We will give a more explicit description of $\mathrm{K}_1(A)$.

Definition 2.20. *Let A be a unital C^*-algebra. Denote by $\mathrm{K}_1^u(A)$ the abelian group with one generator for each unitary matrix in $\mathrm{GL}_n(A)$, subject to the following relations.*

1. *If $u, v \in \mathrm{GL}_n(A)$ can be joined by a path of unitaries in $\mathrm{GL}_n(A)$ then $[u] = [v]$.*
2. *$[1] = [0]$.*
3. *$[u] + [v] = [u \oplus v]$*

For unitaries $u, v \in \mathrm{GL}_n(A)$ we write $u \sim v$ if u and v can be joined by a path of unitaries. Then $u \oplus 1 \sim 1 \oplus u$ by using

$$R_t \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} R_t^*, \quad R_t = \begin{pmatrix} \cos\frac{\pi t}{2} & \sin\frac{\pi t}{2} \\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{pmatrix}.$$

Furthermore

$$u \oplus v \sim uv \oplus 1 \sim vu \oplus 1, \quad u \oplus u^* \sim 1 \oplus 1$$

$$[u] + [v] = [u \oplus v] = [uv \oplus 1] = [uv],$$

so addition in $\mathrm{K}_1^u(A)$ corresponds to matrix product.

Proposition 2.21. *For a unital C^*-algebra A*

$$\mathrm{K}_1^u(A) \cong \mathrm{K}_0(SA) = \mathrm{K}_1(A).$$

2.6 Products

For any unital C*-algebras A_1, A_2 there exists a bilinear associative product

$$\times\colon \mathrm{K}_i(A_1) \times \mathrm{K}_j(A_2) \to \mathrm{K}_{i+j}(A_1 \otimes_{min} A_2)$$

defined as follows.

1. If q_1, q_2 are projections in $M_k(A_1)$, $M_p(A_2)$, then $q_1 \otimes q_2$ is a projection in $M_{kp}(A_1 \otimes_{min} A_2)$ using $M_k(\mathbb{C}) \otimes M_p(\mathbb{C}) \cong M_{kp}(\mathbb{C})$.

2. This gives rise to the product

$$\mathrm{K}_0(A_1) \otimes \mathrm{K}_0(A_2) \to \mathrm{K}_0(A_1 \otimes A_2).$$

3. This extends to nonunital algebras.

4. Now use suspension and the isomorphism $S^i A_1 \otimes S^j A_2 \cong S^{i+j}(A_1 \otimes A_2)$ to get

$$\mathrm{K}_i(A_1) \otimes \mathrm{K}_j(A_2) \to \mathrm{K}_{i+j}(A_1 \otimes A_2).$$

2.7 Bott periodicity

Let $b \in \mathrm{K}_2(\mathbb{C}) = \mathrm{K}_0(C_0(\mathbb{R}^2))$ be the Bott generator. Taking the exterior product with b defines a map

$$\beta_A\colon \mathrm{K}_0(A) \to \mathrm{K}_0(A \otimes C_0(\mathbb{R}^2)) = \mathrm{K}_2(A).$$

Theorem 2.22 (Bott periodicity). *For every C*-algebra A, the map β_A is an isomorphism.*

Proof. We shall use the Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0$$

Proposition 2.23. *The tensor product of a short exact sequence*

$$0 \to \mathcal{I}_1 \to A_1 \to A_1/\mathcal{I}_1 \to 0$$

with a C-algebra A_2 i.e. a sequence*

$$0 \to \mathcal{I}_1 \otimes A_2 \to A_1 \otimes A_2 \to A_1/\mathcal{I}_1 \otimes A_2 \to 0$$

remains exact if either

1. *the surjection $A_1 \to A_1/\mathcal{I}_1$ has completely positive section $s\colon A_1/\mathcal{I}_1 \to A_2$, or*

2. *A_2 is nuclear.*

A linear map $f\colon A \to B$ of C*-algebras is *positive* if and only if $f(x) \geq 0$ for all $x \geq 0$. It is *completely positive* if and only if $f_n\colon M_n(A) \to M_n(B)$, $(a_{ij}) \mapsto (f(a_{ij}))$ is positive for all n.

Proposition 2.24. *The Toeplitz extension has completely positive section* $C(S^1) \to \mathcal{T}$, $f \mapsto T_f$.

Remark that the map $f \mapsto T_f$ is not an algebra homomorphism.

Using the two propositions above we get that for every C*-algebra A ther is an exact sequence.

$$0 \to \mathcal{K} \otimes A \to \mathcal{T} \otimes A \to C(S^1) \otimes A \to 0.$$

The boundary map of this sequence is

$$\partial \colon \mathrm{K}_1(C(S^1) \otimes A) \to \mathrm{K}_0(\mathcal{K} \otimes A) \cong \mathrm{K}_0(A).$$

Regard S^1 as a one-point compactification of $\mathbb{R}$. Restrict to $C_0(\mathbb{R}) \otimes A$. Then we have

$$\alpha_A \colon \mathrm{K}_2(A) = \mathrm{K}_1(C_0(\mathbb{R}) \otimes A) \to \mathrm{K}_0(A).$$

We will prove, after Atiyah, that α_A is an inverse to β_A with respect to the exterior product. The proof depends on the following formal properties of α_A

1. $\alpha_{\mathbb{C}}(b) = 1$. If u is a unitary-valued function on S^1, then $\alpha_{\mathbb{C}} \colon [u] \to \operatorname{Index} T_u$ is the minus winding number of u. Furthermore $b = (1, 1, \bar{z}) \mapsto 1$.

2. for all A, B the following diagram is commutative

$$\begin{tikzcd}
\mathrm{K}_2(A) \otimes \mathrm{K}_0(B) \arrow[r] \arrow[d, "\alpha_A \otimes \mathrm{id}"'] & \mathrm{K}_2(A \otimes B) \arrow[d, "\alpha_{A \otimes B}"] \\
\mathrm{K}_0(A) \otimes \mathrm{K}_0(B) \arrow[r] & \mathrm{K}_0(A \otimes B)
\end{tikzcd}$$

(α_A is right linear over $\mathrm{K}_0(B)$, $\alpha_{A \otimes B}(x \times y) = \alpha_A(a) \times Y$).

We have from (1) that $\alpha_A \beta_A = \mathrm{id}$ for $A = \mathbb{C}$. In general if $x \in \mathrm{K}_0(A)$ then from (2)

$$\alpha_A \beta_A(x) = \alpha_A(b \times x) = \alpha_A(b \times x) = \alpha_{\mathbb{C}}(b) \times x = 1 \times x = x.$$
$$\alpha_{\mathbb{X} \otimes A}(b \times x) = \alpha_{\mathbb{C}}(b) \times x = 1 \times x = x.$$

Thus β_A is injective. The idea of Atiyah's proof is to use $\alpha_A \beta_A = \mathrm{id}$ to prove that $\beta_A \alpha_A = \mathrm{id}$. Consider two flip isomorphisms:

$$\sigma \colon A \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes A$$
$$\tau \colon C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2)$$

which interchange the first and last terms in the tensor products.

For any $y \in \mathrm{K}_0(A \otimes C_0(\mathbb{R}^2))$

$$\tau_*(b \times y) = \sigma_*(y) \times b.$$

The map induced by τ on K-theory is the identity. Now

$$y = \alpha_{A \otimes C_0(\mathbb{R}^2)}(\beta_{A \otimes C_0(\mathbb{R}^2)}(y)) = \alpha_{A \otimes C_0(\mathbb{R}^2)}(b \times y) = \alpha_{A \otimes C_0(\mathbb{R}^2)}(\sigma_*(y) \times b) = \alpha_A(\sigma_*(y)) \times b$$

Applying σ_* to both sides we obtain

$$\sigma_*(y) = \sigma_* \beta_A \alpha_A \sigma_*(y).$$

But $\sigma_*^2 = \mathrm{id}$ and y was arbitrary, so $\beta_A \alpha_A = \mathrm{id}$. $\square$

2.8 Cuntz's proof of Bott periodicity

We will give another proof of Bott periodicity, due to Cuntz. Let E be a functor on some class of C*-algebras which is

1. homotopy invariant,
2. half exact,
3. stable.

Then one can define higher E-functors E_n, $n \geq 0$. Moreover E is additive, that is if $\phi_1, \phi_2 \colon A \to B$ are C*-algebra morphisms such that $\phi_1(A)\phi_2(A) = 0$ then $\phi_1 + \phi_2 \colon A \to B$ is a C*-algebra morphism and $E(\phi_1 + \phi_2) = E(\phi_1) + E(\phi_2)$.

Theorem 2.25 (Cuntz)**.** *Let E be a functor with these properties. Then E satisfies Bott periodicity $E_2(A) \cong E_0(A)$ for every C*-algebra for which E is defined.*

Proof. We start with Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\sigma} C(S^1) \to 0$$

Define $p \colon \mathcal{T} \to \mathbb{C}$ as the composition

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{\sigma} & C(S^1) & \xrightarrow{\varepsilon_1} & \mathbb{C} \\ & & & & \\ T_f & \longmapsto & f & \longmapsto & f(1) \end{array}$$

Then p has a right inverse $j \colon \mathbb{C} \to \mathcal{T}$. We want to prove, that $E(p) \colon E(\mathcal{T}) \to E(\mathbb{C})$, $E(j) \colon E(\mathbb{C}) \to E(\mathcal{T})$ are inverses of each other. The easy part is $\mathrm{id} = E(p \circ j) = E(p) \circ E(j)$ because $p \circ j \colon \mathbb{C} \to \mathbb{C}$ is the identity map.

Proposition 2.26. *The maps $E(j) \colon E(\mathbb{C}) \to E(\mathcal{T})$ and $E(p) \colon E(\mathcal{T}) \to E(\mathbb{C})$ estabilish an isomorphism $E(\mathbb{C}) \cong E(\mathcal{T})$. Moreover for any C*-algebra the maps*

$$\begin{aligned} &\mathrm{id}_A \otimes j \colon A = A \otimes \mathbb{C} \to A \otimes \mathcal{T} \\ &\mathrm{id}_A \otimes p \colon A \otimes \mathcal{T} \to A \end{aligned}$$

estabilish an isomorphism $E(A) \cong E(A \otimes \mathcal{T})$.

Granted the proposition, the proof proceeds as follows. The extension

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{T}_0 \arrow[r, "i"] & \mathcal{T} \arrow[r, "p"] & \mathbb{C} \arrow[r] \arrow[l, bend left, "j"] & 0 \end{tikzcd}$$

where by definition $\mathcal{T}_0 = \ker p$, is split and the sequence

$$0 \to A \otimes \mathcal{K} \to A \otimes \mathcal{T}_0 \to A \otimes C(S^1) \to 0$$

is exact. By proposition $E(\mathcal{T}_0 \otimes A) = 0$, so $E_0(A \otimes \mathcal{K}) \cong E_1(A \otimes C_0(\mathbb{R})) = E_2(A)$. $\square$

2.9 The Mayer–Vietoris sequence

Assume we have the pull-back diagram

$$\begin{tikzcd} A \arrow[r, "q_2"] \arrow[d, "q_1"'] & A_2 \arrow[d, "p_2"] \\ A_1 \arrow[r, "p_1"'] & B \end{tikzcd}$$

$$A = \{(a_1, a_2) \in A_1 \oplus A_2 \mid p_1(a_1) = p_2(a_2)\}.$$

Then there is an exact sequence

$$\begin{tikzcd} \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(A_1) \oplus \mathrm{K}_0(A_2) \arrow[r] & \mathrm{K}_0(B) \arrow[d] \\ \mathrm{K}_1(B) \arrow[u] & \mathrm{K}_1(A_1) \oplus \mathrm{K}_1(A_2) \arrow[l] & \mathrm{K}_1(A) \arrow[l] \end{tikzcd}$$

We have only to assume that at least one of p_1, p_2 is surjective.

Example 2.27. For $n \geq 2$ the K-theory of Cuntz algebra O_n is

$$\mathrm{K}_0(O_n) = \mathbb{Z}/(n-1)\mathbb{Z},$$
$$\mathrm{K}_1(O_n) = 0.$$

From these computations it follows that $O_n \not\simeq O_m$.

Example 2.28. Noncommutative torus A_θ has the following K-theory

$$\mathrm{K}_0(A_\theta) = \mathbb{Z} \oplus \mathbb{Z}$$
$$\mathrm{K}_1(A_\theta) = \mathbb{Z} \oplus \mathbb{Z}$$

Example 2.29. For the free group on two generators F_2 the map

$$C^*(F_2) \to C^*_r(F_2)$$

induces an isomorphism in K-theory (K-amenability) which gives $\mathrm{K}_0(C^*_r(F_2))$, $\mathrm{K}_1(C^*_r(F_2))$.

2.10 Completely positive maps

Lemma 2.30. *Let A and B be C^*-algebras and let $\varphi : A \to B$ be a linear map. Then φ is c.p. if and only if for all n and all $a_1, \dots, a_n \in A$ the matrix*

$$\begin{bmatrix} \varphi(a_1^*a_1) & \varphi(a_1^*a_2) & \cdots & \varphi(a_1^*a_n) \\ \varphi(a_2^*a_1) & \varphi(a_2^*a_2) & \cdots & \varphi(a_2^*a_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(a_n^*a_1) & \varphi(a_n^*a_2) & \cdots & \varphi(a_n^*a_n) \end{bmatrix}$$

is a positive element of $M_n(B)$.

Proof. The lemma is a consequence of the fact that any positive element $x \in M_n(A)$ is a sum of elements of the form

$$\begin{bmatrix} a_1^*a_1 & a_1^*a_2 & \cdots & a_1^*a_n \\ a_2^*a_1 & a_2^*a_2 & \cdots & a_2^*a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^*a_1 & a_n^*a_2 & \cdots & a_n^*a_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^* \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}.$$

Indeed, if $y \in M_n(A)$ and $x = y^*y$ then writing

$$y = \begin{bmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & \ddots & \vdots \\ y_{n,1} & \cdots & y_{n,n} \end{bmatrix}$$

we have

$$x = \begin{bmatrix} y_{1,1}^* & \cdots & y_{n,1}^* \\ \vdots & \ddots & \vdots \\ y_{1,n}^* & \cdots & y_{n,n}^* \end{bmatrix} \begin{bmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & \ddots & \vdots \\ y_{n,1} & \cdots & y_{n,n} \end{bmatrix}$$

so that

$$x_{k,l} = \sum_{r=1}^{n} y_{r,k}^* y_{r,l}.$$

In other words $x = u_1 + \cdots + u_n$, where

$$u_r = \begin{bmatrix} y_{r,1}^* y_{r,1} & \cdots & y_{r,1}^* y_{r,n} \\ \vdots & \ddots & \vdots \\ y_{r,n}^* y_{r,1} & \cdots & y_{r,n}^* y_{r,n} \end{bmatrix}.$$

□

Corollary 2.31. *A map $\varphi : A \to B(\mathcal{H})$ is c.p. if and only if for any n and any $a_1, \dots, a_n \in A$, $\xi_1, \dots, x_n \in \mathcal{H}$ the number*

$$\sum_{i,j=1}^{n} (\xi_i | \varphi(a_i^* a_j)\xi_j)$$

is positive.

Theorem 2.32 (Stinespring)**.** *Let $\varphi : A \to B(\mathcal{H})$ be a unital c.p. map. Then there exists a Hilbert space K, a representation π of A on K and an isometry $V : K \to \mathcal{H}$ such that*

$$\varphi(a) = V^* \pi(a) V$$

for all $a \in V$.

Proof. Let $A \odot \mathcal{H}$ be the algebraic tensor product of A and $\mathcal{H}$. We define a sesquilinear form $(\cdot|\cdot)$ on $A \odot \mathcal{H}$ by

$$\left(\sum_i a_i \xi_i \,\middle|\, \sum_j b_j \eta_j \right) = \sum_{i,j} (\xi_i | \varphi(a_i^* b_j) \eta_j)$$

and let $\mathcal{N} = \{X \in A \odot \mathcal{H} \mid (X|X) = 0\}$ Now $A \odot \mathcal{H}$ is an A module under

$$a(b \otimes \eta) = ab \otimes \eta$$

and $\mathcal{N}$ is a submodule (because $a_i^* a^* a a_j \leq \|a\|^* a_i^* a_j$). Clearly $\mathcal{K} = (A \odot \mathcal{H})/\mathcal{N}$ is a pre-Hilbert space. We let K be the completion of $\mathcal{K}$.

The action of A on $\mathcal{K}$ can be extended to action by bounded operators on K. We denothe the operator on K corresponding to $a \in A$ by $\pi(a)$. It is straightforward to check that π is a $*$-representation of A. Let us also define $V : \mathcal{H} \to K$ by $V\xi = 1_A \otimes \xi \in \mathcal{K} \subset K$. It is easy to see that

$$\pi(a)V\xi = (a \otimes \xi)$$

for all $a \in A$ and $\xi \in \mathcal{H}$.

The map V^* can be decomposed as $V^* = V^{-1}P$, where P is the projection of $\mathcal{H}$ onto

$$VH = \overline{\operatorname{span}\{1_A \otimes \xi \mid \xi \in \mathcal{H}\}}.$$

Moreover we have a formula for P, namely if (ξ_n) is an orthonormal basis of $\mathcal{H}$ and $X \in K$ then

$$PX = \sum (1_A \otimes \xi_n | X)\,(1_A \otimes \xi_n).$$

Now take X of the form $X = a \otimes \xi$. We have

$$\begin{aligned} PX &= \sum (1_A \otimes \xi_n | X)\,(1_A \otimes \xi_n) \\ &= \sum (1_A \otimes \xi_n | a \otimes \xi)\,(1_A \otimes \xi_n) \\ &= \sum (\xi_n | \varphi(a)\xi_n)\,(1_A \otimes \xi_n) \\ &= 1_A \otimes \left(\sum (\xi_n | \varphi(a)\xi_n)\,\xi_n\right) \\ &= 1_A \otimes \varphi(a)\xi. \end{aligned}$$

This shows that

$$\big(V^*\pi(a)V\big)\xi = \varphi(a)\xi$$

for all $x \in A$ and $a \in A$. ☐

Let us remark that all maps of the form $\varphi(a) = v^*\rho(a)v$, where ρ is a $*$-homomorphism into $\mathrm{B}(K)$ and $v \in \mathrm{B}(\mathcal{H}, K)$ are completely positive. Indeed, the map $\varphi_n : M_n(A) \to M_n\big(\mathrm{B}(K)\big) = \mathrm{B}(K^n)$ maps

$$a = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}$$

to

$$\begin{bmatrix} v^* & & \\ & \ddots & \\ & & v^* \end{bmatrix} \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} v & & \\ & \ddots & \\ & & v \end{bmatrix} = (1 \otimes v)^* \big[(\mathrm{id}_{M_n} \otimes \rho)(a)\big](1 \otimes v) \geq 0.$$

If ρ is unital and V is an isometry then φ is unital.

2.11 The Toeplitz extension

The Hilbert space $L^2(\mathbb{T})$ (with normalized Lebesgue measure) has orthonormal basis $(\phi_n)_n \in \mathbb{Z}$, where

$$\phi_n(\lambda) = \lambda^n$$

for all $\lambda \in \mathbb{T}$. Define the *Hardy projection* $P \in \mathrm{B}\big(L^2(\mathbb{T})\big)$ by

$$P\phi_n = \begin{cases} \phi_n & n \geq 0, \\ 0 & n < 0 \end{cases}$$

and let $\mathcal{H} = PL^2(\mathbb{T})$. Let p denote the projection P considered as a map $L^2(\mathbb{T}) \to \mathcal{H}$ and let j be the incusion of $\mathcal{H}$ into $L^2(\mathbb{T})$, so that $j = p^*$. Finally for $f \in \mathrm{C}(\mathbb{T})$ denote by M_f the operator of multiplication by f on $L^2(\mathbb{T})$.

Now for $f \in \mathrm{C}(\mathbb{T})$ the *Toeplitz operator* T_f of f is defined as

$$T_f = pM_f j.$$

We have $T_f \in B(\mathcal{H})$ and the C*-subalgebra generated in $B(\mathcal{H})$ by all such operators is called the *Toeplitz algebra* and is denoted by $\mathcal{T}$. One can easily show that $\mathcal{T}$ coincides with the C*-subalgebra of $B(\mathcal{H})$ generated by the isometry $s = T_z$, where $z \in \mathrm{C}(\mathbb{T})$ is the identity function: $z(\lambda) = \lambda$.

We have $s^*s = 1_{\mathcal{H}}$ while ss^* is the projection onto $\overline{\mathrm{span}\{\phi_n \mid n \geq 1\}}$. Therefore $1 - ss^* = |\phi_0)\,(\phi_0|$ and

$$s^m(1 - ss^*)s^{n*} = |\phi_m)\,(\phi_n| \tag{2.1}$$

so that all finite dimensional operators on $\mathcal{H}$ are contained in $\mathcal{T}$. It follows that the ideal $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$ is also contained in $\mathcal{T}$.

Let us analyse the C*-algebra $\mathcal{T}/\mathcal{K}(\mathcal{H})$. Since $1 - ss^* \in \mathcal{K}(\mathcal{H})$ we see that the image u of s in the quotient algebra is unitary. Since $\mathcal{T}/\mathcal{K}(\mathcal{H})$ is generated by u (because $\mathcal{T}$ is generated by s) we see that $\mathcal{T}/\mathcal{K}(\mathcal{H})$ is commutative.

Lemma 2.33. $\mathrm{Sp}\, u = \mathbb{T}$.

Proof. For any $\lambda \in \mathbb{T}$ consider $v_\lambda \in B(\mathcal{H})$ given by $v_\lambda \phi_n = \lambda^n \phi_n$. We have

$$sv_\lambda\phi_n = \lambda^n s\phi_n = \lambda^n \phi_{n+1}$$

and

$$v_\lambda s\phi_n = v_\lambda\phi_{n+1} = \lambda^{n+1}\phi_{n+1} = \lambda s v_\lambda \phi_n.$$

It follows that $v_\lambda s v_\lambda^* = \lambda s$.

Clearly the inner automorphism Ad_{v_λ} of $B(\mathcal{H})$ leaves $\mathcal{T}$ and $\mathcal{K}(\mathcal{H})$ invariant. Therefore it descends to an automorphism of $\mathcal{T}/\mathcal{K}(\mathcal{H})$. This shows that $\mathrm{Sp}\, u$ is invariant under all rotations, so it must be $\mathbb{T}$. □

It follows immediately from Lemma 2.33 that $\mathcal{T}/\mathcal{K}(\mathcal{H})$ is isomorphic to $\mathrm{C}(\mathbb{T})$.

Let $\pi : \mathcal{T} \to \mathrm{C}(\mathbb{T})$ be the quotient map sending s to u followed by identification of u with the canonical generator z of $\mathrm{C}(\mathbb{T})$ and let σ be the map $\mathrm{C}(\mathbb{T}) \ni f \mapsto T_f \in \mathcal{T}$. It's simple to see that σ is a positive map ($\sigma(f)$ is PM_fP restricted to $\mathcal{H}$) and thus continuous. We will now check that

$$\pi \circ \sigma = \mathrm{id}_{\mathrm{C}(\mathbb{T})}. \tag{2.2}$$

Equality (2.2) follows from the fact that $\pi\big(\sigma(z)\big) = z$ and that $\pi \circ \sigma$ is a $*$-homomorphism of unital algebras. The only nontrivial fact is multiplicativity of $\pi \circ \sigma$. This, however follows from the fact that for any $f, g \in \mathrm{C}(\mathbb{T})$ we have

$$T_f T_g - T_{fg} \in \mathcal{K}(\mathcal{H}).$$

Indeed,

$$\begin{aligned} T_f T_g &= pM_f jpM_g j \\ &= pM_f\big(M_g j + jpM_g j - M_g j\big) \\ &= pM_f M_g j + pM_f(jpM_g - M_g)j. \end{aligned}$$

For g a polynomial the operator $jpM_g - M_g$ i finite dimensional, so for $g \in \mathrm{C}(\mathbb{T})$ we have

$$jpM_g - M_g \in \mathcal{K}\big(L^2(\mathbb{T})\big).$$

Therefore $T_f T_g = T_{fg}$ modulo compact operators.

The above argument shows that we have an exact sequence with a positive splitting:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r] & \mathcal{T} \arrow[r, "\pi"] & \mathrm{C}(\mathbb{T}) \arrow[r] \arrow[l, bend right, "\sigma"'] & 0 \end{tikzcd}$$

which is called the *Toeplitz extension*.

Let us identify $\mathrm{C}_0\big(]0,1[\big)$ with the ideal

$$\big\{f \in \mathrm{C}(\mathbb{T}) \mid f(1) = 0\big\} \tag{2.3}$$

and let $\mathcal{T}_0$ be the pre-image under π of this ideal. We have $\mathcal{K}(\mathcal{H}) \subset \mathcal{T}_0$ bcause the image of $1 - ss^*$ under π vanishes at 1 and $\mathcal{K}$ coincides with the two sided ideal generated in $\mathcal{T}$ by $1 - ss^*$ (cf. (2.1)). If σ_0 is the restriction of σ to $\mathrm{C}_0\big(]0,1[\big)$ identified with (2.3) then we have the following morphism of positively split extensions:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r] \arrow[d, equal] & \mathcal{T}_0 \arrow[r, "\pi"] \arrow[d, hook] & \mathrm{C}\big(]0,1[\big) \arrow[r] \arrow[l, bend right, "\sigma_0"'] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r] & \mathcal{T} \arrow[r, "\pi"] & \mathrm{C}(\mathbb{T}) \arrow[r] \arrow[l, bend right, "\sigma"'] & 0 \end{tikzcd}$$

2.12 The Wold decomposition

Let $\mathcal{H}$ be a Hilbert space and let $v \in B(\mathcal{H})$ be an isometry: $v^*v = 1_{\mathcal{H}}$. For $n \in \mathbb{Z}_+$ let $\mathcal{H}_n = v^n\mathcal{H}$ (so that $\mathcal{H}_0 = \mathcal{H}$) and let

$$\mathcal{H}_\infty = \bigcap_{n \in \mathbb{Z}_+} \mathcal{H}_n.$$

We have $\mathcal{H}_\infty = \mathcal{H}_\infty$ by the very definition of $\mathcal{H}_\infty$ (note that $\mathcal{H}_i \supset \mathcal{H}_{i+1}$), so if we let $u\big|_{\mathcal{H}_\infty}$ then u is a unitary on $\mathcal{H}_\infty$.

Let us now note that for any n the spaces $\mathcal{H}_n \ominus \mathcal{H}_{n+1}$ and $\mathcal{H}_{n+1} \ominus \mathcal{H}_{n+2}$ are isomorphic (namely v maps one isomorphically onto the other). Therefore their dimensions are the same. We can choose an orthonormal basis (ξ_ι) of $\mathcal{H}_0 \ominus \mathcal{H}_1$ and then for each n the system $(v^n\xi_\iota)$ will be an orthonormal basis of $\mathcal{H}_n \ominus \mathcal{H}_{n+1}$. Since

$$\mathcal{H} = \Big(\bigoplus_{n\in\mathbb{Z}_+}(\mathcal{H}_n \ominus \mathcal{H}_{n+1})\Big) \oplus \mathcal{H}_\infty$$

we can decompose v into $w \oplus u$, where w acts on

$$\bigoplus_{n\in\mathbb{Z}_+}(\mathcal{H}_n \ominus \mathcal{H}_{n+1}).$$

This last Hilbert space is cearly isomorphic to $\ell^2(\mathbb{N}) \otimes (\mathcal{H}_0 \ominus \mathcal{H}_1)$ and with this identification $w = s \otimes 1$, where s is the shift operator on $\ell^2(\mathbb{N})$ considered in Section **??**.

We have this shown that any isometry v is unitarily equivalent to $(s\otimes 1)\oplus u$, where u is unitary and s is the unilateral shift on $\ell^2(\mathbb{N})$. This can be easily used to prove that v generates a C^*-algebra isomorphich to the Toeplitz algebra. In other words the Toeplitz algebra is the universal C^*-algebra generated by an isometry. To see this in very concrete terms take the map $\pi : \mathcal{T} \to \mathrm{C}(\mathbb{T})$ sending s to the canonical generator of $\mathrm{C}(\mathbb{T})$. Composing this with the map sending the generator of $\mathrm{C}(\mathbb{T})$ to u we obtain a mapping $\Psi : \mathcal{T} \to \mathrm{C}^*(u)$. Now let $\Psi : \mathcal{T} \to \mathrm{C}^*(v)$ send s to w (which is unitari;y equiva;ent to $s \otimes 1$). Clearly

$$\Phi \oplus \Psi : \mathcal{T} \ni s \longmapsto \Phi(s) \oplus \Psi(s) \in \mathrm{C}^*(v)$$

is an isomorphism of C^*-algebras.

2.13 Cuntz's proof of Bott periodicity

Theorem 2.34. *Let E be a functor on the category of C^*-algebras with $*$-homomorphisms to abelian groups which is*

- *homotopy invariant,*
- *half-exact,*
- *stable.*

Define maps $\xi : \mathcal{T} \to \mathbb{C}$ and $\eta : \mathbb{C} \to \mathcal{T}$ by

$$\xi = \delta_1 \circ \pi$$

where $\pi : \mathcal{T} \to \mathrm{C}(\mathbb{T})$ and δ_1 is the evaluation at $1 \in \mathbb{T}$ and

$$\eta(1) = 1 \in \mathcal{T}.$$

Then the maps

$$\begin{tikzcd} E(\mathcal{T}) \arrow[r, bend left, "E(\xi)"] & E(\mathbb{C}) \arrow[l, bend left, "E(\eta)"] \end{tikzcd}$$

are mutualy inverse isomorphism.

Proof. Since $\xi \circ \eta = \mathrm{id}_{\mathbb{C}}$ we only need to check that $E(\eta) \circ E(\xi) = \mathrm{id}_{E(\mathcal{T})}$.

Let $e \in \mathcal{K}$ be the matrix

$$\begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and let $\kappa : \mathcal{T} \to \mathcal{K} \otimes \mathcal{T}$ be the map

$$\mathcal{T} \ni x \longmapsto e \otimes x = \begin{bmatrix} x & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \in \mathcal{K} \otimes \mathcal{T}.$$

Then by stability of E the map $E(\kappa)$ is an isomorphism $E(\mathcal{T}) \to E(\mathcal{K} \otimes \mathcal{T})$. Note also that

$$\kappa \circ \eta \circ \xi : \mathcal{T} \ni s \longmapsto e \otimes 1 \in \mathcal{K} \otimes \mathcal{T}.$$

Let us identify $\mathcal{K}$ with the ideal in $\mathcal{T}$ in such a way that $e = 1 - ss^*$ (cf. Section 2.11) and let $\widehat{\mathcal{T}}$ be the C*-algebra generated by $\mathcal{K} \otimes \mathcal{T}$ and $\mathcal{T} \otimes 1$ inside $\mathcal{T} \otimes \mathcal{T}$.

Define

$$w_0 = s(1-e) \otimes 1 + e \otimes s.$$

Then w_0 is an isometry:

$$\begin{aligned} w_0^* w_0 &= \big((1-e)s^* \otimes 1 + e \otimes s^*\big)\big(s(1-e) \otimes 1 + e \otimes s\big) \\ &= (1-e)s^*s(1-e) \otimes 1 + es(1-e) \otimes s^* + (1-e)s^*e \otimes s + e \otimes s^*s \\ &= (1-e) \otimes 1 + (1-ss^*)s(1-e) \otimes s^* + (1-e)s^*(1-ss^*) \otimes s + e \otimes 1 = 1 \otimes 1 \end{aligned}$$

where we used the fact that

$$(1-ss^*)s = s^*(1-ss^*) = 0. \tag{2.4}$$

In the same way we show that

$$w_1 = s(1-e) \otimes 1 + e \otimes 1$$

is an isometry. Moreover we can write

$$w_0 = u_0(s \otimes 1), \qquad w_1 = u_1(s \otimes 1),$$

where

$$\begin{aligned} u_0 &= s(1-e)s^* \otimes 1 + es^* \otimes s + se \otimes s^* + e \otimes e, \\ u_1 &= s(1-e)s^* \otimes 1 + es^* \otimes 1 + se \otimes 1, \end{aligned}$$

(use (2.4) again). The same trick shows after straightforward computations that u_0 and u_1 are selfadjoint unitaries. For $i = 1, 2$ we can construct paths $(u_i^t)_{t \in [0, \frac{1}{2}]}$ connecting u_i to the identity of $\mathcal{T} \otimes \mathcal{T}$:

$$u_i^t = \tfrac{1}{2}(1 + u_i) - \tfrac{\exp(2\pi i t)}{2}(1 - u_i)$$

thus getting a path $(u_t)_{t \in [0,1]}$ connecting u_0 to u_1:

$$u_t = \begin{cases} u_0^t & 0 \le t \le \frac{1}{2}, \\ u_1^t & \frac{1}{2} \le t \le 1. \end{cases}$$

Let us also note that this is a path of unitary operators: for $i = 1, 2$ we have

$$\begin{aligned}\left(\tfrac{1}{2}(1+u_i) - \tfrac{\exp(2\pi i t)}{2}(1-u_i)\right)^* &\left(\tfrac{1}{2}(1+u_i) - \tfrac{\exp(2\pi i t)}{2}(1-u_i)\right)\\ &= \tfrac{1}{4}(1+u_i)^2 + \tfrac{1}{4}(1-u_i)^2\\ &\quad - \tfrac{1}{4}(1+u_i)(1-u_i)2\cos(2\pi t)\\ &= \tfrac{1}{2}(1+u_i) + \tfrac{1}{2}(1-u_i) = 1\end{aligned}$$

and, of course, the same in the opposite order.

Consider the path

$$w_t = u_t(s \otimes 1), \qquad (t \in [0,1])$$

of elements of $\widehat{\mathcal{T}}$. Then

$$\begin{aligned}w_t^* w_t &= (s^* \otimes 1)u_t^* u_t (s \otimes 1) = 1 \otimes 1,\\ w_t w_t^* &= u_t(s \otimes 1)(s^* \otimes 1)u_t^* = u_t(ss^* \otimes 1)u_t^* \neq 1 \otimes 1.\end{aligned}$$

Therefore for any $t \in [0,1]$ the C*-algebra generated by w_t inside $\widehat{\mathcal{T}}$ is isomorphic to $\mathcal{T}$. Let $\alpha_t : \mathcal{T} \to \widehat{\mathcal{T}}$ be this isomorphism.

We have thus constructed a family of homomorphisms $\alpha_t : \mathcal{T} \to \widehat{\mathcal{T}}$ $(t \in [0,1])$ such that

$$\begin{aligned}\alpha_0(s) &= w_0 = s(1-e) \otimes 1 + e \otimes s,\\ \alpha_1(s) &= w_1 = s(1-e) \otimes 1 + e \otimes 1.\end{aligned}$$

Also we have the following exact sequence:

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{T} \longrightarrow \widehat{\mathcal{T}} \xrightarrow{\widehat{\pi}} \mathrm{C}(\mathbb{T}) \longrightarrow 0,$$

where $\widehat{\pi}$ maps $k \otimes x + y \otimes 1 \in \widehat{\mathcal{T}}$ to $\pi(y)$ (cf. Section 2.11).

Let $\overline{\mathcal{T}}$ be the pull back:

$$\begin{tikzcd} \overline{\mathcal{T}} \arrow[r] \arrow[d] & \mathcal{T} \arrow[d, "\pi"] \\ \widehat{\mathcal{T}} \arrow[r, "\widehat{\pi}"] & \mathrm{C}(\mathbb{T}) \end{tikzcd}$$

so that

$$\overline{\mathcal{T}} = \left\{(r, z) \in \widehat{\mathcal{T}} \oplus \mathcal{T} \mid \widehat{\pi}(r) = \pi(z)\right\},$$

The algebra $\overline{\mathcal{T}}$ contains $\mathcal{K} \otimes \mathcal{T}$ as an ideal:

$$\mathcal{K} \otimes \mathcal{T} \cong \left\{(X, 0) \in \widehat{\mathcal{T}} \oplus \mathcal{T} \mid X \in \mathcal{K} \otimes \mathcal{T} \subset \widehat{\mathcal{T}}\right\} \subset \overline{\mathcal{T}}$$

and we let γ be this inclusion of $\mathcal{K} \otimes \mathcal{T}$ into $\overline{\mathcal{T}}$. Further let ρ be the projection from $\overline{\mathcal{T}}$ onto the second coordinate:

$$\rho : \overline{\mathcal{T}} \ni (r, z) \longmapsto z \in \mathcal{T}.$$

Then ρ is onto, because for any $z \in \mathcal{T}$ the we have $(z \otimes 1, z) \in \overline{\mathcal{T}}$. Moreover

$$\begin{aligned}\ker \rho &= \left\{(r, z) \in \overline{\mathcal{T}} \mid z = 0\right\},\\ &= \left\{(r, 0) \in \widehat{\mathcal{T}} \oplus \mathcal{T} \mid \widehat{\pi}(r) = 0\right\},\\ &= \left\{(k \otimes x + y \otimes 1, 0) \in \widehat{\mathcal{T}} \oplus \mathcal{T} \mid \pi(y) = 0\right\} = \mathcal{K} \otimes \mathcal{T}.\end{aligned}$$

Therefore we have an exact sequence

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{T} \xrightarrow{\gamma} \overline{\mathcal{T}} \xrightarrow{\rho} \mathcal{T} \longrightarrow 0.$$

This sequence is split by $\lambda : \mathcal{T} \ni s \mapsto (s \otimes 1, s) \in \overline{\mathcal{T}}$.

By split exactness of E we have

$$E\big(\overline{\mathcal{T}}\big) = E\big(\gamma(\mathcal{K} \otimes \mathcal{T})\big) \oplus E(\mathcal{T}).$$

Let us define $\beta_t : \mathcal{T} \to \overline{\mathcal{T}}$ by

$$\beta_t(x) = \big(\alpha_t(x), x\big).$$

To see that the element $\big(\alpha_t(x), x\big)$ belongs to $\overline{\mathcal{T}}$ note that $\widehat{\pi} \circ \alpha_t$ maps s to

$$\widehat{\pi}\big(u_t(s \otimes 1)\big) = \widehat{\pi}(u_t)\widehat{\pi}(s \otimes 1) = \widehat{\pi}(u_t)\pi(s).$$

Moreover it is easy to see that $\widehat{\pi}(u_0) = \widehat{\pi}(u_0) = 1_{\mathrm{C}(\mathbb{T})}$, so by definition of (u_t) we have $\widehat{\pi}(u_t) = 1$ for all t. Since $\mathcal{T}$ is generated by s, it follws that $\widehat{\pi} \circ \alpha_t = \pi$ for all t.

Now

$$\begin{aligned} \beta_0(s) &= (w_0, s) = \big(s(1-e) \otimes 1 + e \otimes s, s\big), \\ \beta_0(s) &= (w_1, s) = \big(s(1-e) \otimes 1 + e \otimes 1, s\big). \end{aligned}$$

Let us extract the "common summand" of both these maps, i.e. let $\beta : \mathcal{T} \ni s \mapsto \big(s(1-e) \otimes 1, s\big) \in \overline{\mathcal{T}}$. Then

$$\begin{aligned} \beta_0(s) &= \beta(s) + \big(\kappa(s), 0\big) = \beta(s) + (\gamma \circ \kappa)(s), \\ \beta_0(s) &= \beta(s) + \big((\kappa \circ \eta \circ \xi)(s), 0\big) = \beta(s) + (\gamma \circ \kappa \circ \eta \circ \xi)(s). \end{aligned}$$

It's an easy exercise to see that β_0 and β_1 are homotopic, so that $E(\beta_1) = E(\beta_0) : E(\mathcal{T}) \to E\big(\overline{\mathcal{T}}\big)$ by homotopy invariance of E. Then we note that the homomorphisms β and $\gamma \circ \kappa$ have ranges that multipliy to 0. This means that $\beta + \gamma \circ \kappa$ is a homomorphism (this we know since it is equal to β_0) and

$$E(\beta_0) = E(\beta + \gamma \circ \kappa) = E(\beta) + E(\gamma \circ \kappa) = E(\beta) + E(\gamma)E(\kappa) \tag{2.5}$$

(see remarks following the proof). The same reasoning applied to β and $\gamma \circ \kappa \circ \eta \circ \xi$ gives

$$E(\beta_1) = E(\beta + \gamma \circ \kappa \circ \eta \circ \xi) = E(\beta) + E(\gamma \circ \kappa \circ \eta \circ \xi) = E(\beta) + E(\gamma)E(\kappa \circ \eta \circ \xi). \tag{2.6}$$

We know that $E(\gamma)$ is the inclusion of $E(\mathcal{K} \otimes \mathcal{T})$ into

$$E\big(\overline{\mathcal{T}}\big) = E\big(\gamma(\mathcal{K} \otimes \mathcal{T})\big) \oplus E(\mathcal{T}) = E(\gamma)\big(E(\mathcal{K} \otimes \mathcal{T})\big) \oplus E(\mathcal{T}).$$

Moreover, it is clear from (2.5) and (2.6) that $E(\beta_0) - E(\beta)$ and $E(\beta_1) - E(\beta)$ map $E(\mathcal{T})$ into this summand of $E\big(\overline{\mathcal{T}}\big)$. Therefore, looking again at (2.5) and (2.6) we see that

$$E(\kappa) = E(\kappa \circ \eta \circ \xi)$$

as maps $E(\mathcal{T}) \to E(\mathcal{K} \otimes \mathcal{T})$. Now by stability of E the map $E(\kappa)$ is an isomorphism, so we can cancel it to obtain

$$E(\eta \circ \xi) = \mathrm{id}_{E(\mathcal{T})}.$$

□

Let us explain one device we used in the above proof. Consider two $*$-homomorphisms ϕ_1, ϕ_2 from a C*-algebra A to C*-algebra B and let E be a functor as considered in Theorem 2.34. Assume that for any $a_1, a_2 \in A$ we have

$$\phi_1(a_1)\phi_2(a_2) = 0.$$

Then $\phi_1 + \phi_2$ is a $*$-homomorphism $A \to B$ and $E(\phi_1 + \phi_2) = E(\phi_1) + E(\phi_2)$. Indeed, let j be the obvious (incective) map $\phi_1(A) \oplus \phi_2(A) \to B$ and for $i = 1, 2$ let

$$p_i : \phi_1(A) \oplus \phi_2(A) \longrightarrow \phi_i(A) \subset \phi_1(A) \oplus \phi_2(A)$$

be the canonical projection. Note that $E(p_1) + E(p_2) = \mathrm{id}_{E(\phi_1(A) \oplus \phi_2(A))}$ and therefore

$$\begin{aligned} E(\phi_1 + \phi_2) &= E(j)\big(E(p_1) + E(p_2)\big)E(\phi_1 + \phi_2) \\ &= E(j)E\big(p_1 \circ (\phi_1 + \phi_2)\big) + E(j)E\big(p_2 \circ (\phi_1 + \phi_2)\big) \\ &= E(\phi_1) + E(\phi_2). \end{aligned}$$

Chapter 3

Hilbert modules

3.1 Definitions

Suppose that A is a commutative unital C*-algebra, that is $A = C(X)$ for some compact Hausdorff topological space X. Suppose that F is a Hermitian vector bundle over X. Let E be all the continuous sections (defined over X) of the Hermitian vector bundle F. Then E is a $C(X)$-module and has a $C(X)$-valued inner product

$$\langle \xi, \eta \rangle(t) = \langle \xi(t), \eta(t) \rangle.$$

Definition 3.1. *If A is a C*-algebra (not necessarily unital or commutative), then an* inner product A-module *is a right A-module E with a compatible scalar multiplication*

$$\lambda(xa) = (\lambda x)a = x(\lambda a), \ \lambda \in \mathbb{C}, \ x \in E, \ a \in A$$

together with a map (inner product) $E \times E \to A$ such that

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
2. $\langle z, \alpha y \rangle = \langle x, y \rangle \alpha$
3. $\langle y, x \rangle = \langle x, y \rangle^*$
4. $\langle x, x \rangle \geq 0$ *(in A) and if* $\langle x, x \rangle = 0$ *then* $x = 0$.

There is a Cauchy-Schwartz inequality for $x, y \in E$

$$\langle y, x \rangle \langle x, y \rangle \leq \| \langle x, x \rangle \| \langle y, y \rangle.$$

Define a norm of $x \in E$ by $\|x\| := \|\langle x, x \rangle\|^{\frac{1}{2}}$. Then there is an inequality

$$\|\langle x, y \rangle\| \leq \|x\| \|y\|.$$

Definition 3.2. *If an inner product A-module E is complete with respect to $\| \cdot \|$ then it is called a Hilbert A-module.*

Example 3.3. A is a Hilbert A-module with respect to

$$\langle x,\, y\rangle = x^*y,\ \|x\|_H = \|x\|_A.$$

Similarly A^n is a Hilbert A-module with respect to

$$\langle x,\, y\rangle = \sum_{i=1}^n x_i^* y_i.$$

Example 3.4. If $\{E_i\}_{i=1}^n$ is a finite family of Hilbert A-modules, then $\bigoplus_{i=1}^n E_i$ is a Hilbert A-module with respect to

$$\langle x,\, y\rangle = \sum_{i=1}^n x_i^* y_i.$$

If $\{E_i\}_{i\in I}$ is an arbitrary family of Hilbert A-modules, then $\bigoplus_{i\in I} E_i$ is the space of sequences $(x_i)_{i\in I}$ such that $\sum_{i\in I}\langle x_i,\, x_i\rangle$ converges in A. Then

$$\langle x,\, y\rangle = \sum_{i\in I} x_i^* y_i.$$

converges by Cauchy-Schwartz inequality.

Example 3.5. If $\mathcal{H}$ is a Hilbert space, then the algebraic tensor product $\mathcal{H}\otimes_{alg} A$ has A-valued inner product

$$\langle \xi\otimes a,\, \eta\otimes b\rangle = \langle \xi,\, \eta\rangle a^*b,\ \xi,\eta\in H,\ a,b\in A.$$

The completion with respect to a Hilbert A-module norm is a Hilbert A-module denoted by $\mathcal{H}\otimes A$. If $\{e_i\}$ is an orthonormal basis for $\mathcal{H}$, then $\mathcal{H}\otimes A\cong\bigoplus A_i$. When $\mathcal{H}$ is infinite dimensional, separable, then $\mathcal{H}\otimes A$ is denoted by $\mathcal{H}_A$.

Suppose E, F are Hilbert A-modules. Denote by $\mathcal{L}(E,F)$ the set of bounded, adjointable maps $t\colon E\to F$ that is such that there exists $t^*\colon F\to E$ for which

$$\langle tx,\, y\rangle_F = \langle x,\, t^*y\rangle_E,\ x\in E,\ y\in F.$$

For this to make sense, t needs to be A-linear, $t(xa) = t(x)a$. Not every bounded A-linear map has an adjoint (for example the inclusion $\{f\in C([0,1])\mid f(1) = 0\}\hookrightarrow C([0,1])$).

There is a composition

$$\mathcal{L}(E,F)\times\mathcal{L}(F,G)\to\mathcal{L}(E,G),$$
$$(t,s)\mapsto s\circ t.$$

It follows that $\mathcal{L}(E,E)$ is a C*-algebra.

Let E, F be Hilbert A-modules, $x\in E$, $y\in F$. Define for $z\in F$

$$\theta_{x,y}\colon F\to E,\quad \theta_{x,y}(z) = x\langle y,\, z\rangle.$$

Then $\theta_{x,y}\in\mathcal{L}(E,F)$, $(\theta_{x,y})^* = \theta_{y,x}$ and $\theta_{x,y}\theta_{u,v} = \theta_{x\langle x,\, y\rangle v} = \theta_{x,v\langle u,\, y\rangle}$. For $t\in\mathcal{L}(E,G)$, $s\in\mathcal{L}(G,F)$

$$t\theta_{x,y} = \theta_{tx,y},\quad \theta_{x,y}s = \theta_{x,s^*y}.$$

Denote by $\mathcal{K}(E,F)$ the closed linear span of $\{\theta_{x,y}\}$. We write $\mathcal{K}(E)$ for $\mathcal{K}(E,E)$, which is an analogue of compact operators.

Example 3.6. If $E = A$, then $\mathcal{K}(A) = A$ and the isomorphism is given by

$$\theta_{a,b} \mapsto m_{ab^*} \text{ (left multiplication)}$$

$$\theta_{1,1} = \mathrm{id}\colon A \to A.$$

If A is unital, then $\mathcal{K}(A) \cong \mathcal{L}(A)$ and every $t \in \mathcal{L}(A)$ acts by $t(1)$.

Example 3.7. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{K}(H \otimes A) = \mathcal{K}(H) \otimes A$, where $\mathcal{K}(\mathcal{H})$ is the usual space of compact operators. Apply

Proposition 3.8. *Assume A is unital, E a Hilbert A-module. then the following are equivalent*

1. *E is a finitely generated projective A-module.*
2. *$\mathcal{K}(E) \cong \mathcal{L}(E)$.*
3. *The identity map on E is compact.*
4. *$\mathrm{id}\colon E \to E$ is of finite rank.*

Proposition 3.9. *Let A, B, C be C*-algebras such that A is an ideal in B and let E be a Hilbert C-module. Suppose that $\alpha\colon A \to \mathcal{L}(E)$ is a nondegenerate *-homomorphism ($A \cdot E$ is dense in E). Then α extends uniqualy to a *-homomorphism $\bar{\alpha}\colon B \to \mathcal{L}(E)$. If α is injective and A is essential in B, then $\bar{\alpha}$ is injective.*

Proof. Let e_j be an approximate unit for A. For $b \in B$, $a_1, \dots, a_n \in A$, $\xi_1, \dots, \xi_n \in E$

$$\begin{aligned} \|\sum_{i=1}^{n} \alpha(ba_i)\xi_i\| &\leq \lim_j \|\sum_{i=1}^{n} \alpha(be_j a_i)\xi_i\| \\ &= \lim_j \|\alpha(be_j) \sum_{i=1}^{n} \alpha(a_i)\xi_i\| \\ &\leq \|b\| \|\sum_{i=1}^{n} \alpha(a_i)\xi_i\|. \end{aligned}$$

The map

$$\sum_{i=1}^{n} \alpha(a_i)\xi_i \mapsto \sum_{i=1}^{n} \alpha(ba_i)\xi_i$$

is well defined and continuous.

Since α is non-degenerate, it extends by continuity to a bounded map $\bar{\alpha}(b)$ on E. Similar argument shows that $\bar{\alpha}(b^*)$ is an adjoint for $\bar{\alpha}(b)$. □

Apply this when $C = E = A$, and $\alpha\colon A \to \mathcal{L}(A)$ is the canonical embedding. Then any C*-algebra B which contains A as an essential ideal embeds in $\mathcal{L}(A)$.

If B is a maximal essential extension of A (A is an essential ideal in B and if A is also an essential ideal in C, then $\mathrm{id}\colon A \to A$ extends to an embedding $\beta\colon C \to B$), then there is an injection $\beta\colon \mathcal{L}(A) \to B$ whose restriction to A is the identity map.

By proposition, the canonical embedding $\alpha\colon A \to \mathcal{L}(A)$ has an injective extension $\bar{\alpha}\colon B \to \mathcal{L}(A)$. We can apply the proposition again to A as an ideal in $\mathcal{L}(A)$. Then α has a unique extension to a *-homomorphism $\mathcal{L}(A) \to \mathcal{L}(A)$. There are two maps

$$\mathrm{id}, \bar{\alpha}\beta\colon \mathcal{L}(A) \to \mathcal{L}(A)$$

and $\bar{\alpha}\beta = \mathrm{id}$, so $\bar{\alpha}$ is surjective. $\mathcal{L}(A)$ is a unique maximal essential extension of A so $\mathcal{L}(A) = M(A)$.

Theorem 3.10. *Let A be a C*-algebra. Then*

1. *$\mathcal{L}(A)$ is an essential extension of $\mathcal{K}(A)$ which is maximal in the above sense.*
2. *If a C*-algebra B is maximal essential extension of A, then we have a *-isomorphism $B \xrightarrow{\cong} \mathcal{L}(A)$ whose restriction to A is the canonical map $A \mapsto \mathcal{K}(A)$.*

Proposition 3.11. *Let A, C be C*-algebras and E a Hilbert c-module. Suppose $\alpha\colon A \to \mathcal{L}(E)$ is a nondegenerate injective *-homomorphism and let B be the idealiser of α in $\mathcal{L}E$,*

$$B := \{s \in \mathcal{L}(E) \mid s\mathcal{L}(A) \subseteq \mathcal{L}(A),\ \mathcal{L}(A)s \subseteq \mathcal{L}(A)\}.$$

*Then α extends to a *-isomomorphism*

$$M(A) \xrightarrow{\cong} B.$$

Theorem 3.12 (Kasparov)**.** *If E is a Hilbert module then $\mathcal{L}(E) \cong M(\mathcal{K}(E))$.*

Proof. The inclusion map $i\colon \mathcal{K}(E) \to \mathcal{L}(E)$ is nondegenerate and the idealiser of $\mathcal{K}(E)$ is $\mathcal{L}(E)$. □

Example 3.13. For $A = \mathbb{C}$ we have $M(\mathcal{K}(\mathcal{H})) = B(\mathcal{H})$ and an exact sequence

$$0 \to \mathcal{K}(A) \to M(A) \to M(A)/\mathcal{K}(A) \to 0.$$

We call $Q(A) := M(A)/A$ the *outer multiplier algebra*.

Definition 3.14. *The* stable multiplier algebra

$$M^s(A) := M(A \otimes \mathcal{K}),$$

and the quotient

$$Q^s(A) := M(A \otimes \mathcal{K})/A \otimes \mathcal{K}$$

is the stable outer multiplier algebra.

Proposition 3.15. *For any C*-algebra A*

$$\mathrm{K}_0(M^s(A)) = \mathrm{K}_1(M^s(A)) = 0.$$

Proof. Let v_i be a sequence of projections in $1 \otimes B(\mathcal{H})$ with orthogonal ranges. If p is any projection in $M^s(A)$, then let $q := \sum_i v_i p v_i^*$

$$w := \begin{pmatrix} 0 & 0 \\ v_1 & \sum_i v_{i+1} v_i^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

The sums $\sum_i v_i p v_i^*$ and $\sum_i v_{i+1} v_i^*$ converge in $A \otimes \mathcal{K}$.

$$w^* w = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \quad w w^* = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$

so $[p] + [q] = [q]$ in $\mathrm{K}_0(M^s(A))$.

For K_1 there is a similar argument and the Cuntz-Higson theorem that $\mathrm{U}(M^s(A))$ is contractible. □

For any C*-algebra A there is an isomorphism

$$\mathrm{K}_i(A) \xrightarrow{\cong} \mathrm{K}_{i-1}(Q^s(A)).$$

3.2 Examples

Here we recall the main examples that we shall use in what follows; see [l-e95] for more information.

Example 3.16. Let A be C*-algebra. We define a Hilbert A-module structure on $\mathcal{H} = A^n$ by

$$\begin{aligned} (a_1, \dots, a_n) + (b_1, \dots, b_n) &= (a_1 + b_1, \dots, a_n + b_n), \\ (a_1, \dots, a_n)a &= (a_1 a, \dots, a_n a), \\ \langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle &= a_1^* b_1 + a_2^* b_2 + \dots a_n^* b_n. \end{aligned}$$

Example 3.17. Let

$$\mathcal{H} = \{(a_1, a_2, \dots) \mid \sum_{j=1}^{\infty} a_j^* a_j \text{ is norm-convergent in } A\}$$

with the operations

$$\begin{aligned} (a_1, a_2, \dots) + (b_1, b_2, \dots) &= (a_1 + b_1, a_2 + b_2, \dots), \\ (a_1, a_2, \dots)a &= (a_1 a, a_2 a, \dots), \\ \langle (a_1, a_2, \dots), (b_1, b_2, \dots) \rangle &= \sum_{j=1}^{\infty} a_j^* b_j. \end{aligned}$$

Then $\mathcal{H}$ is a Hilbert A-module.

Example 3.18. Let G be a locally compact Hausdorff second countable topological group. Fix a left-invariant Haar measure dg for G. Let A be a G-C*-algebra. Denote

$$L^2(G, A) := \{f \colon G \to A \mid \int_G g^{-1} f(g)^* f(g) dg \text{ is norm-convergent in } A\}.$$

Then $L^2(G, A)$ is a Hilbert A-module with operations

$$\begin{aligned}(f+h)g &= f(g)+h(g),\\ (fa)(g) &= f(g)[ga],\\ \langle f,\, h\rangle &= \int_G g^{-1}f(g)^*h(g)dg.\end{aligned}$$

Definition 3.19. *An A-module map $T\colon \mathcal{H} \to \mathcal{H}$ is* adjointable *if there exists an A-module map $T^*\colon \mathcal{H} \to \mathcal{H}$ with*

$$\langle Tu,\, v\rangle = \langle u,\, T^*v\rangle$$

for all $u, v \in \mathcal{H}$.

If T^* exists, then it is unique, and $\sup_{\|u\|=1} \|Tu\| < \infty$. Set

$$B(\mathcal{H}) := \{T\colon A \to A \mid \|T \text{ is adjointable}\}.$$

Then $B(\mathcal{H})$ is a C*-algebra with operations

$$\begin{aligned}(T+S)u &= Tu+Su\\ (ST)(u) &= S(Tu)\\ (T\lambda)u &= (Tu)\lambda\\ \|T\| &= \sup_{\|u\|=1}\|Tu\|\end{aligned}$$

for $u \in \mathcal{H}$, $\lambda \in \mathbb{C}$.

3.3 Kasparov stabilization theorem

A Hilbert B-module E is countably generated if there exists a countable subset $X \subset E$ such that the smallest closed submodule of E containing X is E.

Theorem 3.20. *For every countably generated Hilbert B-module E there is an isomorphism*

$$\mathcal{H}_B \oplus B \cong \mathcal{H}_B.$$

Proof. A variant of Gram-Schmidt orthogonalization. There exists $u \in \mathcal{L}(\mathcal{H}_B \oplus B, \mathcal{H}_B)$ such that $u^*u = 1_{\mathcal{H}_B\oplus B}$, $uu^* = 1_{\mathcal{H}_B}$. It implies that for every countably generated B-module H there exists a porjection $p \in \mathcal{L}(\mathcal{H}_B)$ such that $E \cong p\mathcal{H}_B$. ☐

3.4 Morita equivalence

Recall:

- A C*-algebra A is stable if and only if $A \cong A \otimes \mathcal{K}$.
- Two C*-algebras A, B are stably isomorphic if and only if $A\otimes\mathcal{K} \cong B\otimes\mathcal{K}$.
- A Hilbert A-module E is full if and only if $\langle E,\, E\rangle$ is dense in A.

Suppose we have a C*-algebra, E, F are Hilbert A-modules. The space of compact operators $\mathcal{K}(E, F)$ from E to F is a right $\mathcal{K}(E)$-module and a left $\mathcal{K}(F)$-module with respect to the natural composition of maps.

$$\begin{tikzcd} E \arrow[r, "\beta"] \arrow[rr, bend left, "f\beta"] & E \arrow[r, "f"] \arrow[rr, bend right, "\alpha f"'] & F \arrow[r, "\alpha"] & E \end{tikzcd}$$

Let $B = \mathcal{K}(E)$, $G = \mathcal{K}(E, F)$. Then G is a right B-module and has a B-valued inner product

$$\langle s,\, t\rangle_B := s^*t, \quad s, t \in G.$$

Proposition 3.21. *Let A be a C*-algebra and E, F Hilbert A-modules. If E is full, then*

$$\mathcal{K}_B(G) \cong \mathcal{K}_A(F),$$
$$\mathcal{L}_B(G) \cong \mathcal{L}_A(F).$$

Proof. Let $t \in \mathcal{L}_A(F)$. The map $\alpha(t)\colon u \mapsto tu$, $u \in G$, is adjointable

$$\langle tu,\, v\rangle_B = (tu)^*v = u^*t^*v = \langle u,\, t^*v\rangle_B$$

so $\alpha(t) \in \mathcal{L}_B(G)$. Thus the left $\mathcal{L}_A(F)$-module structure on G provides a map $\alpha\colon \mathcal{L}_A(F) \to \mathcal{L}_B(G)$ which is a *-homomorphism. If $\alpha(t) = 0$ then $tu = 0$ for all $u \in G$. In particular

$$t\theta_{z,x}(y) = 0, \quad x, y \in E,\ z \in F$$

so $tz\langle x,\, y\rangle = 0$.

Now suppose E is full, so

$$\overline{F\langle E,\, E\rangle} = \overline{FA} = F.$$

Since $tF\langle E,\, E\rangle = \{0\}$ implies $t = 0$, we have that α is surjective.

Let $x, y \in E$, $z, w \in F$, $s = \theta_{z,x}$, $t = \theta_{w,y}$. Then $s, t \in G$ and $\alpha(\theta_{z\langle x,\, y\rangle, w}) = \theta_{s,t}$. Since G is generated as a normed linear space by the elements of the form s, t, and *-homomorphisms between C*-algebras have closed range, it follows that $\alpha(\mathcal{K}_A(F)) \supset \mathcal{K}_B(G)$.

On the other hand if E is full then elements of the form $\theta_{z\langle x,\, y\rangle, w}$ generate $\mathcal{K}_A(F)$, so $\alpha(\mathcal{K}_A(F)) \subset \mathcal{K}_B(G)$. We can now restrict α to $\mathcal{K}_A(F)$ to get $\mathcal{K}_A(F) \cong \mathcal{K}_B(G)$.

For the second statement we use the fact that if algebras are isomorphic, then their multiplier algebras are also isomorphic. ☐

Definition 3.22. *Two C*-algebras are* Morita equivalent*, $A \sim_M B$ if and only if there is a full Hilbert A-module E such that $B \cong \mathcal{K}_A(E)$ (strong Morita equivalence due to Rieffel).*

Proposition 3.23. *Morita equivalence is an equivalence relation.*

Proof. 1. Reflexive: $A \cong \mathcal{K}_A(A)$.

2. Symmetric: by proposition ($F = A$) if $B \cong \mathcal{K}_A(E)$ and $G = \mathcal{K}_A(E, A)$ as B-modules, then $A \cong \mathcal{K}_B(G)$.

3. Transitive: suppose $B \cong \mathcal{K}_A(E)$, $C \cong \mathcal{K}_B(F)$, E-full Hilbert A-module, F-fill Hilbert B-module. If $\iota\colon B \to \mathcal{L}_A(E)$ let $G := F \otimes_i E$. Then G is a full Hilbert A-module and $\iota_*\colon C \xrightarrow{\cong} \mathcal{K}_A(G)$.

□

Theorem 3.24. *Two σ-unital C*-algebras are Morita equivalent if and only if they are stably isomorphic.*

Proof. For any C*-algebra A

$$\mathcal{K}_A(\mathcal{H}_A) = \mathcal{K}_A(\mathcal{H} \otimes A) \cong \mathcal{K}_{\mathbb{C}}(\mathcal{H}) \otimes \mathcal{K}_A(A) = \mathcal{K} \otimes A$$

so $A \sim_M \mathcal{K} \otimes A$. If A and B are stably isomorphic then

$$A \sim_M \mathcal{K} \otimes A \cong \mathcal{K} \otimes B \sim_M B$$

so $A \sim B$ (we do not need σ-unitality here).

Suppose that $A \sim_M B$ and let $B \cong \mathcal{K}_A(E)$. Then if A, B are σ-unital

$$\mathcal{K}_{\otimes} B \cong \mathcal{K}_A(\mathcal{H} \otimes E) \cong \mathcal{K}_A(\mathcal{H}_A) \cong \mathcal{K} \otimes A.$$

□

3.5 Tensor products of Hilbert modules

1. *Outer tensor products* For $i = 1, 2$ let B_i be a C*-algebras and E_i a Hilbert B_i-module. The Hilbert $B_1 \otimes_{min} B_2$-module $E_1 \otimes E_2$ is by definition the completion of the algebraic tensor product $E_1 \otimes_{alg} E_2$ in the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}$, where for $\xi_i, \eta_i \in E_i$

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle := \langle \xi_1, \eta_1 \rangle \otimes \langle \xi_2, \eta_2 \rangle.$$

2. *Inner tensor products* Let A, B be two C*-algebras, E_1 a Hilbert A-module, E_2 a Hilbert B-module, and $\pi\colon A \to \mathcal{L}(E_2)$ a *-homomorphism. The Hilbert B-module $E_1 \otimes_\pi E_2$ (also denoted by $E_1 \otimes_A E_2$) is the Hausdorff completion of the algebraic tensor product $E_1 \otimes_{alg} E_2$ with respect to the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}$, where for $\xi_i, \eta_i \in E_i$

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle := \langle \xi_2, \pi(\langle \xi_1, \eta_1 \rangle)\eta_1 \rangle.$$

The action of B is given by $(\xi_1 \otimes \xi_2)b := \xi_1 \otimes \xi_2 b$. Note that for $a \in A$, $\xi_1 \in E_1$, $\xi_2 \in E_2$ we have $\xi_1 \otimes \pi(a)\xi_2 = \xi_1 a \otimes \xi_2$.

Chapter 4

Fredholm modules and Kasparov's K-homology

4.1 Fredholm modules

If P and Q are bounded operators on a Hilbert space we shall write $P \sim Q$ when they differ by a compact operator. We assume that A is a separable C*-algebra, not necessarily unital.

Definition 4.1. *An (ungraded)* Fredholm module *over A is given by the following data:*

1. *a separable Hilbert space $\mathcal{H}$,*
2. *a representation $\rho\colon A \to B(\mathcal{H})$,*
3. *an operator F on $\mathcal{H}$ such that for all $a \in A$*

$$(F^2 - 1)\rho(a) \sim 0$$
$$(F - F^*)\rho(a) \sim 0$$
$$F\rho(a) - \rho(a)F \sim 0.$$

The representation ρ is not required to be non-degenerate.

Definition 4.2. *Aa* $\mathbb{Z}_2$-graded Fredholm module *over A is given by the same data as in definition (4.1) plus the following additional structure:*

1. *the Hilbert space is equipped with the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$,*
2. *for each $a \in \mathcal{H}$, $\rho(a)$ is even, $\rho(a) = \rho^+(a) \oplus \rho^-(a)$,*

$$\rho(a) = \begin{pmatrix} \rho^+(a) & 0 \\ 0 & \rho^-(a) \end{pmatrix}$$

   where $\rho^\pm$ is a representation on $\mathcal{H}^\pm$,

3. *F is odd,*

$$F = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}, \quad u\colon \mathcal{H}^+ \to \mathcal{H}^-,\ v\colon \mathcal{H}^- \to \mathcal{H}^+.$$

The operators u, v are not independent: V is essentially the adjoint of u. We can rewrite the conditions of the original definition as follows

$$\begin{aligned}(uv-1)\rho^-(a) &\sim 0\\ (vu-1)\rho^+(a) &\sim 0\\ (u-v^*)\rho^+(a) &\sim 0\\ u\rho^+(a) &\sim \rho^-(a)u.\end{aligned}$$

Let $p \in \mathbb{N}$.

Definition 4.3. *A p-graded Fredholm module is a Fredholm module (H, ρ, F) as above for which there exist operators $\varepsilon_1, \ldots, \varepsilon_p$ such that*

$$\varepsilon_j = -\varepsilon_j^*, \quad \varepsilon_j^2 = -1, \quad , \varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0, ; i \neq j.$$

Example 4.4. Fredholm modules over $\mathbb{C}$. Assume that $\rho\colon \mathbb{C} \to B(\mathcal{H})$ is the unique unital representation. Then an ungraded Fredholm module is given by an essentially selfadjoint Fredholm operator F. This characterisation follows from Atkinson's theorem. Recall we defined a Fredholm operator to be an operator F such that $\ker F$, $\ker F^*$ are finite dimensional.

Theorem 4.5 (Atkinson). *Let $F \in B(\mathcal{H})$. Then then the following are equivalent*

1. *F is Fredholm.*
2. *The image of F in $\mathcal{Q}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is invertible.*
3. *There exist $G \in B(\mathcal{H})$ such that $1 - FG$, $1 - GF$ are compact.*

A graded Fredholm module is given by an essentially selfadjoint operator F of the form

$$F = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$$

where u and v are Fredholm and $u \sim v^*$. By definition $\operatorname{Index}(F) = \operatorname{Index}(u)$.

Example 4.6. The pseudodifferential operator extension. Let M be a smooth manifold without boundary (not necessarily compact). Let S^*M be the cosphere bundle of M: take the cotangent bundle of M, delete the zero section (zero cotangent vectors), identify non-zero cotangent vectors which differ only by multiplication by a positive scalar (if M is equipped qith a Riemannian metric then S^*M can be identified with the space of unit length cotangent vectors).

There is an extension

$$0 \to \mathcal{K}(L^2(M)) \to \Psi DO(M) \to C_0(S^*M) \to 0$$

The outline of the construction is as follows. If M is an opent subset of $\mathbb{R}^n$, then suppose that σ is a complex valued function on T^*M which has the property (homogenity):

$$\sigma(x, t\xi) = \sigma(x, \xi), \ t \geq 1, \ |\xi| \geq 1.$$

Assume that σ is compactly supported in the M-direction, i.e. $\sigma(x, \xi)$ vanishes when x is outside some compact subset of M. Then the linear map $D_\sigma\colon C_c^\infty(M) \to C_c^\infty(M)$ given by the integral formula

$$D_\sigma f(x) := \frac{1}{(2\pi)^n} \int \sigma(x, \xi)\hat{f}(\xi) e^{i\langle x,\, \xi\rangle} d\xi,$$

where $\hat{f}$ denotes the Fourier transform of f, is an example of a pseudodifferential operator. Because σ is homogeneous, it defines a function σ_0 on the cosphere bundle S^*M, which is called the *symbol* of the operator D_σ.

Proposition 4.7. *The operator D_σ extends by continuity to a bounded linear operator on $L^2(M)$. The map which associates to each D_σ its symbol σ_0 extends to a *-homomorphism form the C*-algebra $\Psi DO(M)$ generated by all D_σ onto the C*-algebra $C_0(S^*M)$.*

The map $\Psi DO(M) \to C_0(S^*M)$ is called the *symbol map.*

This proposition gives the extension when M is an open subset of $\mathbb{R}^n$. The extension to manifolds is done as follows. If $M \subseteq \mathbb{R}^n$ is open and $g \in C_c^\infty(M)$ then the multiplication operator M_g is a pseudodifferential operator associated with the function $\sigma(x, \xi) = g(x)$, so $M_g \in \Psi DO(M)$. Next we use the invariance of pseudodifferential operators under smooth changes of coordinates. If $\Psi\colon M \to M'$ is a diffeomorphism of open sets in $\mathbb{R}^n$, then the transform under Ψ of an operator in $\Psi DO(M)$ with symbol σ_0 is an operator in $\Psi DO(M')$ with symbol $\Phi_*(\sigma_0)$ ($\Phi\colon M \to M'$ induces $C_c(M') \to C_c(M)$ by composition. Get unitary $u\colon L^2(M') \to L^2(M)$ by multiplying by $\sqrt{\mathrm{Jac}(f)}$ and then $T \in B(L^2(M)) \mapsto u^*TU \in B(L^2(M'))$). So we can define $\Psi DO(M)$ for any smooth manifold (using invariance plus partition of unity) to be a C*-algebra consisting of those $T \in B(L^2(M))$ such that

1. $\lim \|TM_{g_n} - T\| = 0 = \lim \|M_{g_n}T - T\|$ for some approximate unit g_n for $C_0(M)$
2. T commutes with $C_0(M)$ modulo compact operators
3. for each coordinate chart U and each $g \in C_0(U)$, the operator M_gTM_g belongs to $\Psi DO(M)$.

Symbol of T is well defined as an element of $C_0(S^*M)$.

The operator D_σ is Fredholm if and only if σ_0 is nowhere zero. Let $D \in M_k(\Psi DO(M))$ be a system of psedudodifferential operators whose symbol is a unitary matrix-valued function on S^*M. Then

$$\mathcal{H} = L^2(M)^k \oplus L^2(M)^k,\ F = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

together with a representation $\rho\colon C(M) \to B(L^2(M))$ by multiplication operators is a graded Fredholm module over $C(M)$.

This construction generalises Atiyah's definition of Ell. There is a pairing with K-theory. For a projection $p \in M_k(C(M))$

$$F_p := \begin{pmatrix} 0 & \rho(p)D^* \\ \rho(p)D & 0 \end{pmatrix}$$

is an operator on $\mathcal{H} = \rho(p)L^2(M)^k \oplus \rho(p)L^2(M)^k$, and

$$\langle [p],\, [D] \rangle := \mathrm{Index}(\rho(p)D\rho(p)).$$

Let A be a C*-algebra and $(\mathcal{H}, F)$ a Fredholm module over A. It extends to $M_n(A)$ and $\mathcal{H}_n := \mathcal{H} \otimes \mathbb{C}^n$, $F_n := F \otimes \mathrm{id}_n$.

Proposition 4.8. *Let $(\mathcal{H}, F)$ be a Fredholm module over A. There exists a unique additive map $\phi\colon \mathrm{K}_0(A) \to \mathbb{Z}$ such that for every projection $e \in M_n(A)$ we have $\phi([e]) = \mathrm{Index}(T)$, where $T\colon e\mathcal{H}_n^+ \to e\mathcal{H}_n^-$ is defined by $Tx = eF_nx$ for all $x \in \mathcal{H}_n^+$.*

4.2 Commutator conditions

In the definition of Fredholm module $(\mathcal{H}, F)$ we have a condition $[F, \rho(a)] \in \mathcal{K}$ for all $a \in A$. In Kasparov K-homology A has to be a separable C*-algebra. For more subtle invariants, Connes allows Fredholm modules over *-algebras $\mathcal{A}$, not necessarily C*-algebras. Most useful condition is that $[F, a] \in \mathcal{L}^p(\mathcal{H})$ for some $p \geq 1$. There is a fine balance to be struck here: the class of algebras we allow for Fredholm modules should still have a meaningful K-theory, fairly close to the K-theory for C*-algebras. Ideally we want K-theory with the same formal properties as K-theory for C*-algebras. Note that the K-theory for such algebras needs to be developed from scratch. A sensible class of C*-algebras may be determined using the following

Proposition 4.9 (Connes)**.** *Let $\mathcal{A}$ be an involutive algebra, $(\mathcal{H}, F)$ an $(n+1)$-summable Fredholm module over $\mathcal{A}$ with the parity of n. Let A be the C*-algebra closure of $\mathcal{A}$ (in its action on $\mathcal{H}$). Let $\bar{\mathcal{A}}$ be the smallest involutive subalgebra of A containing $\mathcal{A}$ and stable under holomorphic functional calculus. Then $(\mathcal{H}, F)$ is an $(n+1)$-summable Fredholm module over $\bar{\mathcal{A}}$.*

From this one deduces that it is sufficient to restrict attention to local C*-algebras (pre C*-algebras).

Proposition 4.10. *Let $\mathcal{A}$ be a pre C*-algebra (local C*-algebra). Then*

1. *Any Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$ extends by continuity to a Fredholm module over the associated C*-algebra A.*
2. *The inclusion $\mathcal{A} \hookrightarrow A$ is an isomorphism on K-theory.*

Proposition 4.11 (Connes)**.** *Suppose that $(\mathcal{H}, F)$ is a 1-summable Fredholm module and γ is an involution im[plementing the $\mathbb{Z}/2$-grading on $\mathcal{H}$. Then the map*

$$\tau\colon A \to \mathbb{C},\ a \mapsto \tfrac{1}{2}\,\mathrm{Tr}(\gamma F[F, a])$$

is a trace on A.

Proof. Define

$$\mathcal{A} := \{a \in A \mid [F, a] \in \mathcal{L}^1(\mathcal{H})\},\ \mathcal{A} \subset A.$$

We have

$$\begin{aligned}
\gamma F[F, a] &= \gamma a - \gamma F a F \\
&= \gamma a + F a \gamma F \\
&= a\gamma F^2 + F a \gamma F - F\gamma F + F a \gamma F \\
&= [F, a]\gamma F,
\end{aligned}$$

where we use $F^2 = 1$ or the equalities are modulo $\mathcal{K}$. Next

$$\begin{aligned}\tau(ab) &= \tfrac{1}{2}\operatorname{Tr}(\gamma F[F, ab]) \\ &= \tfrac{1}{2}\operatorname{Tr}(\gamma F[F, a]b + \gamma Fa[F, b]) \\ &= \tfrac{1}{2}\operatorname{Tr}([F, a]\gamma Fb + [F, b]\gamma Fa) \\ &= \tau(ba).\end{aligned}$$

□

We call τ the *character* of the Fredholm module $(\mathcal{H}, F)$.

Theorem 4.12 (Connes)**.** *Let A be a unital C^*-algebra equipped with a faithful positive trace τ, $\tau(1) = 1$. Let $(\mathcal{H}, F)$ be a Fredholm module over A such that*

$$\mathcal{A} := \{a \in A \mid [F, a] \in \mathcal{L}^1(\mathcal{H})\}$$

is a dense subalgebra of A and the restriction of τ to $\mathcal{A}$ is the character of the Fredholm module $(\mathcal{H}, F)$. Then A contains no nontrivial idempotents.

Proof. $\mathcal{A}$ is a subalgebra of A stable under holomorphic functional calculus. The inclusion $\mathcal{A} \hookrightarrow A$ induces an isomorphism $\mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(A)$. The trace τ takes integer values (this is the index map). If e is a projection then $\tau(e) = 0, 1$. Because τ is faithful $e = 0, 1$. □

Example 4.13. Let F_2 be the nonabelian free group on two generators. It acts on a tree (1-dimensional simplicial complex with no loops). Let Δ_0 be the set of vertices and Δ_1 be the set of edges. Denote by $[x, y]$ for $x, y \in \Delta_0$ the set of vertices on the unique path from x to y, and by x_0 the origin. For all $x \in \Delta_0 \setminus \{x_0\}$ let $\beta(x) \in \Delta_1$ be the unique edge containing x in $[x, x_0]$.

Lemma 4.14.

1. *The map $\beta\colon \Delta_0 \to \Delta_1$ is a bijection.*
2. *For a fixed $g \in F_2$, the set of $x \in \Delta_0$ such that $g\beta(g^{-1}x) \neq \beta(x)$ is finite and equals $[x_0, gx_0]$.*

Proof.

1. The inverse is given by

$$\beta^{-1}(\text{edge } u) := \text{vertex of } u \text{ further from } x_0.$$

2. $g\beta(g^{-1}x)$ is the edge containing x and lying in $[gx_0, x]$. Suppose $x \notin [gx_0, x]$.

□

Define a map $\mathcal{F}\colon l^2(\Delta_0) \to l^2(\Delta_1)$ by

$$\mathcal{F}\delta_x := \begin{cases}\delta_{\beta_x} & \text{for } x \neq x_0 \\ 0 & \text{for } x = x_0\end{cases}$$

Proposition 4.15.

1. *$\mathcal{F}$ is an operator of index 1, $\mathcal{F}\mathcal{F}^* = 1$, $\mathcal{F}^*\mathcal{F} = 1 - p_{x_0}$, where $p_{x_0} : l^2(\Delta_0) \to \mathbb{C}\delta_{x_0}$.*

2. *Let π_0, π_1 be actions of F_2 on $l^2(\Delta_0)$, $l^2(\Delta_1)$. For all $g \in F_2$ the operator $\pi_1(g)\mathcal{F} - \mathcal{F}\pi_0(g)$ is of finite rank and $(l^2(\Delta_0) \oplus l^2(\Delta_1), \mathcal{F})$ is a Fredholm module.*

Let

$$\mathcal{A} := \{a \in C_r^*(F_2) \mid [\mathcal{F}, a] \in l^2(\Delta_0) \oplus l^2(\Delta_1)\}.$$

By the proposition $\mathbb{C}[F_2] \subset \mathcal{A}$, so $\mathcal{A}$ is dense in $C_r^*(F_2)$. Now from the Connes theorem one obtains the proof of the Kadison-Kaplansky conjecture

Theorem 4.16. *The algebra $C_r^*(F_2)$ has no nontrivial idempotents.*

Fredholm modules of this type can be constructed for any locally compact group acting on a tree (Julg, Vallette).

Theorem 4.17. *Let G be any locally compact group acting on a tree such that the stabiliser of any vertes is amenable. Then G is K-amenable.*

4.3 Quantised calculus of one variable

Let f be a function on $\mathbb{R}$. Find function algebras for which $df := [F, f]$ has a given regularity. Take $\mathcal{H} = L^2(\mathbb{R})$. The Hilbert transform is given by

$$(F\xi)(s) = \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{|s-t|>\varepsilon} \frac{\xi(t)}{s-t} dt.$$

We have $F^2 = 1$ and $[F, f]$ is the operator on $L^2(\mathcal{H})$ associated to the kernel

$$k(s,t) = \frac{f(s) - f(t)}{s - t}$$

This can be transported to S^1 by some conformal map. Then we obtain a Fredholm module given by the data

$$\mathcal{H} = L^2(S^1), \quad F = 2P - 1,$$

where $P : L^2(S^1) \to H^2(S^1)$ is the orthogonal projection onto the Hardy space.

For any $f \in L^\infty(S^1)$

- $[F, f]$ is a finite rank operator if and only if f is a rational function.
- $[F, f]$ is compact if and only if f is of vanishing mean oscillation, that is for
$$M_a f := \sup_{|I| \le a} \frac{1}{|I|} \int_I |f - I(t)| dt,$$
where $I(t) = \frac{1}{|I|} \int f dx$, we have $\lim_{a \to 0} M_a f = 0$.
- $[F, f]$ is in $\mathcal{L}^p(\mathcal{H})$ if and only if f is in Besov space $B_p^{\frac{1}{p}}$, that is
$$\int\int |f(x+t) - 2f(x) + f(x-t)|^p t^{-2} dx dt < \infty.$$

4.4 Quantised differential calculus

Let $(\mathcal{A}, H, F)$ be a Fredholm module over an involutive algebra $\mathcal{A}$, n integer ≥ 0. We assume that the Fredholm module is even for n even and odd for n odd. In either case it is $(n+1)$-summable: $[F, a] \in \mathcal{L}^{n+1}(\mathcal{H})$ for all $a \in \mathcal{H}$.

For $k = 0$, put $\Omega^0 = \mathcal{A}$. For $k > 0$

$$\Omega^k := \operatorname{span}\{a^0[F, a^1] \ldots [F, a^k] \mid a^j \in \mathcal{A}\}.$$

By Hölder inequality, $\Omega^k \subset \mathcal{L}^{\frac{n+1}{k}}(\mathcal{H})$. Put $\Omega^* := \bigoplus_{k\geq 0} \Omega^k$. The product in Ω^* is the operator product. We use the Leibniz rule for $[F, -]$ to check that if $\omega \in \Omega^k$ and $\omega \in \Omega^{k'}$ then $\omega\omega' \in \Omega^{k+k'}$

$$a^0[F, a^1] \ldots [F, a^k]a^{k+1} = \sum_{j=1}^{k-1} (-1)^{k-j} a^0[F, a^1] \ldots [F, a^j a^{j+1}] \ldots [F, a^{k+1}]+$$

$$+(-1)^k a^0 a^1[F, a^2] \ldots [F, a^{k+1}].$$

It is a differential graded algebra (DGA) with differential $d\colon \Omega^k \to \Omega^{k+1}$ given by the graded commutator

$$d\omega = [F, \omega] = F\omega - (-1)^{|\omega|}\omega F.$$

It is a graded derivation, that is

$$d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{|\omega_1|} d\omega_2,\ d^2 = 0.$$

4.5 Closed graded trace

We will define a supertrace $\operatorname{Tr}_s\colon \Omega^n \to \mathbb{C}$. If T is an operator on $\mathcal{H}$ such that $FT + TF \in \mathcal{L}^1(\mathcal{H})$ then put

$$\operatorname{Tr}'(T) := \tfrac{1}{2} \operatorname{Tr}(F(FT + TF)).$$

If $T \in \mathcal{L}^1(\mathcal{H})$, then put

$$\operatorname{Tr}'(T) := \operatorname{Tr}(T).$$

Now define $\operatorname{Tr}_s(\omega)$ for $\omega \in \Omega^n$

$$\operatorname{Tr}_s(\omega) := \begin{cases} \operatorname{Tr}'(\omega) & \text{for } n \text{ odd} \\ \operatorname{Tr}'(\gamma\omega) & \text{for } n \text{ even} \end{cases}$$

where γ is the involution implementing the $\mathbb{Z}/2$-grading on $\mathcal{H}$.

Proposition 4.18. Tr_s *is a closed graded trace.*

1. $\operatorname{Tr}_s(d\omega) = 0$
2. *If* $\omega \in \Omega^k$, $\omega' \in \Omega^{k'}$, $k + k' = n$, *then*

$$\operatorname{Tr}_s(\omega\omega') = (-1)^{kk'} \operatorname{Tr}_s(\omega'\omega)$$

Proof. In the odd case
$$F\omega + \omega F = [F, \omega] = d\omega,$$
and in the even case
$$F\gamma\omega + \gamma\omega F = -\gamma F\omega + \gamma\omega F = -\gamma[F, \omega] = -\gamma d\omega,$$
so for $\omega = d\eta$, $\mathrm{Tr}_s(\omega) = 0$.

For the trace condition, take $\omega \in \Omega^k$, $\omega' \in \Omega^{k'}$, $k + k' = n$, n odd.

$$\begin{aligned}
\mathrm{Tr}_s(\omega\omega') &= \tfrac{1}{2}\,\mathrm{Tr}(Fd(\omega\omega')) \\
&= \tfrac{1}{2}\,\mathrm{Tr}(F(d\omega)\omega' + (-1)^{|\omega|}F\omega(d\omega)) \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|}(d\omega)F\omega' + (-1)^{|\omega|}F\omega d\omega') \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|+1}F\omega' d\omega + (-1)^{|\omega|}F\omega) \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|+1}F\omega' d\omega + (-1)^{|\omega|+|\omega'|+1}F(d\omega')\omega) \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|+1}F\omega' d\omega + (-1)^{|\omega|+|\omega'|+1}F(d(\omega'\omega) + (-1)^{|\omega'|+1}\omega'(d\omega))) \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|+|\omega'|+1}Fd(\omega'\omega)) \\
&= \tfrac{1}{2}\,\mathrm{Tr}(Fd(\omega'\omega)).
\end{aligned}$$

If $|\omega| + |\omega'| = n$ and n is odd, then $|\omega|$, ω' cannot both be odd, so $|\omega||\omega'| = 0$ mod 2 and
$$\mathrm{Tr}_s(\omega\omega') = (-1)^{|\omega||\omega'|}\,\mathrm{Tr}_s(\omega'\omega) = \mathrm{Tr}_s(\omega'\omega).$$
The even case is very similar, use the extra condition $F\gamma = -\gamma F$. □

Definition 4.19. *The* character *of the cycle* $(A, \mathcal{H}, F)$ *is the cyclic cocycle*
$$\tau_n(a^0, a^1, \ldots, a^n) = \mathrm{Tr}_s\, a^0[F, a^1] \ldots [F, a^n].$$

Difficult problem: provide an explicit formula for this cocycle in terms of data defining the Fredholm module. The cyclic cocycle seems to depend on n, but in fact there is no problem here.

Recall Connes' periodicity operator $S\colon \mathrm{HC}^n(\mathcal{A}) \to \mathrm{HC}^{n+2}(\mathcal{A})$,
$$\ldots \to \mathrm{HH}^{n-1}(\mathcal{A}, \mathcal{A}) \to \mathrm{HC}^n(\mathcal{A}) \xrightarrow{S} \mathrm{HC}^{n+2}(\mathcal{A}) \to \mathrm{HH}^{n+2}(\mathcal{A}, \mathcal{A}) \to \ldots$$

Proposition 4.20. *The characters* τ_{n+2q} *satisfy*
$$\tau_{m+2} = -\frac{2}{m+2} S\tau_m,\ m = n + 2q,$$
so the cocycles together determine a class in periodic cyclic homology
$$\mathrm{HP}^*(\mathcal{A}) = \varinjlim(\mathrm{HC}^n(\mathcal{A}), S)$$

Definition 4.21. *Let* $(\mathcal{A}, \mathcal{H}, F)$ *be a finitely summable Fredholm module over an involutive algebra* $\mathcal{A}$*. The* Chern character $\mathrm{ch}_*(\mathcal{H}, F) \in \mathrm{HP}^*(\mathcal{A})$ *is the periodic cyclic homology class whose components are the following cyclic cocycles for large enough* n*:*
$$(-1)^{\frac{n(n-1)}{2}}\Gamma\left(\frac{n}{2} + 1\right)\mathrm{Tr}'(\gamma a^0[F, a^1] \ldots [F, a^n])$$

for n even (even Fredholm module), and

$$\sqrt{2i}(-1)^{\frac{n(n-1)}{2}}\Gamma\left(\frac{n}{2}+1\right)\mathrm{Tr}'(\gamma a^0[F,a^1]\dots[F,a^n])$$

for n odd.

Remark 4.22. Let ΩA be the universal differential graded algebra, $\mathbb{N}$-graded. It is also $\mathbb{Z}/2$-graded algebra with respect to the Fedosov product

$$\omega_1\circ\omega_2=\omega_1\omega_2+(-1)^{|\omega_1|}d\omega_1 d\omega_2.$$

Supertraces $\mathrm{Tr}\colon \Omega A\to\mathbb{C}$ are linear maps which satisfy the suspension conditions.

Theorem 4.23 (Connes, Cuntz-Quillen). *There is one-to-one correspondence between (harmonic) periodic cocycles and supertraces on ΩA.*

$$(\Omega A, b, B)\to \text{ (entire) cyclic type homology theories.}$$

4.6 Index pairing formula

From Atiyah and Kasparov we have the following result:

Proposition 4.24. *Let $\mathcal{A}$ be an involutive algebra, $(\mathcal{H},F)$ a Fredholm module over $\mathcal{A}$. For $q\in\mathbb{N}$ let $(\mathcal{H}_q,F_q)$ be the Fredholm module over $M_q(\mathcal{A})=\mathcal{A}\otimes M_q(\mathbb{C})$, $\mathcal{H}_q=\mathcal{H}\otimes\mathbb{C}^q$, $F_q=F\otimes\mathrm{id}_q$. Extend the action of $\mathcal{A}$ on $\mathcal{H}$ to a unital action of $\widetilde{\mathcal{A}}$.*

1. *In the even case: let γ be the involution for $\mathbb{Z}/2$-grading and $e\in M_q(\widetilde{\mathcal{A}})$ be a projection. Then the operator*

$$\pi_q^-(e)F_q\pi_q^+(e)\colon \pi_q^+(e)\mathcal{H}_q^+\to\pi_q^-(e)\mathcal{H}_q^-$$

is Fredholm. There is an additive map

$$\varphi\colon\ \mathrm{K}_0(\mathcal{A})\to\mathbb{Z}$$

given by

$$\varphi([e]):=\mathrm{Index}(\pi_q^-(e)F_q\pi_q^+(e)).$$

2. *In the odd case: let $u\in\mathrm{GL}_q(\widetilde{\mathcal{A}})$, $E=\frac{1+F}{2}$. Then*

$$E_q\pi_q(u)E_q\colon E_q\mathcal{H}_q\to E_q\mathcal{H}_q$$

is Fredholm. There is an additive map

$$\varphi([u]):=\mathrm{Index}(E_q\pi_q(u)E_q).$$

When A is a C*-algebra, $\mathrm{K}_1(A)$ in 2. is the topological K-theory $\mathrm{K}_1^{top}(A)$ as defined before.

In both cases, the index map depends only on the class

$$[(\mathcal{H},F)]\in\mathrm{KK}_i(A,\mathbb{C})=\mathrm{K}^i(A),$$

the K-homology of A. This can be regarded as a pairing

$$\mathrm{K}_i(A)\times\mathrm{K}^i(A)\to\mathbb{Z}.$$

Proposition 4.25. *For $x \in \mathrm{K}_i(A)$*

$$\varphi(x) = \langle x,\, \mathrm{ch}_*(\mathcal{H}, F)\rangle = \langle \mathrm{ch}(x),\, \mathrm{ch}(\mathcal{H}, F)\rangle.$$

On the right hand side in the proposition we have a pairing between K-theory and cyclic sohomology. A more symmetric formula would use a complementary Chern character on K-homology. Since Connes' construction, formulae were given for Chern characters in K-theory with values in $\mathrm{HP}_*(\mathcal{A})$.

The pairing has simple definition. Let $\tau \in \mathrm{HC}^n(\mathcal{A})$. Take $\tau \otimes \mathrm{Tr}\colon M_k(\mathcal{A}) \to \mathbb{C}$ for every k,

$$\tau \otimes \mathrm{Tr}(a^0 \otimes T^0, a^1 \otimes T^1, \dots, a^n \otimes T^n) = \tau(a^0, \dots, a^n)\,\mathrm{Tr}(T^0, \dots .T^n).$$

Then

$$\langle [e],\, [\tau]\rangle = \frac{1}{m!}(\tau \otimes \mathrm{Tr})(e, e, \dots, e).$$

All this is explained in Quillen's higher traces paper.

4.7 Kasparov's K-homology

Let $(\rho, \mathcal{H}, F)$ be a Fredholm module, $U\colon \mathcal{H}' \to \mathcal{H}$ be a unitary isomorphism (preserving the grading if there is one). Then $(U^*\rho U, \mathcal{H}', U^*FU)$ is also a Fredholm module unitarily equivalent to $(\rho, \mathcal{H}, F)$.

Definition 4.26. *Suppose $(\rho, \mathcal{H}, F_t)$ is a family of Fredholm modules parametrized by $t \in [0, 1]$, $\mathcal{H}$ is fixed Hilbert space, and F_t varies with t. If the function $t \mapsto F_t$ is norm continuous, then we say that the family defines an* operator homotopy *between the Fredholm modules $(\rho, \mathcal{H}, F_0)$ and $(\rho, \mathcal{H}, F_1)$ an that these two Fredholm modules are* Operator homotopic*.*

Definition 4.27. *If $(\rho, \mathcal{H}, F)$ and $(\rho, \mathcal{H}, F')$ are Fredholm modules on $\mathcal{H}$ and $(F - F')\rho(a)$ is compact for all $a \in A$, then we call F a* compact perturbation *of F'.*

Compact perturbation impliest operator homotopy - the linear path from F to F' defines an operator homotopy.

One can perform a compact perturbation to make F exactly self adjoint, $F \mapsto \frac{1}{2}(F + F^*)$.

Definition 4.28. K-homology *of a C^*-algebra A, $\mathrm{K}^p(A)$, is an abelian group with one generator $[x]$ for each unitary equivalence class of graded Fredholm modules over A with the following relations:*

1. *If x_0 and x_1 are operator homotopic graded Fredholm modules, then $[x_0] = [x_1] \in \mathrm{K}^p(A)$.*

2. *If x_0 and x_1 are graded Fredholm modules then $[x_0 \oplus x_1] = [x_0] + [x_1]$ in $\mathrm{K}^p(A)$, where $x_0 \oplus x_1 = (A, \mathcal{H}_0 \oplus \mathcal{H}_1, \rho_0 \oplus \rho_1, F_0 \oplus F_1)$.*

We have $p = 0$ for graded, and $p = 1$ for ungraded Fredholm modules.

Remark 4.29. p-graded Fredholm modules give rise to lower K-homology $\mathrm{K}^{-p}(A)$ for all $p \in \mathbb{N}$.

$\mathrm{K}^p(A)$ is a contravariant functor in A. If $\alpha\colon A' \to A$ is a *-homomorphism, and $(\rho, \mathcal{H}, F)$ is a Fredholm A-module, then $(\rho \circ \alpha, \mathcal{H}, F)$ is a Fredholm A'-module. We have an induced map

$$\alpha^*\colon \mathrm{K}^p(A) \to \mathrm{K}^p(A').$$

Definition 4.30. *A Fredholm module $(\rho, \mathcal{H}, F)$ is* degenerate *if and only if*

$$\begin{aligned}[\rho(a), F] &= 0\\ \rho(a)(F^2 - 1) &= 0\\ \rho(a)(F - F^*) &= 0\end{aligned}$$

for all $a \in A$.

Proposition 4.31. *The class of a degenerate Fredholm module is zero in* $\mathrm{K}^p(A)$.

Proof. Let $x = (\rho, \mathcal{H}, F)$ be a degenerate Fredholm module. Then

$$x' := (\rho', \mathcal{H}', F'), \quad \mathcal{H}' := \bigoplus_{i=1}^{\infty}, \quad F' := \bigoplus_{i=0}^{\infty} F, \quad \rho' := \bigoplus_{i=0}^{\infty} \rho.$$

This is a Fredholm module, since x is degenerate. But $x \oplus x'$ is unitarily equivalent to x', so $[x \oplus x'] = [x] + [x'] = [x']$ and $[x] = 0$. □

Lemma 4.32. *For a graded Fredholm module $(\rho, \mathcal{H}, F)$ denote by $(\rho^{op}, \mathcal{H}^{op}, -F^{op})$ the Fredholm module with the opposite grading. This is the additive inverse to $(\rho, \mathcal{H}, F)$.*

Proof. Let

$$F_t := \begin{pmatrix} \cos(\frac{\pi}{2}t)F & \sin(\frac{\pi}{2}t)\mathrm{Id} \\ \sin(\frac{\pi}{2}t)\mathrm{Id} & -\cos(\frac{\pi}{2}t)F \end{pmatrix},$$

$$F_0 = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is the operator homotopy on $\mathcal{H} \oplus \mathcal{H}^{op}$ from $F_0 = F \oplus (-F^{op})$ to the degenerate $F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. □

Example 4.33. $\mathrm{K}^0(\mathbb{C}) = \mathbb{Z}$. If $(\rho, \mathcal{H}, F)$ is a Fredholm module over $\mathbb{C}$, then $\rho(1) =: p$ is a projection in $B(\mathcal{H})$ and up to compact perturbation $(\rho, \mathcal{H}, F)$ is the direct sum of $(\rho, p\mathcal{H}, pFp)$ and $(\rho, (1-p)\mathcal{H}, (1-p)F(1-p))$. The second module carries the zero action of $\mathbb{C}$. The first is determined by pFp. Put

$$(\rho, \mathcal{H}, F) \mapsto \mathrm{Index}(pFp).$$

This gives a homomorphism $\mathrm{K}^0(\mathbb{C}) \to \mathbb{Z}$. Since an essentially unitary operator with index zero is a compact perturbation of a unitary, this map is an isomorphism.

Lemma 4.34. *Let $(\rho, \mathcal{H}, F)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Fredholm module. Assume that there exists a self adjoint odd-graded involution $E\colon \mathcal{H} \to \mathcal{H}$ which commutes with ρ (the action of A) and anticommutes with F. Then the Fredholm module $(\rho, \mathcal{H}, F)$ represents the zero element in* $\mathrm{K}^0(A)$.

Proof. Let $F_t := \cos(t)F + \sin(t)E$. This is an operator homotopy from F to the degenerate operator E. □

In particular putting tho ungraded Fredholm modules together produces a degenerate Fredholm module. Conversely, if we ignore $\mathbb{Z}/2\mathbb{Z}$-grading on an even Fredholm module then the resulting odd Fredholm module represents the zero element. A possible argument is as follows. A $\mathbb{Z}/2\mathbb{Z}$-graded module is given by the data $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$,

$$F = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho(a) & 0 \\ 0 & \rho(a) \end{pmatrix}.$$

We construct an operator homotopy

$$F_t = \begin{pmatrix} \cos(\frac{\pi}{2}t)\mathrm{Id} & \sin(\frac{\pi}{2}t)v \\ \sin(\frac{\pi}{2}t)u & \cos(\frac{\pi}{2}t)\mathrm{Id} \end{pmatrix}$$

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}.$$

For this we need to assume that F_1 is an involution.

Chapter 5

Boundary maps in K-homology

5.1 Relative K-homology

Definition 5.1. *Let J be an ideal in A. A* relative Fredholm module *for $(A, A/J)$ is a triple $(\rho, \mathcal{H}, F)$, where*

1. *$\mathcal{H}$ is a separable Hilbert space*
2. *$\rho\colon A \to B(\mathcal{H})$ is a *-representation*
3. *for all $a \in A$, $j \in J$*

$$\begin{aligned} (F^2 - 1)\rho(j) &\sim 0 \\ (F - F^*)\rho(j) &\sim 0 \\ [F, \rho(a)] &\sim 0 \end{aligned}$$

One defines also the graded version.

The relative Fredholm modules generate *relative K-homology* $\mathrm{K}^p(A, A/J)$. The natural map

$$\mathrm{K}^p(A, A/J) \to \mathrm{K}^p(J)$$

is an isomorphism (excision).

To any extension of separable C*-algebras one can associate an exact sequence of lenght six

$$\begin{tikzcd} \mathrm{K}^1(A/J) \arrow[r] & \mathrm{K}^1(A) \arrow[r] & \mathrm{K}^1(J) \arrow[d] \\ \mathrm{K}^0(J) \arrow[u] & \mathrm{K}^0(A) \arrow[l] & \mathrm{K}^0(A/J) \arrow[l] \end{tikzcd}$$

We can give an explicit description of the boundary maps in this six term exact sequence.

5.2 Semi-split extensions

There is ono-to-one correspondence between extensions of A by $\mathcal{K}(\mathcal{H})$ and unitary equivalence classes of *-homomorphisms $\phi\colon A \to \mathcal{Q}(\mathcal{H})$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r] & B(\mathcal{H}) \arrow[r, "\pi"] & \mathcal{Q}(\mathcal{H}) \arrow[r] & 0
\end{tikzcd}$$

Definition 5.2. *A unital injective extension $\phi\colon A \to \mathcal{Q}(\mathcal{H})$ is semi-split if there is another unital extension $\phi'\colon A \to \mathcal{Q}(\mathcal{H})$ such that $\phi \oplus \phi'$ is split extension.*

Definition 5.3. *Let the extension*

$$0 \to J \to A \to A/J \to 0$$

be semi-split by a completely positive map $\widetilde{A/J} \to \widetilde{A}$. Let $\rho\colon A \to B(\mathcal{H})$ be a representation of A on a separable Hilbert space $\mathcal{H}$. A Stinespring dilation *associates to the above data is a *-homomorphism*

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} : A/J \to B(\mathcal{H} \oplus \mathcal{H}'),$$

where $\mathcal{H}'$ is a separable Hilbert space and $\psi_{11}(x) = \rho(s(x))$.

The existence of such extension follows from Stinespring's theorem.

Theorem 5.4 (Stinespring)**.** *A unital linear map $\sigma\colon A \to B(\mathcal{H})$ is absolutely positive if and only if there are*

1. *an isometry $v\colon \mathcal{H} \to \mathcal{H}$*
2. *a nondegenerate representation $\rho\colon A \to B(\mathcal{H})$ such that $\sigma(a) = v^*\rho(a)v$*

In $\mathbb{Z}/2\mathbb{Z}$-graded case one applies this to each component separately.

Proposition 5.5. *Take an extension as above. Let $(\rho, \mathcal{H}, F)$ be a selfadjoint relative Fredholm module (graded or ungraded). Let $\psi\colon A/J \to B(\mathcal{H} \oplus \mathcal{H}')$ be a Stinespring dilaton. Then the boundary maps are given by*

1. *$\partial\colon \mathrm{K}^0(A, A/J) \to \mathrm{K}^1(A/J)$: the cycle $(\rho, \mathcal{H}, F)$ is graded Fredholm module. Assume that F^2 is a projection (this can always be done). Let $Q_\pm$ be the components of the projection $1 - F^2$ on $\mathcal{H}^\pm$. Then the projections*

   $$\begin{pmatrix} Q_\pm & 0 \\ 0 & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H}')$$

   commute modulo compacts with $\psi(x)$ for all $x \in A/J$ and so define ungraded Fredholm modules. Their difference represents a class of $\partial[\rho, \mathcal{H}, F]$ (if $\rho\colon A \to B(\mathcal{H})$, $P \in B(\mathcal{H})$ is a projection such that $[P, \rho(a)] \in \mathcal{K}$ for all $a \in A$, then $(\rho, \mathcal{H}, F = 2P - 1)$ is an ungraded Fredholm module over A).

2. *$\partial\colon \mathrm{K}^1(A, A/J) \to \mathrm{K}^0(A/J)$: the cycle $(\rho, \mathcal{H}, F)$ is ungraded Fredholm module. Then*

   $$\begin{pmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H}')$$

   is unitary on $\mathcal{H} \oplus \mathcal{H}'$ commuting with ψ. The corresponding Fredholm module represents a class of $\partial[\rho, \mathcal{H}, F]$.

5.3 Schrödinger pairs

Recall that we call an operator $X \in B(\mathcal{H})$ *contractive* if and only if $\|X\| \leq 1$. For a selfadjoint contractive operator X we define X^b to be a commutative C*-subalgebra of $B(\mathcal{H})$ consisting of all $\psi(X)$ for $\psi \in C_0(-1,1)$.

Definition 5.6. *Let X, Y be contractive operators on $\mathcal{H}$. The pair (X,Y) is a* Schrödinger *pair if and only if*

1. *Y commutes with X^b modulo $\mathcal{K}(\mathcal{H})$.*
2. *$X^b \cdot Y^b \subseteq \mathcal{K}(\mathcal{H})$.*

We call (X,Y) a graded Schrödinger pair *if the commutator in 1 is graded.*

We call (X,Y) a strong Schrödinger pair *if Y commutes with X.*

Example 5.7. Let $\mathcal{H} = L^2(\mathbb{R})$, X multiplies by $x \mapsto \frac{x}{\sqrt{1+x^2}}$, and Y multiplies the Fourier transform by $\xi \mapsto \frac{\xi}{\sqrt{1+\xi^2}}$

$$(Xf)(x) := \frac{x}{\sqrt{1+x^2}} f(x)$$
$$(Y\hat{f})(\xi) := \frac{\xi}{\sqrt{1+\xi^2}} \hat{f}(\xi)$$

These are the position and momentum operators in quantum mechanics and (X,Y) is a strong Schrödinger pair.

Example 5.8. Let $(\rho, \mathcal{H}, F)$ be an ungraded Fredholm module over J, where J is an ideal in some C*-algebra A and ρ extends to A. If a is an element of A such that $a^2 - 1 \in J$, then $X = \rho(a)$, $Y = F$ constitute Schrödinger pair.

If the extension of ρ makes $(\rho, \mathcal{H}, F)$ into a relative Fredholm module (i.e. $[F, \rho(a)] \sim 0$ for all $a \in A$) then (X,Y) is a strong Schrödinger pair.

Example 5.9. Let $\mathcal{H} = L^2([-1,1])$. Define operators

$$(T\psi)(x) = \psi(-x),$$
$$(S\psi)(x) = x\psi(x).$$

If $f \in C_0(-1,1)$ then $f(T) = 0$ and (T,S) is a Schrödinger pair. If f is an odd function on $[-1,1]$ such that $f(-1) = f(1) = 0$, then

$$[f(S), T] = -2f(S)T$$

which is not compact. Thus (S,T) is not a Schrödinger pair.

Definition 5.10. *Let (X,Y) be a Schrödinger pair. The* Schrödinger operator *is given by*

$$V(X,Y) := \varepsilon X + (1 - X^2)^{\frac{1}{2}} Y,$$

where $\varepsilon = i$ in the ungraded, and $\varepsilon = 1$ in the graded case.

Proposition 5.11. *Let (X,Y) be a (graded) strong Schrödinger pair. Then*

1. *in the ungraded case the Schrödinger operator*

$$V(X,Y) = iX + (1 - X^2)^{\frac{1}{2}} Y$$

is essentially unitary and so Fredholm,

2. *in the graded case the Schrödinger operator*

$$V(X,Y) = X + (1 - X^2)^{\frac{1}{2}} Y$$

is essentially selfadjoint graded and Fredholm.

Proposition 5.12.

1. *In the ungraded case*

$$\operatorname{Index} V(X,Y) = \operatorname{Index} V(Y,X).$$

2. *In the graded case*

$$\operatorname{Index} V(X,Y) = -\operatorname{Index} V(Y,X).$$

Proposition 5.13. *If X, Y commute modulo compacts with a representation of C^*-algebra B then*

1. *in the graded case*

$$[V(X,Y)] = [V(Y,X)] \in \mathrm{K}^0(B),$$

2. *in the ungraded case*

$$[V(X,Y)] = -[V(Y,X)] \in \mathrm{K}^1(B).$$

Proof.

$$\begin{aligned} V(X,Y)^2 - 1 &\sim X^2 + (1 - X^2)Y^2 - 1 \\ &= -(1 - X^2)(1 - Y^2) \in \mathcal{K}(\mathcal{H}), \end{aligned}$$

because $XY + YX \sim 0$. Next

$$V(X,Y)V(Y,X) + V(Y,X)V(X,Y) \sim 2(Y(1 - X^2)^{\frac{1}{2}} Y + X(1 - Y^2)^{\frac{1}{2}} X) \geq 0,$$

so Fredholm modules associated with $V(X,Y)$ and $V(Y,X)$ are homotopic, which is a consequence of the following

Proposition 5.14. *If $(\rho, \mathcal{H}, F_0)$ and $(\rho, \mathcal{H}, F_1)$ are (graded) Fredholm modules such that $\rho(a)(F_0F_1 + F_1F_0)\rho(a^*)$ are positive modulo compacts for all $a \in A$, then F_0 and F_1 are operator homotopic.*

□

Recall the index map on K-homology

$$\operatorname{Index}\colon \mathrm{K}^p(A) \to \mathbb{Z}, \quad (\rho, \mathcal{H}, F) \mapsto \inf F.$$

If $F = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$, then $\operatorname{Index} F = \operatorname{Index} u$.

Lemma 5.15. *Let (X,Y) be a Schrödinger pair on an ungraded Hilbert space $\mathcal{H}$. Put $P_Y := \frac{1}{2}(1+Y)$. Then the operator*

$$W_1(X,Y) := e^{i\pi X}P_Y - (1-P_Y)$$

is essentially unitary and Fredholm. Furthermore

$$\operatorname{Index} W_1(X,Y) = \operatorname{Index} V(X,Y).$$

Proof. Denote for convinience $S := \sin(\frac{\pi}{2}X)$. From the definition of Schrödinger pair we know that

- Y commutes with S^b modulo compacts,
- $(1-S^2)(1-Y^2)$ is compact.

Write

$$\begin{aligned}
e^{-i\frac{\pi}{2}X}W_1(X,Y) &= e^{i\frac{\pi}{2}X}P_Y - e^{-i\frac{\pi}{2}X}(1-P_Y)\\
&= (\cos(\frac{\pi}{2}X) + i\sin(\frac{\pi}{2}X))P_Y - (\cos(\frac{\pi}{2}X) - i\sin(\frac{\pi}{2}X))(1-P_Y)\\
&= ((1-S^2)^{\frac{1}{2}} + iS)P_Y - ((1-S^2)^{\frac{1}{2}} - iS)(1-P_Y)\\
&= ((1-S^2)^{\frac{1}{2}} + iS)(\tfrac{1}{2}(1+Y)) - ((1-S^2)^{\frac{1}{2}} - iS)(\tfrac{1}{2}(1-Y))\\
&= iS + (1-S^2)^{\frac{1}{2}}Y\\
&= V(S,Y).
\end{aligned}$$

Thus the operator $W_1(X,Y)$ is essentially unitary and homotopic to $V(S,Y)$ through the path

$$t \mapsto e^{-it\frac{\pi}{2}X}W_1(X,Y),\ t \in [0,1],$$

so $[W_1(X,Y)] = [V(S,Y)]$. But S is homotopic to X via a path $X_t := tX + (1-t)S$, and (X_t,Y) are a Schrödinger pairs for all t. This gives a homotopy $V(X_t,Y)$ and $[V(X,Y)] = [V(S,Y)] = W_1(X,Y)$. □

Lemma 5.16. *Let (X,Y) be a graded Schrödinger pair on a graded Hilbert space $\mathcal{H}$. Suppose also that*

1. *$Q_X = 1 - X^2$ is a projection*
2. *there exists a self adjoint involution Y_0 on $\mathcal{H}$ which commutes with Q_X modulo the compacts.*

Then the operator $W_2(X,Y) = YQ_X + Y_0(1-Q_X)$ is essentially self-adjoint, graded, Fredholm, and

$$\operatorname{Index} W_2(X,Y) = \operatorname{Index} V(X,Y).$$

Proof. Let

$$X_t := \cos\left(\frac{\pi}{2}t\right)X + \sin\left(\frac{\pi}{2}t\right)Y_0(1-Q_X).$$

Then for all $t \in [0,1]$ we have

$$\begin{aligned} X_t(YQ_X) &\sim 0 \\ (YQ_X)X_t &\sim 0 \\ X_t^2 &\sim 1 - Q_X \end{aligned}$$

Indeed:

$$\begin{aligned} X_t^2 &= \cos^2\left(\frac{\pi}{2}t\right) X^2 + \sin^2\left(\frac{\pi}{2}t\right) Y_0(1-Q_X)Y_0(1-Q_X) \\ &\quad + \cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right) XY_0(1-Q_X) + \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right) Y_0(1-Q_X)X \\ &= \cos^2\left(\frac{\pi}{2}t\right) X^2 + \sin^2\left(\frac{\pi}{2}t\right)(1-Q_X)^2. \end{aligned}$$

The operator Q_X is a projection onto $\ker X$. The path $t \mapsto YQ_X + X_t$ gives an operator homotopy from $V(X,Y)$ to $W_2(X,Y)$. Indeed:

$$Q_X X_t = \cos\left(\frac{\pi}{2}t\right) \underbrace{Q_X X}_{0} + \sin\left(\frac{\pi}{2}t\right) Q_X Y_0(1-Q_X) \in \mathcal{K}(\mathcal{H}).$$

Recall that $V(X,Y) = X + (1-X^2)^{\frac{1}{2}}Y$. Thus for

$$\begin{aligned} t = 0: &\quad YQ_X + X_t = YQ_x + X \sim V(X,Y), \\ t = 1: &\quad YQ_X + Y_0(1-Q_X) = W_2(X,Y). \end{aligned}$$

□

5.4 The index pairing

Proposition 5.17 (odd case)**.** *Let A be a unital C^*-algebra and suppose given*

1. *an ungraded unital Fredholm module $(\rho, \mathcal{H}, F)$ over A,*
2. *a unitary u in a matrix algebra $M_k(A)$ over A.*

Let $P_k = 1 \otimes \frac{1}{2}(1+F)\colon \mathcal{H}^k \to \mathcal{H}^k$ and $U = (1 \otimes \rho)(u)\colon \mathcal{H}^k \to \mathcal{H}^k$ be a unitary operator. Then:

1. *the operator $W := P_kUP_k - (1-P_k)\colon \mathcal{H}^k \to \mathcal{H}^k$ is essentially unitary, so Fredholm.*
2. *The Fredholm index of $W = P_kUP_k - (1-P_k)$ depends only on $[U] \in \mathrm{K}_1(A)$ and $[P] \in \mathrm{K}^1(A)$. This index gives a pairing*

$$\mathrm{K}_1(A) \times \mathrm{K}^1(A) \to \mathbb{Z}$$

$$([u],[F]) \mapsto \operatorname{Index}(P_kUP_k - (1-P_k)).$$

Assume that $F^2 = 1$ so that P_k is a projection. Then $\mathcal{H}^k = \operatorname{im} P_k \oplus \operatorname{im}(1-P_k)$ and $W = P_k U P_k \oplus (-\mathrm{Id})$ with respect to this decomposition. The second summand has index zero, and this is precisely the pairing that was defined before.

By definition of Fredholm module P_k and U commute modulo compacts $\mathcal{K}(\mathcal{H}^k)$ and

$$W^*W \sim P_k U^* U P_k + (1 - P_k) \sim 1.$$

Thus $W^*W \sim 1$ and similarly $WW^* \sim 1$. The map $([u], [F]) \mapsto \operatorname{Index} W$ is additive and stable under compact perturbations and homotopies.

Proposition 5.18 (even case)**.** *Let $(\rho, \mathcal{H}, F)$ be a graded unital Fredholm module over A and let $p \in M_k(A)$ be a projection. Put $P = 1 \otimes \rho(p) \colon \mathcal{H}^k \to \mathcal{H}^k$ (projection) and write*

$$F = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$$

relative to the graded decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Then

1. *the operator $P(1 \otimes u)P \colon P(\mathbb{C}^n \otimes \mathcal{H}^+) \to P(\mathbb{C}^n \otimes \mathcal{H}^-)$ is essentially unitary and so Fredholm.*

2. *the pairing*

$$(p, F) \mapsto \operatorname{Index} P(1 \otimes u)P$$

depends only on the K-theory class of $[p]$ and K-homology class of $(\rho, \mathcal{H}, F)$.

Example 5.19. Let $\alpha \colon A \to \mathbb{C}$ be a *-homomorphism. Define $(\rho, \mathcal{H}, F)$ by $\mathcal{H} = \mathbb{C} \oplus 0$, $\rho = \alpha \oplus 0$, $F = 0$. The index pairing with this Fredholm module gives a homomorphism $i_\alpha \colon \mathrm{K}_0(A) \to \mathbb{Z}$ which by definition sends a projection p to the index of the zero operator $0 \colon \operatorname{im} \alpha(p) \to 0$, hence the index pairing gives the same map as

$$\alpha_* \colon \mathrm{K}_0(A) \to \mathrm{K}_0(\mathbb{C}).$$

Theorem 5.20. *Let J be an ideal in a separable C^*-algebra A for which the quotient map $A \to A/J$ is semi-split. We denote by ∂_0, ∂_1 the connecting homomorphisms in K-theory and by ∂^0, ∂^1 the connecting homomorphisms in K-homology.*

If $x \in \mathrm{K}_0(A/J)$ and $y \in \mathrm{K}^1(J)$ then

$$\langle \partial_0 x, y, y \rangle = -\langle x, \partial^1 y \rangle.$$

If $x \in \mathrm{K}_1(A/J)$ and $y \in \mathrm{K}^0(J)$ then

$$\langle \partial_1 x, y \rangle = \langle x, \partial^0 y \rangle.$$

Proof. The six term exact sequences in K-theory and K-homology:

$$\begin{tikzcd}
\mathrm{K}_0(J) \arrow[r] & \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(A/J) \arrow[d, "\partial_0"] \\
\mathrm{K}_1(A/J) \arrow[u, "\partial_1"] & \mathrm{K}_1(A) \arrow[l] & \mathrm{K}_1(J) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd} \mathrm{K}^0(A/J) \arrow[r] & \mathrm{K}^0(A) \arrow[r] & \mathrm{K}^0(J) \arrow[d, "\partial^0"] \\ \mathrm{K}^1(J) \arrow[u, "\partial^1"] & \mathrm{K}^1(A) \arrow[l] & \mathrm{K}^1(A/J) \arrow[l] \end{tikzcd}$$

We shall assume that A/J is unital. The strategy is to construct a Schrödinger operator V and show using two diefferent deformation arguments that

$$\operatorname{Index} V = \langle \partial_{0,1} x,\, y\rangle,$$
$$\operatorname{Index} V = \mp\langle x,\, \partial^{1,0} y\rangle$$

Case 1.

Step 1. Suppose we are given a short exact sequence of separable C*-algebras

$$0 \to J \to A \to A/J \to 0.$$

Let $(\rho, \mathcal{H}, F)$ be an ungraded Fredholm module for J. Let $p \in M_k(A/J)$ be a projection and let $a \in M_n(A)$ be a lift of p. Then

$$X = (1 \otimes \rho)(2a-1), \quad Y = 1 \otimes F$$

form a Schrödinger pair. If $(\rho, \mathcal{H}, F)$ is a relative Fredholm module for $(A, A/J)$ then (X, Y) is a strong Schrödinger pair. The map

$$(p, F) \mapsto \operatorname{Index} V(X, Y)$$

defines pairings

$$\mathrm{K}_0(A/J) \otimes \mathrm{K}^1(J) \to \mathbb{Z}$$
$$\mathrm{K}_0(A/J) \otimes \mathrm{K}^1(A, A/J) \to \mathbb{Z}$$

which are compatible with the excision isomorphism $\mathrm{K}^1(A, A/J) \xrightarrow{\cong} \mathrm{K}^1(J)$. For $x \in \mathrm{K}_0(A/J)$, $y \in \mathrm{K}^1(J)$ denote this pairing by $x \cdot y$.

Step 2. $x \cdot y = -\langle x, \partial^1 y\rangle$.
Assume that $x = [p]$ with $p \in A/J$ (similar arguments works for matrices) and $y = [(\rho, \mathcal{H}, F)]$.

$$\partial^1 y = \left[\left(\psi, \mathcal{H}\oplus\mathcal{H}', \begin{pmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{pmatrix}\right)\right],$$

where $\psi\colon A/J \to B(\mathcal{H}\oplus\mathcal{H}')$ is a representation obtained from a completely positive section $s\colon \widetilde{A/J} \to \widetilde{A}$ by composing with ρ and then applying Stinespring's dilation.
Put

$$\widehat{X} = \psi(2p-1) \in B(\mathcal{H}\oplus\mathcal{H}'), \quad \widehat{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Then $X_{11} = \rho(2a-1)$, $a \in A$ is a lift of $p \in A/J$. This is the operator which appears in the definition of Schrödinger pairing.
If $Y = F$ then $X = X_{11}$ and Y form a strong Schrödinger pair and $x \cdot y = \operatorname{Index} V(X, Y)$. Now

$$\widehat{Y} = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}, \quad \widehat{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

form a Schrödinger pair (not strong). Now

$$(\widehat{X})^2 = \psi((2p-1)^2) = 1$$

$$X_{12}X_{21} = 1 - X_{11}^2 = 1 - X^2 \in \rho(J).$$

Furthermore $X = \rho(2a-1)$, and $2a-1$ lifts $2p-1$, $(2p-1)^2 = 1$, so $(2a-1)^2 = 1 + j$, where $j \in J$. Now $X^2 = \rho((2a-1)^2) = 1 + \rho(j)$ and we get the required compactness conditions from the definition of Fredholm module. Indeed, $1 - Y^2 = 1 - F^2$, so

$$(1-F^2)\rho(j) = (1-F^2)(1-X^2) \in \mathcal{K}(\mathcal{H}).$$

Essentially the same calculation will show that

$$V(\widehat{Y}, \widehat{X}) \sim \begin{pmatrix} V(X,Y) & 0 \\ 0 & 1 \end{pmatrix}.$$

By the proposition

$$\operatorname{Index} V(\widehat{Y}, \widehat{X}) = \operatorname{Index} V(Y,X) = -\operatorname{Index} V(X,Y) = -x \cdot y.$$

If $P_{\widehat{X}} = \frac{1}{2}(\widehat{X}+1) = \psi(p)$ then using the formula for $\partial^1 y$

$$\begin{aligned}\langle x,\, \partial^1 y\rangle &= \operatorname{Index}\left(\begin{pmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{pmatrix} P_{\widehat{X}} - (1 - P_{\widehat{X}})\right) \\ &= \operatorname{Index}(e^{i\pi F} P_{\widehat{X}} - (1 - P_{\widehat{X}})) \\ &= \operatorname{Index} W_1(\widehat{Y}, \widehat{X}).\end{aligned}$$

But we have seen that $\operatorname{Index}(W_1(\widehat{Y}, \widehat{X})) = \operatorname{Index}(V(\widehat{Y}, \widehat{X}))$ so

$$\langle x,\, \partial^1 y\rangle = \operatorname{Index} W_1(\widehat{Y}, \widehat{X}) = \operatorname{Index}(V(\widehat{Y}, \widehat{X})) = \operatorname{Index}(V(X,Y)) = -x{\cdot}y.$$

Step 3. As before, assume that a projection $p \in A/J$ has a lift to a self adjoint $a \in A$, and that y is represented by $(\rho, \mathcal{H}, F)$. Put $X := \rho(2a-1)$, $Y := F$. The boundary map in K-theory gives

$$\partial_0 x = [e^{2\pi i a}] \in M_k(\widetilde{J}).$$

The index pairing is

$$\langle \partial_0 x,\, y\rangle = \operatorname{Index}((1 \otimes \rho)e^{2\pi i a} P_Y + (1 - P_Y)),$$

where $P_Y = \frac{1}{2}(1+Y)$. Put $T := -e^{-i\pi X} P_Y + (1-P_Y) = -W_1(X,Y)$. Then

$$\langle \partial_0 x,\, y\rangle = \operatorname{Index}(T) = \operatorname{Index}(W_1(X,Y)) = \operatorname{Index}(V(X,Y)) = x \cdot y,$$

so

$$\langle \partial_0 x,\, y\rangle = -\langle x,\, \partial^1 y\rangle.$$

Case 2.

Step 1. In the graded case take a short exact sequence

$$0 \to J \to A \to A/J \to 0$$

and a graded Fredholm module $(\rho, \mathcal{H}, F)$ for $(A, A/J)$. Let $u \in M_n(A/J)$ be a unitary matrix and $a \in M_n(A)$ a lift of u to A. Then

$$X = \begin{pmatrix} 0 & (1 \otimes \rho)(a^*) \\ (1 \otimes \rho)(a) & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 \otimes F & 0 \\ 0 & -1 \otimes F \end{pmatrix}$$

on a graded Hilbert space $\mathbb{C}^n \otimes \mathcal{H} \oplus \mathbb{C}^n \otimes \mathcal{H}^{op}$ form a strong Schrödinger pair. Furthermore

$$(u, F) \mapsto \mathrm{Index}(V(X, Y))$$

defines a bilinear pairing $\mathrm{K}_1(A/J) \times \mathrm{K}^0(A, A/J) \to \mathbb{Z}$ denoted again by $x \cdot y$.

Step 2. For $x \in \mathrm{K}_1(A/J)$, $y \in \mathrm{K}^0(A, A/J)$ we have $x \cdot y = \langle x, \partial^0 y \rangle$. Assume that $y = [(\rho, \mathcal{H}, F)]$ is a graded relative module for $\mathrm{K}^0(A, A/J)$. Use the descritption of boundary map $\partial^0 \colon \mathrm{K}^0(J) \to \mathrm{K}^1(A/J)$, so assume that $(\rho, \mathcal{H}, F)$ is paritally isometric i.e. $Q := 1 - F^2$ is a projection with graded components $Q^{\pm}$. Then $\partial^0[y] = [Q^+] - [Q^-]$.
Assume that u is a unitary in A/J representing x. Define

$$X := \begin{pmatrix} 0 & \rho(a^*) \\ \rho(a) & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}^{op}$. Then by definition $x \cdot y = \mathrm{Index}(V(X, Y)) = \mathrm{Index}(V(Y, X))$. Put

$$Q_Y := 1 - Y_2 = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

The operator $X_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is odd and commutes modulo compacts with Y^2. By Lemma 6.10 for $W_2(Y, X) := XQ_Y + X_0(1 - Q_Y)$

$$\mathrm{Index}(V(Y, X)) = \mathrm{Index}(W_2(Y, X)) = \langle x, [Q^+] \rangle - \langle x, [Q^-] \rangle = \langle x, \partial^0 y \rangle.$$

Step 3. Assume that A is unital. Let $y \in \mathrm{K}^0(A, A/J)$ be represented by a graded relative Fredholm module $(\rho, \mathcal{H}, F)$, and $x \in \mathrm{K}_1(A/J)$ represented by a unitary $u \in A/J$. Then u lifts to a partial isometry $a \in A$. Put

$$X := \begin{pmatrix} 0 & \rho(a^*) \\ \rho(a) & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}.$$

Then $x \cdot y = \mathrm{Index}(V(X, Y))$, and

$$Q_X := 1 - X^2 = \begin{pmatrix} \rho(1 - a^*a) & 0 \\ 0 & \rho(1 - aa^*) \end{pmatrix}$$

is a projection onto $\ker X$. We have furthermore

$$V(X, Y) = X + (1 - X^2)Y = (1 - Q_X)X(1 - Q_X) + Q_X Y Q_X,$$

$$\mathrm{Index}(V(X,Y)) = \mathrm{Index}(Q_X Y Q_X).$$

Using the boundary formula for the boundary map in K-theory

$$\partial_1 x = [1 - a^* a] - [1 - a a^*] \in \mathrm{K}_0(J)$$

we get

$$\mathrm{Index}(Q_X Y Q_X) = \langle \partial_1 x,\, y\rangle.$$

□

5.5 Product of Fredholm operators

The construction of the index pairing by means of Schrödinger operators is a special case of the Kasparov product.

Let F_1, F_2 be graded Fredholm operators, $F_i = \begin{pmatrix} 0 & U_i^* \\ U_i & 0 \end{pmatrix}$, on a graded Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. A graded Fredholm operator F on $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ (graded Hilbert space product) is *aligned* with F_i if $F(F_i \widehat{\otimes} 1) + (F_i \widehat{\otimes} 1)F \geq 0$ modulo compacts.

Proposition 5.21. *Let F_i be graded Fredholm operator on $\mathcal{H}_i$, $i = 1, 2$. There exist graded Fredholm operators on $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ which are simultaneously aligned with F_1 and F_2. Any two such operators have the same index.*

Proof. Define $F \in B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ by $F := F_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} F_2$. Then

$$F(F_i \widehat{\otimes} 1) + (F_i \widehat{\otimes} 1)F = 2(F_i^2 \otimes 1) \geq 0,$$

so F is aligned with both F_1, F_2. Moreover

$$\mathrm{Index}(F) = \mathrm{Index}(F_1) \cdot \mathrm{Index}(F_2),$$

so F is a good model for the product of F_i, but we need $F^2 - 1 \sim 0$.

Lemma 5.22. *Let F_i are graded Fredholm operators on $\mathcal{H}_i$, and N_i a positive operators on $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ such that*

1. $N_1^2 + N_2^2 = 1$,
2. $[N_i, F_j \widehat{\otimes} 1] \sim 0$, $i, j = 1, 2$,
3. $N_i (F_i \widehat{\otimes} 1)^2 \sim N_i$.

Then

$$F := N_1^{\frac{1}{2}} (F_1 \widehat{\otimes} 1) N_1^{\frac{1}{2}} + N_2^{\frac{1}{2}} (1 \widehat{\otimes} f_2) N_2^{\frac{1}{2}}$$

is an odd Fredholm operator aligned with F_1, F_2. Moreover $F^2 \sim 1$.

□

Let $(\rho_i, \mathcal{H}_i, F_i)$ be graded Fredholm modules over C*-algebras A_i, $i = 1, 2$. Define a representation of $A_1 \otimes A_2$ on $B(\mathcal{H})$, $\mathcal{H} = \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$:

$$\rho\colon A_1 \otimes A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2), \quad \rho(a_1 \otimes a_2) := \rho_1(a_1)\rho_2(a_2).$$

We say that Fredholm module $(\rho, \mathcal{H}, F)$ is aligned with $(\rho_i, \mathcal{H}_i, F_i)$ if

$$\rho(a)(F(F_i \widehat{\otimes} 1) + (F_i \widehat{\otimes} 1)F)\rho(a^*) \geq 0 \mod \mathcal{K}(\mathcal{H})$$

for all $a \in A_1 \otimes A_2$, and if $\rho(a)F$ derives $\mathcal{K}(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ for all $a \in A_1 \otimes A_2$, that is

$$[\rho(a)F, K_1 \otimes T_2] \in \mathcal{K}_1(\mathcal{H}_1) \otimes B(\mathcal{H}_2), \quad \forall K_1 \otimes T_2 \in \mathcal{K}(\mathcal{H}_1) \otimes B(\mathcal{H}_2).$$

Proposition 5.23. *Let $(\rho_i, \mathcal{H}_i, F_i)$ be graded Fredholm modules over separable C^*-algebras A_1 and A_2. There exist Fredholm modules F which are aligned with F_1 and F_2. Moreover the operator homotopy class of such an F is determined uniquely by the operator homotopy classes of F_1 and F_2.*

The hard part is to prove existence of such Fredholm modules.

Definition 5.24. *The module F from the proposition is the* product *of F_1 and F_2.*

Chapter 6

Equivariant KK-theory

6.1 K-homology revisited

K-homology was discussed in depth in Chapter 4; here we recall the main points in a form that is suitable for our purposes and prepares the ground for a discussion of equivariant K-homology, which will be our main objective here. For more information see [b-b98] or [hr00].

Let A be a separable C*-algebra (A has a countable dense subset). We will define generalized elliptic operators over A in the odd and even case.

Definition 6.1 (odd case)**.** *A generalized odd elliptic operator over* A *is a triple* $(\mathcal{H}, \psi, T)$ *such that*

1. *$\mathcal{H}$ is a separable Hilbert space,*
2. *$\psi\colon A \to B(\mathcal{H})$ is a *-homomorphism,*
3. *$T \in B(\mathcal{H})$*

and

$$T = T^*, \quad \psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1 - T^2) \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$.

*We will denote the set of such triples by $\mathcal{E}^1(A)$. If $\varphi\colon A \to B$ is a *-homomorphism then there is an induced map*

$$\varphi^*\colon \mathcal{E}^1(B) \to \mathcal{E}^1(A), \quad \varphi^*(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi \circ \varphi, T).$$

Example 6.2. $S^1 := \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1^2 + t_2^2 = 1\}$, $A = C(S^1)$, $\psi\colon C(S^1) \to B(L^2(S^1))$

$$\psi(\alpha)(u) = \alpha u, \quad \alpha \in C(S^1),\ u \in L^2(S^1),$$

$$(\alpha u)(\lambda) = \alpha(\lambda)u(\lambda), \quad \lambda \in S^1.$$

The Dirac operator D of S^1 is $-i\frac{\partial}{\partial\theta}$. If we take a basis $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ of $L^2(S^1)$, then

$$D(e^{in\theta}) = \left(-i\frac{\partial}{\partial\theta}\right)(e^{in\theta}) = ne^{in\theta}.$$

Set $T = D(I + DD)^{-\frac{1}{2}}$. Then

$$T(e^{in\theta}) = \frac{n}{\sqrt{1+n^2}} e^{in\theta},$$

and $(L^2(S^1), \psi, T) \in \mathcal{E}^1(C(S^1))$.

We will define odd K-homology of A by

$$\mathrm{K}^1(A) := \mathcal{E}^1(A)/\sim (= \mathrm{KK}(A, \mathbb{C})),$$

where the relation $\sim$ is homotopy, which is defined below.

Definition 6.3. *Let $\xi = (\mathcal{H}, \psi, T)$, $\eta = (\mathcal{H}', \psi', T')$ be elements of $\mathcal{E}^1(A)$. We say that ξ is* isomorphic *to η, $\xi \cong \eta$ if there exists a unitary operator $U\colon \mathcal{H} \to \mathcal{H}'$ with commutativity in the diagrams*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "U"] \arrow[d, "T"'] & \mathcal{H}' \arrow[d, "T'"] \\ \mathcal{H} \arrow[r, "U"'] & \mathcal{H}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[r, "U"] \arrow[d, "\psi(a)"'] & \mathcal{H}' \arrow[d, "\psi'(a)"] \\ \mathcal{H} \arrow[r, "U"'] & \mathcal{H}' \end{tikzcd}$$

for all $a \in A$.

Definition 6.4. *We say that $\xi = (\mathcal{H}, \psi, T), \eta = (\mathcal{H}', \psi', T') \in \mathcal{E}^1(A)$ are* strictly homotopic *if there exists a continuous function $[0,1] \to B(\mathcal{H})$, $t \mapsto T_t$ such that*

1. *$T_0 = T$,*
2. *for all $t \in [0,1]$, $(\mathcal{H}, \psi, T_t) \in \mathcal{E}^1(A)$,*
3. *$(\mathcal{H}, \psi, T_1) \cong (\mathcal{H}', \psi', T')$.*

Definition 6.5. *We say that a generalized elliptic operator $(\mathcal{H}, \psi, T) \in \mathcal{E}^1(A)$ is* degenerate *if and only if*

$$\psi(a)T - T\psi(a) = 0, \quad \psi(a)(I - T^2) = 0, \quad \text{for all } a \in A.$$

Definition 6.6. *We say that $\xi = (\mathcal{H}, \psi, T), \eta = (\mathcal{H}', \psi', T') \in \mathcal{E}^1(A)$ are* homotopic, *$\xi \sim \eta$, if and only if there exist degenerate generalized elliptic operators $\widetilde{\xi}$, $\widetilde{\eta}$ with $\xi \oplus \widetilde{\xi}$ strictly homotopic to $\eta \oplus \widetilde{\eta}$.*

Definition 6.7. *The* odd K-homology *of a C^*-algebra A is defined as the group of homotopy classes of generalized odd elliptic operators,*

$$\mathrm{K}^1(A) := \mathcal{E}^1(A)/\sim.$$

It is an abelian group with respect to

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T')$$

with inverse defined by

$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, -T).$$

*If $\varphi\colon A \to B$ is a *-homomorphism, then there is an induced homomorphism of abelian groups*

$$\varphi^*\colon \mathrm{K}^1(B) \to \mathrm{K}^1(A), \quad \varphi^*(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi \circ \varphi, T).$$

Now we will define even elliptic operators and $\mathrm{K}^0(A)$. The discussion in Chapter 4 presents a definition of the even K-homology in terms of graded Hilbert spaces. Here we present an alternative definition that does not use $\mathbb{Z}/2\mathbb{Z}$-grading. There is a canonical isomorphism of abelian groups between the even K-homology defined in Chapter 4 and the group defined here.

Definition 6.8 (even case)**.** *A* generalized even elliptic operator *over A is a triple $(\mathcal{H}, \psi, T)$ such that*

1. *$\mathcal{H}$ is a separable Hilbert space,*
2. *$\psi\colon A \to B(\mathcal{H})$ is a *-homomorphism,*
3. *$T \in B(\mathcal{H})$*

and

$$\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1 - TT^*) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1 - T^*T) \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$.

We will denote the set of such triples by $\mathcal{E}^0(A)$.

Definition 6.9. *The* even K-homology *of a C^*-algebra A is the group of homotopy classes of generalized even elliptic operators,*

$$\mathrm{K}^0(A) := \mathcal{E}^0(A)/\sim .$$

It is an abelian group with respect to

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T')$$

with inverse defined by

$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, -T).$$

*If $\varphi\colon A \to B$ is a *-homomorphism, then there is an induced homomorphism of abelian groups*

$$\varphi^*\colon\ \mathrm{K}^0(B) \to \mathrm{K}^0(A), \quad \varphi^*(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi \circ \varphi, T).$$

Note that the basic difference between the even and odd cases is that in the odd case the operator T is self-adjoint and in the even case the operator T is not required to be self-adjoint.

6.2 Equivariant K-homology of spaces

Definition 6.10. *Let X be a Hausdorff topological space. Let G be a locally compact Hausdorff topological group. Assume given a continuous action*

$$G \times X \longrightarrow X$$

of G on X. This action is proper *if the following two conditions are satisfied:*

1. *The quotient space (with the quotient topology) X/G is Hausdorff.*

2. *For each $x \in X$ there exists (U, H, ρ) such that U is a G-invariant open neighbourhood of x, H is a compact subgroup of G, and $\rho\colon U \to G/H$ is a continuous G-equivariant map. (Here G/H is given the quotient topology.)*

Let G and X be as in the above definition. Assume that the action of G on X is proper. Given $x \in X$, choose (U, H, ρ) as in the above definition. Set $S = \rho^{-1}(eH)$, where e is the identity element of G. Observe that S is preserved by the action of H. Map $G \times_H S$ to U by $(g, s) \mapsto gs$. This is a G-equivariant homeomorphism from $G \times_H S$ onto U. Thus a proper G-space is locally induced from an H-space, where H is a compact subgroup of G.

In the literature, there are various definitions of proper action. These definitions are compared in [b-h03],[cem01]. For instance, another definition of proper action is:

Definition 6.11. *The action $G \times X \to X$ is* proper *if and only if the map*

$$G \times X \to X \times X, \quad (g, x) \mapsto (gx, x)$$

is proper. (For locally compact Hausdorff spaces a continuous map is proper if and only if the preimage of any compact set is compact.)

All the various definitions of proper action agree if both the group G and the space X are locally compact Hausdorff and second countable.

Examples 6.12.

1. Any (continuous) action by a compact group is proper.
2. If G acts simplicially on a simplicial complex X, then the action is proper if and only if the vertex stabilizers are compact.

Let X be a locally compact space equipped with a proper action by a locally compact group G.

Definition 6.13. *A* generalized elliptic G-equivariant operator *on X is a triple (U, π, F) such that*

- *U is a unitary representation of G on a Hilbert space $\mathcal{H}$ ($g \mapsto U_g$),*
- *π is a *-homomorphism from $C_0(X)$ to $B(\mathcal{H})$ which is covariant, that is*

  $$\pi(gf) = U_g \pi(f) U_g^*$$

  for all $f \in C_0(X)$, where $(gf)(x) = f(g^{-1}x)$,
- *F is a bounded self adjoint operator which is G-equivariant, that is $U_g F = F U_g$, and*

  $$\pi(f)(F^2 - 1), \quad [\pi(f), F]$$

  are compact for all $f \in C_0(X)$.

Definition 6.14. *Two cycles $\alpha_0 = (U_0, \pi_0, F_0)$, $\alpha_1 = (U_1, \pi_1, F_1)$ are* operator homotopic, *$\alpha_0 \sim_h \alpha_1$, if and only if $U_0 = U_1$, $\pi_0 = \pi_1$ and there exists a path $t \mapsto F_t$, $t \in [0, 1]$ such that $\alpha_t = (U_0, \pi_0, F_t)$ is a generalized G- elliptic operator.*

We say that α_0, α_1 are equivalent, *$\alpha_0 \sim \alpha_1$, if and only if there exist degenerate operators β_0, β_1 such that $\alpha_0 \oplus \beta_0 \sim_h \alpha_1 \oplus \beta_1$, up to unitary equivalence.*

Definition 6.15. *The* equivariant K-homology *groups of X are defined by*

- $\mathrm{K}_0^G(X) =$ *equivalence classes of $\mathbb{Z}/2\mathbb{Z}$-graded G- elliptic operators, that is*

$$U_g = \begin{pmatrix} U_g^+ & 0 \\ 0 & U_g^- \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi^+ & 0 \\ 0 & \pi^- \end{pmatrix}, \quad F = \begin{pmatrix} 0 & p^* \\ p & 0 \end{pmatrix},$$

- $\mathrm{K}_1^G(X) =$ *equivalence classes of G- elliptic operators.*

Remark 6.16. Kasparov uses a weaker form of homotopy. He allows the representations to vary as well, but proves that the resulting theory is isomorphic to the one defined here.

This construction is functorial with respect to G-equivariant proper maps between G-spaces. If $h\colon X \to Y$ is such a map, then it induces $h^*\colon C_0(Y) \to C_0(X)$, $h^*(f) = f \circ h$. The induced map $h_*\colon \mathrm{K}_0^G(X) \to \mathrm{K}_0^G(Y)$ sends a cycle (U, π, F) over $C_0(X)$ to the cycle $(U, \pi \circ h^*, F)$, so the theory is covariant.

Proposition 6.17 (Kasparov)**.** *If $f, g\colon X \to Y$ are proper G-homotopic maps, then*

$$f_* = g_*\colon \mathrm{K}_j^G(X) \to \mathrm{K}_j^G(Y).$$

Example 6.18. Let $X = \mathbb{R}$, $G = \mathbb{Z}$ act on X by translations $(x, m) \mapsto x+m$. Let $\mathcal{H} = L^2(\mathbb{R})$, π a representation of $C_0(\mathbb{R})$ on $L^2(\mathbb{R})$ by pointwise multiplication. The Fourier transform sends the unbounded operator $D = -i\frac{d}{dt}$ to the multiplication by the dual variable λ on $L^1(\widehat{\mathbb{R}})$. Let $\widehat{F}$ be the operator of multiplication by $\operatorname{sign}(\lambda)$ and let F be the operator obtained from $\widehat{F}$ by the inverse Fourier transform (Hilbert transform)

$$(Ff)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt,$$

where the integral is considered in the sense of principal value. Then $F^2 = 1$, $FU_n = U_nF$ for all $n \in \mathbb{Z}$, and $[\pi(f), F]$ is compact. This data gives a generator for $\mathrm{K}_1^{\mathbb{Z}}(\mathbb{R})$.

Example 6.19. Let $G = \{e\}$, the trivial one-element group, $X = S^1$. Denote $e_n := e^{2\pi i n\theta}$. Define F by

$$F(e_n) = \begin{cases} e_n & n > 0, \\ -e_n & n < 0, \\ 0 & n = 0. \end{cases}$$

Then $1 - F^2$ is a rank one projection onto the subspace of constant functions in $L^2(S^1)$. For $f \in C(S^1)$ define $\pi(f) \in B(L^2(S^1))$ by $(\pi f)(u) = fu$ where for $u \in L^2(S^1)$

$$(fu)(x) = f(x)u(x).$$

Let

$$A = \{f \in C(S^1) \mid [\pi(f), F] \text{ is compact}\}.$$

Then $A = C(S^1)$. Indeed, $[\pi(e_1), F]$ is an operator of rank 2, so A contains the *-subalgebra generated by e_1, which is the algebra of trigonometric polynomials, and these are dense in $C(S^1)$.

The operator F is the sign of the unbounded operator $D = -i\frac{d}{d\theta}$, $D(e_n) = 2\pi n e_n$. That is

$$F = \operatorname{sign} D = \frac{D}{|D|} \text{ on } \mathbb{C}^{\perp}.$$

This data gives a generator for $\mathrm{K}_1(S^1)$. There is a descent map

$$\mathrm{K}_j^{\mathbb{Z}}(\mathbb{R}) \to \mathrm{K}_j(S^1).$$

In degree one it sends the generator of $\mathrm{K}_1^{\mathbb{Z}}(\mathbb{R})$ to the generator of $\mathrm{K}_1(\mathbb{R})$ just described.

Proposition 6.20. *If G acts freely and properly then*

$$\mathrm{K}_j^G(X) \cong \mathrm{K}_j(X/G) \quad (= \mathrm{KK}^j(C_0(X/G)), \mathbb{C})).$$

Proof. We shall outline the proof for the special case when the group G is discrete. In this special case there is a (canonical) isomorphism

$$\mathrm{K}_j^G(X) \cong \mathrm{KK}^j(C_0(X) \rtimes G, \mathbb{C}),$$

which is an example of a descent map. Then use freeness to prove the Morita equivalence

$$C_0(X) \rtimes G \sim_{Morita} C_0(X/G)$$

which gives an isomorphism

$$\mathrm{KK}^j(C_0(X) \rtimes G, \mathbb{C}) \cong \mathrm{KK}^j(C_0(X/G)), \mathbb{C}).$$

□

Example 6.21. If $X = \mathrm{pt}$, and G is compact, then $\mathrm{K}_0^G(\mathrm{pt})$ is the additive group of the representation ring $\mathrm{R}(G)$, and $\mathrm{K}_1^G(\mathrm{pt}) = 0$.

$$[U_0] - [U_1] \in \mathrm{R}(G) \mapsto (U_0 \oplus U_1, \mathbb{C}, 0) \in \mathrm{K}_0^G(\mathrm{pt}),$$
$$(U, \mathbb{C}, F = \begin{pmatrix} 0 & p^* \\ p & 0 \end{pmatrix}) \mapsto \mathrm{Index}_G(F) = \ker(p) - \ker(p^*)$$

regarded as an element of $\mathrm{R}(G)$.

If Y is a Hausdorff topological space with proper G-action, then we define

$$\mathrm{K}_j^G(Y) := \varinjlim_{X \subset Y} \mathrm{K}_j^G(X),$$

where the limit is taken over the inductive system of G-compact subsets of Y (i.e. with compact quotient X/G). This is *G-equivariant K-homology with G-compact supports.*

6.3 Equivariant K-homology of C*-algebras

Let G be a locally compact Hausdorff second countable group, and $\mathcal{H}$ a separable Hilbert space. Denote the set of unitary operators on $\mathcal{H}$ by

$$\mathcal{U}(\mathcal{H}) := \{U \in B(\mathcal{H}) \mid UU^* = U^*U = I\}$$

Definition 6.22. *A* unitary representation *of G is a group homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H})$ such that for each $v \in \mathcal{H}$ the map $G \to \mathcal{H}$, $g \mapsto \pi(g)v$ is a continuous map from G to $\mathcal{H}$.*

Definition 6.23. *A G-*C*-algebra *is a C*-algebra A with a given continuous action*

$$G \times A \to A$$

by automorphisms. The continuity condition is that for each $a \in A$ the map $G \to A$, $g \mapsto ga$ is a continuous map from G to A.

Example 6.24. Let X be a locally compact G-space. Then G acts on $C_0(X)$ by

$$(g\alpha)(x) = \alpha(g^{-1}x),\ g \in G,\ \alpha \in C_0(X),\ x \in X.$$

This makes $C_0(X)$ a G-C*-algebra.

Let A be a (separable) G-C*-algebra.

Definition 6.25. *A* covariant representation *of A is a triple $(\mathcal{H}, \psi, \pi)$ such that*

- *$\mathcal{H}$ is a separable Hilbert space,*
- *$\psi\colon A \to B(\mathcal{H})$ is a *-homomorphism,*
- *$\pi\colon G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G,*
- *and*
$$\psi(ga) = \pi(g)\psi(a)\pi(g^{-1})$$
for all $g \in G$, $a \in A$.

Definition 6.26. Equivariant odd K-homology $\mathrm{K}^1_G(A)$ *of a G-C*-algebra A is the group of homotopy classes of quadruples $(\mathcal{H}, \psi, T, \pi)$, where $(\mathcal{H}, \psi, \pi)$ is a covariant representation of A, and $T \in B(\mathcal{H})$ is such that*

$$\begin{aligned} T &= T^*, \\ \pi(g)T - T\pi(g) &\in \mathcal{K}(\mathcal{H}), \\ \psi(a)T - T\psi(a) &\in \mathcal{K}(\mathcal{H}), \\ \psi(a)(1 - T^2) &\in \mathcal{K}(\mathcal{H}) \end{aligned}$$

for all $g \in G$, $a \in A$.

$$\begin{aligned} \mathcal{E}^1_G(A) &= \{(\mathcal{H}, \psi, \pi, T)\}, \\ \mathrm{K}^1_G(A) &= \{(\mathcal{H}, \psi, \pi, T)\}/\sim . \end{aligned}$$

The equivalence relation $\sim$ is homotopy and will be precisely defined below.

Example 6.27. Let $G = \mathbb{Z}$, $X = \mathbb{R}$, $A = C_0(\mathbb{R})$. Consider the action by translations

$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}, \quad (n, t) \mapsto n + t.$$

Let $\mathcal{H} = L^2(\mathbb{R})$. Define $\psi\colon A \to B(\mathcal{H})$ by

$$\psi(\alpha)u = \alpha u, \quad \alpha u(t) = \alpha(t)u(t), \quad \alpha \in C_0(\mathbb{R}),\ u \in L^2(\mathbb{R}),\ t \in \mathbb{R}.$$

The representation $\pi\colon \mathbb{Z} \to \mathcal{U}(L^2(\mathbb{R}))$ is defined by

$$(\pi(n)u)(t) := u(t-n).$$

As an operator on $L^2(\mathbb{R})$ we take $-i\frac{d}{dx}$. It is not a bounded operator on $L^2(\mathbb{R})$, but we can"normalize" it to obtain a bounded operator T. Since $-i\frac{d}{dx}$ is self-adjoint, one can use the functional calculus, so that T can be taken to be the function $\frac{x}{\sqrt{1+x^2}}$ applied to $-i\frac{d}{dx}$,

$$T := \left(\frac{x}{\sqrt{1+x^2}}\right)(-i\frac{d}{dx}).$$

Equivalently, T can be constructed using Fourier transform. Let $\mathcal{M}_x$ be the operator of "multiplication by x"

$$(\mathcal{M}_x f)(x) = xf(x).$$

The Fourier transform converts $-i\frac{d}{dx}$ to $\mathcal{M}_x$ i.e., the diagram

$$\begin{tikzcd} L^2(\mathbb{R}) \arrow[r, "\mathcal{F}"] \arrow[d, "-i\frac{d}{dx}"'] & L^2(\mathbb{R}) \arrow[d, "\mathcal{M}_x"] \\ L^2(\mathbb{R}) \arrow[r, "\mathcal{F}"] & L^2(\mathbb{R}) \end{tikzcd}$$

commutes, where $\mathcal{F}$ denotes the Fourier transform. Let $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ be the operator of "multiplication by $\frac{x}{\sqrt{1+x^2}}$". Then

$$\left(\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}} f\right)(x) = \frac{x}{\sqrt{1+x^2}} f(x),$$

and $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ is a bounded operator

$$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Now, T is the unique bounded operator $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that the following diagram commutes:

$$\begin{tikzcd} L^2(\mathbb{R}) \arrow[r, "\mathcal{F}"] \arrow[d, "T"'] & L^2(\mathbb{R}) \arrow[d, "\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}"] \\ L^2(\mathbb{R}) \arrow[r, "\mathcal{F}"] & L^2(\mathbb{R}) \end{tikzcd}$$

Then

$$(L^2(\mathbb{R}), \psi, \pi, T) \in \mathcal{E}^1_{\mathbb{Z}}(C_0(\mathbb{R})).$$

Definition 6.28. Equivariant even K-homology $\mathrm{K}^0_G(A)$ *of a G-C^*-algebra A is the group of homotopy classes of quadruples $(\mathcal{H}, \psi, T, \pi)$, where $(\mathcal{H}, \psi, \pi)$ is a covariant representation of A, and $T \in B(\mathcal{H})$ is such that*

$$\pi(g)T - T\pi(g) \in \mathcal{K}(\mathcal{H}),$$
$$\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}),$$
$$\psi(a)(1 - T^*T) \in \mathcal{K}(\mathcal{H}),$$
$$\psi(a)(1 - TT^*) \in \mathcal{K}(\mathcal{H})$$

for all $g \in G$, $a \in A$.

$$\mathcal{E}^0_G(A) = \{(\mathcal{H}, \psi, \pi, T)\},$$
$$\mathrm{K}^0_G(A) = \{(\mathcal{H}, \psi, \pi, T)\}/\sim .$$

The equivalence relation $\sim$ is homotopy and will be precisely defined below.

If A, B are G-C*-algebras, and $\varphi\colon A \to B$ is a G-equivariant *-homomorphism, then $\varphi^*\colon \mathcal{E}^j_G(B) \to \mathcal{E}^j_G(A)$ for $j = 0, 1$ is given by

$$\varphi^*(\mathcal{H}, \psi, \pi, T) \mapsto (\mathcal{H}, \psi \circ \varphi, \pi, T).$$

Addition in $\mathrm{K}^j_G(A)$ is direct sum

$$(\mathcal{H}, \psi, \pi, T) + (\mathcal{H}', \psi', \pi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', \pi \oplus \pi', T \oplus T'),$$

and the inverse is

$$-(\mathcal{H}, \psi, \pi, T) = (\mathcal{H}, \psi, \pi, -T).$$

6.4 Kasparov's bifunctor: KK-theory

Definition 6.29. *Let A, B be C*-algebras. An A-B-bimodule is a pair $(\mathcal{E}, \psi)$ where $\mathcal{E}$ is a Hilbert B-module, and a *-homomorphism $\psi\colon A \to \mathcal{L}(\mathcal{E})$ is given.*

Denote by $E(A, B)$ the set of triples $(\mathcal{E}, \psi, F)$, where $(\mathcal{E}, \psi)$ is an (A, B)-bimodule, $F \in \mathcal{L}(E)$, and for all $a \in A$

$$\psi(a)(F^2 - 1) \in \mathcal{K}(\mathcal{E}), \quad [\psi(a), F] \in \mathcal{K}(\mathcal{E}).$$

The triple $(\mathcal{E}, \psi, F)$ is *degenerate* if for all $a \in A$ $\psi(a)(F^2 - 1) = 0$, $[\psi(a), F] = 0$. Denote by $D(A, B)$ the set of degenerate triples.

The *addition* on $E(A, B)$ is defined by

$$(\mathcal{E}, \psi, F) + (\mathcal{E}', \psi', F') = (\mathcal{E} \oplus \mathcal{E}', \psi \oplus \psi', F \oplus F').$$

A *homotopy* in $E(A, B)$ is an element of $E(A, B[0, 1])$. In some sense it is a map $[0, 1] \to E(A, B)$. A homotopy in which the bimodule $(\mathcal{E}, \psi)$ is fixed and the operator F varies in a norm continuous way is called an operator homotopy.

Definition 6.30. *The group* $\mathrm{KK}(A, B)$ *is defined to be the set of homotopy classes in $E(A, B)$ modulo degenerate elements.*

The construction is functorial in both variables, that is

- If $f\colon A_1 \to A_2$, $(\mathcal{E}, \psi, F) \in E(A_2, B)$, then $(\mathcal{E}, \psi \circ f, F) \in E(A_1, B)$ and this induces
$$f^*\colon \mathrm{KK}(A_2, B) \to \mathrm{KK}(A_1, B).$$
- If $g\colon B_1 \to B_2$, $(\mathcal{E}, \psi, F) \in E(A, B_1)$, then define $\psi \otimes 1\colon A \to \mathcal{L}(\mathcal{E} \otimes_g B_2)$,
$$(\psi \otimes 1)(a)(\xi \otimes b) = \psi(a)\xi \otimes b,$$
then $(\mathcal{E} \otimes_g B_2, \psi \otimes 1, F \otimes 1) \in E(A, B_2)$. This induces
$$g_*\colon \mathrm{KK}(A, B_1) \to \mathrm{KK}(A, B_2).$$

Let A be a C*-algebra, $\mathcal{H}$ a Hilbert module, $u, v \in \mathcal{L}(\mathcal{H})$. Denote

$$\theta_{u,v} \in \mathcal{L}(\mathcal{H}), \quad \theta_{u,v}(\xi) = u\langle v, \xi\rangle, \quad \theta^*_{u,v} = \theta_{v,u}.$$

The $\theta_{u,v}$ are the *rank one* operators on $\mathcal{H}$ [b-b98]. A *finite rank* operator on $\mathcal{H}$ is any $T \in \mathcal{L}(\mathcal{H})$ such that T is a finite sum of $\theta_{u,v}$.

$$T = \theta_{u_1,v_1} + \theta_{u_2,v_2} + \ldots + \theta_{u_n,v_n}.$$

The compact operators $\mathcal{K}(\mathcal{H})$ are defined as the norm closure in $\mathcal{L}(\mathcal{H})$ of the space of finite rank operators. It is an ideal in $\mathcal{L}(\mathcal{H})$.

We say that $\mathcal{H}$ is *countably generated* if in $\mathcal{H}$ there is a countable (or finite) set such that the A-module generated by this set is dense in $\mathcal{H}$.

Let A, B be C*-algebras, $\varphi\colon A \to B$ a *-homomorphism, and $\mathcal{H}$ a Hilbert A-module. We will define $\mathcal{H} \otimes_A B$ which will be a Hilbert B-module. First form the algebraic tensor product $\mathcal{H} \odot_A B$. It is a right B-module

$$(h \otimes b)b' = h \otimes bb', \ h \in \mathcal{H}, \ b, b' \in B.$$

Now define B-valued inner product $\langle -, -\rangle$ on $\mathcal{H} \odot_A B$ by

$$\langle h \otimes b, h' \otimes b'\rangle = b^*\varphi(\langle h, h'\rangle)b'.$$

Set

$$\mathcal{N} := \{\xi \in \mathcal{H} \odot_A B \mid \langle \xi, \xi\rangle = 0\}.$$

It is a B-submodule of $\mathcal{H} \odot_A B$, and $\mathcal{H} \odot_A B/\mathcal{N}$ is a pre-Hilbert B-module.

Definition 6.31. *$\mathcal{H} \otimes_A B$ is the completion of $\mathcal{H} \odot_A B/\mathcal{N}$.*

Let A, B be separable C*-algebras, $\mathcal{E}^1(A, B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a countably generated Hilbert B-module, $\psi\colon A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphism, $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned} T &= T^*, \\ \psi(a)(I - T^2) &\in \mathcal{K}(\mathcal{H}), \\ \psi(a)T - T\psi(a) &\in \mathcal{K}(\mathcal{H}), \end{aligned}$$

for all $a \in A$.

We say that $(\mathcal{H}_0, \psi_0, T_0)$, $(\mathcal{H}_1, \psi_1, T_1) \in \mathcal{E}^1(A, B)$ are *isomorphic* if there exists an isomorphism of Hilbert B-modules $\Phi\colon \mathcal{H}_0 \to \mathcal{H}_1$ with

$$\Phi\psi_0(a) = \psi_1(a)\Phi, \ \text{for all } a \in A, \ \Phi T_0 = T_1\Phi.$$

Let A, B, D be separable C*-algebras, $\varphi\colon B \to D$ a *-homomorphism. There is an induced map

$$\varphi_*\colon \mathcal{E}^1(A, B) \to \mathcal{E}^1(A, D),$$
$$\varphi_*(\mathcal{H}, \psi, T) = (\mathcal{H} \otimes_B D, \psi \otimes_B I, T \otimes_B I),$$

where I is the identity operator of D.

Consider two maps $\rho_0, \rho_1\colon C([0,1], B) \to B$, $\rho_0(f) = f(0)$, $\rho_1(f) = f(1)$. We say that $(\mathcal{H}_0, \psi_0, T_0)$, $(\mathcal{H}_1, \psi_1, T_1) \in \mathcal{E}^1(A, B)$ are *homotopic* if there exists $(\mathcal{H}, \psi, T) \in \mathcal{E}^1(A, C([0,1], B))$ with $(\rho_j)_*(\mathcal{H}, \psi, T) \cong (\mathcal{H}_j, \psi_j, T_j)$.

For the even case, consider $\mathcal{E}^0(A,B) = \{(\mathcal{H},\psi,T)\}$, where $\mathcal{H}$ is a countably generated Hilbert B-module, $\psi\colon A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphism, and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned}\psi(a)T - T\psi(a) &\in \mathcal{K}(\mathcal{H}),\\ \psi(a)(I - T^*T) &\in \mathcal{K}(\mathcal{H}),\\ \psi(a)(I - TT^*) &\in \mathcal{K}(\mathcal{H}),\end{aligned}$$

for all $a \in A$.

Definition 6.32. *We define the* KK-theory *of A, B as*

$$\begin{aligned}\mathrm{KK}^0(A,B) &:= \mathcal{E}^0(A,B)/\sim,\\ \mathrm{KK}^1(A,B) &:= \mathcal{E}^1(A,B)/\sim,\end{aligned}$$

where the relation $\sim$ is homotopy. $\mathrm{KK}^j(A,B)$ *is an abelian group with addition and additive inverse given by*

$$\begin{aligned}(\mathcal{H},\psi,T) + (\mathcal{H}',\psi',T') &= (\mathcal{H}\oplus\mathcal{H}',\psi\oplus\psi',T\oplus T')\\ -(\mathcal{H},\psi,T) &= \begin{cases}(\mathcal{H},\psi,T^*) & j=0,\\ (\mathcal{H},\psi,-T) & j=1.\end{cases}\end{aligned}$$

Note that in both the even and the odd case the Hilbert B-module $\mathcal{H}$ is not $\mathbb{Z}/2\mathbb{Z}$-graded. In Kasparov's papers (**??**) the Hilbert B-module $\mathcal{H}$ is $\mathbb{Z}/2\mathbb{Z}$-graded. The abelian groups defined here (i.e. without $\mathbb{Z}/2\mathbb{Z}$-grading on the Hilbert B-module $\mathcal{H}$) are isomorphic to the Kasparov groups. Without the $\mathbb{Z}/2\mathbb{Z}$-grading the difference between the even and odd cases is that in the odd case the operator T is self-adjoint, and in the even case the operator T is not required to be self-adjoint.

6.5 Equivariant KK-theory

Let A be a G-C*-algebra.

Definition 6.33. *A G-*Hilbert A-module *is a Hilbert A-module $\mathcal{H}$ with a given continuous action $G \times \mathcal{H} \to \mathcal{H}$, $(g,v) \mapsto gv$ such that*

$$\begin{aligned}g(u+v) &= gu + gv\\ g(ua) &= (gu)(ga)\\ \langle gu,\, gv\rangle &= g\langle u,\, v\rangle\end{aligned}$$

for $u, v \in \mathcal{H}$, $g \in G$, $a \in A$. Continuity here means that for each $u \in \mathcal{H}$, $g \mapsto gu$ is a continuous map $G \to \mathcal{H}$.

Remark 6.34. If $A = \mathbb{C}$ then the action of G on $\mathbb{C}$ must be the trivial action, because $\mathbb{C}$ has no nontrivial *-automorphisms. Thus if $A = \mathbb{C}$ a Hilbert A-module is a Hilbert space together with a given unitary representation $G \to \mathcal{U}(\mathcal{H})$.

For each $g \in G$, denote by L_g the map $L_g \colon \mathcal{H} \to \mathcal{H}$, $L_g(v) = gv$. Note that L_g might not be in $\mathcal{L}(\mathcal{H})$. But if $T \in \mathcal{L}(\mathcal{H})$, then $L_g T L_g^{-1} \in \mathcal{L}(\mathcal{H})$. Thus $\mathcal{L}(\mathcal{H})$ is a G-C*-algebra with $gT = L_g T L_g^{-1}$.

Example 6.35. If A is a G-C*-algebra, n positive integer. Then A^n is a G-Hilbert A-module with $g(a_1, a_2, \dots, a_n) = (ga_1, ga_2, \dots, a_n)$.

Let A, B be separable G-C*-algebras, $\mathcal{E}^1(A, B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a G-Hilbert B-module (countably generated), $\psi \colon A \to \mathcal{L}(B)$ is a *-homomorphism with

$$\psi(ga) = g\psi(a),\ g \in G, a \in A,$$

and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned} T &= T^* \\ gT - T &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)T - T\psi(a) &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)(I - T^2) &\in \mathcal{K}(\mathcal{H}) \end{aligned}$$

for all $g \in G$, $a \in A$.

In the even case we take $\mathcal{E}^0(A, B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a G-Hilbert B-module (countably generated), $\psi \colon A \to \mathcal{L}(B)$ is a *-homomorphism with

$$\psi(ga) = g\psi(a),\ g \in G, a \in A,$$

and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned} gT - T &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)T - T\psi(a) &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)(I - T^*T) &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)(I - TT^*) &\in \mathcal{K}(\mathcal{H}) \end{aligned}$$

for all $g \in G$, $a \in A$.

Definition 6.36. *We define the* equivariant KK-theory *of A, B as*

$$\begin{aligned} \mathrm{KK}^0_G(A, B) &:= \mathcal{E}^0(A, B)/\sim \\ \mathrm{KK}^1_G(A, B) &:= \mathcal{E}^1(A, B)/\sim \end{aligned}$$

where the relation $\sim$ is homotopy. $\mathrm{KK}^j_G(A, B)$ *is an abelian group with addition and additive inverse given by*

$$\begin{aligned} (\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') &= (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T') \\ -(\mathcal{H}, \psi, T) &= \begin{cases} (\mathcal{H}, \psi, T^*) & j = 0, \\ (\mathcal{H}, \psi, -T) & j = 1. \end{cases} \end{aligned}$$

6.6 Kasparov product

Theorem 6.37 (Kasparov)**.** *Let A, B, C be separable G-C^*-algebras. Then there is a biadditive* pairing *for $i, j \in \mathbb{Z}/2\mathbb{Z}$*

$$\mathrm{KK}^i_G(A, B) \times \mathrm{KK}^j_G(B, C) \to \mathrm{KK}^{i+j}_G(A, C)$$

If D is a separable G-C^*-algebra, then there is the extension of scalars *homomorphism*

$$\tau_D\colon KK_G^i(A,B) \to KK_G^i(A \otimes D, B \otimes D)$$

and the descent homomorphism

$$j_G\colon KK_G^i(A,B) \to KK^i(A \rtimes_r G, B \rtimes_r G).$$

Chapter 7

Topological applications

7.1 The Chern character

Let A be a C*-algebra with unit 1_A. Consider the topological groups $\mathrm{GL}(n, A)$ and embeddings

$$\mathrm{GL}(n, A) \hookrightarrow \mathrm{GL}(n+1, A)$$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & 0 \\ 0 & \dots & 0 & 1_A \end{pmatrix}$$

Each $\mathrm{GL}_n(A)$ is topologized by the norm of A. Then

$$\mathrm{GL}(A) = \varinjlim_{n\to\infty} \mathrm{GL}(n, A) = \bigcup_{n=1}^{\infty} \mathrm{GL}(n, A)$$

with the direct limit topology. Define the K-theory groups

$$\mathrm{K}_j(A) := \pi_{j-1}(\mathrm{GL}(A)), \quad j = 1, 2, 3, \dots.$$

Bott periodicity states that $\Omega^2 \mathrm{GL}(A) \sim \mathrm{GL}(A)$ (see [b-r59]), so $\mathrm{K}_j(A) \cong \mathrm{K}_{j+2}(A)$ for $j = 0, 1, 2, \dots$. Thus in fact we have two groups $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$. This formulation of Bott periodicity may appear a little different from the one presented in 2.7, but is in fact equivalent to it.

If A is not unital, then we can adjoin a unit,

$$0 \to A \to \widetilde{A} \to \mathbb{C} \to 0$$

and define

$$\mathrm{K}_0(A) := \ker(\mathrm{K}_0(\widetilde{A}) \to \mathrm{K}_0(\mathbb{C})),$$
$$\mathrm{K}_1(A) := \mathrm{K}_1(\widetilde{A}).$$

If $\varphi\colon A \to B$ is a *-homomorphism, then there is an induced homomorphism of abelian groups $\mathrm{K}_j(A) \to \mathrm{K}_j(B)$.

Example 7.1. $\mathbb{C}$ is a C*-algebra, $\|\lambda\| = |\lambda|$, $\lambda^* = \bar{\lambda}$.

Theorem 7.2 ([b-r59]).

$$\mathrm{K}_j(\mathbb{C}) = \begin{cases} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$$

Theorem 7.3 ([b-r59]).

$$\pi_j(\mathrm{GL}(n,\mathbb{C})) = \begin{cases} 0 & j \text{ even} \\ \mathbb{Z} & j \text{ odd} \end{cases}$$

for $j = 0, 1, \ldots, 2n-1$.

For a locally compact Hausdorff topological space one defines a topological K-theory with compact supports (Atiyah-Hirzebruch)

$$\mathrm{K}^j(X) := \mathrm{K}_j(C_0(X)).$$

If X is compact Hausdorff then $\mathrm{K}^0(X)$ is the Grothendieck group of complex vector bundles on X.

There is the Chern character [h-f56]

$$\mathrm{ch}\colon \mathrm{K}^j(X) \to \bigoplus_l \mathrm{H}_c^{j+2l}(X;\mathbb{Q}), \quad j = 0, 1.$$

Theorem 7.4. *For any locally compact Hausdorff topological space* X

$$\mathrm{ch}\colon \mathrm{K}^j(X) \to \bigoplus_l \mathrm{H}_c^{j+2l}(X;\mathbb{Q})$$

is a rational isomorphism, i.e.

$$\mathrm{ch}\colon \mathrm{K}^j(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \bigoplus_l \mathrm{H}_c^{j+2l}(X;\mathbb{Q})$$

is an isomorphism for $j = 0, 1$.

As the target for the Chern character, we can use Čech cohomology, Alexander-Spanier cohomology or representable cohomology (all with compact supports). Note that it is the direct sum, not the direct product, of the cohomology groups which is used.

7.2 K-theory of the reduced group C*-algebra

We recall that a representation of C*-algebra A is a *-homomorphism

$$\varphi\colon A \to B(\mathcal{H}),$$

where $\mathcal{H}$ is a Hilbert space.

An imprecise and heuristic point of view on the reduced C*-algebra $C_r^*(G)$ of a locally compact group G is that there exists a locally compact Hausdorff topological space $\widehat{G}_r$. The space $\widehat{G}_r$ has one point for each distinct (i.e., non-equivalent) irreducible unitary representation of G which is weakly contained in the (left) regular representation of G. The space $\widehat{G}_r$ is known as the support of the Plancherel measure or *the reduced unitary dual* of G. We remark that the space of non-equivalent unitary representations of G which are weakly contained in the left regular representation of G is canonically bijective to the space of distinct irreducible representations of $C_r^*(G)$. The K-theory $\mathrm{K}_*(C_r^*G)$ can be viewed as the topological K-theory with compact supports of $\widehat{G}_r$.

Example 7.5. For $G = \mathrm{SL}(2,\mathbb{R})$ we have $\widehat{G}_r$:

If a compact group G acts on $\mathbb{C}$ by a C*-automorphisms, then it must act trivially, since $\mathbb{C}$ has no nontrivial *-automorphisms. We will prove the following:

Theorem 7.6. *For a compact group G there is an isomorphism*

$$\mathrm{K}_0(C_r^*(G)) \cong \mathrm{R}(G).$$

The key element in the proof is the Peter-Weyl theorem:

Theorem 7.7 (Peter-Weyl)**.** *If G is a compact, Hausdorff, second countable topological group, then every irreducible unitary representation of G is finite dimensional.*

Proof. Let $\rho\colon G \to \mathrm{U}(\mathcal{H})$ be an irreducible representation on a separable Hilbert space $\mathcal{H}$. Choose a projection p on $\mathcal{H}$, $p \neq 0$, $p = p^*$ with finite dimensional range. Let

$$T := \int_G \rho(g)p\rho(g)^* dg,$$

where dg is a Haar measure. Then

- T commutes with $\rho(g)$ for all $g \in G$,
- $T = T^*$, $T \geq 0$, $T \neq 0$,
- T is compact operator, $T \in \mathcal{K}(\mathcal{H})$.

The structure theorem for compact selfadjoint positive operators gives

$$\mathrm{sp}(T) := \{a_n \in \mathbb{R} \mid a_n \to 0\},$$

where each a_n is an eigenvalue with finite dimensional eigenspace. In particular any compact nonzero selfadjoint operator has a finite dimensional eigenspace. For T this eigenspace has to be preserved by the group action, so ρ has to be finite dimensional if it is irreducible. ∎

Proof. (of Theorem 7.6) Notice that for a compact group $C_r^*(G) = C^*(G)$ (there is only one C*-algebra for a compact group). Second countability of G implies that the irreducible unitary representations of G (up to equivalence) form a countable set. There is a C*-isomorphism

$$C^*(G) \cong \bigoplus_{\sigma \in \mathrm{Irrep}(G)} A_\sigma,$$

where each A_σ is a finite dimensional C*-algebra, which is isomorphic to $M_n(\mathbb{C})$, $n = \dim \sigma$. Hence

$$\mathrm{K}_j(C^*(G)) \cong \bigoplus_{\sigma \in \mathrm{Irrep}(G)} \underbrace{\mathrm{K}_j(A_\sigma)}_{\mathrm{K}_j(\mathbb{C})} \cong \begin{cases} \mathrm{R}(G) & \text{for } j = 0, \\ 0 & \text{for } j = 1. \end{cases}$$

□

7.3 Reduced crossed product

Let A be a G-C*-algebra. Denote

$$C_c(G, A) = \{f \colon G \to A \mid f \text{ is continuous and has compact support}\}$$

Then $C_c(G, A)$ is an algebra with operations

$$\begin{aligned} (f + h)(g) &= f(g) + h(g) \\ (f\lambda)(g) &= f(g)\lambda \\ (f * h)(g_0) &= \int_G f(g)[gh(g^{-1}g_0)]dg \end{aligned}$$

for $g, g_0 \in G$, $\lambda \in \mathbb{C}$. The operation $*$ is the *twisted convolution*. There is an injection of algebras $C_c(G, A) \to \mathcal{L}(L^2(G, A))$.

$$f \mapsto T_f, \quad T_f(u) = f * u$$

$$(f * u)(g_0) = \int_G f(g)(gu(g^{-1}g_0))dg.$$

Definition 7.8. *The* reduced crossed product *C*-algebra $C_r(G, A)$ is the completion of $C_c(G, A)$ in $\mathcal{L}(L^2(G, A))$ with respect to the norm $\|f\| = \|T_f|$ (see [p-g79]).*

Example 7.9. Let G be a finite group and A a G-C*-algebra. Give G the Haar measure in which every element has mass 1. Then

$$C_r^*(G, A) = \{\sum_{\gamma \in \Gamma} a_\gamma[\gamma] \mid a_\gamma \in A\}$$

with the following operations

$$\left(\sum_{\gamma\in\Gamma} a_\gamma[\gamma]\right) + \left(\sum_{\gamma\in\Gamma} b_\gamma[\gamma]\right) = \sum_{\gamma\in\Gamma}(a_\gamma + b_\gamma)[\gamma]$$

$$(a_\gamma[\gamma])(b_\beta[\beta]) = a_\gamma(\gamma b_\beta)[\gamma\beta]$$

$$\left(\sum_{\gamma\in\Gamma} a_\gamma[\gamma]\right)^* = \sum_{\gamma\in\Gamma}(\gamma^{-1}a_\gamma^*)[\gamma^{-1}]$$

$$\lambda\left(\sum_{\gamma\in\Gamma} a_\gamma[\gamma]\right) = \sum_{\gamma\in G}(\lambda a_\gamma)[\gamma]$$

for $\gamma \in G$, $\lambda \in \mathbb{C}$.

Let X be a locally compact G-space. Then $C_0(X)$ is a G-C*-algebra with

$$(gf)(x) = f(g^{-1}x), \quad f \in C_0(X),\ g \in G,\ x \in X.$$

We will denote $C^*_r(G, C_0(X))$ by $C^*_r(G, X)$. We ask about the K-theory of this C*-algebra. If G is compact, then $\mathrm{K}_j(C^*_r(G, X))$ is the Atiyah-Segal group $\mathrm{K}^j_G(X)$, $j = 0, 1$. Hence for G non-compact $\mathrm{K}_j(C^*_r(G, X))$ is the natural extension of the Atiyah-Segal theory to the case when G is non-compact.

We say that the G-space is G-compact if and only if the quotient space X/G is compact. If X is a proper G-compact G-space, then an equivariant $\mathbb{C}$-vector bundle E on X determines an element $[E] \in \mathrm{K}_0(C^*_r(G, X))$.

Theorem 7.10 (W. Lück, B. Oliver [lo01])**.** *If* Γ *is a (countable) discrete group and* X *is a proper* Γ*-compact* Γ*-space, then* $\mathrm{K}_0(C^*_r(\Gamma, X))$ *is the Grothendieck group of* Γ*-equivariant* $\mathbb{C}$*-vector bundles on* X*.*

7.4 $\mathrm{KK}^0_G(\mathbb{C},\mathbb{C})$

If G is a compact group then $\underline{E}G = \mathrm{pt}$ and $\mathrm{K}_0(C^*_r(G)) = \mathrm{R}(G)$ - the representation ring of G. We obtain $\mathrm{R}(G)$ as the Grothendieck group of the category of finite dimensional (complex) representations of G. It is a free abelian group with one generator for each distinct (i.e. nonequivalent) irreducible representation of G.

Theorem 7.11. *For a compact group G there is an isomorphism*

$$\mathrm{KK}^0_G(\mathbb{C},\mathbb{C}) \cong \mathrm{R}(G).$$

Proof. Given $(\mathcal{H}, \psi, T, \pi) \in \mathcal{E}^0_G(\mathbb{C})$ within the equivalence relation on $\mathcal{E}^0_G(\mathbb{C})$ we may assume that

$$T\pi(g) - \pi(g)T = 0, \tag{7.1}$$

because we can average T over the compact group G

$$T' := \int_G \pi(g)T\pi(g)^* dg = 0,$$

$$\begin{aligned} T - T' &= T - \int_G \pi(g)T\pi(g)^* dg \\ &= \int_G (T - \pi(g)T\pi(g)^*)dg \in \mathcal{K}(\mathcal{H}), \end{aligned}$$

because $\int_G T dg = T$ since we normalize Haar measure.

Furthermore we can assume that

$$\psi(\lambda) = \lambda \mathrm{Id}. \tag{7.2}$$

Indeed, $\psi\colon \mathbb{C} \to B(\mathcal{H})$ is a *-homomorphism, and $\psi(1)$ is a selfadjoint projection. For all $\lambda \in \mathbb{C}$

$$\psi(\lambda) = \lambda\psi(1), \quad p := \psi(1).$$

$\mathcal{H}$ splits into $p\mathcal{H} \oplus (1-p)\mathcal{H}$, and

$$\begin{aligned} Tp - pT &\in \mathcal{K}(\mathcal{H}), \\ T(1-p) - (1-p)T &\in \mathcal{K}(\mathcal{H}). \end{aligned}$$

Compare T to $pTp \oplus (1-p)T(1-p)$, to see that on $(1-p)\mathcal{H}$ ψ is 0.

The only nontrivial condition on $(\mathcal{H}, \psi, T, \pi)$ is

$$\begin{aligned} I - T^*T &\in \mathcal{K}(\mathcal{H}), \\ I - TT^* &\in \mathcal{K}(\mathcal{H}). \end{aligned}$$

These conditions imply that T is Fredholm, that is

$$\begin{aligned} \dim_{\mathbb{C}}(\ker T) &< \infty, \\ \dim_{\mathbb{C}}(\operatorname{coker} T) &< \infty. \end{aligned}$$

The spaces $\ker T$ and $\operatorname{coker} T$ are finite dimensional representations of G. We have

$$\mu(\mathcal{H}, \psi, T, \pi) = \ker T - \operatorname{coker} T \in \mathrm{R}(G).$$

First we will prove the surjectivity of the map $\mathrm{KK}^0_G(\mathbb{C},\mathbb{C}) \to \mathrm{R}(G)$. Let $V \in \mathrm{R}(G)$ be a finite dimensional irreducible unitary representation. Consider the countable direct sums $\bigoplus V$ and $\bigoplus \pi$. Let T be the shift

$$(v_1, v_2, \ldots) \mapsto (v_2, v_3, \ldots).$$

Then $\ker T = V$ (first copy), and $\operatorname{coker} T = 0$.

Given $(\mathcal{H}, \psi, T, \pi) \in \mathcal{E}^0_G(\mathbb{C})$, we can assume as above that $T\pi(g) - \pi(g)T = 0$ for all $g \in G$ and that $\psi(\lambda) = \lambda \mathrm{Id}$ for all $\lambda \in \mathbb{C}$. The Hilbert space $\mathcal{H}$ is canonically a direct sum of isotypical components of π. Consider one isotypical component; this is of the form $V \otimes U$, where V is an irreducible representation of G and U is a Hilbert space. Choosing an orthonormal basis for U, we have that the isotypical component is of the form $V \oplus V \oplus V \ldots$. With respect to this direct sum decomposition, the operator T is a matrix of operators. However, each of these operators is an intertwining operator for the irreducible representation V and therefore, by Schur's lemma, is a complex number. In other words, the restriction of T to the isotypical component is of the form $\mathrm{Id} \otimes F$, where Id is the identity operator of V, and F is a Fredholm operator on U. The Theorem now follows from the well known proposition that two Fredholm operators on a Hilbert space are homotopic (through Fredholm operators) if and only if they have the same index. $\square$

7.5 Topological K-theory of Γ

Let Γ be a (countable) discrete group. Consider pairs (M, E) such that M is a C^∞-manifold without boundary, with a given smooth proper co-compact action of Γ and a given Γ-equivariant Spin^c-structure, and E is a Γ-equivariant vector bundle on M. We introduce an equivalence relation on such pairs [bd82], which, by definition, is the equivalence relation $\sim$ generated by the three elementary moves

- Bordism,
- Direct sum - disjoint union,
- Vector bundle modification.

Then we define the *topological K-theory of* Γ as

$$\mathrm{K}_0^{top}(\Gamma) \oplus \mathrm{K}_1^{top}(\Gamma) = \{(M, E)\}/\sim .$$

Addition will be disjoint union

$$(M, E) + (M', E') = (M \cup M', E \cup E').$$

The main result of this section is:

Theorem 7.12 (P. Baum, N. Higson, T. Schick [bhs07])**.** *There is a natural isomorphism of abelian groups*

$$\tau\colon\ \mathrm{K}_j^{top}(\Gamma) \to \mathrm{K}_j^{\Gamma}(\underline{\mathrm{E}}\Gamma)$$

for $j = 0, 1$.

We now describe the equivalence relation $\sim$ in detail. We say that (M, E) is *isomorphic* to (M', E') if and only if there exist a Γ-equivariant diffeomorphism $\psi\colon M \to M'$ preserving the Γ-equivariant Spin^c-structures on M, M' with $\psi^* E' \cong E$. Then the three elementary moves, which by definition generate the relation, can be described as follows.

- *Bordism:* we say that (M_0, E_0) is bordant to (M_1, E_1) if and only if there exists (W, E) such that
  1. W is a C^∞-manifold with boundary, with a given smooth proper co-compact action of Γ;
  2. W has a given Γ-equivariant Spin^c-structure;
  3. E is a Γ-equivariant vector bundle on W;
  4. $(\partial W, E|_{\partial W}) \cong (M_0, E_0) \cup (-M_1, E_1)$.
- *Direct sum - disjoint union:* if E, E' are Γ-equivariant vector bundles on M, then
$$(M, E) \cup (M, E') \sim (M, E \oplus E').$$

- *Vector bundle modification:* let F be a Γ-equivariant Spin^c vector bundle on M. Assume that for every fiber F_p we have $\dim_{\mathbb{R}}(F_p) = 0 \mod 2$. Consider the one-dimensional Γ-equivariant trivial bundle $\mathbf{1} = M \times \mathbb{R}$, $\gamma(p,t) = (\gamma p, t)$. Let $S(F \oplus \mathbf{1})$ be the unit sphere bundle of $F \oplus \mathbf{1}$. $F \oplus \mathbf{1}$ is a Γ-equivariant Spin^c vector bundle with odd dimensional fibers. Let Σ be the spinor bundle for $F \oplus 1$

$$\pi\colon\ \mathbb{C}\mathrm{l}(F_p \oplus \mathbb{R}) \otimes \Sigma_p \to \Sigma_p.$$

Decompose $\pi^*\Sigma = \beta_+ \oplus \beta_-$. Then

$$(M, E) \sim (S(F \oplus 1), \beta_+ \otimes \pi^* E).$$

In the bordism elementary move we are using the standard fact that if W is a Spin^c-manifold with boundary ∂W then, in a canonical way, ∂W is again a Spin^c-manifold, i.e., ∂W 'inherits' a Spin^c-structure from the Spin^c-structure of W.

In the vector bundle modification elementary move $S(F \oplus \mathbf{1})$ is given the Spin^c-structure determined by the Spin^c-structure of M and the Spin^c-vector bundle F.

7.6 The Baum-Connes conjecture

Let G be a locally compact, Hausdorff, second countable (the topology of G has a countable base) group. Examples are:

- Lie groups with $\pi_0(G)$ finite - $\mathrm{SL}(n, \mathbb{R})$,
- p-adic groups - $\mathrm{SL}(n, \mathbb{Q}_p)$,
- adelic groups - $\mathrm{SL}(n, \mathbb{A})$,
- discrete groups - $\mathrm{SL}(n, \mathbb{Z})$.

For a group G we have the reduced C*-algebra of G, denoted by C_r^*G. The problem is to compute its K-theory $\mathrm{K}_j(C_r^*G)$, $j = 0, 1$ [bch93].

Conjecture 3 (P. Baum - A. Connes)**.** *For all locally compact, Hausdorff, second countable groups G*

$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}G) \to \mathrm{K}_j(C_r^*G)$$

is an isomorphism for $j = 0, 1$.

In the previous part, the reduced C*-algebra was defined in the context of discrete groups; for convenience, we recall here the definition of this algebra for locally compact groups.

Example 7.13. Let G be a locally compact Hausdorff second countable topological group. Fix a left-invariant Haar measure dg for G, that is for all continuous $f\colon G \to \mathbb{C}$ with compact support

$$\int_G f(\gamma g)dg = \int_G f(g)dg$$

for all $\gamma \in G$.

Let L^2G be the following Hilbert space

$$L^2G = \{u\colon G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty\}$$

$$\langle u, v\rangle = \int_G \overline{u(g)}v(g)dg, \quad u, v \in L^2G.$$

Let $\mathcal{L}(L^2G)$ be the C*-algebra of all bounded operators $T\colon L^2G \to L^2G$. Let

$$C_cG = \{f\colon G \to \mathbb{C} \mid f \text{ is continuous, and has compact support}\}.$$

Then C_cG is an algebra

$$\begin{aligned}(\lambda f)g &= \lambda(fg), \quad \lambda \in \mathbb{C},\ g \in G\\ (f+h)g &= fg + hg\\ (f * h)g_0 &= \int_G f(g)h(g^{-1}g_0)dg, \quad g_0 \in G.\end{aligned}$$

There is an injection of algebras

$$0 \to C_cG \to B(L^2G)$$

given by $f \mapsto T_f$, $T_f(u) = f * u$, $u \in L^2G$,

$$(f * u)g_0 = \int_G f(g)u(g^{-1}g_0)dg, \quad g_0 \in G.$$

Define the *reduced C*-algebra* C_r^*G of G as the closure of $C_cG \subset B(L^2G)$ in the operator norm. C_r^*G is a sub-C*-algebra of $B(L^2G)$.

We will also need the following commutative C*-algebra of functions on a locally compact space.

Example 7.14. Let X be a locally compact Hausdorff topological space, and $X^+ = X \cup \{p_\infty\}$ its one-point compactification. Define

$$C_0(X) := \{\alpha\colon X^+ \to \mathbb{C} \mid \alpha \text{ is continuous}, \alpha(p_\infty) = 0\},$$

$$\|\alpha\| = \sup_{p \in X} |\alpha(p)|, \quad \alpha^*(p) = \overline{\alpha(p)}.$$

with operations

$$\begin{aligned}(\alpha+\beta)(p) &= \alpha(p) + \beta(p),\\ (\alpha\beta)(p) &= \alpha(p)\beta(p),\\ (\lambda\alpha)(p) &= \lambda\alpha(p),\ \lambda \in \mathbb{C}.\end{aligned}$$

If X is compact, then

$$C_0(X) := C(X) = \{\alpha\colon X \to \mathbb{C} \mid \alpha \text{ is continuous}\},$$

Definition 7.15. *A subalgebra A of $B(\mathcal{H})$ is a C*-algebra of operators if and only if*

1. *A is closed with respect to the operator norm.*

2. *If $T \in A$, then the adjoint operator $T^* \in A$.*

The Gelfand-Naimark theorem (see Theorem 1.8) asserts that any C*-algebra is isomorphic, as a C*-algebra, to a C*-algebra of operators. In particular, let A be a commutative C*-algebra. Then A is (canonically) isomorphic to $C_0(X)$ where X is the space of maximal ideals of A. Thus a non-commutative C*-algebra can be viewed as a "noncommutative locally compact Hausdorff topological space".

We have an equivalence of the following categories

- Commutative C*-algebras. The set of morphisms between two commutative algebras A and B is the set of all $*$-homomorphisms $\phi : A \to B$;

- Locally compact Hausdorff topological spaces with morphisms from X to Y being continuous maps $f\colon X^+ \to Y^+$ with $f(p_\infty) = q_\infty$. Here X^+, Y^+ are the one-point compactifications of X, Y, and p_∞, q_∞ are the points at infinity. Note that $f\colon X^+ \to Y^+$ is not required to map X to Y.

The functor which which gives the equivalence of categories assigns to any locally compact Hausdorff topological space X the C*-algebra $C_0(X)$. This functor reverses the direction of morphisms and so, strictly speaking, is an equivalence of categories between the category of commutative C*-algebras and the opposite category to the category of locally compact Hausdorff topological spaces.

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Part VII

Galois structures

by

Tomasz Brzeziński

George Janelidze

Tomasz Maszczyk

Based on the lectures of:

- Tomasz Brzeziński
  (*Department of Mathematics, Swansea University, Singleton Park, Swansea SA2 8PP, UK*)
  – Chapters 4, 5, 6, 7.

- George Janelidze
  (*Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa*)
  – Chapters 3, 8.

- Tomasz Maszczyk
  (*Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, 00956 Warszawa, Poland; Institute of Mathematics, University of Warsaw, Banacha 2, 02097 Warszawa, Poland*)
  – Chapters 1, 2.

With additional lectures by:

- Gabriella Böhm – Section 6.5

- Piotr M. Hajac – Introduction VII, Section **??**.

Introduction

Galois theory appears in mathematics as a part of algebra - theory of field extensions, and topology - theory of coverings.

$$\begin{tikzcd}
\text{Finite field extensions} \arrow[dr] & & \text{Coverings of spaces} \arrow[dl] \\
& \text{Galois theory} \arrow[dl] \arrow[dr] & \\
\text{Categorical} \arrow[dr] & & \text{Noncommutative} \arrow[dl] \\
& ? &
\end{tikzcd}$$

The categorical and noncommutative approach to Galois theory meet at a place which will be described in the rest of the lectures.

0.7 Principal actions and finite fibre bundles

Let X be a set with group action $X \times G \to X$, that is

$$(xg)g' = x(gg'), \quad xe = x, \quad \text{for all } a \in X,\ g, g' \in G,\ e \in G \text{ a unit.}$$

There is a map

$$\begin{aligned} F \colon X \times G &\to X \times X \\ (x, g) &\mapsto (x, xg) \end{aligned}$$

We say that the action is *free* if F is injective.

$$(F(x, g) = F(y, g)) \Longleftrightarrow (x = y \wedge xgh^{-1} = y)$$

Example 0.16. Kronecker foliation on the 2-torus $\mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2$.

If the action is free, then the image of F is $X \times_{X/G} X$. If x, y belong to the same orbit in X/G, then there is a unique $g \in G$ such that $y = xg$. We can define a *translation map*

$$\begin{aligned} \hat{\tau} \colon X \times_{X/G} X &\to G \\ (x, y) &\mapsto g \text{ such that } xg = y. \end{aligned}$$

We can view $\hat{\tau}$ as an "inverse" of F on the image.

Properties of $\hat{\tau}$

- $\hat{\tau}(xg, yh) = g^{-1}\hat{\tau}(x, y)h$
- $x\hat{\tau}(x, y) = y$
- $\hat{\tau}(x, y)\hat{\tau}(y, z) = \hat{\tau}(x, z)$
- $\hat{\tau}(x, y)^{-1} = \hat{\tau}(y, x)$

Assume from now on that $X \times G \to X$ is continuous. Then F is continuous. If the action is free, then there exists $\hat{\tau}$ but it need not be continuous (for example Kronecker foliation). The translation map is always continuous if $X = \tilde{G}$ - topological group, and G is a closed subgroup of $\tilde{G}$ and the action is by multiplication. Then $\hat{\tau}(x, y) = x^{-1}y$.

Example 0.17. Palais foliation of $\mathbb{R}^2$. The action of $\mathbb{R}$ is by the flow of vertically invariant and unit vector field

$$(x, y) \mapsto (\cos(x), \sin(x)).$$

It is free, locally trivial, $\hat{\tau}$ is continuous but it is not proper (the quotient map is not Hausdorff).

Definition 0.18. *A continuous map $X \xrightarrow{f} Y$ is proper if for all Z the map $X \times X \xrightarrow{(f,\mathrm{id})} Y \times Z$ is closed.*

If X is Hausdorff and Y is locally compact Hausdorff then f is proper if and only if for all compact $K \subseteq Y$ the preimage $f^{-1}(K)$ is compact. Action is called proper if F is proper. If G is compact, its action is always proper, $F^{-1}(X, XG) = X \times G$.

Proposition 0.19. *Let $X \times G \to X$ be free and continuous. It is proper if and only if $\hat{\tau}$ is continuous and $X \times_{X/G} X$ is closed in $X \times X$.*

Cartan equality:

$$\text{principal} = \text{free} + \text{proper}.$$

A triple (X, π, M), where $X \xrightarrow{\pi} M$ is a continuous surjection is called a *bundle*. Any continuous group action yields a bundle $(X, \pi, X/G)$.

Definition 0.20. *A* principal bundle *is a quadruple (X, π, M, G) such that*

1. *(X, π, M) is a bundle, and G is a topological group acting continuously on X from the right.*
2. *The action of G on X is principal (i.e. free and proper).*
3. *$\pi(x) = \pi(y)$ if and only if there exists $g \in G$ such that $y = xg$ (the fibres are the orbits of G).*
4. *The induced map $X/G \to M$ is a homomorphism.*

0.8 Compact principal bundles as principal comodule algebras

Definition 0.21. *An H-*comodule algebra *P (existence of S^{-1} is assumed) is principal if*

1. *The canonical map*

$$\begin{aligned} \mathrm{can}\colon P \otimes P &\to P \otimes H \\ p \otimes q &\mapsto (p \otimes 1)\Delta_R(q) = pq_{(0)} \otimes q_{(1)} \end{aligned}$$

   is bijective (Galois condition).

2. *For*

$$P^{\mathrm{co}H} := \{p \in P \mid \Delta_R(p) = p \otimes 1\}$$

   there exists $s \in \mathrm{Hom}^H_{P^{\mathrm{co}H}}(P, P^{\mathrm{co}H} \otimes P)$ such that $ms = \mathrm{id}$ (equivariant projectivity).

There are known correspondences

(A)

$$\text{Compact coverings} \cong \text{Commutative principal C*-comodule algebras}$$

$$\begin{gathered} P = C(X),\ H = C(G),\ P^{\mathrm{co}H} = C(X/G) \\ \Delta_R(p)(x, g) = p(xg). \end{gathered}$$

(B)

$$\begin{aligned} \text{Compact U(1)-principal bundles} \cong{} & \text{Commutative unital C*-algebras } C(X) \\ & \text{such that the } \mathbb{C}[\mathbb{Z}]\text{-comodule algebra} \\ & P := \{f \in C(X) \mid \Delta_R(f) \in C(X) \otimes_{alg} \mathbb{C}[\mathbb{Z}]\} \\ & \subseteq C(\mathrm{U}(1)). \end{aligned}$$

Conjecture 4. *There is a correspondence*

$$\begin{aligned} \textit{Compact principal bundles} \cong{} & \textit{Commutative unital C*-algebras } C(X) \\ & \textit{such that } \mathcal{O}(G)\textit{-comodule algebra} \\ & P := \{f \in C(X) \mid \Delta_R(f) \in C(X) \otimes_{alg} \mathcal{O}(G)\} \textit{ is principal.} \end{aligned}$$

Principality of P implies principality of X, but the converse is an open problem.

Chapter 1

Galois theory

Galois theory is a language to speak about various phenomena in algebra, arithmetic and geometry. It helps to deal with the problems of solving polynomial equations and possibility of geometric constructions.

1.1 Fields

Definition 1.1. *A* field $\mathbb{F}$ *is an abelian group (addition) such that the set* $\mathbb{F}^* = \{x \in \mathbb{F} \mid x \neq 0\}$ *is equipped with a structure of an abelian group (multiplication) which distributes over addition.*

Definition 1.2. *A* field $\mathbb{F}$ *is a commutative ring without nontrivial ideals.*

Definition 1.3. *A* field $\mathbb{F}$ *is a commutative division ring.*

Examples 1.4.

1. $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

2. $\mathbb{Q}(\sqrt{D})$, where D is not a perfect square, that is the equation $x^2 = D$ has no rational solutions. In another words $\mathbb{Q}(\sqrt{D})$ is the smallest field containing $\mathbb{Q}$ and $\sqrt{D} \in \mathbb{R}, \mathbb{C}$. All even powers of $\sqrt{D}$ belong to $\mathbb{Q}$, and all odd powers of $\sqrt{D}$ are nontrivial multiples of $\sqrt{D}$. Thus for every polynomial $f \in \mathbb{Q}[X]$ we have $f(\sqrt{D}) = a + b\sqrt{D}$, where $a, b \in \mathbb{Q}$. The inverse of an element $a + b\sqrt{D}$, $a, b \in \mathbb{Q}$ is given by

$$\frac{1}{a + b\sqrt{D}} = \frac{a}{a^2 - b^2 D} - \frac{b}{a^2 - b^2 D}\sqrt{D},$$

   so in fact for every rational function $f \in \mathbb{Q}(X)$ we have $f(\sqrt{D}) = a + b\sqrt{D}$, where $a, b \in \mathbb{Q}$.

3. Rational functions in one variable $\mathbb{Q}(X)$, and in n variables $\mathbb{Q}(X_1, \ldots, X_n)$.

4. $\mathbb{F}_p$ - classes of integers modulo prime p. There exist also a field $\mathbb{F}_{p^n}$ for every $n > 0$, of p^n elements, unique up to isomorphism and all finite fields are of this form.

1.2 Morphisms of fields

Definition 1.5. *A* morphism of fields $\phi\colon \mathbb{F} \to \mathbb{F}'$ *is a homomorphism of rings.*

Morphism of fields $\phi\colon \mathbb{F} \to \mathbb{F}'$ is always injective, because

$$1_{\mathbb{F}'} = \phi(1_{\mathbb{F}}) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}) = \phi(x)\phi(x)^{-1},$$

so $\phi(x) \neq 0$ for every $x \in \mathbb{F}$.

There is always a ring homomorphism $\phi\colon \mathbb{Z} \to \mathbb{F}$. We have two cases

1. ϕ is injective: then $\phi(\mathbb{Z}) \subset \mathbb{F}$ generates a subfield isomorphic to $\mathbb{Q}$, and we say that $\mathbb{F}$ has characteristic 0, $\mathrm{char}(\mathbb{F}) = 0$.

2. ϕ is not injective: then there exists the smallest positive integer $p > 0$ such that $\phi(p) = 0$. It is a prime number, because if $p = ab$, $1 < a, b < p$ then we would have

   $$0 = \phi(p) = \phi(a)\phi(b) \neq 0.$$

   In this case $\phi(\mathbb{Z}) \subset \mathbb{F}$ generates a subfield isomorphic to $\mathbb{F}_p$, and we say that $\mathbb{F}$ has characteristic p, $\mathrm{char}(\mathbb{F}) = p$.

Definition 1.6. *A field* $\mathbb{E}$ *is an* extension *of the field* $\mathbb{F}$ *if* $\mathbb{F}$ *is a subfield of* $\mathbb{E}$.

We write $\mathbb{E}/\mathbb{F}$ or draw

$$\begin{array}{c}\mathbb{E}\\ \Big| \\ \mathbb{F}\end{array}$$

Corollary 1.7. *If* $\mathbb{E}$ *is an extension of* $\mathbb{F}$ *then*

- $\mathrm{char}(\mathbb{E}) = \mathrm{char}(\mathbb{F})$*,*
- $\mathbb{E}$ *is a vector space over* $\mathbb{F}$*.*

Definition 1.8.

1. *The* degree $[\mathbb{E} : \mathbb{F}]$ *of an extension* $\mathbb{E}/\mathbb{F}$ *is defined as* $\dim_{\mathbb{F}}(\mathbb{E})$*.*
2. $\mathbb{E}$ *is a* finite extension *of* $\mathbb{F}$ *if* $[\mathbb{E} : \mathbb{F}] < \infty$*.*

Examples 1.9. 1. $[\mathbb{Q}(\sqrt{D}) : \mathbb{Q}] = 2$ with $\{1, \sqrt{D}\}$ as basis over $\mathbb{Q}$.

2. $[\mathbb{C} : \mathbb{R}] = 2$ with $\{1, i\}$ as basis over $\mathbb{R}$.
3. $[\mathbb{Q}(x) : \mathbb{Q}] = \infty$ with $\{1, x, x^2, \ldots\}$ being an infinite linearly independent system.
4. $[\mathbb{R} : \mathbb{Q}] = \infty$ with $\{1, e, e^2, \ldots\}$ being an infinite linearly independent system, where $e \approx 2.72\ldots$ is the Euler number.

Linear dependence of powers of $e \in \mathbb{E}$ over $\mathbb{F} \subset \mathbb{E}$ is nothing else but a polynomial equation

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0.$$

Note that in the first two examples the degree is equal to the minimal degree of a polynomial equation satisfied by the adjoint element

$$\sqrt{D}: \quad x^2 - D = 0, \qquad i: \quad x^2 + 1 = 0.$$

1.3 Polynomials

Denote by $\mathbb{F}[X]$ the ring of polynomials in one variable X. It is an *integral domain*, that is if $f(X), g(X) \in \mathbb{F}[X]$ are nonzero polynomials, then $f(X)g(X) \neq 0$. It is also a *Euclidean domain*, that is for all $f(X), g(X) \in \mathbb{F}[X]$ there are unique polynomials $q(X), r(X) \in \mathbb{F}[X]$ such that

$$f(X) = g(X)q(X) + r(X),$$

where either $r(X) = 0$ or $\deg(r(X)) < \deg(g(X))$.

Corollary 1.10. *For any two nonzero polynomials $f(X), g(X) \in \mathbb{F}[X]$ there is their greatest common divisor*

$$\gcd(f(X), g(X)) = a(X)f(X) + b(X)g(X).$$

Corollary 1.11. *Every ideal in $\mathbb{F}[X]$ is principal, that is of the form $(f(X))$.*

Corollary 1.12. *Every nonconstant polynomial $f(X) \in \mathbb{F}[X]$ can be factored as*

$$f(X) = uf_1(X) \dots f_k(X),$$

where $f_i(X)$ are monic, irreducible, and $u \in \mathbb{F}^*$. This factorisation is essentially unique.

There is an important construction of field extensions from irreducible polynomials.

Proposition 1.13. *Let $f(X) \in \mathbb{F}[X]$ be an irreducible of degree d. Then*

$$\mathbb{E} := \mathbb{F}[X]/(f(X))$$

is an extension of degree d.

Proof. First we prove that the classes of $1, x, x^2, \dots, x^{d-1}$ form a basis of $\mathbb{E}$ over $\mathbb{F}$. Every polynomial $g(X) \in \mathbb{F}[X]$ can be presented as

$$g(X) = f(X)q(X) + r(X),$$

where $r(X) = 0$ or $\deg(r(X)) < d$. Thus $g(X)$ is a combination of $1, x, \dots, x^{d-1}$, and classes of $1, x, \dots, x^{d-1}$ generate $\mathbb{E}$.

Every linear combination of classes of $1, x, \dots, x^{d-1}$ is a polynomial of degree less than $d = \deg(f(X))$, so classes of $1, x, \dots, x^{d-1}$ are linearly independent. Observe that $\mathbb{E}$ is an integral domain - it is a consequence of the unique factorisation property for $\mathbb{F}[X]$ and an assumption that f is irreducible.

The proof will be finished if we prove the following lemma

Lemma 1.14. *Every finite dimensional commutative $\mathbb{F}$-algebra $\mathbb{E}$ which is an integral domain is a field.*

Proof. Take $e \in \mathbb{E}^*$. There exists a linear dependence among elements $1, e, e^2, \dots$ since $\mathbb{E}$ is of finite dimension over $\mathbb{F}$. We can divide by the monomial of the lowest degree to obtain

$$1 + f_1 e + f_2 e^2 + \dots + f_n e^n = 0,$$
$$e(-f_1 - f_2 e - \dots - f_n e^{n-1}) = 1,$$

so e has an inverse. □

□

Corollary 1.15 (Kronecker). *Let $f(X) \in \mathbb{F}[X]$ be any nonconstant polynomial. Then there exists an extension $\mathbb{E}/\mathbb{F}$ in which $f(X)$ has a root.*

Proof. We can assume that $f(X)$ is irreducible. Then take

$$\mathbb{E} = \mathbb{F}[X]/(f(X))$$

The root of $f(X)$ in $\mathbb{E}$ is the class of $X \in \mathbb{F}[X]$. □

Definition 1.16. *Let $e \in \mathbb{E}$ be algebraic over $\mathbb{F}$. Then the monic irreducible polynomial $f_e(X) \in \mathbb{F}[X]$ such that $f_e(e) = 0$ is determined uniquely (as the monic generator of the ideal $\{f(X) \in \mathbb{F}[X] \mid f(e) = 0\}$) and is called the* minimal polynomial *of e.*

Lemma 1.17. *Let $e \in \mathbb{F}$ be algebraic over $\mathbb{F}$. Then the canonical map*

$$\varphi \colon \mathbb{F}[X]/(f_e(X)) \to \mathbb{F}(e) \subset \mathbb{E}, \quad x \mapsto e$$

is an isomorphism.

Proof. Because $f_e(e) = 0$ the map φ is well defined. It is enough to prove that

$$\dim_{\mathbb{F}} (\mathbb{F}[X]/(f_e(X))) = \dim_{\mathbb{F}} (\mathbb{F}(e))$$

By definition

$$\dim_{\mathbb{F}} (\mathbb{F}[X]/(f_e(X))) = \deg(f_e(X)).$$

Also

$$\dim_{\mathbb{F}} (\mathbb{F}(e)) = \deg(f_e(X)),$$

because $f_e(x)$ is a monic polynomial of lowest degree vanishing at e. □

1.4 Automorphisms of fields

If $G \subset \mathrm{Aut}(\mathbb{E})$ is a subgroup then $\mathbb{E}^G \subset \mathbb{E}$ is a subfield.

Definition 1.18. *Let $G = \{g_1, \ldots, g_n\} \subset \mathrm{Aut}(\mathbb{E})$. We define a* trace

$$\mathrm{Tr}_G \colon \mathbb{E} \to \mathbb{E}^G, \quad \mathrm{Tr}_G(e) = \sum_{g \in G} g(e).$$

Trace Tr_G is an $\mathbb{E}^G$-linear map.

Theorem 1.19 (Dedekind). *If $g_1, \ldots, g_n$ are pairwise distinct automorphisms of $\mathbb{E}$, they are linearly independent over $\mathbb{E}$ as $\mathbb{E}$-valued functions on $\mathbb{E}$.*

Proof. Induction by n. If $n = 1$ then $g_1 \neq 0$ since it is an automorphism.

Take pairwise distinct automorphisms $g_1, \ldots, g_{n+1}$. If they were linearly dependent then for instance

$$g_{n+1} = e_1 g_1 + \ldots + e_n g_n$$

with at least one $e_i \neq 0$. We would have

$$g_{n+1}(e)(e_1 g_1(e') + \ldots + e_n g_n(e')) = g_{n+1}(e) g_{n+1}(e') = g_{n+1}(ee') =$$

$$= e_1 g_1(ee') + \ldots + e_n g_n(ee') = e_1 g_1(e) g_1(e') + \ldots + e_n g_n(e) g_n(e').$$

Hence

$$e_1 g_{n+1}(e) g_1 + \ldots + e_n g_{n+1}(e) g_n = e_1 g_1(e) g_1 + \ldots e_n g_n(e) g_n.$$

But $g_1, \ldots, g_n$ are linearly independent so

$$g_{n+1}(e) e_1 = e_1 g_1(e), \ldots, g_{n+1}(e) e_n = e_n g_n(e),$$

$$g_1 e_1 = g_{n+1} e_1, \ldots, g_n e_n = g_{n+1} e_n$$

which means that for at least one i we would have $g_i = g_{n+1}$, contradiction. □

Corollary 1.20. *If* $|G| < \infty$ *then* $\mathrm{Tr}_G \neq 0$.

Proof. If $\mathrm{Tr}_G = 0$, that is $\mathrm{Tr}_G(e) = 0$ for all $e \in \mathbb{E}$ then by definition

$$\sum_{g \in G} g(e) = \left(\sum_{g \in G} g \right)(e) = 0$$

that is $\sum_{g \in G} g = 0$, which contradicts linear independence. □

Theorem 1.21. *Let* G *be a group of automorphisms of* $\mathbb{E}$*. Assume that at least one of numbers* $|G|$, $[\mathbb{E} : \mathbb{E}^G]$ *is finite. Then they are equal.*

Proof.

1. Assume $|G| < \infty$, $G = \{g_1, \ldots, g_n\}$. Take $e_1, \ldots, e_m \in \mathbb{E}$, where $m > n$. Let $(e'_1, \ldots, e'_m)$ be a nonzero solution of the system

   $$\sum_{j=1}^{n} g_i^{-1}(e_j) e'_j = 0.$$

   We can assume that $\mathrm{Tr}_G(e'_1) \neq 0$. Then

   $$\underbrace{\sum_{i=1}^{n} \sum_{j=1}^{m} e_j g_i(e'_j)}_{=\sum_{j=1}^{n} e_j \mathrm{Tr}_G(e'_j)} = \sum_{i=1}^{n} g_i \left(\sum_{j=1}^{m} g_i^{-1}(e_j) e'_j \right) = 0,$$

   so $e_1, \ldots, e_m$ are linearly dependent over $\mathbb{E}^G$ if $m > n$ which means that $[\mathbb{E} : \mathbb{E}^G] \leq n = |G|$.

2. Assume $[\mathbb{E} : \mathbb{E}^G] < \infty$. Take a basis $e_1, \ldots, e_N$ of $\mathbb{E}$ over $\mathbb{E}^G$. Let $(e'_1, \ldots, e'_M)$, $N < M \leq G$ be a nonzero solution of the system of equations

   $$\sum_{j=1}^{M} e'_j g_j(e_i) = 0.$$

Then for all $e \in \mathbb{E}$

$$\sum_{j=1}^{M} e'_j g_j(e) = 0,$$

$$\sum_{j=1}^{M} e'_j g_j = 0$$

which contradicts Dedekind theorem (1.19). Thus $[\mathbb{E} : \mathbb{E}^G] \geq |G|$.

Together 1 and 2 give $[\mathbb{E} : \mathbb{E}^G] = |G|$. □

Definition 1.22. *An algebraic extension $\mathbb{E}/\mathbb{F}$ is called Galois if there exists a subgroup $G \subset \operatorname{Aut}(\mathbb{E})$ such that $\mathbb{F} = \mathbb{E}^G$.*

Theorem 1.23. *Let $\mathbb{E}/\mathbb{F}$ be an algebraic extension. Then it is Galois if and only if $\mathbb{F} = \mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})}$.*

Proof. Assume that there exists group G such that $\mathbb{E}^G = \mathbb{F}$. Then

$$\begin{tikzcd}
G \subset \operatorname{Gal}(\mathbb{E}/\mathbb{F}) \arrow[r, Rightarrow] & \mathbb{E}^G \arrow[d, equal] & \supset & \mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})} \arrow[d, equal] \\
& \mathbb{F} & = & \mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})}
\end{tikzcd}$$

so $\mathbb{E}^G = \mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})}$. □

Remark 1.24. This G may not be equal $\operatorname{Gal}(\mathbb{E}/\mathbb{F})$.

Corollary 1.25. *If $[\mathbb{E} \colon \mathbb{F}]$ is finite then it is Galois if and only if $|\operatorname{Gal}(\mathbb{E}/\mathbb{F})| = [\mathbb{E} : \mathbb{F}]$.*

Proof. If $\mathbb{F} = \mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})}$ then

$$|\operatorname{Gal}(\mathbb{E}/\mathbb{F})| = |\operatorname{Gal}(\mathbb{E}/\mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})})| = [\mathbb{E} : \mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})}] = [\mathbb{E} : \mathbb{F}].$$

If $|\operatorname{Gal}(\mathbb{E}/\mathbb{F})| = [\mathbb{E} : \mathbb{F}]$ then $\mathbb{F} \subset \mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})} \subset \mathbb{E}$.

To finish the proof we need the following:

Lemma 1.26. *Assume $\mathbb{F} \subset \mathbb{E} \subset \mathbb{D}$. Then provided finiteness*

$$[\mathbb{D} : \mathbb{E}] = [\mathbb{D} : \mathbb{E}][\mathbb{E} : \mathbb{F}]$$

Proof. Let $\{d_1, \ldots, d_n\}$ be a basis of $\mathbb{E}/\mathbb{F}$. It is enough to show that $\{d_i e_j\}$ is a basis of $\mathbb{D}/\mathbb{F}$. Let

$$d = \sum_i d_i \tilde{e}_i, \quad \tilde{e}_i = \sum_j e_i f_{ij}.$$

Then

$$d = \sum_{i,j} d_i e_j f_{ij},$$

so $\{d_i e_j\}$ span $\mathbb{D}/\mathbb{F}$. If $\sum_{i,j} d_i e_j f_{ij} = 0$ then

$$\sum_i d_i \underbrace{\left(\sum_j e_j f_{ij} \right)}_{\in \mathbb{E}} = 0$$

which gives a contradiction. □

Now

$$[\mathbb{E}^{\mathrm{Gal}(\mathbb{E}/\mathbb{F})} : \mathbb{F}] = \frac{[\mathbb{E} : \mathbb{F}]}{[\mathbb{E} : \mathbb{E}^{\mathrm{Gal}(\mathbb{E}/\mathbb{F})}]} = 1,$$

so $\mathbb{E}^{\mathrm{Gal}(\mathbb{E}/\mathbb{F})} = \mathbb{F}$. □

Definition 1.27. *A field extension* $\mathbb{E}/\mathbb{F}$ *is* normal *if* $\mathbb{E}$ *contains all roots of minimal polynomials of all elements in* $\mathbb{F}$ *which are algebraic over* $\mathbb{F}$.

Lemma 1.28. *Let* $\mathbb{E}/\mathbb{F}$ *be algebraic i.e.* $\mathbb{F} \subset \mathbb{E} \subset \overline{\mathbb{F}}$, *and let* $\mathbb{E}/\mathbb{F}$ *be normal. Then for every embedding over* $\mathbb{F}$

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "\varphi"] & & \overline{\mathbb{F}} \\ & \mathbb{F} \arrow[ul] \arrow[ur] & \end{tikzcd},$$

one has $\varphi(\mathbb{E}) = \mathbb{E}$.

Proof. Take $e \in \mathbb{E}$, $f(e) = 0$, so $f(\varphi(e)) = 0$. Hence φ maps the set of roots of every $f(X) \in \mathbb{F}[X]$ in $\mathbb{E}$ into the set of all roots of $f(X)$. Thus $\mathbb{E} = \mathbb{F}(\text{roots(family of polynomials)})$. The homomorphism φ transforms roots of this family into the roots of its image.

$$\begin{aligned} \varphi(\mathbb{E}) &= \varphi(\mathbb{F}(\text{roots(family of polynomials)})) \\ &= \mathbb{F}(\varphi(\text{roots(family of polynomials)})) \\ &= \mathbb{F}(\text{roots(family of polynomials)}) \\ &= \mathbb{E}. \end{aligned}$$

□

Definition 1.29. *A field extension* $\mathbb{E}/\mathbb{F}$ *is* separable *if every* $e \in \mathbb{F}$ *is a single root of its minimal polynomial.*

Definition 1.30. *An extension* $\mathbb{E}/\mathbb{F}$ *is a* splitting field *of* $f(X) \in \mathbb{F}[X]$ *if*

$$f(X) = c(X - e_1) \dots (X - e_n) \in \mathbb{E}[X]$$

and such decomposition is impossible in $\mathbb{F}'[X]$ *for any proper subfield* $\mathbb{F} \subset \mathbb{F}' \subset \mathbb{E}$.

All splitting fields of a given polynomial are isomorphic over $\mathbb{F}$.

1.5 Extending isomorphisms

Lemma 1.31. *Let* $\sigma_0 \colon \mathbb{F}_1 \to \mathbb{F}_2$ *be an isomorphism of fields. Let* $f_1(X) \in \mathbb{F}_1[X]$ *be irreducible and* $\mathbb{E}_1 = \mathbb{F}_1(e_1)$, *where* $f_1(e_1) = 0$. *Let* $\mathbb{E}_2 = \mathbb{F}_2(e_2)$, *where* $f_2(e_2) = 0$ *for* $f_2(X) = \sigma_0(f_1(X))$. *Then* σ_0 *extends to a unique isomorphism* $\sigma \colon \mathbb{E}_1 \to \mathbb{E}_2$ *with* $\sigma(e_1) = e_2$.

Proof. Extend σ_0 to $\sigma_0\colon \mathbb{F}_1[X] \to \mathbb{F}_2[X]$. The polynomial $f_1(X)$ is irreducible if and only if $f_2(X)$ is irreducible. By the Kronecker theorem (1.15)

$$\mathbb{F}_i[X]/(f_i)(X) \cong \mathbb{F}_i(e_i) = E_i, \quad i = 1, 2.$$

□

Lemma 1.32. *Let $\sigma_0\colon \mathbb{F}_1 \to \mathbb{F}_2$ be an isomorphism. Let $f_1(X) \in \mathbb{F}_1[X]$ and $f_2(X) = \sigma_0(f_1(X)) \in \mathbb{F}_2[X]$. Let $\mathbb{E}_i$ be splitting field of $f_i(X)$. Then σ_0 extends to an isomorphism of $\sigma\colon \mathbb{E}_1 \to \mathbb{E}_2$.*

Proof. Factor $f_1(X)$ into k irreducibles in $\mathbb{F}_1[X]$ and consider $d := \deg(f_1(X)) - k$. The proof goes by induction on d. If $d = 0$, then $f_1(X)$ is a product of linear factors, $\mathbb{E}_1 = \mathbb{F}_1$, $\mathbb{E}_2 = \mathbb{F}_2$, and $\sigma = \sigma_0$.

Suppose $d > 0$. Then $f_1(X)$ has an irreducible factor of degree > 1. Take a root $e_1 \in \mathbb{E}_1$ of $g_1(X) \in \mathbb{F}_1[X]$, and a root $e_2 \in \mathbb{E}_2$ of $\sigma_0(g_1(X)) \in \mathbb{F}_2[X]$. Then $\mathbb{F}_i(e_i) \in \mathbb{E}_i$, $i = 1, 2$, and by the previous lemma (1.31) there is an isomorphism

$$\tilde{\sigma}_0\colon \mathbb{F}_1(e_1) \to \mathbb{F}_2(e_2)$$

with $\tilde{\sigma}_0\big|_{\mathbb{F}_1} = \sigma_0$ and $\tilde{\sigma}_0(e_1) = e_2$. Take now $\tilde{\mathbb{F}}_1 := \mathbb{F}_1(e_1)$ instead of $\mathbb{F}_1$. Consider $\tilde{f}_i(X) = f_i(X) \in \tilde{\mathbb{F}}_i[X]$. Now $g_1(X) \in \mathbb{F}_1[X]$ has a linear factor $(X - e_1)$. Thus $\tilde{f}_1(X)$ has $\tilde{k} > k$ irreducible factors in $\tilde{\mathbb{F}}_1[X]$. Thus $\tilde{d} = \deg \tilde{f}_1(X) - \tilde{k} < d$. Now $\mathbb{E}_i$ is still a splitting field of a polynomial $\tilde{f}_i(X) \in \tilde{\mathbb{F}}_i[X]$, so $\tilde{\sigma}_0$ extends to some $\sigma\colon \mathbb{E}_1 \cong \mathbb{E}_2$. □

Theorem 1.33. *An algebraic extension $\mathbb{E}/\mathbb{F}$ is Galois if and only if it is normal and separable.*

Proof.

- Assume that $\mathbb{E}/\mathbb{F}$ is Galois, that is there exists group $G < \operatorname{Aut}(\mathbb{E})$ such that $\mathbb{F} = \mathbb{E}^G$. It is enough to prove that the minimal polynomial $f_e(X) \in \mathbb{F}[X]$ of any $e \in \mathbb{E}$ splits into pairwise distinct linear factors in $\mathbb{E}[X]$.

  Because $f_e(e) = 0$ we have for all $g \in G$ that $f_e(g(e)) = gf_e(e) = 0$, so $|Ge| < \infty$ as the number of roots is finite. Say $Ge = \{g_1(e), \ldots, g_r(e)\}$. Define $f(X) := (X - g_1(e)) \ldots (X - g_r(e))$. For all $g \in G$ we have $g(f(X)) = f(X)$, so $f(X) \in \mathbb{F}[X]$. Since all roots of f are pairwise distinct roots of f_e we have that $f | f_e$. But f_e is monic irreducible, so $f = f_e$. This implies that f_e splits as desired.

- Assume now that $\mathbb{E}/\mathbb{F}$ is separable and normal. Take $e \in \mathbb{E} \setminus \mathbb{F}$ and its minimal polynomial $f_e(X)$. In $\mathbb{E}$ $f_e(X)$ splits as $f_e(X) = (X - e_1) \ldots (X - e_r)$. Assume that $e_1 := e \notin \mathbb{F}$, so $\deg(f_e(X)) > 0$. There must be another root $e_2 \neq e_1$ of $f_e(X)$. There is an isomorphism $\mathbb{F}(e_1) \to \mathbb{F}(e_2)$ which is id on $\mathbb{F}$ and sends e_1 to e_2. It extends to $\overline{\mathbb{F}(e_1)} = \overline{\mathbb{F}} \to \overline{\mathbb{F}} = \overline{\mathbb{F}(e_2)}$ (nonconstructive axiom of choice). Since $\mathbb{E}/\mathbb{F}$ is normal this isomorphism restricts to $g \in \operatorname{Gal}(\mathbb{E}/\mathbb{F})$ such that $g(e_1) = e_2 \neq e_1$. There are no elements of $\mathbb{E} \setminus \mathbb{F}$ which are fixed by $\operatorname{Gal}(\mathbb{E}/\mathbb{F})$, so $\mathbb{E}^{\operatorname{Gal}(\mathbb{E}/\mathbb{F})} = \mathbb{F}$ and $\mathbb{E}/\mathbb{F}$ is a Galois extension.

□

Corollary 1.34. *Extension $\mathbb{E}/\mathbb{F}$ is finite Galois if and only if it is a splitting field of a separable polynomial $f(X) \in \mathbb{F}[X]$.*

Proof. We know that $\mathbb{E}/\mathbb{F}$ is finite Galois if and only if it is finite, normal, and separable. In fact $\mathbb{E}/\mathbb{F}$ is finite and normal if and only if $\mathbb{E}$ is a splitting field of some $f(X) \in \mathbb{F}[X]$. Indeed, if $\mathbb{E}/\mathbb{F}$ is finite and normal, then we can take all roots of a family of polynomials and choose a linearly independent (finite) subset of roots generating $\mathbb{E}/\mathbb{F}$. They are roots of some finite number of polynomials $f_1(X), \dots, f_n(X) \in \mathbb{F}[X]$. Then $\mathbb{E}$ is a splitting field of $f(X) = f_1(X) \cdot \ldots \cdot f_n(X)$. The reverse implication is obvious from the definition of normality. Finally $\mathbb{E}/\mathbb{F}$ is separable if and only if $f(X) = f_1(X) \cdot \ldots \cdot f_n(X)$ is separable. ☐

1.6 The fundamental theorem of Galois theory

Theorem 1.35. *Let $\mathbb{E}/\mathbb{F}$ be a finite Galois extension, $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Then*

1. *There is a one-to-one correspondence between intermediate fields $\mathbb{F} \subset \mathbb{F}' \subset \mathbb{E}$ and subgroups $G \supset G' \supset \{1\}$ given by*

$$\mathbb{F}' := \mathbb{E}^{G'}$$

2. *Extension $\mathbb{F}'/\mathbb{F}$ is normal if and only if G' is a normal subgroup of G. This is the case if and only if $\mathbb{F}'/\mathbb{F}$ is Galois. In this case $\mathrm{Gal}(\mathbb{F}'/\mathbb{F}) \cong G/G'$.*

3. *For each $\mathbb{F} \subset \mathbb{F}' \subset \mathbb{E}$*

$$[\mathbb{F}' : \mathbb{F}] = [G : G']$$
$$[\mathbb{E} : \mathbb{F}'] = |G'|$$

Remark 1.36.

1. If $\mathbb{F} \subset \mathbb{F}' \subset \mathbb{F}'' \subset \mathbb{E}$ then $G' \supset G''$.

2. Extension $\mathbb{E}/\mathbb{F}'$ is always Galois with $\mathrm{Gal}(\mathbb{E}/\mathbb{F}') = G'$.

3. If an extension $\mathbb{E}/\mathbb{F}$ is separable then $\mathbb{F}'/\mathbb{F}$ is separable. Thus $\mathbb{F}'/\mathbb{F}$ is normal if and only if it is Galois.

4. From the proof we will get that if $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) = G$, then $G/G' = \mathrm{Gal}(\mathbb{F}'/\mathbb{F})$ in the case $\mathbb{F}'/\mathbb{F}$ is Galois.

Proof.

1. Define a map

$$\phi\colon \{\text{subgroups of } G\} \to \{\text{intermediate fields}\}$$
$$G' \mapsto \mathbb{E}^{G'}$$

- ϕ is injective: $G' \neq G'' \implies \mathbb{E}^{G'} \neq \mathbb{E}^{G''}$

  Lemma 1.37. $\mathrm{Gal}(\mathbb{E}/\mathbb{E}^{G'}) = G'$.

Proof. $\mathbb{E}^{G'} = \mathbb{E}^{\mathrm{Gal}(\mathbb{E}/\mathbb{E}^{G'})}$ because $\mathbb{E}/\mathbb{E}^{G'}$ is Galois. Furthermore

$$|G'| = [\mathbb{E} : \mathbb{E}^{G'}] = [\mathbb{E} : \mathbb{E}^{\mathrm{Gal}(\mathbb{E}/\mathbb{E}^{G'})}] = |\operatorname{Gal}(\mathbb{E}/\mathbb{E}^{G'})|$$

and $G' \subset \operatorname{Gal}(\mathbb{E}/\mathbb{E}^{G'})$, so $G' = \operatorname{Gal}(\mathbb{E}/\mathbb{E}^{G'})$. □

By lemma if $\mathbb{E}^{G'} \subset \mathbb{E}^{G''}$ then $G'' = \operatorname{Gal}(\mathbb{E}/\mathbb{E}^{G'}) \subset G'$. Hence if $\mathbb{E}^{G'} = \mathbb{E}^{G''}$ then $G' = G''$.

- ϕ is surjective. Indeed, let $\mathbb{F} \subset \mathbb{F}' \subset \mathbb{E}$, $G' = \operatorname{Gal}(\mathbb{E}/\mathbb{F}') \subset \operatorname{Gal}(\mathbb{E}/\mathbb{F}) = G$. If $\mathbb{E}/\mathbb{F}$ is Galois then $\mathbb{E}$ is a splitting field of a separable polynomial with coefficients in $\mathbb{F}$, $f(X) \in \mathbb{F}[X] \subset \mathbb{F}'[X]$. Thus $\mathbb{E}$ is a splitting field of $f(X) \in \mathbb{F}'[X]$, so $\mathbb{E}/\mathbb{F}'$ is Galois and $\mathbb{F}' = \mathbb{E}^{G'}$.

2. Suppose $G' \triangleleft G$, $\mathbb{F}' := \mathbb{E}^{G'}$. Then $\mathbb{E}/\mathbb{F}'$ is a Galois extension. Take $g \in \operatorname{Gal}(\mathbb{E}/\mathbb{F})$. Then $g(\mathbb{F}') = \mathbb{E}^{gGg^{-1}} = \mathbb{E}^{G'} = \mathbb{F}'$. This gives the restriction map

$$\begin{aligned} \mathrm{Res}\colon \operatorname{Gal}(\mathbb{E}/\mathbb{F}) = G &\to \operatorname{Gal}(\mathbb{F}'/\mathbb{F}) \\ g &\mapsto g|_{\mathbb{F}'} \\ \ker(\mathrm{Res}) &= \operatorname{Gal}(\mathbb{E}/\mathbb{F}') = G \\ \operatorname{im}(\mathrm{Res}) &= G/G' \end{aligned}$$

We want to prove that Res is onto. Let $\tilde{g} \in \operatorname{Gal}(\mathbb{F}'/\mathbb{F})$. We know that $\mathbb{E}$ is a splitting field of some polynomial $f(X) \in \mathbb{F}'[X]$, so $\tilde{g}\colon \mathbb{F}' \to \mathbb{F}'$ extends to $g\colon \mathbb{E} \xrightarrow{\cong} \mathbb{E}$, $g|_{\mathbb{F}} = \mathrm{id}$. Thus $g \in \operatorname{Gal}(\mathbb{E}/\mathbb{F})$ and $g|_{\mathbb{F}'} = \tilde{g}$ and Res is onto. Hence $\operatorname{Gal}(\mathbb{F}'/\mathbb{F}) \cong G/G'$.

Suppose the converse, that is $\mathbb{F}'/\mathbb{F}$ is Galois. Then $\mathbb{F}'$ is a splitting field of some separable polynomial $f(X) \in \mathbb{F}[X]$ with roots (distinct by separability) $e_1, \ldots, e_n \in \mathbb{F}' \subset \mathbb{E}$ and $\mathbb{F}' = \mathbb{F}(e_1, \ldots, e_n) \subset \mathbb{E}$. Take $g \in \operatorname{Gal}(\mathbb{E}/\mathbb{F}) = G$. We have $g(f(X)) = f(X)$, so g permutes the set of roots $\{e_1, \ldots, e_n\}$. Hence $\mathbb{E}^{G'} = \mathbb{F}' = g(\mathbb{F}') = \mathbb{E}^{gG'g^{-1}}$. By 1, $G' = gG'g^{-1}$, so $G' \triangleleft G$.

3. If $\mathbb{E}/\mathbb{F}'$ is Galois extension, then

$$[\mathbb{E} : \mathbb{F}'] = |\operatorname{Gal}(\mathbb{E}/\mathbb{F}')| = |G'|,$$

$$[\mathbb{E} : \mathbb{F}] = [\mathbb{E} : \mathbb{F}] \cdot [\mathbb{F}' : \mathbb{F}], \quad |G| = |G'| \cdot [G : G'].$$

Hence $[\mathbb{F}' : \mathbb{F}] = [G : G']$ and $[\mathbb{E} : \mathbb{F}'] = |G'|$.

□

Corollary 1.38. *If*

$$\begin{tikzcd} & \mathbb{F}' \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbb{F} \arrow[dr, dash] & & \mathbb{E} \arrow[dl, dash] \\ & \mathbb{F}'' & \end{tikzcd}$$

are field extensions, then the following are equivalent

1. $g(\mathbb{F}') = \mathbb{F}''$ *for some* $g \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$
2. $g\,\mathrm{Gal}(\mathbb{E}/\mathbb{F}')g^{-1} = \mathrm{Gal}(\mathbb{E}/\mathbb{F}'')$

Definition 1.39. *An abstract group G acts transitively on a set S if for all elements $s, s' \in S$ there is $g \in G$ such that $s' = g(s)$.*

Proposition 1.40. *Let $\mathbb{E}/\mathbb{F}$ be finite Galois extension, so $\mathbb{E}$ is a splitting field of a separable polynomial $f(X) \in \mathbb{F}[X]$. Then $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ is isomorphic to a subgroup of the permutation group on the roots of $f(X)$. If $f(X)$ is irreducible then this action is transitive.*

Proof. Take the set of roots of $f(X)$, $S := \{e_1, \dots, e_n\}$. Let $g \in G$ such that $g(f(X)) = f(X)$, so g permutes S. We have $\mathbb{E} = \mathbb{F}(e_1, \dots, e_n)$, and if for all i $g(e_i) = e_i$, then $g = \mathrm{id}$. This means that G embeds in the group of permutations of S.

Take now $e_i \neq e_j$. If $f(X)$ is irreducible then there exists an isomorphism $\sigma_0 \colon \mathbb{F}(e_i) \xrightarrow{\cong} \mathbb{F}(e_j)$, such that $\sigma_0|_{\mathbb{F}} = \mathrm{id}$, $\sigma_0(e_i) = e_j$. Hence σ_0 extends to $g \colon \mathbb{E} \to \mathbb{E}$, $g|_{\mathbb{F}} = \mathrm{id}$, so there exists an element $g \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ such that $g(e_i) = e_j$. □

1.7 The normal basis theorem

We know that if $\mathbb{E}/\mathbb{F}$ is finite Galois then $[\mathbb{E} : \mathbb{F}] = |\,\mathrm{Gal}(\mathbb{E}/\mathbb{F})|$.

Definition 1.41. *If $\mathbb{E}/\mathbb{F}$ is finite Galois then a basis $\{e_1, \dots, e_n\}$ is called* normal *if there exists $e \in \mathbb{E}$ such that $e_i = g_i(e)$ for $\{g_1, \dots, g_n\} = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$.*

Theorem 1.42 (Normal Basis theorem)**.** *If $\mathbb{F}$ is infinite and $\mathbb{E}/\mathbb{F}$ is finite Galois then $\mathbb{E}$ has a normal basis over $\mathbb{F}$.*

The proof uses some additional results.

Lemma 1.43. *If $\mathbb{E}/\mathbb{F}$ is a Galois extension of degree n, $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) = \{g_1, \dots, g_n\}$, then $\{e_1, \dots, e_n\} \subset \mathbb{E}$ is a basis over $\mathbb{F}$ if and only if the matrix $\{g_i(e_j)\}$ is non-singular.*

Proof. If $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{E}/\mathbb{F}$, then if for some $(c_1, \dots, c_n) \neq 0 \in \mathbb{F}^n$ and all $j = 1, \dots, n$

$$\sum_{i=1}^{n} c_i g_i(e_j) = 0$$

we get that for all $e \in E$

$$\sum_{i=1}^{n} c_i g_i(e) = 0,$$

which contradicts Dedekind theorem (1.19).

On the other hand if $\sum_{j=1}^{n} c_j e_j = 0$ is a nontrivial linear dependence then for all i

$$\sum_{i=1}^{n} c_j g_i(e_j) = 0$$

which means that $\{g_i(e_j)\}$ is singular. □

Lemma 1.44. *Let $\mathbb{F}$ be infinite, $\mathbb{E}/\mathbb{F}$ an extension. If $f(c_1, \dots, c_n) = 0$ for all $(c_1, \dots, c_n) \in \mathbb{F}^n$ and $f(X_1, \dots, X_n) \in \mathbb{E}[X_1, \dots, X_n]$, then $f(X_1, \dots, X_n) = 0$.*

Proof. Induction by n. If $f(e_1) = 0$ for infinitely many e_1, then $f(X_1) = 0$.

Let $n > 1$. Then we can write

$$f(X_1, \dots, X_n) = \sum_{k=0}^{n} f_k(X_1, \dots, X_{n-1}) X_n^k.$$

Take $(c_1, \dots, c_{n-1})$ such that $f(c_1, \dots, c_{n-1}, X_n) = 0$. Then for all k we have $f_k(c_1, \dots, c_{n-1}) = 0$, and by the inductive step $f_k(X_1, \dots, X_{n-1}) = 0$, so $f(X_1, \dots, X_n) = 0$. □

The next result we need is a generalization of the Dedekind theorem, provided $\mathbb{F}$ is infinite.

Theorem 1.45. *Let $\mathbb{F}$ be infinite, $\mathbb{E}/\mathbb{F}$ finite extension, $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) = \{g_1, \dots, g_n\}$. Then $g_1, \dots, g_n$ are algebraically independent i.e. for all $e \in \mathbb{E}$ if for some $f(X_1, \dots, X_n) \in \mathbb{E}[X_1, \dots, X_n]$ we have $f(g_1(e), \dots, g_n(e))$, then $f(X_1, \dots, X_n) = 0$.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathbb{E}$ over $\mathbb{F}$. By the first lemma $\{g_i(e_j)\}$ is nonsingular. Let $e = \sum_{j=1}^n c_j e_j$, so $g_i(e) = \sum_{j=1}^n c_j g_i(e_j)$. Suppose $f(g_1(e), \dots, g_n(e)) = 0$ for all $e \in \mathbb{E}$. After substitution

$$f(\dots, \sum_{j=1}^{n} c_j g_i(e_j), \dots) = 0,$$

for all $e \in \mathbb{E}$, so from the second lemma

$$f(\dots, \sum_{j=1}^{n} X_j g_i(e_j), \dots) = 0.$$

Since $X_i \mapsto \sum_{j=1}^n X_j g_i(e_j)$ is an automorphism of $\mathbb{E}[X_1, \dots, X_n]$ we get that $f(\dots, X_i, \dots) = 0$. □

Proof. (of the Normal Basis Theorem (1.42)) Let $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) = \{g_1, \dots, g_n\}$. Take a matrix

$$A_{ij} = X_k \quad \text{if} \quad g_i g_j = g_k.$$

Denote its determinant by $d(X_1, \dots, X_n) := \det(A_{ij}) \in \mathbb{E}[X_1, \dots, X_n]$. Then $d(1, \dots, 1) = \pm 1 \neq 0$ because each X_k appears only once in every row and every column. Hence $d(X_1, \dots, X_n) \neq 0$.

Let $e_j := g_j(e)$, $A^e_{ij} := g_k(e)$ if $g_i g_j = g_k$. Then $A^e_{ij} = g_i g_j(e) = g_i(e_j)$, and $\det(A^e_{ij}) = d(g_1(e), \dots, g_n(e))$. By the previous theorem there exists $e \in \mathbb{E}$ such that

$$d(g_1(e), \dots, g_n(e)) \neq 0.$$

By the first lemma $\{e_1, \dots, e_n\}$ is a normal basis. □

1.8 Hilbert's 90 theorem

Definition 1.46. *Let $\mathbb{E}/\mathbb{F}$ be finite Galois, $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$.*

- *The* norm $\mathrm{N}_{\mathbb{E}/\mathbb{F}} \colon \mathbb{E} \to \mathbb{F}$ *is given by*

$$\mathrm{N}_{\mathbb{E}/\mathbb{F}}(e) := \prod_{g \in G} g(e).$$

- *The* trace $\mathrm{Tr}_{\mathbb{E}/\mathbb{F}} \colon \mathbb{E} \to \mathbb{F}$ *is given by*

$$\mathrm{Tr}_{\mathbb{E}/\mathbb{F}} := \sum_{g \in G} g(e).$$

Theorem 1.47. *Let $\mathbb{E}/\mathbb{F}$ be a finite Galois extension with cyclic Galois group $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$ generated by $g \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Then the following sequences of abelian groups*

1.
$$\mathbb{E}^* \xrightarrow{\partial} \mathbb{E}^* \xrightarrow{\mathrm{N}_{\mathbb{E}/\mathbb{F}}} \mathbb{F}, \quad \partial(e) = \frac{e}{g(e)}$$

2.
$$\mathbb{E} \xrightarrow{\partial} \mathbb{E} \xrightarrow{\mathrm{Tr}_{\mathbb{E}/\mathbb{F}}} \mathbb{F}, \quad \partial(e) = e - g(e)$$

are exact.

Proof.

1. By the Dedekind theorem $\{g, g^2, \ldots, g^{n-1}, g^n = 1\}$ are linearly independent over $\mathbb{E}$. For every $(e_1, \ldots, e_n) \in \mathbb{E}^n$ there exists $\tilde{\tilde{e}} \in \mathbb{E}$ such that

$$\tilde{e} := \sum_{i=1}^{n} e_i g^i(\tilde{\tilde{e}}) \neq 0.$$

Take
$$\begin{cases} e_i & := \ eg(e) \ldots g^{i-1}(e), \ i = 1, \ldots n-1, \\ e_n & := \ 1 = \mathrm{N}_{\mathbb{E}/\mathbb{F}}(e) \end{cases}$$

We have

$$\begin{aligned}
eg(e_i) &= eg(e)\dots g^i(e) = e_{i+1},\ i = 1,\dots,n-1\\
eg(e_n) &= e = e_1\\
eg(\tilde{e}) &= \sum_{i=1}^{n} eg(e_i)g^{i+1}(\tilde{\tilde{e}})\\
&= \sum_{i=1}^{n-1} \underbrace{eg(e_i)}_{e_{i+1}} g^{i+1}(\tilde{\tilde{e}}) + \underbrace{eg(e_n)}_{e_1}\underbrace{g^{n+1}}_{g}(\tilde{\tilde{e}})\\
&= \sum_{i=1}^{n-1} e_{i+1}g^{i+1}(\tilde{\tilde{e}}) + e_1 g(\tilde{\tilde{e}})\\
&= \sum_{i=2}^{n} e_i g^i(\tilde{\tilde{e}}) + e_1 g(\tilde{\tilde{e}})\\
&= \sum_{i=1}^{n} e_i g^i(\tilde{\tilde{e}})\\
&= \tilde{e}.
\end{aligned}$$

Hence $e = \frac{\tilde{e}}{g(\tilde{e})}$.

2. By the Dedekind theorem (1.19) there is $\tilde{\tilde{e}} \in \mathbb{E}$ such that $\mathrm{Tr}_{\mathbb{E}/\mathbb{F}}(\tilde{\tilde{e}}) = 1$.

$$\begin{aligned}
\tilde{e} &:= eg(\tilde{\tilde{e}}) + (e+g(e))g^2(\tilde{\tilde{e}}) + \ldots + (e+g(e)+\ldots+g^{n-2}(e))g^{n-1}(\tilde{\tilde{e}})\\
\tilde{e} - g(\tilde{e}) &= e(\underbrace{\tilde{\tilde{e}} + g(\tilde{\tilde{e}}) + \ldots + g^{n-1}(\tilde{\tilde{e}})}_{=\mathrm{Tr}_{\mathbb{E}/\mathbb{F}}(\tilde{\tilde{e}})=1}) - (\underbrace{e + g(e) + \ldots + g^{n-1}(e)}_{=\mathrm{Tr}_{\mathbb{E}/\mathbb{F}}(e)=0})\tilde{\tilde{e}}\\
&= e.
\end{aligned}$$

□

Chapter 2

Hopf–Galois extensions

2.1 Canonical map

Theorem 2.1. *Let $\mathbb{E}/\mathbb{F}$ be a finite Galois extension, $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Then*

$$\begin{aligned} \mathrm{can}\colon \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} &\to \mathrm{Map}(G, \mathbb{E}), \\ e_1 \otimes e_2 &\mapsto (g \mapsto e_1 g(e_2)) \end{aligned}$$

is bijective.

Proof. Let $G = \{g_1, \dots, g_n\}$. Observe that can is left $\mathbb{E}$-linear and

$$\dim_{\mathbb{E}}(\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) = \dim_{\mathbb{F}}(\mathbb{E}) = [\mathbb{E} : \mathbb{F}],$$

$$\dim_{\mathbb{E}}(\mathrm{Map}(G, \mathbb{E})) = |G|.$$

By Galois theory these dimensions are equal. It is enough to prove that can is injective. Let $\sum \tilde{e}_i \otimes e_i \in \ker(\mathrm{can})$, where $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{E}/\mathbb{F}$. After applying the canonical map we get that for all $g_j \in G$

$$\sum_{i=1}^{n} \tilde{e}_i g_j(e_i) = 0.$$

By the Dedekind theorem (1.19) $g_j(e_i)$ are nonsingular, so all $\tilde{e}_i$ are zero, and $\ker(\mathrm{can}) = \{0\}$. $\square$

Theorem 2.2. *If $\mathbb{E}/\mathbb{F}$ is a finite Galois extension, $G < \mathrm{Gal}(\mathbb{E}/\mathbb{F})$, then*

$$\mathrm{can}\colon \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \to \mathrm{Map}(G, \mathbb{E}),$$

$$e_1 \otimes e_2 \to (g \mapsto e_1 g(e_2))$$

is well defined, and the following implication holds:

$$\mathrm{can} \text{ is bijective } \implies \mathbb{F} = \mathbb{E}^G.$$

Proof. We have

$$\underbrace{\dim_{\mathbb{F}}(\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E})}_{[\mathbb{E}:\mathbb{F}]^2} = \underbrace{\dim_{\mathbb{F}}(\mathrm{Map}(G, \mathbb{E}))}_{|G|[\mathbb{E}:\mathbb{F}]}$$

Hence $[\mathbb{E} : \mathbb{F}] = |G| = [\mathbb{E} : \mathbb{E}^G]$. If $\mathbb{F} \subset \mathbb{E}^G$, then $[\mathbb{E} : \mathbb{F}] = [\mathbb{E} : \mathbb{E}^G][\mathbb{E}^G : \mathbb{F}]$, so $[\mathbb{E}^G : \mathbb{F}] = 1$, that is $\mathbb{F} = \mathbb{E}^G$. $\square$

Corollary 2.3. *If $\mathbb{E}/\mathbb{F}$ is a finite extension, $G < \mathrm{Gal}(\mathbb{E}/\mathbb{F})$, then $\mathbb{E}/\mathbb{F}$ is Galois if and only if* can *is bijective.*

What algebraic structures are involved in can?
On $\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}$ there is a structure of a bimodule over $\mathbb{E}$

$$e(e_1 \otimes e_2) = ee_1 \otimes e_2$$
$$(e_1 \otimes e_2)e = e_1 \otimes e_2 e$$

If one wants can to be a bimodule map, then $\mathrm{Map}(G, \mathbb{E})$ has to be equipped with the following bimodule structure

$$(e\varphi)(g) = e\varphi(g)$$
$$(\varphi e)(g) = \varphi(g)g(e)$$

2.2 Coring structure

Definition 2.4. *(C, Δ, ε) is called a* coring *over $\mathbb{E}$ if C is a bimodule over $\mathbb{E}$ equipped with bimodule maps $\Delta\colon C \to C \otimes_{\mathbb{E}} C$ (a* comultiplication*), and $\varepsilon\colon C \to \mathbb{E}$ (a* counit*) such that the following diagrams commute*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes_{\mathbb{E}} C \arrow[d, "\mathrm{id}\otimes\Delta"] \\
C \otimes_{\mathbb{E}} C \arrow[r, "\Delta\otimes\mathrm{id}"] & C \otimes_{\mathbb{E}} C \otimes_{\mathbb{E}} C
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & C \otimes_{\mathbb{E}} C \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
C \otimes_{\mathbb{E}} C \arrow[r, "\varepsilon\otimes\mathrm{id}"] & C
\end{tikzcd}$$

On $\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}$ there is a coring structure given by

$$\Delta(e_1 \otimes e_2) := (e_1 \otimes 1) \otimes (1 \otimes e_2) \in (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}),$$
$$\varepsilon(e_1 \otimes e_2) := e_1 e_2.$$

The following diagrams commute

$$\begin{tikzcd}
\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \arrow[r, "\Delta"] \arrow[d, "\Delta"] & (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \arrow[d, "\mathrm{id}\otimes\Delta"] \\
(\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \arrow[r, "\Delta\otimes\mathrm{id}"] & (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
(\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \arrow[r, "\varepsilon\otimes\mathrm{id}"] & \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}
\end{tikzcd}$$

so $(\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}, \Delta, \varepsilon)$ is a coring.

On $\mathrm{Map}(G, \mathbb{E})$ there is also a canonical comultiplication Δ induced by the group law $G \times G \to G$.

$$\begin{tikzcd}
\mathrm{Map}(G,\mathbb{E}) \arrow[rr, dashed, "\Delta"] \arrow[dr] & & \mathrm{Map}(G,\mathbb{E}) \otimes_{\mathbb{E}} \mathrm{Map}(G,\mathbb{E}) \arrow[dl, "\cong"] \\
& \mathrm{Map}(G \times G, \mathbb{E}) &
\end{tikzcd}$$

The isomorphism $\mathrm{Map}(G,\mathbb{E}) \otimes_{\mathbb{E}} \mathrm{Map}(G,\mathbb{E}) \to \mathrm{Map}(G \times G, \mathbb{E})$ is given by

$$\varphi_1 \otimes \varphi_2 \mapsto ((g_1, g_2) \mapsto \varphi_1(g_1) g_1(\varphi_2(g_2))).$$

The counit is induced by the neutral element $g_0 \in G$

$$\varepsilon\colon \mathrm{Map}(G,\mathbb{E}) \to \mathbb{E}, \quad \varphi \mapsto \varphi(g_0).$$

Altogether these give a coring structure on $\mathrm{Map}(G, \mathbb{E})$.

Proposition 2.5. *The canonical map* $\mathrm{can}\colon \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \to \mathrm{Map}(G, \mathbb{E})$ *is a homomorphism of corings over* $\mathbb{E}$.

Proof. We have to check compatibility with comultiplication, that is commutativity of the diagram

$$\begin{tikzcd}
\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \arrow[r, "\mathrm{can}"] \arrow[dd, "\Delta"'] & \mathrm{Map}(G,\mathbb{E}) \arrow[d] \\
& \mathrm{Map}(G \times G, \mathbb{E}) \arrow[d] \\
(\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \arrow[r, "\mathrm{can} \otimes \mathrm{can}"] & \mathrm{Map}(G,\mathbb{E}) \otimes_{\mathbb{E}} \mathrm{Map}(G,\mathbb{E})
\end{tikzcd}$$

We have

$$\begin{tikzcd}
e_1 \otimes e_2 \arrow[r, mapsto] \arrow[ddd, mapsto] & (g \mapsto e_1 g(e_2)) \arrow[d, mapsto] \\
& ((g_1, g_2) \mapsto e_1 g_1 g_2(e_2)) \arrow[d, equal] \\
& ((g_1, g_2) \mapsto e_1 g_1(g_2(e_2))) \\
(e_1 \otimes 1) \otimes (1 \otimes e_2) \arrow[r, mapsto] & (g_1 \mapsto e_1) \otimes (g_2 \mapsto g_2(e_2)) \arrow[u, mapsto]
\end{tikzcd}$$

Next we check the compatibility with the counit that is commutativity of the diagram

$$\begin{tikzcd}
\mathbb{E}\otimes_{\mathbb{F}}\mathbb{E} \arrow[r, "\mathrm{can}"] \arrow[d, "\varepsilon"'] & \mathrm{Map}(G,\mathbb{E}) \arrow[d, "\varepsilon"] \\
\mathbb{E} \arrow[r] & \mathbb{E}
\end{tikzcd}$$

We have

$$\begin{tikzcd}
e_1\otimes e_2 \arrow[r, mapsto] \arrow[d, mapsto] & (g\mapsto e_1g(e_2)) \arrow[d, mapsto] \\
e_1e_2 \arrow[r, mapsto] & e_1e_2 = e_1g_0(e_2)
\end{tikzcd}$$

□

We will use the Sweedler notation for comultiplication $\Delta\colon C\to C\otimes_{\mathbb{E}} C$

$$\Delta(c) = \sum_i c_{1i}\otimes c_{2i} =: c_{(1)}\otimes c_{(2)}.$$

Proposition 2.6. *Let* (C,Δ,ε) *be a coring over* $\mathbb{E}$. *Then* $\mathrm{Hom}_{\mathbb{E}}(C,\mathbb{E})$ *is a ring with multiplication given by*

$$(\varphi_1\varphi_2)(c) := \varphi_1(c_{(1)}\varphi_2(c_{(2)}))$$

and unit ε.

Examples 2.7.

1. $\mathrm{Hom}_{\mathbb{E}}(\mathbb{E}\otimes_{\mathbb{F}}\mathbb{E},\mathbb{E}) = \mathrm{Hom}_{\mathbb{F}}(\mathbb{E},\mathbb{E}) = \mathrm{End}_{\mathbb{F}}(\mathbb{E})$ with composition of morphisms as multiplication, and identity as the unit.
2. For finite G

$$E\rtimes G \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{E}}(\mathrm{Map}(G,\mathbb{E}),\mathbb{E})$$

$$\sum_{g\in G} e_g x_g \mapsto (\varphi\mapsto \sum_i e_i\varphi(g_i)),\quad x_g e = g(e)x_g$$

Corollary 2.8. *The canonical map of corings* $\mathrm{can}\colon \mathbb{E}\otimes_{\mathbb{F}}\mathbb{E}\to\mathrm{Map}(G,\mathbb{E})$ *induces a ring homomorphism*

$$\mathrm{Hom}_{\mathbb{E}}(\mathrm{can},\mathbb{E})\colon E\rtimes G\to \mathrm{End}_{\mathbb{F}}(\mathbb{E}).$$

Proposition 2.9. *Let* $\mathbb{E}/\mathbb{F}$ *be a finite extension,* $G\subset \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. *Then* $\mathbb{E}/\mathbb{F}$ *is Galois if and only if* $\mathrm{Hom}_{\mathbb{E}}(\mathrm{can},\mathbb{E})$ *is bijective.*

Proof. If $\mathbb{E}/\mathbb{F}$ is Galois, then can is bijective, so $\mathrm{Hom}(\mathrm{can},\mathbb{E})$ is bijective.

Apllying $\mathrm{Hom}(-,\mathbb{E})$ to $\mathrm{Hom}(\mathrm{can},\mathbb{E})$ we obtain can again by finite dimension over $\mathbb{F}$. □

Remark 2.10. $\mathrm{End}_{\mathbb{F}}(\mathbb{E})$ is a matrix algebra with entries in $\mathbb{F}$. If $\mathbb{E}/\mathbb{F}$ is Galois then $\mathbb{E}\rtimes G$ is Morita equivalent to $\mathbb{F}=\mathbb{E}^G$ (i.e. the category $\mathbb{E}^G-\mathbf{Mod}$ is equivalent to $\mathbb{E}\rtimes G-\mathbf{Mod}$). It is the cornerstone of noncommutative geometry. If G is not finite, then $\mathbb{E}^G$ can be pathological and then one can take its noncommutative replacement $\mathbb{E}\rtimes G$.

Example 2.11. Let $\mathbb{E} = \mathbb{C}((X))$ be the field of rational complex functions. Take $G = \mathbb{Z}$ generated by $g(X) := 2X$. Each $e \in \mathbb{E}$ can be written as

$$e = \frac{a_{-n}}{x^n} + \frac{a_{-n+1}}{x^{n-1}} + \ldots + a_0 + a_1 x + \ldots$$

If g fixes e, then $a_i = 0$ for $i \neq 0$, so $\mathbb{E}^G = \mathbb{C}$. On the other hand $\mathbb{E} \rtimes G = \mathbb{C}((X)) \rtimes G$.

2.3 Hopf–Galois field extensions

Assume $[\mathbb{E} : \mathbb{F}] < \infty$, $G < \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Then

$$\mathrm{Map}(G, \mathbb{E}) = \mathbb{E} \otimes_{\mathbb{F}} \mathrm{Map}(G, \mathbb{F}).$$

$\mathrm{Map}(G, \mathbb{F})$ is an $\mathbb{F}$-algebra with pointwise multiplication and it is also a coalgebra with comultiplication

$$\begin{tikzcd}
\mathrm{Map}(G,\mathbb{F}) \arrow[rr, dashed, "\Delta"] \arrow[dr] & & \mathrm{Map}(G,\mathbb{F}) \otimes_{\mathbb{F}} \mathrm{Map}(G,\mathbb{F}) \arrow[dl, "\cong"] \\
& \mathrm{Map}(G \times G, \mathbb{F}) &
\end{tikzcd}$$

$$\varphi_1 \otimes \varphi_2 \mapsto ((g_1, g_2) \mapsto \varphi_1(g_1)\varphi_2(g_2))$$
$$\varepsilon \colon \mathrm{Map}(G, \mathbb{F}) \to \mathbb{F}, \quad \varphi \mapsto \varphi(g_0).$$

There is also a coinverse map

$$S \colon \mathrm{Map}(G, \mathbb{F}) \to \mathrm{Map}(G, \mathbb{F}), \quad \varphi \mapsto (g \mapsto \varphi(g^{-1}))$$

Fact 2.12. *The comultiplication, counit, and coinverse are homomorphisms of (commutative)* $\mathbb{F}$*-algebras.*

This fact motivates the following definition:

Definition 2.13. *An* $\mathbb{F}$*-algebra* H *is called* Hopf algebra *if it has a coassociative counital comultiplication* Δ*, and the coinverse* S *such that the following diagram is commutative*

$$\begin{tikzcd}
& \mathcal{H} \arrow[dl, "\Delta"'] \arrow[d] \arrow[dr, "\Delta"] & \\
\mathcal{H} \otimes \mathcal{H} \arrow[dd, "S \otimes \mathrm{id}"'] & \mathbb{F} \arrow[ddd] & \mathcal{H} \otimes \mathcal{H} \arrow[dd, "\mathrm{id} \otimes S"] \\
& & \\
\mathcal{H} \otimes \mathcal{H} \arrow[dr, "m"'] & & \mathcal{H} \otimes \mathcal{H} \arrow[dl, "m"] \\
& \mathcal{H} &
\end{tikzcd}$$

The action of G in $\mathbb{E}$ defines the coaction of $\mathrm{Map}(G,\mathbb{F})$ on $\mathbb{E}$, i.e.

$$\begin{aligned} \mathbb{E} &\mapsto \mathbb{E} \otimes_{\mathbb{F}} \mathrm{Map}(G,\mathbb{F}) = \mathrm{Map}(G,\mathbb{E}) \\ e &\mapsto (g \mapsto g(e)) \end{aligned}$$

compatible as follows with the comultiplication Δ

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] & & \mathbb{E} \otimes_{\mathbb{F}} \mathrm{Map}(G,\mathbb{F}) \arrow[d, "\mathrm{id}\otimes\Delta"] \\ \mathbb{E} \otimes_{\mathbb{F}} \mathrm{Map}(G,\mathbb{F}) \arrow[rr] & & \mathbb{E} \otimes_{\mathbb{F}} \mathrm{Map}(G,\mathbb{F}) \otimes_{\mathbb{F}} \mathrm{Map}(G,\mathbb{F}) \end{tikzcd}$$

Remark 2.14. For any $\mathbb{K} \subset \mathbb{F} \subset \mathbb{E}$, $\mathbb{F}/\mathbb{K}$ finite, one can take another Hopf algebra $\mathrm{Map}(G,\mathbb{K})$ and obtain

$$\mathrm{can}\colon \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \to \mathbb{E} \otimes_{\mathbb{K}} \mathrm{Map}(G,\mathbb{K}).$$

In the coring approach there is one canonical coring $\mathrm{Map}(G,\mathbb{E})$ related to the action of G on $\mathbb{E}$, which in the Hopf approach can be realized by many Hopf algebras $\mathrm{Map}(G,\mathbb{K})$ defined over subfields $\mathbb{F} \subset \mathbb{F}$. Even when we fix $\mathbb{K} = \mathbb{F}$ after replacing $\mathrm{Map}(G,\mathbb{F})$ by an arbitrary abstract Hopf algebra over $\mathbb{F}$ theory is not as complete as in the group case.

For every group G

$$\mathrm{Map}(G,\mathbb{F}) = \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}G,\mathbb{F}),$$

where $\mathbb{F}G$ is the group algebra of G. $\mathbb{F}G$ is also a Hopf algebra with comultiplication obtained from the diagonal map $G \mapsto G \times G$, $g \mapsto (g,g)$,

$$\Delta\colon \mathbb{F}G \to \mathbb{F}G \otimes \mathbb{F}G, \quad g \mapsto g \otimes g,$$

counit $\varepsilon\colon \mathbb{F}G \to \mathbb{F}$ sending all group elements to $1 \in \mathbb{F}$, and coinverse obtained from group inverse $g \mapsto g^{-1}$.

Dualization $\mathrm{Hom}(-,\mathbb{F})$ transforms the coalgebra structure of $\mathbb{F}G$ into the algebra structure of $\mathrm{Map}(G,\mathbb{F})$. If $|G| < \infty$ then

$$\mathbb{F}G \cong \mathrm{Hom}_{\mathbb{F}}(\mathrm{Map}(G,\mathbb{F}),\mathbb{F})$$

transforms the coalgebra structure of $\mathrm{Map}(G,\mathbb{E})$ into the algebra structure of $\mathbb{F}G$. From the point of view of $\mathbb{F}G$ the canonical map looks like

$$\begin{aligned} \mathrm{can}\colon \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} &\to \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}G,\mathbb{E}) \\ e_1 \otimes e_2 &\mapsto (h \mapsto e_1 h e_2), \end{aligned}$$

where $\mathbb{F}G$ acts on $\mathbb{E}$ in the following way

$$h(e_1 e_2) := h_{(1)}(e_1) h_{(2)}(e_2).$$

Fixed subfield can also be defined in terms of this action

$$\mathbb{E}^{\mathbb{F}G} := \{e \in \mathbb{E} \mid \forall\, h \in \mathbb{F}G he = \varepsilon(h)e\} = \mathbb{E}^G$$

Replacing $\mathbb{F}G$ by an arbitrary Hopf algebra H we obtain

$$\begin{aligned}\mathrm{can}\colon \mathbb{E}\otimes_{\mathbb{F}}\mathbb{E} &\to \mathrm{Hom}_{\mathbb{F}}(\mathcal{H},\mathbb{E})\\ e_1\otimes e_2 &\mapsto (h\mapsto e_1he_2).\end{aligned}$$

where H acts on $\mathbb{E}$ in the same manner

$$h(e_1e_2) := h_{(1)}(e_1)h_{(2)}(e_2).$$

To extend the Galois theory to this case we need a notion of a Hopf subalgebra of H.

Definition 2.15. *$\mathcal{H}'\subset\mathcal{H}$ is a Hopf subalgebra of* H *if the inclusion is a homomorphism of Hopf algebras.*

Theorem 2.16 (Chase-Sweedler)**.** *Let $\mathbb{E}/\mathbb{F}$ be Hopf–Galois with respect to the action of a cocommutative Hopf algebra* H*. Then*

$$\phi\colon \{\mathcal{H}'\subset\mathcal{H}\mid \mathcal{H}' \text{ is Hopf subalgebra of } \mathcal{H}\}\to\{\mathbb{F}'\mid \mathbb{F}\subset\mathbb{F}'\subset\mathbb{E} \text{ subfield}\}$$

$$\mathcal{H}'\mapsto \mathbb{E}^{\mathcal{H}'}$$

is injective and inclusion reversing.

Note that the claim is about injectivity only. Another distinction comparing with classical Galois theory is that the Hopf algebra making a given extension Hopf–Galois is not unique.

Example 2.17. [gp87]Let $\mathbb{F}=\mathbb{Q}$, $\mathbb{E}=\mathbb{Q}(\sqrt[4]{2})$, $\omega :=\sqrt[4]{2}$

$$\mathcal{H} := \mathbb{Q}[c,s]/(c^2+s^2-1,\ cs)$$

with the comultiplication

$$\begin{aligned}\Delta\colon \mathcal{H} &\to \mathcal{H}\otimes_{\mathbb{F}}\mathcal{H}\\ c &\mapsto c\otimes c - s\otimes s\\ s &\mapsto c\otimes s + s\otimes c,\end{aligned}$$

counit

$$\begin{aligned}\varepsilon\colon \mathcal{H} &\to \mathbb{F}\\ c &\mapsto 1\\ s &\mapsto 0,\end{aligned}$$

and coinverse

$$\begin{aligned}S\colon \mathcal{H} &\to \mathcal{H}\\ c &\mapsto c\\ s &\mapsto -s.\end{aligned}$$

The action $\mathcal{H}\otimes_{\mathbb{F}}\mathbb{E}\to\mathbb{E}$ is given in a table

	1	ω	ω^2	ω^4
c	1	0	$-\omega^2$	0
s	0	$-\omega$	0	ω^3

Then $\mathbb{E}/\mathbb{F}$ is H-Galois.

Example 2.18. Let $\mathbb{F} = \mathbb{Q}$, $\mathbb{E} = \mathbb{Q}(\sqrt[4]{2})$, $\omega := \sqrt[4]{2}$

$$\tilde{\mathcal{H}} := \mathbb{Q}[\tilde{c}, \tilde{s}]/(\tilde{c}^2 + \tilde{s}^2 - 1,\ \tilde{c}\tilde{s})$$

with the comultiplication

$$\begin{aligned} \Delta\colon \tilde{\mathcal{H}} &\to \tilde{\mathcal{H}} \otimes_{\mathbb{F}} \tilde{\mathcal{H}} \\ \tilde{c} &\mapsto \tilde{c} \otimes \tilde{c} - \tfrac{1}{2}\tilde{s} \otimes \tilde{s} \\ \tilde{s} &\mapsto \tilde{c} \otimes \tilde{s} + \tilde{s} \otimes \tilde{c}, \end{aligned}$$

counit

$$\begin{aligned} \varepsilon\colon \mathcal{H} &\to \mathbb{F} \\ \tilde{c} &\mapsto 1 \\ \tilde{s} &\mapsto 0, \end{aligned}$$

and coinverse

$$\begin{aligned} S\colon \mathcal{H} &\to \mathcal{H} \\ \tilde{c} &\mapsto \tilde{c} \\ \tilde{s} &\mapsto -\tilde{s}. \end{aligned}$$

The action $\tilde{\mathcal{H}} \otimes_{\mathbb{F}} \mathbb{E} \to \mathbb{E}$ is given in a table

	1	ω	ω^2	ω^4
$\tilde{c}$	1	0	$-\omega^2$	0
$\tilde{s}$	0	ω^3	0	-2ω

Then $\mathbb{E}/\mathbb{F}$ is $\tilde{\mathcal{H}}$-Galois.

Example 2.19. Note that $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not normal, because the minimal polynomial of $\sqrt[4]{2}$ is $X^4 - 2$, and it has imaginary roots $\pm i\sqrt[4]{2} \notin \mathbb{Q}(\sqrt[4]{2}) \subset \mathbb{R}$. Hence it is not Galois in a classical sense. However it is Hopf–Galois.

Example 2.20. There are separable field extensions which are not Hopf–Galois at all. For example no field extension $\mathbb{E}/\mathbb{F}$, $[\mathbb{E}/\mathbb{F}] = 5$ with $\operatorname{Gal}(\widetilde{\mathbb{E}}/\mathbb{F}) = 5$ (where $\widetilde{\mathbb{E}}$ denotes the normal closure of $\mathbb{F} \subset \mathbb{E} \subset \overline{\mathbb{F}}$) can be Hopf–Galois.

What can be said about separable Hopf–Galois extensions?

Definition 2.21. *If S is a set, then a subgroup of* $\operatorname{Perm}(S)$ *is called* regular *if it is transitive with trivial stabilizers.*

Let $\widetilde{\mathbb{E}}$ be a normal closure of $\mathbb{E}$ in $\overline{\mathbb{F}}$, so $\operatorname{Gal}(\widetilde{\mathbb{E}}/\mathbb{E}) \subset \operatorname{Gal}(\widetilde{\mathbb{E}}/\mathbb{F})$. Denote

$$S := \operatorname{Gal}(\widetilde{\mathbb{E}}/\mathbb{E}) / \operatorname{Gal}(\widetilde{\mathbb{E}}/\mathbb{F}) \quad \text{(left cosets)}$$

Theorem 2.22. *The following conditions are equivalent:*

1. *There is a Hopf $\mathbb{F}$-algebra* H *such that $\mathbb{E}/\mathbb{F}$ is* H-*Hopf–Galois.*
2. *There is a regular subgroup $N \subset \operatorname{Perm}(S)$ such that* $\operatorname{Gal}(\widetilde{\mathbb{E}}/\mathbb{F}) = \operatorname{Perm}(S)$ *normalizes N.*

Proposition 2.23. *The following conditions are equivalent:*

1. *There exists a Galois extension $\mathbb{F}'/\mathbb{F}$ such that $\mathbb{F}' \otimes_{\mathbb{F}} \mathbb{E}$ is a field containing $\widetilde{\mathbb{E}}$.*
2. *There exists a Galois extension $\mathbb{F}'/\mathbb{F}$ such thae $\mathbb{F}' \otimes_{\mathbb{F}} \mathbb{E} = \widetilde{\mathbb{E}}$.*
3. $\mathrm{Gal}(\widetilde{\mathbb{E}}/\mathbb{E})$ *has a normal complement $N \subset \mathrm{Gal}(\widetilde{\mathbb{E}}/\mathbb{F})$.*
4. *There exists a normal subgroup $N \subset \mathrm{Gal}(\widetilde{\mathbb{E}}/\mathbb{F})$ which is regular in* $\mathrm{Perm}(S)$.

Definition 2.24. *If $\mathbb{E}/\mathbb{F}$ is finite and one of the conditions (1)-(4) is fulfilled then this extension is called* almost classical.

Theorem 2.25 ([gp87])**.** *If $\mathbb{E}/\mathbb{F}$ is almost classicaly Galois, then there is a Hopf algebra* H *such that $\mathbb{E}/\mathbb{F}$ is* H*-Hopf–Galois and the map*

$$\phi\colon \{\mathcal{H}' \subset \mathcal{H} \mid \mathcal{H}' \textit{ is Hopf subalgebra of } \mathcal{H}\} \to \{\mathbb{F}' \mid \mathbb{F} \subset \mathbb{F}' \subset \mathbb{E} \textit{ subfield}\},$$
$$\mathcal{H}' \mapsto \mathbb{E}^{\mathcal{H}'}$$

is bijective.

However, even for classical Galois extensions one cannot expect that for such H, making this extension Hopf–Galois, the image of ϕ contains all intermediate subfield.

Theorem 2.26 ([gp87])**.** *Any classical Galois extension $\mathbb{E}/\mathbb{F}$ can be endowed with an* H*- Galois structure such that the image of ϕ consists of normal intermediate extensions $\mathbb{F} \subset \mathbb{F}' \subset \mathbb{E}$.*

Example 2.27. Let $\mathbb{F} = \mathbb{Q}$, $\mathbb{E} = \mathbb{Q}(\omega, \xi)$, where $\omega = \sqrt[3]{2}$ and $\xi = \frac{\sqrt{3}+i}{2}$. It is known that the extension $\mathbb{E}/\mathbb{F}$ is Galois with $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) = S_3$. But there exists a Hopf algebra

$$\mathcal{H} := \underbrace{\mathbb{Q}\langle c, s, t\rangle}_{\text{noncommutative variables}} /(c(c-1)(c+1),\ 2c^2+st+ts-2,\ cs,\ sc,\ ct,\ tc,\ s^2,\ t^2)$$

The comultiplication is given by

$$\begin{aligned}\Delta\colon \mathcal{H} &\to \mathcal{H} \otimes_{\mathbb{F}} \mathcal{H}\\ c &\mapsto c \otimes c + \tfrac{1}{2}(s \otimes t + t \otimes s)\\ s &\mapsto c \otimes s + s \otimes c + \tfrac{1}{2} t \otimes t\\ t &\mapsto c \otimes t + t \otimes c + s \otimes s\end{aligned}$$

H is a Hopf algebra making $\mathbb{E}/\mathbb{F}$ Hopf–Galois, where action $\mathcal{H} \otimes_{\mathbb{F}} \mathbb{E} \to \mathbb{E}$ is given in the table

	1	ω	ξ
c	1	0	ξ^2
s	0	ω^2	0
t	0	0	0

In the image of ϕ one obtains only normal intermediate extensions.

2.4 Torsors

Let G be a group, X a set, $X \times G \to X$ right action $(x, g) \mapsto xg$. We assume that neutral element acts trivially $xg_0 = x$, and that $x(g_1g_2) = (xg_1)g_2$.

Example 2.28.

1. $X = \emptyset$ or $X = *$, a one element set.

2. $X = G$, $G \times G \to G$ group composition.

3. $X = \{1, 2, \dots, n\}$, $G = S_n$ acting by permutations.

Definition 2.29. *A G-torsor is a G-set which is isomorphic to G in the category of G-sets.*

Theorem 2.30. *The following conditions are equivalent*

1. *X is a G-torsor.*

2. *For all $x, y \in X$ there is a unique $g \in G$ such that $xg = y$.*

3. *For all $x \in X$ the map $g \mapsto xg$ gives an isomorphism $G \cong X$ of G-sets.*

4. *The map $X \times G \to X \times X$, $(x, g) \mapsto (x, xg)$ is bijective.*

We are mainly interested in algebraic sets.

Definition 2.31. *If $I = \sqrt{I} \lhd \mathbb{F}[X_1, \dots, X_n]$ is a radical ideal of an algebraic set $X \subset \overline{\mathbb{F}}^n$, then we form a* coordinate ring *of X*

$$\mathcal{O}(X) = \mathbb{F}[X_1, \dots, X_n]/I$$

Definition 2.32. *If $\mathbb{F} \subset \mathbb{E}$ is an algebraic field extension then $X(\mathbb{E})$ is the set of* $\mathbb{E}$-points *of X.*

Fact 2.33. *If X, Y are algebraic sets corresponding to $\mathbb{F}$-algebras $\mathcal{O}(X)$, $\mathcal{O}(Y)$ respectively then*

$$\mathcal{O}(X \times_{\mathbb{F}} Y) = \mathcal{O}(X) \otimes_{\mathbb{F}} \mathcal{O}(Y)$$

$$(X \times_{\mathbb{F}} Y)(\mathbb{E}) = X(\mathbb{E}) \times Y(\mathbb{E})$$

Example 2.34. Let $\mathcal{O}(X) := \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$. The real points $X(\mathbb{R})$ form a circle in $\mathbb{R}^2$. But an algebraic set can have complex points, which are the complex solutions of $X^2 + Y^2 = 1$.

Definition 2.35. *A* morphism of algebraic sets $X \to Y$ *over $\mathbb{F}$ is a homomoprhism of $\mathbb{F}$-algebras $\mathcal{O}(Y) \to \mathcal{O}(X)$.*

This gives a map $X(\mathbb{E}) \to Y(\mathbb{E})$ for every algebraic extension $\mathbb{E}/\mathbb{F}$.

Example 2.36. $\mathrm{GL}_n(\mathbb{E})$ - set of $\mathbb{E}$-points of general linear group over $\mathbb{F}$.

$$\mathcal{O}(\mathrm{GL}_n) = \mathbb{F}[X_{11}, \dots, X_{ij}, \dots, X_{nn}; \det([X_{ij}]_{i,j=1}^n)^{-1}]$$

A linear algebraic group $G \subset \mathrm{GL}_n$ is defined by polynomial relations $f_1(X), \dots, f_r(X)$, $X = [X_{ij}]_{i,j=1}^n$. A matrix $A \in G(\mathbb{E})$ if and only if $f_1(A) = 0, \dots, f_r(A) = 0$. Here are the examples of linear algebraic groups:

1. GL_n, $f_1(X) = 0$.
2. SL_n, $f_1(X) = \det(X) - 1$.
3. O_n, $\{A^T A = I\}$.
4. UT_n, $f_{ij}(X) = X_{ij}$ for $i > j$.

If G is an algebraic group, then $\mathcal{H} = \mathcal{O}(G)$ is a Hopf algebra with the pointwise multiplication and comultiplication induced by the composition in G,

$$\Delta\colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes_{\mathbb{F}} \mathcal{O}(G) \cong \mathcal{O}(G \times_{\mathbb{F}} G).$$

If $\mathbb{E}/\mathbb{F}$ is any field extension then an $\mathbb{F}$-homomorphism $\mathcal{O}(G) \to \mathbb{E}$ is determined by a subgroup $G(\mathbb{E}) \subset \mathrm{GL}_n(\mathbb{E})$.

Consider a group action of G on X. The compatibility conditions can be shown using diagrams

$$\begin{tikzcd} X \times_{\mathbb{F}} G \times_{\mathbb{F}} G \arrow[r] \arrow[d] & X \times_{\mathbb{F}} G \arrow[d] \\ X \times_{\mathbb{F}} G \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \times * \arrow[d] \arrow[dr] & \\ X \times_{\mathbb{F}} G \arrow[r] & X \end{tikzcd}$$

These diagrams can be dualized

$$\begin{tikzcd} \mathcal{O}(X) \otimes_{\mathbb{F}} \mathcal{O}(G) \otimes_{\mathbb{F}} \mathcal{O}(G) & \mathcal{O}(X) \otimes_{\mathbb{F}} \mathcal{O}(G) \arrow[l] \\ \mathcal{O}(X) \otimes_{\mathbb{F}} \mathcal{O}(G) \arrow[u] & \mathcal{O}(X) \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}(X) \otimes * \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \mathcal{O}(X) \otimes_{\mathbb{F}} \mathcal{O}(G) & \mathcal{O}(X) \arrow[l] \end{tikzcd}$$

Algebraic G-action on an algebraic set induces a coaction of the Hopf algebra $\mathcal{O}(G)$ on an algebra $\mathcal{O}(X)$. Then X is a G-torsor if the map

$$\begin{tikzcd} \mathcal{O}(X) \otimes_{\mathbb{F}} \mathcal{O}(X) \arrow[r] \arrow[d, equal] & \mathcal{O}(X) \otimes_{\mathbb{F}} \mathcal{O}(X) \arrow[d, equal] \\ \mathcal{O}(X \times_{\mathbb{F}} X) & \mathcal{O}(X \times_{\mathbb{F}} G) \end{tikzcd}$$

is induced by the canonical map

$$X \times_{\mathbb{F}} X \leftarrow X \times_{\mathbb{F}} G$$

On $\mathbb{E}$-points it is given by

$$\begin{tikzcd} (X \times_{\mathbb{F}} X)(\mathbb{E}) \arrow[d, equal] & (X \times_{\mathbb{F}} G)(\mathbb{E}) \arrow[l] \arrow[d, equal] \\ X(\mathbb{E}) \times X(\mathbb{E}) & X(\mathbb{E}) \times G(\mathbb{E}) \end{tikzcd}$$

$$(x, xg) \longleftarrow\!\mapsfrom (x, g)$$

Example 2.37. If $\mathbb{E}/\mathbb{F}$ is a finite Galois extension then the algebraic set X over $\mathbb{F}$ corresponding to an $\mathbb{F}$-algebra $\mathbb{E} = \mathcal{O}(X)$ is a G-torsor where G is a linear algebraic group corresponding to an $\mathbb{F}$-algebra $\mathcal{O}(G) = \mathrm{Map}(\mathrm{Gal}(\mathbb{E}/\mathbb{F}), \mathbb{F})$. Note that $X(\mathbb{F}) = \emptyset$, $X(\mathbb{E})$ is a finite set of cardinality equal to the degree of the extension $[\mathbb{E} : \mathbb{F}]$. If $\mathbb{E}$ is a splitting field of $f(X) \in \mathbb{F}[X]$, then $X(\mathbb{E})$ is the set of roots of f.

2.5 Crossed homomorphisms and G-torsors

Let $\mathbb{E}/\mathbb{F}$ be a finite Galoios extension, and G linear algebraic group over $\mathbb{F}$.

Definition 2.38. *A crossed homomorphism is a map*

$$\varphi\colon \mathrm{Gal}(\mathbb{E}/\mathbb{F}) \to G(\mathbb{E})$$

satisfying $\varphi(g_1g_2) = \varphi(g_1)g_1(\varphi(g_2))$. Two crossed morphisms φ, φ' are said to be equivalent *if $\varphi'(g) = \psi\varphi(g)\psi^{-1}$ for some $\psi \in G(\mathbb{E})$.*

A crossed homomorphism φ gives rise to a torsor as follows. On $\mathbb{E} \otimes_{\mathbb{F}} \mathcal{O}(G)$ we have an obvious $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$-action, and we define a φ-twisted action by

$$g \cdot (e \otimes h) := g(e) \otimes \varphi(g^{-1})^* h \in \mathbb{E} \otimes_{\mathbb{F}} \mathcal{O}(G)$$

Then the fixed $\mathbb{F}$-subalgebra $(\mathbb{E} \otimes_{\mathbb{F}} \mathcal{O}(G))^{\mathrm{Gal}(\mathbb{E}/\mathbb{F})}$ is a coordinate ring $\mathcal{O}(X)$ for a G-torsor X with the G-action induced by the restriction $\mathcal{O}(X) \to \mathcal{O}(X) \otimes_{\mathbb{F}} \mathcal{O}(G)$ of the comultiplication

$$\mathbb{E} \otimes_{\mathbb{F}} \mathcal{O}(G) \to (\mathbb{E} \otimes_{\mathbb{F}} \mathcal{O}(G)) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathcal{O}(G)).$$

Definition 2.39. *W say that the extension $\mathbb{E}/\mathbb{F}$ trivializes a G-torsor X if after the base extension $\mathbb{E}/\mathbb{F}$ we have an isomorphism.*

$$\mathbb{E} \otimes_{\mathbb{F}} \mathcal{O}(X) \cong \mathbb{E} \otimes_{\mathbb{F}} \mathcal{O}(G).$$

Theorem 2.40. *The isomorphism classes of G-torsors over $\mathbb{F}$ trivializable by the extension $\mathbb{E}/\mathbb{F}$ correspond bijectively to the equivalence classes of crossed homomorphisms* $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) \to G(\mathbb{E})$.

Example 2.41. Let $G = \mathrm{GL}_1 = \overline{\mathbb{F}}^*$ (invertible elements). Then the set of nonzero vectors in any one dimensional vector space over $\mathbb{F}$ is a G-torsor X over $\mathbb{F}$. Then the set of isomorphism classes of such torsors correspond bijectively to the set of isomorphism classes of one dimensional vector spaces over $\mathbb{F}$.

The set of equivalence classes of crossed homomorphisms $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) \to G(\mathbb{E})$ is $\mathrm{H}^1(\mathrm{Gal}(\mathbb{E}/\mathbb{F}); \mathbb{E}^*)$, which is 0 by the Hilbert's 90'th theorem (1.47).

Example 2.42. Let $\mathbb{F} = \mathbb{C}$, $\mathcal{O}(G) = \mathbb{C}[g, h, g^{-1}, h^{-1}]$,

$$\mathcal{O}(X) = \mathbb{C}\langle x, y, x^{-1}, y^{-1}\rangle/(xy = qyx), \quad q \in \mathbb{C}^*.$$

2.6 Descent theory

Let $\mathbb{E}/\mathbb{F}$ be a field extension. Given an algebraic object A defined over $\mathbb{F}$ (vector space, quadratic space, algebra, coalgebra, Hopf algebra etc.) one can construct an algebraic object $\mathbb{E} \otimes_{\mathbb{F}} A$ defined over $\mathbb{E}$. The aim of descent theory is to say something about what happens when we go in the opposite direction. For example given $a_{\mathbb{E}} \in A_{\mathbb{E}} := \mathbb{E} \otimes_{\mathbb{F}} A$ we can ask what conditions guarantee that $a_{\mathbb{E}} = 1 \otimes a_{\mathbb{F}}$.

Example 2.43. If $\mathbb{E}/\mathbb{F}$ is finite Galois extension then taking $A_{\mathbb{F}} = \mathbb{F}$ we obtain $A_{\mathbb{E}} = \mathbb{E} \otimes_{\mathbb{F}} \mathbb{F} = \mathbb{E}$. The answer in this case is: this happens if and only if $g(a) = a$ for all $g \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$.

Another problem consists in the question when a given $A_{\mathbb{E}}$ defined over $\mathbb{E}$ is of the form $A_{\mathbb{E}} = \mathbb{E} \otimes_{\mathbb{F}} A_{\mathbb{F}}$. This is called a problem of forms of algebraic structures.

Definition 2.44. *$A'_{\mathbb{F}}$ is called $\mathbb{E}$-form of $A_{\mathbb{F}}$ if $\mathbb{E} \otimes_{\mathbb{F}} A'_{\mathbb{F}} \cong \mathbb{E} \otimes_{\mathbb{F}} A_{\mathbb{F}}$.*

Example 2.45. Let $\mathbb{F} = \mathbb{R}$, $\mathbb{E} = \mathbb{C}$, $\mathcal{H} = \mathbb{R}\mathbb{Z}$ (group algebra). Define

$$\begin{aligned} \mathcal{H}' &:= \mathbb{R}[c, s]/(c^2 + s^2 - 1) \\ \Delta(c) &= c \otimes c - s \otimes s \\ \Delta(s) &= c \otimes s + s \otimes c \end{aligned}$$

Then

$$a := 1 \otimes c + i \otimes s = c + is \in \mathbb{E} \otimes_{\mathbb{F}} \mathcal{H}'$$

is invertible with inverse $a^{-1} = c - is \in \mathbb{E} \otimes_{\mathbb{F}} \mathcal{H}'$. Hence $c, s \in \mathbb{E} \otimes_{\mathbb{F}} \mathcal{H}'$, and

$$\mathbb{E} \otimes_{\mathbb{F}} \mathcal{H}' = \mathbb{C}[a, a^{-1}] \cong \mathbb{C}\mathbb{Z} \cong \mathbb{E} \otimes_{\mathbb{F}} \mathcal{H}$$

Note that H and $\mathcal{H}'$ are not isomorphic over $\mathbb{R}$, because their groups of real points are different:

$$\mathrm{Hom}(\mathcal{H}, \mathbb{R}) \cong \mathbb{R}^*$$

with only two elements of finite order $\{1, -1\}$, and

$$\mathrm{Hom}(\mathcal{H}', \mathbb{R}) \cong \mathrm{U}(1)$$

with infinitely many elements of finite order.

Theorem 2.46 ([hp85])**.** *Let Γ be a finitely generated group with finite isomorphism group G. Then there is a bijection between the set of isomorphism classes of G-Galois extensions of $\mathbb{F}$ and the set of Hopf algebra forms of $\mathcal{H} = \mathbb{F}\Gamma$. This associates with each G-Galois extension $\mathbb{E}$ of $\mathbb{F}$ the Hopf algebra*

$$\mathcal{H}' := \left\{ \sum_{\gamma \in \Gamma} c_\gamma \gamma \in \mathbb{E}\Gamma \mid \forall\, g \in G \ \sum_{\gamma \in \Gamma} g(c_\gamma) g(\gamma) = \sum_{\gamma \in \Gamma} c_\gamma \gamma \right\}$$

which is an $\mathbb{E}$-form of $\mathbb{E}\Gamma$ by the isomorphism

$$\begin{aligned} \mathbb{E} \otimes_{\mathbb{F}} \mathcal{H}' &\to \mathbb{E} \otimes_{\mathbb{F}} \mathcal{H} = \mathbb{E}\Gamma \\ e \otimes \sum_{\gamma \in \Gamma} c_\gamma \gamma &\mapsto \sum_{\gamma \in \Gamma} e c_\gamma \otimes \gamma \end{aligned}$$

Example 2.47. Let $G = \mathbb{Z}/2$, g-generator, $\mathbb{C} \hookrightarrow \mathbb{H} = \{z_0 + z_1 j \mid jz = \bar{z}j,\ j^2 = -1\}$.

$$g(z_0 + z_1 j) := z_0 - z_1 j = i(z_0 + z_1 j)i^{-1}$$

Then $\mathbb{H}^G = \mathbb{C} \subset \mathbb{H}$, and

$$\mathrm{can}\colon \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H} \cong \mathrm{Map}(G, \mathcal{H})$$

$$q_1 \otimes q_2 \mapsto (g \mapsto q_1 g(q_2))$$

2.7 Splitting of polynomials with roots in noncommutative algebras

It is interesting to see a cyclic property of splittings of a polynomial with noncommutative coefficients into linear factors $(X - a_k)$ with a_k's commuting with coefficients. We will show that any cyclic permutation of linear factors gives the same result and all a_k are roots of that polynomial. It implies that although the set of a_k's appearing in a splitting of a polynomial with commutative coefficients in some noncommutative extension does not determine the splitting (in general), the cyclic order consisting of roots appearing in the splitting does. It is an interesting example of cyclic symmetry "in nature". From the point of view of algebraic geometry this can be understood as the principle saying that a finite geometric cycle defined over a noncommutative algebra should be regarded not as a set (with multiplicities) but rather as a cyclic order (with multiplicities).

Let $f(X) = f_n X^n + f_{n-1} X^{n-1} + \cdots + f_0 \in A[X]$ be a polynomial with coefficients in a commutative unital ring A. Suppose there is given a splitting of $f(X)$ in $A[X]$

$$f(X) = f_n (X - a_1) \cdots (X - a_n). \tag{2.1}$$

Then by the substitution homomorphism argument one sees that all a_k's are roots of $f(X)$ and by commutativity of $A[X]$ any permutation of them defines the same splitting. Therefore the problem of splitting of a given polynomial reduces to the problem of finding the set of its roots. This fact is fundamental for Galois theory and algebraic geometry.

In the case of noncommutative coefficients of a given polynomial the situation is much worse. First of all, a given splitting does not reduces to the set of elements a_k, since we cannot permute linear factors because of noncommutativity of $A[X]$. Moreover, if $a \in A$ is not central in A then the substitution homomorphism of rings

$$\mathbb{Z}[X] \to A, \quad X \mapsto a \tag{2.2}$$

does not extend to a homomorphism of A-algebras

$$A[X] \to A, \quad X \mapsto a, \tag{2.3}$$

because X is central in $A[X]$. This means that one can not use the substitution A-algebra homomorphism argument to prove that elements $a_1, \ldots, a_n$ appearing in the decomposition

$$f(X) = f_n (X - a_1) \cdots (X - a_n) \tag{2.4}$$

are roots of $f(X)$. The problem of such splittings in terms of relationships between coefficients of a given polynomial with a generic set of its (left or right) roots and elements a_k (so called pseudoroots) was related to quadratic algebras with structure encoded by graphs in [gr96][grw01][ggrw05][rswxx]. However, these relationships are much more complicated than in the commutative case and make sense only if some elements of the algebra are invertible.

The interest for splitting polynomials in noncommutative algebras started in 1921 when Wedderburn proved [w-jhm21] that any minimal polynomial $f(X) \in K[X]$ of an element of a central division algebra A algebraic over the center K of A splits in $A[X]$ into linear factors which can be permuted cyclically and every pseudoroot appearing in this splitting is a root of $f(X)$. This fact was very helpful in determining the structure of division algebras of small order [w-s02] and found many other applications (see e.g. [ll04] for references).

However, under the assumption that coefficients (which do not have to commute one with each other) commute with pseudoroots (which do not have to commute one with each other), the situation is much closer to the commutative case. We show that then pseudoroots are roots and any cyclic permutation of them gives the same splitting. This means that instead of finite sets of commutative roots (ordered n-tuples up to all permutations) we obtain finite cyclically ordered sets (ordered n-tuples up to all cyclic permutations) of noncommutative roots.

Theorem 2.48 ([m-txx]). *Let A be a unital ring and $A^{a_1,\dots,a_n}$ be its subring of elements commuting with $a_1, \dots, a_n \in A$. If $f(X) \in A^{a_1,\dots,a_n}[X]$ splits in $A[X]$ as follows*

$$f(X) = f_n(X - a_1)(X - a_2)\cdots(X - a_n) \tag{2.5}$$

then

$$f(X) = f_n(X - a_n)(X - a_1)\cdots(X - a_{n-1}) \tag{2.6}$$

and

$$f(a_1) = \cdots = f(a_n) = 0. \tag{2.7}$$

Corollary 2.49 ([m-txx]). *For any splitting as in Theorem 1 substitution homomorphisms*

$$\begin{aligned} A^{a_1,\dots,a_n}[X] &\to A^{a_1,\dots,a_n}[a_k] \subset A^{a_k} \subset A \\ X &\mapsto a_k \end{aligned} \tag{2.8}$$

define a ring homomorphism

$$A^{a_1,\dots,a_n}[X]/(f(X)) \to A^{a_1,\dots,a_n}[a_1] \times \cdots \times A^{a_1,\dots,a_n}[a_n] \tag{2.9}$$

where cyclic permutations of roots in the splitting correspond to cyclic permutations of factors in the cartesian product on the right hand side.

Example 2.50 ([m-txx]). Let A be the ring of upper triangular 2×2 matrices over a nonzero commutative ring K and take $f(X) = X^2(X-1) \in K[X] \subset A[X]$. Although it has a double root in K it can be split in A as follows

$$f(X) = (X - a_1)(X - a_2)(X - a_3), \tag{2.10}$$

where

$$a_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.11}$$

are pairwise distinct roots in A. This can be viewed as a kind of resolution of singularity by passing to a noncommutative extension.

The linear factors can be cyclically permuted, but none two of them can be transposed, because

$$[a_1, a_2] = [a_2, a_3] = [a_3, a_1] = -a_2 \neq 0. \tag{2.12}$$

In spite of the fact that A is not a field and $f(X)$ is not separable this example shares many properties with splitting of a separable polynomial in its Galois extension, if the Galois group is replaced by the endomorphism monoid. We will assume only that K is a domain. Although the set of roots and the set of cycles of roots appearing in possible splittings are both infinite it turns out that [m-txx]

1. The set of all roots is the union of supports of all cycles.
2. One can choose a finite number of roots whose translates cover the set of all roots.
3. One can choose a finite set of roots containing translates of all roots.
4. One can choose a finite number of cycles whose translates cover the set of all cycles.
5. One can choose a finite set of cycles containing translates of all cycles.

Points (2)-(5) replace the fact that for a separable polynomial there is only one cycle of roots contained in its splitting field and this cycle is an orbit of the Galois group. Instead of strict equivalence of roots induced by the transitive Galois action we have the action of endomorphisms preserving some partial order on roots [m-txx].

Chapter 3

Galois theory in general categories

3.1 Introduction

This chapter describes a purely-categorical approach to Galois theory whose first version was proposed in [j-g84] as a generalization of A. R. Magids Galois theory of commutative rings [m-ar74]. It is, however, important to note here that:

Magids approach is itself on the one hand a generalization of the commutative-ring reduction of A. Grothendieck's Galois–Poincaré theory [g-a71] and on the other hand a generalization of Galois theory of commutative rings due to S. U. Chase, D. K. Harrison, and A. Rosenberg [chr65]. A reasonable historical overview would also require at least mentioning [ag60], [ab59], [cs69], [j-gj66], [vz66], and [vz69].

The approach of [j-g84] was presented slightly differently in [j-g89-1] and then extended in [j-g90] and again in [j-g91-2]. Further developments in various directions include [bj97, bj99, bj04], [cj96, cj02, cjkp97, cjm96], [e-t07, egl08, g-m04, g-m07, gr07, gj03], [j-g89-2], [j-g91-1], [j-g92], and [j-g08, jk94, jk97, jk00-1, jk00-2, jmt98, jss93, js99, jt99-1, jt99-2]; some of them are briefly described in [bj01], [g-m04], and [j-g04] (see also references there).

Apart from the commutative-ring-theoretic motivation, categorical Galois theory has an important topos-theoretic motivation (based on the geometric/topological motivation), provided by [b-m80, b-m82, bd80], which itself generalizes A. Grothendieck's and C. Chevalley's approach. Giving details here would require mentioning many books and articles devoted to covering maps and the fundamental group. The same can be said about the algebraic-geometric side of the story involving étale coverings of schemes and the étale fundamental group.

There are many investigations of other kinds of abstract Galois theories, especially in topos theory, still to be compared with what we describe (see e.g. [jss93] and [js99], and what they say about A. Joyal's and M. Tierneys Galois theory [jt84] and about the Tannaka duality respectively). Some topos-theoretic comparison results are contained in [b-m04] and [bl03].

Section 3.2 can simply be omitted by those readers who have no doubts about the importance of category theory; however, it should be useful to others as it presents an important motivation well known but not mentioned in many textbooks. Sections 3.3–3.11 will tell almost nothing new to those readers who are familiar with the corresponding material from S. Mac Lanes book [m-s71]. Sections 3.12 and 3.13 also present material well known to category-theorists, even though it is not present in [m-s71]. Sections 3.14,3.15 and 3.20 describe the main notions and the main result (fundamental theorem) of categorical Galois theory respectively, while the intermediate Sections 3.16–3.19 describe the main examples. And Section 3.21 shows what does that fundamental theorem give in what should be considered as (the) classical cases. Sections 8.1–8.3 (Appendix) attempt to make this chapter self-contained.

3.2 How do categories appear in modern mathematics?

The question “How do categories appear in modern mathematics?” has many answers; this section is devoted to only one of them, far away from the original answer visible in the joint work of S. Eilenberg and S. Mac Lane, and our presentation is very brief of course.

First, thinking of mathematics as the study of abstract mathematical structures, such as groups, rings, topological spaces, etc., we ask: what is a mathematical structure in general? And, having Bourbaki structures in mind, we might answer:

- We begin with two finite collections of sets: constant sets $E_1, \ldots, E_m$ and variable sets $X_1, \ldots, X_n$.
- We build a scale, which is a sequence of sets obtained from the sets above by taking finite products and power sets, and by iterating these operations.
- A type is a uniformly defined subset $T(X_1, \ldots, X_n)$ of a set in such a scale, and a structure of that type on the sets $X_1, \ldots, X_n$ is an element s in $T(X_1, \ldots, X_n)$; one then also says that $(X_1, \ldots, X_n, s)$ is a structure of the type T. Making the term “uniformly” precise would be a long story, which we omit; let us only mention that considering various structures of a given type T, we will fix the sets $E_1, \ldots, E_m$, but not the sets $X_1, \ldots, X_n$ - which explains why we write $T(X_1, \ldots, X_n)$ and not $T(E_1, \ldots, E_m, X_1, \ldots, X_n)$.

For the readers not familiar with Bourbaki structures it might be helpful to consider the following simple examples, where, as for most basic mathematical structures, we have $m = 0$ and $n = 1$:

Example 3.1.

1. A topology on a set X is an element of the set

   $$\begin{aligned} T &= T(X) \\ &= \{\tau \in PP(X) \mid \tau \text{ is closed under arbitrary unions and finite intersections}\}, \end{aligned}$$

   where $P(X)$ denotes the power set of X;

2. a binary operation on a set X is an element of the set

$$T = T(X) = \{m \in P(X \times X \times X)) \mid m \text{ determines a map } X \times X \to X\}.$$

It turns out that every mathematical structure ever considered in mathematics can indeed be presented an $(X_1, \dots, X_n, s)$ above, and moreover, using the fact that arbitrary bijections $f_1 : X_1 X_1', \dots, f_n : X_n X_n'$ induce a bijection $T(f_1, \dots, f_n) : T(X_1, \dots, X_n))T(X_1', \dots, X_n')$, it is easy to define a general notion of an isomorphism for structures of the same type:

Definition 3.2. *Let $(X_1, \dots, X_n, s)$ and $(X_1', \dots, X_n', s')$ be mathematical structures of the same type T; an* isomorphism

$$(f_1, \dots, f_n)\colon (X_1, \dots, X_n, s) \to (X_1', \dots, X_n', s'),$$

is a family of bijections $f_1\colon X_1 \to X_1', \dots, f_n\colon X_n \to X_n'$ with $T(f_1, \dots, f_n)(s) = s'$.

However, we are not able to define structure preserving maps (=homomorphisms) in general. The best we can do, is:

Definition 3.3. *Let T be a type. For structures $(X_1, \dots, X_n, s)$ and $(X_1', \dots, X_n', s')$ of the same type T, a* map

$$(f_1, \dots, f_n)\colon (X_1, \dots, X_n, s)) \to (X_1', \dots, X_n', s'),$$

is a family of maps $f_1\colon X_1 \to X_1', \dots, f_n\colon X_n \to X_n'$. A class $\mathbf{M}$ of such maps is said to be a class of morphisms*, if it satisfies the following conditions:*

1. *If*
$$(f_1, \dots, f_n)\colon (X_1, \dots, X_n, s) \to (X_1', \dots, X_n', s')$$
and
$$(f_1', \dots, f_n')\colon (X_1', \dots, X_n', s') \to (X_1'', \dots, X_n'', s'')$$
are in $\mathbf{M}$, then so is
$$(f_1' f_1, \dots, f_n' f_n)\colon (X_1, \dots, X_n, s) \to (X_1'', \dots, X_n'', s'');$$

2. *the class of invertible morphisms in $\mathbf{M}$ coincides with the class of isomorphisms in the sense of Definition 3.2.*

Accordingly, our study of the structures of a given type T will depend on the chosen class $\mathbf{M}$ of morphisms - suggesting that it is a study of a new structure whose "elements" are structures of the type T and the elements of $\mathbf{M}$. And such a new structure is first of all a category of course, but is it merely a category? Would not replacing our T and $\mathbf{M}$ with an abstract category trivialize our study? In other words, is abstract category theory powerful enough to express deep properties of classical mathematical structures and simple enough to clarify those properties and to help proving them? Answering these questions seriously, and especially saying well-motivated "yes" to the last one, is not what we can do in a few page section of these notes. But the following definition, of one of the oldest categorical definitions, due to S. Mac Lane, should give some initial indication of the remarkable power of the categorical approach:

Definition 3.4. *The* product *of two objects A and B in a category $\mathbf{C}$ is an object $A\times B$ in $\mathbf{C}$ together with two morphisms $\pi_1 : A\times B \to A$ and $\pi_2 : A\times B \to B$, such that for every object C and morphisms $f\colon C \to A$ and $g\colon C \to B$, there exists a unique morphism $h\colon C \to A \times B$ making the diagram*

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd} \tag{3.1}$$

commute, i.e. satisfying $\pi_1 h = f$ and $\pi_2 h = g$.

This so simple definition is equivalent to the familiar ones in essentially all important categories of interest in algebra and geometry/topology, and the same is true for its dual, which is:

Definition 3.5. *The* coproduct *of two objects A and B in a category $\mathbf{C}$ is an object $A+B$ in $\mathbf{C}$ together with two morphisms $\iota_1\colon A \to A+B$ and $\iota_2 :\ B \to A+B$, such that for every object C and morphisms $f\colon A \to C$ and $g\colon B \to C$, there exists a unique morphism $h\colon A+B \to C$ making the diagram*

$$\begin{tikzcd} A \arrow[dr, "f"'] & A + B \arrow[l, "\iota_1"'] \arrow[r, "\iota_2"] \arrow[d, dashed, "h"] & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \tag{3.2}$$

commute, i.e. satisfying $h\iota_1 = f$ and $h\iota_2 = g$.

Furthermore, these categorical definitions give a new insight into our understanding of very first mathematical concepts, such as multiplication and addition of natural numbers, intersection, product, and union of sets, and conjunction and disjunction in mathematical logic. In particular they make addition dual to multiplication and make disjoint union more natural than the ordinary one. In simple words, everyone knows that, say,

$$a + b = b + a \text{ and } ab = ba \text{ (for natural } a \text{ and } b),$$

but only category theory tells us that these equalities are special cases of a single result!

3.3 Isomorphism and equivalence of categories

The purpose of this section is to list and prove basic properties of isomorphisms and equivalences of categories. We assume that the readers are familiar with:

- Isomorphisms in general categories: they compose, they have uniquely determined inverses that are isomorphisms themselves, and they determine the isomorphism relation d on the set of objects of the given category; and that relation is an equivalence relation.

- Isomorphisms of categories: the following condition on a functor $F\colon \mathbf{A} \to \mathbf{B}$ are equivalent:

(a) F is an isomorphism;

(b) F is bijective on objects and on morphisms;

(c) F is bijective on objects and fully faithful (recall that "fully faithful" means "bijective of Hom sets").

- Isomorphism of functors: a natural transformation $\tau\colon \mathbf{F} \to \mathbf{G}$ of functors $\mathbf{A} \to \mathbf{B}$ is an isomorphism if and only if the morphism $\tau_A\colon F(A) \to G(A)$ is an isomorphism for each object A in $\mathbf{A}$. The isomorphism relation is a congruence on the category of all categories, i.e. if (F, F') and (G, G') are composable pairs of functors, then $F \approx F'$ and $G \approx G'$ implies $FF' \approx GG'$.

Theorem 3.6. *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor, G_0 a map from the set $\mathbf{A}_0$ of objects in $\mathbf{A}$ to the set $\mathbf{B}_0$ of objects in $\mathbf{B}$, and , $\tau = (\tau_A\colon F(A) \to G_0(A))_{A\in\mathbf{A}_0}$ a family of isomorphisms. Then there exists a unique functor $G\colon \mathbf{A} \to \mathbf{B}$, for which G_0 is the object function and $\tau\colon \mathbf{F} \to \mathbf{G}$ is an (iso)morphism.*

Proof. On the one hand $\tau\colon \mathbf{F} \to \mathbf{G}$ is an isomorphism if and only if for each morphism $\alpha\colon A \to A'$ in $\mathbf{A}$, we have $G(\alpha) = \tau_{A'}F(\alpha)\tau_A^{-1}$, and on the other hand it is easy to check that sending $\alpha\colon A \to A'$ to $\tau_{A'}F(\alpha)\tau_A^{-1}$ determines a functor $\mathbf{A} \to \mathbf{B}$ whose object function is G_0. ☐

Remark 3.7. 1. Since G_0 above is completely determined by the family $\tau = (\tau_A)_{A\in\mathbf{A}_0}$, the assumptions of Theorem 3.6 should be understood as "given $F\colon \mathbf{A} \to \mathbf{B}$ and, for each object A in $\mathbf{A}$, an isomorphism τ_A from $F(A)$ to somewhere".

2. Theorem 3.6 has an interesting application: Starting from an arbitrary isomorphism $\theta\colon X \to Y$ in a category $\mathbf{A}$, we apply this theorem to $\mathbf{B} = \mathbf{A}$, $F = 1_{\mathbf{A}}$, and

$$\tau_A = \begin{cases} \theta\colon X \to Y, & \text{if } A = X; \\ \theta^{-1}\colon Y \to X, & \text{if } A = Y; \\ 1_A\colon A \to A, & \text{if } X \neq A \neq Y; \end{cases} \tag{3.3}$$

it is easy to see that the resulting functor $G\colon \mathbf{A} \to \mathbf{A}$ is an isomorphism (for, use Theorem 3.8(c) below, and the fact that a functor is an isomorphism if and only it is bijective on objects and fully faithful). This in fact explains how to interchange isomorphic objects in any categorical construction.

Given a functor $F\colon \mathbf{A} \to \mathbf{B}$ and objects A and A' in $\mathbf{A}$, let us write

$$F_{A,A'}\colon \operatorname{Hom}_{\mathbf{A}}(A, A') \to \operatorname{Hom}_{\mathbf{B}}(F(A), F(A')) \tag{3.4}$$

for the induced map between the Hom sets $\operatorname{Hom}_{\mathbf{A}}(A, A')$ and $\operatorname{Hom}_{\mathbf{B}}(F(A), F(A'))$. As in fact already observed in the proof of Theorem 3.6, given an isomorphism

$\tau\colon F \to G$, the diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathbf{B}}(F(A), F(A')) \arrow[dd, shift left, "f \mapsto \tau_{A'} f \tau_A^{-1}"] \\
\mathrm{Hom}_{\mathbf{A}}(A, A') \arrow[ur, "F_{A,A'}"] \arrow[dr, "G_{A,A'}"'] & \\
 & \mathrm{Hom}_{\mathbf{B}}(G(A), G(A')) \arrow[uu, shift left, "g \mapsto \tau_{A'}^{-1} g \tau_A"]
\end{tikzcd} \tag{3.5}$$

commutes. Since its vertical arrows are bijections, we obtain:

Theorem 3.8. *If F and G are isomorphic functors, then:*

1. *F is faithful (=all $F_{A,A'}$'s above are injective) if and only if so is G;*
2. *F is full (=all $F_{A,A'}$'s above are surjective) if and only if so is G;*
3. *F is fully faithful (=all $F_{A,A'}$'s above are bijective) if and only if so is G.*

Definition 3.9. *An* equivalence of categories $\mathbf{A}$ *and* $\mathbf{B}$ *is a system consisting of functors*

$$\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B} \quad \textit{and isomorphisms } \alpha\colon 1_{\mathbf{A}} \to GF, \quad \beta\colon 1_{\mathbf{B}} \to FG$$

we will also say that $(F, G, \alpha, \beta)\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ is a category equivalence, and (briefly) that $F\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ is a category equivalence.

Remark 3.10.

1. If $F\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ is a category isomorphism, then it is a category equivalence;
2. If
$$(F, G, \alpha, \beta)\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$$
is a category equivalence, then so is
$$(G, F, \beta, \alpha,)\colon \mathbf{B} \times \mathbf{B} \to \mathbf{A};$$
3. If
$$(F, G, \alpha, \beta)\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}, \text{ and } (H, I, \gamma, \delta)\colon \mathbf{B} \times \mathbf{B} \to \mathbf{C}$$
are category equivalences, then so is
$$(HF, GI, (G\gamma F)\alpha, (H\beta I)\delta) : \mathbf{A} \times \mathbf{A} \to \mathbf{C},$$
where $G\gamma F\colon GF \to GIHF$ and $H\beta I\colon HI \to HFGI$ denote natural transformations defined by $(G\gamma F)_A = G(\gamma_{F(A)})$ and $(H\beta I)_C = H(_\beta I(C))$ respectively.
4. As follows from the previous assertions, the category equivalence determines an equivalence relation on the collection of all categories; we will simple write $A \sim B$ when there exists a category equivalence $\mathbf{A} \times \mathbf{A} \to \mathbf{B}$.

5. If $F\colon \mathbf{A}\times\mathbf{A} \to \mathbf{B}$ is a category equivalence and $F' \approx F$, then $F'\colon \mathbf{A}\times\mathbf{A} \to \mathbf{B}$ also is a category equivalence.

The next definition will later help us describe the relationship between isomorphisms and equivalences of categories precisely.

Definition 3.11. *A category* $\mathbf{S}$ *is said to be a* skeleton*, if for objects* A *and* B *in* $\mathbf{S}$*, we have:*

$$A \approx B \implies A = B;$$

for an arbitrary category $\mathbf{C}$*, we say that* $\mathbf{S}$ *is a (the) skeleton of* $\mathbf{C}$ *and write* $\mathbf{S} = \mathrm{Skel}(\mathbf{C})$ *if* $\mathbf{S}$ *is a full subcategory in* $\mathbf{C}$*, and the inclusion functor* $\mathbf{S} \to \mathbf{C}$ *is a category equivalence.*

This definition immediately suggests to ask, if every category has a skeleton, and if the skeleton of a category is uniquely (up to an isomorphism?) determined. These questions are answered below.

Lemma 3.12. *If* $F\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ *is a category equivalence, then* F *is fully faithful and essentially (=up to isomorphism) bijective on objects, i.e.:*

1. *for objects* A *and* A' *in* $\mathbf{A}$*,* $F(A) \approx F(A') \implies A \approx A'$ *(essential injectivity);*
2. *for each object* B *in* $\mathbf{B}$*, there exists an object* A *in* $\mathbf{A}$ *with* $F(A) \approx B$ *(essential surjectivity).*

Proof. Let $(F, G, \alpha, \beta)\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ a category equivalence involving F. As follows from Theorem 3.8(c) applied to $1_{\mathbf{A}} \approx GF$, the functor GF is fully faithful. Therefore the composite

$$\mathrm{Hom}_{\mathbf{A}}(A, A') \xrightarrow{F_{A,A'}} \mathrm{Hom}_{\mathbf{B}}(F(A), F(A')) \xrightarrow{GF(A),F(A')} \mathrm{Hom}_{\mathbf{A}}(GF(A), GF(A'))$$

is a bijection for all objects A and A' in $\mathbf{A}$, from which we conclude:

- F is faithful;
- since F is always faithful in such a situation, G is also faithful by 3.10(b);
- since G is faithful, $G_{F(A),F(A')}$ is always injective;
- since $F_{A,A'}$ and $G_{F(A),F(A')}$ are injective and their composite is bijective, $F_{A,A'}$ is bijective too.

That is, F is fully faithful. Essential bijectivity on objects is obvious:

$$F(A) \approx F(A') \implies A \approx GF(A) \approx GF(A') \approx A',$$

and $F(A) \approx B$ for $A = G(B)$. □

Remark 3.13.

1. In fact the crucial properties here are fully faithfulness and essential surjectivity, since it is easy to show that a fully faithful functor is always essentially injective on objects. Indeed, if $F\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ is fully faithful, and $\beta : F(A) \to F(A')$ is an isomorphism in $\mathbf{B}$, then we can choose $\alpha\colon A \to A'$ with $F(\alpha) = \beta$ and $\alpha'\colon A' \to A$ with $F(\alpha') = \beta^{-1}$ - and these chosen morphisms will be inverse to each other since so are their images under F.

2. Proving essential injectivity of the functor F in (a) we in fact also proved another important property of a fully faithful functor, which is reflection of isomorphisms. It says: if $F(\alpha)$is an isomorphism, then so is α.

From Remark 3.10(a), Lemma 3.12, and Remark 3.13 we obtain:

Lemma 3.14. *The following conditions on a functor F between skeletons are equivalent:*

1. *F is a category equivalence;*
2. *F is fully faithful and essentially bijective on objects;*
3. *F is fully faithful and essentially surjective on objects;*
4. *F is an isomorphism.*

Remark 3.15.

1. It is not, however, true of course that $G = F^{-1}$ for any equivalence $(F, G, \alpha, \beta) \colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ between skeletons.
2. As follows from 3.10(d) and 3.14(a)-(d), skeletons of equivalent categories are always isomorphic. In particular so are every two skeletons of the same category.

Theorem 3.16. *Every category has a skeleton.*

Proof. Given a category $\mathbf{A}$, we choose:

- an object in each isomorphism class of objects in $\mathbf{A}$, and for any object A in $\mathbf{A}$, the chosen object isomorphic to A will be denoted by $\Phi(A)$;
- an isomorphism $\varphi_A \colon A \to \Phi(A)$, assuming for simplicity that $\varphi_{\Phi(A)} = 1_{\Phi(A)}$;
- $\Phi \colon \mathbf{A} \to \mathbf{A}$ to be the functor obtained from the identity functor of A and the family $(\varphi_A)_{A \in \mathbf{A}_0}$ as in Theorem 3.6 (see also Remark 3.7(a)), making $\varphi \colon 1_{\mathbf{A}} \to \Phi$ an isomorphism;
- $\mathbf{S}$ to be the full subcategory in $\mathbf{A}$ with object all $\Phi(A)(A \in \mathbf{A}_0)$;
- $F \colon \mathbf{S} \to \mathbf{A}$ to be the inclusion functor;
- $G \colon \mathbf{A} \to \mathbf{S}$ defined by $FG = \Phi$ (which indeed defines a functor since the image of Φ is inside $\mathbf{S}$), making $GF = 1_{\mathbf{S}}$, since $\varphi_{\Phi(A)} = 1_{\Phi(A)}$ for all objects A in $\mathbf{A}_0$.

Here S is a skeleton and $(F, G, 1_{1_{\mathbf{S}}}, \varphi) \colon \mathbf{S} \to \mathbf{A}$ is a category equivalence. ∎

Theorem 3.17. *1. A functor is a category equivalence if and only if it is fully faithful and essentially surjective on objects.*

2. Two categories are equivalent if and only if they have isomorphic skeletons.

Proof. 1. : Suppose $F\colon \mathbf{A}\times\mathbf{A}\to\mathbf{B}$ is fully faithful and essentially surjective on objects. Consider the diagram

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[d, shift left, "L"] & & B \arrow[d, shift left, "N"] \\ \mathrm{Skel}(A) \arrow[u, shift left, "K"] \arrow[rr, shift left, "NFK"] & & \mathrm{Skel}(B) \arrow[u, shift left, "M"] \arrow[ll, shift left, "(NFK)^{-1}"] \end{tikzcd}$$

in which:

- the vertical arrows determine equivalences $\mathbf{A}\sim\mathrm{Skel}(\mathbf{A})$ and $\mathbf{B}\sim\mathrm{Skel}(\mathbf{B})$, which exist by Theorem 3.16.
- the composite NFK is fully faithful and essentially surjective on objects, because so are N, F, and K; therefore NFK is an isomorphism by Lemma 3.14(c)-(d).

Using Remark 3.10 we conclude that $MNFKL$ is a category equivalence, and then that since $MNFKL\approx 1_{\mathbf{B}}F1_{\mathbf{A}}=F$, so is F. The "only if" part is Lemma 3.12.

2. : Again, just use Remark 3.10, Lemma 3.14, and the square diagram above (although the "only if" part has already been proved: see Remark 3.15(b)). □

3.4 Yoneda lemma and Yoneda embedding

The purpose of this section is to describe fully faithful functors

$$\mathbf{C}\xrightarrow{Y}\mathbf{Sets}^{\mathbf{C}^{op}}\xrightarrow{G}(\mathbf{Cat}\downarrow\mathbf{C}) \tag{3.6}$$

where $\mathbf{C}$ is an arbitrary category, $\mathbf{Sets}^{\mathbf{C}^{op}}$ is the category of functors $\mathbf{C}^{op}\to\mathbf{Sets}$, and $(\mathbf{Cat}\downarrow\mathbf{C})$ is the comma category of the category $\mathbf{Cat}$ of all categories over the category $\mathbf{C}$ (i.e. the category of pairs $(\mathbf{D},P)$, where $\mathbf{D}$ is a category and $P\colon\mathbf{D}\to\mathbf{C}$ a functor. As we will see, the fully faithfulness of Y will follow from

Theorem 3.18 (Yoneda lemma)**.** *For any functor $T\colon\mathbf{C}^{op}\to\mathbf{Sets}$ and any object C in $\mathbf{C}$, the map*

$$\mathrm{Nat}(\mathrm{Hom}_{\mathbf{C}}(-,C),T)\to T(C),\quad \tau\mapsto\tau_C(1_C) \tag{3.7}$$

from the set $\mathrm{Nat}(\mathrm{Hom}_C(-,C),F)$, of natural transformations from $\mathrm{Hom}_C(-,C)$ to T, to the set $T(C)$ is bijective.

Proof. Let us denote the map above by α and define a map

$$\beta\colon T(C)\to\mathrm{Nat}(\mathrm{Hom}_C(-,C),T)$$

by

$$\beta(t)_A(f)=T(f)(t)\text{ for a }t\in T(C)\text{ and a morphism}f\colon A\to C\text{ in }\mathbf{C}.$$

We are going to show that α and β are inverse to each other. We have

$$\alpha\beta(t) = \beta(t)_C(1_C) = T(1_C)(t) = t \text{ for each } t \in T(C),$$

proving that $\alpha\beta$ is the identity map of $T(C)$. On the other hand, for $\tau : \mathrm{Hom}_C(-, C) \to T$ and $f\colon \mathbf{A} \to \mathbf{C}$, we have

$$\beta\alpha(\tau)_A(f) = T(f)(\alpha(\tau)) = T(f)(\tau_C(1_C)) = \tau_A(\mathrm{Hom}_{\mathbf{C}}(f, C)(1C)) = \tau_A(f),$$

where the last equality is visible in the naturality square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(C, C) \arrow[r, "\tau_C"] \arrow[d, "\mathrm{Hom}_{\mathbf{C}}(f,C)"'] & T(C) \arrow[d, "T(f)"] \\
\mathrm{Hom}_{\mathbf{C}}(A, C) \arrow[r, "\tau_A"'] & T(A),
\end{tikzcd}$$

and the equality $\beta\alpha(\tau)A(f) = \tau_A(f)$ (for all f) implies that $\beta\alpha$ is the identity map of $\mathrm{Nat}(\mathrm{Hom}_{\mathbf{C}}(-, C), T)$. □

Consider the special case of this theorem in which the functor T is of the form $T = \mathrm{Hom}_C(-, C')$ for some C' in $\mathbf{C}$. Then the bijection of Theorem 3.18 together with its inverse become

$$\mathrm{Nat}(\mathrm{Hom}_{\mathbf{C}}(-, C), \mathrm{Hom}_{\mathbf{C}}(-, C')) \underset{t \mapsto (f \mapsto tf)}{\overset{\tau \mapsto \tau_C(1_C)}{\rightleftarrows}} \mathrm{Hom}_{\mathbf{C}}(C, C'), \tag{3.8}$$

where $t \mapsto (f \mapsto tf)$ means that $t\colon C \to C'$ is sent to the natural transformation

$$\tau\colon \mathrm{Hom}_{\mathbf{C}}(-, C) \to \mathrm{Hom}_{\mathbf{C}}(-, C') \text{ defined by } \tau_A(f) = tf.$$

However this map $\mathrm{Hom}_{\mathbf{C}}(C, C') \to \mathrm{Nat}(\mathrm{Hom}_{\mathbf{C}}(-, C), \mathrm{Hom}_{\mathbf{C}}(-, C'))$ is the same as $Y_{C,C'}$, where

$$Y\colon \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}} \text{ is the functor defined by } Y(C) = \mathrm{Hom}_{\mathbf{C}}(-, C),$$

i.e. the functor corresponding to the functor $\mathrm{Hom} : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Sets}$ via the canonical category isomorphism

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{C}^{op} \times \mathbf{C}, \mathbf{Sets}) \approx \mathrm{Hom}_{\mathbf{Cat}}(C, \mathbf{Sets}^{\mathbf{C}^{op}}). \tag{3.9}$$

Therefore Theorem 3.18 gives

Corollary 3.19. *The functor*

$$Y\colon \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}} \textit{ defined by } Y(C) = \mathrm{Hom}_{\mathbf{C}}(-, C) \tag{3.10}$$

is fully faithful.

The functor Y above is usually called the Yoneda embedding (for $\mathbf{C}$), while the functor $G : \mathbf{Sets}^{\mathbf{C}^{op}} \to (\mathbf{Cat} \downarrow \mathbf{C})$ we are going to introduce now has no name; a somewhat artificial name would be "the discrete form of Grothendieck construction".

For a functor $T : \mathbf{C}^{op} \to \mathbf{Sets}$, the category $\mathrm{E}(T)$ is defined as the category of pairs (A, a), where A is an object in $\mathbf{C}$ and a is an element $T(A)$; in this category, a morphism

$$f\colon (A, a) \to (B, b) \text{ is a morphism } f\colon A \to B \text{ in } \mathbf{C} \text{ with } T(f)(b) = a.$$

We define the functor

$$G : \mathbf{Sets}^{\mathbf{op}} \to (\mathbf{Cat} \downarrow \mathbf{C}) \text{ by } G(T) = (\mathrm{E}(T), P_T),$$

where $P_T\colon \mathrm{E}(T) \to$ is the forgetful functor, sending $f\colon (A, a) \to (B, b)$ to $f\colon A \to B$. In order to see how exactly is G defined on morphisms, let us describe morphisms in $(\mathbf{Cat} \downarrow \mathbf{C})$ of the form $\Phi : (\mathrm{E}(T), P_T) \to (\mathrm{E}(U), PU)$:

Such a morphism is a functor $\Phi\colon \mathrm{E}(T) \to \mathrm{E}(U)$ making the diagram

$$\begin{tikzcd}
\mathrm{E}(T) \arrow[rr, "\Phi"] \arrow[dr, "P_T"'] & & \mathrm{E}(U) \arrow[dl, "P_U"] \\
& \mathbf{C} &
\end{tikzcd}$$

commute. At the level of objects this means that, for each (A, a) in $\mathrm{E}(T)$, $\Phi(A, a)$ should a pair whose first component is A. This means that to give the object function of Φ is to give a fimily of maps $\varphi = (\varphi_A : T(A) \to U(A))_{A \in \mathbf{A}_0}$ and define Φ on objects by $\Phi(A, a) = (A, \varphi_A(a))$. After that, again, since the diagram above commutes, on morphisms Φ must be defined by

$$\Phi(f\colon (A, a) \to (B, b)) = f\colon (A, \varphi_A(a)) \to (B, \varphi_B(b)).$$

This simply means that the images of morphisms are uniquely determined, but the fact that Φ is indeed defined on morphisms puts the following condition on the family φ: if f is a morphism from (A, a) to (B, b), then it also must be a morphism from $(A, \varphi_A(a))$ to $(B, \varphi_B(b))$. And since f is a morphism from (A, a) to (B, b) if and only if $a = T(f)(b)$, this means that every $f\colon A \to B$ must be a morphism from $(A, \varphi_A T(f)(b))$ to $(B, \varphi_B(b))$ for each b in $T(B)$. In other words, for every $f\colon A \to B$ in $\mathbf{A}$, we must have $\varphi_A T(f) = U(f)\varphi_B$, which is the same as to say that φ is a natural transformation from T to U.

That is, we can define

$$G : \mathbf{Sets}^{\mathbf{C}^{op}} \to (\mathbf{Cat} \downarrow \mathbf{C}) \text{ by } G(\varphi : T \to U) = \Phi : (\mathrm{E}(T), P_T) \to (\mathrm{E}(U), P_U) \tag{3.11}$$

In the notation above (omitting routine verification of preservation of composition and identity morphisms), and this makes it fully faithful.

3.5 Representable functors and discrete fibrations

Definition 3.20.

1. *A functor* $T\colon \mathbf{C}^{op} \to \mathbf{Sets}$ *is said to be* representable *if it is isomorphic to a functor of the form* $Y(C) = \mathrm{Hom}_{\mathbf{C}}(-, C)$ *for some object* C *in* $\mathbf{C}$.

2. *A functor $P\colon \mathbf{D} \to \mathbf{C}$ is said to be a discrete fibration, if the diagram*

$$\begin{tikzcd}
\mathbf{D}_1 \arrow[r] \arrow[d, "P_1"'] & \mathbf{D}_0 \arrow[d, "P_0"'] \\
\mathbf{C}_1 \arrow[r] & \mathbf{C}_0
\end{tikzcd}$$

in which the horizontal arrows are the codomain maps of $\mathbf{D}$ and $\mathbf{C}$, and the vertical arrows are the morphism function and the object function of P respectively, is a pullback.

This section is devoted to the following two theorems:

Theorem 3.21. *A functor $T\colon \mathbf{C}^{op} \to \mathbf{Sets}$ is representable if and only if the category* $\mathrm{E}(T)$ *has a terminal object. Moreover, a natural transformation τ :* $\mathrm{Hom}_{\mathbf{C}}(-, C) \to T$ *is an isomorphism if and only if the pair (C, t), in which t is the image of τ under the map (3.7), is a terminal object in* $\mathrm{E}(T)$.

Proof. The following assertions are obviously equivalent:

1. $\tau : \mathrm{Hom}_{\mathbf{C}}(-, C) \to T$ is an isomorphism;
2. $\tau_A : \mathrm{Hom}_{\mathbf{C}}(A, C) \to T(A)$ is a bijection for each object A in $\mathbf{C}$;
3. for every object A in $\mathbf{C}$ and every $a \in T(A)$ there exists a unique morphism $f\colon A \to C$ with $\tau_A(f) = a$;
4. for every object A in $\mathbf{C}$ and every $a \in T(A)$ there exists a unique morphism $f\colon A \to C$ with $T(f)\tau_C(1_C) = a$;
5. for every object (A, a) in $\mathrm{E}(T)$ there exists a unique morphism from (A, a) to $(C, \tau_C(1C))$;
6. $(C, \tau_C(1_C))$ is a terminal object in $\mathrm{E}(T)$.

And since $(C, \tau_C(1_C))$ is exactly the image of τ under the map (3.7), this completes the proof. □

Theorem 3.22. *A functor $P\colon \mathbf{D} \to \mathbf{C}$ is a discrete fibration, if and only if the object $(\mathbf{D}, P)$ of $(\mathbf{Cat} \downarrow \mathbf{C})$ is isomorphic to $G(T) = (\mathrm{E}(T), P_T)$, for some functor $T\colon \mathbf{C}^{op} \to \mathbf{Sets}$.*

Proof. "If": We have to prove that $(\mathrm{E}(T), P_T)$ is always a discrete fibration. This means to prove that for every morphism $f\colon A \to B$ in $\mathbf{C}$ and every $b \in T(B)$, there exists a unique $a \in T(A)$ for which f is a morphism from (A, a) to (B, b). However this is trivial since f is a morphism from (A, a) to (B, b) if and only if $a = T(f)(b)$.

"Only if": Assuming that $P\colon \mathbf{D} \to \mathbf{C}$ is a discrete fibration, we define a functor $T\colon \mathbf{C}^{op} \to \mathbf{Sets}$ as follows:

- For an object C in $\mathbf{C}$, we take $T(C)$ to be the set of objects D in $\mathbf{D}$ with $P(D) = C$.

- For a morphism $f\colon A \to B$ in $\mathbf{C}$, and an element b in $T(B)$, which in fact an object in $\mathbf{D}$ with $P(b) = B$, we take g to be the morphism g in $\mathbf{D}$, with $P(g) = f$ and codomain of g equal to b. The existence and uniqueness of such a g follows from the fact that the diagram is a pullback. We then take $T(f)(b)$ to be the domain of g.

Accordingly the procedure of defining $T(f)$'s (for all f) displays as

$$\begin{tikzcd} T(f)(b) \arrow[r, dashed] & b \\ A \arrow[r] & B \end{tikzcd}$$

and it is easy to see that it indeed defines a functor $T\colon \mathbf{C}^{op} \to \mathbf{Sets}$ in such a way that $(\mathrm{E}(T), P_T)$ becomes isomorphic to $(\mathbf{D}, P)$. $\square$

3.6 Adjoint functors

Adjoint functors will be defined at the end of this section via several equivalent kinds of data that will be described before.

Definition 3.23. *Let $U\colon \mathbf{A} \to \mathbf{X}$ be a functor and X an object in $\mathbf{X}$. A* universal arrow $X \to U$ *is a pair $(F(X), \eta_X)$ in which $F(X)$ is an object in $\mathbf{A}$ and $\eta_X : X \to UF(X)$ a morphism in $\mathbf{X}$ with the following universal property: for every object A in $\mathbf{A}$ and every morphism $u\colon X \to U(A)$ in $\mathbf{X}$ there exists a unique morphism $f\colon F(X) \to A$ making the diagram*

$$\begin{tikzcd} UF(X) \arrow[dr, "U(f)"] & \\ & U(A) \\ X \arrow[uu, "\eta_X"] \arrow[ur, "u"'] & \end{tikzcd} \tag{3.12}$$

commute.

Theorem 3.24. *Let $U\colon \mathbf{A} \to \mathbf{X}$ be a functor and $((F(X), \eta_X))_{X \in \mathbf{X}_0}$ a family of universal arrows $X \to U$ given for each object X in $\mathbf{X}$. Then there exists a unique functor $F\colon \mathbf{X} \to \mathbf{A}$ for which the family $((F(X), \eta_X))_{X \in \mathbf{X}_0}$ determines a natural transformation $\eta\colon 1_{\mathbf{X}} \to UF$.*

Proof. Given a morphism $h\colon X \to Y$ in $\mathbf{X}$, we can define $F(h)\colon F(X) \to F(Y)$ as the unique morphism making the diagram commute for $A = F(Y)$ and $u = \eta_Y h$. Since the commutativity in this case is equivalent to the commutativity of the naturality square

$$\begin{tikzcd} UF(X) \arrow[r, "UF(h)"] \arrow[d, "\eta_X"] & UF(Y) \arrow[d, "\eta_Y"] \\ X \arrow[r, "h"'] & Y, \end{tikzcd} \tag{3.13}$$

this proves the theorem. $\square$

Remark 3.25. 1. The universal property given in Definition 3.23 can be equivalently reformulated as: the map

$$\varphi_{X,A} : \mathrm{Hom}_{\mathbf{A}}(F(X), A) \to \mathrm{Hom}_{\mathbf{X}}(X, U(A)), \text{ defined by } \varphi_{X,A}(f) = U(f)\eta_X, \tag{3.14}$$

is a bijection for each object A in $\mathbf{A}$. Moreover, since this map is obviously natural in A, that universal property can also be reformulated as: the natural transformation

$$\varphi_{X,-} : \mathrm{Hom}_{\mathbf{A}}(F(X), -) \to \mathrm{Hom}_{\mathbf{X}}(X, U(-)), \text{ defined by } \varphi_{X,A}(f) = U(f)\eta_X, \tag{3.15}$$

is an isomorphism. Furthermore, let

$$\varphi_{X,-} : \mathrm{Hom}_{\mathbf{A}}(F(X), -) \to \mathrm{Hom}_{\mathbf{X}}(X, U(-)) \tag{3.16}$$

be an arbitrary isomorphism. Then, for any $f\colon F(X) \to A$, using the naturality square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{A}}(F(X), F(X)) \arrow[r, "\varphi_{X,F(X)}"] \arrow[d, "\mathrm{Hom}_{\mathbf{A}}(F(X),f)"] & \mathrm{Hom}_{\mathbf{X}}(X, U(F(X))) \arrow[d, "\mathrm{Hom}_{\mathbf{X}}(X,U(f))"] \\
\mathrm{Hom}_{\mathbf{A}}(F(X), A) \arrow[r, "\varphi_{X,A}"'] & \mathrm{Hom}_{\mathbf{X}}(X, U(A)),
\end{tikzcd} \tag{3.17}$$

we obtain

$$\begin{aligned}
\varphi_{X,A}(f) &= \varphi_{X,A}\,\mathrm{Hom}_{\mathbf{A}}(F(X), f)(1F(X)) \\
&= \mathrm{Hom}_{\mathbf{X}}(X, U(f))\varphi_{X,F(X)}(1_{F(X)}) \\
&= U(f)\varphi_{X,F(X)}(1_{F(X)}).
\end{aligned}$$

Therefore we have one more reformulation of the universal property given in Definition 3.23, namely: there exists an isomorphism (3.16); and with this reformulation η_X and $\varphi_{X,-}$ determine each other by

$$\varphi_{X,A}(f) = U(f)\eta_X \text{ and } \eta_X = \varphi_{X,F(X)}(1_F(X)). \tag{3.18}$$

2. The relationship between η_X and $\varphi_{X,-}$ can be seen of course as a special case of the statement dual to Theorem 3.21, but we omit details here.

3. Suppose η_X, or, equivalently, $\varphi_{X,-}$ is given for every object X in $\mathbf{X}$. Then, by Theorem 3.24, there is a unique way to make F a functor $\mathbf{X} \to \mathbf{A}$, so that the family $((F(X), \eta_X))_{X\in\mathbf{X}_0}$ determines a natural transformation $\eta\colon 1_{\mathbf{X}} \to UF$. And it is easy to check that this will also make $\varphi_{X,-}$ natural in $\mathbf{X}$, yielding a natural isomorphism

$$\begin{tikzcd}
& \mathbf{A}^{op} \times \mathbf{A} \arrow[dr, "\mathrm{Hom}_{\mathbf{A}}"] & \\
\mathbf{X}^{op} \times \mathbf{A} \arrow[ur, "F^{op}\times 1"] \arrow[dr, "1\times U"'] & & \mathbf{Sets} \\
& \mathbf{X}^{op} \times \mathbf{X} \arrow[ur, "\mathrm{Hom}_{\mathbf{X}}"'] &
\end{tikzcd} \tag{3.19}$$

Moreover, the "φ approach" shows that the unique functoriality of F is actually a consequence of the fact that the Yoneda embedding $\mathbf{C}^{op} \to \mathbf{Sets}^{\mathbf{C}}$ is fully faithful. Indeed, given a morphism $h\colon X \to Y$ in $\mathbf{X}$, the naturality square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{A}}(F(Y),-) \arrow[r, "\varphi_{Y,-}"] \arrow[d, "\mathrm{Hom}_{\mathbf{A}}(F(h),-)"'] & \mathrm{Hom}_{\mathbf{X}}(Y,U(-)) \arrow[d, "\mathrm{Hom}_{\mathbf{X}}(h,U(-))"] \\
\mathrm{Hom}_{\mathbf{A}}(F(X),-) \arrow[r, "\varphi_{X,-}"'] & \mathrm{Hom}_{\mathbf{X}}(X,U(-)),
\end{tikzcd} \tag{3.20}$$

determines $\mathrm{Hom}_{\mathbf{A}}(F(h),-)$, and since the Yoneda embedding $\mathbf{C}^{op} \to \mathbf{Sets}^{\mathbf{C}}$ is fully faithful, $\mathrm{Hom}_{\mathbf{A}}(F(h),-)$ determines $F(h)$.

From Remark 3.25 we obtain

Theorem 3.26. *For a functor $U\colon \mathbf{A} \to \mathbf{X}$, the following kinds of data uniquely determine each other:*

1. *a family $((F(X),\eta_X))_{X\in\mathbf{X}_0}$ of universal arrows $X \to U$ given for each object X in $\mathbf{X}$;*
2. *a functor $F\colon \mathbf{X} \to \mathbf{A}$ and a natural transformation $\eta\colon 1_{\mathbf{X}} \to UF$ such that $(F(X),\eta_X)$ is a universal arrow $X \to U$ for each object X in $\mathbf{X}$;*
3. *a family $(F(X))_{X\in\mathbf{X}_0}$ of objects in $\mathbf{A}$ and a family*
$$(\varphi_{X,-}\colon \mathrm{Hom}_{\mathbf{A}}(F(X),-) \to \mathrm{Hom}_{\mathbf{X}}(X,U(-)))_{X\in\mathbf{X}_0}$$
of isomorphisms given for each object X in $\mathbf{X}$;
4. *a functor $F\colon \mathbf{X} \to \mathbf{A}$ and an isomorphism (3.19).*

Moreover, the η_X of (1) corresponds to the η_X of (2), the $\varphi_{X,-}$ of (3) corresponds to (the X-component) of φ of (4), and these η_X and $\varphi_{X,-}$ corresponding to each other via (3.18).

The data in (4) shows certain dual symmetry between U and F, and suggests to dualize Definition 3.23 and Theorem 3.26 as follows:

Definition 3.27. *Let $F\colon \mathbf{X} \to \mathbf{A}$ be a functor and A an object in $\mathbf{A}$. A* universal arrow $F \to A$ *is a pair $(U(A),\varepsilon_A)$ in which $U(A)$ is an object in $\mathbf{X}$ and $\varepsilon_A : FU(A) \to A$ a morphism in $\mathbf{A}$ with the following universal property: for every object X in $\mathbf{X}$ and every morphism $f\colon F(X) \to A$ in $\mathbf{A}$ there exists a unique morphism $u\colon X \to U(A)$ making the diagram*

$$\begin{tikzcd}
FU(A) \arrow[dd, "\varepsilon_A"'] & \\
& F(X) \arrow[ul, "F(u)"'] \arrow[dl, "f"] \\
A &
\end{tikzcd} \tag{3.21}$$

commute.

Theorem 3.28. *For a functor $F\colon \mathbf{X} \to \mathbf{A}$, the following kinds of data uniquely determine each other:*

1. *a family $((U(A), \varepsilon_A))_{A\in\mathbf{A}_0}$ of universal arrows $F \to A$ given for each object A in $\mathbf{A}$;*
2. *a functor $U\colon \mathbf{A} \to \mathbf{X}$ and a natural transformation $\varepsilon\colon FU \to 1_{\mathbf{A}}$ such that $(U(A), \varepsilon_A)$ is a universal arrow $F \to A$ for each object A in $\mathbf{A}$;*
3. *a family $(U(A))_{A\in\mathbf{A}_0}$ of objects in $\mathbf{X}$ and a family*
$$(\psi_{-,A} : \mathrm{Hom}_{\mathbf{X}}(-, U(A)) \to \mathrm{Hom}_{\mathbf{A}}(F(-), A))_{A\in\mathbf{A}_0}$$
of isomorphisms given for each object A in $\mathbf{A}$;
4. *a functor $U\colon \mathbf{A} \to \mathbf{X}$ and an isomorphism*

$$\begin{tikzcd}
 & \mathbf{A}^{op} \times \mathbf{A} \arrow[dr, "\mathrm{Hom}_{\mathbf{A}}"] & \\
\mathbf{X}^{op} \times \mathbf{A} \arrow[ur, "F^{op}\times 1"] \arrow[dr, "1\times U"'] & & \mathbf{Sets} \\
 & \mathbf{X}^{op} \times \mathbf{X} \arrow[ur, "\mathrm{Hom}_{\mathbf{X}}"'] &
\end{tikzcd} \tag{3.22}$$

Moreover, the ε_A of (1) corresponds to the ε_A of (2), the $\psi_{-,A}$ of (3) corresponds to (the A-component) of ψ of (4), and these ε_A and $\psi_{-,A}$ corresponding to each other via
$$\psi_{X,A(u)} = \varepsilon_A F(u) \text{ and } \varepsilon_A = \psi_{U(A),A}(1_{U(A)}). \tag{3.23}$$

Remark 3.29. The data described in Theorem 3.26(d) is obviously identical to the data described in Theorem 3.28(d): just take φ and ψ inverse to each other. Therefore these two theorems actually describe eight equivalent kinds of data.

Remark 3.29 is not the end of this story: although eight is a large number, it is good to add at least one more, which is purely equational. For, we observe:

- Having functors $U\colon \mathbf{A} \to \mathbf{X}$ and $F\colon \mathbf{X} \to \mathbf{A}$, and merely natural transformations $\eta : 1_{\mathbf{X}} \to UF$ and $\varepsilon\colon FU \to 1_{\mathbf{A}}$, we can still define natural transformations φ and ψ as in 3.19 and in 3.22 respectively.
- Under no conditions on η and ε, those φ and ψ will also be merely natural transformations independent from each other. But requiring them to be each other's inverses and reformulating this requirement in terms of η and ψ will give us a new equivalent form of the desired data, which is purely equational.
- Requiring that φ and ψ be each other's inverses means to require
$$\psi_{X,A}\varphi_{X,A}(f) = f \text{ and } \varphi_{X,A}\psi_{X,A(u)} = u,$$
for each $f\colon F(X) \to A$ in $\mathbf{A}$ and each $u\colon X \to U(A)$ in $\mathbf{X}$. But then Yoneda lemma (Theorem 3.18) tells us that it suffices to have these equalities for $f = 1_{F(X)}\colon F(X) \to F(X)$ and $u = 1_{U(A)}\colon U(A) \to U(A)$.

- Thus, we are interested in

$$\psi_{X,F(X)}\varphi_{X,F(X)}(1_{F(X)}) = 1_{F(X)}$$

and

$$\varphi_{U(A),A}\psi_{U(A),A}(1_{U(A)}) = 1_{U(A)}.$$

Translated into the language of η and φ, these equations become

$$\varepsilon_{F(X)}F(\eta_X) = 1_{F(X)} \text{ and } U(\varepsilon_A)\eta_{U(A)} = 1_{U(A)}, \tag{3.24}$$

and we obtain:

Theorem 3.30. *Let $U\colon \mathbf{A} \to \mathbf{X}$ and $F\colon \mathbf{X} \to \mathbf{A}$ be functors and $\eta\colon 1_{\mathbf{X}} \to UF$ and $\varepsilon\colon FU \to 1_{\mathbf{A}}$ natural transformations. The following conditions are equivalent:*

1. *$(F(X), \eta_X)$ is a universal arrow $X \to U$ for each object X in $\mathbf{X}$, and ε is the corresponding family of morphisms, i.e. $U(\varepsilon_A)\eta_{U(A)} = 1_{U(A)}$ for every object A in $\mathbf{A}$;*

2. *$(U(A), \varepsilon_A)$ is a universal arrow $F \to A$ for each object A in $\mathbf{A}$, and η is the corresponding family of morphisms, i.e. $\varepsilon_{F(X)}F(\eta_X) = 1_{F(X)}$ for every object X in $\mathbf{X}$;*

3. *the equalities $\varepsilon_{F(X)}F(\eta_X) = 1_{F(X)}$ and $U(\varepsilon_A)\eta_{U(A)} = 1_{U(A)}$ hold for every object X in $\mathbf{X}$ and every object A in $\mathbf{A}$.*

Remark 3.31. Using the standard notation for composing functors and natural transformations, the equalities (3.25) (for all X and A) are displayed as commutative diagrams

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FUF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} UFU \arrow[d, "U\varepsilon"'] & U \arrow[l, "\eta U"'] \arrow[dl, equal] \\ U & \end{tikzcd} \tag{3.25}$$

and called triangular identities.

Definition 3.32.

Let $U\colon \mathbf{A} \to \mathbf{X}$ and $F\colon \mathbf{X} \to \mathbf{A}$ be functors, $\eta\colon 1_{\mathbf{X}} \to UF$ and $\varepsilon\colon FU \to 1_{\mathbf{A}}$ be natural transformations satisfying the triangular identities, and φ and ψ be as in Theorems 3.26 and 3.28 respectively. We will say that:

1. $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ is an *adjunction*; however, we might also omit either η or ε, or replace them with either φ or ψ;

2. F is the left adjoint (of U), U is the right adjoint (of F), η is the *unit* of adjunction, and ε is the *counit* of adjunction.

3.7 Monoidal categories

In this section we introduce monoidal categories with some examples and related concepts.

Definition 3.33. *A* monoidal category *is a system* $(\mathbf{C}, I, \otimes, \alpha, \lambda, \rho)$ *in which:*

1. $\mathbf{C}$ *is a category;*
2. I *is an object in* $\mathbf{C}$*;*
3. $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *is a functor, written as* $\otimes(A, B) = A \otimes B$*;*
4. $\alpha = (\alpha_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C)_{A,B,C\in\mathbf{C}}$*,* $\lambda = (\lambda_A\colon A \to I \otimes A)_{A\in\mathbf{C}}$*, and* $\rho = (\rho_A\colon A \to A \otimes I)_{A\in\mathbf{C}}$ *are natural isomorphisms making the diagrams commute:*

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[r, "\alpha"] & (A \otimes I) \otimes B \\
A \otimes B \arrow[u, "1\otimes\lambda"] \arrow[r, equal] & A \otimes B, \arrow[u, "\rho\otimes 1"']
\end{tikzcd} \tag{3.26}$$

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha\otimes 1"']
\end{tikzcd} \tag{3.27}$$

Here and below we write just α *instead of* $\alpha_{A,B,C}$ *for short; it is also often useful to write* $(\mathbf{C}, I, \otimes, \alpha, \lambda, \rho) = (\mathbf{C}, I, \otimes) = (\mathbf{C}, \otimes) =$*. A monoidal category* $(\mathbf{C}, I, \otimes, \alpha, \lambda, \rho)$ *is said to be strict if* $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ *for all* A, B, C*;* $I \otimes A = A = A \otimes I$ *for all* A*; and* α*,* λ*, and* ρ *are the identity morphisms.*

Example 3.34. Any monoid $\mathbf{M} = (\mathbf{M}, e, m)$ can be regarded as a strict monoidal category $(\mathbf{C}, I, \otimes)$, in which $\mathbf{C}$ is the underlying set $\mathbf{M}$ regarded as a discrete category (i.e. a category with no non-identity arrows), $I = e$, and $\otimes = m$.

Example 3.35. Any category $\mathbf{X}$ yields the strict monoidal category $\mathrm{End}(\mathbf{X}) = (\mathrm{End}(\mathbf{X}), 1_{\mathbf{X}}, -)$ of functors $\mathbf{X} \to \mathbf{X}$, where $1_{\mathbf{X}}$ is the identity functor $\mathbf{X} \to \mathbf{X}$ and $\otimes$ is the composition of functors.

Example 3.36. If $\mathbf{C}$ is a category with finite products, then $(\mathbf{C}, I, \otimes, \alpha, \lambda, \rho)$, in which $I = \mathbf{1}$ is a terminal object in $\mathbf{C}$, $\otimes = \times$ is a (chosen) binary product operation, and α, λ, ρ arise from the canonical isomorphisms $A \times (B \times C) \cong (A \times B) \times C$, $A \cong \mathbf{1} \times A$, $A \cong A \times \mathbf{1}$ respectively, is a monoidal category. Such a monoidal structure is said to be cartesian.

Example 3.37. An internal graph G in a category is a diagram of the form

$$\begin{tikzcd}
G_1 \arrow[r, shift left, "d_G"] & G_0 \arrow[l, shift left, "c_G"]
\end{tikzcd}$$

in $\mathbf{C}$. For a fixed object O, the internal graphs G in $\mathbf{C}$ with $G_0 = O$ are called internal O-graphs in $\mathbf{C}$, and their category will be denoted by $\mathbf{Graphs}(, \mathbf{O})$; a morphism $f\colon G \to H$ in $\mathbf{Graphs}(,\mathbf{O})$ is a morphism $f\colon G1 \to H1$ in $\mathbf{C}$ with $d_H f = d_G$ and $c_H f = c_G$. When $\mathbf{C}$ has chosen pullbacks, this category becomes a monoidal category $(\mathbf{Graphs}(,\mathbf{O}), \mathbf{I}, \otimes, \alpha, \lambda, \rho)$ as follows:

1. I has $I_0 = I_1 = O$ and $d_I = c_I = 1_O$;

2. $\otimes$ is defined as the span composition, i.e. for G and H in $\mathbf{Graphs}(\mathbf{C}, O)$, $G \otimes H$ is defined by $(G \otimes H)_1 = G_1 \times_O H_1$, $d_{G\otimes H} = d_H \pi_2$, and $c_{G\otimes H} = c_G \pi_1$ via the diagram

$$\begin{tikzcd} & & G_1 \times_O H_1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ & G_1 \arrow[dl, "c_G"'] \arrow[dr, "d_G"] & & H_1 \arrow[dl, "c_G"'] \arrow[dr, "d_G"] & \\ O & & O & & O \end{tikzcd} \tag{3.28}$$

in which diamond part is the chosen pullback of the pair (d_G, c_G).

3. α, λ, and ρ arise from the appropriate canonical isomorphisms.

In the special case in which $O = 1$ is a terminal object in $\mathbf{C}$, the pullbacks we need become binary products, and the monoidal category we obtain coincides with the one from Example 3.36.

Example 3.38. Dualizing Example 3.36, if $\mathbf{C}$ is a category with finite coproducts, then $(\mathbf{C}, I, \otimes, \alpha, \lambda, \rho)$, in which $I = 0$ is an initial object in $\mathbf{C}$, $\otimes = +$ is a (chosen) binary coproduct operation, and α, λ, ρ arise from the canonical isomorphisms $A + (B + C) \cong (A + B) + C$, $A \cong 0 + A$, $A \cong A + 0$ respectively, is a monoidal category.

Example 3.39. Let R be a commutative ring, and $\mathbf{C}$ the category of R-modules. Then $(, \mathbf{I}, \otimes, \alpha, \lambda, \rho)$, in which $I = R$, $\otimes$ the usual tensor product over R, and α, λ, ρ the usual natural isomorphisms, forms a monoidal category.

Definition 3.40. *Let* $= (\mathbf{C}, \mathbf{I}, \otimes, \alpha, \lambda, \rho)$ *and* $\mathbf{C}' = (\mathbf{C}', I, \otimes, \alpha, \lambda, \rho)$ *be monoidal categories (we use the prime sign ' only for* $\mathbf{C}$*, although the* I*,* $\otimes$*, etc. in* $\mathbf{C}$ *and in* $\mathbf{C}'$ *are not, of course, supposed to be the same). A* monoidal functor $F = (F, \theta, \phi) : \mathbf{C} \to \mathbf{C}'$ *consists of*

1. *an ordinary functor* $F\colon \mathbf{C} \to \mathbf{C}'$*;*

2. *a morphism* $\theta\colon I \to F(I)$ *in* $\mathbf{C}'$*;*

3. *a natural transformation* $\phi = (\phi_{A,B} : F(A) \otimes \mathbf{F}(B) \to F(A \otimes B))_{A,B\in\mathbf{C}}$ *making the diagrams*

$$\begin{tikzcd} F(A) \otimes (F(B) \otimes F(C)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\phi"'] & (F(A) \otimes \mathbf{F}(B)) \otimes F(C) \arrow[d, "\phi\otimes 1"] \\ F(A) \otimes (F(B \otimes C)) \arrow[d, "\phi"'] & (F(A \otimes B)) \otimes F(C) \arrow[d, "\phi"] \\ F(A \otimes (B \otimes C)) \arrow[r, "F(\alpha)"'] & F((A \otimes B) \otimes C), \end{tikzcd} \tag{3.29}$$

$$\begin{tikzcd} I \otimes F(A) \arrow[d, "\theta\otimes 1"'] & F(A) \arrow[l, "\lambda"'] \arrow[d, "F(\lambda)"] \\ F(I) \otimes F(A) \arrow[r, "\phi"'] & F(I \otimes A), \end{tikzcd} \tag{3.30}$$

$$\begin{tikzcd} F(A) \otimes I \arrow[d, "1\otimes\theta"'] & F(A) \arrow[l, "\rho"'] \arrow[d, "F(\rho)"'] \\ F(A) \otimes F(I) \arrow[r, "\phi"'] & F(A \otimes I), \end{tikzcd} \tag{3.31}$$

commute. A monoidal functor $F = (F, \theta, \phi)$ is said to be strong if θ and ϕ are isomorphisms, and strict if moreover $F(I) = I$, $F(A) \otimes \mathbf{F}(B) = F(A \otimes B)$ for all A and B, and θ and ϕ are the identity morphisms.

Definition 3.41. *Let $F_i = (F_i, \theta_i, \phi_i)\colon \mathbf{C} \to \mathbf{C}'$ $(i = 1, 2)$ be monoidal functors. A* monoidal natural transformation $\tau\colon F_1 \to F_2$ *is an ordinary natural transformation $\tau\colon F_1 \to F_2$ such that the diagrams*

$$\begin{tikzcd} I \arrow[r, "\theta_1"] \arrow[d, equal] & F_1(I) \arrow[d, "\tau"] \\ I \arrow[r, "\theta_2"'] & F_2(I), \end{tikzcd} \tag{3.32}$$

$$\begin{tikzcd} F_1(A) \otimes F_1(B) \arrow[r, "\phi_1"] \arrow[d, "\tau\otimes\tau"'] & F_1(A \otimes B) \arrow[d, "\tau"] \\ F_2(A) \otimes F_2(B) \arrow[r, "\phi_2"'] & F_2(A \otimes B) \end{tikzcd} \tag{3.33}$$

commute.

Several examples of monoidal functors are used as definitions of important concepts. Two of them will be given here with further cases considered in the next sections.

Definition 3.42. *Let $\mathbf{C}$ be monoidal category and $\mathbf{X}$ a category. A $\mathbf{C}$-action on $\mathbf{X}$ is a monoidal functor $\mathbf{C} \to \mathrm{End}(\mathbf{X})$, where $\mathrm{End}(\mathbf{X})$ is as in Example 3.35.*

Equivalently such a $\mathbf{C}$-action can be defined as a functor $\mathbf{C} \times \mathbf{X} \to \mathbf{X}$, which we will write as $(\mathbf{C}, \mathbf{X}) \mapsto \otimes\mathbf{X}$, equipped with natural transformations $\theta = (\theta_X : X \to I \otimes X))_{X \in \mathbf{X}}$ and $\phi = (\phi_{A,B,X} : A \otimes (B \otimes X) \to (A \otimes B) \otimes X)_{A,B \in, \mathbf{X} \in \mathbf{X}}$ making the diagrams

$$\begin{tikzcd} A \otimes (B \otimes (C \otimes X)) \arrow[r, equal] \arrow[d, "1\otimes\phi"'] & A \otimes (B \otimes (C \otimes X)) \arrow[d, "\phi"] \\ A \otimes ((B \otimes C) \otimes X) \arrow[d, "\phi"'] & (A \otimes B) \otimes (C \otimes X) \arrow[d, "\phi"'] \\ (A \otimes (B \otimes C)) \otimes X \arrow[r, "\alpha\otimes 1"'] & ((A \otimes B) \otimes C) \otimes X, \end{tikzcd} \tag{3.34}$$

$$\begin{tikzcd} A\otimes X \arrow[r, equal] \arrow[d, "\theta"'] & A\otimes X \arrow[d, "\lambda\otimes 1"] \\ I\otimes(A\otimes X) \arrow[r, "\phi"'] & (I\otimes A)\otimes X, \end{tikzcd} \tag{3.35}$$

$$\begin{tikzcd} A\otimes X \arrow[r, equal] \arrow[d, "1\otimes\theta"'] & A\otimes X \arrow[d, "1\otimes\rho"] \\ A\otimes(I\otimes X) \arrow[r, "\phi"'] & (A\otimes I)\otimes X, \end{tikzcd} \tag{3.36}$$

commute.

Definition 3.43. *Let* $\underline{\mathbf{1}}$ *be the trivial monoid considered as a monoidal category. A* monoidal functor *from it to an arbitrary monoidal category* $\mathbf{C}$ *can be presented as a triple* $M = (M, e, m)$, *in which* M *is an object in* $\mathbf{C}$ *and* $e\colon I \to M$ *and* $m\colon M\otimes M \to M$ *morphisms in* $\mathbf{C}$ *making the diagram*

$$\begin{tikzcd} M\otimes(M\otimes M) \arrow[r, "\alpha"] \arrow[d, "1\otimes m"'] & (M\otimes M)\otimes M \arrow[r, "m\otimes 1"] & M\otimes M \arrow[d, "m"] & M \arrow[l, "(e\otimes 1)\lambda"'] \arrow[d, "(1\otimes e)\rho"] \arrow[dl, equal] \\ M\otimes M \arrow[rr, "m"'] & & M & M\otimes M \arrow[l, "m"'] \end{tikzcd} \tag{3.37}$$

commute. Such a triple is called a monoid *in* $\mathbf{C}$.

Moreover, a monoidal natural transformation $\tau\colon (M_1, e_1, m_1) \to (M_2, e_2, m_2)$ being a morphism $\tau\colon M_1 \to M_2$ in $\mathbf{C}$ with $e_1 = e_2$ and $\tau m_1 = m_2(\tau\otimes\tau)$, is nothing but a monoid homomorphism in $\mathbf{C}$. So, the monoids in $\mathbf{C}$ form a category $\mathrm{Mon}(\mathbf{C})$, which is the category $\mathbf{MonCat}(\underline{\mathbf{1}}, \mathbf{C})$ of monoidal functors $\underline{\mathbf{1}} \to$. In particular this immediately tells us that every monoidal functor $F = (F, \theta, \phi) : \mathbf{C} \to \mathbf{C}'$ induces a functor $\mathrm{Mon}(F)\colon \mathrm{Mon}(\mathbf{C}) \to \mathrm{Mon}(\mathbf{C}')$, which sends (M, e, m) to the composite

$$\underline{\mathbf{1}} \xrightarrow{(M,e,m)} \mathbf{C} \xrightarrow{(F,\theta,\phi)} \mathbf{C}'$$

considered as a monoid in $\mathbf{C}'$.

3.8 Monads and algebras

In this section we introduce monads, algebras over monads, and free algebras; we also introduce a very general notion of a monoid action as a "general example".

Definition 3.44. *A* monad *on a category* $\mathbf{X}$ *is a monoid in the monoidal category* $\mathrm{End}(\mathbf{X})$ *of Example 3.35. Explicitly, a monad on* $\mathbf{X}$ *is a triple* $T = (T, \eta, \mu)$, *in which* $T\colon \mathbf{X} \to \mathbf{X}$ *is a functor and* $\eta\colon 1_{\mathbf{X}} \to T$ *and* $\mu\colon T^2 \to T$ *natural transformations making the diagram*

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[d, "T\eta"] \arrow[dl, equal] \\ T^2 \arrow[r, "\mu"'] & T & T \arrow[l, "\mu"] \end{tikzcd} \tag{3.38}$$

Definition 3.45. *Let $T = (T, \eta, \mu)$ be a monad on a category $\mathbf{X}$. A T-*algebra *(or an algebra over T) is a pair (X, ξ), in which X is an object in $\mathbf{X}$ and $\xi\colon T(X) \to X$ a morphism making the diagram*

$$\begin{tikzcd}
T^2(X) \arrow[r, "\mu T"] \arrow[d, "T(\xi)"'] & T(X) \arrow[d, "\xi"'] & X \arrow[l, "\eta_X"'] \arrow[dl, equal] \arrow[d, "T\eta"] \\
T(X) \arrow[r, "\mu"'] & X & {}
\end{tikzcd} \tag{3.39}$$

commute. A morphism $h\colon (X, \xi) \to (X', \xi')$ of T-algebras is a morphism $h\colon X \to X'$ making the diagram

$$\begin{tikzcd}
T(X) \arrow[r, "T(h)"] \arrow[d, "\xi"'] & T(X') \arrow[d, "\xi'"] \\
X \arrow[r, "h"'] & X
\end{tikzcd} \tag{3.40}$$

commute. The category of T-algebras will be denoted by $\mathbf{X}^T$.

Theorem 3.46. *Let $T = (T, \eta, \mu)$ be a monad on a category $\mathbf{X}$, and let $U^T\colon \mathbf{X}^T \to \mathbf{X}$ be the forgetful functor defined by $U^T(X, \xi) = X$. Then:*

1. *for each object X in $\mathbf{X}$, the pair $(T(X), \mu_X)$ is a T-algebra;*

2. *the functor $F^T\colon X \to X^T$, defined by $F^T(X) = (T(X), \mu_X)$ is a left adjoint of U^T. The unit and counit of the adjunction are $\eta\colon 1_{\mathbf{X}} \to T = U^T F^T$ and $\varepsilon\colon F^T U^T \to 1_{\mathbf{X}^T}$ defined by $\varepsilon_{(T(X), \mu_X)} = \mu_X$ respectively.*

Proof. 1. We have to prove the commutativity of

$$\begin{tikzcd}
T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] & T^2(X) \arrow[d, "\mu_X"'] & X \arrow[l, "\eta_{T(X)}"'] \arrow[dl, equal] \\
T^2(X) \arrow[r, "\mu_X"'] & X &
\end{tikzcd} \tag{3.41}$$

but it follows from the commutativity of (3.38).

2. The square part of (3.41) insures that putting

$$\varepsilon_{(T(X), \mu_X)} = \mu_X$$

determines a natural transformation $\varepsilon\colon F^T U^T \to 1_{\mathbf{X}^T}$, and it is easy to see that η and ε satisfy the triangular identities.

□

Example 3.47.

Let $\mathbf{X}$ be a category equipped with an action of a monoidal category $\mathbf{C}$. According to Definition 3.42, such an action is simply a monoidal functor $F\colon \mathbf{C} \to \mathrm{End}(\mathbf{X})$, and, like every monoidal functor, it induces a functor $\mathrm{Mon}(F)\colon \mathrm{Mon}(\mathbf{C}) \to \mathrm{Mon}(\mathrm{End}(\mathbf{X}))$. Therefore every monoid $M = (M, e, m)$ in $\mathbf{C}$ determines a monad on $\mathbf{X}$; the algebras over that monad are called M-actions, and their category is denoted by $\mathbf{X}^M$. Explicitly, such an M-action is a pair $(\mathbf{X}, \xi)$, in which $\xi\colon M \otimes X \to X$ is a morphism in $\mathbf{X}$ making the diagram

$$\begin{tikzcd}
M \otimes (M \otimes X) \arrow[r, "\phi"] \arrow[d, "1\otimes\xi"'] & (M \otimes M) \otimes X \arrow[r, "m\otimes 1"] & M \otimes X \arrow[d, "\xi"'] & X \arrow[l, "(e\otimes 1)\theta"'] \arrow[dl, equal] \\
M \otimes X \arrow[rr, "\xi"'] & & X &
\end{tikzcd}$$

commute. Here θ and ϕ are as in (3.34)-(3.36).

Remark 3.48.

1. According to G. M. Kelly, an "M-action" is the right name not for a pair (X, ξ) above, but just for its structure morphism ξ.
2. Example 3.47 is at the same time a "generalization". Indeed, starting from an arbitrary monad T on X, we can consider T-algebras as T-actions in the sense of Example 3.47, putting $=$ End$(\mathbf{X})$ and considering the identity momoidal functor End$(\mathbf{X}) \to$ End$(\mathbf{X})$ as the action of End$(\mathbf{X})$ on $\mathbf{X}$.

3.9 More on adjoint functors and category equivalences

This section contains additional observations on adjoint functors and category equivalence; some them will be explicitly used later, while others simply help to understand the concepts involved. We begin with

Remark 3.49.

1. It is easy to see that $(F, U, \eta, \xi)\colon \mathbf{X} \to \mathbf{A}$ is an adjunction if and only if so is
$$(U^{op}, F^{op}, \varepsilon^{op}, \eta^{op})\colon \mathbf{X}^{op} \to \mathbf{A}^{op}$$
(in the obvious notation). Therefore every general property of adjoint functors has its dual, where the left and the right adjoints exchange their roles (see e.g. Theorems 3.53 and 3.54 below).
2. Since in an adjunction $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$, $\eta_X\colon X \to UF(X)$ is a universal arrow $X \to U$ for each object $X \in \mathbf{X}$, the functor U alone determines such an adjunction uniquely up to an isomorphism; dually, the same is true for F.
3. It is easy to see that adjunctions compose: if $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ and $(G, V, \theta, \zeta)\colon \mathbf{Y} \to \mathbf{X}$ are adjunctions, then so is
$$(FG, VU, (V\eta G)\theta, \varepsilon(F\zeta U))\colon \mathbf{Y} \to \mathbf{A}.$$
(cf. 3.10(c)).
4. Let
$$\begin{tikzcd}
A \arrow[r, shift left, "K"] \arrow[d, shift left, "K'"] & B \arrow[l, shift left, "M"] \arrow[d, shift left, "L"] \\
B' \arrow[u, shift left, "M'"] \arrow[r, shift left, "L'"] & C \arrow[l, shift left, "N'"] \arrow[u, shift left, "N"]
\end{tikzcd} \tag{3.42}$$

a diagram of functors in which M, M', N, and N' are the left adjoints of K, K', L, and L' respectively. Then, as easily follows from (b) and (c), we have $LK \approx L'K' \equiv MN \approx M'N'$.

Lemma 3.50. *Every fully faithful functor reflects isomorphisms, i.e. under such a functor only isomorphisms are sent to isomorphisms.*

Proof. Let $U\colon \mathbf{A} \to \mathbf{X}$ be a fully faithful functor with $U(f\colon A \to B)$ being an isomorphism. Since U is full, $U(f)^{-1} = U(g)$ for some $g\colon B \to A$ in $\mathbf{A}$. Then since $U(gf) = 1_{U(A)}$, $U(fg) = 1_{U(B)}$, and U is faithful, we obtain $gf = 1_A$ and $fg = 1_B$, which shows that $f\colon A \to B$ is an isomorphism. □

Definition 3.51. *An adjunction $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ is said to be an* adjoint equivalence *if η and ε are isomorphisms.*

Theorem 3.52. *Let $U\colon \mathbf{A} \to \mathbf{X}$ be a category equivalence, F_0 a map from the set $\mathbf{X}_0$ of objects in $\mathbf{X}$ to the set $\mathbf{A}_0$ of objects in $\mathbf{A}$, and $\eta = (\eta_X\colon X \to UF_0(X))_{X\in\mathbf{X}_0}$ a family of isomorphisms. Then there exists a unique functor $F\colon \mathbf{X} \to \mathbf{A}$ and a unique natural transformation $\varepsilon\colon FU \to 1_{\mathbf{A}}$, for which F_0 is the object function of F and $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ is an adjunction. Moreover, that adjunction is always an adjoint equivalence.*

Proof. Since U is fully faithful (by Lemma 3.12) and each $\eta_X\colon X \to UF_0(X)$ is an isomorphism, it is easy to see that $\eta_X\colon X \to UF_0(X)$ is a universal arrow $X \to U$ for each object X in $\mathbf{X}$. After that the first assertion of the theorem follows from Remark 3.29 (see also Definition 3.32). Next, since η_Xs are isomorphisms, so are $U(\varepsilon_A)$s (by the second identity in (3.25)), and by Lemma 3.50 this implies that ε is an isomorphism. □

Theorem 3.53. *Let $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ be an adjunction. Then:*

1. *U is faithful if and only if ε is an epimorphism;*
2. *U is full if and only if ε is a split monomorphism;*
3. *and therefore U is fully faithful if and only if ε is an isomorphism.*

Proof. For two arbitrary objects A and B in $\mathbf{A}$, consider the diagram

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathbf{X}}(U(A),U(B)) \arrow[dd, shift left, "u\mapsto\varepsilon_B F(u)"] \\
\mathrm{Hom}_{\mathbf{A}}(A,B) \arrow[ur, "U_{A,B}"] \arrow[dr, "\mathrm{Hom}_{\mathbf{A}}(\varepsilon_A,B)"'] & \\
& \mathrm{Hom}_{\mathbf{A}}(FU(A),B) \arrow[uu, shift left, "f\mapsto U(f)\eta_{U(A)}"]
\end{tikzcd} \tag{3.43}$$

where the vertical arrows are bijections inverse to each other (since they are $\psi_{U(A),B}$ and $\varphi_{U(A),B}$ respectively: see (3.18) and (3.23)). Since the left-hand vertical arrow is bijective and makes the triangle commute (by naturality of ε), we have:

1. $U_{A,B}$ is injective $\mathrm{Hom}_{\mathbf{A}}(\varepsilon_A, B)$ is injective;

2. $U_{A,B}$ is surjective $\mathrm{Hom}_{\mathbf{A}}(\varepsilon_A, B)$ is surjective;

3. $U_{A,B}$ is bijective $\mathrm{Hom}_{\mathbf{A}}(\varepsilon_A, B)$ is bijective.

Since $\mathrm{Hom}_{\mathbf{A}}(\varepsilon_A, B)$ is injective, surjective, or bijective if and only if ε_A is an epimorphism, split monomorphism, or isomorphism respectively, this completes the proof. □

Dually, we obtain:

Theorem 3.54. *Let $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ be an adjunction. Then:*

1. *F is faithful if and only if η is a monomorphism;*

2. *F is full if and only if η is a split epimorphism;*

3. *and therefore F is fully faithful if and only if η is an isomorphism.*

- which helps to prove the following:

Theorem 3.55. *The following conditions on an adjunction $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ are equivalent:*

1. *$(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ is an adjoint equvalence;*

2. *F and U are fully faithful;*

3. *F is fully faithful and U reflects isomorphisms;*

4. *η is an isomorphism and U reflects isomorphisms;*

5. *U is fully faithful and F reflects isomorphisms;*

6. *ε is an isomorphism and F reflects isomorphisms.*

Proof. $(a) \implies (b)$, $(c) \implies (d)$, and $(e) \implies (f)$ follow from Theorems 3.54(a)-(c) and 3.53(a)-(c). $(b) \implies (c)$ and $(b) \implies (e)$ follow from Lemma **??**. Therefore it suffices to prove the implications $(d) \implies (a)$ and $(f) \implies (a)$. Moreover, since these implications are dual to each other, it suffices to prove only one of them, say, $(d) \implies (a)$. For, consider the second identity $U(\varepsilon_A)\eta_{U(A)} = 1_{U(A)}$ in (3.25). Assuming that η is an isomorphism, we conclude that so is $U(\varepsilon_A)$ for each A, and, when U reflects isomorphisms, this implies that ε is an isomorphism, as desired. □

3.10 Remarks on coequalizers

The remarks on coequalizers we make in this section are presented as a definition and an example:

Definition 3.56. 1. *A* coequalizer *diagram in a given category is a diagram of the form*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd} \tag{3.44}$$

in which $hf = hg$, and for every morphism $h'\colon B \to C'$ with $h'f = h'g$, there exists a unique morphism $k\colon C \to C'$ with $kh = h'$. We will then also say that h is the coequalizer of the pair (f, g).

2. *A morphism that occurs in a coequalizer diagram as the morphism h occurs in (3.44) is called a* regular epimorphism.

3. *A coequalizer diagram is said to be* absolute*, if it is preserved by any functor, i.e. if its image under any functor is a coequalizer diagram.*

Example 3.57. 1. Consider a split fork, i.e. a diagram of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend right=60, "j"'] \arrow[r, "h"] & C \arrow[l, bend right=60, "i"'] \end{tikzcd} \tag{3.45}$$

in which $hf = hg$, $hi = 1_C$, $fj = 1_B$, and $gj = ih$. In each such diagram f, g, and h form a coequalizer diagram. Indeed, given $h' : B \to C'$ with $h'f = h'g$, it is easy to see that there is a unique morphism $k\colon C \to C'$ with $kh = h'$: just take $k = h'i$, which gives

$$kh = h'ih = h'gj = h'fj = h',$$

and the uniqueness follows from the fact that h is a split epimorphism. Since the conditions imposed on the diagram (3.45) were purely equational and therefore are "preserved" by every functor, this also proves that f, g, and h form an absolute coequalizer diagram.

2. An arbitrary split epimorphism $h\colon B \to C$ can be involved in a split fork, namely in

$$\begin{tikzcd} B \arrow[r, shift left, "1_B"] \arrow[r, shift right, "ih"'] & B \arrow[l, bend right=60, "1_B"'] \arrow[r, "h"] & C \arrow[l, bend right=60, "i"'] \end{tikzcd} \tag{3.46}$$

where i is a splitting, i.e. a morphism from C to B with $hi = 1_C$. Therefore every split epimorphism is a regular epimorphism.

3. For a monad $T = (T, \eta, \mu)$ on a category $\mathbf{X}$, any T-algebra (X, ξ) determines the following split fork in $\mathbf{X}$:

$$\begin{tikzcd} T^2(X) \arrow[r, shift left, "\mu_X"] \arrow[r, shift right, "T(\xi)"'] & T(X) \arrow[l, bend right=60, "\eta_{T(X)}"'] \arrow[r, "\xi"] & X \arrow[l, bend right=60, "\eta_X"'] \end{tikzcd} \tag{3.47}$$

3.11 Monadicity

In this section we discuss the relationship between adjunctions and monad.

Theorem 3.58. *For every adjunction $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$, the triple $T = (T, \eta, \mu)$ defined by*

- $T = UF$,
- *η of (T, η, μ) is the same as η of $(F, U, \eta, \varepsilon)$,*

- $\mu = U\varepsilon F$, *i.e.* $\mu = (\mu_X : T^2(X) \to T(X))_{X \in \mathbf{X}_0}$ *is defined by* $\mu_X = U(\varepsilon_{F(X)})$,

is a monad on $\mathbf{X}$.

Proof. □

For the triple above, and any object X in $\mathbf{X}$, the X-component of the diagram (3.38) becomes

$$\begin{tikzcd}
UFUFUF(X) \arrow[r, "U(\varepsilon_{FUF(X)})"] \arrow[d, "1\otimes m"'] & UFUF(X) \arrow[r, "\eta_{UF(X)}"] & UF(X) \arrow[d, "m"'] & M \arrow[l, "UFU(\varepsilon_{F(X)})"'] \arrow[d, "UF(\eta_X)"] \arrow[dl, equal] \\
UFUF(X) \arrow[rr, "m"'] & & UF(X) & UFUF(X) \arrow[l, "m"']
\end{tikzcd} \tag{3.48}$$

and its left-hand square commutes by the naturality of while the triangles commute by the triangular identities (3.25).

Example 3.59. Starting from an arbitrary monad $T = (T, \eta, \mu)$ on a category $\mathbf{X}$, we obtain the forgetful-free adjunction $(F^T, U^T, \eta^T, \varepsilon^T) : \mathbf{X} \to \mathbf{X}^T$ described in Theorem 3.46. It is easy to see that the corresponding monad on $\mathbf{X}$ is the same as the original monad $T = (T, \eta, \mu)$. This tells us that every monad can be obtained from an adjunction as in Theorem 3.58. Since this result is originally due to S. Eilenberg and J. Moore, the category $\mathbf{X}^T$ is often called the Eilenberg-Moore category (of algebras over T). Note also, that using only free T-algebras, i.e. the T-algebras of the form $F^T(X) = (T(X), \mu_X)$ we could also obtain an adjunction whose corresponding monad is $T = (T, \eta, \mu)$. Furthermore, since such an algebra $(T(X), \mu_X)$ is fully determined by its underlying object X, the full subcategory in $\mathbf{X}^T$ with objects all free T-algebras can be described as the so-called Kleisli category of T, whose objects are the same as the objects in $\mathbf{X}$. In detail:

- The category Kleisli(T) is defined as the category with the same objects as the in $\mathbf{X}$, and a morphism $f : X \to Y$ being a morphism $f : X \to T(Y)$ in $\mathbf{X}$; the composite of morphisms $f : X \to Y$ and $g : Y \to Z$ in Kleisli(T) is the composite

$$X \xrightarrow{f} T(Y) \xrightarrow{T(g)} T^2(Z) \xrightarrow{\mu_Z} T(Z)$$

in $\mathbf{X}$.

- The forgetful functor U: Kleisli(T) $\to \mathbf{X}$ is defined by $U(f : X \to Y) = \mu_Z T(f) : T(X) \to T(Y)$, and free functor $F : \mathbf{X} \to$ Kleisli(T) is defined by $F(f : X \to Y) = \eta_Y f : X \to T(Y)$, considered as a morphism from X to Y in Kleisli(T).

- And the monad obtained from adjunction as in Theorem 10.1 is again the same as the original monad $T = (T, \eta, \mu)$ (a result due to H. Kleisli).

It is now natural ask, to what extend is it possible to recover the adjunction $(F, U, \eta, \varepsilon) : \mathbf{X} \to \mathbf{A}$ from the monad $T = (T, \eta, \mu)$ in the situation of Theorem 3.58? In order to formulate this question properly, we need:

Theorem 3.60.

$(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ and $T = (T, \eta, \mu)$ be as in Theorem 3.58. Then there exists a unique functor $K\colon \mathbf{A} \to \mathbf{X}^T$ with $U^T K = U$ and $KF = FT$.

Proof. Existence: Simply define K by

$$K(A) = (U(A), U(\varepsilon_A)). \tag{3.49}$$

To prove that $(U(A), U(\varepsilon_A))$ is indeed a T-algebra is to prove that the diagram

$$\begin{tikzcd} UFUFU(A) \arrow[r, "U(\varepsilon_{FU(A)})"] \arrow[d, "1\otimes m"'] & UFU(A) \arrow[r, "\eta_{U(A)}"] & U(A) \arrow[d, "m"'] & M \arrow[l, "UFU(\varepsilon_A)"'] \arrow[dl, equal] \\ UFU(A) \arrow[rr, "m"'] & & U(A) & \end{tikzcd} \tag{3.50}$$

commutes, which, for the left-hand square, follows from the naturality of , and, for the triangle, follows from the second identity in (3.25) (cf. (3.48)). Defining K by (3.49), we also obviously have $U^T K = U$, and $KF = FT$ since $KF(X) = (UF(X), U((F(X))) = (T(X),)X) = FT(X)$.

Uniqueness: Let $H\colon \mathbf{A} \to \mathbf{X}^T$ be a functor satisfying $U^T H = U$ and $HF = FT$. Since $U^T H = U$, such a functor must be given by $H(A) = (U(A), \varepsilon_A)$ for some natural transformation $\xi\colon UFU \to U$. On the other hand, since $HF = FT$, we must have $\xi_{F(X)} = \mu_X = U(\varepsilon_{F(X)})$. . After that, comparing the naturality square from (3.50) with the naturality square

$$\begin{tikzcd} UFUFU(A) \arrow[r, "U(\varepsilon_{FU(A)})=\xi_{FU(A)}"] \arrow[d, "UFU(\varepsilon_A)"'] & UFU(A) \arrow[d, "U(\varepsilon_A)"] \\ UFU(A) \arrow[r, "\xi_A"'] & U(A) \end{tikzcd}$$

we obtain $\xi_A UFU(\varepsilon_A) = U(\varepsilon_A)UFU(\varepsilon_A)$, which implies $\xi_A = U(\varepsilon_A)$, since $UFU(\varepsilon_A)$ is a split epimorphism by the second identity in (3.25). □

Definition 3.61. *Let $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ and $T = (T, \eta, \mu)$ be as in Theorems 3.58 and 3.60. Then:*

1. *the functor $K\colon \mathbf{A} \to \mathbf{X}^T$ as in Theorem 3.60 is called the comparison functor;*

2. *the functor $U\colon \mathbf{A} \to \mathbf{X}$ is said to be monadic if the functor $K\colon \mathbf{A} \to \mathbf{X}^T$ above is a category equivalence.*

Accordingly, saying that an adjunction $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ can be recovered from the corresponding monad $T = (T, \eta, \mu)$ on $\mathbf{X}$ should be understood as saying that the functor U is monadic. In order to formulate some of the monadicity results, we will need the following construction containing long calculations:

Let $(F, U, \eta, \varepsilon)\colon \mathbf{X} \to \mathbf{A}$ and $T = (T, \eta, \mu)$ be as above, and suppose that for every T-algebra (X, ξ), the pair $(\varepsilon_{F(X)}, F(\xi))$ has a coequalizer in $\mathbf{A}$. Then the comparison functor $K\colon \mathbf{A} \to \mathbf{X}^T$ has a left adjoint forming an adjunction $(L, K, \eta', \varepsilon')\colon \mathbf{X} \to \mathbf{A}$ that can be described as follows:

1. For a T-algebra (X, ξ), the object $L(X, \xi)$ is defined via the coequalizer diagram

$$\begin{tikzcd} FUF(X) \arrow[r, shift left, "\varepsilon_{F(X)}"] \arrow[r, shift right, "F(\xi)"'] & F(X) \arrow[r, "\pi(X,\xi)"] & L(X, \xi) \end{tikzcd} \tag{3.51}$$

2. For a morphism $h\colon (X, \xi) \to (X', \xi')$ of T-algebras, we form the diagram

$$\begin{tikzcd} FUF(X) \arrow[r, shift left, "\varepsilon_{F(X)}"] \arrow[r, shift right, "F(\xi)"'] \arrow[d, "FUF(h)"'] & F(X) \arrow[r, "\pi(X,\xi)"] \arrow[d, "F(h)"] & L(X, \xi) \arrow[d, "L(h)"] \\ FUF(X') \arrow[r, shift left, "\varepsilon_{F(X')}"] \arrow[r, shift right, "F(\xi')"'] & F(X') \arrow[r, "\pi(X',\xi')"] & L(X, \xi') \end{tikzcd} \tag{3.52}$$

   of solid arrows, in which

$$\begin{aligned} \pi_{(X',\xi')} F(h) \varepsilon_{F(X)} &= \pi_{(X',\xi')} \varepsilon_{F(X')} FUF(h) \\ &= \pi_{(X',\xi')} F(\xi') FUF(h) \\ &= \pi(X', \xi') F(h) F(\xi) \end{aligned}$$

   implies the existence and uniqueness of the dotted arrow making the right-hand square commute. This determines a functor $L\colon \mathbf{X}^T \to A$.

3. We then define $\eta'_{(X,\xi)}\colon (X, \xi) \to KL(X, \xi) = (UL(X, \xi), U(\varepsilon_{L(X,\xi)}))$ as the composite $U(\pi_{(X,\xi)})\eta_X$, which we can do since the diagram

$$\begin{tikzcd} UF(X) \arrow[r, "UF(\eta_X)"] \arrow[d, "\xi"'] & UFUF(X) \arrow[r, "UFU(\pi_{(X,\xi)})"] & UFUL(X, \xi) \arrow[d, "U(\varepsilon_{L(X,\xi)})"'] \\ X \arrow[r, "\eta_X"'] & UF(X) \arrow[r, "U(\pi_{(X,\xi)})"] & UL(X, \xi) \end{tikzcd} \tag{3.53}$$

commutes. Indeed, we have

$$
\begin{aligned}
U(\varepsilon_{L(X,\xi)})UFU(\pi_{(X,\xi)}))UF(\eta_X) &= U(\varepsilon_{L(X,\xi)}FU(\pi_{(X,\xi)}))F(\eta_X))\\
&\quad \text{(by functoriality of U)}\\
&= U(\pi_{(X,\xi)})\varepsilon_{F(X)}F(\eta_X))\\
&\quad \text{(by naturality of } \varepsilon)\\
&= U(\pi_{(X,\xi)}))\\
&\quad \text{(by the first identity in (3.25))}\\
&= U(\pi_{(X,\xi)}))U(\varepsilon_{F(X)})\eta_{UF(X)}\\
&\quad \text{(by the second identity in (3.25)}\\
&\qquad \text{applied to } A = F(X))\\
&= U(\pi_{(X,\xi)})\varepsilon_{F(X)})\eta_{UF(X)}\\
&\quad \text{(by functoriality of } U)\\
&= U(\pi_{(X,\xi)})F(\xi))\eta_{UF(X)}\\
&\quad \text{(since (3.51) is a coequalizer diagram)}\\
&= U(\pi_{(X,\xi)}))UF(\xi)\eta_{UF(X)}\\
&\quad \text{(by functoriality of } U)\\
&= U(\pi_{(X,\xi)}))\eta_X\xi\\
&\quad \text{(by naturality of } \eta).
\end{aligned}
$$

4. To show that $\eta'_{(X,\xi)}$ is a universal arrow $(X,\xi) \to K$ is to show that for every morphism $k\colon (X,\xi) \to (U(A), U(\varepsilon_A))$ there exists a unique morphism $l\colon L(X,\xi) \to A$ with
$$U(l)U(\pi(X,\xi))\eta_X = k.$$
Since $U(l)U(\pi_{(X,\xi)}) = U(l\pi_{(X,\xi)})$ and (F,U,η,ε) is an adjunction, this is the same as to show that there exists a unique morphism $l\colon L(X,\xi) \to A$ with $l\pi_{(X,\xi)} = \varepsilon_A F(k)$. Since (3.51) is a coequalizer diagram this simply means to show that
$$\varepsilon_A F(k)\varepsilon_{F(X)} = \varepsilon_A F(k)F(\xi). \tag{3.54}$$
For, we have
$$
\begin{aligned}
\varepsilon_A F(k)F(\xi) &= \varepsilon_A F(k\xi) \text{ (by functoriality of } F)\\
&= \varepsilon_A F(U(\varepsilon_A)UF(k))\\
&\quad \text{(since } k\colon (X,\xi) \to (U(A), U(\varepsilon_A)) \text{is a morphism of } T\text{-algebras)}\\
&= \varepsilon_A FU(\varepsilon_A F(k)) \text{ (by functoriality of } U)\\
&= \varepsilon_A F(k)\varepsilon_{F(X)} \text{ (by naturality of } \varepsilon),
\end{aligned}
$$
as desired.

5. In particular, for an object A in $\mathbf{A}$, the morphism $\varepsilon'_A\colon LK(A) \to A$ is the

unique morphism $L((U(A), U(\varepsilon_A)) \to A$ making the diagram

$$\begin{tikzcd}
FUFU(A) \arrow[r, shift left, "\varepsilon_{FU(A)}"] \arrow[r, shift right, "FU(\varepsilon_A)"'] & FU(A) \arrow[rr, "\pi(U(A),\varepsilon_A)"] \arrow[dr, "\varepsilon_A"'] & & L((U(A), U(\varepsilon_A)) \arrow[dl, "\varepsilon'_A"] \\
& & A &
\end{tikzcd} \tag{3.55}$$

commute.

Remark 3.62. As an intermediate result of the calculation in (3), we have

$$U(\pi_{(X,\xi)})\eta_X \xi = U(\pi_{(X,\xi)}) \tag{3.56}$$

for every T-algebra $(T(X), \xi)$. Since $\eta'_{(X,\xi)} \colon (X, \xi) \to KL(X, \xi)$ was defined (in (3.52)) as $U(\pi_{(X,\xi)})\eta_X$, this equality together with Example 3.57 tell us that $\eta'_{(X,\xi)}$ considered as a morphism in $\mathbf{X}$ is the unique morphism making the diagram

$$\begin{tikzcd}
UFUF(X) \arrow[r, shift left, "U(\varepsilon_{F(X)})=\mu_X"] \arrow[r, shift right, "UF(\xi)=T(\xi)"'] & UF(X) \arrow[rr, "\xi"] \arrow[dr, "U(\pi(X,\xi))"'] & & L((U(A), U(\varepsilon_A)) \arrow[dl, "\pi_{(X,\xi)}=U(\pi_{(X,\xi)})\eta_X"'] \\
& & A &
\end{tikzcd} \tag{3.57}$$

commute.

Theorem 3.63. *For $(F, U, \eta, \varepsilon) \colon \mathbf{X} \to \mathbf{A}$ and $T = (T, \eta, \mu)$ as above the following conditions are equivalent:*

1. *the functor $U \colon \mathbf{A} \to \mathbf{X}$ is monadic;*

2. *the functor U preserves the coequalizer diagram (3.51) for every T-algebra (X, ξ), and, for every object A in $\mathbf{A}$, the morphism ε_A is the coequalizer of the pair $(\varepsilon_{FU(A)}, FU(\varepsilon_A))$;*

3. *the functor U reflects isomorphisms and preserves the coequalizer diagram (3.51) for every T-algebra (X, ξ);*

4. *the functor U reflects isomorphisms, and every pair (f, g) of parallel morphisms in $\mathbf{A}$, for which the pair $(U(f), U(g))$ has an absolute coequalizer, has a coequalizer preserved by U.*

Proof. • Since $U^T K = U$, and $U^T \colon \mathbf{X}^T \to \mathbf{X}$ obviously reflects isomorphisms, U reflects isomorphisms if and only if K does.

- As follows from Remark 3.62 and the fact that the top part of the diagram 3.57 is a coequalizer diagram (see Example 3.57), the functor U preserves the coequalizer diagram (3.51) if and only if $\eta'_{(X,\xi)} \colon (X, \xi) \to KL(X, \xi)$ is an isomorphism.

- As follows from 3.55, the morphism ε_A is the coequalizer of the pair $(\varepsilon_{FU(A)}, FU(\varepsilon_A))$ if and only if $\varepsilon'_A : LK(A) \to A$ is an isomorphism.
- This proves $(a) \implies (b)$ and makes $(b) \implies (c)$ a consequence of Theorem 3.55 (in fact a consequence of the last argument in its proof).
- Since the pair $(U(\varepsilon_{F(X)}), UF(\xi)) = (\mu_X, T(\xi))$ involved in (3.57) is a part of a split fork (3.47), (d) implies (c).
- After this all we need to prove is that if (f, g) of parallel morphisms in $\mathbf{X}^T$, for which the pair (f, g) has an absolute coequalizer in $\mathbf{X}$, then (f, g) has a coequalizer in $\mathbf{X}^T$ preserved by U^T. For, consider the diagram

$$\begin{tikzcd} T(X) \arrow[r, shift left, "T(f)"] \arrow[r, shift right, "T(g)"'] \arrow[d] & T(Y) \arrow[r, "T(h)"] \arrow[d] & T(Z) \arrow[d] \\ X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & Z \end{tikzcd} \tag{3.58}$$

where: h is the coequalizer of (f, g) in $\mathbf{X}$; the left-hand and the middle vertical arrow are the domain and the codomain of f (and of g) respectively in the category $\mathbf{X}^T$; and the dotted arrow is determined by the fact that the top row in (3.58) is a coequalizer diagram (since h is the absolute coequalizer of (f, g) in $\mathbf{X}$). Using the fact that not only T but also T^2 preserves the equalizer of (f, g), it is easy to check that the dotted arrow determines a T-algebra structure on Z and then makes h is the coequalizer of (f, g) in $\mathbf{X}^T$ and this coequalizer is trivially preserved by U^T.

□

Remark 3.64.

1. Condition (3.63)(d) can modified by asking the pair $(U(f), U(g))$ to be a split coequalizer (i.e. to be a part of a split fork) instead of an absolute one. As one can see from the argument proving $(d) \implies (c)$ of Theorem 3.63, this follows from the fact that the diagram (3.47) is a split fork.
2. The pair $(\varepsilon_{F(X)}, F(\xi))$ involved in (3.51) is reflexive, which means that $\varepsilon_{F(X)}$ and $F(\xi)$ are split epimorphisms with a common splitting which is $F(\eta_X)$. Therefore using the same arguments as in the proof of Theorem 3.63, we can prove the following: if a functor admits a left adjoint, reflects isomorphisms, and preserves coequalizers of reflexive pairs, then it is monadic.

3.12 Internal precategory actions

This section presents generalized versions of very first concepts of internal category theory need for the purposes of categorical Galois theory.

Definition 3.65. *An* internal precategory *in a category* $\mathbf{X}$ *is a diagram*

$$P = (P_0, P_1, P_2, d, c, e, m) = \begin{tikzcd} P_2 \arrow[r, bend left=60, "p"] \arrow[r, "m"] \arrow[r, bend right=60, "q"'] & P_1 \arrow[r, bend left=60, "d"] \arrow[r, bend right=60, "c"'] & P_0 \arrow[l, "e"'] \end{tikzcd} \tag{3.59}$$

in $\mathbf{X}$ *with* $de = 1 = ce$*,* $dp = cq$*,* $dm = dq$*, and* $cm = cp$*. An internal precategory in* **Sets** *is simply called a precategory.*

Example 3.66. Any (small) category $\mathbf{C}$ can be regarded as a precategory; it is then to be displayed as

$$\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, d, c, e, m) = \begin{tikzcd} \mathbf{C}_2 \arrow[r, bend left=60, "p"] \arrow[r, "m"] \arrow[r, bend right=60, "q"'] & \mathbf{C}_1 \arrow[r, bend left=60, "d"] \arrow[r, bend right=60, "c"'] & \mathbf{C}_0 \arrow[l, "e"'] \end{tikzcd} \tag{3.60}$$

where:

- $\mathbf{C}_0$ is the set of objects in $\mathbf{C}$;
- $\mathbf{C}_1$ is the set of morphisms in $\mathbf{C}$;
- d and c are the domain map and the codomain map respectively, i.e. $d(f) = x$ and $c(f) = y$ if and only if f is a morphism from x to y;
- $\mathbf{C}_2 = \{(g, f) \mid d(g) = c(f)\}$ is the set of composable pairs of morphisms in $\mathbf{C}$;
- p and q are the projection maps, i.e. $p(g, f) = g$ and $q(g, f) = f$.

Example 3.66 suggests:

Definition 3.67. *An* internal category *in a category* $\mathbf{X}$ *with pullbacks is an internal precategory* $\mathbf{C}$ *in* $\mathbf{X}$*, in which the diagram formed by* d*,* c*,* p*,* q *is a pullback (yielding* $\mathbf{C}_2 = \mathbf{C}_1 \times_{\mathrm{d,c)}} \mathbf{C}_1$*) and the diagram*

$$\begin{tikzcd} \mathbf{C}_1 \times_{(d,c)} \mathbf{C}_1 \times_{(d,c)} \mathbf{C}_1 \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] & \mathbf{C}_1 \times_{(d,c)} \mathbf{C}_1 \arrow[r, "\langle ec,1\rangle"] \arrow[d, "m"'] & \mathbf{C}_1 \arrow[d, "\langle 1,ed\rangle"] \arrow[dl, equal] \\ \mathbf{C}_1 \times_{(d,c)} \mathbf{C}_1 \arrow[r, "m"'] & \mathbf{C}_1 & \mathbf{C}_1 \times_{(d,c)} \mathbf{C}_1 \arrow[l, "m"] \end{tikzcd} \tag{3.61}$$

commutes.

Remark 3.68. 1. Comparing diagrams (3.61) and (3.37) makes clear that an internal category C in $\mathbf{X}$ is nothing but a monoid in the monoidal category $(\mathbf{Graphs}(\mathbf{X}, O), I, \otimes, \alpha, \lambda, \rho)$, described in Example 3.37, for $O = C_0$.

2. An internal category in **Sets** is of course the same as an ordinary (small) category.

The readers familiar with simplicial sets might prefer to consider precategories as truncated simplicial sets, and present Example 3.66 via the notion of nerve of a category. According to this approach, but also independently of it, given a precategory P, it is convenient to use displays like

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & z \\ & y \arrow[ur, "g"'] & \end{tikzcd}, \quad \begin{tikzcd} x \arrow[loop above, "e(x)"] \end{tikzcd} \tag{3.62}$$

for t in P_2, $g = p(t)$, $f = q(t)$, $h = m(t)$, $x = d(f) = d(h)$, $y = d(g) = c(f)$, and $z = c(g) = c(h)$. Note that these displays "remember" all identities required in Definition 3.65. Thinking of internal precategories as generalized categories, we are going now to generalize functors. In fact there are several concepts to be introduced, and the first obvious step is to define precategory morphisms as the corresponding diagram morphisms, which brings us to

Definition 3.69. *Let P and P' be internal precategories in* $\mathbf{X}$*. A* morphism $\varphi\colon \mathbf{P} \to \mathbf{P'}$ *is a diagram in* $\mathbf{X}$ *of the form*

$$\begin{tikzcd} P_2 \arrow[r, bend left=50, "p"] \arrow[r, "m"] \arrow[r, bend right=50, "q"'] \arrow[d, "\varphi_2"'] & P_1 \arrow[r, bend left=50, "d"] \arrow[r, bend right=50, "c"'] \arrow[d, "\varphi_1"'] & P_0 \arrow[l, "e"'] \arrow[d, "\varphi_0"'] \\ P'_2 \arrow[r, bend left=50, "p'"] \arrow[r, "m'"] \arrow[r, bend right=50, "q'"'] & P'_1 \arrow[r, bend left=50, "d'"] \arrow[r, bend right=50, "c'"'] & P'_0 \arrow[l, "e'"'] \end{tikzcd} \tag{3.63}$$

which reasonably commutes, i.e. has $\varphi_0 d = d'\varphi_1$, $\varphi_0 c = c'\varphi_1$, $\varphi_1 e = e'\varphi_0$, $\varphi_1 p = p'\varphi_2$, $\varphi_1 q = q'\varphi_2$, and $\varphi_1 m = m'\varphi_2$. A morphism $\varphi\colon \mathbf{P} \to \mathbf{P'}$ above is said to be

1. *a* discrete fibration *if the squares ($\varphi_0 c = c'\varphi_1$ and $\varphi_1 p = p'\varphi_2$ in (3.63) are pullbacks;*

2. *a* discrete opfibration *if the squares ($\varphi_0 d = d'\varphi_1$ and $\varphi_1 q = q'\varphi_2$ in (3.63) are pullbacks.*

Remark 3.70. It is easy to show that if $\varphi\colon \mathbf{P} \to \mathbf{P'}$ is a discrete fibration and P' is an internal category, then P also is an internal category. On the other hand, if P and P' were internal categories, then $\varphi'\colon \mathbf{P} \to \mathbf{P'}$ is a discrete fibration whenever just the square $\varphi_0 c = c'\varphi_1$ in (3.63) is a pullback. Therefore discrete fibrations of internal categories in **Sets** are the same as ordinary ones defined in Definition 3.20(b).

Next, we need "functors" $P \to \mathbf{X}$, and since this concept is less obvious, let us begin with the case $\mathbf{X} = \mathbf{Sets}$:

Definition 3.71. *Let P be a precategory. Then:*

1. *For a category C, a prefunctor P C is a precategory morphism P C, where C is regarded as a precategory in the same way as in Example 3.66.*

2. *A P-action is a diagram*

$$A = (A_0, \pi, \xi) = \ P_1 \times_{P_0} A_0 \xrightarrow{\xi} A_0 \xrightarrow{\pi} P_0, \tag{3.64}$$

where $P_1 \times_{P_0} A_0 = P_1 \times_{(d,\pi)} A_0$ is the pullback of d and π, ξ is written as $\xi(f, a) = fa$, and

$$\pi(fa) = c(f),\ e(x)a = a,\ ha = g(fa) \tag{3.65}$$

in the situation (3.62) whenever $\pi(a) = x$.

Remark 3.72. 1. When P is a category (see Example 3.66), the equalities (3.65) are to be rewritten as

$$\pi(fa) = c(f),\ 1_x a = a,\ (gf)a = g(fa). \tag{3.66}$$

That is, when P is a category, a P-action is nothing but a functor from P to **Sets**. To be absolutely precise, we should say that in that case there is a canonical equivalence between the category of P-actions and the category of functors from P to **Sets**.

- The general case reduces to the case of categories. Indeed: Let

$$L\colon \text{Precategories} \to \text{Categories} \tag{3.67}$$

be the left adjoint of the inclusion functor from the category of categories to the category of precategories. Explicitly, for a precategory $P = (P_0, P_1, P_2, d, c, e, m)$, the category $L(P)$ is the quotient category $\mathrm{Pa}(G)/\sim$, where:

- $\mathrm{Pa}(G)$ is the free category ("the category of paths") on the underlying graph $G = (P_0, P_1, d, c)$ of P; that is, the objects of $\mathrm{Pa}(G)$ are the elements of P_0, and a morphism $x \to y$ is a finite (possibly empty) sequence $(f_0, \ldots, f_n)$, in which $d(f_n) = x$, $c(f_i) = d(f_{i-1})$ (for $i = 1, \ldots, n$), and $c(f_0) = y$.
- $\sim$ is the smallest congruence on $\mathrm{Pa}(G)$, for which $e(x) \sim 1_x$ and $m(t) \sim p(t)q(t)$ for each x in P_0 and t in P_2.

Requiring $m(t) \sim p(t)q(t)$ here is of course the same to require $h \sim gf$ in the situation (3.62), and the category of P-actions can be identified with the category of $L(P)$-actions.

2. The category of P-actions is canonically equivalent to the category of prefunctors $P \to \mathbf{Sets}$. This can be either shown directly, or deduced from (a) and (b), since the category of prefunctors $P \to \mathbf{Sets}$ is obviously canonically isomorphic to the category of functors $L(P) \to \mathbf{Sets}$.

Internalizing now Definition 3.71(b) we arrive at:

Definition 3.73. *Let $P = (P_0, P_1, P_2, d, c, e, m)$ be an internal precategory in a category $\mathbf{X}$ with pullbacks. A P-*action *is a diagram*

$$A = (A_0, \pi, \xi) = \ P_1 \times_{P_0} A_0 \xrightarrow{\xi} A_0 \xrightarrow{\pi} P_0, \tag{3.68}$$

where $P_1 \times_{P_0} A_0 = P_1 \times_{(d,\pi)} A_0$ is the pullback of d and π, and the diagram

$$\begin{tikzcd}
P_2 \times_{(dq,\pi)} A_0 \arrow[r, "\langle p,q\rangle\times 1"] \arrow[d, "m\times 1"'] & P_1 \times_{(d,c)} P_1 \times_{(d,\pi)} A_0 \arrow[r, "1\times\xi"] & P_1 \times_{(d,\pi)} A_0 \arrow[d, "\xi"'] & A_0 \arrow[l, "\langle e\pi,1\rangle"'] \arrow[dl, equal] \\
P_1 \times_{(d,\pi)} A_0 \arrow[rr, "\xi"'] \arrow[d, "\mathrm{proj}_1"'] & & A_0 \arrow[dl, "\pi"] & \\
P_1 \arrow[r, "c"'] & P_0 & &
\end{tikzcd} \tag{3.69}$$

commutes. The category of P-actions will be denoted by $\mathbf{X}^P$.

Remark 3.74. When P is an internal category, the diagram 3.69 becomes

$$\begin{tikzcd}
P_1 \times_{(d,c)} P_1 \times_{(d,\pi)} A_0 \arrow[r, "1\times\xi"] \arrow[d, "m\times 1"'] & P_1 \times_{(d,\pi)} A_0 \arrow[d, "\xi"'] & A_0 \arrow[l, "\langle e\pi,1\rangle"'] \arrow[dl, equal] \\
P_1 \times_{(d,\pi)} A_0 \arrow[r, "\xi"'] \arrow[d, "\mathrm{proj}_1"'] & A_0 \arrow[d, "\pi"'] & \\
P_1 \arrow[r, "c"'] & P_0 &
\end{tikzcd} \tag{3.70}$$

This makes a P-action a special case of an M-action in the sense of Example 3.47. Specifically:

we take the monoidal category $\mathbf{C}$ of Example 3.47 to be $(\mathbf{Graphs}(\mathbf{X}, O), I, \otimes, \alpha, \lambda, \rho)$;

the role of $\mathbf{X}$ in Example 3.47 will be played by the comma category $(\mathbf{X} \downarrow P_0)$ (of pairs $A = (A_0, \pi)$, where $\pi \colon A_0 \to P_0$ is a morphism in $\mathbf{X}$);

the C-action on $(\mathbf{Graphs}(\mathbf{X}, O), I, \otimes, \alpha, \lambda, \rho)$ is defined in the obvious way using $P \otimes A = (P_1 \times_{(d,\pi)} A_0, c(\mathrm{proj}_1))$ defined via

$$\begin{tikzcd}
 & & P_1 \times_{(d,\pi)} A_0 \arrow[dl, "\mathrm{proj}_1"'] \arrow[dr, "\mathrm{proj}_2"] & \\
 & P_1 \arrow[dl, "c"'] \arrow[dr, "d"'] & & A_0 \arrow[dl, "\pi"] \\
P_0 & & P_0 &
\end{tikzcd} \tag{3.71}$$

then since P becomes a monoid in $(\mathbf{Graphs}(\mathbf{X},O),I,\otimes,\alpha,\lambda,\rho)$, we have the category $(\mathbf{X}\downarrow P_0)^P$ of P-actions in the sense of Example 3.47, and it coincides with the category $\mathbf{X}^P$ of P-actions in the sense of Definition 3.73.

We end this section with a natural (dual) internal-precategorical version of the results of Section 3.5 concerning discrete fibrations:

Theorem 3.75. *Let $P=(P_0,P_1,P_2,d,c,e,m)$ be an internal precategory in a category $\mathbf{X}$ with pullbacks and $A=(A_0,\pi,xi)$ a P-action. Then the diagram*

$$\begin{tikzcd}
P_2\times_{(dq,\pi)}A_0 \arrow[r, "m\times 1"] \arrow[r, bend left=40, "(1\times\xi)(\langle p,q\rangle\times 1)"] \arrow[r, bend right=40, "q\times 1"'] \arrow[d, "\mathrm{proj}_1"'] & P_1\times_{(d,\pi)}A_0 \arrow[r, bend left=40, "\mathrm{proj}_2"] \arrow[r, bend right=40, "\xi"'] \arrow[d, "\mathrm{proj}_1"'] & A_0 \arrow[l, "\langle e\pi,1\rangle"'] \arrow[d, "\pi"] \\
P_2 \arrow[r, "m"] \arrow[r, bend left=40, "p"] \arrow[r, bend right=40, "q"'] & P_1 \arrow[r, bend left=40, "d"] \arrow[r, bend right=40, "c"'] & P_0 \arrow[l, "e"']
\end{tikzcd} \tag{3.72}$$

is a discrete opfibration. Moreover, sending A to the so defined opfibration determines an equivalence between the category $\mathbf{X}^P$ of P-actions and the category $\mathbf{DisOpfib}(P)$ of discrete opfibrations over P (i.e. the category of discrete opfibrations $?\to P$ considered as a full subcategory of the comma category $((Precategories\ in\ \mathbf{X})\downarrow P)$).

Proof is a routine calculation.

3.13 Descent via monadicity and internal actions

In this section we develop a simplified approach to Grothendieck descent theory suitable for our purposes. Let $p\colon E\to B$ be a fixed morphism in a category $\mathbf{C}$ with pullbacks. Consider the diagram

$$\begin{tikzcd}
(\mathbf{C}\downarrow B) \arrow[rr, "K"] \arrow[dr, shift left, "p^*"] & & (\mathbf{C}\downarrow E)^T \arrow[dl, shift left, "U"] \\
& (\mathbf{C}\downarrow E) \arrow[ul, shift left, "p!"] \arrow[ur, shift left, "F"] &
\end{tikzcd} \tag{3.73}$$

in which:

- $p!$ is defined as the composition with p, i.e. by $p!(D,\delta)=(D,p\delta)$;
- $p*$ is the pullback-along-p (change-of-base functor determined by p), and we will write $p^*(A,\alpha)=(E\times_{(p,\alpha)}A,\mathrm{proj}_1)=(E\times_B A,\mathrm{proj}_1)$;
- it is to see that $p!$ is the left adjoint p^*, and T denotes the corresponding monad on $(\mathbf{C}\downarrow E)$;

- $(\mathbf{C}\downarrow E)^T$ is the category of T-algebras and U, F, and K the corresponding forgetful functor, free functor, and comparison functor respectively.

Explicitly:

- a T-algebra is a diagram

$$(D,\delta,\zeta) = \ E\times_{(p,p\delta)} D \xrightarrow{\zeta} D \xrightarrow{\delta} E, \tag{3.74}$$

  for which the diagram

$$\begin{tikzcd}
E\times_{(p,p)} E\times_{(p,p\delta)} D \arrow[r, "1\times\zeta"] \arrow[d, "\langle \mathrm{proj}_1{,}\mathrm{proj}_3\rangle"'] & E\times_{(p,p\delta)} D \arrow[d, "\zeta"'] & D \arrow[l, <->, "\langle\delta{,}1\rangle"'] \arrow[dl, equal] \\
E\times_{(p,p\delta)} D \arrow[r, "\zeta"] \arrow[d, "\mathrm{proj}_1"'] & D \arrow[dl, "\delta"] & \\
E & &
\end{tikzcd} \tag{3.75}$$

  commutes;

- the functor U is defined by $U(D,\delta,\zeta)=(D,\delta)$;

- the functor F is defined by $F(D,\delta) = (E\times_{(p,p\delta)} D, \mathrm{proj}_1, \langle \mathrm{proj}_1, \mathrm{proj}_3\rangle)$, where (here and below) proj_i $(i=1,2,3)$ are suitable projections;

- the functor K is defined by $K(A,\alpha) = (E\times_{(p,\alpha)} A, \mathrm{proj}_1, \langle \mathrm{proj}_1, \mathrm{proj}_3\rangle)$.

The diagrams (3.74) and (3.75) look almost similar to the diagrams (3.68) and (3.69) (see also (3.70)), and in fact they are special cases of those. For, let us take ($\mathbf{X}=\mathbf{C}$ and) P to be the internal category

$$\mathrm{Eq}(p) = \quad \begin{tikzcd}
E\times_B E\times_B E \arrow[r, bend left, "\langle \mathrm{proj}_1{,}\mathrm{proj}_2\rangle"] \arrow[r, "\langle \mathrm{proj}_1{,}\mathrm{proj}_3\rangle"] \arrow[r, bend right, "\langle \mathrm{proj}_2{,}\mathrm{proj}_3\rangle"'] \arrow[d, Rrightarrow, "\mathrm{proj}_1"'] & E\times_B E \arrow[r, bend left, "\mathrm{proj}_2"] \arrow[r, bend right, "\mathrm{proj}_1"'] \arrow[d, Rrightarrow, "\mathrm{proj}_1"'] & E \arrow[l, "\langle 1{,}1\rangle"'] \\
E\times_{(p,p)} E\times_{(p,p)} E & E\times_{(p,p)} E &
\end{tikzcd} \tag{3.76}$$

and write (D,δ) instead of (A_0,π) in (3.68) and (3.69). Then (3.68) becomes

$$(E\times_{(p,p)} E)\times_{(\mathrm{proj}_2,\delta)} D \xrightarrow{\xi} D \xrightarrow{\delta} E, \tag{3.77}$$

and a straightforward calculation proves:

Theorem 3.76.

For an object (D, δ) in $(\mathbf{C} \downarrow E)$, the morphism

$$\bar{\delta} = \langle \mathrm{proj}_1, \mathrm{proj}_3 \rangle \colon (E \times_{(p,p)} E) \times_{(\mathrm{proj}_2,\delta)} D \to E \times_{(\mathrm{proj}_2,\delta)} D \to E \times_{(p,p\delta)} D \tag{3.78}$$

is an isomorphism and (D, δ, ζ) is a T-algebra if and only if $(D, \delta, \zeta\bar{\delta})$ is an $\mathrm{Eq}(p)$-action. Moreover, sending (D, δ, ζ) to $(D, \delta, \zeta\bar{\delta})$ determines a category isomorphism

$$(\mathbf{C} \downarrow E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)}. \tag{3.79}$$

Remark 3.77.

1. As the notation obviously suggests, $\mathrm{Eq}(p)$ is nothing but the right internal version of the equivalence on E determined by p. Moreover, of course there are suitable notions of an internal groupoid, an internal preorder, an internal equivalence relation, and the opposite internal category to a given one, for which:

   - every internal groupoid is isomorphic to its opposite internal groupoid;
   - a morphism of internal groupoids is a discrete fibration if and only it is a discrete opfibration;
   - an internal preorder is the same as an internal category whose domain morphism and codomain morphism are jointly monic;
   - an internal equivalence relation is the same as an internal groupoid that is an internal preorder. In particular we do not need to be too careful in distinguishing $\mathrm{Eq}(p)$ from its opposite internal equivalence relation.

2. Every morphism $\varphi\colon \mathbf{P} \to \mathbf{P}'$ of internal precategories in $\mathbf{C}$ obviously determines an induced functor $\mathbf{C}^{\varphi} : \mathbf{C}^{P'} \to \mathbf{C}^{P}$, and this determines a pseudofunctor (where "pseudo" refers to preservation of composition and identities only up to "good" isomorphisms; omitting details let us just mention that this is similar to "preservation" of by monoidal functors)

   $$\mathbf{C}^{?} : \mathbf{Precat}(\mathbf{C})^{op} \to \mathbf{Cat}, \tag{3.80}$$

   where $\mathbf{Precat}(\mathbf{C})$ and $\mathbf{Cat}$ denote the category of internal precategories in $\mathbf{C}$ and the category of categories respectively. In particular applying this pseudofunctor to the commutative diagram

   $$\begin{tikzcd} \mathrm{Eq}(1_E) = E \arrow[rr, "p"] \arrow[dr] & & B = \mathrm{Eq}(1_B) \\ & \mathrm{Eq}(p) \arrow[ur] & \end{tikzcd} , \tag{3.81}$$

   (in the obvious notation) and identifying $\mathbf{C}^{\mathrm{Eq}(1_E)}$ and $\mathbf{C}^{\mathrm{Eq}(p)}$ with $(\mathbf{C} \downarrow E)$ and $(\mathbf{C} \downarrow B)$ respectively, we obtain a diagram

   $$\begin{tikzcd} (\mathbf{C} \downarrow B) \arrow[rr] \arrow[dr] & & (\mathbf{C} \downarrow E) \\ & \mathbf{C}^{\mathrm{Eq}(p)} \arrow[ur] & \end{tikzcd} \tag{3.82}$$

that can be identified with

$$\begin{tikzcd}
(\mathbf{C}\downarrow E) \arrow[rr] \arrow[dr] & & (\mathbf{C}\downarrow E) \\
& (\mathbf{C}\downarrow E)^T \arrow[ur] &
\end{tikzcd} \tag{3.83}$$

via the isomorphism (3.79).

Definition 3.78. *A morphism $p\colon E\to B$ in a category $\mathbf{C}$ with pullbacks is said to be:*

1. *a* descent morphism *if the comparison functor $K\colon(\mathbf{C}\downarrow B)\to(\mathbf{C}\downarrow E)^T$ is fully faithful;*

2. *an* effective descent morphism *if the functor $p*$ is monadic, i.e. if K is an equivalence of categories; in this and in the more general situation considered in later sections we will also say that $p\colon E\to B$ is a monadic extension.*

3.14 Galois structures and admissibility

Admissible Galois structures introduced in this section are the basic categorical structures for Galois theory in general categories.

Definition 3.79.

A *Galois structure* is a system $(\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\Phi)$, in which

$$(I,H,\eta,\varepsilon)\colon \mathbf{C}\to\mathbf{X} \tag{3.84}$$

is an adjunction, and $\mathbf{F}$ and Φ a class of morphisms in $\mathbf{C}$ and in $\mathbf{X}$ respectively, satisfying the following conditions:

1. $I(F)\subseteq\Phi$ and $H(\Phi)\subseteq\mathbf{F}$.

2. The category $\mathbf{C}$ admits pullbacks along morphisms from $\mathbf{F}$, and the class $\mathbf{F}$ is pullback stable; similarly, the category $\mathbf{X}$ admits pullbacks along morphisms from Φ, and the class Φ, is pullback stable. Furthermore, the classes $\mathbf{F}$ and Φ contain all isomorphisms in $\mathbf{C}$ and $\mathbf{X}$ respectively.

Given a Galois structure $\Gamma=(\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\Phi)$ and an object B in $\mathbf{C}$, there is an induced adjunction

$$(I^B,H^B,\eta^B,\varepsilon^B)\colon \mathbf{F}(B)\to\Phi(I(B)), \tag{3.85}$$

in which:

- $\mathbf{F}(B)$ is the full subcategory in $(\mathbf{C}\downarrow B)$ with objects all pairs (A,α) with $\alpha\colon A\to B$ in $\mathbf{F}$;

- similarly $\Phi(I(B))$ is the full subcategory in $(\mathbf{X}\downarrow I(B))$ with objects all pairs (X,φ) with $\varphi\colon X\to I(B)$ in Φ;

- $I^B(A,\alpha) = (I(A), I(\alpha))$;
- $H^B(X,\varphi) = (B \times_{HI(B)} H(X), \mathrm{proj}_1)$ is defined via the pullback

$$\begin{tikzcd} B \times_{HI(B)} H(X) \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"'] & H(X) \arrow[d, "H(\varphi)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd} \tag{3.86}$$

- $(\eta^B)(A,\alpha) = \langle \alpha, \eta_A \rangle \colon A \to B \times_{HI(B)} H(A)$;
- $(\varepsilon^B)(X,\varphi)$ is the composite

$$I(B \times_{HI(B)} H(X)) \xrightarrow{I(\mathrm{proj}_2)} IH(X) \xrightarrow{\varepsilon_X} X \tag{3.87}$$

where proj_2 is as in (3.14).

Using the notation above we introduce

Definition 3.80. *An object B in $\mathbf{C}$ is said to be* admissible *if $B\colon I^B H^B \to 1_{\Phi(I(B))}$ is an isomorphism. If this is the case for each B in $\mathbf{C}$, then we say that the Galois structure $\Gamma = (\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \Phi)$ is* admissible.

Obvious but important:

Proposition 3.81. *If $\varepsilon\colon IH \to 1_{\mathbf{X}}$ is an isomorphism, then the following conditions on an object B in $\mathbf{C}$ are equivalent:*

1. *B is admissible;*
2. *the functor $H^B\colon \Phi(I(B)) \to \mathbf{F}(B)$ is fully faithful;*
3. *the functor I preserves all pullbacks of the form 3.14.*

Remark 3.82. From now on $\Gamma = (\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ will denote a fixed admissible Galois structure in which $\varepsilon\colon IH \to 1_{\mathbf{X}}$ is an isomorphism, and so the equivalent conditions of Proposition 3.81 hold.

More precisely, we will freely use this convention in Sections 3.15 and 3.20, and it will hold true in all examples of Sections 3.16-3.19, which we will prove there.

3.15 Monadic extensions and coverings

In this section we introduce the main notions of categorical Galois theory (using convention from Remark 3.82). Given a morphism $p\colon E \to B$ in $\mathbf{C}$, pulling back along p determines a functor

$$p{*}\colon \mathbf{F}(B) \to \mathbf{F}(E), \tag{3.88}$$

and the composition with p determines a functor

$$p!\colon \mathbf{F}(E) \to \mathbf{F}(B), \tag{3.89}$$

which is the left adjoint of $p{*}$.

Definition 3.83. *A pair (E, p), in which $p\colon E \to B$ is morphism in* $\mathbf{C}$*, or a morphism $p\colon E \to B$ itself, is said to be a* monadic extension *of B if the following conditions hold:*

1. *If (D, δ) is in $\mathbf{F}(E)$, then $(D, p\delta)$ is in $\mathbf{F}(B)$;*
2. *the functor $p*\colon \mathbf{F}(B) \to \mathbf{F}(E)$ is monadic.*

We are now ready to introduce our main definition:

Definition 3.84.
1. *An object (A, α) in $\mathbf{F}(B)$ is said to be a trivial covering (of B) if the morphism $(\eta_B)_{(A,\alpha)}\colon (A, \alpha) \to H^B I^B(A, \alpha)$ is an isomorphism, or, equivalently, the diagram*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\alpha"'] & HI(A) \arrow[d, "HI(\alpha)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd} \tag{3.90}$$

is a pullback.

2. *An object (A, α) in $\mathbf{F}(B)$ is said to be* split *over a monadic extension (E, p) of B if $p*(A, \alpha)$ is a trivial covering.*
3. *An object (A, α) in $\mathbf{F}(B)$ is said to be a* covering *of B if there exists a monadic extension (E, p) of B such that (A, α) is split over (E, p). We will then also say that $W\colon \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ is a covering morphism.*

According to this definition we have

$$\mathbf{TrivCov}(B) = \mathbf{Spl}(B, 1_B) \subseteq \mathbf{Cov}(B) = \bigcup_{(E,p)} \mathbf{Spl}(E, p) \subseteq \mathbf{F}(B), \tag{3.91}$$

where:

- $\mathbf{TrivCov}(B)$ is the full subcategory in $\mathbf{F}(B)$ with objects all trivial coverings of B;
- $\mathbf{Spl}(E, p)$ is the full subcategory in $\mathbf{F}(B)$ of all objects split over (E, p);
- $\mathbf{Cov}(B)$ is the full subcategory in $\mathbf{F}(B)$ with objects all coverings of B;
- the union of $\mathbf{Spl}(E, p)$'s in 3.91 is taken over all monadic extensions (E, p) of B.

Remark 3.85. The following simple properties of coverings are useful:

1. Since $\varepsilon^B\colon I^B H^B \to 1_{\Phi(I(B))}$ is always an isomorphism, an object (A, α) in$\mathbf{F}(B)$ is a trivial covering if and only if $(A, \alpha) \approx H^B(X, \varphi)$ for some (X, φ) in $\Phi(I(B))$.

2. For every morphism $\beta\colon B' \to B$, the functor, $\beta^*\colon \mathbf{F}(B) \to \mathbf{F}(B')$ sends trivial coverings to trivial coverings, and the functor I preserves pullbacks along trivial coverings. To see this, consider the cube diagram

$$\begin{tikzcd}
A' \arrow[rr] \arrow[dr] \arrow[dd] & & H(X) \arrow[dr] \arrow[dd] & \\
& A \arrow[rr] \arrow[dd] & & HI(A) \arrow[dd] \\
B' \arrow[rr] \arrow[dr] & & HI(B) \arrow[dr] & \\
& B \arrow[rr] & & HI(B),
\end{tikzcd} \tag{3.92}$$

where the left-hand face is a pullback, $A \to B$ is a trivial covering, the right-hand face is the H-image of the pullback formed by the I-images of $B' \to B$ and $A \to B$, and the arrows connecting the left-hand and right-hand faces are canonical morphisms. In this diagram all vertical faces are pullbacks, and, by the admissibility, X can be identified with $I(A')$. This implies our assertions above.

3. As follows from (b), for monadic extensions (E, p) and (E', p'), we have $\mathbf{Spl}(E, p) \subseteq \mathbf{Spl}(E', p')$ whenever p' factors through p.

4. Using some further arguments one can show that the union in 3.91 is in fact directed.

3.16 Categories of abstract families

In this section we present an example of an admissible Galois structure, which will later help us to present the classical Galois theory as a special case of the categorical one. We take $\mathbf{X}$ to be a full subcategory of the category of sets, closed under finite limits, and $\mathbf{A}$ an arbitrary category that has a terminal object 1.

Definition 3.86. *The category* $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ *of* families of objects *in* $\mathbf{A}$ *with index sets in* $\mathbf{X}$ *has:*

1. *its objects all families $A = (A_i)_{i \in I(A)}$ of objects A_i in $\mathbf{A}$ with $I(A)$ in $\mathbf{X}$;*

2. *a morphism $A \to B$ in $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ is a pair (f, α), in which $f\colon I(A) \to I(B)$ is a map of sets and is a family of morphisms $\alpha = (\alpha_i\colon A_i \to B_{f(i)})_{i \in I(A)}$ in $\mathbf{A}$.*

Sending $(f, \alpha)\colon \mathbf{A} \to \mathbf{B}$ to $f\colon I(A) \to I(B)$ determines a functor $I\colon \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}) \to \mathbf{X}$, with the right adjoint H defined by $H(X) = (A_i)_{i \in I(A)}$, where $I(A) = X$ and $A_i = 1$ for all i. This can easily be checked either directly, or using the following obvious facts

- Sending A to $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ determines a 2-functor

$$\mathrm{Fam}_{\mathbf{X}}\colon \mathbf{Cat} \to \mathbf{Cat}, \tag{3.93}$$

  where **Cat** is the 2-category of all categories.

- $\mathrm{Fam}_{\mathbf{X}}(\mathbf{1})$ is canonically isomorphic to $\mathbf{X}$, where $\mathbf{1}$ denotes a (the) one-morphism category (=the terminal object in the category of all categories).

- The unique functor $A \to \mathbf{1}$ has the right adjoint sending the unique object of $\mathbf{1}$ to the terminal $\mathbf{1}$ object in $\mathbf{A}$, and the functor $I\colon \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}) \to \mathbf{X}$ above is nothing but the composite $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A}) \to \mathrm{Fam}_{\mathbf{X}}(\mathbf{1}) \to \mathbf{X}$.

It is then easy to prove:

Theorem 3.87. *Let $\Gamma = (\mathrm{Fam}_{\mathbf{X}}(\mathbf{A}), \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ be a Galois structure, in which $I\colon \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}) \to \mathbf{X}$ and $H\colon \mathbf{X} \to \mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ are as above, with suitable η and ε, Φ the class of all morphisms in $\mathbf{X}$, and $\mathbf{F}$ an arbitrary class of morphisms in $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ containing $H(\Phi)$ and satisfying condition (b) in Definition 3.79. Then $\varepsilon\colon IH \to 1_{\mathbf{X}}$ is an isomorphism and Γ is admissible.*

Theorem 3.88. *Let $\Gamma = (\mathrm{Fam}_{\mathbf{X}}(\mathbf{A}), \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ be as in Theorem 3.87, and $(f, \alpha)\colon \mathbf{A} \to \mathbf{B}$ be in $\mathbf{F}$. Then $(A, (f, \alpha))$ is a trivial covering of $\mathbf{B}$ if and only if $i\colon A_i \to B_{f(i)}$ is an isomorphism for each $i \in I(A)$.*

3.17 Coverings in classical Galois theory

In this section we describe the relationship between the separable/Galois extensions in classical Galois theory and covering morphisms of categorical Galois theory.

Here, K denotes a field, $\mathbf{C}$ the opposite category of commutative unitary K-algebras that are finite-dimensional as K-vector spaces, and $\mathbf{X}$ the category of finite sets. We define here $I\colon \mathbf{C} \to \mathbf{X}$ by

$$I(A) = \text{ the set of minimal (non-zero) idempotents in } \mathbf{A}; \tag{3.94}$$

that is $I(A)$ consists of all $e \in A$ such that $e^2 = e \neq 0$ and $e'^2 = e' \neq 0 \neq ee'$ implies $ee' = e$. Sending A to the family $(A_e)_{e \in I(A)}$ determines a category equivalence

$$\mathbf{C} \sim \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}), \tag{3.95}$$

where $\mathbf{A}$ is the full subcategory in $\mathbf{C}$ with objects all (commutative unitary) K-algebras with no non-trivial idempotents, i.e. no elements e with $e^2 = e$ and $0 \neq e \neq 1$. Moreover, the functor $I\colon \mathbf{C} \to \mathbf{X}$ above is a special case of the one defined in the previous section up to the equivalence 3.95. Using this fact and Theorem 3.87 we obtain:

Theorem 3.89. *Let $I\colon \mathbf{C} \to \mathbf{X}$ be as above, $H\colon \mathbf{X} \to \mathbf{C}$ the right adjoint of I defined therefore by*

$$\begin{aligned} H(X) &= \underbrace{K \times \ldots \times K}_{\text{coproduct in } C \text{ of } K \text{ with itself ``}X\text{-times''}} \\ &= \text{the } K\text{-algebra of all maps from } X \text{ to } K, \end{aligned} \tag{3.96}$$

η and ε the unit and counit of adjunction, and $\mathbf{F}$ and Φ the classes of all morphisms in $\mathbf{C}$ and in $\mathbf{X}$ respectively. Then $\varepsilon\colon IH \to 1_{\mathbf{X}}$ is an isomorphism and $\Gamma = (\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ is an admissible Galois structure.

Next, using Theorem 3.63, we easily prove:

Theorem 3.90. *A morphism $p\colon E \to B$ in $\mathbf{C}$, in which B is a field, is a monadic extension if and only if E is a non-zero ring. In particular this is the case whenever E is a field.*

Proof. The functor $p*\colon \mathbf{F}(B) \to \mathbf{F}(E)$, whose monadicity we have to prove for a non-zero E, is the same as the functor $E \otimes_B (-)$:

$$(\text{Commutative unitary } B\text{-algebras})^{op} \to (\text{Commutative unitary E-algebras})^{op}. \tag{3.97}$$

According to Theorem 3.63 it suffices to prove that this functor reflects isomorphisms and preserves coequalizers. Moreover, since the coequalizers in the categories involved are the same as equalizers of algebras, and since those are calculated via the corresponding equalizers of underlying modules, we only need to prove that the functor

$$E \otimes_B (-) : B\text{-modules} \to E\text{-modules} \tag{3.98}$$

reflects isomorphisms and is (left) exact, which is obvious since B is a field. □

Now we are ready to prove:

Theorem 3.91. *Let $K \subseteq B \subseteq E$ be finite (=finite-dimensional over B) field extensions and $A = (A, \alpha)$ a B-algebra (in particular $p\colon B \to A$ is a ring homomorphism and B acts on A via $ba = \alpha(b)a$. Out of the following three conditions, the first two are always equivalent, and the third always follows from them and implies them when $B \subseteq E$ is a Galois extension:*

1. *(A, α) belongs to $\mathbf{Spl}(E, p)$ (where $\mathbf{Spl}(E, p)$ is defined with respect to the Galois structure described in Theorem 3.89) with p being the inclusion map $B \to E$ considered as a morphism $E \to B$ in $\mathbf{C}$;*

2. *$E \otimes_B A \approx E \times \ldots \times E$ (a finite product of K-algebras = a finite coproduct in $\mathbf{C}$);*

3. *$A \approx E_1 \times \ldots \times E_n$ for some natural n (0 is not excluded), where $B \subseteq E_i \subseteq E$ $(i = 1, \ldots, n)$ (and therefore $E_1, \ldots, E_n$ are field extensions of B).*

Proof.

$(a) \Leftrightarrow (b)$ easily follows, using the equivalence 3.95, from Theorem 3.91 and the fact that $E \otimes_B A$ considered as an object in $\mathbf{C}$ is the same as $p*(A, \alpha)$.

$(b) \Leftrightarrow (c)$ ($(b) \implies (c)$ always, and $(c) \implies (b)$ when $B \subseteq E$ is a Galois extension) is well known in classical algebra, and we only sketch the proof here:

$(b) \implies (c)$ (b) implies that A has no nilpotent elements. Therefore $A \approx E_1 \times \ldots \times E_n$ as B-algebras, for some field extensions $E_1, \ldots, E_n$ of B, say, by the Wedderburn Theorem. After that in order to show that $E_1, \ldots, E_n$ can be chosen among the subextensions of $B \subseteq E$, it suffices to show that each of $E_1, \ldots, E_n$ admits a B-algebra homomorphism into E. This, however, immediately follows from $E \otimes_B A \approx E \times \ldots \times E$ and $A \approx E_1 \times \ldots \times E_n$.

$(c) \implies (b)$ when $B \subseteq E$ is a Galois extension: Since a finite product of B-algebras satisfying (b) itself obviously satisfies (b), we can assume from the beginning that A is a B-subalgebra in E. Moreover, since $B \subseteq E$ is a Galois extension, there is a polynomial $u \in B[x]$ that splits into linear factors $u = \prod_{i=1}^{m}(x - a_i)$ with $a_i = a_j \implies i = j$, and has $B[x]/uB[x]$. Therefore

$$\begin{aligned} E \otimes_B A \approx E \otimes_B (B[x]/uB[x]) &\approx E[x]/uE[x] \\ &\approx E[x]/(\prod_{i=1}^{m}(x - a_i)) \\ &\approx \prod_{i=1}^{m}(E[x]/(x - a_i)E[x]) \\ &\approx E \times \ldots \times E \quad (m \text{ times}), \end{aligned}$$

as desired.

$\square$

In fact the connection with classical Galois theory goes much further, and provides categorical proofs for many of its results. Let us mention just two of them that are "almost corollaries" of Theorem 3.91:

Theorem 3.92. *Let $K \subseteq B \subseteq E$ be finite field extensions and p the inclusion map $B \subseteq E$ considered as a morphism $E \to B$ in $\mathbf{C}$. Then the following conditions are equivalent:*

1. *(E, p) belongs to $\mathbf{Spl}(E, p)$;*
2. *$B \subseteq E$ is a Galois extension.*

Theorem 3.93. *Let $K \subseteq B$ be a finite field extensions and $A = (A, \alpha)$ a B-algebra as above. Then the following conditions are equivalent:*

1. *(A, α) is a covering of B;*
2. *there exists a finite field extension $B \subseteq E$, such that (A, α) belongs to $\mathbf{Spl}(E, p)$, where $\mathbf{Spl}(E, p)$ is as in Theorem **??**;*
3. *there exists a finite Galois field extension $B \subseteq E$, such that (A, α) belongs to $\mathbf{Spl}(E, p)$, where $\mathbf{Spl}(E, p)$ is as in Theorem **??**;*
4. *$A = (A, \alpha)$ is a commutative separable B-algebra;*
5. *$A = (A, \alpha)$ is a finite product of finite separable field extensions of B.*

3.18 Covering spaces in algebraic topology

The purpose of this section is to present classical covering maps of locally connected topological spaces as covering morphisms in the sense of categorical Galois theory.

Therefore we take here $\mathbf{C}$ to be the category of locally connected topological spaces and $\mathbf{X}$ the category of sets. And we define the functor $I \colon \mathbf{C} \to \mathbf{X}$ by

$$I(A) = \pi_0(A) \quad (\text{the set of connected components of } A). \tag{3.99}$$

Sending spaces to the families of their connected components determines a category equivalence

$$\mathbf{C} \sim \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}), \tag{3.100}$$

where $\mathbf{A}$ is the category of connected locally connected topological spaces. Moreover, the functor $I\colon \mathbf{C} \to \mathbf{X}$ above is a special case of the one defined in Section 3.16. Using this fact and Theorem 3.87 we easily obtain:

Theorem 3.94. *Let $I\colon \mathbf{C} \to \mathbf{X}$ be as above, $H\colon \mathbf{X} \to \mathbf{C}$ the inclusion functor, η and ε the unit and counit of adjunction, $F = \mathbf{Etale}$ the class of local homeomorphisms (=etale maps) of locally connected topological spaces, and the class of all morphisms in $\mathbf{X}$. Then $\varepsilon\colon IH \to 1_{\mathbf{X}}$ is an isomorphism and $\Gamma = (\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ is an admissible Galois structure.*

A brief story of monadic extensions and coverings with respect to this Galois structure is:

Theorem 3.95. *A morphism $p\colon E \to B$ in $\mathbf{C}$ is a monadic extension if and only if it is a surjective local homeomorphism.*

Proof. "If": Assuming that $p\colon E \to B$ is a surjective local homeomorphism, we have to prove that the functor 3.88, which we write here as

$$p{*}\colon \mathbf{Etale}(B) \to \mathbf{Etale}(E), \tag{3.101}$$

is monadic. We observe:

1. Since the class of local homeomorphisms is closed under composition the functor 3.101 has a left adjoint.

2. A morphism $f\colon (A, \alpha) \to (A', \alpha')$ in $\mathbf{Etale}(B)$ is an isomorphism if and only if the map $f\colon \mathbf{A} \to \mathbf{A}'$ is bijective; this easily implies that, for a surjective p, the functor 3.101 reflects isomorphisms.

3. When $\alpha\colon A \to B$ is a local homeomorphism, the local connectedness of B implies the local connectedness of A. Therefore $\mathbf{Etale}(B)$ can be identified, up to a category equivalence, with the topos of sheaves (of sets) over the space B. The same is true for E, and the functor 3.101 can be identified with the inverse image functor

$$p{*}\colon \mathbf{Shv}(B) \to \mathbf{Shv}(E) \tag{3.102}$$

between the toposes of sheaves. Since the functor 3.102 has a (well-known) right adjoint, namely the direct image functor

$$p{*}\colon \mathbf{Shv}(E) \to \mathbf{Shv}(B), \tag{3.103}$$

we conclude that it preserves all coequalizers. Indeed, it is easy to show that any left adjoint functor preserves all colimits, and in particular all coequalizers.

4. The desired monadicity follows from (1), (2), (3), and Theorem 3.63.

"Only if": When (E, p) is a monadic extension, p must be a local homeomorphism by Definition 3.83(a) (applied to $\delta = 1_E$). Therefore we only need to prove that p is surjective. For, consider the objects $(B, 1_B)$ and $(p(E), \text{inclusion})$ in $\mathbf{F}(B) = \mathbf{Etale}(B)$; note that $(p(E), \text{inclusion})$ is indeed in $\mathbf{F}(B)$ since p is open. Since the functor $p*$ reflects isomorphisms and sends the canonical map $(p(E), \text{inclusion}) \to (B, 1_B)$ to an isomorphism, we must have $p(E) = B$, as desired. □

Lemma 3.96. *Suppose B (in $\mathbf{C}$) is connected. Then the following conditions on an object (A, α) in $\mathbf{F}(B) = \mathbf{Etale}(B)$ are equivalent:*

1. *(A, α) is a trivial covering of B (in the sense of Definition 3.84(a));*
2. *A is a disjoint union of open subsets, each of which is mapped homeomorphically on B by .*

Proof. This is an easy corollary of Theorem 3.88. . □

Theorem 3.97. *The following conditions on an object (A, α) in $\mathbf{F}(B) = \mathbf{Etale}(B)$ are equivalent:*

1. *(A, α) is a covering of B (in the sense of Definition 3.84(c));*
2. *every element b in $\mathbf{B}$ has an open neighbourhood U for which the pair*
$$(\alpha^{-1}(U), \ \textit{the map } \alpha^{-1}(U) \to U \ \textit{induced by } \alpha) \tag{3.104}$$
is a trivial covering of U (in the sense of Definition 3.84(a));
3. *the same as (b), but with U required to be connected;*
4. *(A, α) is a covering space over B in the classical sense, i.e. every element in $\mathbf{B}$ has an open neighbourhood whose inverse image is a disjoint union of open subsets, each of which is mapped homeomorphically on it by α.*

Proof.

$(a) \implies (b)$ easily follows from the "only if" part of Theorem 3.95, and $(b) \implies (a)$ can easily be deduced from the same theorem and the following simple observation:

For each b in $\mathbf{B}$, let U_b be a chosen open neighbourhood of b, let E be the topological coproduct of all these neighbourhoods, and let $p\colon E \to B$ be the map induced by the family of inclusion maps $U_b \to B$ (for all b in $\mathbf{B}$). Then p is a local homeomorphism.

$(b) \implies (c)$ follows from the local connectedness of B and $(c) \implies (b)$ is trivial.

(d) follows from the local connectedness of B and Lemma 3.96. .

□

3.19 Central extensions of groups

The purpose of this section is to present central extensions of groups as covering morphisms in the sense of categorical Galois theory.

Accordingly $\mathbf{C}$ will denote now the category of groups, $\mathbf{X}$ the category of abelian groups, and $I\colon \mathbf{C} \to \mathbf{X}$ the left adjoint of the inclusion $\mathbf{X} \to \mathbf{C}$, which will plays the role of H. That is:

- From the viewpoint of universal algebra I is the abelianization functor sending groups to their quotients determined by the identity $xy = yx$, we could write

$$I(A) = A/R, \text{ where } R \text{ is the congruence generated by } \{(a,b) \in A\times A \mid ab = ba\}. \tag{3.105}$$

- From the viewpoint of group theory I is to be defined by

$$I(A) = A/[A, A], \tag{3.106}$$

where [A,A] is the commutator of A with itself.

- From the viewpoint of homological algebra I is to be defined by

$$I(A) = \mathrm{H}_1(A; \mathbb{Z}), \tag{3.107}$$

where $\mathrm{H}_1(A; \mathbb{Z})$ is the first homology group of A with coefficients in the additive group of integers, on which A acts trivially.

The Galois structure $\Gamma = (\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ that we fix in this section will have $\mathbf{C}$, $\mathbf{X}$, I, H as above, with the canonical η and ε, and F and Φ being the classes of surjective homomorphisms of groups and of abelian groups respectively. The morphism $\varepsilon\colon IH \to 1_{\mathbf{X}}$ is obviously an isomorphism here, but the admissibility needs a little proof:

Theorem 3.98. $\Gamma = (\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ *is admissible.*

Proof. Consider the pullback 3.86, which now becomes

$$\begin{tikzcd} B \times_{B/[B,B]} X \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"'] & X \arrow[d, "\varphi"] \\ B \arrow[r, "\eta_B"'] & B/[B,B] \end{tikzcd} \tag{3.108}$$

We need to prove that $\mathrm{proj}_2\colon B \times_{B/[B,B]} X \to X$ has the universal property of the abelianization of $B \times_{B/[B,B]} X$, or, equivalently, that the kernel $\ker(\mathrm{proj}_2)$ of this morphism is contained in $[B \times_{B/[B,B]} X, B \times_{B/[B,B]} X]$. We observe that any element k in $\ker(\mathrm{proj}_2)$ is of the form $k = (b, 1)$, where b is in $[B, B]$, and so we can present it as

$$k = ([b_1, b_1'] \dots [b_n, b_n'], 1) = ([b_1, b_1'], 1) \dots ([b_n, b_n'], 1). \tag{3.109}$$

Since φ is surjective, there exist $x_1, \dots, x_n, x_1', \dots, x_n'$ in $\mathbf{X}$ with $\varphi(x_1) = b_1[B, B]$, $\dots$, $\varphi(x_n) = b_n[B, B]$, $\varphi(x_1') = b_1'[B, B]$, $\dots$, $\varphi(x_n') = b_n'[B, B]$; and since X is abelian, we have $[x_1, x_1'] = \dots = [x_n, x_n'] = 1$. Therefore

$$k = ([b_1, b_1'], [x_1, x_1']) \dots ([b_n, b_n'], [x_n, x_n']) = [(b_1, x_1), (b_1', x_1')] \dots [(b_n, x_n), (b_n', x_n')], \tag{3.110}$$

which shows that k is in $[B \times_{B/[B,B]} X, B \times_{B/[B,B]} X]$, as desired. $\square$

Remark 3.99. The surjectivity of φ played a crucial role in the proof of Theorem 3.58. Indeed, taking $X = 0$ in 3.51, we would obtain $\ker(\mathrm{proj}_2) \to [B, B]$, but at the same time $[B \times_{B/[B,B]} X, B \times_{B/[B,B]} X] \approx [[B, B], [B, B]]$ (canonically).

Next, the monadic extensions:

Theorem 3.100. *A morphism $p \colon E \to B$ in $\mathbf{C}$ is a monadic extension if and only if it is surjective.*

Proof. "If": According to Remark 3.64(b), it suffices to prove that, for a surjective p, the functor $p* : \mathbf{F}(B) \to F(E)$ reflects isomorphisms and preserves coequalizers of reflexive pairs. However, it is an easy exercise to show that the coequalizers of reflexive pairs of group homomorphisms are calculated as in the category of sets - which reduces the problem to the case of sets, where the proof becomes another easy exercise.

The "only if" part follows from Definition 3.83(a) (applied to $\delta = 1_E$). □

In order to characterize coverings we will also need the following almost obvious fact:

Lemma 3.101. *For a pullback diagram*

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "v"'] & B \end{tikzcd} \tag{3.111}$$

with α and δ surjective, the conditions (a) and (b) below are related as follows: (a) always implies (b), and (b) implies (a) whenever v is surjective.

1. *(A, α) is a central extension of B (i.e. $ka = ak$ for all k in $\ker(\alpha)$ and all a in $\mathbf{A}$);*

2. *(D, δ) is a central extension of E.*

- after which we are ready to prove

Theorem 3.102. *The following conditions on an object (A, α) in $\mathbf{F}(B)$ are equivalent:*

1. *(A, α) is a covering of B;*

2. *(A, α) is a central extension of B.*

Proof.

$(a) \implies (b)$ follows from (the "only if" part of) Theorem 3.100 and Lemma 3.101.

$(b) \implies (a)$ As follows from the "if" part of Theorem 3.100, (A, α) is a monadic extension of B. Consider the object

$$\alpha^*(A, \alpha) = (A \times_B A, \mathrm{proj}_1) \tag{3.112}$$

in $\mathbf{F}(A)$. It has $\ker(\mathrm{proj}_1)$ canonically isomorphic to $\ker(\alpha)$, and proj_1 is a split epimorphism. Being central by Lemma 3.101 $(a) \implies (b)$, it is therefore isomorphic to

$$\alpha^*(A,\alpha) = (A \times \ker(\alpha), \text{ the first projection}), \tag{3.113}$$

after which we only need to observe:

The object 3.113 is a trivial covering of A since $(\ker(\alpha), \ker(\alpha) \to 0)$ is a trivial covering of 0, and the class of trivial coverings is pullback stable by Remark 3.85(b).

□

3.20 The fundamental theorem of Galois theory

In this section we formulate and prove the fundamental theorem of categorical Galois theory.

Let $\Gamma = (\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ be a fixed abstract Galois structure satisfying convention from Remark 3.82. We begin by considering various induced adjunctions:

We can obviously look at the category of internal precategories in $\mathbf{C}$ as the functor category $\mathbf{C}^\tau$, where τ is the free category determined by the graph

$$\begin{tikzcd} 2 \arrow[r, "m"] \arrow[r, bend left=60, "p"] \arrow[r, bend right=60, "q"'] & 1 & \arrow[l, "e"'] \arrow[l, bend right=60, "d"', leftarrow] \arrow[l, bend left=60, "c", leftarrow] \end{tikzcd} \tag{3.114}$$

and the identities $de = 1 = ce$, $dp = cq$, $dm = dq$, and $cm = cp$ as in Definition 3.65. And then the category of internal precategories in $\mathbf{C}$ becomes nothing but the functor category $\mathbf{C}^\tau$. Our adjoint functors between $\mathbf{C}$ and $\mathbf{X}$ induce adjoint functors between $\mathbf{C}$ and $\mathbf{X}$, which we will display as

$$(I^\tau, H^\tau, \eta^\tau, \varepsilon^\tau)\colon \mathbf{C}^\tau \to \mathbf{X}^\tau. \tag{3.115}$$

Using also F^τ = the class of all κ in $\mathbf{C}^\tau$ with κ_0, κ_1, and κ_2 in $\mathbf{F}$, and the similarly defined ,,, we obtain the induced Galois structure

$$\Gamma^\tau = (\mathbf{C}^\tau, \mathbf{X}^\tau, I^\tau, H^\tau, \eta^\tau, \varepsilon^\tau, \mathbf{F}^\tau, \mathbf{\Phi}^\tau) \tag{3.116}$$

for internal precategories. After that we take an object P in $\mathbf{C}^\tau$, and construct a further induced adjunction in the same way as the adjunction 3.85 was constructed out of an object B in $\mathbf{C}$; we display it as

$$(I^P, H^P, \eta^P, \varepsilon^P)\colon \mathbf{F}^\tau(P) \to \Phi^\tau(IP). \tag{3.117}$$

where we write IP instead of $I^\tau(P)$, since $I^\tau(P)$ is nothing but the composite of $P\colon \tau \to \mathbf{C}$ with $I\colon \mathbf{C} \to \mathbf{X}$. From Remark 3.85(b) we obtain:

Lemma 3.103. *If (Q,κ) is a discrete opfibration over P, in which ,0 is a trivial covering, then 1 and 2 also are trivial coverings, and $I^P(Q,\kappa)$ is a discrete opfibration over $I^\tau(P)$.*

Corollary 3.104. *The adjunction 3.117 induces an equivalence between:*

1. *the full subcategory in $\mathbf{F}^{\tau}(P)$ with objects all (Q,κ) that are discrete opfibrations with κ_0 being a trivial covering, and*
2. *the full subcategory in $\Phi^{\tau}(I^{\tau}(P))$ with objects all objects in it that are discrete opfibrations.*

Identifying now discrete opfibrations with actions (see Theorem 3.75), we obtain

Theorem 3.105. *The adjunction 3.117 induces an equivalence between:*

1. *the full subcategory $\mathbf{Triv}(\mathbf{C}^P)$ in $\mathbf{C}^P$ with objects all $A = (A_0, \pi, \zeta)$ in $\mathbf{C}^P$, in which is a trivial covering;*
2. *the full subcategory $\mathbf{X}^{IP} \cap \Phi$ in $\mathbf{X}^{IP}$ with objects all $X = (X_0, \pi, \xi)$ in $\mathbf{X}^P$, in which π, is in Φ.*

- after which we are ready to prove:

Theorem 3.106 ("The fundamental theorem of Galois theory)**.** *Let $\Gamma = (\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{\Phi})$ be a fixed abstract Galois structure satisfying convention from Remark 3.82 as above, and let (E,p) be a monadic extension of an object B in $\mathbf{C}$. Then sending an object (A,α) in $\mathbf{Spl}(E,p)$ to the triple $(I(E \times_B A), I(\mathrm{proj}_1), I(\mathrm{proj}_1) \times I(\mathrm{proj}_2))$, determines a category equivalence*

$$\mathbf{Spl}(E,p) \to \mathbf{X}^{I(Eq(p))} \cap \Phi \tag{3.118}$$

(denoting by proj_i $(i = 1,2)$ suitable projections, in particular using proj_1 for both $E \times_B A \to E$ and $E \times_B E \to E$, and using the notation of Theorem 3.105 for $P = \mathrm{Eq}(p)$).

Proof. All we need is to consider the diagram

$$\begin{tikzcd} \mathbf{F}(B) \arrow[r] & \mathbf{F}(E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)} \cap \mathbf{F} \\ \mathbf{Spl}(E,p) \arrow[u] \arrow[r, dashed] & \mathbf{Triv}(\mathbf{C}^{\mathrm{Eq}(p)}) \sim \mathbf{X}^{I(\mathrm{Eq}(p))} \cap \Phi \arrow[u] \end{tikzcd} \tag{3.119}$$

in which:

1. T is the monad determined by the monadic functor $p^* : \mathbf{F}(B) \to \mathbf{F}(E)$, and $\mathbf{F}(B) \to \mathbf{F}(E)^T$ is the comparison functor, which is a category equivalence since p^* is monadic.
2. $\mathbf{F}(E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)} \cap \mathbf{F}$ is the isomorphism established in the same way as the isomorphism 3.79 in Theorem 3.6. It therefore sends a T-algebra (D, δ, ζ) to the triple $(D, \delta, \zeta\bar{\delta})$, where $\delta = \langle \mathrm{proj}_1, \mathrm{proj}_3 \rangle \colon (E \times_{(p,p)} \times E) \times_{(\mathrm{proj}_2, \delta)} D \to E \times_{(p, p\delta)} D$ as in Theorem 3.76.
3. Calculating the composite $\mathbf{F}(B) \to \mathbf{F}(E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)} \cap \mathbf{F}$ we easily conclude that it sends an object (A, α) to the triple $(E \times_B A, \mathrm{proj}_1, \mathrm{proj}_1 \times \mathrm{proj}_2)$, where proj_i $(i = 1,2)$ are the same as in the formulation of the theorem.

4. The vertical arrows are the inclusion functors.

5. (A, α) belongs to $\mathbf{Spl}(E, p)$ exactly when $(E \times_B A, \mathrm{proj}_1)$ is a trivial covering. Therefore (3) tells us that the composite $\mathbf{F}(B) \to \mathbf{F}(E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)} \cap \mathbf{F}$ determines the dotted arrow in 3.62, and that that arrow is an equivalence of categories.

6. $\mathbf{Triv}(\mathbf{C}^{\mathrm{Eq}(p)}) \sim \mathbf{X}^{I(\mathrm{Eq}(p))} \cap \Phi$ is the equivalence described in Theorem **??** (for $P = \mathrm{Eq}(p)$).

7. The desired equivalence is the composite of the equivalences $\mathbf{Spl}(E, p) \sim \mathbf{Triv}(\mathbf{C}^{\mathrm{Eq}(p)})$ and $\mathbf{Triv}(\mathbf{C}^{\mathrm{Eq}(p)}) \sim \mathbf{X}^{I(\mathrm{Eq}(p))} \cap \Phi$.

□

Remark 3.107.

1. According to this theorem it is good to write

$$\mathrm{Gal}(E, p) = I(\mathrm{Eq}(p)), \tag{3.120}$$

and call this internal precategory the Galois pregroupoid of the monadic extension (E, p). Here "pregroupoid" (rather than "precategory") refers to a certain extra structure, that makes $I(\mathrm{Eq}(p))$ a groupoid whenever it is a category. And in fact it is a groupoid whenever (E, p) is normal, which means that (E, p) belongs to $\mathbf{Spl}(E, p)$. Other reasonable synonyms of "normal" are *Galois covering* and *regular covering*. Furthermore, for a normal (E, p), $I(\mathrm{Eq}(p))$ is a group if and only if E is connected, i.e. $I(E)$ is a terminal object in $\mathbf{X}$.

2. There is also a reasonable way to define fundamental groupoids as "the largest" Galois groupoids.

3.21 Back to the classical cases

In this section we consider the simplest applications the fundamental theorem of categorical Galois theory.

The classical form of the fundamental theorem of Galois theory is usually formulated as follows:

Theorem 3.108. *Let $B \subseteq E$ be a finite Galois field extension, and $\mathrm{Aut}_B(E)$ its Galois group. Then:*

1. *The correspondences*

$$F \mapsto \mathrm{Aut}_F(E) \quad \text{and} \quad H \mapsto E^H = \{x \in E \mid g \in H \implies g(x) = x\} \tag{3.121}$$

determine inverse to each other and inclusion reversing bijections between the lattice $\mathrm{Sub}(E/B)$ of field subextensions of $B \subseteq E$, and the lattice $\mathrm{Sub}(\mathrm{Aut}_B(E))$ of subgroups in $\mathrm{Aut}_B(E)$.

2. *If $B \subseteq F$ is a field subextension of $B \subseteq E$, then every B-algebra homomorphism from F to E extends to a B-algebra automorphism of E.*

3. *A field subextension $B \subseteq F$ of $B \subseteq E$ is a Galois extension if and only if its corresponding subgroup $\mathrm{Aut}_F(E)$ is a normal subgroup in $\mathrm{Aut}_B(E)$. In this case every B-algebra automorphism of E restricts to a B-algebra automorphism of F, yielding a short exact sequence*

$$0 \to \mathrm{Aut}_F(E) \to \mathrm{Aut}_B(E) \to \mathrm{Aut}_B(F) \to 0 \tag{3.122}$$

of groups.

How does this theorem follow from Theorem 3.106?
Answering this question requires a number of simple observations:

1. Every statement of Theorem 3.108 is a statement about purely-categorical properties of the category $\mathrm{Sub}(E/B)$ of subextensions of the field extension $B \subseteq E$. The only thing that needs an explanation here, is that E itself can be defined categorically as a special object in $\mathrm{Sub}(E/B)$. For, just observe that it is the only weak terminal object (i.e. the only object that admits morphisms from all other objects into it).

2. Moreover, it turns out that the category $\mathrm{Sub}(E/B)^{op}$ is equivalent to the category of transitive (=one-orbit) $\mathrm{Aut}_B(E)$-sets - which is known as Grothendieck's form of the fundamental theorem of Galois theory - and every statement of Theorem 3.108 follows from this fact.

3. Furthermore, it is sufficient to know that $\mathrm{Sub}(E/B)^{op}$ is equivalent to the category of transitive G-sets for some monoid G, because this fact itself implies that G is isomorphic to $\mathrm{Aut}_B(E)$. Indeed:

   - We know that $\mathrm{Sub}(E/B)$ has a unique weakly terminal object, namely E, and that the endomorphism monoid of this object is $\mathrm{Aut}_B(E)$.
   - On the other hand G acts on itself via its multiplication, and this object is weakly initial in the category of transitive G-sets; and its endomorphism monoid is isomorphic to G.
   - Therefore the equivalence of $\mathrm{Sub}(E/B)^{op}$ to the category of transitive G-sets implies that G is isomorphic to $\mathrm{Aut}_B(E)$.

4. Let us now apply Theorem 3.106 to the situation considered in Section 3.17. As follows from the equivalence $(a) \implies (c)$ in Theorem 3.91, which assumes that $B \subseteq E$ is a Galois extension and $p\colon E \to B$ is the inclusion map $B \to E$, in that situation we have

$$\mathbf{Spl}(E, p) \sim \mathrm{Fam}_{\mathbf{X}}(\mathrm{Sub}(E/B)^{op}). \tag{3.123}$$

   At the same time Theorem 3.106 tells us that the category $\mathbf{Spl}(E, p)$ is equivalent to the category of finite G-sets for some finite monoid G - namely for $G = L(I(\mathrm{Eq}(p)))$, where L is the functor 3.67, and $L(I(\mathrm{Eq}(p)))$ is indeed a monoid since $I(E)$ has only one element.

5. As follows from (4), $\mathrm{Sub}(E/B)^{op}$ must be equivalent to the category of transitive G-sets, as desired. Therefore Theorem 3.108 indeed follows from Theorem 3.106.

The situation with covering spaces is very similar: many standard text books in algebraic topology show how the connected covering spaces of a "good" space B are "classified" via subgroups of the fundamental group of B by proving a theorem similar to Theorem 3.76, usually not showing the categorical result behind, which is:

Theorem 3.109. *Let B be a connected locally connected topological space, admitting a universal covering space (E, p) over it. Then the category of covering spaces over B is equivalent to the category of* $\mathrm{Aut}(E, p)$*-sets.*

- and this theorem can easily be obtained as a corollary of Theorem 3.106, using the results of Section 3.18. Recall, however, that what is called a universal covering space of B is in fact a weakly initial object in the category of non-empty covering spaces over B, and that "weakness" can be avoided by using pointed spaces.

Applying Theorem 3.106 to the situation considered in Section 3.19, we obtain, in particular, a description of the category $\mathbf{Centr}(B)$ of central extension of an arbitrary group B. The full explanation would involve some homological algebra and internal category theory (in "nice" categories), which would take us too far. Therefore let us just mention that it becomes especially simple when B is perfect, i.e. when $[B, B] = B$: in this case

$$\mathbf{Centr}(B) \sim (\mathbf{Ab} \downarrow \mathrm{H}_2(A; \mathbb{Z})), \tag{3.124}$$

which presents the second homology group $\mathrm{H}_2(A; \mathbb{Z})$ as a certain "Galois group", and implies the well-known result saying that every perfect group has a universal central extension.

Finally, let us mention one less familiar examples of Galois theories very briefly; being less familiar it was, however, the original motivating example for categorical Galois theory:

Example 3.110. The system $(\mathbf{C}, \mathbf{X}, I, H, \eta, \varepsilon, \mathbb{F}, \Phi)$ described below is an admissible Galois structure in which ε is an isomorphism:

- $\mathbf{C}$ is the opposite category of commutative unitary rings;
- $\mathbf{X}$ is the opposite category of (unitary) Boolean rings, or, equivalently, the opposite category of Boolean algebras; up to a category equivalence we can identify $\mathbf{X}$ with the category of Stone spaces (=profinite topological spaces = compact totally disconnected Hausdorff spaces = compact 0-dimensional Hausdorff spaces = compact topological spaces in which every two points can be separated by a closed-and-open subset);
- $I \colon \mathbf{C} \to \mathbf{X}$ is sending rings to the Boolean rings of their idempotents, or, considering $\mathbf{X}$ as the category of Stone spaces, I is defined by

$$\begin{aligned} I(A) = {} & \text{Boolean spectrum of } A \\ = {} & \text{Stone space of the Boolean algebra of idempotents in } A \\ = {} & \text{the space of connected components of the Zariski spectrum of } A; \end{aligned}$$

- $H \colon \mathbf{X} \to \mathbf{C}$ is defined by

$$H(X) = \mathrm{Hom}(X, \mathbb{Z}), \tag{3.125}$$

where X is any object in $\mathbf{X}$ considered as a topological space, $\mathbb{Z}$ is the ring of integers equipped with the discrete topology, and $\mathrm{Hom}(X, Z)$ is set of continuous maps $X \to \mathbb{Z}$ with the ring structure induced by the ring structure of $\mathbb{Z}$;

- and are defined accordingly, and $\mathbf{F}$ and Φ are the classes of all morphisms in $\mathbf{C}$ and $\mathbf{X}$ respectively.

The covering morphisms with respect to this Galois structure are the same as what A. R. Magid calls componentially locally strongly separable algebras; they are defined as follows:

1. a commutative (unitary) algebra S over a commutative (unitary) ring R is said to be *separable* if it is projective as an $S \otimes_R S$-module;
2. a commutative separable R-algebra S is said to be *strongly separable* if it is projective as an R-module;
3. an R-algebra S is said to be locally strongly separable if every finite subset in it is contained in a strongly separable R-subalgebra;
4. a commutative R-algebra S is said to be componentially locally strongy separable if its all Boolean localizations S_x are locally strongly separable R_x-algebras; here, for a maximal ideal x of the Boolean ring of idempotents in R, the Boolean localizations S_x is defined as the quotient algebra S/S_x.

And these componentially locally strongly separable algebras were the most general algebras involved in Magid's separable Galois theory of commutative rings. For a field extension $B \subseteq E$ we have:

$$\begin{aligned} E \text{ is a separable } B\text{-algebra} &\Leftrightarrow E \text{ is a strongly separable } B\text{-algebra} \\ &\Leftrightarrow E \text{ is a finite separable extension of } B. \end{aligned}$$

Chapter 4

Comonads and Galois comodules of corings

The aim of the remaining lectures is to study Galois structures which arise in differential non-commutative geometry, in particular to show, how Galois conditions encode geometric notions such as principal (and associated vector) bundles. The Galois condition which arises in this context is very closely related to (co)monadicity described earlier. To make better connection with the preceding sections we start with the category theory considerations.

We use the following notational conventions. The identity morphism for an object X is denoted by X (though occasionally we write id for clarity). We do not write composition symbol $\circ$ when composing functors. Given a natural transformation δ between functors F and G, δ_X denotes corresponding morphism $F(X) \to G(X)$. For any other functors H, K (composable with F or G, respectively) $H\delta$ means the natural transformation $HF \to HG$ given on objects X as $H(\delta_X)$, while δK means the transformation $FK \to GK$ given on objects as $\delta_{K(X)}$.

4.1 Comonads

Definition 4.1. *A* comonad *on a category* $\mathbf{A}$ *is a triple* $G = (G, \delta, \sigma)$*, where* $G\colon \mathbf{A} \to \mathbf{A}$ *is a functor* $\delta\colon G \to GG$*,* $\sigma\colon G \to \mathrm{id}_{\mathbf{A}}$ *are natural transformations such that the following diagrams*

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & GG \arrow[d, "G\delta"] \\ GG \arrow[r, "\delta G"'] & GGG \end{tikzcd}, \qquad \begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, equal] & GG \arrow[d, "G\sigma"] \\ GG \arrow[r, "\sigma G"'] & G \end{tikzcd}$$

commute. The transformation δ *is called a* comultiplication*, and* σ *is called a* counit.

Comonads form a category. A morphism between comonads $G \to G'$ is a natural transformation $\varphi\colon G \to G'$ rendering commutative the following diagrams

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[d, "\delta"'] & & G' \arrow[d, "\delta'"] \\ GG \arrow[r, "G\varphi"'] & GG' \arrow[r, "\varphi G'"'] & G'G' \end{tikzcd} , \qquad \begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\sigma"'] & & G' \arrow[dl, "\sigma'"] \\ & \mathrm{id}_{\mathbf{A}} & \end{tikzcd} .$$

Definition 4.2. *A* coalgebra over a comonad $G = (G, \delta, \sigma)$ *is a pair* (A, ρ^A) *consisting of an object* A *of* $\mathbf{A}$ *and a morphism* $\rho^A \colon A \to G(A)$*, such that the following diagrams commute*

$$\begin{tikzcd} A \arrow[r, "\rho^A"] \arrow[d, "\rho^A"'] & G(A) \arrow[d, "\delta_A"] \\ G(A) \arrow[r, "G(\rho^A)"'] & GG(A) \end{tikzcd} , \qquad \begin{tikzcd} A \arrow[r, "\rho^A"] \arrow[dr, equal] & G(A) \arrow[d, "\sigma_A"] \\ & A \end{tikzcd} .$$

A morphism of coalgebras $(A, \rho^A) \to (B, \rho^B)$ *is a morphism* $f \colon A \to B$ *in* $\mathbf{A}$ *compatible with the structure maps* ρ^A, ρ^B *in the sense of the commutativity of the following diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\rho^A"'] & B \arrow[d, "\rho^B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd} .$$

The category of coalgebras of G *is often referred to as the* Eilenberg-Moore category *and is denoted by* $\mathbf{A}_G$.

Dually to comonads one considers *monads* F on a category $\mathbf{A}$ and their Eilenberg-Moore category of algebras $\mathbf{A}^F$; see Section 3.8.

The introduction of the Eilenberg-Moore category allows one to realise a close relationship between adjoint functors and comonads. Any adjoint pair of functors $L \colon \mathbf{A} \to \mathbf{B}$, $R \colon \mathbf{B} \to \mathbf{A}$ (L is the left adjoint of R) gives rise to a comonad (G, δ, σ) on $\mathbf{B}$, where $G = LR$, $\delta = L\eta R$ (that is $\delta_B = L(\eta_{R(B)})$), $\sigma = \psi$ and η is the unit of adjunction (L, R), and ψ the counit of adjunction.

Given a comonad (G, δ, σ) on $\mathbf{A}$, there is an adjunction

$L \colon \mathbf{A}_G \to \mathbf{A}$,	the forgetful functor,
$R \colon \mathbf{A} \to \mathbf{A}_G$,	the free coalgebra functor defined by $R(A) = (G(A), \delta_A)$

Similarly, if (L, R) is an adjoint pair of functors, then $F = RL$ is a monad on $\mathbf{M}_A$. Conversely, given a comonad F on $\mathbf{A}$, the free algebra functor $\mathbf{A} \to \mathbf{A}^F$ is the left adjoint of the forgetful functor $\mathbf{A}^F \to \mathbf{A}$.

4.2 Comonadic triangles and descent theory

The correspondence between pairs of adjoint functors and comonads leads to the following fundamental question: What is the relationship between a category on which a pair of adjoint functors is defined and a category of coalgebras of a given comonad. The situation is summarised in

Definition 4.3. *Take categories* $\mathbf{A}$*,* $\mathbf{B}$*, a comonad* G *on* $\mathbf{A}$ *and adjoint functors* $L\colon \mathbf{B} \to \mathbf{A}$*,* $R\colon \mathbf{A} \to \mathbf{B}$*. A triangle of categories and functors*

$$\begin{tikzcd} \mathbf{B} \arrow[rr, "K"] \arrow[dr, shift left, "L"] & & \mathbf{A}_G \arrow[dl, "U_G"] \\ & \mathbf{A} \arrow[ul, shift left, "R"] & \end{tikzcd}$$

where U_G *is the forgetful functor is called a* G-comonadic triangle *provided* $U_G K = L$ *. The functor* K *is referred to as a* comparison functor*.*

We would like to study, when the comparison functor K is an equivalence. First, we need to find an equivalent description of comparison functors.

Proposition 4.4. *Fix categories* $\mathbf{A}$*,* $\mathbf{B}$*, a comonad* G *on* $\mathbf{A}$*, and adjoint functors* $L\colon \mathbf{B} \to \mathbf{A}$*,* $R\colon \mathbf{A} \to \mathbf{B}$*. There is a one-to-one correspondence between comparison functors* K *in comonadic triangles made of* G*,* L *and* R*, and comonad morphisms* $\varphi\colon LR \to G$*.*

Proof. Given φ define a natural transformation

$$\beta\colon L \xrightarrow{L\eta} LRL \xrightarrow{\varphi L} GL, \quad \beta_B\colon L(B) \to G(L(B)), \quad \beta_B = \varphi_{L(B)} \circ L(\eta_B),$$

where η is the unit of adjunction (L, R). Then the functor $K\colon \mathbf{B} \to \mathbf{A}_G$ is given by $B \mapsto (L(B), \beta_B)$. Conversely, given $K\colon B \to (K(B), \rho^{K(B)})$ define

$$\beta\colon L \to GL, \text{ by } \beta_B = \rho^{K(B)}.$$

Then

$$\varphi\colon LR \xrightarrow{\beta R} GLR \xrightarrow{G\psi} G,$$

where ψ is the counit of adjunction (L, R), is the required morphism of comonads. ☐

Proposition 4.5. *In the set-up of Proposition 4.4, If* $\mathbf{B}$ *has equalisers, then* K *has a right adjoint* $D\colon \mathbf{A}_G \to \mathbf{B}$ *defined by the equaliser*

$$\begin{tikzcd} D(A, \rho^A) \arrow[r, "\mathrm{eq}_A"] & R(A) \arrow[r, shift left, "\alpha_A"] \arrow[r, shift right, "R(\rho^A)"'] & RG(A), \end{tikzcd}$$

where

$$\alpha\colon R \xrightarrow{\eta R} RLR \xrightarrow{R\varphi} RG.$$

Proof. The unit of the adjoint pair (K, D) is given by $\widehat{\eta}_B$ in the diagram:

$$\begin{tikzcd}
DK(B) \arrow[r] & RL(B) \arrow[r, shift left, "\alpha_{L(B)}"] \arrow[r, shift right, "R(\beta_B)"'] & RGL(B) \\
& B\, . \arrow[ul, dashed, "\widehat{\eta}_B"] \arrow[u, "\eta_B"'] &
\end{tikzcd}$$

The existence of such $\widehat{\eta}_B$ follows by the universal property of equalisers. The counit of the adjoint pair (K, D) is given by $\widehat{\psi}$ in the diagram

$$\begin{tikzcd}
KD(A, \rho^A) \arrow[r, "L(\mathrm{eq}_A)"] \arrow[dr, dashed, "\widehat{\psi}_{(A,\rho^A)}"'] & LR(A) \arrow[r, shift left, "L(\alpha_A)"] \arrow[r, shift right, "LR(\rho^A)"'] \arrow[d, "\psi_A"] & LRG(A) \\
& A\, . &
\end{tikzcd}$$

Note that $\widehat{\psi}_{(A,\rho^A)}$ is a composite, the universal property of an equaliser is not used here. $\square$

Recall that a *contractible equaliser* of two morphisms $g, h \colon B \to G$ is a morphism $f \colon A \to B$ fitting into the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[l, bend left, "i"] & G\, , \arrow[l, bend left, "j"]
\end{tikzcd}$$

with two maps i, j such that

$$i \circ f = A, \quad j \circ g = B, \quad j \circ h = f \circ i, \quad g \circ f = h \circ f;$$

compare Example 3.57. For objects A in $\mathbf{A}$ consider a contractible equaliser

$$\begin{tikzcd}
R(A) \arrow[r, "\alpha_A"] & RG(A) \arrow[r, shift left, "\alpha_{G(A)}"] \arrow[r, shift right, "R(\delta_A)"'] \arrow[l, bend left, "R(\sigma_A)"] & RG^2(A). \arrow[l, bend left, "RG(\sigma_A)"]
\end{tikzcd}$$

In view of the universal property of equalisers this implies that

$$\alpha_A = \mathrm{eq}_{G(A)}, \quad R(A) = D(G(A), \delta_A),$$

hence

$$\begin{aligned}
\widehat{\psi}_{(G(A),\delta_A)} &= \psi_{G(A)} \circ L(\mathrm{eq}_{G(A)}) \\
&= \psi_{G(A)} \circ L(\alpha_A) \\
&= \psi_{G(A)} \circ LR(\varphi_A) \circ L(\eta_{R(A)}) \\
&= \varphi_A,
\end{aligned}$$

where the last equality follows by one of the triangular equalities for the unit and counit of an adjunction. Since a functor which has a right adjoint is full and faithful if and only if the counit of adjunction is a (natural) isomorphism, this simple calculation of $\widehat{\psi}$ immediately establishes the following

Proposition 4.6. *If D is full and faithful, then φ is an isomorphism of comonads.*

The problem of finding when K is an equivalence is equivalent to studying the comonadicity of L. Thus the Beck monadicity theorem yields (see Theorem 3.63)

Theorem 4.7. *Consider a comonadic triangle in Definition 4.3. If $\mathbf{B}$ has equalisers, then K is an equivalence if and only if φ is an isomorphism, L preserves equalisers that define D, and L reflects isomorphisms.*

Comonadic triangles encode (and generalise) the typical setup of *descent theory*. Let T be a monad on a category $\mathbf{B}$, and let $L\colon \mathbf{B} \to \mathbf{A}$, $R\colon \mathbf{A} \to \mathbf{B}$ be a pair of adjoint functors. Setting $G = LR$ one obtains the following comonadic triangle

$$\begin{tikzcd} \mathbf{B} \arrow[rr, "K"] \arrow[dr, shift left, "F^T"] & & (\mathbf{B}^T)_G \arrow[dl, "U_G"] \\ & \mathbf{B}^T \arrow[ul, shift left, "U^T"] & \end{tikzcd} .$$

Here K is the standard comparison functor corresponding to $\varphi = \mathrm{id}$. $(\mathbf{B}^T)_G$ is known as the category of *descent data*. We say that this triangle is of *descent type* whenever K is full and faithful, and we say that it defines an *effective descent* when K is an equivalence. The standard descent theory studies effective descent in specific situations (such as, e.g. arise in algebraic geometry).

4.3 Comonads on a category of modules. Corings.

Let A, B be associative and unital algebras over a commutative ring k, with multiplication operations denoted by μ_A, μ_B and units 1_A, 1_B (understood both as elements or linear maps $k \to A$, $k \to B$), respectively. Denote by $\mathbf{M}_A$, $\mathbf{M}_B$ the categories of right modules over A and B. Categories of modules are additive and have colimits (they are abelian categories), and we would like to study functors which preserve these structures. Such functors are fully characterised by the Eilenberg-Watts theorem.

Theorem 4.8 (Eilenberg-Watts)**.** *Let $F\colon \mathbf{M}_A \to \mathbf{M}_B$ be an additive functor that preserves colimits. Then $F(A)$ is an (A, B)-bimodule and*

$$F \cong - \otimes_A F(A), \qquad F(M) \cong M \otimes_A F(A) .$$

We would like to study comonads (G, δ, σ) on $\mathbf{M}_A$ such that G preserves colimits. By Theorem 4.8, $G \cong - \otimes_A G(A)$. Let $\mathcal{C} := G(A)$, so $\mathcal{C}$ is an A-bimodule. Next we explore consequences of the fact that δ, σ are natural transformations. For any $M \in \mathbf{M}_A$, $m \in M$, consider a morphism in $\mathbf{M}_A$

$$l_m\colon A \to M, \quad a \mapsto ma .$$

The naturality of δ implies

$$\begin{tikzcd}
\mathcal{C} \cong A \otimes_A \mathcal{C} \arrow[rr, "l_m \otimes_A \mathcal{C}"] \arrow[d, "\delta_A"] & & M \otimes_A \mathcal{C} \arrow[d, "\delta_M"] \\
\mathcal{C} \otimes_A \mathcal{C} \cong A \otimes_A \mathcal{C} \otimes_A \mathcal{C} \arrow[rr, "l_m \otimes_A \mathcal{C} \otimes_A \mathcal{C}"] & & M \otimes_A \mathcal{C} \otimes_A \mathcal{C}\,.
\end{tikzcd}$$

The evaluation of this diagram at $1_A \otimes_A c$, $c \in \mathcal{C}$ gives

$$\delta_M(m \otimes_A c) = m \otimes_A \delta_A(c).$$

Hence $\delta_M = M \otimes_A \delta_A$. This means, in particular, that δ_A is a left A-linear, hence an A-bilinear map (it is right A-linear as a morphism in $\mathbf{M}_A$). Similarly, $\sigma_M = M \otimes_A \sigma_A$. Let

$$\Delta_{\mathcal{C}} := \delta_A \colon \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}\,, \qquad \varepsilon_{\mathcal{C}} := \sigma_A \colon \mathcal{C} \to A\;.$$

Then diagrams for coassociativity of δ and counitality of σ are equivalent to the following commutative diagrams

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta_{\mathcal{C}}"] \arrow[d, "\Delta_{\mathcal{C}}"'] & \mathcal{C} \otimes_A \mathcal{C} \arrow[d, "\Delta_{\mathcal{C}} \otimes_A \mathcal{C}"] \\
\mathcal{C} \otimes_A \mathcal{C} \arrow[r, "\mathcal{C} \otimes_A \Delta_{\mathcal{C}}"'] & \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta_{\mathcal{C}}"] \arrow[d, "\Delta_{\mathcal{C}}"'] \arrow[dr, "\cong"] & \mathcal{C} \otimes_A \mathcal{C} \arrow[d, "\mathcal{C} \otimes_A \varepsilon_{\mathcal{C}}"] \\
\mathcal{C} \otimes_A \arrow[r, "\varepsilon_{\mathcal{C}} \otimes_A \mathcal{C}"'] & A \otimes_A \mathcal{C} \cong \mathcal{C} \otimes_A A\;.
\end{tikzcd}
\tag{4.1}$$

Definition 4.9. *(compare Definition 2.4) An A-bimodule $\mathcal{C}$ together with A-bilinear maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$, $\varepsilon_{\mathcal{C}} : \mathcal{C} \to A$ satisfying (4.1) is called an* A-coring *(pronounced:* co-ring*).* $\Delta_{\mathcal{C}}$ *is called the* comultiplication *and* $\varepsilon_{\mathcal{C}}$ *is called the* counit *of $\mathcal{C}$.*

A morphism of A-corings $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}}) \to (\mathcal{D}, \Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}})$ is an A-bimodule map $f \colon \mathcal{C} \to \mathcal{D}$ such that the following diagrams commute

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "\Delta_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\Delta_{\mathcal{D}}"] \\
\mathcal{C} \otimes_A \mathcal{C} \arrow[r, "f \otimes_A f"'] & \mathcal{D} \otimes_A \mathcal{D}\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[dr, "\varepsilon_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\varepsilon_{\mathcal{D}}"] \\
 & A\;.
\end{tikzcd}$$

Using arguments similar to those establishing the correspondence between corings and (tensor functor) comonads, one easily finds that f arises as (and gives rise to) a morphism of comonads (evaluated at A). We have thus established bijective correspondences:

$$\begin{tikzcd}
A\text{-corings} \arrow[d, leftrightarrow] \\
\text{Additive comonads on } \mathbf{M}_A \text{ that preserve colimits} \arrow[d, leftrightarrow] \\
\text{Additive comonads on } \mathbf{M}_A \text{ that have a right adjoint} \arrow[d, leftrightarrow] \\
\text{Additive monads on } \mathbf{M}_A \text{ that have a left adjoint}\,.
\end{tikzcd}$$

This last correspondence follows by the fact that the right adjoint of a comonad is a monad and vice versa. The correspondence between corings and comonads is explicitly given by

$$(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}}) \leftrightarrow (- \otimes_A \mathcal{C}, - \otimes_A \Delta_{\mathcal{C}}, - \otimes_A \varepsilon_{\mathcal{C}}).$$

Definition 4.10. *Let $\mathcal{C}$ be an A-coring, M be a right A-module and let $\rho^M : M \to M \otimes_A \mathcal{C}$ be a right A-module map. A pair (M, ρ^M) is called a* right $\mathcal{C}$-comodule *if and only if the following diagrams commute*

$$\begin{tikzcd}
M \arrow[r, "\rho^M"] \arrow[d, "\rho^M"'] & M \otimes_A \mathcal{C} \arrow[d, "\rho^M \otimes_A \mathcal{C}"] \\
M \otimes_A \mathcal{C} \arrow[r, "M \otimes_A \Delta_{\mathcal{C}}"'] & M \otimes_A \mathcal{C} \otimes_A \mathcal{C}\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\rho^M"] \arrow[dr, "\cong"] & M \otimes_A \mathcal{C} \arrow[d, "M \otimes_A \varepsilon_{\mathcal{C}}"] \\
& M \otimes_A A\,.
\end{tikzcd}$$

The map ρ^M is called a coaction.

(M, ρ^M) is a $\mathcal{C}$-comodule if and only if (M, ρ^M) is a coalgebra for the corresponding comonad $G = (-\otimes_A \mathcal{C}, -\otimes_A \Delta_{\mathcal{C}}, -\otimes_A \varepsilon_{\mathcal{C}})$. A morphism in $(\mathbf{M}_A)_G$ is a right A-module map $f \colon M \to N$ rendering the following diagram commutative

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\rho^M"'] & N \arrow[d, "\rho^N"] \\
M \otimes_A \mathcal{C} \arrow[r, "f \otimes_A \mathcal{C}"'] & N \otimes_A \mathcal{C}\,.
\end{tikzcd}$$

The category of right $\mathcal{C}$-comodules (i.e. the category of coalgebras of $(- \otimes_A \mathcal{C}, - \otimes_A \Delta_{\mathcal{C}}, - \otimes_A \varepsilon_{\mathcal{C}})$) is denoted by $\mathbf{M}^{\mathcal{C}}$. Morphisms between comodules (M, ρ^M) and (N, ρ^N) are denoted by $\mathrm{Hom}^{\mathcal{C}}(M, N)$.

Left comodules are defined symmetrically as coalgebras of the comonad $(\mathcal{C} \otimes_A -, \Delta_{\mathcal{C}} \otimes_A -, \varepsilon_{\mathcal{C}} \otimes_A -)$ on the category of left A-modules.

As an example of a coring we can study corings associated to a module.

Example 4.11. Take algebras A, B and look at functors $\mathbf{M}_B \to \mathbf{M}_A$ preserving colimits. By the Eilenberg-Watts theorem (Theorem 4.8) such functors have

the tensor form, i.e. there is a (B, A)-bimodule M such that the functor is of the form
$$- \otimes_B M \colon \mathbf{M}_B \to \mathbf{M}_A.$$
For $N \in \mathbf{M}_A$, morphisms $\mathrm{Hom}_A(M, N)$ form a right B-module by
$$(f \cdot b)(m) = f(bm), \qquad f \in \mathrm{Hom}_A(M, N),\ m \in M,\ b \in B.$$
Since the functor $\mathrm{Hom}_A(M, -) \colon \mathbf{M}_A \to \mathbf{M}_B$ is the right adjoint to $- \otimes_B M$, there is a comonad
$$G = \mathrm{Hom}_A(M, -) \otimes_B M \colon \mathbf{M}_A \to \mathbf{M}_A\ ,$$
with comultiplication
$$\begin{gathered}\delta_N \colon \mathrm{Hom}_A(M, N) \otimes_B M \to \mathrm{Hom}_A(M, \mathrm{Hom}_A(M, N) \otimes_B M) \otimes_B M,\\ f \otimes_B m \mapsto [m' \mapsto f \otimes_B m'] \otimes_B m\ ,\end{gathered}$$
and counit
$$\sigma_N \colon \mathrm{Hom}_A(M, N) \otimes_B M \to N\,, \qquad f \otimes_B m \mapsto f(m)\,.$$
G preserves colimits if M is finitely generated and projective as a right A-module, i.e. there exists $e = \sum_i e_i \otimes_A \xi_i$, $e_i \in M$, $\xi_i \in M^* := \mathrm{Hom}_A(M, A)$, $i = 1, 2, \ldots, n$ such that, for all $m \in M$,
$$m = \sum_i e_i \xi_i(m).$$
In this case
$$\mathrm{Hom}_A(M, N) \cong N \otimes_A M^*, \qquad G = - \otimes_A M^* \otimes_B M.$$
Hence $\mathcal{C} = M^* \otimes_B M$ is an A-coring with comultiplication and counit
$$\Delta_{\mathcal{C}}(\xi \otimes_B m) = \xi \otimes_B e \otimes_B m, \qquad \varepsilon_{\mathcal{C}}(\xi \otimes_B m) = \xi(m).$$
$G(A) = M^* \otimes_B M$ is called a *(finite) comatrix coring.*

4.4 Galois comodules for corings

We start with an A-coring $\mathcal{C}$ and take a category of right A-modules $\mathbf{A} := \mathbf{M}_A$. Denote by $G = - \otimes_A \mathcal{C}$ the corresponding comonad on $\mathbf{A}$. The category $\mathbf{A}_G$ of G-coalgebras is thus the same as the category of $\mathcal{C}$-comodules $\mathbf{M}^{\mathcal{C}}$. Take a comodule $(M, \rho^M) \in \mathbf{M}^{\mathcal{C}}$ and set $B = \mathrm{End}^{\mathcal{C}}(M) = \mathrm{Hom}^{\mathcal{C}}(M, M)$. This is an algebra with respect to composition of morphisms and M is a left B-module by evaluation ($b \cdot m = b(m)$ for $b \in B$, $m \in M$). Furthermore, the definition of comodule morphisms imply that ρ^M is a left B-linear map. Set $\mathbf{B} := \mathbf{M}_B$. Since ρ^M is a left B-linear map, there is a functor
$$K \colon \mathbf{M}_B \to \mathbf{M}^{\mathcal{C}}, \qquad V \mapsto (V \otimes_B M, V \otimes_B \rho^M). \tag{4.2}$$

Note that for the forgetful functor $U^{\mathcal{C}}\colon \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$, $U^{\mathcal{C}}K(V) = V \otimes_B M$. Thus there is a comonadic triangle

$$\begin{tikzcd}
\mathbf{M}_B \arrow[rr, "K"] \arrow[dr, shift left, "-\otimes_B M"] & & \mathbf{M}^{\mathcal{C}} \arrow[dl, "U^{\mathcal{C}}"] \\
& \mathbf{M}_A \arrow[ul, shift left, "\mathrm{Hom}_A(M,-)"] &
\end{tikzcd}$$

By Proposition 4.4 there is a comonad morphism

$$\varphi\colon\ \mathrm{Hom}_A(M,-) \otimes_B M \to -\otimes_A \mathcal{C}.$$

Recall that, for all $N \in \mathbf{M}_A$, the counit ψ of the tensor-hom adjunction $(-\otimes_B M, \mathrm{Hom}_A(M,-))$ in provided by the evaluation map

$$\psi_N\colon\ \mathrm{Hom}_A(M,N) \otimes_B M \to N, \qquad f \otimes_B m \mapsto f(m).$$

Therefore,

$$\begin{aligned}
\varphi_N &= G(\psi_N) \circ \rho^{KR(N)},\\
f \otimes_B m \mapsto{}& (\psi_N \otimes_A \mathcal{C})(\rho^{K(\mathrm{Hom}_A(M,N))}(f \otimes_B m))\\
&= (\psi_N \otimes_A \mathcal{C})(f \otimes_B \rho^M(m)) = (f \otimes_A \mathcal{C})(\rho^M(m))\,.
\end{aligned}$$

Writing

$$\rho^M(m) = m_{(0)} \otimes_A m_{(1)}\ ,$$

(summation implicit) we obtain

$$\varphi_N(f \otimes_B m) = f(m_{(0)}) \otimes_A m_{(1)}\ . \tag{4.3}$$

Definition 4.12. *A right $\mathcal{C}$-comodule (M, ρ^M) is called a* Galois comodule *if and only if the natural transformation φ determined by all the maps φ_N (4.3) is a natural isomorphism.*

If M is finitely generated and projective as a right A-module, then the comonad $\mathrm{Hom}_A(M,-) \otimes_B M$ comes from the comatrix coring $-\otimes_A M^* \otimes_B M$. The fact that φ is a comonad morphism is equivalent to the fact that φ_A is a coring morphism. Write

$$\mathrm{can}_M := \varphi_A\colon M^* \otimes_B M \to \mathcal{C}\,, \qquad \xi \otimes_B m \mapsto \xi(m_{(0)})m_{(1)}\,.$$

The map can_M is called the *canonical map.*

Definition 4.13. *A Galois comodule (M, ρ^M) such that M is finitely generated projective as a right A-module is called a* finite Galois comodule.

The Galois property of a finite Galois comodule is entirely encoded in the properties of the canonical map. More precisely,

Lemma 4.14. *A right $\mathcal{C}$-comodule (M, ρ^M) with M finitely generated projective as a right A-module is a Galois comodule if and only if the canonical map can_M is an isomorphism of A-corings.*

Since the category of modules has equalisers, the comparison functor K in (4.2) has a right adjoint D; see Proposition 4.5. Recall that D is defined by the diagram

$$\begin{tikzcd} D(N,\rho^N) \arrow[r, "\mathrm{eq}_N"] & R(N) \arrow[r, shift left, "\alpha_N"] \arrow[r, shift right, "R(\rho^N)"'] \arrow[d, equal] & RG(N) \arrow[d, equal] \\ & \mathrm{Hom}_A(M,N) & \mathrm{Hom}_A(M, N\otimes_A \mathcal{C}) . \end{tikzcd}$$

The equalised maps can be explicitly computed as

$$R(\rho^N)(f) = \mathrm{Hom}_A(M,\rho^N)(f) = \rho^N \circ f,$$

and

$$\alpha_N(f) = R(\varphi_N)(\eta_{R(N)}(f)) = \mathrm{Hom}_A(M,\varphi_N)(f\otimes_B -) = (f\otimes_A \mathcal{C})\circ \rho^N.$$

Therefore, D can be identified with the comodule homomorphism functor, i.e. for all right $\mathcal{C}$-comodules (N,ρ^N),

$$D(N,\rho^N) = \mathrm{Hom}^{\mathcal{C}}(M,N).$$

In order to state the conditions under which the comparison functor K, and thus also the constructed functor D, is an equivalence we need to recall the notions of flatness and faithful flatness. Consider a sequence of right B-module maps

$$V \longrightarrow V' \longrightarrow V''. \tag{4.4}$$

For any left B-module M there is then also the following sequence

$$V\otimes_B M \longrightarrow V'\otimes_B M \longrightarrow V''\otimes_B M. \tag{4.5}$$

The module M is said to be *flat* if the exactness of any sequence (4.4) implies exactness of the corresponding sequence (4.5). The module M is said to be *faithfully flat* if its flat and, for any sequence of modules (4.4), the exactness of (4.5) implies exactness of (4.4).

Combining the discussion of comodules in this and preceding sections with Beck's monadicity theorem (see Proposition 4.6 and Theorem 4.7) one derives the main characterisation of Galois comodules in terms of equivalences of categories.

Theorem 4.15 (The finite Galois comodule structure theorem)**.** *Let (M,ρ^M) be a comodule over a coring $\mathcal{C}$ such that M is finitely generated projective as a right module over A. Then the following conditions are equivalent:*

1. *the functor $\mathrm{Hom}^{\mathcal{C}}(M,-) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_B$ is fully faithful and $\mathcal{C}$ is flat as a right A-module;*
2. *M is flat as a left B-module and (M,ρ^M) is Galois comodule.*

Furthermore the following conditions are equivalent:

1. *$\mathrm{Hom}^{\mathcal{C}}(M,-)$ is an equivalence of categories and $\mathcal{C}$ is flat as a right A-module;*
2. *M is faithfully flat as a left B-module and (M,ρ^M) is Galois comodule.*

4.5 A Galois condition motivated by algebraic geometry

The notion of a Galois comodule presented in Section 4.4 is considered to be standard in the theory of corings; see [bw03]. Recently, motivated by an approach to non-commutative algebraic geometry through monoidal categories, Maszczyk introduced a different Galois condition in [m-t07]. We describe this condition here and compare it with the one studied in Section 4.4.

Start with a morphism of A-corings $\gamma\colon \mathcal{D} \to \mathcal{C}$. Then $\mathcal{D}$ is a $\mathcal{C}$-bicomodule (i.e. it has both left and right $\mathcal{C}$-coaction such that the left coaction is a morphism of right $\mathcal{C}$-comodule) via

$$\begin{aligned}&(\mathcal{D} \otimes_A \gamma) \circ \Delta_{\mathcal{D}} \quad \text{(right } \mathcal{C}\text{-coaction)}\ ,\\ &(\gamma \otimes_A \mathcal{D}) \circ \Delta_{\mathcal{D}} \quad \text{(left } \mathcal{C}\text{-coaction)}\ .\end{aligned}$$

Consider the k-module of $\mathcal{C}$-bicomodule maps $B :={}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(\mathcal{D}, \mathcal{C})$. Then B is an algebra with the product of $b, b' \in B$ given by

$$bb' = (\mathcal{C} \otimes_A \varepsilon_{\mathcal{C}}) \circ (b \otimes_A b') \circ \Delta_{\mathcal{D}},$$

i.e. explicitly

$$bb' : d \mapsto d_{(1)} \otimes_A d_{(2)} \mapsto b(d_{(1)}) \otimes_A b'(d_{(2)}) \mapsto b(d_{(1)})\varepsilon_{\mathcal{C}}(b'(d_{(2)}))\ .$$

Furthermore, $\mathcal{D}$ is a B-bimodule. Define

$$\overline{\mathcal{D}} := \mathcal{D}/[\mathcal{D}, B], \qquad p\colon \mathcal{D} \to \overline{\mathcal{D}}\ .$$

Then $\overline{\mathcal{D}}$ is an A-coring with the structure induced by p from that of A-coring $\mathcal{D}$, and there is a commutative triangle of coring maps:

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & \mathcal{C} \\ & \overline{\mathcal{D}}\ . \arrow[ur, "\overline{\gamma}"'] & \end{tikzcd}$$

Following [m-t07], $\mathcal{C}$ is said to be a *Galois coring* if the map $\overline{\gamma}$ is an isomorphism. The above triangle of coring maps induces two functors

$$F\colon \mathbf{M}^{\mathcal{D}} \to \mathbf{M}^{\overline{\mathcal{D}}}, \qquad (M, \rho^M) \mapsto (M, (M \otimes_A p) \circ \rho^M),$$

and

$$G\colon \mathbf{M}^{\mathcal{D}} \to \mathbf{M}^{\overline{\mathcal{C}}}, \qquad (M, \rho^M) \mapsto (M, (M \otimes_A \gamma) \circ \rho^M).$$

Any right $\mathcal{D}$-comodule (M, ρ^M) defines two comonadic triangles

$$\begin{tikzcd} \mathbf{M}_B \arrow[rr, "\overline{K}=-\otimes_B F(M)"] \arrow[dr, shift left, "-\otimes_B M"] & & \mathbf{M}^{\overline{\mathcal{D}}} \arrow[dl, "U^{\overline{\mathcal{D}}}"] \\ & \mathbf{M}_A, \arrow[ul, shift left, "\mathrm{Hom}_A(M,-)"] & \end{tikzcd}$$

with the corresponding (to $\overline{K}$) morphism of comonads

$$\overline{\varphi}(f \otimes_B m) = (f \otimes_A p)(\rho^M(m)), \qquad f \in \mathrm{Hom}_A(M, N)\,,\ m \in M\,,$$

and

$$\begin{tikzcd}
\mathbf{M}_B \arrow[rr, "K=-\otimes_B G(M)"] \arrow[dr, shift left, "-\otimes_B M"] & & \mathbf{M}^{\mathcal{C}} \arrow[dl, "U^{\mathcal{D}}"] \\
& \mathbf{M}_A, \arrow[ul, shift left, "\mathrm{Hom}_A(M,-)"] &
\end{tikzcd}$$

with the corresponding (to K) morphism of comonads

$$\varphi(f \otimes_B m) = (f \otimes_A \gamma)(\rho^M(m)), \qquad f \in \mathrm{Hom}_A(M, N)\,,\ m \in M\,.$$

Proposition 4.16 (G. Böhm)**.** *Assume that $\overline{\varphi}$ is an isomorphism (i.e. that $F(M)$ is a Galois $\overline{\mathcal{D}}$-comodule). Then φ is an isomorphism (i.e. $G(M)$ is a Galois $\mathcal{C}$-comodule) if and only if $\overline{\gamma}$ is an isomorphism of corings (i.e. $\mathcal{C}$ is Galois in the sense of [m-t07]).*

References for this chapter are: [bm05], [b-j67], [bw03], [d-e70], [em65], [g-j06], [g-a60], [m-s71], [m-t07], [m-b06], [s-m75], [w-c60].

Chapter 5

Hopf–Galois extensions of non-commutative algebras

In this lecture we introduce the key notions in the Galois theory of Hopf algebras or in the algebraic approach to non-commutative principal bundles. We also show how Hopf–Galois extensions fit into the theory of Galois comodules of corings described in Chapter 4.

From now on, k denotes a field, and all algebras etc. are over k. The tensor product over k is denoted by $\otimes$.

5.1 Coalgebras and Sweedler's notation

Definition 5.1. *A* coalgebra *is a vector space C with k-linear maps $\Delta_C : C \to C \otimes C$, $\varepsilon_C : C \to k$ such that the following diagrams commute*

$$
\begin{tikzcd}
C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & C \otimes C \arrow[d, "C\otimes\Delta_C"] \\
C \otimes C \arrow[r, "\Delta_C\otimes C"'] & C \otimes C \otimes C,
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\Delta_C"] \arrow[d, "C"', "\Delta"] \arrow[dr, equal] & C \otimes C \arrow[d, "\varepsilon_C\otimes C"] \\
C \otimes C \arrow[r, "C\otimes\varepsilon_C"'] & C
\end{tikzcd}
$$

Δ_C is called a comultiplication *and ε_C is called a* counit.

In other words, a k-coalgebra is the same as a k-coring (when a vector space is viewed as a symmetric k-bimodule). Following this identification of k-coalgebras as k-corings one defines C-comodules as comodule of the k-coring C. (The reader should notice that we use the term *coalgebra* here in the sense different from that in Chapter 4.)

The idea of comultiplication is somewhat counter-intuitive: out of a single element of a vector spaces, a family of elements is produced. Heyneman and Sweedler developed a shorthand notation which proves very useful in explicit computations that involve comultiplications and counits. The Sweedler notation for comultiplication is based on omitting unnecessary summation range, index and sign, and then employing the coassociativity of comultiplication (the first of diagrams in Definition 5.1) to relabel indices by consecutive numbers. Given an element $c \in C$, we write

$$\begin{aligned}\Delta_C(c) &= \sum_{i=1}^{n} c^i_{(1)} \otimes c^i_{(2)} \\ &= \sum_i c^i_{(1)} \otimes c^i_{(2)} \\ &= \sum c_{(1)} \otimes c_{(2)} \\ &= c_{(1)} \otimes c_{(2)}.\end{aligned}$$

The coassociativity of comultiplication means that the two ways to compute the result of two applications of Δ give the same result:

$$\begin{aligned}(C \otimes \Delta_C) \circ \Delta_C(c) &= (C \otimes \Delta_C)(c_{(1)} \otimes c_{(2)}) \\ &= c_{(1)} \otimes \Delta_C(c_{(2)}) = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} \\ (\Delta_C \otimes C) \circ \Delta_C(c) &= (\Delta_C \otimes C)(c_{(1)} \otimes c_{(2)}) \\ &= \Delta_C(c_{(1)}) \otimes c_{(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}\,.\end{aligned}$$

We can order all indices appearing in above expressions (and in all expressions involving multiple application of Δ_C) in the following way. Remove the brackets, put 0. in front of the index and then arrange them in increasing order. In this way we obtain

$$0.1 < 0.21 < 0.22, \qquad 0.11 < 0.12 < 0.2\,.$$

The coassociativity of Δ_C tells us that we do not need to care about exact labels but only about their increasing order. Hence we can relabel:

$$c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}.$$

Exercise 5.2. *Compute and check labelling for all three applications of Δ to an element $c \in C$.*

In terms of the Sweedler notation, the counitality of the comultiplication or the second of the diagrams in Definition 5.1 comes out as

$$c_{(1)}\varepsilon_C(c_{(2)}) = \varepsilon_C(c_{(1)})c_{(2)} = c.$$

Example 5.3. Let X be a set, $C = kX$ – the linear span of X (elements of X form a basis of the vector space kX). Define the comultiplication and counit by

$$\Delta_C(x) = x \otimes x, \qquad \varepsilon_C(x) = 1, \qquad \text{for all } x \in X.$$

Remark: for any coalgebra C an element $c \in C$ such that $\Delta_C(c) = c \otimes c$, $\varepsilon_C(c) = 1$ is called a *group-like* element.

Example 5.4. Consider the trigonometric identities

$$\begin{aligned}\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y), \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y),\end{aligned}$$

and the values of sine and cosine at the origin $\sin(0) = 0$, $\cos(0) = 1$. We can abstract from these expressions the variables x and y and use the trignometric identities to define the comultiplication, and values at 0 to define the counit. Thus we consider a two-dimensional coalgebra C with a basis $\{\sin, \cos\}$, and comultiplication and counit

$$\Delta_C(\sin) = \sin \otimes \cos + \cos \otimes \sin, \qquad \Delta_C(\cos) = \cos \otimes \cos - \sin \otimes \sin,$$
$$\varepsilon_C(\sin) = 0, \qquad \varepsilon_C(\cos) = 1.$$

This coalgebra is often referred to as the *trigonometric coalgebra*.

Example 5.5. Let G be a monoid with unit e, $\mathcal{O}(G)$ algebra of functions $G \to k$. If G is finite we take all functions, and if G is an algebraic group then we take polynomial (or representative) functions. $\mathcal{O}(G)$ is a coalgebra with comultiplication and counit

$$\Delta_{\mathcal{O}(G)}(f)(g \otimes g') = f(gg'), \qquad \varepsilon_{\mathcal{O}(G)}(f) = f(e).$$

5.2 Bialgebras and comodule algebras

In addition to comultiplication and counit, coalgebras in Examples 5.4 and 5.5 can be equipped with the structure of an algebra in a way that is compatible with the coalgebra structure.

Definition 5.6. *A* bialgebra *is a vector space H such that:*
(a) H is an algebra with multiplication μ_H and unit 1_H;
(b) H is a coalgebra with comultiplication Δ_H and counit ε_H;
(c) Δ_H and ε_H are algebra maps, i.e. the following diagrams commute

$$\begin{tikzcd}
H \otimes H \arrow[rr, "\mu_H"] \arrow[d, "\Delta_H \otimes \Delta_H"'] & & H \arrow[d, "\Delta_H"] \\
H \otimes H \otimes H \otimes H \arrow[r, "H \otimes \mathrm{flip} \otimes H"] & H \otimes H \otimes H \otimes H \arrow[r, "\mu_H \otimes \mu_H"] & H \otimes H,
\end{tikzcd}$$

$$\begin{tikzcd}
H \otimes H \arrow[rr, "\mu_H"] \arrow[dr, "\varepsilon_H \otimes \varepsilon_H"'] & & H \arrow[dl, "\varepsilon_H"] \\
& k, &
\end{tikzcd}$$

and $\Delta_H(1_H) = 1_H \otimes 1_H$ *and* $\varepsilon_H(1_H) = 1$.

Explicitly, in terms of the Sweedler notation the first of diagrams in Definition 5.6 reads, for all $h, h' \in H$,

$$\Delta_H(hh') = h_{(1)}h'_{(1)} \otimes h_{(2)}h'_{(2)} \ .$$

Example 5.7. Let G be a monoid with unit e, and let $H = kG$ – the linear span of G. The multiplication is the monoid multiplication extended linearly, i.e. $\mu_H : g \otimes g' \mapsto gg'$, for all $g, g' \in G$, unit $1_H = e$, the comultiplication is given by $\Delta_H(g) = g \otimes g$, and the counit by $\varepsilon_H(g) = 1$ (see Example 5.3). With these structures kG is a bialgebra.

Example 5.8. Let G be a monoid with unit e, and let $H = \mathcal{O}(G)$ – the functions $G \to k$; see Example 5.5. H is an algebra by the pointwise multiplication $\mu_H(f \otimes f')(g) = f(g)f'(g)$, and with the unit $1_H(g) = 1$. The comultiplication is given by $\Delta_H(f)(g \otimes g') = f(gg')$, and the counit by $\varepsilon_H(f) = f(e)$ as in Example 5.5. With these operations H is a bialgebra. For example:

(i) Functions on the two element group $G = \mathbb{Z}_2$. As a vector space

$$\mathcal{O}(\mathbb{Z}_2) = k^2 \text{ with basis } e_1, e_2\,,$$

$$e_1(1) = 1, \quad e_1(-1) = 0, \qquad e_2(1) = 0, \quad e_2(-1) = 1.$$

The comultiplication derived from the rule described above comes out as

$$\Delta_H(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2, \qquad \Delta_H(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1.$$

The pointwise multiplication is $e_1e_1 = e_1$, $e_1e_2 = e_2e_1 = 0$, $e_2e_2 = e_2$.

(ii) Functions on the circle group $G = S^1 = \mathrm{U}(1)$, $k = \mathbb{C}$. As an algebra $\mathcal{O}(\mathrm{U}(1))$ is isomorphic to the algebra of Laurent polynomials,

$$\mathcal{O}(\mathrm{U}(1)) \cong \mathbb{C}[X, X^{-1}].$$

The comultiplication is given on generators by $\Delta_H(X) = X \otimes X$, $\Delta_H(X^{-1}) = X^{-1} \otimes X^{-1}$ (and is extended multiplicatively to the whole of $\mathbb{C}[X, X^{-1}]$).

Definition 5.9. *Given a bialgebra H, a right H-*comodule algebra *is a quadruple $(A, \mu_A, 1_A, \rho^A)$, where*

(a) $(A, \mu_A, 1_A)$ is a k-algebra with multiplication μ_A and unit 1_A;

(b) (A, ρ^A) is a right H-comodule (with coaction $\rho^A\colon A \to A \otimes H$);

(c) the coaction ρ^A is an algebra map, when $A \otimes H$ is viewed as a tensor product algebra

$$(a \otimes h)(a' \otimes h') = aa' \otimes hh', \qquad 1_{A\otimes H} = 1_A \otimes 1_H\ .$$

That is the following diagram commutes

$$\begin{tikzcd}
A \otimes A \arrow[r, "\rho^A\otimes\rho^A"] \arrow[dd, "\mu_A"'] & A \otimes H \otimes A \otimes H \arrow[d, "A\otimes\mathrm{flip}\otimes H"] \\
 & A \otimes A \otimes H \otimes H \arrow[d, "\mu_A\otimes\mu_H"] \\
A \arrow[r, "\rho^A"] & A \otimes H,
\end{tikzcd}$$

and $\rho^A(1_A) = 1_A \otimes 1_H$.

An alternative definition of a bialgebra can be given by considering the structure of the category of comodules of a coalgebra H. A coalgebra H is a bialgebra if and only if the category of right H-comodules, $\mathbf{M}^H$, is a monoidal category and the forgetful functor from $\mathbf{M}^H$ to vector spaces is strongly monoidal (i.e. the monoidal operation in $\mathbf{M}^H$ is the same as the tensor product of vector spaces). If H is a bialgebra and (M, ρ^M) and (N, ρ^N) are H-comodules, then $(M \otimes N, \rho^{M\otimes N})$ is an H-comodule with the coaction

$$M \otimes N \xrightarrow{\rho^M \otimes \rho^N} M \otimes H \otimes N \otimes H \xrightarrow{M \otimes \mathrm{flip} \otimes H} M \otimes N \otimes H \otimes H \xrightarrow{M \otimes N \otimes \mu_H} M \otimes N \otimes H.$$

With this interpretation a right H-comodule algebra is simply an algebra in the monoidal category of right H-comodules.

Similarly to comultiplication, in explicit expressions and calculations it is useful to use Sweedler's notation for comodules. Let $(A, \rho^A) \in \mathbf{M}^H$. For all $a \in A$, we write omitting the sum sign and summation indices

$$\rho^A(a) = a_{(0)} \otimes a_{(1)}.$$

Note that all the elements $a_{(0)}$ are in A, while all the $a_{(1)}$ are in H. The comodule property $(A \otimes \Delta_H) \circ \rho^A(a) = (\rho^A \otimes H) \circ \rho^A(a)$ can be written as

$$a_{(0)} \otimes a_{(1)(1)} \otimes a_{(1)(2)} = a_{(0)(0)} \otimes a_{(0)(1)} \otimes a_{(1)} =: a_{(0)} \otimes a_{(1)} \otimes a_{(2)}\ .$$

In general, after relabelling according to the same rules as for comultiplication, symbols with positive Sweedler indices are elements of the Hopf algebra H. The compatibility condition from Definition 5.9 can be written as

$$(aa')_{(0)} \otimes (aa')_{(1)} = a_{(0)}a'_{(0)} \otimes a_{(1)}a'_{(1)}\ .$$

Example 5.10. Since the comultiplication in a bialgebra is an algebra map, the pair (H, Δ_H) is a right comodule algebra. One often refers to (H, Δ_H) as a *(right) regular comodule.*

Example 5.11. Let G be a group, $H = kG$. Then A is an H-comodule algebra if and only if A is a G-graded algebra

$$A = \bigoplus_{g\in G} A_g, \quad A_g A_{g'} \subseteq A_{gg'}, \quad 1_A \in A_e.$$

If $a \in A_g$, then define

$$\rho^A(a) = a \otimes g.$$

Since $1_A \in A_e$, $\rho^A(1_A) = 1_A \otimes e = 1_A \otimes 1_H$.

Take $a \in A_g$, $a' \in A_{g'}$. Then $aa' \in A_{gg'}$, hence $\rho^A(aa') = aa' \otimes gg'$ as needed.

Example 5.12. Let $H = \mathcal{O}(G)$ for a monoid G. For a G-set X, take $A = \mathcal{O}(X)$ and identify $\mathcal{O}(X) \otimes \mathcal{O}(G)$ with $\mathcal{O}(X \times G)$. Then A is an H-comodule algebra with respect to

$$\rho^A(f)(x, g) = f(xg), \qquad \forall x \in X, g \in G.$$

5.3 Hopf–Galois extensions and Hopf algebras

Definition 5.13. *If A is a right H-comodule algebra (of a bialgebra H), define the set of* coinvariants *(or* coaction invariants*) as*

$$A^{\mathrm{co}H} := \{b \in A \mid \rho^A(b) = b \otimes 1_H\}\,.$$

Coinvariants $A^{\mathrm{co}H}$ are a subalgebra of A, because ρ^A is an algebra map. Furthermore

$$A^{\mathrm{co}H} = \{b \in A \mid \text{for all } a \in A,\ \rho^A(ba) = b\rho^A(a)\}\,.$$

A is an $A^{\mathrm{co}H}$-bimodule, and ρ^A is a left $A^{\mathrm{co}H}$-module map. The coaction ρ^A is also a right $A^{\mathrm{co}H}$-module map, when $A \otimes H$ has the right multiplication given by $(a \otimes h) \cdot a' = aa' \otimes h$.

Example 5.14. Take a regular comodule algebra (H, Δ_H); see Example 5.10. Then H has trivial coaction invariants, i.e.

$$H^{\mathrm{co}H} = k \cdot 1_H\ .$$

Indeed, since $\Delta_H(1_H) = 1_H \otimes 1_H$, $k1_H \subseteq H^{\mathrm{co}H}$. On the other hand if

$$h_{(1)} \otimes h_{(2)} = h \otimes 1_H\ ,$$

then apply $\varepsilon_H \otimes H$ to get $h = \varepsilon_H(h)1_H$, hence $h \in k1_H$.

Definition 5.15. *A right H-comodule algebra is called a* Hopf–Galois extension *(of the coinvariants $B := A^{\mathrm{co}H}$) if the* canonical map

$$\mathrm{can}\colon A \otimes_B A \to A \otimes H, \qquad a \otimes_B a' \mapsto a\rho^A(a')\,.$$

is bijective (an isomorphism of left A-modules and right H-comodules).

Example 5.16. Let G be a group, $H = kG$, and $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra. Then A is Hopf–Galois extension if and only if it is strongly graded, i.e., for all $g, g' \in G$,

$$A_g A_{g'} = A_{gg'}, \qquad B = A^{\mathrm{co}H} = A_e.$$

In this case, for all $a \in A_g$, $a' \in A_{g'}$

$$\mathrm{can}\colon a' \otimes_B a \mapsto a'a \otimes g,$$
$$\mathrm{can}^{-1}\colon a' \otimes g \mapsto \sum_i a'\overline{a_i} \otimes_B a_i.$$

where $a_i \in A_g$, $\overline{a_i} \in A_{g^{-1}}$ are such that $\sum_i \overline{a_i} a_i = 1_A$.

As explained in Example 5.10 (H, Δ_H) is a right H-comodule algebra. It is thus tempting to ask the following

Question 3. When is (H, Δ_H) a Hopf–Galois extension by H?

Since the coinvariants $H^{\mathrm{co}H}$ of (H, Δ_H) coincide with the ground field k (see Example 5.14), Question 3 is equivalent to determining, when

$$\mathrm{can}_H\colon H \otimes H \to H \otimes H, \qquad h' \otimes h \mapsto h'h_{(1)} \otimes h_{(2)}\,,$$

is an isomorphism.

Lemma 5.17. *(H, Δ_H) is a Hopf–Galois extension if and only if there is a map $S\colon H \to H$ such that*

$$h_{(1)}S(h_{(2)}) = \varepsilon_H(h)1_H = S(h_{(1)})h_{(2)}\,.$$

Such a map S is called an antipode.

Proof. If such a map S exists, then the inverse of the canonical map is given by

$$\mathrm{can}^{-1}(h' \otimes h) = h'S(h_{(1)}) \otimes h_{(2)} .$$

Conversely, if can^{-1} exists, then the linear map

$$S = (H \otimes \varepsilon_H) \circ \mathrm{can}^{-1} \circ (1_H \otimes H) ,$$

has the required properties. □

Definition 5.18. *A bialgebra with an antipode is called a* Hopf algebra*.*

The antipode is an anti-algebra and anti-coalgebra map, and plays the role similar to the mapping which to each element of a group assigns its inverse (and hence can be heuristically understood as a generalised inverse).

Examples 5.19.

1. If G is a group, then kG is a Hopf algebra with the antipode $S\colon kG \to kG$ given on G by $g \mapsto g^{-1}$.
2. Similarly, for a group G, the antipode on $\mathcal{O}(G)$ is given by
$$S\colon f \mapsto [g \mapsto f(g^{-1})] .$$

5.4 Cleft extensions

Take an algebra B and a Hopf algebra H. Let $A = B \otimes H$ and consider it as a right H-comodule with coaction

$$\rho^A\colon B \otimes H \to B \otimes H \otimes H, \qquad \rho^A = B \otimes \Delta_H .$$

Suppose furthermore that $B \otimes H$ is an algebra with multiplication and unit

$$(b \otimes h)(b' \otimes h') = bb' \otimes hh' , \qquad 1_B \otimes 1_H .$$

This makes $(B \otimes H, \mathrm{B} \otimes \Delta_H)$ into an H-comodule algebra. Clearly

$$A^{\mathrm{co}H} = (B \otimes H)^{\mathrm{co}H} = \{b \otimes 1_H \mid b \in B\} \cong B.$$

The canonical map is

$$\begin{aligned} \mathrm{can}\colon (B \otimes H) \otimes_B (B \otimes H) \cong B \otimes H \otimes H \to B \otimes H \otimes H, \\ b \otimes h' \otimes h \mapsto b \otimes h'h_{(1)} \otimes h_{(2)}, \end{aligned}$$

and hence is bijective with the inverse

$$\begin{aligned} \mathrm{can}^{-1}\colon B \otimes H \otimes H \to B \otimes H \otimes H , \\ b \otimes h' \otimes h \mapsto b \otimes h'S(h_{(1)}) \otimes h_{(2)}. \end{aligned}$$

Therefore, $B \otimes H$ is a Hopf–Galois extension (of B). More generally, one can study Hopf–Galois extensions built on the comodule $(B \otimes H, \mathrm{B} \otimes \Delta_H)$.

Definition 5.20. *Let A be a Hopf–Galois extension of $B = A^{\mathrm{co}H}$. A is said to have a* normal basis property *if $A \cong B \otimes H$ as a left B-module and right H-comodule.*

Proposition 5.21. *Let (A, ρ^A) be a right H-comodule algebra, and let $B = A^{\mathrm{co}H}$. The following statements are equivalent:*

1. *A is a Hopf–Galois extension with a normal basis property.*
2. *There exists a map $j\colon H \to A$ such that:*
   (a) *j is a right H-comodule map, i.e. the following diagram*

$$\begin{tikzcd} H \arrow[r, "j"] \arrow[d, "\Delta_H"] & A \arrow[d, "\rho^A"] \\ H \otimes H \arrow[r, "j \otimes H"'] & A \otimes H \end{tikzcd}$$

   is commutative;

   (b) *j is convolution invertible, i.e. there exists a linear map $\tilde{j}\colon H \to A$ such that, for all $h \in H$,*

$$j(h_{(1)})\tilde{j}(h_{(2)}) = \tilde{j}(h_{(1)})j(h_{(2)}) = \varepsilon_H(h)1_A.$$

Proof. (2) $\implies$ (1) We prove that the inverse of the canonical map can has the following form

$$\mathrm{can}^{-1}\colon a \otimes h \mapsto a\tilde{j}(h_{(1)}) \otimes_B j(h_{(2)})\,.$$

In one direction, starting with can^{-1}, we compute

$$\begin{aligned} \mathrm{can}(a\tilde{j}(h_{(1)}) \otimes j(h_{(2)})) &= a\tilde{j}(h_{(1)})j(h_{(2)(1)}) \otimes h_{(2)(2)} \\ &= a\tilde{j}(h_{(1)(1)})j(h_{(1)(2)}) \otimes h_{(2)} = a \otimes h. \end{aligned}$$

The first equality follows by the fact that the coaction ρ^A is an algebra map and by the colinearity of j (condition 2(a) in Proposition 5.21). The final equality is a consequence of condition 2(b). The proof that the composite $\mathrm{can}^{-1} \circ \mathrm{can}$ is the identity map is slightly more involved. First note that

$$\rho^A(\tilde{j}(h)) = \tilde{j}(h_{(2)}) \otimes S(h_{(1)}). \tag{5.1}$$

This is verified in a few steps. Start with the equality

$$1_A \otimes S(h_{(1)}) \otimes \tilde{j}(h_{(2)}) = \tilde{j}(h_{(1)})j(h_{(2)}) \otimes S(h_{(3)}) \otimes \tilde{j}(h_{(4)})\,,$$

which is a consequence of condition 2(b) (and the definition of a counit). Then apply $\rho^A \otimes H \otimes A$ and use the multiplicativity of ρ^A and right H-colinearity of j to obtain

$$1_A \otimes 1_H \otimes S(h_{(1)}) \otimes \tilde{j}(h_{(2)}) = \tilde{j}(h_{(1)})_{(0)}j(h_{(2)}) \otimes \tilde{j}(h_{(1)})_{(1)}h_{(3)} \otimes S(h_{(4)}) \otimes \tilde{j}(h_{(5)})\,.$$

Next multiply elements in H and use the definition of the antipode to reduce above equality to

$$1_A \otimes S(h_{(1)}) \otimes \tilde{j}(h_{(2)}) = \tilde{j}(h_{(1)})_{(0)}j(h_{(2)}) \otimes \tilde{j}(h_{(1)})_{(1)} \otimes \tilde{j}(h_{(3)})\,.$$

Finally, equality (5.1) is obtained by multiplying elements in A and then using the convolution inverse property 2(b). Equation (5.1) implies that, for all $a \in A$,

$$a_{(0)}\tilde{j}(a_{(1)}) \in B = A^{\mathrm{co}H}\,. \tag{5.2}$$

To verify this claim, simply apply ρ^A to $a_{(0)}\tilde{j}(a_{(1)})$, use the multiplicativity of ρ^A, covariance property (5.1) and the definition of the antipode to obtain

$$\begin{aligned}\rho^A(a_{(0)}\tilde{j}(a_{(1)})) &= a_{(0)}\tilde{j}(a_{(2)})_{(0)} \otimes a_{(1)}\tilde{j}(a_{(2)})_{(1)}\\ &= a_{(0)}\tilde{j}(a_{(3)}) \otimes a_{(1)}S(a_{(2)}) = a_{(0)}\tilde{j}(a_{(1)}) \otimes 1_H\,.\end{aligned}$$

Property (5.2) is used to compute $\mathrm{can}\circ\mathrm{can}^{-1}$:

$$\begin{aligned}\mathrm{can}\circ\mathrm{can}^{-1} : a' \otimes_B a &\mapsto a'a_{(0)} \otimes a_{(1)}\\ &\mapsto a' \underbrace{a_{(0)}\tilde{j}(a_{(1)})}_{\in B} \otimes_B j(a_{(2)})\\ &= a' \otimes_B a_{(0)}\tilde{j}(a_{(1)})j(a_{(2)})\\ &= a' \otimes_B a_{(0)}\varepsilon_H(a_{(1)})\\ &\text{(by property 2(b) in Proposition 5.21)}\\ &= a' \otimes_B a\,.\end{aligned}$$

This completes the proof that A is a Hopf–Galois extension. We need to show that it has the normal basis property, i.e. that A is isomorphic to $B \otimes H$. Consider the map

$$\theta\colon B \otimes H \to A\,, \qquad b \otimes h \mapsto bj(h)\,.$$

This is clearly a left B-module map. It is also right H-colinear since ρ^A is left linear over the coaction invariants and j is right H-colinear by assumption (2)(a). The inverse of θ is

$$\theta^{-1}\colon A \to B \otimes H, \qquad a \mapsto a_{(0)}\tilde{j}(a_{(1)}) \otimes a_{(2)}\,.$$

The verification that θ^{-1} is the inverse of θ makes use of assumption 2(b) and is left to the reader.

(1) $\implies$ (2) Given a left B-linear, right H-colinear isomorphism $\theta\colon B\otimes H \xrightarrow{\cong} A$, define

$$j\colon H \to A, \qquad h \mapsto \theta(1_B \otimes h)\,.$$

Since θ is right H-colinear, so is j. The convolution inverse of j is

$$\tilde{j}\colon H \xrightarrow{1_A\otimes H} A \otimes H \xrightarrow{\mathrm{can}^{-1}} A \otimes_B A \xrightarrow{A\otimes_B\theta^{-1}} A \otimes_B B \otimes H \cong A \otimes H \xrightarrow{A\otimes\varepsilon_H} A\,.$$

□

Verification of property 2(b) is left to the reader.

Definition 5.22. *A comodule algebra A such that there is a convolution invertible right H-comodule map $j\colon H \to A$ is called a* cleft extension *(of $A^{\mathrm{co}H}$). The map j is called a* cleaving map.

Since 1_H is a grouplike element $j(1_H)\tilde{j}(1_H) = 1_A$, so $j(1_H) \neq 0$, and a cleaving map can always be normalised so that $j(1_H) = 1_A$. Proposition 5.21 establishes a one-to-one correspondence between cleft extensions and Hopf–Galois extensions with a normal basis property. Note finally that the isomorphism $\theta : A \to B \otimes H$ can be used to generate an algebra structure on $B \otimes H$. In this way one obtains an example of a *twisted tensor product* or *crossed product* algebra.

5.5 Hopf–Galois extensions as Galois comodules

In this section we would like to make a connection between Hopf–Galois extensions and Galois comodules of a coring described in Chapter 4.

Take a bialgebra H. Let A be a right H-comodule algebra, i.e. an algebra $(A, \mu_A, 1_A)$ and an H-comodule (A, ρ^A) such that $\rho^A \colon A \to A \otimes H$ is an algebra map. Then $\mathcal{C} = A \otimes H$ is an A-bimodule with the following A-actions

$$a{\cdot}(a'{\otimes}h) = aa'{\otimes}h \ \text{ (left } A\text{-action)}, \qquad (a'{\otimes}h){\cdot}a = a'a_{(0)}{\otimes}ha_{(1)} \ \text{ (right } A\text{-action)}\,.$$

Furthermore, $\mathcal{C}$ is an A-coring with counit $\varepsilon_{\mathcal{C}} = A \otimes \varepsilon_H$ and comultiplication

$$\Delta_{\mathcal{C}} \colon A \otimes H \to (A \otimes H) \otimes_A (A \otimes H) \cong A \otimes H \otimes H, \qquad a \otimes h \mapsto a \otimes \Delta_H(h)\,.$$

A is a right $\mathcal{C}$-comodule with the coaction

$$A \to A \otimes_A (A \otimes H) \cong A \otimes H, \qquad a \mapsto \rho^A(a) = a_{(0)} \otimes_A (1_A \otimes a_{(1)})\,.$$

In other words, once the identification of $A \otimes_A \mathcal{C}$ with $\mathcal{C} = A \otimes H$ is taken into account, A is a $\mathcal{C}$-comodule by the same coaction by which A is an H-comodule. Next we need to compute the endomorphism ring of the right $\mathcal{C}$-comodule (A, ρ^A). $B = \mathrm{End}^{\mathcal{C}}(A)$ is a subalgebra of A, once the right A-module endomorphisms of A are identified with A by the left multiplication map, i.e.

$$B = \mathrm{End}^{\mathcal{C}}(A) \subseteq A \cong \mathrm{End}_A(A), \qquad A \ni b \mapsto [l_b \colon a \mapsto ba] \in \mathrm{End}_A(A)\,.$$

The element $b \in A$ is an element of the subalgebra B if and only if the corresponding A-linear map l_a is right $\mathcal{C}$-colinear, i.e.

$$\begin{tikzcd}
1_A \arrow[r, mapsto, "\rho^A"] \arrow[d, mapsto, "l_b"'] & 1_A \otimes_A 1_A \otimes 1_H \arrow[d, mapsto, "l_b \otimes_A 1_A \otimes 1_H"] \\
a \arrow[r, mapsto, "\rho^A"'] & \rho^A(b) = b \otimes 1_H .
\end{tikzcd}$$

Hence $l_b \in \mathrm{End}^{\mathcal{C}}(A)$ if and only if $\rho^A(b) = b \otimes 1_H$. This means that the endomorphism algebra $B = \mathrm{End}^{\mathcal{C}}(A)$ coincides with the algebra of H-comodule invariants,

$$B = A^{\mathrm{co}H} = \{a \in A \mid \rho^A(a) = a \otimes 1_H\}\,.$$

Obviously A is a finitely generated projective right A-module and the dual module can be identified with A,

$$A^* := \mathrm{Hom}_A(A, A) \cong A\,.$$

The dual basis for A is

$$e = (l_{1_A} \otimes_A 1_A) = 1_A \otimes_A 1_A \in A \otimes_A A\,.$$

The corresponding comatrix coring is simply the *Sweedler canonical coring* (associated to the inclusion of algebras $B \subseteq A$) $A \otimes_B A$, with the comultiplication and counit

$$\begin{aligned} \Delta_{A\otimes_B A} &\colon a \otimes_B a' \mapsto (a \otimes_B 1_A) \otimes_A (1_A \otimes_B a')\,, \\ \varepsilon_{A\otimes_B A} &\colon a \otimes_B a' \mapsto aa'\,. \end{aligned}$$

The canonical map for the right $\mathcal{C}$-comodule (A, ρ^A) as defined in Section 4.4 comes out as

$$\mathrm{can}_A \colon A\otimes_B \to A \otimes H, \qquad a \otimes_B a' \mapsto l_a(a'_{(0)}) \otimes_A (1 \otimes a'_{(1)}) = a\rho^A(a')\,,$$

and hence it coincides with the canonical map for the right H-comodule algebra (A, ρ^A) as defined in Definition 5.15. Consequently, a right H-comodule algebra A is a Hopf–Galois extension (of $B = A^{\mathrm{co}H}$) if and only if (A, ρ^A) is a (finite) Galois comodule of $\mathcal{C} = A \otimes H$.

Right comodules of $\mathcal{C} = A\otimes H$ are right A-modules M with a map $\rho^M \colon M \to M \otimes_A A \otimes H \cong M \otimes H$, which is a right coaction. The coaction property means that (M, ρ^M) is a right H-comodule. The right A-module property of ρ^M yields the compatibility condition

$$\rho^M(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}\,.$$

Right A-modules and H-comodules M with this compatibility condition are called *relative Hopf modules* and their category is denoted by $\mathbf{M}^H_A$. Thus $\mathbf{M}^H_A$ is isomorphic to the category of right $\mathcal{C} = A \otimes H$-comodules.

The following result is often referred to as an easy part of the Schneider Theorem I.

Theorem 5.23 (Schneider)**.** *Let A be a right H-comodule algebra, $B = A^{\mathrm{co}H}$. The following statements are equivalent:*

1. *A is a Hopf–Galois extension such that A is faithfully flat as a left B-module.*

2. *The functor $-\otimes_B A\colon \mathbf{M}_B \to \mathbf{M}^H_A$ is an equivalence.*

Proof. Take $\mathcal{C} = A \otimes H$, identify right $\mathcal{C}$-comodules with relative Hopf modules $\mathbf{M}^H_A$ and apply the finite Galois comodule theorem, Theorem 4.15. □

References for this chapter are: [bm89], [b-t02], [bw03], [dt86], [m-s93], [s-p04], [s-h90], [s-m69], [s-k01].

Chapter 6

Connections in Hopf–Galois extensions

Geometric aspects of Hopf–Galois extensions are most clearly present in the theory of connections. The aim of this lecture is to outline the main points of this theory.

6.1 Connections

Connections are differential geometric objects. Thus before connections in a Hopf–Galois extension can be defined, one needs to describe what is meant by a differential structure.

Definition 6.1. *A* differential graded algebra *is an* $\mathbb{N}\cup\{0\}$*-graded algebra*

$$\Omega A = \bigoplus_{n=0}^{\infty} \Omega^n A\,,$$

with an operation

$$d\colon \Omega^n A \to \Omega^{n+1} A\,,$$

such that $d\circ d = 0$ *and, for all* $\omega \in \Omega^n A$ *and* $\omega' \in \Omega A$,

$$d(\omega\omega') = d(\omega)\omega' + (-1)^n \omega d(\omega')\,. \tag{6.1}$$

Equation (6.1) *is known as the Leibniz rule.*

The zero-degree part of a differential graded algebra, $\Omega^0 A$, is an algebra which is denoted by A.

Take an algebra $(A, \mu_A, 1_A)$. One associates to A a differential graded algebra ΩA as follows

$$\Omega^1 A := \ker \mu_A = \left\{ \sum_i a_i \otimes a_i' \in A\otimes A \mid \sum a_i a_i' = 0 \right\} \cong A\otimes A/k\,,$$

$$d(a) = 1_A \otimes a - a\otimes 1_A\,,$$

$$\Omega^n A := \Omega^{n-1} A \otimes_A \Omega^1 A\,.$$

The differential d is extended to the whole of ΩA using the Leibniz rule (6.1). $\Omega^1 A$ is an A-bimodule. As an algebra $\Omega A = T_A(\Omega^1 A)$ (the tensor algebra associated to the A-bimodule $\Omega^1 A$. This $(\Omega A, d)$ is called the *universal differential envelope* of A. $(\Omega^1 A, d)$ is known as the *universal differential calculus* on A. We will only work with universal differential calculus (or envelope).

Lemma 6.2. *If (A, ρ^A) is a comodule algebra over a bialgebra H, then $\Omega^1 A$ is a right H-comodule by*

$$\rho^{\Omega^1 A} \colon \Omega^1 A \to \Omega^1 A \otimes H, \qquad \sum_i a_i \otimes a'_i \mapsto \sum_i a_{i(0)} \otimes a'_{i(0)} \otimes a_{i(1)} a'_{i(1)} \,.$$

Furthermore, d is a right H-comodule map. We say that $(\Omega^1 A, d)$ is a covariant differential calculus *on A.*

Proof. To check that $\rho^{\Omega^1 A}$ is well-defined, we need to show that $\mathrm{Im}\rho^{\Omega^1 A} \subseteq \Omega^1 A \otimes H$. Applying $\mu_A \otimes H$ to $\rho^{\Omega^1 A}(\sum_i a_i \otimes a'_i)$ and using the multplicativity of ρ^A we obtain

$$\sum_i a_{i(0)} a'_{i(0)} \otimes a_{i(1)} a'_{i(1)} = \rho^A(\sum_i a_i a'_i) = 0, \quad \text{since } \sum_i a_i a'_i = 0.$$

Furthermore, for all $a \in A$,

$$\begin{aligned} \rho^{\Omega^1 A}(d(a)) &= \rho^{\Omega^1 A}(1_A \otimes a - a \otimes 1_A) \\ &= 1_A \otimes a_{(0)} \otimes a_{(1)} - a_{(0)} \otimes 1_A \otimes a_{(1)} = d(a_{(0)}) \otimes a_{(1)} \,, \end{aligned}$$

i.e. d is a right H-comodule map as required. □

Definition 6.3. *Let (A, ρ^A) be a right H-comodule algebra, $B = A^{\mathrm{co}H}$. The A-subbimodule $\Omega^1_{hor} A$ of $\Omega^1 A$ generated by all $d(b)$, $b \in B$, is called a module of* horizontal one-forms. *Thus:*

$$\Omega^1_{hor} A = A(\Omega^1 B)A = \left\{ \sum_i (a_i \otimes b_i a'_i - a_i b_i \otimes a'_i) \mid a_i, a'_i \in A,\ b_i \in B \right\} .$$

Equivalently, horizontal forms can be defined by the following short exact sequence

$$0 \to \Omega^1_{hor} A \to A \otimes A \to A \otimes_B A \to A \,,$$

where $A \otimes A \to A \otimes_B A$ is the epimorphism defining $A \otimes_B A$.

Definition 6.4. *A* connection *in a Hopf–Galois extension $B \subseteq A$ is a left A-linear map $\Pi \colon \Omega^1 A \to \Omega^1 A$, such that*

(a) $\Pi \circ \Pi = \Pi$,
(b) $\ker \Pi = \Omega^1_{hor} A$,
(c) $(\Pi \otimes H) \circ \rho^{\Omega^1 A} = \rho^{\Omega^1 A} \circ \Pi$.

In other words, a connection is an H-covariant splitting of $\Omega^1 A$ into the horizontal and vertical parts.

6.2 Connection forms

In classical differential geometry connections in a principal bundle are in one-to-one correspondence with connection forms, i.e. differential forms on the total space of the bundle with values in the Lie algebra of the structure group that are covariant with respect to the adjoint action of the Lie algebra. To be able to establish a similar relationship between connections and connection forms in a Hopf–Galois extension we first need to reinterpret the definition of a Hopf–Galois extension in terms of the universal differential envelope.

Definition 6.5. *Let A be a right H-comodule algebra. Set $B = A^{\mathrm{co}H}$ and $H^+ = \ker \varepsilon_H \subseteq H$. Define*

$$\mathrm{ver}\colon \Omega^1 A \to A \otimes H^+, \qquad \sum_i a_i' \otimes a_i \mapsto \sum_i a_i' \rho^A(a_i)\,.$$

The map ver *is called a* vertical lift*.*

Note in passing that ver is well defined (its range is in $A \otimes H^+$), since $\sum_i a_i' a_i = 0$ implies

$$\sum_i a_i' a_{i(0)} \varepsilon_H(a_{i(1)}) = \sum_i a_i' a_i = 0\,.$$

Proposition 6.6. *The following statements are equivalent:*

1. *$B \subseteq A$ is Hopf–Galois extension.*
2. *The sequence*
$$0 \to \Omega^1_{hor} A \to \Omega^1 A \xrightarrow{\mathrm{ver}} A \otimes H^+ \to 0$$
is exact.

Proof. Note that

$$A \otimes A \cong \Omega^1 A \oplus A\,, \qquad A \otimes H \cong A \otimes H^+ \oplus A\,,$$

as left A-modules. This implies that the sequence

$$0 \to \Omega^1_{hor} A \to \Omega^1 A \xrightarrow{\mathrm{ver}} A \otimes H^+ \to 0$$

is exact if and only if the sequence

$$0 \to \Omega^1_{hor} A \to A \otimes A \xrightarrow{\overline{\mathrm{can}}} A \otimes H \to 0$$

is exact. Here $\overline{\mathrm{can}}$ is the lift of the canonical map defined by the commutative diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\pi"] \arrow[dr, "\overline{\mathrm{can}}"'] & & A \otimes_B A \arrow[dl, "\mathrm{can}"] \\ & A \otimes H, & \end{tikzcd}$$

in which π is the defining projection of the tensor product $A \otimes_B A$. Since $\Omega^1_{hor} A = \ker \pi$ (compare Definition 6.3), the second sequence is exact if and only if the canonical map can is bijective. □

The vector space H^+ is a right H-comodule by the *adjoint coaction*,

$$\mathrm{Ad}\colon H^+ \to H^+ \otimes H, \qquad h \mapsto h_{(2)} \otimes S(h_{(1)})h_{(3)}\,.$$

Therefore $A \otimes H^+$ is a right H-comodule by combining ρ^A and Ad, i.e.

$$\rho^{A\otimes H^+}\colon A \otimes H^+ \to A \otimes H^+ \otimes H,$$
$$a \otimes h \mapsto a_{(0)} \otimes h_{(2)} \otimes a_{(1)}S(h_{(1)})h_{(3)}\,.$$

Lemma 6.7. *The vertical lift is a right H-comodule map from $(\Omega^1 A, \rho^{\Omega^1 A})$ to $(A \otimes H^+, \rho^{A\otimes H^+})$. Consequently, the sequence in Proposition 6.6 is a sequence of left A-modules and right H-comodules.*

Proof. The first statement is checked by a direct calculation that is left to the reader as an exercise. The second statement is obvious. □

Definition 6.8. *A* connection form *in a Hopf–Galois extension $B \subseteq A$ is a k-linear map $\omega\colon H^+ \to \Omega^1 A$ such that*
(a) $\rho^{\Omega^1 A} \circ \omega = (\omega \otimes H) \circ \mathrm{Ad}$,
(b) $\mathrm{ver} \circ \omega = 1_A \otimes H^+$.

Theorem 6.9. *Connections in a Hopf–Galois extension $B \subseteq A$ (by a Hopf algebra H) are in bijective correspondence with connection forms. The correspondence is*

$$\omega \mapsto \Pi, \quad \Pi(a'da) = a'a_{(0)}\omega(a_{(1)} - \varepsilon_H(a_{(1)})).$$

Proof. Existence of Π means that $\Omega^1_{hor}A$ is a direct summand of $\Omega^1 A$ as a left A-module and right H-comodule. This is equivalent to the existence of splitting of the left A-module and right H-comodule sequence

$$0 \to \Omega^1_{hor}A \to \Omega^1 A \xrightarrow{\mathrm{ver}} A \otimes H^+ \to 0.$$

In view of the identification

$${}_A\mathrm{Hom}^H(A \otimes H^+, \Omega^1 A) \cong \mathrm{Hom}^H(H^+, \Omega^1 A),$$

any splitting yields an ω with required properties. □

6.3 Strong connections

Recall that given an algebra B and a left B-module Γ, a *connection* in Γ is a k-linear map

$$\nabla\colon \Gamma \to \Omega^1 B \otimes_B \Gamma,$$

such that, for all $b \in B$, $x \in \Gamma$,

$$\nabla(bx) = d(b) \otimes_B x + b\nabla(x).$$

A connection in Γ exists if and only if Γ is a projective B-module (remember that $\Omega^1 B$ is the universal differential calculus) if and only if there exists a left B-module splitting (section) of the multiplication map $B \otimes \Gamma \to \Gamma$.

A general connection in a Hopf–Galois extension $B \subset A$ does not induce a connection in the left B-module A. Only connections which are related to a more restrictive notion of horizontal forms yield connections in modules.

Definition 6.10. *Given a connection* Π *in* $B \subset A$*, the right* H*-comodule map*

$$D\colon A \to \Omega^1_{hor}A, \qquad D := d - \Pi \circ d\,,$$

is called a covariant derivative *corresponding to* Π*. The connection* Π *is called a* strong connection *if* $D(A) \subseteq (\Omega^1 B)A$*.*

Lemma 6.11. *Let* D *be a covariant derivative corresponding to a strong connection in a Hopf–Galois extension* $B \subseteq A$*. Then* D *is a connection in the left* B*-module* A*.*

Lemma 6.11 is a special case of Theorem 6.15 so is left without a proof (for the time being).

Definition 6.12. *A connection form* ω *such that its associated connection is a strong connection is called a* strong connection form*. Thus a strong connection form is a* k*-linear map* $\omega\colon H^+ \to \Omega^1 A$ *characterised by the following properties:*

(a) $\rho^{\Omega^1 A} \circ \omega = (\omega \otimes H) \circ \mathrm{Ad}$,
(b) $\mathrm{ver} \circ \omega = 1_A \otimes H^+$,
(c) $d(a) - \sum a_{(0)}\omega(a_{(1)} - \varepsilon_H(a_{(1)})) \in (\Omega^1 B)A$*, for all* $a \in A$*.*

Definition 6.13. *Let* (A, ρ^A) *be a right* H*-comodule and let* $(V, {}^V\!\rho)$ *be a left* H*-comodule. The* cotensor *product is defined as an equaliser*

$$A\Box_H V \longrightarrow A \otimes V \underset{A\otimes {}^V\!\rho}{\overset{\rho^A\otimes V}{\rightrightarrows}} A \otimes H \otimes V.$$

This means that

$$A\Box_H V = \{\sum_i a_i \otimes v_i \in A \otimes V \mid \sum_i \rho^A(a_i) \otimes v_i = \sum_i a_i \otimes {}^V\!\rho(v_i)\}.$$

The functor $A\Box_H -\colon {}^H\mathbf{M} \to \mathrm{Vect}$ is a left exact functor, and $A\Box_H H \cong A$.

If (A, ρ^A) is a comodule algebra, $B = A^{\mathrm{co}H}$, then $A\Box_H V$ is a left B-module by

$$b(\sum_i a_i \otimes v_i) = \sum_i ba_i \otimes v_i.$$

This defines a functor $A\Box_H -\colon {}^H\mathbf{M} \to {}_B\mathbf{M}$ from the category of left H-comodules to the category of left B-modules.

Definition 6.14. *Given a left* H*-comodule* $(V, {}^V\!\rho)$ *and a Hopf–Galois extension* $B \subseteq A$*, the left* B*-module* $\Gamma := A\Box_H V$ *is called a* module associated *to* $B \subseteq A$*.*

Here Γ plays the role of module of sections of a vector bundle (with a standard fibre V) associated to the non-commutative principal bundle represented by the Hopf–Galois extension $B \subseteq A$. In the case of a cleft extension $B \subseteq A$, $A \cong B \otimes H$, hence $\Gamma \cong (B \otimes H)\Box_H V \cong B \otimes V$, and thus it is a free B-module. More generally,

Theorem 6.15. *If* Π *is a strong connection, then*

$$\nabla\colon A\Box_H V \to \Omega^1 B \otimes_B (A\Box_H V), \quad \nabla = D \otimes V\,,$$

is a connection in the associated left B*-module* $\Gamma = A\Box_H V$*. Consequently* Γ *is a projective* B*-module.*

Proof. Since $D(A) \subseteq (\Omega^1 B)A \cong \Omega^1 B \otimes_B A$ and D is a right H-comodule map, the map ∇ is well defined. For all $b \in B$ and $a \otimes v \in A\Box_H V$ (summation suppressed for clarity) we can compute:

$$\begin{aligned}\nabla(ba \otimes v) &= d(ba) \otimes v - \Pi(d(ba)) \otimes v \\ &= dba \otimes v + bda \otimes v - \Pi(dba) \otimes v - \Pi(bda) \otimes v \\ &= dba \otimes v + bda \otimes v - b\Pi(da) \otimes v = dba \otimes v + b\nabla(a \otimes v),\end{aligned}$$

where the second equality follows by the Leibniz rule and the third one by the left A-linearity of Π and the fact that $(db)a$ is a horizontal form, hence in the kernel of Π. We thus conclude that ∇ is a connection. The last assertion follows since every module admitting a connection (with respect to the universal differential calculus) is projective. □

In general, the associated module $A\Box_H V$ in Theorem 6.15 is not finitely generated as a left A-module, even if V is a finite dimensional vector space. However, if H has a bijective antipode, then $A\Box_H V$ is finitely generated and projective for any finite dimensional V (and, of course, provided that A has a strong connection).

Theorem 6.16 (Dąbrowski-Grosse-Hajac)**.** *A strong connection in a Hopf–Galois extension $B \subseteq A$ by a Hopf algebra H exists if and only if A is H-equivariantly projective as a left B-module, i.e. if and only if there exists a left B-module, right H-comodule section of the multiplication map $\mu_A : B \otimes A \to A$ (section means $s : A \to B \otimes A$ such that $\mu_A \circ s = A$).*

Proof. Given a section $s : A \to B \otimes A$, define a connection by

$$\Pi(a'da) = a'da - a' \otimes a + a's(a) = a's(a) - a'a \otimes 1_A.$$

This map is clearly left A-linear and right H-colinear. It is an idempotent since, using the fact that s is a section of the multiplication map $B \otimes A \to A$, one easily finds that $-a' \otimes a + a's(a) \in \Omega^1_{hor}A$. This also implies that $\ker \Pi \subseteq \Omega^1_{hor}A$. The converse inclusion follows by the left B-linearity of s and the Leibniz rule. Write $s(a) = a^{(1)} \otimes a^{(2)} \in B \otimes A$ (summation suppressed). The splitting property means that $a^{(1)}a^{(2)} = a$, so

$$D(a) = 1_A \otimes a - s(a) = 1_A \otimes a^{(1)}a^{(2)} - a^{(1)} \otimes a^{(2)} = (da^{(1)})a^{(2)} \in (\Omega^1 B)A.$$

If Π is a strong connection, then the splitting of the product is given by

$$s(a) = a \otimes 1_A + \Pi(da).$$

The map s is obviously right H-colinear and the section of the multiplication map. Note that $s(a) = 1_A \otimes a - D(a)$, hence $s(a) \in B \otimes A$ as Π is a strong connection. An easy calculation proves that s is left B-linear. That the above assignments describe mutual inverses is immediate. □

Corollary 6.17. *Let $B \subseteq A$ be a Hopf–Galois extension by H with a strong connection. Then*

1. *A is projective as a left B-module;*

2. *B is a direct summand of A as a left B-module;*

3. *A is faithfully flat as a left B-module.*

Proof. The statement (1) follows by Lemma 6.11 (or is contained in Theorem 6.16). For (2), let $s_L \colon A \to B$ be a k-linear map which is identity on B. Then the map $\mu_A \circ (B \otimes s_L) \circ s$ is a left B-linear splitting of the inclusion $B \subseteq A$. Statements (1) and (2) imply (3); see [r-l88, 2.11.29]. □

To give an example of a strong connection we construct such a connection in a cleft extension; see Section 5.4.

Proposition 6.18. *Let $B \subseteq A$ be a cleft extension, with a cleaving map $j \colon H \to A$ such that $j(1_H) = 1_A$. Write $\tilde{j} : H \to A$ for the convolution inverse of j; see Proposition 5.21. Then*

$$\omega \colon H^+ \to \Omega^1 A, \qquad h \mapsto \tilde{j}(h_{(1)}) \otimes j(h_{(2)}),$$

is a strong connection form.

Proof. First, for all $h \in H^+$, $\tilde{j}(h_{(1)})j(h_{(2)}) = \varepsilon_H(h) = 0$, so ω is well defined. We need to check if ω satisfies conditions (a)-(c) in Definition 6.12. This is done by the following three direct calculations.

$$\begin{aligned}
\rho^{\Omega^1 A} \circ \omega(h) &= \tilde{j}(h_{(1)})_{(0)} \otimes j(h_{(2)})_{(0)} \otimes \tilde{j}(h_{(1)})_{(1)} j(h_{(2)})_{(1)} \\
&= \tilde{j}(h_{(1)(2)}) \otimes j(h_{(2)(1)}) \otimes S(h_{(1)(1)}) h_{(2)(2)} \\
&= \tilde{j}(h_{(2)}) \otimes j(h_{(3)}) \otimes S(h_{(1)}) h_{(4)} \\
&= \tilde{j}(h_{(2)(1)}) \otimes j(h_{(2)(2)}) \otimes S(h_{(1)}) h_{(3)} \\
&= \omega(h_{(2)}) \otimes S(h_{(1)}) h_{(3)} \\
&= \omega \circ \mathrm{Ad}(h).
\end{aligned}$$

The first equality is simply the definition of $\rho^{\Omega^1 A}$, the second uses the H-colinearity of j and its consequence (5.1). Then the Sweedler indices have been rearranged and definitions of the adjoint coaction and ω used. Next,

$$\begin{aligned}
\mathrm{ver}(\omega(h)) &= \mathrm{ver}(\tilde{j}(h_{(1)}) \otimes j(h_{(2)})) \\
&= \tilde{j}(h_{(1)}) j(h_{(2)})_{(0)} \otimes j(h_{(2)})_{(1)} \\
&= \tilde{j}(h_{(1)}) j(h_{(2)(1)}) \otimes h_{(2)(2)} \\
&= \tilde{j}(h_{(1)(1)}) j(h_{(1)(2)}) \otimes h_{(2)} \\
&= \varepsilon(h_{(1)}) 1_A \otimes h_{(2)} \\
&= 1_A \otimes h.
\end{aligned}$$

The second equality is the definition of the vertical lift, then the H-colinearity of j is used and the Sweedler indices rearranged. The penultimate equality is a consequence of property (2)(b) in Proposition 5.21. Finally, using the normalisation of j (and hence also of $\tilde{j}$) one can compute, for all $a \in A$,

$$\begin{aligned}
D(a) &= d(a) - \Pi(d(a)) \\
&= d(a) - a_{(0)} \omega(a_{(1)} - \varepsilon_H(a_{(1)})) \\
&= 1 \otimes a - a \otimes 1 - a_{(0)} \tilde{j}(a_{(1)}) \otimes j(a_{(2)}) + a \tilde{j}(1_H) \otimes j(1_H) \\
&= 1 \otimes a - a_{(0)} \tilde{j}(a_{(1)}) \otimes j(a_{(2)}).
\end{aligned}$$

Since $a_{(0)}\tilde{j}(a_{(1)}) \in B$ (see (5.2) in the proof of Proposition 5.21), we obtain $D(a) \in \Omega^1 B \otimes_B A \subseteq B \otimes A$. □

The normalisation of a cleaving map in Proposition 6.18 is not an essential assumption. If $j(1_H) \neq 1_A$ we can choose

$$\omega(h) = \tilde{j}(h_{(1)}) \otimes j(h_{(2)}) - \tilde{j}(1_H) \otimes j(1_H) + 1_A \otimes 1_A.$$

6.4 The existence of strong connections. Principal comodule algebras

Here we would like to determine, when a Hopf–Galois extension admits a strong connection. In all geometrically interesting situations the antipode S of a Hopf algebra is bijective, hence it is natural to restrict our considerations to this case. If a Hopf algebra H has a bijective antipode, then we make a right H-comodule algebra A into a left H-comodule via

$${}^A\rho\colon A \to H \otimes A, \qquad a \mapsto S^{-1}(a_{(1)}) \otimes a_{(0)}.$$

Theorem 6.19. *If a Hopf algebra H has a bijective antipode, then strong connections in a Hopf–Galois extension $B \subseteq A$ are in bijective correspondence with k-linear maps $\ell\colon H \to A \otimes A$ such that*

(a) $\ell(1_H) = 1_A \otimes 1_A$,

(b) $\overline{\mathrm{can}} \circ \ell = 1_A \otimes H$ (or $\mu_A \circ \ell = 1_A \circ \varepsilon_H$),

(c) $(\ell \otimes H) \circ \Delta_H = (A \otimes \rho^A) \circ \ell$,

(d) $(H \otimes \ell) \circ \Delta_H = ({}^A\rho \otimes A) \circ \ell$.

The correspondence is given by

$$\Pi(a'd(a)) = a'a_{(0)}\ell(a_{(1)}) - a'a \otimes 1.$$

We also refer to ℓ as a strong connection.

Proof. The idea of the proof is to show the relation between ℓ and connection forms. First we comment on two versions of condition (b). Since $(A\otimes\varepsilon_H)\circ\overline{\mathrm{can}} = \mu_A$, the first version of condition (b) immediately implies that $\mu_A \circ \ell = 1_A \circ \varepsilon_H$. The converse follows by the use of colinearity (condition (c)).

So, using the second version of (b), if $\varepsilon_H(h) = 0$, then $\mu_A \circ \ell(h) = 0$. This means that given ℓ one can define a map $\omega_\ell : H^+ \to \Omega^1 A$, by $\omega_\ell(h) = \ell(h)$. Obviously, for all $h \in H^+$, $\mathrm{ver} \circ \omega_\ell(h) = \overline{\mathrm{can}} \circ \ell(h) = 1_A \otimes h$. A straightforward calculation reveals that (c) and (d) imply that $(\omega_\ell \otimes H) \circ \mathrm{Ad} = \rho^{\Omega^1 A} \circ \omega_\ell$. Hence if ℓ exists, the corresponding ω_ℓ is a connection one-form. By Theorem 6.9 there is a connection Π_ℓ in A with the form stated. Using explicit definition of the universal differential, the corresponding covariant derivative comes out as

$$D_\ell(a) = 1_A \otimes a - a_{(0)}\ell(a_{(1)}). \tag{6.2}$$

Now use the fact that A is a right H-comodule algebra, conditions (c) and (d) for ℓ and the fact that the inverse of the antipode S^{-1} is an anti-algebra map to compute

$$\begin{aligned}({}^A\rho \otimes H)(D_\ell(a)) &= 1_H \otimes 1_A \otimes a - a_{(2)}S^{-1}(a_{(1)}) \otimes a_{(0)}\ell(a_{(3)}) \\ &= 1_H \otimes 1_A \otimes a - S^{-1}(a_{(1)}S(a_{(2)})) \otimes a_{(0)}\ell(a_{(3)}) \\ &= 1_H \otimes 1_A \otimes a - 1_H \otimes a_{(0)}\ell(a_{(1)}) = 1_H \otimes D_\ell(a).\end{aligned}$$

This implies that, for all $a \in A$, $D_\ell(a) \in B \otimes A$, i.e. the connection Π_ℓ is strong.

Conversely, given a strong connection Π with connection one-form $\omega : H^+ \to \Omega^1 A$, define $\ell_\omega : H \to A \otimes A$ by $\ell_\omega(h) = \varepsilon_H(h) 1_A \otimes 1_A + \omega(h - \varepsilon_H(h))$. Such an ℓ_ω satisfies (a) and (b) (the latter by condition (b) of Definition 6.12). Now, condition (a) of Definition 6.12 implies that

$$(\ell_\omega \otimes H) \circ \mathrm{Ad} = \rho^{A \otimes A} \circ \ell_\omega, \tag{6.3}$$

where $\rho^{A\otimes A}$ is the diagonal coaction of H on $A \otimes A$, given by the same formula as $\rho^{\Omega^1 A}$. The covariant derivative D corresponding to Π has the same form as in equation (6.2). Since the connection Π is strong,

$$(B \otimes \rho^A)(D(a)) = \rho^{\Omega^1 A}(D(a)), \qquad \forall a \in A.$$

In view of equation (6.2) this means that

$$(A \otimes \rho^A)(a_{(0)} \ell_\omega(a_{(1)})) = \rho^{A\otimes A}(a_{(0)} \ell_\omega(a_{(1)})), \qquad \forall a \in A. \tag{6.4}$$

Putting equations (6.3) and (6.4) together and using defining properties of the antipode one obtains

$$(A \otimes \rho^A)(a_{(0)} \ell_\omega(a_{(1)})) = a_{(0)} \ell_\omega(a_{(1)}) \otimes a_{(2)}, \qquad \forall a \in A. \tag{6.5}$$

Since A is a Hopf–Galois extension, the canonical map is bijective. This means that, for any $h \in H$, there exists $h^{[1]} \otimes_B h^{[2]} \in A \otimes_B A$ (summation implicit) such that $1 \otimes h = \mathrm{can}(h^{[1]} \otimes_B h^{[2]}) = h^{[1]} h^{[2]}{}_{(0)} \otimes h^{[2]}{}_{(1)}$. Hence equation (6.5) implies for all $h \in H$,

$$\begin{aligned}(A \otimes \rho^A)(\ell_\omega(h)) &= (A \otimes \rho^A)(h^{[1]} h^{[2]}{}_{(0)} \ell_\omega(h^{[2]}{}_{(1)})) \\ &= h^{[1]} h^{[2]}{}_{(0)} \ell_\omega(h^{[2]}{}_{(1)}) \otimes h^{[2]}{}_{(2)} = \ell_\omega(h_{(1)}) \otimes h_{(2)}.\end{aligned}$$

Therefore ℓ_ω satisfies property (c). Finally one easily verifies that (c) combined with equation (6.3) imply property (d). □

Theorem 6.20. *Let A be a comodule algebra of H, set $B := A^{\mathrm{co}H}$, and assume that the antipode of H is bijective. Then the following statements are equivalent.*

1. *There exists $\ell \colon H \to A \otimes A$ such that*
   - *(a) $\ell(1_H) = 1_A \otimes 1_A$,*
   - *(b) $\overline{\mathrm{can}} \circ \ell = 1_A \otimes H$ (or $\mu_A \circ \ell = 1_A \circ \varepsilon_H$),*
   - *(c) $(\ell \otimes H) \circ \Delta_H = (A \otimes \rho^A) \circ \ell$,*
   - *(d) $(H \otimes \ell) \circ \Delta_H = ({}^A\rho \otimes A) \circ \ell$.*
2. *A is a faithfully flat (as a left and right B-module) Hopf–Galois extension.*

Proof. (1) $\Longrightarrow$ (2) The inverse of the canonical map is given as the following composite

$$\mathrm{can}^{-1} \colon A \otimes H \xrightarrow{A \otimes \ell} A \otimes A \otimes A \xrightarrow{\mu_A \otimes A} A \otimes A \to A \otimes_B A\,.$$

Since ℓ is a strong connection, A is faithfully flat as a left B-module by Corollary 6.17. By symmetry, A is a left Hopf–Galois extension, ℓ is a strong connection for this left Hopf–Galois extension, hence A is faithfully flat as the right B-module (by the left-handed version of Corollary 6.17).

(2) $\Longrightarrow$ (1) Since A is faithfully flat as a right B-module, for all left H-comodules V, there is a chain of isomorphisms

$$A \otimes_B (A \Box_H V) \cong (A \otimes_B A) \Box V \cong (A \otimes H) \Box_H V \cong A \otimes V.$$

The flatness of A as a B-module is crucial for the first isomorphism since, in general, the cotensor product does not commute with the tensor product. The second isomorphism is obtained by applying the canonical map. One uses this chain of isomorphisms to argue that $A \Box_H -$ is an exact functor as follows. Any exact sequence of left H-comodules $V \to W \to 0$ yields the exact sequence $A \otimes V \to A \otimes W \to 0$. By the constructed isomorphism, the sequence $A \otimes_B (A \Box_H V) \to A \otimes_B (A \Box_H W) \to 0$ is exact, hence also $A \Box_H V \to A \Box_H W \to 0$ is an exact sequence by the faithful flatness of A as a right B-module. Hence $A \Box_H -$ is right exact, and as it is always left exact, it is simply an exact functor.

For a finitely dimensional right H-comodule, (V, ρ^V), the dual vector space $V^* := \mathrm{Hom}_k(V, k)$ is a left H-comodule. Furthermore,

$$A \Box_H V^* \cong \mathrm{Hom}^H(V, A).$$

This implies that $\mathrm{Hom}^H(-, A)$ is exact, i.e. (A, ρ^A) is an injective H-comodule. In other words there is an H-colinear map $\pi \colon A \otimes H \to A$ such that $\pi \circ \rho^A = A$. Denote by ${}_A\mathbf{M}^H$ the category with objects left A-modules M that are also right H-comodules with a left A-linear coaction ρ^M, provided $M \otimes H$ is seen as a left A-module by the *diagonal action*, $a \cdot (m \otimes h) = a_{(0)} m \otimes a_{(1)} h$. Morphisms are maps which are both left A-linear and right H-colinear. For every $(M, \rho^M) \in {}_A\mathbf{M}^H$, there is a right H-colinear retraction of the coaction $\rho^M : M \to M \otimes H$ (i.e. (M, ρ^M) is injective as an H-comodule),

$$\pi_M \colon M \otimes H \to M\,, \qquad m \otimes h \mapsto \pi(1_A \otimes h S^{-1}(m_{(1)})) m_{(0)}\,.$$

Note that the bijectivity of the antipode plays here the most crucial role. The existence of π_M implies that every short exact sequence in ${}_A\mathbf{M}^H$ splits as a sequence in $\mathbf{M}^H$. In particular, $\overline{can} \colon A \otimes A \to A \otimes H$ is an epimorphism in ${}_A\mathbf{M}^H$, where $A \otimes A$ and $A \otimes H$ are comodules with coactions

$$\rho^{A \otimes A}(a \otimes a') = a_{(0)} \otimes a' \otimes a_{(1)},$$
$$\rho^{A \otimes H}(a \otimes h) = a_{(0)} \otimes h_{(2)} \otimes a_{(1)} S(h_{(1)}),$$

and left A-actions provided by the multiplication in A, $a \cdot (a' \otimes a'') = aa' \otimes a''$, $a \cdot (a' \otimes h) = aa' \otimes h$. Therefore, there is an H-colinear section $\alpha \colon A \otimes H \to A \otimes A$ of $\overline{\mathrm{can}}$. The map

$$s \colon A \to B \otimes A, \qquad a \mapsto a_{(0)} \alpha(1_A \otimes a_{(1)})\,,$$

is a left B-module splitting of the multiplication $B \otimes A \to A$. This shows that A is a projective left B-module. It remains to construct a section of the multiplication $B \otimes A \to A$ which is also right H-colinear.

Define a left B-module, right H-comodule map

$$\varphi\colon A\otimes H\to A,\qquad a\otimes h\mapsto a_{(0)}\pi(1_A\otimes S(a_{(1)})h).$$

A left B-module, right H-comodule splitting of the multiplication map $B\otimes A\to A$ is the composite

$$\sigma\colon A\xrightarrow{\rho^A}A\otimes H\xrightarrow{s\otimes H}B\otimes A\otimes H\xrightarrow{B\otimes\varphi}B\otimes A\,.$$

This can be checked as follows. Write

$$s(a)=\underbrace{a^{(1)}}_{\in B}\otimes\underbrace{a^{(2)}}_{\in A}\qquad\text{(summation implicit)}\,,$$

so that $a^{(1)}a^{(2)}=a$, and compute

$$\begin{aligned}
a&\xmapsto{\sigma}a_{(0)}{}^{(1)}\otimes a_{(0)}{}^{(2)}{}_{(0)}\pi(1_A\otimes S(a_{(0)}{}^{(2)}{}_{(1)})a_{(1)})\\
&\xmapsto{\mu_A}a_{(0)}{}^{(1)}a_{(0)}{}^{(2)}{}_{(0)}\pi(1_A\otimes S(a_{(0)}{}^{(2)}{}_{(1)})a_{(1)})\\
&=(a_{(0)}{}^{(1)}a_{(0)}{}^{(2)})_{(0)}\pi(1_A\otimes S((a_{(0)}{}^{(1)}a_{(0)}{}^{(2)})_{(1)})a_{(1)})\\
&=a_{(0)}\pi(1_A\otimes S(a_{(1)})a_{(2)})\\
&=a\pi(1_A\otimes 1_H)\\
&=a,
\end{aligned}$$

where the first equality follows by the left B-linearity of coaction ρ^A, the second one follows by the splitting property of s, the third one is the antipode axiom, and the last equality is a consequence of the fact that the composite $\pi\circ\rho^A$ is the identity on A.

Thus it has been proven that A is a Hopf–Galois extension that is an H-equivariantly projective left B-module. Theorem 6.16 now implies that there exists a strong connection and Theorem 6.19 yields the required map ℓ. $\square$

Definition 6.21. *A comodule algebra of a Hopf algebra H with a bijective antipode which satisfies conditions in Theorem 6.20 is called a* principal comodule algebra.

Principal comodule algebras are a non-commutative version of principal bundles which retains most of the features of the classical (commutative) objects.

Theorem 6.22 (The difficult part of Schneider's theorem)**.** *Let (A,ρ^A) be an H-comodule algebra that is injective as an H-comodule (i.e. there exists a right H-comodule map $\pi : A\otimes H\to A$, such that $\pi\circ\rho^A=A$). Assume that H has bijective antipode, and that lifted canonical map $\overline{\mathrm{can}}$ is surjective. Then A is a principal comodule algebra.*

Proof. Follow the same steps as in the part (2) $\Longrightarrow$ (1) in Theorem 6.20, starting from the existence of π. $\square$

Theorem 6.23. *Let A be a principal comodule algebra, $B=A^{\mathrm{co}H}$. For any finitely dimensional left H-comodule V, the associated B-module $\Gamma:=A\Box_H V$ is finitely generated and projective.*

Proof. By Theorem 6.15 Γ is projective as a left B-module. By arguments in proof of Theorem 6.20 $A \otimes_B \Gamma \cong A \otimes V$. On the other hand $A \otimes V$ is finitely generated as an A-module and A is faithfully flat right B-module, hence Γ is finitely generated as a left B-module. □

Put differently, Theorem 6.23 states that a principal comodule algebra defines a functor

$$A\Box_H - \colon {}^H\mathbf{M}_f \to {}_B\mathbf{P}_f \tag{6.6}$$

form finitely generated H-comodules to finitely generated projective B-modules.

On the other hand, principal comodule algebras can also be understood as monoidal functors. Start with a right H-comodule algebra (A, ρ^A) with coaction invariants B. Since the coaction ρ^A is right B-linear, there is a right B-action on $A\Box_H V$ defined by

$$\left(\sum_i a_i \otimes v_i\right) \cdot b = \sum_i a_i b \otimes v_i \,,$$

i.e. $A\Box_H V$ inherits B-bimodule structure from that in A. Both categories – of B-bimodules, ${}_B\mathbf{M}_B$, and left H-comodules, ${}^H\mathbf{M}$ – are monoidal, where the monoidal structure in ${}_B\mathbf{M}_B$ is the algebraic tensor product over B, while the monoidal structure in ${}^H\mathbf{M}$ is

$${}^{V\otimes W}\rho\colon V\otimes W \xrightarrow{{}^V\rho\otimes{}^W\rho} H\otimes V\otimes H\otimes W \xrightarrow{H\otimes\mathrm{flip}\otimes W} H\otimes H\otimes V\otimes W \xrightarrow{\mu_H\otimes V\otimes W} H\otimes W;$$

see the comments after the definition of a comodule algebra, Definition 5.9. The functor $A\Box_H - \colon {}^H\mathbf{M} \to {}_B\mathbf{M}_B$ is lax monoidal. It is monoidal if A is a Hopf–Galois extension such that A is faithfully flat as a right B-module.

Proposition 6.24 (Schauenburg-Ulbrich)**.** *If H has a bijective antipode, then there is a bijective correspondence between:*

1. *exact monoidal functors ${}^H\mathbf{M} \to_B \mathbf{M}_B$* (fibre functors)*,*
2. *principal comodule algebras.*

Example 6.25. Let A be a Hopf algebra with bijective antipode, and let $A \xrightarrow{\pi} H$ be a surjective map of Hopf algebras. Then A is a right H-comodule algebra with the coaction $\rho^A = (A\otimes\pi)\circ\Delta_A$, and $B = A^{\mathrm{co}H} = \{a \in A \mid a_{(1)} \otimes \pi(a_{(2)}) = a \otimes 1_H\}$. Suppose that there exists an H-bicomodule map $\iota\colon H \to A$ such that $\pi\circ\iota = H$ and $\iota(1_H) = 1_A$. Here H is understood as a left and right H-comodule via the regular coaction Δ_H and A is a left H-comodule by the induced coaction $(\pi\otimes A)\circ\Delta_A$. Then the map

$$\ell\colon H \to A\otimes A, \qquad h \mapsto S(\iota(h)_{(1)}) \otimes \iota(h)_{(2)},$$

satisfies conditions (a)–(d) in Theorem 6.20, so A is a principal H-comodule algebra (with a strong connection ℓ).

Example 6.26. As a particular application of Example 6.25, take A to be the coordinate algebra of functions on the quantum group $\mathrm{SU}_q(2)$. $A = \mathcal{O}(\mathrm{SU}_q(2))$ is generated by the 2×2 matrix of generators $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ subject to relations

$$\alpha\beta = q\beta\alpha,\ \ \alpha\gamma = q\gamma\alpha,\ \ \beta\gamma = \gamma\beta,\ \ \beta\delta = q\delta\beta,\ \ \gamma\delta = q\delta\gamma,$$

$$\delta\alpha - q^{-1}\beta\gamma = 1, \;\; \alpha\delta - q\beta\gamma = 1,$$

where q is a non-zero number. When k is the field of complex numbers and q is real, then $\mathcal{O}(\mathrm{SU}_q(2))$ is a $*$-algebra with

$$\alpha^* = \delta, \qquad \beta^* = -q\gamma, \qquad \gamma^* = -q^{-1}\beta, \qquad \delta^* = \alpha.$$

The algebra $\mathcal{O}(\mathrm{SU}_q(2))$ is a Hopf algebra with coproduct given by

$$\Delta_A(\alpha) = \alpha\otimes\alpha + \beta\otimes\gamma, \qquad \Delta_A(\beta) = \alpha\otimes\beta + \beta\otimes\delta,$$
$$\Delta_A(\gamma) = \gamma\otimes\alpha + \delta\otimes\gamma, \qquad \Delta_A(\delta) = \delta\otimes\delta + \gamma\otimes\beta,$$

and extended to the whole of $\mathcal{O}(\mathrm{SU}_q(2))$ as an algebra map. The counit is

$$\varepsilon_A(\alpha) = \varepsilon_A(\delta) = 1, \qquad \varepsilon_A(\beta) = \varepsilon_A(\delta) = 0,$$

and the antipode

$$S(\alpha) = \delta, \quad S(\beta) = -q^{-1}\beta, \quad S(\gamma) = -q\gamma, \quad S(\delta) = \alpha.$$

Let $H = \mathcal{O}(U(1)) = k[w, w^{-1}]$, a commutative Hopf algebra generated by the group-like elements w, w^{-1} (cf. Example 5.8). If H is made into a $*$-algebra with $w^* = w^{-1}$, then H is the algebra of polynomials on the circle. One easily finds that, similarly to the classical case, the (diagonal) map $\pi : \mathcal{O}(\mathrm{SU}_q(2)) \to H$ defined by

$$\pi(\alpha) = w, \quad \pi(\delta) = w^{-1}, \quad \pi(\beta) = \pi(\gamma) = 0,$$

is a Hopf algebra map. The induced coaction makes A a $\mathbb{Z}$-graded algebra with the grading

$$\deg(\alpha) = \deg(\gamma) = 1, \qquad \deg(\beta) = \deg(\delta) = -1.$$

The coaction invariants $B = A^{coH}$ are simply the degree-zero subalgebra of A. Thus B is generated by $x = -q^{-1}\beta\gamma$, $z = -q^{-1}\alpha\beta$ $z^* = \gamma\delta$. The elements x and z satisfy relations

$$zx = q^2xz, \quad xz^* = q^2z^*x, \quad zz^* = q^2x(1-q^2x), \quad z^*z = x(1-x).$$

(The coefficients are chosen so that for the $*$-algebra case x is real and z^* is the conjugate of z). An abstract algebra generated by x, z, z^* and the above relations is called a *standard* (or *polar*) *Podleś* (or *quantum*) *sphere* and is denoted by $\mathcal{O}(S^2_q)$.

A unital, H-bicolinear map splitting π is defined by

$$\iota\colon \mathcal{O}(U(1)) \to \mathcal{O}(\mathrm{SU}_q(2)), \qquad \iota(1) = 1, \quad \iota(w^n) = \alpha^n, \quad \iota(w^{-n}) = \delta^n.$$

The corresponding strong connection comes out as

$$\begin{aligned}
\ell(w^n) &= \sum_{k=0}^{n} \binom{n}{k}_{q^{-2}} \gamma^{*k}\alpha^{*n-k} \otimes \alpha^{n-k}\gamma^k,\\
\ell(1) &= 1\otimes 1,\\
\ell(w^{*n}) &= \sum_{k=0}^{n} q^{2k}\binom{n}{k}_{q^{-2}} \alpha^{n-k}\gamma^k \otimes \gamma^{*k}\alpha^{*n-k},
\end{aligned}$$

where the *deformed binomial coefficients* are defined for any number ζ by

$$\binom{n}{k}_\zeta = \frac{(\zeta^n - 1)(\zeta^{n-1} - 1)\dots(\zeta^{k+1} - 1)}{(\zeta^{n-k} - 1)(\zeta^{n-k-1} - 1)\dots(\zeta - 1)}.$$

This example describes a non-commutative version of the Hopf fibration with the Dirac monopole connection.

6.5 Separable functors and the bijectivity of the canonical map

The purpose of this section is to present an alternative proof of Theorem 6.22, the so-called 'difficult part' of Schneider's Theorem I in [s-h90]. Our methods were developed in [abm07], where they were used to prove a generalization of Schneider's theorem to Galois extensions by Hopf algebroids (cf. [bs04]).

Definition 6.27. *For any functor $F : \mathbf{A} \to \mathbf{B}$, an object P of $\mathbf{A}$ is said to be* relative projective w.r.t. F *(or shortly F-*projective*) provided that the map $\mathrm{Hom}_{\mathbf{A}}(P, p) : \mathrm{Hom}_{\mathbf{A}}(P, A) \to \mathrm{Hom}_{\mathbf{A}}(P, A')$ is surjective, for any morphism $p : A \to A'$ in $\mathbf{A}$ for which $F(p)$ is a split epimorphism.*

*Dually, an object I of $\mathbf{A}$ is said to be F-*injective *whenever $\mathrm{Hom}_{\mathbf{A}}(i, I) : \mathrm{Hom}_{\mathbf{A}}(A', I) \to \mathrm{Hom}_{\mathbf{A}}(A, I)$ is surjective, for any morphism $i : A \to A'$ in $\mathbf{A}$ for which $F(i)$ is a split monomorphism.*

Example 6.28. For an algebra B over a field k, let $F : \mathbf{M}_B \to \mathbf{M}_k$ be the forgetful functor. In $\mathbf{M}_k$ any epimorphism (i.e. surjective map) splits hence a right B-module M is F-projective if and only if it is a projective B-module, equivalently, if and only if the B-action $M \otimes B \to M$ possesses a right B-module section.

Dually, for a coalgebra C over a field k, let $F : \mathbf{M}^C \to \mathbf{M}_k$ be the forgetful functor. Then a right C-comodule M is F-injective if and only if it is an injective C-comodule i.e. if and only if the C-coaction $M \to M \otimes C$ possesses a right C-comodule retraction.

Proposition 6.29. *Consider a functor $F : \mathbf{A} \to \mathbf{B}$, possessing a right adjoint $G : \mathbf{B} \to \mathbf{A}$. Then for any objects A of $\mathbf{A}$ and B of $\mathbf{B}$, the object $F(A)$ is G-projective and $G(B)$ is F-injective. In particular, if $G(p)$ is a split epimorphism for some morphism $p : B \to F(A)$, then p is a split epimorphism. If $F(i)$ is a split monomorphism for some morphism $i : G(B) \to A$, then i is a split monomorphism.*

Proof. Take any morphism $f : F(A) \to B'$ and a morphism $p : B \to B'$ such that $G(p)$ has a section $\overline{G(p)}$. In terms of the unit u and the counit n of the adjunction (F, G), one constructs $g := n_B \circ F(\overline{G(p)}) \circ F(G(f)) \circ F(u_A) : F(A) \to B$ such that $p \circ g = f$. This proves that $F(A)$ is G-projective. F-injectivity of $G(B)$ is proven symmetrically. ☐

Definition 6.30. *A functor $F : \mathbf{A} \to \mathbf{B}$ is said to be* separable relative to some functor $R : \mathbf{S} \to \mathbf{A}$ *whenever the natural transformation*

$$\mathrm{Hom}_{\mathbf{A}}(-, R(-)) \to \mathrm{Hom}_{\mathbf{B}}(F(-), FR(-)), \qquad f \mapsto F(f) \tag{6.7}$$

(between functors $\mathbf{A}^{op} \times \mathbf{S} \to \mathrm{Set}$) is a split natural monomorphism.

A functor which is separable relative to the identity functor is called simply *separable*, see [nbo89]. The following proposition extends the fact [nbo89, Proposition 1.2] that separable functors reflect split epimorphisms and split monomorphisms.

Proposition 6.31. *If a functor $F : \mathbf{A} \to \mathbf{B}$ is separable relative to some functor $R : \mathbf{S} \to \mathbf{A}$ then, for any morphism f in $\mathbf{S}$, $F(R(f))$ is a split epimorphism (resp. a split monomorphism) if and only if $R(f)$ is a split epimorphism (resp. a split monomorphism).*

Proof. If $R(f)$ has a right (resp. left) inverse then so has $F(R(f))$, trivially. Conversely, if $F(R(f))$ possesses a right (or left) inverse $\overline{F(R(f))}$, then the retraction of the natural transformation (6.7) takes $\overline{F(R(f))}$ to a right (or left) inverse of $R(f)$. □

The following generalization of Rafael's theorem [r-m90, Theorem 1.2] provides criteria for the relative separability of a left adjoint functor.

Theorem 6.32. *Consider an adjunction $(F : \mathbf{A} \to \mathbf{B}, G : \mathbf{B} \to \mathbf{A})$ with unit $u : \mathbf{A} \to GF$ and counit $n : FG \to \mathbf{B}$. Then F is separable relative to some functor $R : \mathbf{S} \to \mathbf{A}$ if and only if $u_{R(-)}$ is a split natural monomorphism.*

Proof. In terms of a natural retraction $\overline{u_{R(-)}}$ of $u_{R(-)}$, a retraction of (6.7) is given by

$$\big(g : F(A) \to F(R(S))\big) \longmapsto \big(\overline{u_{R(S)}} \circ G(g) \circ u_A : A \to R(S)\big).$$

Conversely, in terms of a natural retraction Φ of (6.7), a retraction of $u_{R(-)}$ is constructed as $\Phi(n_{F(R(-))})$. □

The next theorem clarifies the relevance of relative separable functors in relation with Theorem 6.22.

Theorem 6.33. *Let H be a Hopf algebra over a field k. A right H-comodule algebra A is injective as an H-comodule if and only if the forgetful functor $F : \mathbf{M}^H \to \mathbf{M}_k$ is separable relative to the forgetful functor $R : \mathbf{M}_A^H \to \mathbf{M}^H$.*

Proof. Assume first that F is separable relative to R. The H-coaction $\varrho : A \to A \otimes H$ on an H-comodule algebra A is a morphism in $\mathbf{M}_A^H$ such that the monomorphism $F(R(\varrho))$ is split by the counit of H. Thus we conclude by Proposition 6.31 that $R(\varrho)$ is a split monomorphism in $\mathbf{M}^H$ i.e. A is an injective H-comodule (cf. Example 6.28).

Conversely, assume that A is an injective H-comodule i.e the H-coaction $\varrho : A \to A \otimes H$ possesses an H-comodule retraction π. Then for any relative Hopf module $M \in \mathbf{M}_A^H$, the map

$$M \otimes H \to M, \qquad m \otimes h \mapsto m_{(0)}\pi(1_A \otimes S(m_{(1)})h)$$

yields an H-comodule retraction of the H-coaction $M \to M \otimes H$ – that is, of the unit of the adjunction $(F : \mathbf{M}^H \to \mathbf{M}_k, (-) \otimes H : \mathbf{M}_k \to \mathbf{M}^H)$ – which is natural in $M \in \mathbf{M}_A^H$. Thus Theorem 6.32 implies that F is separable relative to R. □

Borrowing ideas from [b-t05, Theorem 4.4], one proves the following.

Proposition 6.34. *Let H be a Hopf algebra over a field k and let A be a right H-comodule algebra. If the lifted canonical map $\overline{\mathrm{can}} : A \otimes A \to A \otimes H$ is a split epimorphism in $\mathbf{M}_A^H$, then A is an H-Galois extension of $B := A^{coH}$ and a projective right B-module.*

Proof. If χ is a section of $\overline{\mathrm{can}}$ in $\mathbf{M}_A^H$, then the following diagram is commutative, with either *simultaneous* choice of the up- or down-pointing vertical arrows.

$$\begin{tikzcd}
(A \otimes A)^{coH} \otimes_B A \arrow[r, "\simeq"] \arrow[d, shift left, "\overline{\mathrm{can}} \otimes_B A"] & A \otimes B \otimes_B A \arrow[r, "\simeq"] & A \otimes A \arrow[d, shift left, "\overline{\mathrm{can}}"] \\
(A \otimes H)^{coH} \otimes_B A \arrow[r, "\simeq"'] \arrow[u, shift left, "\chi \otimes_B A"] & A \otimes_B A \arrow[r, "\mathrm{can}"'] & A \otimes H \arrow[u, shift left, "\chi"]
\end{tikzcd}$$

Thus we can construct can^{-1} as the composite map

$$A \otimes H \xrightarrow{\chi} A \otimes A \xrightarrow{\simeq} (A \otimes A)^{coH} \underset{B}{\otimes} A \xrightarrow{\overline{\mathrm{can}} \otimes_B A} (A \otimes H)^{coH} \underset{B}{\otimes} A \xrightarrow{\simeq} A \underset{B}{\otimes} A .$$

The restriction of χ yields a right B-module section $\chi^{coH} : (A \otimes H)^{coH} \simeq A \to (A \otimes A)^{coH} \simeq A \otimes B$ of the B-action on A. Thus A is a projective right B-module, cf. Example 6.28. □

An alternative proof of Theorem 6.22. Denote by F the forgetful functor $\mathbf{M}^H \to \mathbf{M}_k$ and denote by R the forgetful functor $\mathbf{M}_A^H \to \mathbf{M}^H$.

Regard $A \otimes A$ as a relative Hopf module via the A-action and H-coaction on the second factor. Regard $A \otimes H$ as a relative Hopf module via the diagonal A-action $(a' \otimes h)a := a'a_{(0)} \otimes ha_{(1)}$ and the H-coaction induced by the comultiplication of H. Then $\overline{\mathrm{can}}$ is a morphism in $\mathbf{M}_A^H$ such that $F(R(\overline{\mathrm{can}}))$ is surjective – hence a split epimorphism in $\mathbf{M}_k$. By Theorem 6.33, the functor F is separable relative to R hence $R(\overline{\mathrm{can}}) : A \otimes A \to A \otimes H$ is a split epimorphism in $\mathbf{M}^H$ by Proposition 6.31. Moreover, since the antipode S of the Hopf algebra H is bijective by assumption, the maps

$$H \otimes A \to A \otimes H, \qquad h \otimes a \mapsto a_{(0)} \otimes ha_{(1)} \qquad \text{and}$$
$$A \otimes H \to H \otimes A, \qquad a \otimes h \mapsto hS^{-1}(a_{(1)}) \otimes a_{(0)}$$

are mutually inverse isomorphisms of relative Hopf modules, where the A-action on $H \otimes A$ is given by multiplication in the second factor and the H-coaction is meant to be the diagonal one $h \otimes a \mapsto h_{(1)} \otimes a_{(0)} \otimes h_{(2)}a_{(1)}$. Since the functor R possesses a left adjoint $(-) \otimes A : \mathbf{M}^H \to \mathbf{M}_A^H$, it follows by Proposition 6.29 that the composite map

$$A \otimes A \xrightarrow{\overline{\mathrm{can}}} A \otimes H \xrightarrow{\simeq} H \otimes A$$

is a split epimorphism in $\mathbf{M}_A^H$. Hence also $\overline{\mathrm{can}}$ is a split epimorphism and we conclude by Proposition 6.34 that A is an H-Galois extension of B and a projective right B-module.

In terms of an H-comodule retraction π of the H-coaction ϱ on A, one constructs a right B-module retraction of the inclusion $B \hookrightarrow A$ by putting $a \mapsto \pi(1_A \otimes S^{-1}(a_{(1)}))a_{(0)}$. Thus A is a faithfully flat right B-module, see [r-l88, 2.11.29].

Since the antipode of H is bijective, the coaction

$$a \mapsto a_{(0)} \otimes S^{-1}(a_{(1)})$$

makes the opposite algebra A^{op} a right comodule algebra for the co-opposite Hopf algebra H_{cop} (i.e. the Hopf algebra defined by the same algebra structure in H, the comultiplication $h \mapsto h_{(2)} \otimes h_{(1)}$ and the antipode S^{-1}). The coinvariant subalgebra is $(A^{op})^{coH_{cop}} = B^{op}$. In terms of a right H-comodule retraction π of the right H-coaction on A, a right H_{cop}-comodule retraction of the H_{cop}-coaction on A^{op} is given by $\pi \circ (A \otimes S)$. Hence A^{op} is an injective H_{cop}-comodule. Moreover, the lifted canonical map

$$A^{op} \otimes A^{op} \to A^{op} \otimes H_{cop}, \qquad a \otimes a' \mapsto aa'_{(0)} \otimes S^{-1}(a'_{(1)})$$

differs from $\overline{\mathrm{can}}$ by an isomorphism, hence it is surjective. Consequently, we can repeat the proof for the H_{cop}-comodule algebra A^{op} to conclude that A^{op} is a faithfully flat right B^{op}-module, i.e. A is a faithfully flat left B-module. $\square$

References for this chapter are: [abm07], [bm93], [c-a85], [cq95], [dgh01], [d-y85], [h-pm95], [s-p04], [ssxx], [s-h90], [u-k87].

Chapter 7

Principal extensions and the Chern–Galois character

In the preceding chapter we have explained how a principal comodule algebra induces a functor from the category of finite dimensional comodules of a Hopf algebra to the category of finitely generated and projective modules over the coaction invariant subalgebra. When restricted to isomorphism classes this functor gives a map from the K-group of the Hopf algebra to the K-group of the invariant subalgebra. This can be followed by a map to the cyclic homology (the Chern-Connes character) thus providing one with homological tools for studying (invariants of) Hopf–Galois extensions. The composite mapping is known as the *Chern–Galois character* and we describe its construction (in a slightly more general set-up than the Hopf–Galois theory) in this chapter.

7.1 Coalgebra-Galois extensions

One of the main examples of principal bundles in classical geometry is provided by homogenous spaces of a Lie group. The following example shows how the classical construction of a principal bundle over a homogeneous space is performed in the realm of non-commutative geometry, and how it forces one to go beyond principal comodule algebras if one wants to develop fully an example driven approach to non-commutative principal bundles.

Example 7.1. Let A be a Hopf algebra. A subalgebra $B \subseteq A$ such that

$$\Delta_A(B) \subset A \otimes B$$

is called a left *A-comodule subalgebra*.

If we think of A as of an algebra of functions on a group G, B is an algebra of functions on a homogeneous space of G.

If A is faithfully flat as a left B-module, one can construct B as a coaction invariant subalgebra (this is the non-commutative counterpart of classical identification of a homogeneous space as a quotient space). First, define

$$B^+ = \ker \varepsilon_A \cap B.$$

Then $J := B^+A$ is a right ideal in A, and a *coideal* in A, i.e., for all $x = ba \in J$

$$\begin{aligned}\Delta_A(ba) &= b_{(1)}a_{(1)} \otimes b_{(2)}a_{(2)}\\ &= ba_{(1)} \otimes a_{(2)} - b_{(1)}a_{(1)} \otimes \varepsilon_A(b_{(2)})a_{(2)} + b_{(1)}a_{(1)} \otimes b_{(2)}a_{(2)}\\ &= \underbrace{ba_{(1)} \otimes a_{(2)}}_{\in J\otimes A} + \underbrace{b_{(1)}a_{(1)} \otimes (b_{(2)} - \varepsilon_A(b_{(2)}))a_{(2)}}_{\in A\otimes J}.\end{aligned}$$

Hence $C := A/J$ is a coalgebra and a right A-module, and $\pi\colon A \to C$ is a right A-linear coalgebra epimorphism. Note that J is not an (two-sided) ideal in A, hence in general C is not a quotient algebra (or Hopf algebra) of A. However, since π is a coalgebra map, A is a right C-comodule by

$$\rho^A = (A \otimes \pi) \circ \Delta_A.$$

Define the coaction invariant subalgebra

$$A^{\mathrm{co}C} := \{b \in A \mid \text{for all } a \in A,\ \rho^A(ba) = b\rho^A(a)\}.$$

Since ρ^A is a right A-module map, the coaction invariant subalgebra can be equivalently described as

$$A^{\mathrm{co}C} = \{b \in A \mid \rho^A(b) = b\rho^A(1_A)\}.$$

Note that, for all $b \in B$,

$$\rho^A(b) = b_{(1)} \otimes \pi(b_{(2)}) = b \otimes \pi(1_A) + b_{(1)} \otimes \pi(\underbrace{b_{(2)} - \varepsilon(b_{(2)})}_{\in B^+\subseteq J}) = b \otimes \pi(1_A),$$

so $B \subseteq A^{\mathrm{co}C}$. Faithful flatness implies also that $A^{\mathrm{co}C} \subseteq B$, that is A is an extension of B by a coalgebra C, but not necessarily by a bialgebra or a Hopf algebra, as one would naively expect guided by the classical geometric intuition. The reasons why the non-commutative geometry is reacher (or less rigid) than the classical one lie in the Poisson geometry and the reader is referred to lectures by N. Ciccoli.

The description of quantum homogeneous spaces as invariant subalgebras in Example 7.1 justifies a generalisation of Hopf–Galois extensions in which the symmetry is given by a coalgebra rather than a Hopf algebra.

Definition 7.2. *Let C be a coalgebra and let (A, ρ^A) be a C-comodule. Set*

$$B = A^{\mathrm{co}C} := \{b \in A \mid \textit{for all } a \in A,\ \rho^A(ba) = b\rho^A(a)\}.$$

A is called a coalgebra-Galois extension *if the canonical left A-linear right C-colinear map*

$$\mathrm{can}\colon A \otimes_B A \to A \otimes C, \qquad a \otimes_B a' \mapsto a\rho^A(a'),$$

is bijective.

Although C in a coalgebra-Galois extension does not need to be an algebra (or have an algebra structure compatible with the coaction and the algebra structure of A), nevertheless the fact that A is an algebra gives some more information about C. In particular, since $A \otimes_B A$ is an A-coring (see Section 5.5), also $A \otimes C$ can be made an A-coring via the isomorphism can in Definition 7.2. The coproduct in $A \otimes_B A$ is transported to a coproduct in $A \otimes C$ as

$$\Delta_{A\otimes C} \colon A \otimes C \to (A \otimes C) \otimes_A (A \otimes C) \cong A \otimes C \otimes C, \qquad \Delta_{A\otimes C} = A \otimes \Delta_C\,.$$

Furthermore, the right A-module structure on $A \otimes_B A$ induces a right A-module structure on $A \otimes C$,

$$(1_A \otimes c) \cdot a = \mathrm{can}(\mathrm{can}^{-1}(1_A \otimes c)a).$$

Define

$$\psi \colon C \otimes A \to A \otimes C, \qquad c \otimes a \mapsto \mathrm{can}(\mathrm{can}^{-1}(1_A \otimes c)a).$$

The map ψ is called a *canonical entwining* associated to the coalgebra-Galois extension $B \subseteq A$. The word entwining means that ψ makes the following *bow-tie diagram* commute

$$\begin{tikzcd}
 & C\otimes A\otimes A \arrow[ddl, "\psi\otimes A"'] \arrow[drr, "C\otimes\mu_A"] & & & & C\otimes C\otimes A \arrow[ddr, "C\otimes\psi"] & \\
 & & & C\otimes A \arrow[urr, "\Delta_C\otimes A"] \arrow[dr, "\varepsilon_C\otimes A"] \arrow[dd, "\psi"] & & & \\
A\otimes C\otimes A \arrow[ddr, "A\otimes\psi"'] & & C \arrow[ur, "C\otimes 1_A"] \arrow[dr, "1_A\otimes C"'] & & A & & C\otimes A\otimes C \arrow[ddl, "\psi\otimes C"] \\
 & & & A\otimes C \arrow[ur, "A\otimes\varepsilon_C"'] \arrow[drr, "A\otimes\Delta_C"'] & & & \\
 & A\otimes A\otimes C \arrow[urr, "\mu_A\otimes C"'] & & & & A\otimes C\otimes C. &
\end{tikzcd}$$

The commutativity of this bow-tie diagram for the canonical entwining can be checked by relating $A \otimes C$ to the Sweedler coring $A \otimes_B A$. In particular the right pentagon and the right triangle are a consequence of the definition of ψ in terms of right A-action on $A \otimes C$, while the left pentagon and triangle are responsible for right A-linearity of comultiplication $A \otimes \Delta_C$. An entwining is a special case of a (mixed) *distributive law* (in the sense of J. Beck).

Lemma 7.3. *In a coalgebra-Galois extension $B \subseteq A$,*

$$\rho^A(aa') = a_{(0)}\psi(a_{(1)} \otimes a'), \qquad \textit{for all } a, a' \in A,$$

where ψ is the canonical entwining. This means that (A, ρ^A) is an entwined module *(a comodule of A-coring $A \otimes C$).*

Proof. This is checked by the following calculation which uses the left linearity of can and can^{-1}, and the definition of can,

$$\begin{aligned}
a_{(0)}\psi(a_{(1)} \otimes a') &= a_{(0)}\,\mathrm{can}(\mathrm{can}^{-1}(1 \otimes a_{(1)})a') \\
&= \mathrm{can}(\mathrm{can}^{-1}(a_{(0)} \otimes a_{(1)})a') = \mathrm{can}(1_B \otimes_B aa') = \rho^A(aa').
\end{aligned}$$

□

Lemma 7.3 provides one with an explicit form of the coaction in terms of the canonical entwining

$$\rho^A(a) = 1_{A(0)}\psi(1_{A(1)} \otimes a).$$

To simplify further discussions, we assume that there is a grouplike element $e \in C$ such that

$$\rho^A(1_A) = 1_A \otimes e, \quad \text{so } \rho^A(a) = \psi(e \otimes a).$$

This is, for example, applicable to quantum homogeneous spaces described in Example 7.1, where $e = \pi(1_A)$.

Lemma 7.4. *The coaction invariant subalgebra of A can be equivalently described as*

$$B = \{b \in A \mid \rho^A(b) = b \otimes e\}.$$

Proof. If $b \in A^{\mathrm{co}C}$, then $\rho^A(b) = b\rho^A(1_A) = b \otimes e$. If $\rho^A(b) = b \otimes e$, then, for all $a \in A$,

$$\rho^A(ba) = b_{(0)}\psi(b_{(1)} \otimes a) = b\psi(e \otimes a) = b\rho^A(a).$$

□

Example 7.5. Let H be a Hopf algebra, and let (A, ρ^A) be a Hopf–Galois extension. Then the right action in the A-coring $A \otimes H$ induced from $A \otimes_B A$ is given by $(a' \otimes h)a = a'a_{(0)} \otimes ha_{(1)}$, hence

$$\psi\colon H \otimes A \to A \otimes H, \qquad h \otimes a \mapsto a_{(0)} \otimes ha_{(1)}.$$

Note that this ψ is bijective if and only if the antipode S is bijective. Then

$$\psi^{-1}(a \otimes h) = hS^{-1}a_{(1)} \otimes a_{(0)}.$$

7.2 Principal extensions

While defining principal comodule algebras we assumed that the Hopf algebra has a bijective antipode. Example 7.5 indicates that this assumption translates to coalgebra-Galois extensions into the bijectivity of the canonical entwining ψ. If ψ is bijective, then A is a left C-comodule by

$${}^A\!\rho\colon A \to C \otimes A\,, \qquad a \mapsto \psi^{-1}(a \otimes e) \quad (e \in C \text{ such that } \rho^A(1_A) = 1_A \otimes e).$$

Definition 7.6. *Let $B \subseteq A$ be a coalgebra-Galois extension by a coalgebra C, with a bijective canonical entwining map $\psi : C \otimes A \to A \otimes C$. Assume that $\rho^A(1_A) = 1_A \otimes e$ for a grouplike element $e \in C$. A k-linear map $\ell\colon C \to A \otimes A$ such that*

(a) $\ell(e) = 1_A \otimes 1_A$,
(b) $\mu_A \circ \ell = 1_A \circ \varepsilon_C$,
(c) $(A \otimes \rho^A) \circ \ell = (\ell \otimes C) \circ \Delta_C$,
(d) $({}^A\!\rho \otimes A) \circ \ell = (C \otimes \ell) \circ \Delta_C$,

is called a strong connection *in $B \subseteq A$. A coalgebra extension with a strong connection is called a* principal extension.

Following the same reasoning as in the principal comodule algebra case one proves

Proposition 7.7. *Let $B \subseteq A$ be a principal extension. Then*

1. *A is a C-equivariantly projective left (or right) B-module (i.e. there is a B-module, C-comodule splitting of the product map $B \otimes A \to A$).*

2. *A is a faithfully flat left (or right) B-module.*

3. *B is a direct summand in A as a left B-module.*

In terms of a strong connection the left B-comodule right C-comodule splitting of the multiplication map is $s(a) = a_{(0)}\ell(a_{(1)})$.

Proposition 7.8. *Let $B \subseteq A$ be a principal extension. If $(V, {}^V\rho)$ is a finite dimensional left C-comodule, then $\Gamma := A\Box_C V$ is a finitely generated and projective left B-module.*

Proof. One can follow the same arguments as in the case of a principal comodule algebra. The module Γ has a connection $\Gamma \ni a \otimes v \mapsto 1_A \otimes a \otimes v - a_{(0)}\ell(a_{(1)}) \otimes v$, hence it is a projective B-module. Consider the sequence of isomorphisms

$$A \otimes_B (A\Box_C V) \cong (A \otimes_B A)\Box_C V \cong (A \otimes C)\Box_C V \cong A \otimes V\,.$$

Since $A \otimes V$ is a finitely generated left A-module and B is a faithfully flat right B-module, $\Gamma := A\Box_C V$ is a finitely generated left B-module. $\square$

In view of Proposition 7.8, a principal extension $B \subseteq A$ can be understood as a functor

$$A\Box_C - \colon {}^C\mathbf{M}_f \to_B \mathbf{P}_f$$

from the category of finite dimensional C-comodules to the category of finitely generated projective B-modules. Passing to the Grothendieck group one obtains a map

$$\mathrm{Rep}(C) \to \mathrm{K}_0(B) \xrightarrow{\mathrm{ch}} \mathrm{HC}_{ev}(B),$$

where $\mathrm{Rep}(C)$ is the Grothendieck group of equivalence classes of finite dimensional comodules of C, ch denotes the Chern character, and $\mathrm{HC}_{ev}(B)$ is the even cyclic homology of B. This composite map is known as the *Chern–Galois character* and we will describe it in the Section 7.4.

7.3 Cyclic homology of an algebra and the Chern character

We begin by describing a cyclic homology of an algebra and the Chern character. For any algebra B, consider a bicomplex $\mathrm{CC}_*(B)$:

$$
\begin{tikzcd}
\vdots \arrow[d, "\partial_3"] & \vdots \arrow[d, "-\partial_3'"] & \vdots \arrow[d, "\partial_3"] & \vdots \arrow[d, "-\partial_3'"] & \\
B^{\otimes 3} \arrow[d, "\partial_2"] & B^{\otimes 3} \arrow[l, "\tilde{\tau}_2"'] \arrow[d, "-\partial_2'"] & B^{\otimes 3} \arrow[l, "N_2"'] \arrow[d, "\partial_2"] & B^{\otimes 3} \arrow[l, "\tilde{\tau}_2"'] \arrow[d, "-\partial_2'"] & \cdots \arrow[l, "N_2"'] \\
B^{\otimes 2} \arrow[d, "\partial_1"] & B^{\otimes 2} \arrow[l, "\tilde{\tau}_1"'] \arrow[d, "-\partial_1'"] & B^{\otimes 2} \arrow[l, "N_1"'] \arrow[d, "\partial_1"] & B^{\otimes 2} \arrow[l, "\tilde{\tau}_1"'] \arrow[d, "-\partial_1'"] & \cdots \arrow[l, "N_1"'] \\
B & B \arrow[l, "\tilde{\tau}_0"'] & B \arrow[l, "N_0"'] & B \arrow[l, "\tilde{\tau}_0"'] & \cdots, \arrow[l, "N_0"']
\end{tikzcd}
$$

where

$$
\begin{aligned}
\partial_n'(b_0 \otimes b_1 \otimes \cdots \otimes b_n) &= \sum_{i=0}^{n-1}(-1)^i b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n, \\
\partial_n(b_0 \otimes b_1 \otimes \cdots \otimes b_n) &= \partial_n'(b_0 \otimes b_1 \otimes \cdots \otimes b_n) + (-1)^n b_n b_0 \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_{n-1}, \\
\tau_n(b_0 \otimes \cdots \otimes b_n) &= (-1)^n b_n \otimes b_0 \otimes \cdots \otimes b_{n-1}, \\
\tilde{\tau}_n &= B^{\otimes(n+1)} - \tau_n, \\
N_n &= \sum_{i=0}^{n}(\tau_n)^i.
\end{aligned}
$$

The homology of the bicomplex $\mathrm{CC}_*(B)$ is known as the *cyclic homology* of B and is denoted by $\mathrm{HC}_*(B)$. In case k is a field of characteristic 0, the cyclic homology can be equivalently described as the homology of the *Connes complex* of B defined as

$$
\mathrm{C}_n^\lambda(B) := B^{\otimes(n+1)}/(\mathrm{id} - \tau_n),
$$
$$
\text{boundary: } \delta_n := \text{the quotient of } \partial_n.
$$

The homology of this complex is denoted by $\mathrm{H}^\lambda(B)$. The Chern and Chern–Galois characters can be defined with respect to either of these homologies, hence – for the convenience of the reader – we will describe both these constructions in parallel. The Chern character is a map $\mathrm{ch}\colon \mathrm{K}_0(B) \to \mathrm{HC}_{ev}(B)$ defined as follows. Take a class $[P] \in \mathrm{K}_0(B)$ of a finitely generated projective B-module P. P has a finite dual basis, say $x_i \in P$, $\pi_i \in {}_B\mathrm{Hom}(P, B)$, $i = 1, \ldots, n$. Since, for all $p \in P$,

$$
p = \sum_{i=1}^{n} \pi_i(p) x_i,
$$

the matrix $E := (E_{ij})_{i,j=1}^n := (\pi_j(x_i))_{i,j=1}^n$ is an idempotent with image P. With the idempotent E one associates a $2n$-cycle in $\mathrm{CC}_*(B)$. First define

$$
\widetilde{ch}_n(E) := \sum_{i_1, i_2, \ldots, i_{n+1}} E_{i_1 i_2} \otimes E_{i_2 i_3} \otimes \cdots \otimes E_{i_{n+1} i_1},
$$

and then $2n$-cycle

$$\bigoplus_{l=0}^{2n}(-1)^{\lfloor \frac{l}{2} \rfloor}\frac{l!}{\lfloor \frac{l}{2} \rfloor !}\widetilde{\mathrm{ch}}_l(E).$$

The class of this $2n$-cycle does not depend on the choice of P or E in $[P]$. Hence it defines an abelian group map

$$\mathrm{ch}\colon\ \mathrm{K}_0(B) \to \mathrm{HC}_{ev}(B),$$

known as the *Chern character*.

In the case of the Connes complex, with the idempotent E one associates a $2n$-cycle in $\mathrm{C}^{\lambda}_*(B)$ by taking the quotient of

$$\widetilde{ch}_{2n}(E) := \sum_{i_1,i_2,\ldots,i_{2n+1}} E_{i_1 i_2} \otimes E_{i_2 i_3} \otimes \cdots \otimes E_{i_{2n+1} i_1}.$$

Note that similar construction for even number of factors yields $0 \in \mathrm{C}^{\lambda}_*(B)$. The class of this $2n$-cycle does not depend on the choice of E or P in $[P]$. It is also compatible with the direct sums of P's and additive structure of $\mathrm{C}^{\lambda}_*(B)$. Hence it defines an abelian group map

$$\mathrm{ch}\colon\ \mathrm{K}_0(B) \to \mathrm{H}^{\lambda}_{ev}(B),$$

also known as the *Chern character*.

If $B = C^{\infty}(X)$, then

$$\mathrm{ch}\colon\ \mathrm{K}^0(X) \to \mathrm{H}^{ev}_{\mathrm{dR}}(X), \qquad [E] \mapsto \mathrm{Tr}(EdE \ldots dE),$$

which is simply the Chern character in differential geometry.

7.4 The Chern–Galois character

Let $B \subseteq A$ be a principal extension by a coalgebra C. Take a strong connection ℓ and introduce a Sweedler-type notation for ℓ,

$$\ell(c) = c^{\langle 1 \rangle} \otimes c^{\langle 2 \rangle}. \tag{7.1}$$

Let $(V, {}^V\!\rho)$ be a finite dimensional left C-comodule with a basis $\{v_i\}$. This defines an $n \times n$ matrix of elements $(e_{ij})^n_{i,j=1}$ by

$${}^V\!\rho(v_i) = \sum_{j=1}^{n} e_{ij} \otimes v_j.$$

The trace of $(e_{ij})^n_{i,j=1}$ is known as the *character of the comodule* V. The coassociativity of ${}^V\!\rho$ implies that (e_{ij}) is a *coidempotent matrix*, i.e.,

$$\Delta_C(e_{ij}) = \sum_{l=1}^{n} e_{il} \otimes e_{lj}, \qquad \varepsilon_C(e_{ij}) = \delta_{ij}, \qquad i,j = 1,\ldots,n.$$

Lemma 7.9. *For any* $c \in C$,

$$\ell(c_{(1)})\ell(c_{(2)}) \in A \otimes B \otimes A.$$

Proof. Use the introduced notation for the strong connection (7.1) and apply $A \otimes \rho^A \otimes A$ to $\ell(c_{(1)})\ell(c_{(2)})$ to obtain

$$\begin{aligned} {c_{(1)}}^{\langle 1\rangle} \otimes \rho^A({c_{(1)}}^{\langle 2\rangle}{c_{(2)}}^{\langle 1\rangle}) \otimes {c_{(2)}}^{\langle 2\rangle} &= {c_{(1)}}^{\langle 1\rangle} \otimes {{c_{(1)}}^{\langle 2\rangle}}_{(0)}\psi({{c_{(1)}}^{\langle 2\rangle}}_{(1)} \otimes {c_{(2)}}^{\langle 1\rangle}) \otimes {c_{(2)}}^{\langle 2\rangle} \\ &= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}\psi(c_{(2)} \otimes {c_{(3)}}^{\langle 1\rangle}) \otimes {c_{(3)}}^{\langle 2\rangle} \\ &= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}\psi({}^A\rho({c_{(2)}}^{\langle 1\rangle})) \otimes {c_{(2)}}^{\langle 2\rangle} \\ &= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}\psi(\psi^{-1}({c_{(2)}}^{\langle 1\rangle} \otimes e)) \otimes {c_{(2)}}^{\langle 2\rangle} \\ &= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}{c_{(2)}}^{\langle 1\rangle} \otimes e \otimes {c_{(2)}}^{\langle 2\rangle}. \end{aligned}$$

The first equality follows by the entwined module property of A, Lemma 7.3, the second one is the right colinearity of ℓ (condition (c) in Definition 7.6). The third equality follows by condition (d) in Definition 7.6 (left C-colinearity of a strong connection), next one is the definition of left coaction ${}^A\rho$. Finally, employ Lemma 7.4 to conclude that the middle term in $\ell(c_{(1)})\ell(c_{(2)})$ is an element of the coaction invariant subalgebra. □

Next we describe the Chern–Galois character in Connes' complex.

Theorem 7.10. *Given a finite dimensional C-comodule V and the corresponding coidempotent matrix $\mathbf{e} = (e_{ij})_{i,j=1}^n$, define*

$$\widetilde{\mathrm{chg}}_n(\mathbf{e}) := \sum_{i_1,i_2,\ldots,i_{n+1}} {e_{i_1i_2}}^{\langle 2\rangle}\ell(e_{i_2i_3})\ell(e_{i_3i_4})\ell(e_{i_3i_4})\ldots\ell(e_{i_{n+1}i_1}){e_{i_1i_2}}^{\langle 1\rangle} \in B^{\otimes(n+1)}.$$

Then $\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})$ is a $2n$-cycle in $\mathrm{C}^\lambda(B)$, $\widetilde{\mathrm{chg}}_{2n+1}(\mathbf{e}) = 0$. It does not depend on the choice of a basis for V and it is the same for isomorphic comodules.

Proof. Note that $\widetilde{\mathrm{chg}}_n(\mathbf{e})$ is an element of $B^{\otimes(n+1)}$ by Lemma 7.9. An easy calculation that uses $\mu_A \circ \ell = 1_A \circ \varepsilon_C$ gives

$$\partial_{2n}(\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})) = -\widetilde{\mathrm{chg}}_{2n-1}(\mathbf{e}).$$

Since

$$\tau_{2n-1}(\widetilde{\mathrm{chg}}_{2n-1}(\mathbf{e})) = -\widetilde{\mathrm{chg}}_{2n-1}(\mathbf{e}),$$

$\widetilde{\mathrm{chg}}_{2n-1}(\mathbf{e}) = 0$ in $\mathrm{C}^\lambda(B)$. Thus $\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})$ is a $2n$-cycle in $\mathrm{C}^\lambda(B)$.

The $\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})$ do not depend on the choice of basis and a representative in the isomorphism class of comodules, since they are defined only using the character of the comodule V, $\mathrm{tr}(\mathbf{e}) = \sum_i e_{ii}$. □

Similarly in the full cyclic bicomplex

Theorem 7.11. *Given a finite dimensional C-comodule V and the corresponding coidempotent matrix $\mathbf{e} = (e_{ij})_{i,j=1}^n$, define*

$$\widehat{\mathrm{chg}}_{2n} := \bigoplus_{l=0}^{2n} (-1)^{\lfloor \frac{l}{2} \rfloor} \frac{l!}{\lfloor \frac{l}{2} \rfloor !}\widetilde{\mathrm{chg}}_l(\mathbf{e}).$$

Then $\widehat{\mathrm{chg}}_{2n}$ is a $2n$-cycle in $\mathrm{CC}_*(B)$, and it does not depend on the choice of a basis for V and is the same for isomorphic comodules.

Proof. Since $\mu_A \circ \ell = 1_A \circ \varepsilon_C$, one finds

$$
\begin{aligned}
N_n(\widetilde{\mathrm{chg}}_n(\mathbf{e})) &= (n+1)\widetilde{\mathrm{chg}}_n(\mathbf{e}),\\
\partial_n(\widetilde{\mathrm{chg}}_n(\mathbf{e})) &= \widetilde{\mathrm{chg}}_{n-1}(\mathbf{e}), \quad \text{if } n \text{ is even},\\
\partial'_n(\widetilde{\mathrm{chg}}_n(\mathbf{e})) &= \widetilde{\mathrm{chg}}_{n-1}(\mathbf{e}),\\
\tilde{\tau}_n(\widetilde{\mathrm{chg}}_n(\mathbf{e})) &= 2\widetilde{\mathrm{chg}}_n(\mathbf{e}) \quad \text{if } n \text{ is odd}.
\end{aligned}
$$

This implies that $\widehat{\mathrm{chg}}_{2n}$ is a cycle in $CC_*(B)$ as claimed.

The independence (of the choice of a basis and a representative of an isomorphism class of comodules) follows by the same arguments as in the proof of Theorem 7.10. □

The cycles constructed in Theorem 7.10 or Theorem 7.11 might depend on the choice of a strong connection (at least their form explicitly depends on this choice). The full independence is achieved by going to homology.

Theorem 7.12. *The class of the Chern–Galois cycle* $\widehat{\mathrm{chg}}_{2n}(\mathbf{e})$ *(or* $\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})$ *in the case of the Connes complex) defines a map of abelian groups*

$$
\mathrm{chg}\colon \mathrm{Rep}(C) \to \mathrm{HC}_{ev}(B)\,, \tag{7.2}
$$

known as the Chern–Galois character *of the principal extension* $B \subseteq A$*. The Chern–Galois character is independent of the choice of a strong connection.*

Proof. The independence of chg on the choice of ℓ follows by observing that there is a factorisation

$$
\begin{tikzcd}
\mathrm{Rep}(C) \arrow[rr, "\mathrm{chg}"] \arrow[dr, "A\Box_C -"'] & & \mathrm{HC}_{ev}(B) \\
& \mathrm{K}_0(B) \arrow[ur, "\mathrm{ch}"'] &
\end{tikzcd}
$$

in which both factors are independent of ℓ.

In more detail, an idempotent for the left B-module $\Gamma = A\Box_C V$ is

$$
\mathbf{E} = (E_{(i,p),(j,q)}) := \varphi(l_p(e_{ij})x_q)_{(i,p),(j,q)},
$$

where φ is a left B-module retraction of $B \subseteq A$, which exists since B is a direct summand in A, (e_{ij}) is the coidempotent matrix defining the comodule V, $\{x_q\}$ is a finite basis of the subspace of A generated by the ${e_{ij}}^{\langle 1\rangle}{}_\nu$, where ν is a summation index in $\ell(c) = \sum_\nu c^{\langle 1\rangle}{}_\nu \otimes c^{\langle 2\rangle}{}_\nu$. Finally, $\ell_p = (\xi_p \otimes A) \circ \ell$, where $\{\xi_p\}$ is a dual basis to $\{x_q\}$. Then

$$
\widetilde{\mathrm{chg}}_n(\mathbf{e}) = \widetilde{\mathrm{ch}}_n(\mathbf{E}).
$$

This justifies the stated factorisation property. □

7.5 Example: the classical Hopf fibration

We illustrate the construction of the Chern–Galois character on the classical example of the Hopf fibration. The reader is encouraged to compare this example with its non-commutative counterpart described in Example 6.26. In this example we take $k = \mathbb{C}$, and

$$\mathrm{SU}(2) = \left\{ M = \begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix} \mid w, z \in \mathbb{C},\ \det(M) = 1 \right\}.$$

The condition $\det(M) = 1$ means that $|w|^2 + |z|^2 = 1$, i.e. SU(2) is a 3-sphere.

The algebra of functions on SU(2), $\mathcal{O}(\mathrm{SU}(2))$ is generated by

$$a\colon M \mapsto w, \quad c\colon M \mapsto z, \quad a^*\colon M \mapsto \bar{w}, \quad c^*\colon M \mapsto \bar{z},$$

with the relation

$$(aa^* + cc^*)(M) = w\bar{w} + z\bar{z} = 1.$$

Hence

$$A := \mathcal{O}(\mathrm{SU}(2)) = \mathbb{C}[a, a^*, c, c^*]/(aa^* + cc^* = 1).$$

There is an action of the group U(1) (the unit circle $\{u \in \mathbb{C} \mid |u|^2 = 1\}$) on SU(2):

$$\begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix} \cdot u = \begin{pmatrix} wu & -\overline{zu} \\ zu & \overline{wu} \end{pmatrix}.$$

The algebra $\mathcal{O}(\mathrm{U}(1))$ is generated by

$$x\colon u \mapsto u, \qquad x^*\colon u \mapsto \bar{u},$$

with the relation $xx^* = x^*x = 1$. Hence

$$\mathcal{O}(\mathrm{U}(1)) = \mathbb{C}[x, x^*]/(xx^* = x^*x = 1).$$

As a Hopf algebra

$$H := \mathcal{O}(\mathrm{U}(1)) = \mathbb{C}[\mathbb{Z}], \quad x^n \mapsto n,\ x^{*n} = x^{-n} \mapsto -n.$$

A comodule of $\mathcal{O}(\mathrm{U}(1))$ can be viewed as a $\mathbb{Z}$-graded vector space. In particular, the algebra $\mathcal{O}(\mathrm{SU}(2))$ is $\mathbb{Z}$-graded, $\deg(a) = \deg(c) = 1$, $\deg(a^*) = \deg(c^*) = -1$. In fact it is strongly graded, that is $\mathcal{O}(\mathrm{SU}(2))$ is a Hopf–Galois extension by $\mathcal{O}(\mathrm{U}(1))$. Invariant subalgebra B is a degree 0 part generated by the following three polynomials

$$\xi := aa^* - cc^*, \quad \eta := ac^* + ca^*, \quad \zeta := i(ac^* - ca^*),$$

satisfying $\xi^2 + \eta^2 + \zeta^2 = 1$. This means that B is an algebra of functions on the two-sphere, $B = \mathcal{O}(S^2)$.

Since SU(2) is a group, $A = \mathcal{O}(\mathrm{SU}(2))$ is a Hopf algebra with comultiplication

$$\Delta_A(a) = a \otimes a^* - c^* \otimes c, \qquad \Delta_A(c) = c \otimes a + a^* \otimes c.$$

The $\mathbb{Z}$-grading comes from the Hopf algebra map

$$\pi\colon A \to H, \qquad \pi(a) = x, \quad \pi(a^*) = x^*, \quad \pi(c) = \pi(c^*) = 0.$$

The algebra $\mathcal{O}(S^2)$ is an algebra of functions on a homogenous space. The connection is determined by an H-colinear map (see Example 6.25)

$$\iota\colon H = \mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(\mathrm{SU}(2)) = A, \qquad x^n \mapsto a^n, \quad x^{*n} \mapsto a^{*n}, \quad 1 \mapsto 1.$$

The resulting strong connection form, $\ell(x) = a^* da + c^* dc$, is known as the Dirac monopole connection.

To compute the Chern–Galois character (for line bundles), take smooth functions on SU(2) and define

$$\begin{aligned} A := \widehat{C}(\mathrm{SU}(2)) &= \{f \in C^\infty(\mathrm{SU}(2)) \mid \widehat{\rho}(f) \in C^\infty(\mathrm{SU}(2)) \otimes \mathcal{O}(\mathrm{U}(1))\} \\ &= \bigoplus_{n\in\mathbb{Z}} C^\infty_n(\mathrm{SU}(2)), \end{aligned}$$

where $\widehat{\rho}(f)(x,g) = f(xg)$, and $C^\infty_n(\mathrm{SU}(2))$ is the algebra of smooth functions on S^2 and all polynomials of $\mathbb{Z}$-degree n on SU(2) (recall that $\mathcal{O}(\mathrm{SU}(2))$ is a strongly $\mathbb{Z}$-graded algebra). Then $\widehat{C}(\mathrm{SU}(2))$ is a Hopf–Galois extension of $B := C^\infty(S^2)$ by $H = \mathcal{O}(\mathrm{U}(1)) \cong \mathbb{C}[\mathbb{Z}]$.

For any $n \in \mathbb{Z}$, take a one-dimensional left H-comodule $(V_n, {}^{V_n}\rho)$ with coaction

$${}^{V_n}\rho(v) := x^n \otimes v.$$

Then

$$\Gamma_{-n} = \widehat{C}(\mathrm{SU}(2)) \square_{\mathcal{O}(\mathrm{U}(1))} V_n = C^\infty_n(\mathrm{SU}(2))$$

is a line bundle over S^2. The idempotents for Γ_{-n} coming from the strong connection induced by ι can be written explicitely. For example, for Γ_{-1},

$$\mathbf{E}_{-1} = \begin{pmatrix} aa^* & ac^* \\ ca^* & cc^* \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1+\xi & \eta - i\zeta \\ \eta + i\zeta & 1-\xi \end{pmatrix}.$$

Furthermore, the Chern–Galois character is given by the following diagram

$$\begin{tikzcd} \mathrm{Rep}(\mathrm{U}(1)) = \mathrm{Rep}(\mathcal{O}(\mathrm{U}(1))) \arrow[r] & \mathrm{K}_0(C^\infty(S^2)) \arrow[r] \arrow[d, equal] & \mathrm{H}^\lambda_{ev}(C^\infty(S^2)) \arrow[d] \\ & \mathrm{K}^0(S^2) \arrow[r] & \mathrm{H}_{\mathrm{dR}}(S^2). \end{tikzcd}$$

In particular, the first two terms of the Chern (or Chern–Galois) character come out as

$$\begin{aligned} \mathrm{Tr}(E_{-1}) &= 1, \\ \mathrm{Tr}(E_{-1} dE_{-1} dE_{-1}) &= \tfrac{1}{2}(\xi d\eta \wedge d\zeta + \eta d\zeta \wedge d\xi + \zeta d\xi \wedge d\eta). \end{aligned}$$

Integration over the sphere S^2 gives the Chern number

$$\mathrm{ch}(\Gamma_{-1}) = \frac{1}{2\pi i}\int_{S^2} \mathrm{Tr}(E_{-1} dE_{-1} dE_{-1}) = -1.$$

Similarly, for Γ_{-n} we compute

$$\mathrm{ch}(\Gamma_{-n}) = \frac{1}{2\pi i}\int_{S^2} \mathrm{Tr}(E_{-n} dE_{-n} dE_{-n}) = -n.$$

7.6 Ehresmann cyclic homology

Let P be a principal H-comodule algebra, and $B = P^{\mathrm{co}H}$ its coinvariants algebra.

$$\begin{tikzcd}
 & \mathrm{HC}_{\mathrm{ev}}(P\Box_H P) \arrow[dr] & \\
\mathrm{CoRep}(H) \arrow[ur] \arrow[rr] \arrow[dr, "{[A\Box_H -]}"'] & & \mathrm{H}^{\lambda}_{ev}(B) \\
 & \mathrm{K}_0(B) \arrow[ur] &
\end{tikzcd}$$

Definition 7.13. *Let $\pi\colon X \to M$ be a principal G-bundle. Define the* Ehresmannn groupoid *as the quotient with respect to diagonal action $\widetilde{G} := (X\times X)/G$. If $\pi(y) = \pi(p)$ for some $y, p \in X$, then there exist $\check{\tau}(p, y) \in G$ such that $y = \check{\tau}(p, y)p$. Then we identify*

$$[(x, y)][(p, q)] = [(x, q\check{\tau}(p, y))].$$

Definition 7.14. *For a principal H-comodule algebra P define an* Ehresmannn bialgebroid

$$\widetilde{H} := P\Box_H P = \{p \otimes q \in P \otimes P \mid p_{(0)} \otimes P_{(1)} \otimes q = p \otimes \underbrace{S^{-1}(q_{(1)})}_{q_{(-1)}} \otimes q_{(0)}\}. \quad (7.3)$$

$\widetilde{H}$ is an augmented B-bimodule, that is there is a $B\otimes B^{op}$-linear map $\varepsilon\colon \widetilde{H} \to B$

$$\begin{tikzcd}
P\Box_H P \arrow[r] \arrow[d, "\varepsilon"'] & P \otimes P \arrow[d] \\
B \arrow[r] & P
\end{tikzcd}$$

Define coproduct $\Delta\colon \widetilde{H} \to \widetilde{H} \otimes_B \widetilde{H}$ as

$$\Delta(p \otimes q) := p \otimes q^{[1]}_{(-1)} \otimes q^{[2]}_{(-1)} \otimes q_{(0)},$$

where the indices $[1], [2]$ are Sweedler notation for

$$\mathrm{can}^{-1}(1 \otimes c) = \tau(c) = c^{[1]} \otimes_B c^{[2]}.$$

Then $(\widetilde{H}, \Delta, \varepsilon)$ is a B-coring with the subalgebra structure of $P \otimes P^{op}$. It is a B-bialgebroid.

7.6.1 Precyclic complex

Let M be an ε-augmented B-bimodule. Then for every $n \in \mathbb{N}$ there are maps

$$\begin{aligned}
\delta_i\colon M^{\otimes n+1} &\to M^{\otimes n},\ i = 0, \dots, n,\\
\delta_i(m_0 \otimes \cdots \otimes m_n) &= m_0 \otimes \cdots \otimes m_{i-1} \otimes \varepsilon(m_i)m_{i+1} \otimes \cdots \otimes m_n,\\
t_n\colon M^{\otimes n+1} &\to M^{\otimes n+1},\ t_n(m_0 \otimes \cdots \otimes m_n) = m_n \otimes m_0 \otimes \cdots \otimes m_{n-1}.
\end{aligned}$$

Together they define a precyclic module (no degeneracy maps).

7.6.2 Strong connection

$$H \xrightarrow{\text{flip}\circ l} P\Box_H p \ni c^{\langle 1\rangle} \otimes c^{\langle 2\rangle}$$
$$c^{\langle 1\rangle}_{(1)} \otimes c^{\langle 2\rangle}_{(1)} \otimes c^{\langle 1\rangle}_{(2)} \otimes c^{\langle 2\rangle}_{(2)} \in P \otimes \widetilde{H} \otimes P.$$

The element $c := \sum_i e_{ii}$ enjoys the cyclic property:

$$c_{(1)} \otimes \cdots \otimes c_{(n)} = c_{(n)} \otimes c_{(1)} \otimes \cdots \otimes c_{(n-1)}.$$

Lemma 7.15. *For a coaction $\rho\colon V \to H \otimes V$ the element*

$$E_n(\rho) := c^{\langle 2\rangle}_{(n+1)} \otimes c^{\langle 1\rangle}_{(1)} \otimes \cdots \otimes c^{\langle 2\rangle}_{(n)} \otimes c^{\langle 1\rangle}_{(2)}$$

is a cyclic cocycle in $\widetilde{H}^{n+1}/(\mathrm{id} - (-1)^n t_n)$.

7.6.3 Ehresmann factorisation

Proposition 7.16 (Böhm-Hajac)**.** *There is a factorisation of the Chern–Galois character*

$$\mathrm{chg}_{2n}\colon \mathrm{CoRep}(H) \to \mathrm{HC}_{2n}(P\Box_H P) \xrightarrow{\varepsilon^{\otimes 2n+1}} \mathrm{HC}_{2n}(B).$$

References for this chapter are: [b-j69], [bb05], [bh04], [bm98], [c-a85], [l-j98].

Chapter 8

Appendix: Remarks on functors and natural transformations

8.1 Natural transformations

Natural transformations can be composed vertically and horizontally, and these operations agree via the middle interchange law. For, consider the diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, bend left=60, "F"] \arrow[r, "F'"] \arrow[r, bend right=60, "F''"'] & \mathbf{B} \arrow[r, bend left=60, "G"] \arrow[r, "G'"] \arrow[r, bend right=60, "G''"'] & \mathbf{C} \end{tikzcd} \tag{8.1}$$

of categories, functors, and natural transformations. While the vertical composite $\sigma'\sigma\colon F \to F''$ is defined by

$$(\sigma'\sigma)_A = \sigma'_A\sigma_A\colon F(A) \to F''(A), \tag{8.2}$$

the vertical composite $\tau\sigma\colon GF \to G'F'$ is defined by

$$(\tau\sigma)_A = G'(\sigma_A)\tau_{F(A)} = \tau_{F'(A)}G(\sigma_A)\colon GF(A))G'F'(A) \tag{8.3}$$

(in both cases for all objects A in $\mathbf{A}$); here the equality $G'(\sigma_A)\tau_{F(A)} = \tau_{F'(A)}G(\sigma_A)$ is simply the commutativity of the naturality square

$$\begin{tikzcd} GF(A) \arrow[r, "G(\sigma_A)"] \arrow[d, "\tau_{F(A)}"'] & GF'(A) \arrow[d, "\tau_{F'(A)}"] \\ G'F(A) \arrow[r, "G(\sigma_A)"] & G'F'(A) \end{tikzcd} \tag{8.4}$$

Furthermore, the rows and the columns of (8.4) are in fact components of the natural transformations

$$G\sigma\colon GF \to GF' \text{ defined by } (G\sigma)_A = G(\sigma_A) \tag{8.5}$$

and

$$F\colon GF \to G'F \text{ defined by } (\tau F)_A = \tau_{F(A)} \tag{8.6}$$

respectively. Using these natural transformations, the commutativity of (8.4) for all A in $\mathbf{A}$ can be expressed as the commutativity of

$$\begin{tikzcd} GF \arrow[r, "G\sigma"] \arrow[d, "\tau F"'] & GF' \arrow[d, "\tau F'"] \\ G'F \arrow[r, "G'\sigma"'] & G'F' \end{tikzcd} \tag{8.7}$$

We also have

$$G\sigma = 1_G\sigma \text{ and } F = \tau 1_F, \tag{8.8}$$

and the commutativity of (8.7), written as the equality

$$(1_G\sigma)(\tau 1_F) = (\tau 1_{F'})(1_G\sigma), \tag{8.9}$$

can be deduced from the *middle interchange law*

$$(\tau'\tau)(\sigma'\sigma) = (\tau'\sigma')(\tau\sigma), \tag{8.10}$$

written here for the situation (8.1). Indeed, applying (8.10) to

$$\begin{tikzcd} \mathbf{A} \arrow[r, bend left=60, "F"] \arrow[r, "F"] \arrow[r, bend right=60, "F'"'] & \mathbf{B} \arrow[r, bend left=60, "G"] \arrow[r, "G'"] \arrow[r, bend right=60, "G'"'] & \mathbf{C} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{A} \arrow[r, bend left=60, "F"] \arrow[r, "F'"] \arrow[r, bend right=60, "F'"'] & \mathbf{B} \arrow[r, bend left=60, "G"] \arrow[r, "G"] \arrow[r, bend right=60, "G'"'] & \mathbf{C} \end{tikzcd}$$

we obtain

$$(1_{G'}\tau)(\sigma 1_F) = (1_{G'}\sigma)(\tau 1_F) \text{ and } (\tau 1_G)(1_{F'}\sigma) = (\tau 1_{F'})(1_G\sigma) \tag{8.11}$$

respectively, which gives

$$(1_{G'}\sigma)(\tau 1_F) = (1_{G'}\tau)(\sigma 1_F) = \tau\sigma = (\tau 1_G)(1_{F'}\sigma) = (\tau 1_{F'})(1_G\sigma) \tag{8.12}$$

On the other hand the middle interchange law (8.10) can itself be obtained using the appropriate commutative diagrams of the form (8.7), which is easy to show using the diagram

$$
\begin{tikzcd}
GF \arrow[r, "G\sigma"] \arrow[d, "\tau F"'] & GF' \arrow[r, "G\sigma'"] \arrow[d, "\tau F'"] & GF'' \arrow[d, "\tau F''"] \\
G'F \arrow[r, "G'\sigma"'] \arrow[d, "\tau' F"'] & G'F' \arrow[r, "G'\sigma'"'] \arrow[d, "\tau' F'"] & G'F'' \arrow[d, "\tau' F''"] \\
G''F \arrow[r, "G''\sigma"'] & G''F' \arrow[r, "G''\sigma'"'] & G''F''
\end{tikzcd}
\tag{8.13}
$$

whose four small squares are of the form (8.7) (for various functors involved): one way of doing it is to write

$$
\begin{aligned}
(\tau'\tau)(\sigma'\sigma) &= ((\tau'\tau)F'')(G(\sigma'\sigma)) \\
&\text{(by the definition of the horizontal composite of } \tau'\tau \text{ with } \sigma'\sigma) \\
&= (\tau'F'')(\tau F'')(G\sigma')(G\sigma) \\
&\text{(by obvious properties of the “usual” composition)} \\
&= (\tau'F'')(G'\sigma')(\tau F')(G\sigma) \\
&\text{(by commutativity of the right-hand top square in (8.13))} \\
&= (\tau'\sigma')(\tau\sigma) \\
&\text{(by the definition of the horizontal composites of } \tau' \text{ with } \sigma' \\
&\text{and of } \tau \text{ with } \sigma).
\end{aligned}
$$

Note, however, that good understanding of all these calculations requires seeing horizontal composition as functors

$$\mathbf{Cat}(\mathbf{B},\mathbf{C})) \times \mathbf{Cat}(\mathbf{A},\mathbf{B}) \to \mathbf{Cat}(\mathbf{A},\mathbf{C}), \tag{8.14}$$

where $\mathbf{Cat}(\mathbf{A},\mathbf{B})$ denotes the category of all functors $\mathbf{A} \to \mathbf{B}$, etc.

8.1.1 The Hom functors

For a fixed object A in a category $\mathbf{X}$ one can form the covariant Hom functor

$$\mathrm{Hom}_{\mathbf{X}}(A,-)\colon \mathbf{X} \to \mathbf{Sets}, \tag{8.15}$$

sending a morphism $f\colon \mathbf{X} \to \mathbf{Y}$ of $\mathbf{X}$ to the map

$$\mathrm{Hom}_{\mathbf{X}}(A,f) : \mathrm{Hom}_{\mathbf{X}}(A,X))\,\mathrm{Hom}_{\mathbf{X}}(A,Y) \text{ defined by } \alpha \mapsto f\alpha$$

and the contravariant Hom functor

$$\mathrm{Hom}_{\mathbf{X}}(-,A)\colon \mathbf{X}^{op} \to \mathbf{Sets}, \tag{8.16}$$

sending a morphism $f\colon \mathbf{X} \to \mathbf{Y}$ of $\mathbf{X}$ to the map

$$\mathrm{Hom}_{\mathbf{X}}(f, A)\colon\ \mathrm{Hom}_{\mathbf{X}}(Y, A))\,\mathrm{Hom}_{\mathbf{X}}(X, A) \text{ defined by } \alpha \mapsto \alpha f.$$

Moreover, these two constructions agree in the sense that one can also form the functor

$$\mathrm{Hom}\colon \mathbf{X}^{op} \times \mathbf{X} \to \mathbf{Sets} \tag{8.17}$$

sending a morphism $(f, f')\colon (X, X') \to (Y, Y')$ of $\mathbf{X}^{op} \times \mathbf{X}$ to the map

$$\mathrm{Hom}_{\mathbf{X}}(f, f')\colon\ \mathrm{Hom}_{\mathbf{X}}(X, X') \to \mathrm{Hom}_{\mathbf{X}}(Y, Y') \text{ defined by } \varphi \mapsto f'\varphi f,$$

and we have

$$\mathrm{Hom}_{\mathbf{X}}(A, f) = \mathrm{Hom}_{\mathbf{X}}(1_A, f) \text{ and } \mathrm{Hom}_{\mathbf{X}}(f, A) = \mathrm{Hom}_{\mathbf{X}}(f, 1_A) \tag{8.18}$$

in the situations (8.15) and (8.16), and

$$\mathrm{Hom}_{\mathbf{X}}(f, f') = \mathrm{Hom}_{\mathbf{X}}(Y, f')\,\mathrm{Hom}_{\mathbf{X}}(f, X') = \mathrm{Hom}_{\mathbf{X}}(f, Y')\,\mathrm{Hom}_{\mathbf{X}}(X, f') \tag{8.19}$$

in the situation (8.17).

Note that we use "covariant Hom functor" and "contravariant Hom functor" only as convenient expressions, not as instances of "covariant/contravariant functors" - assuming the convention that there are only functors that always covariant, and a "contravariant functor", say, from $\mathbf{A}$ to $\mathbf{B}$, should either be seen as a functor $\mathbf{A}^op \to \mathbf{B}$ or as functor $\mathbf{A} \to \mathbf{B}^{op}$ (and these two functors are dual to each other). For instance it is important that the contravariant Hom functor $\mathrm{Hom}_{\mathbf{X}}(-, A)$ is defined as a functor $\mathbf{X}^{op} \to \mathbf{Sets}$, and not as a functor $\mathbf{X} \to \mathbf{Sets}^{op}$.

8.2 Limits and colimits

8.2.1 General case

For a graph

$$G = \begin{tikzcd} G_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & G_0 \end{tikzcd} \tag{8.20}$$

we will write, as usually, $f\colon x \to y$ when f is in G_1 and $d(f) = x$ and $c(f) = y$. For a category $\mathbf{C}$ and a diagram $D\colon \mathbf{G} \to \mathbf{C}$ a cone over D is a system $(C, \varphi) = (C, (\varphi_x\colon C \to D(x))_{x \in G_0})$, in which C is an object in $\mathbf{C}$, and $\varphi_x\colon C \to D(x)$ for $x \in G_0$ morphisms in $\mathbf{C}$, making the diagram

$$\begin{tikzcd} & C \arrow[dl, "\varphi_x"'] \arrow[dr, "\varphi_y"] & \\ D(x) \arrow[rr, "D(f)"'] & & D(y) \end{tikzcd} \tag{8.21}$$

commute for every $f\colon x \to y$ in G. A morphism $\gamma\colon (C, \varphi) \to (C', \varphi')$ of cones over D is a morphism $\gamma\colon C \to C'$ in $\mathbf{C}$, making the diagram

$$\begin{tikzcd} C \arrow[rr, "\gamma"] \arrow[dr, "\varphi_x"'] & & C' \arrow[dl, "\varphi_y"] \\ & D(x) & \end{tikzcd} \tag{8.22}$$

commute for every x in G. The category of cones over D will be denoted by $\mathbf{Con}(D)$, and its terminal object

$$\lim D = (\lim D, \pi) \tag{8.23}$$

(provided it exists) is called the *limit* of D. The morphisms π_x are then called the limit projections. There are many important special cases, some of which are listed below.

8.2.2 Products

In the notation above, when G_1 is empty, and therefore the graph G can be identified with the set G_0, we write

$$\lim D = \prod_{x \in G} D(x) = (\prod_{x \in G} D(x), \pi) \tag{8.24}$$

and call this limit the product of the family $(D(x))_{x \in G}$. In particular, it is easy to that:

- When G is empty, $\prod_{x \in G} D(x)$ is nothing but the terminal object in C.
- When $G = \{x\}$ is a one-element set, $\prod_{x \in G} D(x) = D(x)$.
- When G has (exactly) two elements, whose images under D are A and B, we have $\prod_{x \in G} D(x) = A \times B$.

  And more generally, when G has n elements, whose images under D are $A_1, \ldots, A_n$, it is convenient to write $\prod_{x \in G} D(x) = A_1 \times \cdots \times A_n$.

8.2.3 Infima

If $\mathbf{C}$ is an ordered set considered as a category, then for every $D \colon G \to \mathbf{C}$ we have

$$\lim D = \prod_{x \in G_0} D(x) = \bigwedge_{x \in G} D(x) = \inf\{D(x) \mid x \in G_0\}, \tag{8.25}$$

i.e. $\lim D$ is the infimum of the set $\{D(x) \mid x \in G_0\}$ in $\mathbf{C}$.

8.2.4 Equalizers

Let G be a graph that has two objects x and y, and two morphisms from x to y, and let D be the diagram sending those two morphisms to

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \tag{8.26}$$

Then to give a cone over D is to give a morphism $h \colon X \to A$ with $fh = gh$. Therefore the limit of D can be identified with a pair (E, e), in which $e \colon E \to A$ is a morphism in $\mathbf{C}$ such that:

1. $fe = ge$;
2. if $fh = gh$ as above, then there exists a unique morphism $u \colon X \to E$ with $eu = h$.

Such a pair (E, e) is called the equalizer of the pair (f, g).

8.2.5 Pullbacks

Let G be a graph that has three objects x, y, and z, one morphism from x to z, and one morphism from y to z, and let D be the diagram sending those two morphisms to

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{8.27}$$

Then to give a cone over D is to give a morphisms $h\colon X \to A$ and $k\colon X \to B$ with $fh = gk$. Therefore the limit of D can be identified with a triple (P, p, q), in which $p\colon P \to A$ and $q\colon Q \to B$ are morphisms in $\mathbf{C}$ such that:

1. $fp = gq$;
2. if $fh = gk$ as above, then there exists a unique morphism $u\colon X \to P$ with $pu = h$ and $qu = k$.

As suggested by the display

$$\begin{tikzcd} X \arrow[dr, dashed, "u"] \arrow[drr, "k"] \arrow[ddr, "h"'] & & \\ & P \arrow[r, "q"'] \arrow[d, "p"] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd} \tag{8.28}$$

the limit of D is called the *pullback* of f and g. One also says that:

- the square formed by f, g, p, q is a *pullback square*, or a *cartesian square*;
- p is a pullback of g along f, and q is a pullback of f along g;
- P is a fibred product of (A, f) and (B, g) (since indeed, $(P, fp) = (P, gq)$ is the product of (A, f) and (B, g) in the category $(\mathbf{C} \downarrow C)$; another good reason is that, say, for $\mathbf{C} = \mathbf{Sets}$, it turnes out that the fibres of $fp = gq$ are the products of the corresponding fibres of f and g). One also writes $P = A \times (f, g)B = A \times_C B$.

8.2.6 Examples of limits

In many concrete categories, including **Sets**, all varieties of universal algebras, and the category of topological spaces, limits can be constructed as follows: the products are the same as the usual cartesian products, and then

$$\lim D = \{(a_x)_{x \in G_0} \in \prod_{x \in G_0} D(x) \mid D(f)(a_x) = a_y \text{ for each } f\colon x \to y \text{ in } G\}, \tag{8.29}$$

in the notation above, with $\pi_x\colon \lim D \to D(x)$ being induced by the corresponding usual product projection for each x in G_0. In particular the equalizer of a pair (8.26) of parallel morphisms in $\mathbf{C}$ can be identified with

$$\{a \in A \mid f(a) = g(a)\}, \tag{8.30}$$

and for the pullback in (8.28) we can write

$$A \times_{(f,g)} B = \{(a,b)) \in A \times B \mid f(a) = g(b)\} \tag{8.31}$$

8.2.7 Colimits

The colimit of a diagram $D\colon G \to \mathbf{C}$ is the same as the limit of the dual diagram $D^{op}\colon G^{op} \to \mathbf{C}^{op}$. That is, the notion of colimit is simply dual to the notion of limit. And all special limits above have their dual versions: coproducts are dual to products, coequalizers to equalizers, and pushouts to pullbacks. The standard notation is:

- $\operatorname{colim} D$ - for the colimit of a diagram D;
- $\sum_{x\in G} D(x)$, or $\coprod_{x\in G} D(x)$ - for the coproduct of the family $(D(x))_{x\in G}$;
- $A+B = A\amalg B$ for the coproduct of A and B, and accordingly for pushouts.

However the constructions of colimits in familiar categories are usually more complicated than those of limits. When we say that limits in varieties of universal algebras and in the category of topological spaces are "constructed in the same way as in the category of sets", it first of all means that the forgetful functors from all these categories to sets preserve limits (in the obvious sense). This, however, is usually not the case for colimits. Say, for a variety C of universal algebras, the colimit of a diagram $D\colon G \to \mathbf{C}$ can be constructed in several steps as follows:

- we take A to the free algebra on the disjoint union of all $D(x)_{(x\in G_0)}$;
- define the congruence $\sim$ on A as the smallest congruence E for which the composite of the canonical maps $D(x) \to A$ and $A \to A/E$ is a homomorphism of algebras;
- then one can show that $A/\sim$ becomes the colimit of D.

8.3 Galois connections

Definition 8.1. *A* Galois connection *between ordered sets L and M is a pair of maps*

$$L \rightleftarrows M, \tag{8.32}$$

both written as $x \mapsto x*$, and satisfying the following conditions:

$$x \leq y \implies y^* \leq x^* \text{ for all } x \text{ and } y \text{ in } L \text{ and for all } x \text{ and } y \text{ in } M; \tag{8.33}$$

$$x \leq x^{**} \text{ for all } x \text{ in } L \text{ and for all } x \text{ in } M. \tag{8.34}$$

That is, a Galois connection between L and M is nothing but an adjunction $L \to M^{op}$, or, equivalently, an adjunction $M \to L^{op}$. And just as any adjunction $\mathbf{X} \to \mathbf{A}$ determines a monad on $\mathbf{X}$, any Galois connection above determines closure operators on L and on M, both given by

$$c(x) = x^{**}. \tag{8.35}$$

Let us recall here that in general a closure operator on ordered sets is unary operation c satisfying the following conditions:

$$x \leq y \implies c(x) \leq c(y) \tag{8.36}$$
$$x \leq c(x); \tag{8.37}$$
$$cc(x) = c(x). \tag{8.38}$$

And if c is defined via a Galois connection as above, then the conditions (8.36) easily follow from (8.33) and (8.34) of course; the crucial observation is the equality

$$x^{***} = x^{*}, \tag{8.39}$$

in which $x^* \leq x^{***}$ by (8.34) applied to x^*, and $x^{***} \leq x^*$ by (8.33) applied to (8.34).

As usually, an element x is called closed (under a given closure operator c) if $c(x) = x$. From the equality (8.39) we easily conclude:

Theorem 8.2. *Any Galois connection (8.32) induces inverse to each other bijections between the set of closed elements in L and the set of closed elements in M. .*

When L and M are power sets ordered by inclusion, the Galois connections between L and M are nothing but binary relations between the ground sets. More precisely, we have:

Theorem 8.3. *Let X and Y be arbitrary sets and $P(X)$ and $P(Y)$ their power sets. Then:*

1. *For any Galois connection between $P(X)$ and $P(Y)$, and x in $\mathbf{X}$ and y in Y, we have:*
$$x \in \{y\}^* \Leftrightarrow y \in \{x\}^* \tag{8.40}$$

2. *Associating to a Galois connection between $P(X)$ and $P(Y)$ the binary relation $\alpha \subseteq X \times Y$ defined by*
$$\alpha = \{(x, y) \mid x \in \{y\}^*\} = \{(x, y) \mid y \in \{x\}^*\} \tag{8.41}$$
determined a bijection from the set of all Galois connections between $P(X)$ and $P(Y)$ and power set $P(X \times Y)$. The inverse bijection sends) $\alpha \in X \times Y$ to the Galois connection between $P(X)$ and $P(Y)$ defined by
$$A^* = \{y \in Y \mid a \in A \implies (a, y) \in \alpha\} \text{ for } A \subseteq X, \tag{8.42}$$
$$B^* = \{x \in X \mid b \in B \implies (x, b) \in \alpha\} \text{ for } B \subseteq Y. \tag{8.43}$$

Proof.

1. We have

$$\begin{aligned} x \in \{y\}^* \Leftrightarrow \{x\} \subseteq \{y\}^* &\implies \{y\}^{**} \subseteq \{x\}^* (\text{by } (8.33)) \\ &\implies \{y\} \subseteq \{x\}^* (\text{by } (8.34)) \\ y &\implies \{x\}^*. \end{aligned}$$

   Therefore $x \in \{y\}^* \implies y \in \{x\}^*$. Similarly (and "symmetrically") the converse implication also holds.

2. It is easy to see that (8.42) and (8.43) indeed define a Galois connection. That is, we have maps

$$\text{Galois connections between } P(X) \text{ and } P(Y) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} P(X \times Y) \tag{8.44}$$

   sending Galois connections to the corresponding binary relations defined by (8.41) and sending binary relations to the corresponding Galois connections defined by (8.42) and (8.43), and we have to show that $\psi\varphi$ and $\varphi\psi$ are the identity maps.

   To show that $\psi\varphi$ is the identity map is to show that, for every Galois connection between $P(X)$ and $P(Y)$, we have

$$\begin{aligned} A^* &= \{y \in Y \mid a \in A \implies y \in \{a\}^*\} \text{ for } A \subseteq X, \\ B^* &= \{x \in X \mid b \in B \implies x \in \{b\}^*\} \text{ for } B \subseteq Y. \end{aligned}$$

   or, equivalently, to show that

$$A^* = \bigcap_{a \in A} \{a\}^* \text{ for } A \subseteq X, \tag{8.45}$$

$$B* = \bigcap_{b \in B} \{b\}^* \text{ for } B \subseteq Y. \tag{8.46}$$

   We have:

$$y \in A^* \iff \forall\, a \in A((a, y) \in \alpha) \iff \forall\, a \in A(y \in \{a\}^*) \iff y \in \bigcap_{a \in A} \{a\}^*,$$

   which proves (8.45), and (8.46) can be proved similarly.

   To show that $\varphi\psi$ is the identity map is (according to (8.41)) to show that, for every binary relation $\alpha \in X \times Y$, we have

$$\alpha = \{(x, y) \mid x \in \{y\}*\}, \text{ where } \{y\}* = \{x \in X \mid (x, y) \in \alpha\},$$

   i.e.

$$\alpha = \{(x, y) \mid (x, y) \in \alpha\},$$

   which is trivial.

□

Remark 8.4. To construct a closure operator out of a Galois connection via (8.35) is a special case of constructing a monad out of an adjunction. But are there also general theorems about adjoint functors that would give Theorems 8.2 and 8.3 as special cases? Yes, but they are far more sophisticated and we shall not need them here.

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Part VIII

The Baum–Connes Conjecture, Localisation of Categories, and Quantum Groups

by

Paul F. Baum

Ralf Meyer

Based on the lectures of:

- Paul F. Baum
  (*Mathematics Department, McAllister Building The Pennsylvania State University, University Park, PA 16802, USA*)
  – Chapter 2.

- Ralf Meyer
  (*Mathematisches Institut and, Courant Centre "Higher order structures", Georg- August Universität Göttingen, Bunsenstrasse 3-5, 37073 Göttingen, Germany*)
  – Chapters 1, 3.

With additional lectures by:

- Max Karoubi – Section 2.5.

- Piotr M. Sołtan – Section 1.4.

Introduction

Noncommutative topology studies the (algebraic) topology of C*-algebras. More precisely, this means functors from the category of C*-algebras to, say, an Abelian category that are homotopy invariant, exact in a suitable sense, and compatible with Morita equivalence.

The natural setting for noncommutative topology is Gennadi Kasparov's bivariant K-theory. This theory can be described either concretely or abstractly. The concrete description is used for actual computations in examples, while the abstract description explains the fundamental role of the theory. Concretely, Kasparov theory provides a $\mathbb{Z}/2$-graded group $\mathrm{KK}_*(A, B)$ for two separable C*-algebras A and B, which is generated by generalised families of elliptic pseudodifferential operators. The Kasparov product turns these groups into a category. The abstract approach characterises this category uniquely up to isomorphism by a certain universal property. The equivalence between the abstract and concrete definitions of KK goes back to Joachim Cuntz [c-j87].

The universal property explains the central role of Kasparov theory in noncommutative geometry and can also be used to construct various maps and isomorphisms between KK-groups. To illustrate this, we show that there is, up to a sign, only one way to construct a natural boundary map for K-theory Mayer–Vietoris sequences.

Most applications of Kasparov theory actually use some equivariant version of the theory, which is defined for C*-algebras with some extra structure like a group or quantum group action. In this context, we briefly recall some basic results about compact and locally compact quantum groups.

Noncommutative topology has a somewhat different flavour than classical homotopy theory because the structure of the Kasparov category is much simpler than the structure of the stable homotopy category, which is its analogue in the classical setting. Nevertheless, there is some common ground for all of homotopy theory, commutative or non-commutative, and homological algebra. The notion of a triangulated category formalises common properties of these theories. From an operator algebraist's point of view, it formalises the properties of Kasparov theory that are needed to manipulate long exact sequences.

Triangulated categories were invented to study localisation of categories, and this plays an important role in noncommutative topology as well. Roughly speaking, localisation approximates a given functor by a new one with better properties. An example is the Baum–Connes assembly map, which replaces the K-theory of reduced crossed products by a more computable invariant.

The localisation of functors can be described most easily in the presence of complementary pairs of subcategories. Roughly speaking, such a pair splits a triangulated category into two orthogonal subcategories that together generate the whole category. Typically, one of these two categories is already given, but the existence of a complement is unclear. Following [**?**], we provide a sufficient criterion for this that is easy to check in examples. This is useful to construct analogues of the Baum–Connes assembly map for locally compact quantum groups.

The most important tool used here is a variant of homological algebra that still works in triangulated categories. Since a triangulated category does not have a canonical notion of exact sequence, this homological algebra is relative to a class of exact chain complexes which is defined by a homological functor. We carry over notions like projective objects and projective resolutions from homological algebra and use them to define derived functors. Furthermore, there is an Abelian category in which these derived functors may be computed. In many examples, this Abelian approximation to our category can be described explicitly.

The machinery explained here should cover the *topological* tools needed to extend the Baum–Connes conjecture to locally compact quantum groups. The localisation approach makes it easy to construct such a map once we know what we want to localise at. In the classical case, the category to localise at is defined using the family of all compact subgroups. I explain here what I currently believe to be the correct analogue of this family for a locally compact quantum group. This choice is suggested by heuristic arguments and by the few examples that have been treated so far.

The easiest case to consider are "torsion-free" discrete quantum groups. For these, we construct the Baum–Connes assembly map. The dual of a compact Lie group is torsion-free in this sense if and only if it is connected and has simply connected fundamental group. Unpublished work by Christian Voigt shows that deformations of simple compact groups are torsion-free as well. We also sketch a proof of the Baum–Connes conjecture for duals of compact Lie groups of this kind.

Chapter 1

Noncommutative algebraic topology

The starting point of noncommutative algebraic topology was the idea to study C*-algebras via their K-theory and related structures, following Elliott's classification of AF-algebras by their K-theory. These successful applications of K-theory motivated a search for other homology theories for C*-algebras. It turned out that all examples were closely related to K-theory and that many remarkable properties of K-theory like Bott periodicity or Pimsner–Voiculescu exact sequences are, in fact, general features of *all* noncommutative homology theories. Kasparov's bivariant K-theory clarified this issue completely: it is, on the one hand, rather close to K-theory, on the other hand, it is the universal homology theory for C*-algebras.

1.1 What is noncommutative (algebraic) topology?

In this section we will study *topological invariants for C*-algebras* and their properties. These are functors F on the category of C*-algebras and *-homomorphisms, with certain formal properties:

Homotopy invariance A *homotopy* between two *-homomorphisms $f_0, f_1\colon A \to B$ is a *-homomorphism $f\colon A \to C([0,1], B)$ such that $\mathrm{ev}_t \circ f = f_t$ for $t = 0, 1$. Homotopy invariance means that $F(f_0) = F(f_1)$ if f_0, f_1 are homotopic.

Exactness For any C*-algebra extension $I \rightarrowtail E \twoheadrightarrow Q$, the sequence

$$F(I) \to F(E) \to F(Q) \tag{1.1}$$

is exact.

Since KK-theory does not have this property, we also allow functors that are *semi split exact*, that is, the sequence (1.1) is only required exact for semi split extensions, which we now define.

A map $s\colon Q \to E$ is *positive* if $x \geq 0$ implies $s(x) \geq 0$. It is *completely positive* if $M_n(s)\colon M_n(Q) \to M_n(E)$ is positive for all $n \geq 0$. A map

$s\colon Q \to E$ is *contractive* if $\|s\| \leq 1$. We call an extension $I \rightarrowtail E \twoheadrightarrow Q$ *semi-split* if it has a completely positive contractive section $s\colon Q \to E$.

Theorem 1.1 (Section 15.8 in [b-b98]). *The extension $I \rightarrowtail E \twoheadrightarrow Q$ is semi-split if Q is nuclear.*

Split-exact For an extension $I \rightarrowtail E \twoheadrightarrow Q$ to *split*, we require $s\colon Q \to E$ to be a *-homomorphism. We call F *split exact* if for every split extension

$$\begin{tikzcd} I \arrow[r, tail, "i"'] & E \arrow[r, two heads, "p"'] & Q \arrow[l, bend right, "s"'] \end{tikzcd} \tag{1.2}$$

the maps $F(i)$ and $F(s)$ induce an isomorphism $F(I) \oplus F(Q) \cong F(E)$.

K-theory is homotopy invariant, exact and split exact.

Proposition 1.2 (Theorem 21.4.4 in [b-b98]). *Let F be a homotopy invariant and (semi-split) exact functor. Then for any (semi-split) extension $I \rightarrowtail E \twoheadrightarrow Q$ there is a natural long exact sequence*

$$\cdots \to F(S^2Q) \to F(SI) \to F(SE) \to F(SQ) \to F(I) \to F(E) \to F(Q) \tag{1.3}$$

where $SA := C_0((0,1), A)$ is the suspension functor.

Therefore, if a functor is semi-split exact and homotopy invariant, then it is split exact.

Morita invariance or C*-stability We call F *C*-stable* if for all C*-algebras A the corner embedding

$$A \to \mathcal{K}(l^2\mathbb{N}) \otimes A$$

induces an isomorphism $F(A) \cong F(\mathcal{K} \otimes A)$.

We call two C*-algebras A, B *Morita equivalent* if there exists a two-sided Hilbert module ${}_A\mathcal{H}_B$ over $A^{\mathrm{op}} \otimes B$ such that

$$({}_A\mathcal{H}_B) \otimes_B ({}_B\mathcal{H}^*_A) \cong {}_AA_A$$
$$({}_B\mathcal{H}^*_A) \otimes_A ({}_A\mathcal{H}_B) \cong {}_BB_B$$

Theorem 1.3 (Brown–Douglas–Rieffel). *Two separable C*-algebras A, B are Morita equivalent, $A \sim_M B$, if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.*

Therefore, C*-stability implies that $F(A) \cong F(B)$ if A and B are Morita equivalent. Morita equivalence and C*-stability are special features of the non-commutative world.

Definition 1.4. *A* topological invariant *for C*-algebras is a functor $F\colon \mathbf{C^*{-}alg} \to \mathbf{Ab}$ which is C*-stable, split exact, semi-split exact, and homotopy invariant.*

Theorem 1.5 (Higson). *If $F\colon \mathbf{C^*{-}alg} \to \mathbf{Ab}$ is C*-stable and split exact then it is homotopy invariant.*

See [**?**] for a simple proof of this theorem.

Actually, any topological invariant has many more formal properties like Bott periodicity, Pimsner–Voiculescu exact sequences for crossed product by $\mathbb{Z}$, Connes–Thom isomorphisms for crossed products by $\mathbb{R}$, Mayer–Vietoris sequences.

Bott periodicity states that $F(S^2A) \cong F(A)$ with a specified isomorphism. The proof uses two extensions

$$\mathcal{K} \rightarrowtail \mathcal{T} \twoheadrightarrow C(\mathrm{U}(1)) \quad (\text{ Toeplitz extension}),$$

$$C_0((0,1)) \to C_0((0,1]) \xrightarrow{\mathrm{ev}_1} \mathbb{C} \quad (\text{cone extension}).$$

These two extensions are related because $\mathrm{U}(1) \setminus \{1\} \cong (0,1)$. Thus we pull back the Toeplitz extension as follows:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, tail] \arrow[d, equal] & \mathcal{T} \arrow[r, two heads] & C(\mathrm{U}(1)) \\
\mathcal{K} \arrow[r, tail] & \mathcal{T}_0 \arrow[r, two heads] \arrow[u, "\subset"] & C_0(\mathrm{U}(1) \setminus \{1\}), \arrow[u, "\subset"]
\end{tikzcd}$$

and use the extension in the second row. The long exact sequence in Proposition 1.3 yields a boundary map

$$F(S^2A) \to F(\mathcal{K} \otimes A) \cong F(A)$$

Bott periodicity asserts that this natural map is invertible for any topological invariant F.

Corollary 1.6. *For any topological invariant F and any semi-split extension $I \rightarrowtail E \twoheadrightarrow Q$, there is a cyclic six-term exact sequence*

$$\begin{tikzcd}
F(I) \arrow[r] & F(E) \arrow[r] & F(Q) \arrow[d] \\
F(SQ) \arrow[u] & F(SE) \arrow[l] & F(SI) \arrow[l]
\end{tikzcd}.$$

Let F be a topological invariant and let A be a C*-algebra. Then $D \mapsto F(A \otimes D)$ is also a topological invariant. Therefore, Bott periodicity is equivalent to the assertion that $F(\mathbb{C}) \cong F(C_0(\mathbb{R}^2))$ for all topological invariants F.

1.2 Kasparov KK-theory

Kasparov's bivariant K-theory, also called KK-theory or Kasparov theory, explains why topological invariants have these nice properties and why they are so close to K-theory. Kasparov theory associates a $\mathbb{Z}/2$-graded Abelian group $\mathrm{KK}_*(A,B)$ to any pair of separable C*-algebras (A,B). Both $B \mapsto \mathrm{KK}(A,B)$ and $A \mapsto \mathrm{KK}(A,B)$ are topological invariants.

There is a natural product

$$\begin{aligned} \mathrm{KK}(A,B) \otimes \mathrm{KK}(B,C) &\to \mathrm{KK}(A,C) \\ (x,y) &\mapsto x \otimes_B y \end{aligned}$$

This turns Kasparov theory into a category, which we also denote by KK. Since any *-homomorphism $A \to B$ has a class in $\mathrm{KK}(A, B)$, we get a functor $\mathbf{C^*{-}alg} \to \mathrm{KK}$. This functor is a topological invariant as well. The universal property of Kasparov theory asserts that it is the universal topological invariant:

Definition 1.7. $\mathbf{C^*{-}alg} \to \mathrm{KK}$ *is the universal split exact, C*-stable* (*homotopy*) *functor.*

This means that KK is an additive category—so that split exactness makes sense—and that the canonical functor $\mathbf{C^*{-}alg} \to \mathrm{KK}$ is split exact, C*-stable, and therefore homotopy invariant; moreover, any other functor F from (separable) C*-algebras to some additive category $\mathcal{C}$ factors uniquely through KK:

$$\begin{tikzcd} \mathbf{C^*{-}alg} \arrow[r] \arrow[dr, "F"'] & \mathrm{KK} \arrow[d, dotted] \\ & \mathcal{C} \end{tikzcd}$$

This abstract point of view explains why KK-theory is so important. To get a useful theory, we also need a concrete description of KK.

This uses a certain set of *cycles* for $\mathrm{KK}_0(A, B)$; *homotopies* of such cycles are cycles for $\mathrm{KK}_0(A, C([0,1], B))$, and the group $\mathrm{KK}_0(A, B)$ is defined as the set of homotopy classes of cycles. Cycles consist of

- a Hilbert B-module $\mathcal{E}$ that is $\mathbb{Z}/2$-graded, that is, $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$
- a *-homomorphism $\varphi\colon A \to B(\mathcal{E})^{\mathrm{even}}$
- an adjointable operator $F \in B(\mathcal{E})^{\mathrm{odd}}$

such that

- $F = F^*$ (or $(F - F^*)\varphi(a) \in \mathcal{K}(\mathcal{E})$ for all $a \in A$)
- $F^2 = 1$ (or $(F^2 - 1)\varphi(a) \in \mathcal{K}(\mathcal{E})$ for all $a \in A$)
- $[F, \varphi(a)] \in \mathcal{K}(\mathcal{E})$ for all $a \in A$.

The sum of two cycles is their direct sum.

In the odd case, we may define

$$\mathrm{KK}_1(A, B) \cong \mathrm{KK}_0(A, SB) \cong \mathrm{KK}_0(SA, B).$$

More concretely, we get $\mathrm{KK}_1(A, B)$ if we drop the $\mathbb{Z}/2$-grading in the definition of KK_0.

Kasparov uses Clifford algebras to unify KK_0 and KK_1 and extend the definition to the real case. We do not treat the real case here but mention the following result, which often allows to reduce problems in real K-theory to problems in complex K-theory

Theorem 1.8 ([**?**]). *Let $A^{\mathbb{R}}$ and $B^{\mathbb{R}}$ be real C*-algebras. Let $A^{\mathbb{C}} = A^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $B^{\mathbb{C}} = B^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be their complexifications. There is a natural map*

$$\mathrm{KK}^{\mathbb{R}}(A^{\mathbb{R}}, B^{\mathbb{R}}) \to \mathrm{KK}^{\mathbb{C}}(A^{\mathbb{C}}, B^{\mathbb{C}}), \quad f^{\mathbb{R}} \mapsto f^{\mathbb{C}}.$$

Moreover, $f^{\mathbb{R}}$ is invertible if and only if $f^{\mathbb{C}}$ is invertible. In particular $B^{\mathbb{R}} \sim 0$ if and only if $B^{\mathbb{C}} \sim 0$.

1.2.1 Relation between the abstract and concrete descriptions

Take a cycle $X = (\mathcal{E}, \varphi, F)$ for $\mathrm{KK}_1(A, B)$. Form $E_X = \mathcal{K}(\mathcal{E}) + \varphi(A)(\frac{1+F}{2})$. This is a C*-algebra because, modulo $\mathcal{K}(\mathcal{E})$, $P := \frac{1+F}{2}$ is a projection which commutes with $\varphi(A)$. By construction, there is a C*-algebra extension

$$\mathcal{K}(\mathcal{E}) \rightarrowtail E_X \twoheadrightarrow A', \qquad A' = E_X/\mathcal{K}(\mathcal{E})$$

and a *-homomorphism $\varphi' \colon A \to A'$, $a \mapsto P\varphi(a) \mod \mathcal{K}(\mathcal{E})$; moreover, $\mathcal{K}(\mathcal{E}) \sim_M I \triangleleft B$, where I is the ideal generated by $(\xi|\eta)$ for $\xi, \eta \in \mathcal{E}$. We may assume that $\mathcal{E}$ is full—even $\mathcal{E} = l^2\mathbb{N} \otimes B$ is possible by Kasparov's Stabilisation Theorem

$$\mathcal{E} \oplus (l^2\mathbb{N} \otimes B) \cong l^2\mathbb{N} \otimes B.$$

We may also assume that φ' is injective as a map to $B(\mathcal{E})/\mathcal{K}(\mathcal{E})$, so that $A' \cong A$. Under these assumptions, we get a C*-algebra extension

$$\mathcal{K} \otimes B \rightarrowtail E_X \twoheadrightarrow A,$$

which is semi-split by $a \mapsto P\varphi(a)P$.

Conversely, this process can be inverted (using Stinespring's Theorem): any semi-split extension $\mathcal{K} \otimes B \rightarrowtail E \twoheadrightarrow A$ yields a class in $\mathrm{KK}_1(A, B)$.

This identifies $\mathrm{KK}_1(A, B)$ with the set of homotopy classes of semi-split extensions of A by $\mathcal{K} \otimes B$. A deep result of Kasparov replaces homotopy invariance by a more rigid equivalence relation: unitary equivalence after adding split extensions. Two extensions are unitarily equivalent if there is a commuting diagram

$$\begin{tikzcd}
\mathcal{K} \otimes B \arrow[r, tail] \arrow[d, "\mathrm{Ad}(u)"'] & E_1 \arrow[r, two heads] \arrow[d, "\cong"] & A \arrow[d, equal] \\
\mathcal{K} \otimes B \arrow[r, tail] & E_2 \arrow[r, two heads] & A
\end{tikzcd}$$

with a unitary multiplier u of $\mathcal{K} \otimes B$.

Corollary 1.9. *For any topological invariant F there is a map*

$$\mathrm{KK}_1(Q, I) \otimes F_{k+1}(Q) \to F_k(I),$$

where $F_k(A) := F(S^k A)$.

Proof. Use the boundary map from Proposition 1.2 for the extension associated to a class in $\mathrm{KK}_1(Q, I)$. □

A similar construction works in the even case. Assuming $F^2 = 1$ and $F = F^*$, we write

$$\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-, \quad \varphi = \varphi^+ \oplus \varphi^-, \quad F = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}.$$

The unitary u yields an isomorphism $\mathcal{E}_- \cong \mathcal{E}_+$. As above, we may assume that $\mathcal{E}$ is full, so that $\mathcal{K}(\mathcal{E}_\pm) \sim_M B$. The homomorphisms

$$\varphi^+ \colon A \to B(\mathcal{E}^+), \quad \mathrm{Ad}(u) \circ \varphi^- \colon A \to B(\mathcal{E}^+)$$

have the property that

$$\varphi^+(a) - \mathrm{Ad}(u)\varphi^-(a) \in \mathcal{K}(\mathcal{E}^+)$$

for all $a \in A$. Let $E = \mathcal{K}(\mathcal{E}^+) + \varphi^+(A)$, then we get an extension

$$\mathcal{K}(\mathcal{E}^+) \rightarrowtail E \twoheadrightarrow A.$$

This extension splits in two different ways via φ^+ and $\mathrm{Ad}(u) \circ \varphi^-$.

Let F be a topological invariant, then split exactness yields

$$F(E) \cong F(B) \oplus F(A),$$

$$F(\varphi^+) - F(\mathrm{Ad}(u) \circ \varphi^-) \colon F(A) \to F(B) \subseteq F(E).$$

Hence we get a map

$$\mathrm{KK}_0(A, B) \otimes F(A) \to F(B).$$

This observation leads us to the following definition of Joachim Cuntz (see [c-j87]):

Definition 1.10. *Let A and D be C*-algebras and let B be an ideal in D. A* quasi-homomorphism *from A to B (via D) is a pair of *-homomorphisms $f, g \colon A \to D$ with $(f - g)(A) \subseteq B$.*

The discussion above shows that any quasi-homomorphism from A to B induces a map $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$. Among all quasi-homomorphisms there is a universal one: let $A * A$ be the free product C*-algebra of two copies of A. Let $qA \triangleleft A*A$ be the kernel of the homomorphism $\mathrm{id}_A * \mathrm{id}_A \colon A*A \to A$. The resulting C*-algebra extension $qA \rightarrowtail A * A \twoheadrightarrow A$ has two canonical sections: the canonical embeddings $i_1, i_2 \colon A \to A * A$. Thus i_1 and i_2 form a quasi-homomorphism from A to qA via $A * A$. By the universal property of free products, any quasi-homomorphism from A to B generates a *-homomorphism $qA \to B$. The converse implication only holds for non-degenerate *-homomorphisms $qA \to B$, using the extension to multiplier algebras. Nevertheless, this is good enough to prove that $\mathrm{KK}_0(A, B)$ is naturally isomorphic to the group $[qA, B \otimes \mathcal{K}]$ of homotopy classes of *-homomorphisms $qA \to B \otimes \mathcal{K}$.

This can be interpreted as follows. The canonical *-homomorphisms $\pi_A \colon qA \to A$ and $\iota_B \colon B \to B \otimes \mathcal{K}$ are KK-equivalences. A given element $f \in \mathrm{KK}_0(A, B)$ factors as $f = \iota_B^{-1} \circ \hat{f} \circ \pi_A^{-1}$ for a *-homomorphism $\hat{f} \colon qA \to B \otimes \mathcal{K}$, that is, the following diagram commutes in KK:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "\iota_B", "\sim"'] \\ qA \arrow[u, "\pi_A"', "\sim"] \arrow[r, "\hat{f}"] & B \otimes \mathcal{K} \end{tikzcd}$$

Consider two extensions

$$C \rightarrowtail E_2 \twoheadrightarrow B, \qquad B \rightarrowtail E_1 \twoheadrightarrow A$$

These give a map

$$F(A) \to F(S^{-2}C) \cong F(C).$$

The miracle of the Kasparov product is that this composite map is described by a quasi-homomorphism from A to C, that is, by a class in $\mathrm{KK}_0(A, C)$. This is the point where special features of C*-algebras are used. Extensions of KK to non-C*-algebras either do not have a product (like Vincent Lafforgue's theory for Banach algebras) or they use extensions of arbitrary length (like Joachim Cuntz's kk).

1.2.2 Relation with K-theory

KK-theory is very close to K-theory. If some construction gives a map $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$ it probably gives a class in $\mathrm{KK}_*(A, B)$. For many C*-algebras, we may compute KK purely in terms of K-theory.

Theorem 1.11. $\mathrm{KK}_*(\mathbb{C}, A) \cong \mathrm{K}_*(A)$.

The proof requires the concrete description of KK. We only hint at how the map is constructed for $A = \mathbb{C}$.

Definition 1.12. *An operator F is* Fredholm *if* $\ker(F)$ *and* $\operatorname{coker}(F)$ *have finite dimension.*

The operator F in the definition of Kasparov cycles is something like a Fredholm operator. A cycle in $\mathrm{KK}_0(\mathbb{C}, \mathbb{C})$ consists of a Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and an operator $F\colon \mathcal{H}_+ \to \mathcal{H}_-$ with $FF^* - \mathrm{id} \in \mathcal{K}$, $F^*F - \mathrm{id} \in \mathcal{K}$; thus F is Fredholm.

The index map gives an isomorphism

$$\begin{aligned} \mathrm{Index}\colon\ \mathrm{KK}_0(\mathbb{C}, \mathbb{C}) &\xrightarrow{\cong} \mathbb{Z} \\ \mathrm{Index}(F) &= \dim(\ker F) - \dim(\operatorname{coker} F). \end{aligned}$$

In the odd case we have $\mathrm{KK}_1(\mathbb{C}, \mathbb{C}) = 0$.

The Kasparov product yields a canonical map

$$\gamma\colon\ \mathrm{KK}_*(A, B) \to \mathrm{Hom}(\mathrm{K}_* A, \mathrm{K}_* B).$$

Next we relate the kernel of this map to $\mathrm{Ext}^1(K_{*+1}A, K_*B)$.

Represent $\alpha \in \mathrm{KK}_1(Q, I)$ by a C*-algebra extension $\alpha = [I \rightarrowtail E \twoheadrightarrow Q]$. Assume $\gamma(\alpha) = 0$. There is an exact sequence

$$\begin{tikzcd} \mathrm{K}_0(I) \arrow[r] & \mathrm{K}_0(E) \arrow[r] & \mathrm{K}_0(Q) \arrow[d, "\gamma(\alpha)"] \\ \mathrm{K}_1(Q) \arrow[u, "\gamma(\alpha)"] & \mathrm{K}_1(E) \arrow[l] & \mathrm{K}_1(I) \arrow[l] \end{tikzcd}$$

Since $\gamma(\alpha) = 0$, we get an extension of $\mathbb{Z}/2$-graded Abelian groups

$$\mathrm{K}_*(I) \rightarrowtail \mathrm{K}_*(E) \twoheadrightarrow \mathrm{K}_*(Q).$$

This construction defines a natural map

$$\mathrm{KK}_*(A, B) \supseteq \ker \gamma \to \mathrm{Ext}^1(\mathrm{K}_{*+1}(A), \mathrm{K}_*(B)).$$

The following Universal Coefficient Theorem shows that this map together with γ often describe KK completely.

Theorem 1.13. *Let $\mathcal{B}$ be the smallest category of separable C^*-algebras closed under suspensions, semi-split extensions,* KK*-equivalence, tensor products, and containing $\mathbb{C}$. Then there exists a natural exact sequence*

$$\operatorname{Ext}^1(\mathrm{K}_{*+1} A, \mathrm{K}_* B) \rightarrowtail \mathrm{KK}_*(A, B) \to \operatorname{Hom}(\mathrm{K}_* A, \mathrm{K}_* B) \tag{1.4}$$

for $A \in \mathcal{B}$ and any C^*-algebra B.

The class $\mathcal{B}$ to which this theorem applies is quite big and contains, for instance, all commutative C*-algebras. As a consequence:

Corollary 1.14. *Let X and Y be locally compact spaces and let Y be a topological invariant for C^*-algebras. If $\mathrm{K}^*(X) \cong \mathrm{K}^*(Y)$, then $F(C_0(X) \cong F(C_0(Y))$.*

Proof. There is an isomorphism

$$\alpha\colon\ \mathrm{K}^*(C_0(X)) \cong \mathrm{K}^*(X) \xrightarrow{\cong} \mathrm{K}^*(Y) \cong \mathrm{K}^*(C_0(Y))$$

By the Universal Coefficient Theorem, α lifts to $\widehat{\alpha} \in \mathrm{KK}_0(C_0(X), C_0(Y))$. Since $\operatorname{Ext}^1 \circ \operatorname{Ext}^1 = 0$, we know that $\widehat{\alpha}$ is invertible. Since KK is universal, $F(\widehat{\alpha})$ is invertible for any topological invariant F. □

As a result, most of the interesting and complicated information in classical homotopy theory is lost when we pass to C*-algebras: only K-theory remains visible. There is not much intersection between classical and non-commutative topology.

The analogies and contrasts between homotopy theory and noncommutative topology are summarized in the following table:

Homotopy theory	*Noncommutative topology*
Spaces	C*-algebras
Stable homotopy category	KK
Stable homotopy groups of spheres	Morphisms from $\mathbb{C}$ to $\mathbb{C}$ in KK
$\pi_*^s(S^0) = \mathrm{Mor}_*(\mathrm{pt}, \mathrm{pt})$	$\mathrm{KK}^*(\mathbb{C}, \mathbb{C}) = \mathbb{Z}[\beta, \beta^{-1}]$, $\deg(\beta) = 2$
	Bott periodicity
Homology $\mathrm{H}_*(-)$	K-theory $\mathrm{K}_*(-)$
Adams spectral sequence	Universal Coefficient Theorem for KK
Interesting topology—no analysis	Simple topology—interesting analysis

The Adams spectral sequence applies to arbitrary objects of the stable homotopy category, its result is still quite complicated, so that stable homotopy groups are hard to compute. In contrast, the Universal Coefficient Theorem does not apply to all objects of KK, but whenever it applies, KK is straightforward to compute.

1.2.3 Index maps and Mayer–Vietoris sequences

Consider an extension of C*-algebras

$$I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q.$$

There are long exact sequences in K-theory and in K-homology:

$$\begin{tikzcd} \mathrm{K}_0(I) \arrow[r] & \mathrm{K}_0(E) \arrow[r] & \mathrm{K}_0(Q) \arrow[d, "\partial"] \\ \mathrm{K}_1(Q) \arrow[u, "\partial"] & \mathrm{K}_1(E) \arrow[l] & \mathrm{K}_1(I), \arrow[l] \end{tikzcd} \tag{1.5}$$

$$\begin{tikzcd} \mathrm{K}^0(Q) \arrow[r] & \mathrm{K}^0(E) \arrow[r] & \mathrm{K}^0(I) \arrow[d, "\delta"] \\ \mathrm{K}^1(I) \arrow[u, "\delta"] & \mathrm{K}^1(E) \arrow[l] & \mathrm{K}^1(Q), \arrow[l] \end{tikzcd} \tag{1.6}$$

and we have pairings between K-theory and K-homology. We are going to prove that

$$-\langle \partial(x),\, y\rangle = \langle x,\, \delta(y)\rangle \qquad \text{for all } x \in \mathrm{K}_1(Q),\ y \in \mathrm{K}^0(I), \tag{1.7}$$

using only formal properties of the boundary maps. This illustrates the power of the universal property of KK.

Theorem 1.15. *Let $\partial\colon \mathrm{K}_1(Q) \to \mathrm{K}_0(I)$ and $\delta\colon \mathrm{K}^0(I) \to \mathrm{K}^1(Q)$ be* natural *boundary maps for morphisms of (semi-split) extensions. There is $\varepsilon \in \{\pm 1\}$ such that*

$$\langle \partial(x),\, y\rangle = \varepsilon\langle x,\, \delta(y)\rangle$$

for all (semi-split) extensions and all $x \in \mathrm{K}_1(Q)$, $y \in \mathrm{K}^0(I)$.

Remark 1.16. The sign ε is fixed by looking at the extension $\mathcal{K} \rightarrowtail \mathcal{T} \twoheadrightarrow C(S^1)$ and the generators of $\mathrm{K}_1(C(S^1)) = \mathbb{Z}$ and $\mathrm{K}^0(\mathcal{K}) = \mathbb{Z}$. With the usual conventions, the isomorphism $\mathrm{K}^1(C(S^1)) \cong \mathrm{Hom}(\mathrm{K}_1(C(S^1))) \cong \mathbb{Z}$ maps

$$[\mathcal{K} \rightarrowtail \mathcal{T} \rightarrowtail C(S^1)] \mapsto -1 \in \mathbb{Z}.$$

Even more, up to a sign there is only one *natural* boundary map:

Theorem 1.17. *Let $\partial\colon \mathrm{K}_{*+1}(Q) \to \mathrm{K}_*(I)$ be a* natural *boundary map. There is $\varepsilon \in \{\pm 1\}$ such that for all semi-split extensions, $\varepsilon \cdot \partial$ is the composition*

$$\mathrm{K}_{*+1}(Q) \cong \mathrm{KK}_{*+1}(\mathbb{C}, Q) \to \mathrm{KK}_*(\mathbb{C}, I) \cong \mathrm{K}_*(I)$$

where the middle map is the Kasparov product with the class of the extension in $\mathrm{KK}_1(Q, I)$. The same holds in K-homology.

Along the way, we will recall some details of the proof that KK is semi-split exact. These will also play a role when we introduce the triangulated category structure on KK. It is possible to treat extensions that are not semi-split in a similar way: this amounts to replacing KK by E-theory everywhere. Since we do not discuss E-theory here, we added the semi-splitness assumption.

Before we prove these theorems, we consider the more general situation of Mayer–Vietoris sequences for pullback diagrams. Consider the category of pullback diagrams

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d] & B \arrow[d] \\ A' \arrow[r, two heads] & B', \end{tikzcd}$$

that is, $A = A' \oplus_{B'} B$ and the map $A' \to B'$ is a semi-split surjection; then so is the map $A \to B$. Morphisms are natural transformations of such diagrams, of course. In this context, we expect that the K-theories of our four C*-algebras are related by a *Mayer–Vietoris sequence*, that is, an exact sequence of the form

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(A') \oplus \mathrm{K}_0(B) \arrow[r] & \mathrm{K}_0(B') \arrow[d, "\delta_0"] \\
\mathrm{K}_1(B') \arrow[u, "\delta_1"] & \mathrm{K}_0(A') \oplus \mathrm{K}_0(B) \arrow[l] & \mathrm{K}_0(A). \arrow[l]
\end{tikzcd}$$

Here the horizontal maps are, up to signs, induced by the four maps in our pullback diagram. There are several ways to construct the boundary maps δ_0 and δ_1, and one may wonder whether these yield the same result. The answer is that, once again, its naturality already determines the boundary map uniquely up to a sign:

Theorem 1.18. *Let $d\colon \mathrm{K}_*(B') \to \mathrm{K}_{*+1}(A)$ be a natural boundary map for Mayer–Vietoris sequences. There are two signs $\varepsilon_0, \varepsilon_1 \in \{\pm 1\}$ such that, for any pullback diagram, $\varepsilon_* \cdot d$ is the composition*

$$\mathrm{K}_*(B') \xrightarrow{\delta} \mathrm{K}_*(\ker(A' \to B')) \cong \mathrm{K}_*(\ker(A \to B)) \to \mathrm{K}_*(A)$$

Remark 1.19. The signs are fixed by looking at the special pullback square

$$\begin{tikzcd}
C_0((0,1)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
C_0((0,1]) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

and its suspension.

We will prove Theorem 1.18 by reducing it to Theorem 1.17. This part of the argument works for any homological invariant instead of K-theory.

Let F be a homological functor on separable C*-algebras and let $d_{A',B}\colon F_1(B') \to F_0(A)$ be a transformation that is natural with respect to morphisms of pullback diagrams. We compare any given square to a simpler one:

$$\begin{tikzcd}
\ker p' \arrow[r, two heads] \arrow[d] & 0 \arrow[d] & & A \arrow[r, two heads] \arrow[d] & B \arrow[d] \\
A' \arrow[r, two heads, "p'"] & B' & \longrightarrow & A' \arrow[r, two heads] & B'
\end{tikzcd}$$

Let $d_{A',0}$ and $d_{A',B}$ be the boundary maps for these two pullback diagrams. Naturality yields a commuting diagram

$$\begin{tikzcd}
F_1(B') \arrow[r, "d_{A',0}"] \arrow[d, equal] & F_0(\ker p') \arrow[d, "F_0(i)"] \\
F_1(B') \arrow[r, "d_{A',B}"] & F_0(A);
\end{tikzcd}$$

it shows that $d_{A',0}$ determines $d_{A',B}$; here i is the canonical embedding $\ker p' \to A$. The category of pullback diagrams with $B = 0$ is equivalent to the category of semi-split C*-algebra extensions. Hence our first reduction step shows that a natural boundary map for Mayer–Vietoris sequences is determined by a natural boundary map for C*-algebra extensions. Thus Theorem 1.18 follows from Theorem 1.17, and we may consider extensions from now on.

Next we compare a given extension with a *mapping cylinder extension*:

$$\begin{tikzcd} I \arrow[r, tail, "i"] \arrow[d, "\alpha"] & E \arrow[r, two heads, "p"] \arrow[d, "\beta"] & Q \arrow[d, equal] \\ C_p \arrow[r, tail, "\subset"] & Z_p \arrow[r, two heads, "\mathrm{ev}_0"] & Q, \end{tikzcd} \tag{1.8}$$

where

$$C_p := \{(e, q) \in E \oplus C_0((0, 1], Q) \mid p(e) = q(1)\},$$
$$Z_p := \{(e, q) \in E \oplus C([0, 1], Q) \mid p(e) = q(1)\},$$

and the projection $Z_p \to Q$ is evaluation at 0. The embedding $\beta\colon E \to Z_p$ maps $e \mapsto \bigl(e, \operatorname{const} p(e)\bigr)$; this restricts to a map $\alpha\colon I \to C_p$, $x \mapsto (x, 0)$.

The projection onto the first factor provides another map $\beta^{[-1]}\colon Z_p \to E$ with $\beta^{[-1]} \circ \beta = \mathrm{id}_E$. It is easy to check that $\beta \circ \beta^{[-1]}$ is homotopic to the identity map on Z_p, so that β is a homotopy equivalence and hence acts by an invertible map on any topological invariant. Using exactness and the Five Lemma, it follows that the map $\alpha\colon I \to C_p$ must act by an invertible map as well. More precisely, α is a KK-equivalence if our extension is semi-split.

Incidentally, the quickest way to derive the exactness of KK in both variables is by reversing this argument (see [?]). It is comparatively easy to get a long exact sequence for KK in both variables for the second row in (1.8). It is shown by hand that the map $I \to C_p$ is a KK-equivalence, and this yields long exact sequences for the first row in (1.8).

Now we return to a natural boundary map for C*-algebra extensions. Since the canonical map α above is a KK-equivalence, the naturality diagram

$$\begin{tikzcd} F_1(Q) \arrow[r, "d_E"] \arrow[d, equal] & F_0(I) \arrow[d, "F(\alpha)", "\cong"'] \\ F_1(Q) \arrow[r, "d_{Z_p}"] & F_0(C_p) \end{tikzcd}$$

shows that the boundary maps d_E and d_{Z_p} for the two rows in (1.8) determine each other uniquely. In particular, $d_E = F_0(\alpha)^{-1} \circ d_{Z_p}$.

In the next reduction step, we compare the mapping cylinder extension to the cone extension over Q:

$$\begin{tikzcd} C_p \arrow[r, tail] & Z_p \arrow[r, two heads] & Q \\ SQ \arrow[r, tail] \arrow[u, "\subset"] & \tilde{C}Q \arrow[r, two heads] \arrow[u, "\subset"] & Q \arrow[u, equal] \end{tikzcd}$$

with $\tilde{C}Q := C_0([0,1), Q)$, which is isomorphic to the usual cone $C_0((0,1], Q)$. Since this involves a reflection on the ideal SQ, which acts as -1 on topological invariants, we get $d_{\tilde{C}Q} = -d_{CQ}$, where the latter denotes the boundary map of the usual cone extension $SQ \rightarrowtail CQ \twoheadrightarrow Q$.

Once again, the naturality of the boundary map yields

$$d_{Z_p} = F_0(j) \circ d_{\tilde{C}Q} = -F_0(j) \circ d_{CQ}.$$

where j denotes the canonical embedding $SQ \to C_p$. As a consequence, a natural boundary map for extensions or for Mayer–Vietoris sequences is determined uniquely once we specify it for the cone extensions $SQ \rightarrowtail CQ \twoheadrightarrow Q$. Since CQ is contractible, any topological invariant vanishes on CQ, so that the boundary map $d_{CQ}\colon F_1(Q) \to F_0(SQ)$ is invertible. Notice also that d_{CQ} is a natural transformation between two functors on the category of C*-algebras because mapping Q to the cone extension is a functor from the category of C*-algebras to the category of (semi-split) C*-algebra extensions.

The definition $F_1(Q) := F_0(SQ)$ provides us with an identical natural transformation $d^0_{CQ}\colon F_1(Q) \to F_0(SQ)$; this choice produces a natural boundary map for extensions. Any other natural transformation must be of the form $d_{CQ} = d^0_{CQ} \circ \Phi$ where $\Phi\colon F_1(Q) \to F_1(Q)$ is an invertible natural transformation. Conversely, any invertible natural transformation may arise here because $d^0_{CQ} \circ \Phi$ is a natural boundary map.

Now we restrict attention to K-theory. To finish the proof, we must show that the only invertible natural transformations $\mathrm{K}_j(Q) \to \mathrm{K}_j(Q)$ for $j = 0, 1$ are $\pm\mathrm{id}$. Other homology theories like K-thory with $\mathbb{Z}/p$-coefficients may have more invertible natural transformations. To begin with, our transformation is only natural with respect to *-homomorphism. By the universal property, this implies naturality with respect to all KK-morphisms. On the level of KK, the functors $\mathrm{K}_j(Q)$ become representable because

$$\mathrm{K}_j(Q) \cong \mathrm{KK}_0(\mathbb{C}[j], Q),$$

where $\mathbb{C}[j]$ denotes the j-fold suspension of $\mathbb{C}$.

The Yoneda Lemma now identifies the rings of natural transformations with $\mathrm{KK}_0(\mathbb{C}[j], \mathbb{C}[j]) \cong \mathbb{Z}$. The only invertible elements of this ring are ± 1 as asserted.

As a result, a natural boundary map $d_{CQ}\colon \mathrm{K}_1(Q) \to \mathrm{K}_0(SQ)$ for cone extensions is unique up to a universal sign, which is computed by looking at the case $Q = C_0(\mathbb{R})$. By the arguments above, the same holds for the natural boundary maps for extensions and Mayer–Vietoris sequences. This finishes the proofs of Theorems 1.17 and 1.18 for K-theory. The argument for K-homology is similar. In the beginning, some arrows have to be reversed because K-homology is contravariant. In the final step, we use $\mathrm{K}^*(A) = \mathrm{KK}^*(A, \mathbb{C})$ to identify the ring of natural operations on K-homology with $\mathbb{Z}$ as above.

Our proof also yields formulas for the boundary maps. The Mayer–Vietoris boundary map $d_{A',B}\colon F_1(B') \to F_0(A)$ is obtained by composing the boundary map for the extension $\ker p' \rightarrowtail A' \twoheadrightarrow B'$ with the map induced by the embedding $\ker p' \to A$. The boundary map $\mathrm{K}_{*+1}(Q) \to \mathrm{K}_*(I)$ for a semi-split extension $I \rightarrowtail E \twoheadrightarrow Q$ is, up to a universal sign, the Kasparov product with the class of the extension in $\mathrm{KK}_1(Q, I)$. Here we use that the class of the extension in $\mathrm{KK}_1(Q, I) \cong \mathrm{KK}_0(SQ, I)$ is the Kasparov product of the class of the embedding $SQ \to C_p$ and the inverse of the KK-equivalence $I \to C_p$. This is checked directly in [**?**]; but our argument also shows that it could not be otherwise because up to a sign this is the only natural way to attach a class in $\mathrm{KK}_1(Q, I)$ to the extension.

Finally, since the various boundary maps are unique up to a sign, the index pairings also match up to a sign. Let $x \in \mathrm{K}_1(Q)$, $y \in \mathrm{K}^0(I)$, and let $[E] \in \mathrm{KK}_1(Q, I)$ denote the class of the extension $I \rightarrowtail E \twoheadrightarrow Q$. Let $\circ$ denote Kasparov products. Write $\partial(x) = \varepsilon_\partial [E] \circ x$ and $\delta(y) = \varepsilon_\delta y \circ [E]$ with certain signs. Then

$$\langle x, \delta y\rangle = \delta(y) \circ x = \varepsilon_\delta (y) \circ [E] \circ x \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z},$$
$$\langle \partial(x), y\rangle = y \circ \partial(x) = \varepsilon_\partial (y) \circ [E] \circ x.$$

Hence $\langle x, \delta y\rangle$ and $\langle \partial(x), y\rangle$ agree up to the universal sign $\epsilon_\delta \cdot \epsilon_\partial$, which depends on the signs that appear in the natural boundary maps.

1.3 Equivariant theory

In equivariant bivariant Kasparov theory additional symmetries create interesting topology, making tools from homotopy theory more relevant.

What equivariant situations are being considered?

- Group actions (of locally compact groups)
- Bundles of C*-algebras $(A_x)_{x\in X}$ over some space X
- Locally compact groupoids
- Coactions of locally compact quantum groups (Baaj–Skandalis)
- C*-algebras over non-Hausdorff space (Kirchberg)

In each case, there is an equivariant K-theory with similar properties as the nonequivariant one, with a similar concrete description—simply add an equivariance condition—and a universal property.

We briefly explain what we mean by a groupoid action. Let $\mathcal{G}$ be a groupoid, and A a C*-algebra. We say that $\mathcal{G}$ acts on A and write $\mathcal{G} \curvearrowright A$, if A is a bundle over $\mathcal{G}^0$ and $\mathcal{G}$ acts fiberwise on this bundle. The continuity of the action is expressed by the existence of a bundle isomorphism $\alpha\colon s^*A \to r^*A$, where r and s are the range and source maps of $\mathcal{G}$ ([**?**]).

$$\mathcal{G}^1 \underset{s}{\overset{r}{\rightrightarrows}} \mathcal{G}^0 , \quad s^*A \xrightarrow{\alpha} r^*A, \quad (s^*A)_g = A_{s(g)}.$$

$$g\colon x \to y \implies \alpha_g\colon A_x \to A_y \text{ *-isomorphism}$$

In each equivariant situation, there is a more or less obvious notion of *equivariant* $*$-homomorphism, leading to a category $\mathbf{C^*{-}alg}_G$ whose objects are the C*-algebras with appropriate additional structure and whose morphisms are the equivariant $*$-homomorphisms. For an extension, we now require all maps that occur to be equivariant, and for a split or semi split extension, we also require the section to be equivariant. This leads to the appropriate notions of *split exactness*, *semi split exactness* and *exactness* in the equivariant case.

Stability is a bit more complicated to formulate. Recall that we can reformulate Morita equivalence using linking algebras: given a Hilbert bimodule $\mathcal{H}$

that implements a Morita equivalence between two C*-algebras A and B, the linking algebra D is the algebra of block matrices

$$\begin{pmatrix} A & \mathcal{H} \\ \mathcal{H}^* & B \end{pmatrix},$$

where the multiplication is given by the usual matrix multiplication, combined with the various module structures and inner products. Let p be the projection onto the upper left corner. This is a projection in the multiplier algebra of D, and both p and the complementary projection $1-p$ are full because $\mathcal{H}$ is full both as a left Hilbert A-module and as a right Hilbert B-module. Conversely, any full projection gives rise to a Morita equivalence between the corner pDp and D.

The correct equivariant generalisation of Morita equivalence can be formulated in several equivalent ways. In terms of linking algebras, an equivariant Morita equivalence consists of an object D of $\mathbf{C}^*{-}\mathbf{alg}_G$ with an equivariant projection p and equivariant isomorphisms $A \cong pDp$, $B \cong (1-p)D(1-p)$, such that both p and $1-p$ are full. A functor on $\mathbf{C}^*{-}\mathbf{alg}_G$ is called (equivariantly) *stable* if $F(pDp) \to F(D)$ is invertible whenever p is a full, equivariant projection in the multiplier algebra of D.

Often this criterion can be simplified somewhat, using some appropriate generalisation of Kasparov's Stabilisation Theorem. This usually allows to restrict attention to Hilbert modules of a rather special form. For instance, if we study C*-algebras with a group action, then it suffices to look at Hilbert modules of the form $L^2(G)\otimes A$: a functor is equivariantly C*-stable if and only if the corner embeddings of A and $\mathcal{K}(L^2G)\otimes A$ into $\mathcal{K}(L^2G\oplus\mathbb{C})\otimes A$ induce isomorphisms.

In the cases mentioned above, the bivariant Kasparov theory KK^G is the universal equivariantly split exact and C*-stable functor on $\mathbf{C}^*{-}\mathbf{alg}_G$ (restrict to separable C*-algebras here); that is, any other functor with these two properties factors uniquely through KK^G, that is, we have a unique factorization

$$\begin{tikzcd}
\mathrm{KK}^G(A,B)\times F(A) \arrow[r, dotted] & F(B). \\
\mathrm{Hom}^G(A,B)\times F(A) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

This universal property characterizes KK^G uniquely up to equivalence.

1.4 Quantum groups

The aim of this section is to introduce the definition of a locally compact quantum group. We will arrive at this definition via the example of a classical locally compact group.

1.4.1 Motivation: from groups to multiplicative unitaries

Let G be a locally compact group. Using the right Haar measure on G we can for the Banach algebra $L^1(G)$. For $f,g\in L^1(G)$ the convolution product of f and g is defined as

$$(f\star g)(x) = \int_G f(xy^{-1})g(y)\,dy$$

for all $x \in G$. Let $\mathcal{H}$ denote the Hilbert space $L^2(G)$ defined with respect to the right Haar measure on G. Now for any $f \in L^1(G)$ we can define the right convolution operator $R_f \in B(\mathcal{H})$ by

$$(R_f\psi)(x) = \int_G f(y)\psi(xy)\,dy$$

for all $\psi \in \mathcal{H}$ and $x \in G$. One easily checks that $R_f R_g = R_{f\star g}$ for all $f, g \in L^1(G)$ and $(R_f)^* = R_{f^*}$, where $f^*(x) = \overline{f(x^{-1})}$. The C*-algebra geberated inside $B(\mathcal{H})$ by operators $\{R_f \mid f \in L^1(G)\}$ is the *reduced group* C*-*algebra* if G and is denoted by $\mathrm{C}^*_r(G)$. In the multiplier algebra of $\mathrm{C}^*_r(G)$ we find the right shift operators $\{R_y\}_{y\in G}$ which act as

$$(R_y\psi)(x) = \psi(xy).$$

Let us now define the *Kac-Takesaki* operator of G. This operator is a unitary $W \in \mathrm{B}(\mathcal{H}\otimes\mathcal{H})$ defined as the direst integral of the field of operators $G \ni y \mapsto R_y \in B(\mathcal{H})$ acting on the direct integral of the constant field of Hilbert spaces (the same Hilbert space $\mathcal{H}$ over every point of G):

$$W = \int_G^\oplus R_y\,dy.$$

With the natural identification

$$\int_G^\oplus H\,dy = \mathcal{H}\otimes\mathcal{H} = L^2(G\times G)$$

we find that W acts in the following way

$$(W\Psi)(x,y) = \Psi(xy,y)$$

for all $\Psi \in L^2(G\times G)$ and all $x, y \in G$.

For $\varphi, \psi \in \mathcal{H}$ we denote by $\omega_{\varphi,\psi}$ the functional on $B(\mathcal{H})$ given by

$$\omega_{\varphi,\psi}(a) = (\varphi|a\psi)$$

for all $a \in B(\mathcal{H})$.

Take arbitrary $\xi, \eta \in \mathcal{H}$. The following calculation

$$\begin{aligned}
(\xi\big|(\mathrm{id}\otimes\omega_{\varphi,\psi})(W)\big|\eta) &= (\xi\otimes\varphi|W|\eta\otimes\psi)\\
&= \iint \overline{\xi(x)\varphi(y)}\big(W(\eta\otimes\psi)\big)(x,y)\,dx\,dy\\
&= \iint \overline{\xi(x)\varphi(y)}\eta(xy)\psi(y)\,dx\,dy\\
&= \int \overline{\xi(x)}\left(\int \overline{\varphi(y)}\psi(y)\eta(xy)\,dy\right)dx\\
&= \int \overline{\xi(x)}(R_{\overline{\varphi}\psi}\eta)(x)dx = (\xi\big|R_{\overline{\varphi}\psi}\big|\eta)
\end{aligned}$$

shows that

$$(\mathrm{id}\otimes\omega_{\varphi,\psi})(W) = R_{\overline{\varphi}\psi}.$$

In particular the closed linear span of the set

$$\{(\mathrm{id}\otimes\omega)(W)\mid\omega\in B(\mathcal{H})_*\}$$

coincides with $\mathrm{C}^*_r(G)$.

In a similar way we show that $(\omega_{\varphi,\psi}\otimes\mathrm{id})(W)$ is the operator of multiplication by the function $G\ni x\mapsto(\varphi|R_x|\psi)$. Using the Stone-Weierstrass theorem one can easily show that such functions span a dense subspace of $\mathrm{C}_0(G)$. Therefore the closed span of the set

$$\{(\omega\otimes\mathrm{id})(W)\mid\omega\in B(\mathcal{H})_*\}$$

coinsides with the image of $\mathrm{C}_0(G)$ in the representation by multiplication operators on $\mathcal{H}$.

Let us now note another interesting property of W. We will let W act on $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$ in three ways: W_{12} is $W\otimes\mathbf{1}_{\mathcal{H}}$, W_{23} is $\mathbf{1}_{\mathcal{H}}\otimes W$ and $W_{1,3}$ is the opreator $(\Sigma\otimes\mathbf{1}_{\mathcal{H}})W_{23}(\Sigma\otimes\mathbf{1}_{\mathcal{H}})$, where Σ is the flip operator $\mathcal{H}\otimes\mathcal{H}\ni\xi\otimes\eta\mapsto\eta\otimes\xi\in\mathcal{H}\otimes\mathcal{H}$. With this nothation we can take $\Theta\in\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}=L^2(G\times G\times G)$ and calculate

$$(W_{23}W_{12}\Theta)(x,y,z)=(W_{12}\Theta)(x,yz,z)=\Theta\bigl(x(yz),yz,z\bigr)$$

and

$$(W_{12}W_{13}W_{23}\Theta)(x,y,z)=(W_{13}W_{23}\Theta)(xy,y,z)=(W_{23}\Theta)\bigl((xy)z,y,z\bigr)=\Theta\bigl((xy)z,yz,z\bigr).$$

Therefore

$$W_{23}W_{12}W_{23}^*=W_{12}W_{13}.$$

The above equation is called the *pentagonal equation* for W. Unitary operators satisfying the pentagonal equation are called *multiplicative unitaries*.

The pentagonal equation is related to the fact that the Kac-Takesaki operator W encodes the group multiplication on G. More precisely let us denote by Δ the C*-algebra morphism $C_0(G)\to C_0(G\times G)$ given by

$$\Delta(f)(x,y)=f(xy)$$

for all $f\in\mathrm{C}_o(G)$ and $x,y\in G$ (a C*-algebra morphism is a nondegenerate $*$-homomorphism into the multiplier algebra). If we denote by π the representation of $\mathrm{C}_0(G)$ by multiplication operators on $\mathcal{H}$ and recall that $\mathrm{C}_0(G\times G)=\mathrm{C}_0(G)\otimes\mathrm{C}_0(G)$ then we find that for any $\xi,\eta,\varphi,\psi\in\mathcal{H}$ and $f\in\mathrm{C}_0(G)$

$$\begin{aligned}
&\bigl(\psi\otimes\varphi\big|W\bigl(\pi(f)\otimes\mathbf{1}_{\mathcal{H}}\bigr)W^*\big|\eta\otimes\xi\bigr)\\
&=\iint\overline{\psi(x)\varphi(y)}\bigl(\bigl(\pi(f)\otimes\mathbf{1}_{\mathcal{H}}\bigr)W^*(\eta\otimes\xi)\bigr)(xy,y)\,dx\,dy\\
&=\iint\overline{\psi(x)\varphi(y)}f(xy)\bigl(W^*(\eta\otimes\xi)\bigr)(xy,y)\,dx\,dy\\
&=\iint\overline{\psi(x)\varphi(y)}f(xy)\eta(x)\xi(y)\,dx\,dy\\
&=(\psi\otimes\varphi|(\pi\otimes\pi)\Delta(f)|\eta\otimes\xi)
\end{aligned}.$$

In other words the morphism Δ is on the level of $B(\mathcal{H})$ encoded by the map

$$a\longmapsto W(a\otimes\mathbf{1}_{\mathcal{H}})W^*.$$

Moreover Δ is *coassociative* in the sense that

$$(\Delta\otimes\mathrm{id})\circ\Delta=(\mathrm{id}\otimes\Delta)\circ\Delta.$$

A similar calculation shows that for any $x\in G$

$$W^*(\mathbf{1}_{\mathcal{H}}\otimes R_x)W=R_x\otimes R_x$$

and thus the formula $a\mapsto W^*(\mathbf{1}_{\mathcal{H}}\otimes a)W$ defines a morphism $\mathrm{C}^*_r(G)\to\mathrm{C}^*_r(G)\otimes\mathrm{C}^*_r(G)$. This morphism aslo is coassociative.

Another element of group structure on G which can be recovered from W is the inverse. The operation $G\ni x\mapsto x^{-1}\in G$ defines a map $S:\mathrm{C}_o(G)\to\mathrm{C}_0(G)$ via

$$(Sf)(x)=f(x^{-1}).$$

Let κ denote the map acting on $\pi\bigl(\mathrm{C}_0(G)\bigr)$ such that $\kappa\circ\pi=\pi\circ S$. Then for any $\varphi,\psi\in\mathcal{H}$ we have

$$(\omega_{\varphi,\psi}\otimes\mathrm{id})(W^*)=\kappa\bigl((\omega_{\varphi,\psi}\otimes\mathrm{id})(W)\bigr)$$

as shown by the following calculation: for $\xi,\eta\in\mathcal{H}$ we have

$$\begin{aligned}(\xi\big|(\omega_{\varphi,\psi}\otimes\mathrm{id})(W^*)\big|\eta)&=(\varphi\otimes\xi|W^*|\psi\otimes\eta)\\&=\iint\overline{\varphi(x)\xi(y)}\bigl(W^*(\psi\otimes\eta)\bigr)(x,y)\,dx\,dy\\&=\iint\overline{\varphi(x)\xi(y)}\psi(xy^{-1})\eta(y)\,dx\,dy\\&=\int\overline{\xi(y)}\,\bigl(\varphi\big|R_{y^{-1}}\big|\psi\bigr)\,\eta(y)\,dy=\bigl(\xi\big|\kappa\bigl((\omega_{\varphi,\psi}\otimes\mathrm{id})(W)\bigr)\big|\eta\bigr)\,.\end{aligned}$$

Now let us point to the last important feature of W. We describe this feature using the complex conjugate Hilbert space $\overline{\mathcal{H}}$ to $\mathcal{H}$. However, since $\mathcal{H}=L^2(G)$, we can identify $\mathcal{H}$ and $\overline{\mathcal{H}}$ using the complex conjugation $\mathcal{H}\ni\psi\mapsto\overline{\psi}\in\mathcal{H}$. The feature of W we wish to exhibit is that there exist positive selfadjoint operators Q and $\widehat{Q}$ on $\mathcal{H}$ such that $\ker Q=\ker\widehat{Q}=\{0\}$ and a unitary operator $\widetilde{W}$ on $\overline{\mathcal{H}}\otimes\mathcal{H}$ such that

$$W(\widehat{Q}\otimes Q)W^*=\widehat{Q}\otimes Q$$

and

$$(\psi\otimes\varphi|W|\xi\otimes\eta)=\Bigl(\overline{\xi}\otimes Q\varphi\Big|\widetilde{W}\Big|\overline{\psi}\otimes Q^{-1}\eta\Bigr)$$

for all $\xi,\psi\in\mathcal{H}$, $\varphi\in\mathrm{D}(Q)$ and $\eta\in\mathrm{D}(Q^{-1})$. To see that this is indeed the case take $\widehat{Q}=Q=\mathbf{1}_{\mathcal{H}}$ and $\widetilde{W}=W^*$ (using the identification of $\mathcal{H}$ with $\overline{\mathcal{H}}$ we described).

The property of W described above is called *modularity* of W. In other words W is a modular multiplicative unitary.

All examples of quantum groups considered in today's literature originate from modular multiplicative unitaries. In particular one could define a quantum group as object (A,Δ), where A is a C*-algebra obtained as the closed linear span of the set

$$\bigl\{(\omega\otimes\mathrm{id})(W)\mid\omega\in B(\mathcal{H})_*\bigr\}$$

and Δ is a morphism $A \to A \otimes A$ given by

$$\Delta(a) = W(a \otimes \mathbf{1}_{\mathcal{H}})W^*,$$

where W is some modular multiplicative unitary operator.

Locally compact quantum groups (as defined by Kustermans and Vaes) are examples of quantum groups in the above sense. It is an open question whether all quantum groups coming wrom multiplicative unitaries are locally compact quantum groups.

1.4.2 MNW definition of a locally compact quantum group

The definition of a locally compact quantum group we will give below is taken from the work of Masuda, Nakagami and Woronowicz. It is equivalent to the Kustermans-Vaes definition.

A locally compact quantum group is a pair (A, Δ) consisting of a C*-algebra A and a morphism $\Delta : A \to A \otimes A$ posessing a number of properties we will list below.

The properties of (A, Δ) we require are the following:

1. The morphism Δ is coassociative: $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ and the sets
$$\big\{\Delta(a)(\mathbf{1}_A \otimes b) \mid a, b \in A\big\} \quad \text{and} \quad \big\{(a \otimes \mathbf{1}_A)\Delta(b) \mid a, b \in A\big\}$$
are contained and linearly dense in $A \otimes A$.

2. There exists a strictly faithful, locally finite lower semicontinuous weight h on A such that h is *right invariant*: for all $\varphi \in A^*{}_+$ and all $a \in A_+$ such that $h(a) < \infty$
$$h(\varphi * a) = \varphi(\mathbf{1}_A)h(a).$$

3. There exists a closed linear operator κ on the Banach space A such that the *strong right invariance* of h holds
$$h\big((\varphi * a^*)b\big) = h\big(a^*\big((\varphi \circ \kappa) * b\big)\big) \tag{1.9}$$
for all $\varphi \in A^*$ such that $\varphi \circ \kappa \in A^*$ ans all $a, b \in A$ such that $h(a^*a), h(b^*b) < \infty$.

4. The operator κ has the following decomposition
$$\kappa = R \circ \tau_{\frac{i}{2}},$$
where R is an involutive $*$-antiautomorphism of A (called the *unitary antipode*) and $\tau_{\frac{i}{2}}$ is the analytic continuation of a one parameter group $(\tau_t)_{t \in \mathbb{R}}$ of $*$-automorphisms of A (called the *scaling group*) such that $\tau_t \circ R = R \circ \tau_t$ for all $t \in \mathbb{R}$.

5. There exists a constant $\lambda > 0$ (called the *scaling constant*) such that $h \circ \tau_t = \lambda^t h$ for all $t \in \mathbb{R}$.

The above list demands a number of comments. First of all let us explain the notation "$\varphi * a$". If $\varphi \in A^*$ and $a \in a$ then it makes sense to write $(\mathrm{id}\otimes\varphi)\Delta(a)$. This is because φ can be written as $\varphi' \circ R_b$, where φ' is another continuous functional on A and R_b is the operator of right multiplication by an element $b \in A$ (this follows from *Cohen's factorization theorem*). In particular we can write

$$(\mathrm{id}\otimes\varphi)\Delta(a) = (\mathrm{id}\otimes\varphi')\big(\Delta(a)(\mathbf{1}_A \otimes b)\big)$$

and the element $\Delta(a)(\mathbf{1}_A \otimes b)$ is by assumption contained in $A \otimes A$, so we can apply to it the mapping $(\mathrm{id}\otimes\varphi')$.

The properties of the weight h (i.e. an additive and positive homogeneous map $A_+ \to [0,\infty]$) listed above were

- lower semicontinuity,
- local finiteness, i.e. the fact that the span of elements whose weight is finite is norm-dense in A,
- strict faithfulness, which is the property that for any sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A such that the sequence $\big(h(a_n^*a_n)\big)_{n\in\mathbb{N}}$ is bounded and $\lim\limits_{n\to\infty} h(a_na_n^*) = 0$ the fact that for some $b \in A$ we have

  $$h\big((b-a_n)^*(b-a_n)\big) \leq h(a_n^*a_n)$$

  for all n implies that $b = 0$.

In order to explain the formula describing strong right invariance let us note that the weight h can be extended to a linear functional on the linear span of the set

$$\big\{a^*b \mid h(a^*a), h(b^*b) < \infty\big\}.$$

Moreover using standard facts about completely positive maps (in particular the *Kadison inequality*) one can show that for any positive $\varphi \in A^*$ and $a \in A$ such that $h(a^*a) < \infty$ we have

$$h\big((\varphi * a)^*(\varphi * a)\big) < \infty$$

and since and element of A^* can be written as a linear combination of elements of A_+^* we find that the expression

$$h\big((\psi * a)^* b\big)$$

makes sense for any $\psi \in A^*$ and a, b such that $h(a^*a), h(b^*b) < \infty$. Therefore both sides of (1.9) are well defined numbers and we can demand that they be equal.

Let (A,Δ) be a locally compact quantum group as defined above. One of the main results of the theory of these objects is that the mapping

$$A \otimes A \ni a \longmapsto \Delta(a)(\mathbf{1}_A \otimes b) \in A \otimes A$$

extends to a unitary map $W : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$, where $\mathcal{H}$ is the GNS-Hilbert space for the weight h. This operator W is a modular multiplicative unitary.

Moreover the quantum group (A, Δ) can be recovered from W as described in Subsection 1.4.1:

$$A = \operatorname{span}\left\{(\omega \otimes \mathrm{id})(W) \mid \omega \in B(\mathcal{H})_*\right\}^{-\|\cdot\|} \subset B(\mathcal{H})$$

and

$$\Delta(a) = W(a \otimes \mathbf{1}_A)W^*$$

for all $a \in A$.

1.4.3 More on strong right invariance

The condition of strong right invariance is at first difficult to understand. In order to become more familiar with it we will first look at it from the point of view of Hopf algebras. Then we will see that it holds in the motivating example form Subsection 1.4.1, i.e. for classical locally compact groups.

Let $\mathbb{A}$ be a Hopf $*$-algebra with a right invariant positive integral h. Without getting into details we will assume that the GNS-construction can be performed for h and we have the Hilbert space $\mathcal{H}$ which is the completion of $\mathbb{A}$ in the norm coming from the scalalr product $(a|b) = h(a^*b)$ for $a, b \in \mathbb{A}$.

Consider the map $\Phi : \mathbb{A} \otimes \mathbb{A} \ni a \otimes b \mapsto \Delta(a)(\mathbf{1}_{\mathbb{A}} \otimes b) \in \mathbb{A} \otimes \mathbb{A}$, where Δ is the comultiplication of $\mathbb{A}$. This map has an explicite inverse:

$$\Phi^{-1} : \mathbb{A} \otimes \mathbb{A} \ni (r \otimes s) \longmapsto (\mathrm{id} \otimes \kappa)\big(\Delta(r)\big)(\mathbf{1}_{\mathbb{A}} \otimes s) \in \mathbb{A} \otimes \mathbb{A}$$

where κ is the antipode of $\mathbb{A}$. We now want to take the map Φ to the level of the hilbert space $\mathcal{H}$. Since $\mathbb{A}$ is dense in $\mathcal{H}$, the mappings Φ and Φ^{-1} are densely defined. We would like to have $\Phi^{-1} = \Phi^*$ which means that

$$(h \otimes h)\big(\Phi(a \otimes b)^*(r \otimes s)\big) = (h \otimes h)\big((a \otimes b)^*\Phi^{-1}(r \otimes s)\big)$$

for all $a, b, r, s \in \mathbb{A}$. In other words

$$(h\otimes h)\big((\mathbf{1}_{\mathbb{A}}\otimes b^*)\Delta(a^*)(r\otimes s)\big) = (h\otimes h)\big((a^*\otimes b^*)(\mathrm{id}\otimes\kappa)\big(\Delta(r)\big)(\mathbf{1}_{\mathbb{A}}\otimes s)\big). \tag{1.10}$$

A simple calculation shows that if $\varphi(c) = h(b^*cs)$ then the left hand side of (1.10) is

$$h\big((\varphi * a^*)r\big)$$

while the right hand side is

$$h\big(a^*\big((\varphi \circ \kappa) * r\big)\big).$$

Thereofre the strong right invariance of h is the key to unitarity of W (cf. Subsection 1.4.2).

Now let (A, Δ) be the quantum group coming from a classical group G, i.e. $A = \mathrm{C}_0(G)$ and Δ dualizes the group multiplication. We let h be the weight on A which corresponds to integration with respect to the right Haar measure and define $\kappa : A \to A$ as $\kappa(f)(x) = f(x^{-1})$ foa all $f \in A$ and $x \in G$. The scaling group in this example is trival and $R = \kappa$. Clearly the weight h is right invariant in the sense explained in Subsection 1.4.2.

Let us check the strong right invariance of h: take $f, g \in \mathrm{C}_0(G) \cap L^2(G)$ (this means that $h\big(|f|^2\big), h\big(|g|^2\big) < \infty$ and let φ be a continuous functional on A, so that for any $u \in A$

$$\varphi(u) = \int u(x)\, d\mu_\varphi(x)$$

for some finite measure μ_φ on G. Then for any $v \in A$ the convolution product $\varphi * v$ is the function

$$G \ni s \longmapsto \int v(st)\, d\mu_\varphi(t).$$

We have

$$\begin{aligned}
h\big((\varphi * \overline{f})g\big) &= \int \Big(\int \overline{f(st)}\, d\mu(t)\Big) g(s)\, ds \\
&= \iint \overline{f(st)} g(s)\, d\mu(t)\, ds \\
&= \int \Big(\int \overline{f(st)} g(s)\, ds\Big)\, d\mu(t) \\
&= \int \Big(\int \overline{f(s)} g(st^{-1})\, ds\Big)\, d\mu(t).
\end{aligned}$$

On the other hand $(\varphi \circ \kappa)(u) = \int u(t^{-1})\, d\mu_\varphi(t)$, so that

$$\begin{aligned}
h\big(\overline{f}\big((\varphi \circ \kappa) * g\big)\big) &= \int \overline{f(s)} \Big(\int g(st^{-1})\, d\mu_\varphi(t)\Big) ds \\
&= \iint \overline{f(s)} g(st^{-1})\, d\mu_\varphi(t)\, ds \\
&= \int \Big(\int \overline{f(s)} g(st^{-1})\, ds\Big)\, d\mu(t).
\end{aligned}$$

1.4.4 Actions of quantum groups

In this subsection we would like to propose a definition of an action of a quatum group (A, Δ) on a "quantum space" or simply C*-algebra B. Although many variations on this definition are considered in the literature we feel that there is consensus that actions as definded below are should be included by any reasonable definiton of a quantum group action.

Let (A, Δ) be a quantum group and let B be a C*-algebra. An *action* of (A, Δ) on B is a morphism $\alpha : B \to B \otimes A$ (i.e. a nondegenerate $*$-homomorphism $B \to \mathrm{M}(B \otimes A)$ such that

$$(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$$

and the set

$$\big\{\alpha(b)(\mathbf{1}_B \otimes a) \mid a \in A, : b \in B\big\}$$

is contained and linearly dense in $B \otimes A$.

If (A, Δ) id of the form $\big(\mathrm{C}_0(G), \Delta\big)$, where G is a locally compact group and Δ is the standard comultiplication on $\mathrm{C}_0(G)$ then an action of (A, Δ) on G is the same thing as a continuous action of G on B.

1.5 Some applications of the universal property

Here we use the universal property of KK^G to construct the exterior product in KK^G for a groupoid G and prove that it is graded commutative. Similar arguments yield the descent functor, induction and restriction functors, and some identities between these. All these results can, of course, be proved also using the concrete description of KK^G, but the abstract approach yields these assertions without computing a single Kasparov product. See also [mn06] for more details.

Let G be a locally compact group and let $\mathbf{C^*{-}alg}_G$ be the corresponding category of C*-algebras with a G-action. The minimal tensor product of two G-C*-algebras is again a G-C*-algebra, using the diagonal action of G. This yields a functor

$$\otimes\colon \mathbf{C^*{-}alg}_G \times \mathbf{C^*{-}alg}_G \to \mathbf{C^*{-}alg}_G, \quad (A,B) \mapsto A \otimes B.$$

We claim that this tensor product descends to a tensor product on KK^G:

$$\begin{array}{ccc} \mathbf{C^*{-}alg}_G \times \mathbf{C^*{-}alg}_G & \longrightarrow & \mathbf{C^*{-}alg}_G \\ \downarrow \qquad\quad \downarrow & & \downarrow \\ \mathrm{KK}^G \times \mathrm{KK}^G & \longrightarrow & \mathrm{KK}^G \end{array}$$

We describe this more concretely. Let $\beta \in \mathrm{KK}_0^G(B_1,B_2)$, $\alpha \in \mathrm{KK}_0^G(A_1,A_2)$. First we construct exterior products $\mathrm{id}_{A_j} \otimes \beta \in \mathrm{KK}_0^G(A_j \otimes B_1, A_j \otimes B_2)$ and $\alpha \otimes \mathrm{id}_{B_j} \in \mathrm{KK}_0^G(A_1 \otimes B_j, A_2 \otimes B_j)$ for $j = 1,2$. Then we check that the following diagram commutes:

$$\begin{tikzcd} A_1 \otimes B_1 \arrow[r, "\mathrm{id}_{A_1}\otimes\beta"] \arrow[d, "\alpha\otimes\mathrm{id}_{B_1}"'] & A_1 \otimes B_2 \arrow[d, "\alpha\otimes\mathrm{id}_{B_2}"] \\ A_2 \otimes B_1 \arrow[r, "\mathrm{id}_{A_2}\otimes\beta"'] & A_2 \otimes B_2. \end{tikzcd} \tag{1.11}$$

Thus we may define

$$\alpha \otimes \beta := (\alpha \otimes \mathrm{id}_{B_2}) \circ (\mathrm{id}_{A_1} \otimes \beta) = (\mathrm{id}_{A_2} \otimes \beta) \circ (\alpha \otimes \mathrm{id}_{B_1}).$$

Notice that this involves the Kasparov product. The commutativity of (1.11) means that the exterior product is commutative on the even parts (if we allow odd KK-groups, then the exterior product becomes *graded* commutative).

In the abstract approach we first fix A and consider the functor

$$\mathbf{C^*{-}alg}_G \to \mathbf{C^*{-}alg}_G \to \mathrm{KK}^G, \qquad B \mapsto A \otimes B \mapsto A \otimes B.$$

Since it is equivariantly split exact and stable, it descends to a functor $\mathrm{KK}^G \to \mathrm{KK}^G$ by the universal property, furnishing us with $\mathrm{id}_A \otimes \beta$. The same construction provides $\alpha \otimes \mathrm{id}_B$. In order to understand why (1.11) commutes, we use the naturality of the constructions above.

In general, if $F_1, F_2\colon \mathbf{C^*{-}alg}_G \to \mathcal{D}$ are split exact and stable, and $\Phi\colon F_1 \to F_2$ is a natural transformation, then there exist $\overline{F_1}, \overline{F_2}\colon \mathrm{KK}^G \to \mathcal{D}$ and a natural transformation $\overline{\Phi}\colon \overline{F_1} \to \overline{F_2}$ such that the following diagram commutes for all $\alpha \in \mathrm{KK}^G(A_1, A_2)$

$$\begin{tikzcd} \overline{F_1}(A_1) \arrow[r, "\overline{\Phi}_{A_1}"] \arrow[d, "\overline{F_1}(\alpha)"'] & \overline{F_2}(A_1) \arrow[d, "\overline{F_2}(\alpha)"] \\ \overline{F_1}(A_2) \arrow[r, "\overline{\Phi}_{A_2}"] & \overline{F_2}(A_2) \end{tikzcd}$$

That is, if this diagram commutes whenever α and β are equivariant *-homomorphisms, then it also commutes if we let α, β be KK^G-morphisms. This statement is a part of the universal property of KK^G.

In our case, it is clear that (1.11) commutes if α and β are $*$-homomorphisms. Hence it still commutes if α is in $\mathrm{KK}_0^G(A_1, A_2)$ and β is a $*$-homomorphism. Finally, another application of the same trick shows that it still commutes if both α and β are KK^G-morphisms of even degree. The odd case can then be reduced to the even case using

$$\mathrm{KK}_0^G(A, B) \cong \mathrm{KK}_0^G(C_0(\mathbb{R}) \otimes A, B) \cong \mathrm{KK}_0^G(A, C_0(\mathbb{R}) \otimes B),$$

but here signs may appear in the computation.

Proposition 1.20. *The exterior product coincides with composition product on $\mathrm{KK}_*^G(\mathbb{C}, \mathbb{C})$, turning this into a graded commutative $\mathbb{Z}/2$-graded ring. Furthermore, $\mathrm{KK}_*^G(\mathbb{C}, \mathbb{C})$ acts on $\mathrm{KK}_*^G(A, B)$ for all $A, B \in \mathbf{C}^*\mathbf{-alg}_G$ by exterior product.*

Now we turn to the descent functor $\mathrm{KK}^G \to \mathrm{KK}$. This is obtained from the reduced crossed product functor

$$\mathbf{C}^*\mathbf{-alg}_G \to \mathbf{C}^*\mathbf{-alg}, \qquad A \mapsto G \ltimes_{\mathrm{r}} A.$$

This functor maps equivariantly split exact extensions to split extensions and maps equivariantly Morita equivalent C*-algebras to Morita equivalent C*-algebras. By the universal property, it descends to a functor $\mathrm{KK}^G \to \mathrm{KK}$. This is the *descent functor*

$$\mathrm{KK}^G(A, B) \to \mathrm{KK}(G \ltimes_{\mathrm{r}} A, G \ltimes_{\mathrm{r}} B),$$

which appears in any construction of the Baum–Connes assembly map.

Now let G be a locally compact group, let $\mathcal{H} \leq G$ be a closed subgroup, and let $\mathcal{H} \curvearrowright A$. Then we define

$$\mathrm{Ind}_H^G A := \{f \in C_0(G, A) \mid f(gh) = (\alpha_h f)(g),\ \|f\| \in C_0(G/H)\},$$

on which we let G act by left translation. (On the level of spaces, this induction corresponds to $\mathrm{Ind}_H^G\colon X \mapsto G \times_H X$). Again, it is evident that Ind_H^G defines a functor $\mathbf{C}^*\mathbf{-alg}_H \to \mathbf{C}^*\mathbf{-alg}_G$ that preserves split extensions and Morita equivalences and therefore descends to a functor

$$\mathrm{Ind}_H^G\colon\ \mathrm{KK}^H \to \mathrm{KK}^G.$$

Green's Imprimitivity Theorem asserts that the crossed products $G \ltimes \mathrm{Ind}_H^G A$ and $H \ltimes A$ are Morita equivalent; a similar statement holds for reduced crossed products. The Morita equivalence is easily seen to be natural for equivariant *-homomorphisms. By the universal property, we conclude that the class of the Morita equivalence in $\mathrm{KK}(H \ltimes_{\mathrm{r}} A, G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G A)$ provides a natural transformation between two functors on KK^H.

If $H \leq G$ is open, then there is a natural isomorphism

$$\mathrm{KK}^G(\mathrm{Ind}_H^G A, B) \cong \mathrm{KK}^H(A, \mathrm{Res}_H^G B),$$

that is, the induction and restriction functors are adjoint. This is proved by constructing the unit and counit of the adjunction—these turn out to be certain natural *-homomorphisms between equivariant stabilisations of A and B. The universal property shows that they still produce natural transformations on the KK-level, and the identities that have to be checked already hold on level of equivariant *-homomorphisms. Thus we can also prove this property of KK^G without using its concrete description.

For several purposes, it is desirable to have an analogue of the exterior product also for quantum group coactions. Christian Voigt has recently developed this, following a suggestion by Ryszard Nest. Here we only indicate the idea behind this in the somewhat simpler case of coactions of finite groups.

Let G be finite group and let A and B be algebras with a G-coaction, that is, a grading by G. Then $A \otimes B$ carries a diagonal coaction

$$(A \otimes B)_g = \bigoplus_{h \in G} A_h \otimes B_{h^{-1}g}.$$

We want to equip $A \otimes B$ with a multiplication that is equivariant for the canonical coaction of G on $A \otimes B$. The usual product does not work unless G is commutative, because if $a \in A_h$, $b \in B_g$, then $a \cdot b = b \cdot a \in (A \otimes B)_{hg}$, but we need $b \cdot a \in (A \otimes B)_{gh}$. We must therefore impose a non-trivial commutation relation. We make the Ansatz

$$b_g \cdot a_h := \alpha_g(a_h) \cdot b_g \qquad \text{for } a_h \in A_h,\ b_g \in B_g,$$

where $\alpha_g \colon A \to A$ for $g \in G$ is some linear map. Associativity dictates that $\alpha_g(a_1 \cdot a_2) = \alpha_g(a_1)\alpha_g(a_2)$, and $\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1 g_2}$. It is reasonable also to require $\alpha_1 = \mathrm{id}_A$, so that α is an action of G on A by algebra automorphisms. Finally, covariance dictates that $\alpha_g(A_h) \subseteq A_{ghg^{-1}}$ for all $g, h \in G$. Any action of G on A with these properties yields an associative product on $A \otimes B$ that is compatible with the diagonal coaction.

Up to Morita equivalence, the extra structure α is no serious restriction because it always exists on a stabilisation $E_A := \mathrm{End}(A \otimes \mathbb{C}[G])$ with the coaction of G induced by the tensor product coaction on $A \otimes \mathbb{C}[G]$. Since $A_h \otimes |\delta_g\rangle\langle\delta_l|$ maps $(A \otimes \mathbb{C}[G])_x$ to $A_{xl^{-1}h} \otimes \mathbb{C}[G]_g \subseteq (A \otimes \mathbb{C}[G])_{xl^{-1}hg}$, this coaction is given by

$$(E_A)_g = \sum_{x,y,z \in G,\ x^{-1}yz = g} A_y \otimes |\delta_z\rangle\langle\delta_x|.$$

Let G act on $A \otimes \mathbb{C}[G]$ by the regular representation. This induces an action $\alpha \colon G \times E_A \to E_A$ by conjugation. We check that if $x^{-1}yz = h$, then

$$\alpha_g(A_y \otimes |\delta_z\rangle\langle\delta_x|) = A_y \otimes |\delta_{zg^{-1}}\rangle\langle\delta_{xg^{-1}}| \in (E_A)_{gx^{-1}yzg^{-1}} = (E_A)_{ghg^{-1}}$$

Thus $E_A \otimes B$ carries a canonical algebra structure compatible with the diagonal coaction of of G.

The equivariant Kasparov theory for C*-algebras with a quantum group coaction is developed by Saad Baaj and Georges Skandalis in [**?**]. At least for regular quantum groups, the same arguments as in the group case show that it is the universal split exact stable functor. The following result is only stated in [**?**] for groups and their duals because the duality theory for regular quantum groups was not yet developed at that time:

Theorem 1.21 (Baaj–Skandalis duality)**.** *Let* (C, Δ) *be a strongly regular locally compact quantum group and let* $(\widehat{C}^c, \widehat{\Delta}^c)$ *be the C*-commutant of its dual. Then the functors*

$$A \text{ with } C\text{-coaction} \mapsto A \rtimes_{\mathrm{r}} \widehat{C}^c \text{ with } \widehat{C}^c\text{-coaction}$$
$$B \text{ with } \widehat{C}^c\text{-coaction} \mapsto B \rtimes_{\mathrm{r}} C \text{ with } C\text{-coaction}$$

descend to an equivalence of categories

$$\mathrm{KK}^C \cong \mathrm{KK}^{\widehat{C}^c}.$$

Proof. This theorem follows immediately from the duality theory for regular quantum groups (see [**?**]). The main difficulty is to construct the crossed product functors. Roughly speaking, $\widehat{C}^c$ is the C*-algebra generated by the *right* regular representation of C. We may represent both $B \otimes C$ and $\widehat{C}^c$ by adjointable operators on the Hilbert B-module $B \otimes L^2(C)$. Assuming that the coaction is continuous, the closure of

$$\delta(B) \cdot (1 \otimes \widehat{C}^c) \subseteq M(B \otimes \mathcal{K}(L^2C))$$

is a C*-algebra. By definition, this is the crossed product $B \rtimes_{\mathrm{r}} \widehat{C}^c$. It comes with a canonical $\widehat{C}^c$-coaction, which acts trivially on $\delta(B)$ and by the comultiplication of $\widehat{C}^c$ on $1 \otimes \widehat{C}^c$. Here we follow the notation of Kustermans and Vaes, whereas Baaj and Skandalis denote $\widehat{C}^c$ by $\widehat{C}$. As C*-algebras, $\widehat{C}^c$ and $\widehat{C}$ are isomorphic, but they usually carry different comultiplications.

Strong regularity means, more or less by definition, that there are canonical *-isomorphisms

$$B \rtimes_{\mathrm{r}} \widehat{C}^c \rtimes_{\mathrm{r}} C \cong B \otimes \mathcal{K}(L^2C),$$
$$B \rtimes_{\mathrm{r}} C \rtimes_{\mathrm{r}} \widehat{C}^c \cong B \otimes \mathcal{K}(L^2C),$$

where the first isomorphism is C-equivariant, and the second $\widehat{C}^c$-equivariant.

To pass to equivariant KK, we must merely observe that the crossed product functors $B \mapsto B \rtimes_{\mathrm{r}} \widehat{C}^c$ and $B \mapsto B \rtimes_{\mathrm{r}} C$ preserve equivariantly split extensions and Morita equivalences. Therefore, these functors descend to functors $\mathrm{KK}^C \leftrightarrow \mathrm{KK}^{\widehat{C}^c}$.

The duality isomorphisms above are obviouly natural for equivariant *-homomorphisms; by the universal property, they remain natural for KK-morphisms. Using stability, we get natural equivariant KK-equivalences

$$B \rtimes_{\mathrm{r}} \widehat{C}^c \rtimes_{\mathrm{r}} C \cong B \text{ for } B \in \mathrm{KK}^C,$$
$$B \rtimes_{\mathrm{r}} C \rtimes_{\mathrm{r}} \widehat{C}^c \cong B \text{ for } B \in \mathrm{KK}^{\widehat{C}^c}.$$

Hence the crossed product functors are inverse to each other and provide an equivalence of categories $\mathrm{KK}^{C} \cong \mathrm{KK}^{\widehat{C}^c}$. ☐

Chapter 2

The Baum-Connes conjecture

2.1 Universal G-space for proper actions

Let G be a topological group which is locally compact, Hausdorff and second countable. A G-space is a topological space X with a given continuous action of G

$$G \times X \to X.$$

If X, Y are G-spaces, then a G-map from X to Y is a continuous G-equivariant map $f\colon X \to Y$

$$f(gp) = gf(p), \quad g \in G,\ p \in X.$$

Two G-maps $f_0, f_1\colon X \to Y$ are G-homotopic if they are homotopic through G-maps, i.e. there exists a homotopy $\{f_t\}$, $0 \leq t \leq 1$ with each f_t a G-map.

Definition 2.1. *A G-space X is* proper *if*

- *X is paracompact and Hausdorff,*
- *the quotient space X/G (with the quotient topology) is paracompact and Hausdorff,*
- *for each $p \in X$ there exists a triple (U, H, ρ) such that*
    1. *U is an open neighbourhood of p in X with $gu \in U$ for all $g \in G$, $u \in U$,*
    2. *H is a compact subgroup of G,*
    3. *$\rho\colon U \to G/H$ is a G-map from U to G/H.*

Proposition 2.2 ([cem01])**.** *If X is a locally compact Hausdorff second countable G-space, then X is proper if and only if the map*

$$G \times X \to X \times X, \quad (g, x) \mapsto (gx, x)$$

is proper (i.e. the preimage of any compact set in $X \times X$ is compact).

Definition 2.3. *A* universal G-space for proper actions*, denoted* $\underline{\mathrm{E}}G$ *is a proper G-space such that if X is any proper G-space, then there exists a G-map $f\colon X \to \underline{\mathrm{E}}G$ and any two G-maps from X to $\underline{\mathrm{E}}G$ are G-homotopic.*

Lemma 2.4. *There exists universal G-space for proper actions.*

The space $\underline{\mathrm{E}}G$ is unique up to homotopy. Indeed, if $\underline{\mathrm{E}}G$ and $(\underline{\mathrm{E}}G)'$ are both universal examples for proper actions of G, then there exist G-maps

$$f\colon \underline{\mathrm{E}}G \to (\underline{\mathrm{E}}G)',$$
$$f'\colon (\underline{\mathrm{E}}G)' \to \underline{\mathrm{E}}G,$$

with $f' \circ f$ and $f \circ f'$ G-homotopic to the identity maps of $\underline{\mathrm{E}}G$ and $(\underline{\mathrm{E}}G)'$ respectively. Moreover f and f' are unique up to homotopy.

The following is a set of axioms for $\underline{\mathrm{E}}G$

1. Y is a proper G-space,
2. if H is any compact subgroup of G then there exists $p \in Y$ with $hp = p$ for all $h \in H$,
3. if we view $Y \times Y$ as a G-space with action

   $$g(y_0, y_1) = (gy_0, gy_1),$$

   $$\rho_0, \rho_1 \colon Y \times Y \to Y, \quad \rho_0(y_0, y_1) = y_0, \ \rho_1(y_0, y_1) = y_1,$$

   then ρ_0 and ρ_1 are G-homotopic.

Lemma 2.5. *If Y satisfies the axioms 1,2,3, then Y is an $\underline{\mathrm{E}}G$.*

Example 2.6.

- If G is compact, then $\underline{\mathrm{E}}G = \mathrm{pt}$.
- If G is a Lie group with $\pi_0(G)$ finite, then $\underline{\mathrm{E}}G = G/H$, where H is maximal compact subgroup of G.
- If G is a p-adic group then $\underline{\mathrm{E}}G$ is the affine Bruhat-Tits building for G, denoted by βG.

  Affine Bruhat-Tits building for $\mathrm{SL}(2, \mathbb{Q}_p)$ is the $(p+1)$-regular tree, that is a tree with exactly $p+1$ edges at each vertex.
- If Γ is (countable) discrete group, then

  $$\underline{\mathrm{E}}\Gamma = \{f\colon \Gamma \to [0,1] \mid \{\gamma \in \Gamma \mid f(\gamma) \neq 0\} \text{ is finite}, \sum_{\gamma \in \Gamma} f(\gamma) = 1\}$$

  The action is given by $(\beta f)(\gamma) = f(\beta^{-1}\gamma)$ for $\beta, \gamma \in \Gamma$, $f\colon \Gamma \to [0,1]$. The space $\underline{\mathrm{E}}\Gamma$ is topologized by the metric

  $$d(f,h) = \left(\sum_{\gamma \in \Gamma} |f(\gamma) - h(\gamma)|^2\right)^{\frac{1}{2}}.$$

Definition 2.7. *A subset $\Delta \subset \underline{\mathrm{E}}G$ is G-compact if*

1. $gx \in \Delta$ *for all* $g \in G$, $x \in \Delta$,
2. *the quotient space* Δ/G *is compact.*

Set
$$\mathrm{K}_j^G(\underline{\mathrm{E}}G) = \varinjlim_{\Delta \subset \underline{E}G,\ \Delta \text{ is } G\text{-compact}} \mathrm{K}_j^G(\Delta).$$

$\mathrm{K}_j^G(\underline{E}G)$ is the equivariant K-homology of $\underline{\mathrm{E}}G$ with G-compact supports. There is a map
$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}G) \to \mathrm{K}_j(C_r^*G)$$
$$(\mathcal{H}, \psi, \pi, T) \mapsto \mathrm{Index}(T).$$

If X is a proper G-space with compact quotient X/G, then
$$\mathcal{E}_j^G(X) := \mathcal{E}_G^j(C_0(X)) = \{(\mathcal{H}, \psi, \pi, T)\}$$
and
$$\mathrm{K}_j^G(X) := \mathrm{KK}_G^j(C_0(X), \mathbb{C}) = \{(\mathcal{H}, \psi, \pi, T)\}/\sim,$$
for $j = 0, 1$, this is the Kasparov equivariant K-homology of X. If X, Y are proper G-spaces with compact quotient spaces X/G, Y/G, and $f\colon X \to Y$ is a continuous G-equivariant map, then $f^*\colon C_0(Y) \to C_0(X)$, $f^*(\alpha) = \alpha \circ f$ induces a homomorphism of abelian groups $f_*\colon\ \mathrm{K}_j^G(X) \to \mathrm{K}_j^G(Y)$,
$$(\mathcal{H}, \psi, \pi, T) \mapsto (\mathcal{H}, \psi \circ f^*, \pi, T).$$

The map
$$\mu\colon\ \mathrm{K}_j^G(X) \to \mathrm{K}_j(C_r^*G)$$
is natural, that is there is commutativity in the diagram

$$\begin{tikzcd}
\mathrm{K}_j^G(X) \arrow[rr, "f_*"] \arrow[dr] & & \mathrm{K}_j^G(Y) \arrow[dl] \\
& \mathrm{K}_j(C_r^*G) &
\end{tikzcd}$$

2.2 The Baum-Connes Conjecture

Conjecture 5. *Let G be a locally compact Hausdorff second countable topological group. Then*
$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}G) \to \mathrm{K}_j(C_r^*G)$$
is an isomorphism for $j = 0, 1$.

It is known that the conjecture is true for

- compact groups,
- abelian groups,
- Lie groups ($\pi_0(G)$ finite),

- p-adic groups,
- adelic groups.

It is not known if the conjecture is true for all discrete groups.

Theorem 2.8 ([s-t07])**.** *Let B_n be tha Braid group on n strands, $n \geq 2$. Then BCC is true for B_n.*

Theorem 2.9 ([hk01])**.** *If Γ is a discrete group which is amenable (or a-t-menable), then BCC is true for Γ.*

Theorem 2.10 ([my02], [l-v99], [l-v02])**.** *If Γ is a discrete group which is hyperbolic (in Gromov's sense), then BCC is true for Γ.*

Theorem 2.11 ([l-v02])**.** *If Γ is any discrete co-compact subgroup of $\mathrm{SL}(3, \mathbb{R})$, then BCC is true for Γ.*

Theorem 2.12. *If Γ is any discrete subgroup of $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ or $\mathrm{Sp}(n, 1)$, then BCC is true for Γ.*

There are following corollaries of the Baum-Connes conjecture.

- Novikov conjecture
- Stable Gromov-Lawson-Rosenberg conjecture
- Idempotent conjecture
- Kadison-Kaplansky conjecture
- Mackey analogy
- Construction of the discrete series via Dirac induction (Parthasarathy, Atiyah, Schmidt)
- Homotopy invariance of ρ-invariants ([ps07])

2.2.1 The conjecture with coefficients

Definition 2.13. *A G-C^*-algebra is a C^*-algebra A with a given continuous action of G*

$$G \times A \to A$$

by C^*-algebra automorphisms. The continuity condition is: for each $a \in A$

$$G \to A, \ g \mapsto ga$$

is continuous map from G to A.

Let A be a G-C*-algebra. Form the reduced crossed product C*-algebra $C_r^*(G, A)$. The goal is to determine $\mathrm{K}_j(C_r^*(G, A))$. Let $\mathrm{K}_j^G(\underline{E}G, A)$ denote the equivariant K-homology of $\underline{E}G$ with G-compact supports and coefficients A, that is

$$\mathrm{K}_j^G(\underline{E}G, A) := \lim_{\longrightarrow \Delta \subset \underline{E}G, \, \Delta \ G-\mathrm{compact}} \mathrm{KK}_G^j(C_0(\Delta), A).$$

Conjecture 6 ([?, ?])**.** *Let G be a locally compact Hausdorff second countable topological group, and let A be any G-C^*-algebra, then*

$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}G, A) \to \mathrm{K}_j(C_r^*(G, A))$$

is an isomorphism for $j = 0, 1$.

Conjecture 6 is the Baum-Connes conjecture with coefficients.

Let Γ be a finitely presented discrete group which contains an expander in its Cayley graph. Such a Γ is a counter-example to the conjecture with coefficients. M. Gromov outlined a proof that such a Γ exists. A number of mathematicians are now filling in the details.

Definition 2.14. *We say that the group G is* exact *if for every exact sequence of C^*-algebras*

$$0 \to I \to A \to B \to 0$$

the sequence

$$0 \to C_r^*(G, I) \to C_r^*(G, A) \to C_r^*(G, B) \to 0$$

is exact.

Remark 2.15. It is very hard to find an example of a group which is not exact. Gromov outlined a construction of a discrete group Γ which contains (in the sense of coarse geometry) an *expander graph* in its *Cayley graph*. Such group will not be exact. Gromov's group Γ will be also a counterexample to the Baum-Connes conjecture with coefficients. Consider the Stone-Čech compactification $\beta\Gamma$. Then we can identify $C(\beta\Gamma)$ with $l^\infty(\Gamma)$, and there is an exact sequence

$$0 \to C_0(\Gamma) \to C(\beta\Gamma) \to C(\beta\Gamma - \Gamma) \to 0,$$

which after applying reduced crossed product $\rtimes_r \Gamma$ will not be exact. Gromov's group will be a counterexample to the Baum-Connes conjecture with coefficients because Gromov's group will not be K-theoretically exact. All this suggests that the class of discrete groups for which the Baum-Connes conjecture with coefficients will be valid might be all the discrete exact groups.

2.3 Assembly map

Let $\Delta \subset X$ be a proper G-space.

Definition 2.16. *We say that Δ is G-compact if Δ is G-invariant and Δ/G is compact and Hausdorff.*

We define the equivariant K-homology of $\underline{\mathrm{E}}G$ by means of Kasparov equivariant KK-theory

$$\mathrm{K}_j^G(\underline{\mathrm{E}}G) = \operatorname{colim}_{\substack{\Delta \in \underline{\mathrm{E}}G \\ G\text{-compact}}} \mathrm{KK}_G^j(C_0(\Delta), \mathbb{C}), \tag{2.1}$$

$$\mathrm{K}_j^G(\underline{\mathrm{E}}G, A) = \operatorname{colim}_{\substack{\Delta \in \underline{\mathrm{E}}G \\ G\text{-compact}}} \mathrm{KK}_G^j(C_0(\Delta), A). \tag{2.2}$$

If A, B are separable G-C*-algebras, then there is the *Kasparov descent map*

$$\mathrm{KK}_G^j(A, B) \to \mathrm{KK}^j(C_r^*(G, A), C_r^*(G, B)). \tag{2.3}$$

In the definition of the assembly map

$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}G) \to \mathrm{K}_j(C_r^*(G))$$

we use the Kasparov product and descent map. Recall that if A, B, D are separable G-C*-algebras, then there is a product

$$\mathrm{KK}_G^i(A,B) \otimes \mathrm{KK}_G^j(B,D) \to \mathrm{KK}_G^{i+j}(A,D), \quad i,j = 0,1.$$

Let X be a proper G-compact G-space. We define a map

$$\mathrm{KK}_G^j(C_0(X),\mathbb{C}) \to \mathrm{K}_j(C_r^*(G))$$

as the composition of the Kasparov descent map

$$\mathrm{KK}_G^j(C_0(X),\mathbb{C}) \to \mathrm{KK}^j(C_r^*(G,X), C_r^*(G))$$

and the Kasparov product with

$$\mathbf{1} = X \times \mathbb{C} \in \mathrm{K}_0(C_r^*(G,X)) = \mathrm{KK}^0(\mathbb{C}, C_r^*(G,X)).$$

Recall the definition of equivariant K-homology of $\underline{\mathrm{E}}G$

$$\mathrm{K}_j^G(\underline{E}G) := \varinjlim_{\Delta\subset \underline{E}G,\ \Delta\ G\text{-compact}} \mathrm{KK}_G^j(C_0(\Delta),\mathbb{C}).$$

For each G-compact $\Delta \subset \underline{E}G$ we have

$$\mu\colon\ \mathrm{KK}_G^j(C_0(\Delta),\mathbb{C}) \to \mathrm{K}_j(C_r^*(G)).$$

If Δ, Ω are two G-compact subsets of $\underline{E}G$ with $\Delta \subset \Omega$, then the diagram

$$\begin{tikzcd}
\mathrm{KK}_G^j(C_0(\Delta),\mathbb{C}) \arrow[rr] \arrow[dr] & & \mathrm{KK}_G^j(C_0(\Omega),\mathbb{C}) \arrow[dl] \\
& \mathrm{K}_j(C_r^*(G)) &
\end{tikzcd}$$

commutes, so we obtain

$$\mu\colon\ \mathrm{K}_j^G(\underline{E}G) \to \mathrm{K}_j(C_r^*(G)).$$

If A is a G-C*-algebra then we define the equivariant K-homology of $\underline{E}G$ with coefficients in A by

$$\mathrm{K}_j^G(\underline{\mathrm{E}}G;A) := \varinjlim_{\Delta\subset \underline{E}G,\ \Delta\ G\text{-compact}} \mathrm{KK}_G^j(C_0(\Delta),A).$$

We define also a map

$$\mu\colon\ \mathrm{K}_j(\underline{\mathrm{E}}G;A) \to \mathrm{K}_j(C_r^*(G,A)).$$

as the composition of the Kasparov descent map

$$\mathrm{KK}_G^j(C_0(X),A) \to \mathrm{KK}^j(C_r^*(G,X), C_r^*(G,A))$$

and the Kasparov product with

$$\mathbf{1} = X \times \mathbb{C} \in \mathrm{K}_0(C_r^*(G, X)) = \mathrm{KK}^0(\mathbb{C}, C_r^*(G, X)).$$

For each G-compact $\Delta \subset \underline{E}G$ we have

$$\mu\colon\ \mathrm{KK}_G^j(C_0(\Delta), A) \to \mathrm{K}_j(C_r^*(G, A)).$$

If Δ, Ω are two G-compact subsets of $\underline{E}G$ with $\Delta \subset \Omega$, then the diagram

$$\begin{tikzcd} \mathrm{KK}_G^j(C_0(\Delta), A) \arrow[rr] \arrow[dr] & & \mathrm{KK}_G^j(C_0(\Omega), A) \arrow[dl] \\ & \mathrm{K}_j(C_r^*(G, A)) & \end{tikzcd}$$

commutes, so we obtain

$$\mu\colon\ \mathrm{K}_j^G(\underline{E}G), A) \to \mathrm{K}_j(C_r^*(G, A)).$$

Let A, B be G-C*-algebras. We denote by KK_G the category whose objects are all (separable) G-C*-algebras with morphisms $\mathrm{KK}_G(A, B)$. Let $\varphi \in \mathrm{KK}_G(A, B)$. On the left side of the Baum-Connes conjecture φ induces a map

$$\mathrm{K}_j^G(\underline{\mathrm{E}}G, A) \to \mathrm{K}_j^G(\underline{\mathrm{E}}G, B) \tag{2.4}$$

by the Kasparov product with φ.

For any subgroup $H \leq G$ there is a restriction map

$$\mathrm{KK}_G^0(A, B) \to \mathrm{KK}_H^0(A, B),\ \varphi \mapsto \varphi|_H. \tag{2.5}$$

Theorem 2.17. *Suppose φ is an equivalence when restricted to any compact subgroup $H \leq G$. Then the map 2.4 is an isomorphism.*

$$\begin{tikzcd} \mathrm{K}_j(\underline{\mathrm{E}}G, A) \arrow[r] \arrow[d, equal] & \mathrm{K}_j(\underline{\mathrm{E}}G, B) \arrow[d, equal] \\ \operatorname{colim}_{\substack{\Delta\subset\underline{\mathrm{E}}G \\ G\text{-compact}}} \mathrm{KK}_G^j(C_0(\Delta), A) \arrow[r] & \operatorname{colim}_{\substack{\Delta\subset\underline{\mathrm{E}}G \\ G\text{-compact}}} \mathrm{KK}_G^j(C_0(\Delta), B) \end{tikzcd}$$

Proof. It suffices to assume that X is a G-compact proper G-space, because Kasparov product commutes with colim. For an H-space Y we have an induced G-space $G \times_H Y$, which is a quotient $G \times Y/H$ with respect to the following H-action

$$h(g, y) = (gh^{-1}, hy).$$

By the Mayer–Vietoris sequence, it suffices to prove the theorem for $X = G \times_H S$ with S compact. Indeed, any pullback diagram

$$\begin{tikzcd} A_0 \otimes_B A_1 \arrow[r] \arrow[d] & A_0 \arrow[d, two heads] \\ A_1 \arrow[r] & B \end{tikzcd}$$

gives rise to the six-term exact sequence

$$\begin{tikzcd}
\mathrm{KK}^0(A_0 \otimes_B A_1, \mathbb{C}) \arrow[d] & \mathrm{KK}^0(A_0, \mathbb{C}) \oplus \mathrm{KK}^0(A_1, \mathbb{C}) \arrow[l] & \mathrm{KK}^0(B, \mathbb{C}) \arrow[l] \\
\mathrm{KK}^1(B, \mathbb{C}) \arrow[r] & \mathrm{KK}^1(A_0, \mathbb{C}) \oplus \mathrm{KK}^1(A_1, \mathbb{C}) \arrow[r] & \mathrm{KK}^1(A_0 \otimes_B A_1, \mathbb{C}) \arrow[u]
\end{tikzcd}$$

Now replace $\mathbb{C}$ by D, and consider an equivariant case to get a six-term exact sequence for $\mathrm{KK}_G(-, D)$.

$$\begin{tikzcd}
\mathrm{KK}^0_G(A_0 \otimes_B A_1, D) \arrow[d] & \mathrm{KK}^0_G(A_0, D) \oplus \mathrm{KK}^0_G(A_1, D) \arrow[l] & \mathrm{KK}^0_G(B, D) \arrow[l] \\
\mathrm{KK}^1_G(B, D) \arrow[r] & \mathrm{KK}^1_G(A_0, D) \oplus \mathrm{KK}^1_G(A_1, D) \arrow[r] & \mathrm{KK}^1_G(A_0 \otimes_B A_1, D) \arrow[u]
\end{tikzcd}$$

Using

$$\begin{tikzcd}
C_0(U \cup V) \arrow[r] \arrow[d] & C_0(U) \arrow[d] \\
C_0(V) \arrow[r] & C_0(U \cap V)
\end{tikzcd}$$

we reduce by the five-lemma to the case $X = G \times_H S$, with S compact.

Frobenius reciprocity:

$$\begin{tikzcd}
G & \mathrm{Ind}_H^G(\psi) & \varphi & \mathrm{Hom}_G(\mathrm{Ind}_H^G(\psi), \varphi) \arrow[d, "\cong"] \\
H & \psi & \varphi|_H & \mathrm{Hom}_H(\psi, \varphi|_H)
\end{tikzcd}$$

Frobenius reciprocity in KK_G:

$$\begin{tikzcd}
G & \mathrm{Ind}_H^G(A) & B & \mathrm{KK}^j_G(\mathrm{Ind}_H^G(A), B) \arrow[d, "\cong"] \\
H & A & B & \mathrm{KK}^j_H(A, B)
\end{tikzcd}$$

Induction of H-C*-algebra is a G-C*-algebra $G \times_H A$.

$$\begin{tikzcd}
\mathrm{KK}^j_G(C_0(G \times_H S), A) \arrow[r] \arrow[d, "\cong"'] & \mathrm{KK}^j_G(C_0(G \times_H S), B) \arrow[d, "\cong"'] \\
\mathrm{KK}^j_H(C(S), A) \arrow[r] & \mathrm{KK}^j_H(C(S), B)
\end{tikzcd}$$

□

2.4 Meyer-Nest reformulation of the Baum-Connes conjecture with coefficients

Theorem 2.18. *Let G ba a locally compact Hausdorff second countable topological group. Then the following two statements are equivalent:*

1. *The Baum-Connes assembly map*
$$\mathrm{K}_j^G(\underline{\mathrm{E}}G, D) \to \mathrm{K}_j(C_r^*(G, D)), \quad j = 0, 1, \tag{2.6}$$
is an isomorphism for all G-C^*-algebras D;

2. *Whenever A, B are G-C^*-algebras and $\phi \in \mathrm{KK}_G^0(A, B)$ is such that*
$$\phi_*\colon\ \mathrm{K}_j(C_r^*(H, A)) \to \mathrm{K}_j(C_r^*(H, B)), \quad j = 0, 1, \tag{2.7}$$
is an isomorphism for all compact subgroups H of G, then
$$\phi_*\colon\ \mathrm{K}_j(C_r^*(G, A)) \to \mathrm{K}_j(C_r^*(G, B)), \quad j = 0, 1, \tag{2.8}$$
is an isomorphism.

Proof.

(1) $\implies$ (2) Given $\phi \in \mathrm{KK}_G^0(A, B)$ as in (2), consider the commutative diagram

$$\begin{tikzcd}
\mathrm{K}_j^G(\underline{\mathrm{E}}G, A) \arrow[r] \arrow[d] & \mathrm{K}_j^G(\underline{\mathrm{E}}G, B) \arrow[d] \\
\mathrm{K}_j(C_r^*(G, A)) \arrow[r] & \mathrm{K}_j(C_r^*(G, B))
\end{tikzcd}$$

in which the two vertical arrows are the Baum-Connes assembly maps, and the to horizontal arrows are given by ϕ. The hypothesis on ϕ plus a Mayer–Vietoris argument proves that the upper horizontal arrow is an isomorphism. (1) asserts that the two vertical arrows are isomorphisms, so it now follows that the lower horizontal arrow is an isomorphism.

(2) $\implies$ (1) Given any G-C*-algebra D, Meyer and Nest prove that there is a projective object P in their category and a weak equivalence $\phi \in \mathrm{KK}_G^0(P, D)$. Here "weak equivalence" means that for any compact subgroup H of G the restriction of ϕ to $\mathrm{KK}_H^0(P, D)$ is invertible. In particular

$$\phi_*\colon\ \mathrm{K}_j(C_r^*(H, P)) \to \mathrm{K}_j(C_r^*(H, D)), \quad j = 0, 1,$$

is an isomorphism, so that ϕ satisfies the condition of (2). Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{K}_j^G(\underline{\mathrm{E}}G, P) \arrow[r] \arrow[d] & \mathrm{K}_j^G(\underline{\mathrm{E}}G, D) \arrow[d] \\
\mathrm{K}_j(C_r^*(G, P)) \arrow[r] & \mathrm{K}_j(C_r^*(G, D))
\end{tikzcd}$$

in which the two vertical arrows are the Baum-Connes assembly map, and the two horizontal arrows are given by ϕ. As showed above, the upper horizontal arrow is an isomorphism. Meyer and Nest prove that for any projective object in their category, the Baum-Connes assembly map is an isomorphism. Thus the left vertical arrow is an isomorphism. According to (2) the lower horizontal arrow is an isomorphism. It now follows that the right vertical arrow is an isomorphism.

□

See Corollary 3.65 of Chapter 3 for further information on the equivalence of (1) and (2).

2.5 Real Baum-Connes conjecture

2.5.1 Generalization of Paschke duality

Assume X is a compact (Hausdorff) space, and let $C(X)$ be the ring of continuous functions from X to $\mathbb{C}$. Assume that we have a C*-algebra injective map

$$\psi\colon C(X) \to B(\mathcal{H}),$$

for "large enough" Hilbert space $\mathcal{H}$ (e.g. $\mathcal{H} = L^2(X,\mu)\otimes H$, where H is infinite dimensional Hilbert space). Denote

$$D^*(X) = \{T \in B(\mathcal{H}) \mid T\psi(f) - \psi(f)T \in \mathcal{K}\},$$
$$C^*(X) = \{T \in B(\mathcal{H}) \mid T\psi(f) \in \mathcal{K},\ \psi(f)T \in \mathcal{K}\}.$$

Theorem 2.19 (Paschke)**.** *Let* $\mathrm{K}_*(X)$ *be the K-homology of* X*, that is* $= \mathrm{KK}_*(C(X),\mathbb{C})$*. Then*

$$\mathrm{K}_*(X) \cong \mathrm{K}_{*+1}(D_*(X)/C_*(X)). \tag{2.9}$$

The question is how to describe $\mathrm{KK}_*(C(X),C(Y))$ in terms of K-theory?.

If Γ is a discrete group acting on a locally compact X, then how to describe $\mathrm{K}_*^\Gamma(X)$? Replace ψ by a representation of $C_0(X)\rtimes\Gamma$ on $\mathcal{H} = L^2(X,\mu)\otimes l^2(\Gamma)\otimes H'$ for some infinite dimensional Hilbert space H', and define

$$D^*_\Gamma(X) = \{T \in B(\mathcal{H}) \mid T\psi(T) - \psi(f)T \in \mathcal{K},\ T\gamma = \gamma T,\ \mathrm{supp}(T) \text{ is } \Gamma\text{-compact}\}, \tag{2.10}$$

$$C^*_\Gamma(X) = \{T \in B(\mathcal{H}) \mid T\psi(T) \in \mathcal{K},\ \psi(f)T \in \mathcal{K},\ T\gamma = \gamma T,\ \mathrm{supp}(T) \text{ is } \Gamma\text{-compact}\}. \tag{2.11}$$

$$\tag{2.12}$$

Here $\mathrm{supp}(T) \subseteq X\times X$ is defined by

$$X\times X - \mathrm{supp}(T) = \{(x,y) \in X\times X \mid \exists U \ni x,\ V \ni y, \text{ such that } \psi(f)T\psi(g) \text{ vanish}$$
$$\text{if } f \text{ is supported in } U,\ g \text{ supported in } V\}.$$

The simplest example is to take $X = \mathrm{pt}$, $\Gamma = \{1\}$. Then Paschke dual is the Calkin algebra $B(\mathcal{H})/\mathcal{K}$.

Theorem 2.20 (J. Roe)**.** *There are isomorphisms*

$$\mathrm{K}_*^\Gamma(X) \cong \mathrm{K}_{*+1}(D_\Gamma^*(X)/C_\Gamma^*(X)),$$
$$\mathrm{KO}_*^\Gamma(X) \cong \mathrm{K}_{*+1}(D_\Gamma^*(X,\mathbb{R})/C_\Gamma^*(X,\mathbb{R})).$$

2.5.2 Real K-theory

Recall that $\mathrm{KO}^*(X)$ denote the Grothendieck-Atiyah-Hirzerbruch real K-functor, which is 8-periodic. Its values for $X = \mathrm{pt}$ are:

n	$\mathrm{KO}^{-n}(\mathrm{pt})$
0	$\mathbb{Z}$
1	$\mathbb{Z}/2$
2	$\mathbb{Z}/2$
3	0
4	$\mathbb{Z}$
5	0
6	0
7	0

We use the convention $\mathrm{K}^*(X) = \mathrm{K}_{-*}(C(X))$, e.g.

$$\mathrm{KO}^{-1}(\mathrm{pt}) = \mathrm{K}_1^{top}(\mathbb{R}) = \pi_0(\mathrm{GL}(\mathbb{R}) \cong \mathbb{Z}/2, \tag{2.13}$$
$$\mathrm{KO}^1(pt) = \mathrm{K}_7^{top}(\mathbb{R}) = \pi_6(\mathrm{GL}(\mathbb{R})) = 0. \tag{2.14}$$

Another example is the following

$$\mathrm{KU}(S^1) = \mathrm{K}(C_{\mathbb{C}}(S^1)) = \mathbb{Z}, \tag{2.15}$$
$$\mathrm{KO}(S^1) = \mathrm{K}(C_{\mathbb{R}}(S^1)) = \mathbb{Z} \oplus \mathbb{Z}_2. \tag{2.16}$$

2.5.3 The Baum-Connes map

Consider the following commutative diagram (for X a proper Γ-space and $\mathbb{C} \rtimes \Gamma = C_r^*(\Gamma)$)

$$\begin{tikzcd}
KK_\Gamma^j(C_0(X),\mathbb{C}) \arrow[r, "\varphi"] \arrow[d, equal] & KK^j(C_0(X) \rtimes \Gamma, \mathbb{C} \rtimes \Gamma) \arrow[d, "\psi"] \\
K_j^\Gamma(X) \arrow[r] & KK^j(\mathbb{C}, C_r^*(\Gamma))
\end{tikzcd}$$

In this diagram, φ is Kasparov's descent map [k-g88]. The map ψ is induced by the Kasparov product

$$\theta : KK^0(\mathbb{C}, C_0(X) \rtimes \Gamma) \;\times\; KK^j(C_0(X) \rtimes \Gamma, \mathbb{C} \rtimes \Gamma) \longrightarrow KK^j(\mathbb{C}, C_r^*(\Gamma))$$

More precisely, ψ is defined by $\psi(u) = \theta(p,u)$, where p is a special idempotent defining an element of $KK^0(\mathbb{C}, C_0(X) \rtimes \Gamma) = K_0(C_0(X) \rtimes \Gamma)$. This special idempotent is defined as follows: let $\varphi \in C_0(X)$ such that $\sum_{g\in\Gamma} g(\varphi^2) = 1$, then

$$p = \sum_{g\in G} \varphi g(\varphi)[g].$$

If Γ acts properly on X, then at the end we get

- the Baum-Connes map

$$\mu\colon\ \mathrm{K}_j^{\Gamma}(X) \to \mathrm{K}_j(C_r^*(\Gamma)), \tag{2.17}$$

- the Baum-Connes conjecture for a classifying space $\underline{\mathrm{E}\,\Gamma}$ for proper actions,

$$\mu_{\Gamma}\colon\ \mathrm{colim}_{X\subset\underline{\mathrm{E}\,\Gamma},\ \mathrm{compact}}\,\mathrm{K}_j^{\Gamma}(X) \to \mathrm{K}_j(C_r^*(\Gamma)). \tag{2.18}$$

2.5.4 Interpretation of the Baum-Connes conjecture in terms of Paschke duality

There is a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_j^{\Gamma}(X) \arrow[r, "\mu"] \arrow[d, "\cong"'] & \mathrm{K}_j(C_r^*(\Gamma)) \arrow[d, dashed, "\cong"] \\
\mathrm{K}_{j+1}(D_{\Gamma}^*(X)/C_{\Gamma}^*(X)) \arrow[r, "\partial"] & \mathrm{K}_j(C_{\Gamma}^*(X))
\end{tikzcd}$$

in which the dashed arrow comes from J. Roe theorem about Morita equivalence of $C_r^*(\Gamma)$ and $C_{\Gamma}^*(X)$, and the bottom horizontal map is the usual boundary map for K-theory.

The Baum-Connes conjecture is then equivalent to the fact

$$\mathrm{colim}_{X\subset\underline{\mathrm{E}\,\Gamma},\ \mathrm{compact}}\,\mathrm{K}_j(D_{\Gamma}^*(X)) = 0,\ \text{ for all } j, \tag{2.19}$$

since we have the exact sequence

$$K_{j+1}(D_{\Gamma}^*(X)) \to K_{j+1}(D_{\Gamma}^*(X)/C_{\Gamma}^*(X)) \to K_j(C_{\Gamma}^*(X)) \to K_j(D_{\Gamma}^*(X)).$$

All what have been said so far can be translated into the framework of real K-theory. We have a Baum-Connes map

$$\mu_{\mathbb{R}}^{\Gamma} : KO_j(\underline{E}\Gamma) \to K_j(C_r^*(\Gamma;\mathbb{R})$$

which we conjecture to be an isomorphism also. We can now state our main theorem [bk04]:

Theorem 2.21. *If $\mu_{\infty}^{\Gamma}(\mathbb{C})$ is an isomorphism, then $\mu_{\infty}^{\Gamma}(\mathbb{R})$ is an isomorphism. Hence if Baum-Connes conjecture is true for $\mathbb{C}$, then it is true for $\mathbb{R}$.*

This theorem is a consequence of the following fact: if A is a Banach algebra such that $\mathrm{K}_j(A') = 0$ for $j = 0, 1$, and $A' = A \otimes_{\mathbb{R}} \mathbb{C}$, then $\mathrm{K}_j(A) = 0$ for $j = 0, 1, 2, 3, 4, 5, 6, 7$.

This general fact about the K-theory of real Banach algebras need some results about the KR-theory of Atiyah [a-mf66] which we adapt for our purposes.

For a Banach algebra A (real, not necessary C*) there is a classifying space $\mathbb{K}(A)$ such that

$$\pi_j(\mathbb{K}(A)) = \mathrm{K}_j(A).$$

Denote

$$\mathrm{Proj}_{2n}(A) = \{p \in M_{2n}(A) \mid p^2 = p\}.$$

There is an embedding

$$\mathrm{Proj}_{2n}(A) \to \mathrm{Proj}_{2n+2}(A),\ p \mapsto \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $\mathbb{K}(A) = \operatorname{colim} \mathrm{Proj}_{2n}(A)$.

More geometrically one can consider on a compact space Z A-bundles (generalization of vector bundles) $E \to Z$. Each fiber E_ζ is a finitely generated projective A-module. There is a Grothendieck group $\mathrm{K}_A(Z)$ of A-bundles. For $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we obtain

$$\begin{aligned} \mathrm{K}_{\mathbb{R}}(Z) &= \mathrm{KO}(Z), \\ \mathrm{K}_{\mathbb{C}}(Z) &= \mathrm{KU}(Z), \\ \mathrm{K}_{\mathbb{H}}(Z) &= \mathrm{KSp}(Z) = \mathrm{KO}^4(Z). \end{aligned}$$

Theorem 2.22.

1. $\mathrm{K}_A(Z) = [Z, \mathbb{K}(A)]$,
2. $\mathrm{K}_A(Z) = \mathrm{K}(A(Z))$,

where $A(Z)$ denotes the ring of continuous functions $Z \to A$.

If A is not unital, then

$$\mathrm{K}_A(X) = \ker(\mathrm{K}_{\widetilde{A}}(X) \to \mathrm{K}_C(\mathrm{pt})) = \mathrm{K}(A(X)),\ C = \mathbb{R}, \mathbb{C}, \mathbb{H}.$$

How to go from $\mathbb{K}(A)$ to $\mathbb{K}(A')$ where $A' = A \otimes_{\mathbb{R}} \mathbb{C}$? Main trick is to use

$$\mathrm{K}(A') \xrightarrow{1-t} \mathrm{K}(A'),\ E \mapsto E - \overline{E},$$

where t means complex conjugation.

If $U \xrightarrow{\sigma} V$ is any map, then we can construct a fibration up to homotopy $T \to U \xrightarrow{\sigma} V$, where

$$T = \{(u, s) \mid u \in U,\ s\colon [0,1] \to V,\ s(0) = v_0,\ s(1) = \sigma(u)\}$$

(v_0 is a base point in V).

This way we get a fibration up to homotopy

$$\mathbb{K}\mathrm{SC}(A) \to \mathbb{K}(A') \xrightarrow{1-t} \mathbb{K}(A'),$$

and an exact sequence of homotopy groups

$$\ldots \to \mathrm{K}_{n+1}(A') \to \mathrm{K}_{n+1}(A') \to \pi_n(\mathbb{K}\mathrm{SC}(A)) \to \mathrm{K}_n(A') \to \mathrm{K}_n(A') \to \ldots. \tag{2.20}$$

There is another fibration

$$\mathbb{K}(A') \to \mathbb{K}(A) \times \mathbb{K}(A \otimes_{\mathbb{R}} \mathbb{H}) \to \mathbb{K}\mathrm{SC}(A)$$

which leads to the exact sequence

$$\ldots \to \mathrm{K}_n(A') \to \mathrm{K}_n(A) \oplus \mathrm{KSp}_n(A) \to \mathrm{KSC}_n(A) \to \ldots \tag{2.21}$$

How to construct and prove (2.20) and (2.21)?

Let Z be a compact space with involution $\zeta \mapsto \bar{\zeta}$. Denote by $S^{p,q}$ a sphere in $\mathbb{R}^p \times \mathbb{R}^q$ with the involution $(x, y) \mapsto (-x, y)$. Let $\mathrm{KR}_A(Z) = \mathrm{K}(A(Z))$, where

$$A(Z) = \{f\colon Z \to A \mid f(\bar{\zeta}) = \overline{f(\zeta)}\}.$$

Example 2.23.

- If $Z = S^{1,0}$, that is a two-point space with involution which interchanges them, then

$$\begin{aligned} \mathrm{KR}_A(S^{1,0}) &= \mathrm{K}(A') \\ \mathbb{K}(A') &= \mathbb{K}(A(S^{1,0})). \end{aligned}$$

- If $Z = S^{2,0}$, then

$$\begin{aligned} \mathrm{KR}_A(S^{2,0}) &= \mathrm{KSC}(A), \\ \mathbb{K}\mathrm{SC}(A) &= \mathbb{K}(A(S^{2,0})). \end{aligned}$$

- If $Z = S^{3,0}$, then

$$\begin{aligned} \mathrm{KR}_A(S^{3,0}) &= \mathrm{K}(A) \oplus \mathrm{KSp}(A), \\ \mathbb{K}(A) \times \mathbb{K}\mathrm{Sp}(A) = \mathbb{K}(A(S^{3,0})) \end{aligned}$$

Theorem 2.24.

$$\begin{aligned} &\mathbb{K}(A(S^{3,0})) = \mathbb{K}(A) \times \mathbb{K}\,\mathrm{Sp}(A), \\ &\mathbb{K}(A(S^{p,0})) = \mathbb{K}(A) \times \Omega^{-p-1}(\mathbb{K}(A)) \quad \textit{for } p \geq 3. \end{aligned}$$

Note that $\Omega^4\mathbb{K}(A) = \mathbb{K}(A \otimes_{\mathbb{R}} \mathbb{H})$.

To prove the theorem one uses the following facts:

- There is a homeomorphism $S^{p,0} - S^{q,0} \sim S^{p-q,0} \times \mathbb{R}^{q,0}$, and a fibration

$$\Omega^{-q}(S^{p-q}) \to \mathbb{K}(A(S^{p,0})) \to \mathbb{K}(A(S^{q,0})).$$

From the exact sequence

$$0 \to A(S^{1,0} \times \mathbb{R}^{1,0}) \to A(S^{2,0}) \to A(S^{1,0}) \to 0$$

we get the fibrations

$$\Omega^{-1}\mathbb{K}(A(S^{1,0})) \to \mathbb{K}(A(S^{2,0})) \to \mathbb{K}(A(S^{1,0})),$$

$$\Omega^{-1}\mathbb{K}(A') \to \mathbb{K}\mathrm{SC}(A) \to \mathbb{K}(A') \xrightarrow{1-t} \underbrace{\Omega^{-2}\mathbb{K}(A')}_{=\mathbb{K}(A')}.$$

- Bott periodicity:

$$\begin{aligned} \mathrm{K}(A(\mathbb{R}^{1,1})) &\cong \mathrm{K}(A), \\ \mathrm{K}_{p,q}(A) &\cong \mathrm{K}(A(\mathbb{R}^p \times \mathbb{R}^q)), \\ \mathrm{K}_{p,q}(A) &\cong \mathrm{K}_{p+1,q+1}(A). \end{aligned}$$

Proof is analogous to Atiyah–Bott proof in complex K-theory.

- There is a fibration

$$\Omega^{-p}\mathbb{K}(A) \xrightarrow{\sigma} \mathbb{K}(A) \to \mathbb{K}(A(S^{p,0})) \to \Omega^{-p-1}(\mathbb{K}(A))$$

Theorem 2.25. *For $p \geq 0$, σ is null-homotopic.*

The map σ is induced by the cup-product with an element in $\mathrm{K}_p(\mathbb{R})$, but this element is 0 for $p \geq 3$. The map $\mathrm{K}_n(A) \xrightarrow{\sigma} \mathrm{K}_{n+p}(A)$ is a $\mathrm{K}_*(A)$-module map, i. e. we have the diagram.

$$\begin{tikzcd}
\mathrm{K}_*(\mathbb{R}) \times \mathrm{K}_*(A) \arrow[r] \arrow[d, "\sigma\times 1"'] & \mathrm{K}_*(A) \arrow[d] \\
\mathrm{K}_{*+p}(\mathbb{R}) \times \mathrm{K}_*(A) \arrow[r] & \mathrm{K}_{*+p}(A)
\end{tikzcd}$$

$$0 \to \mathrm{K}_n(A) \to \mathrm{K}_n(A(S^{3,0})) \to \mathrm{K}_{n+3}(A) \to 0.$$

We can give an explicit description of the maps in the sequence:

$$\mathrm{KSC}_n(A) \to \mathrm{K}_n(A) \oplus \mathrm{KSp}_n(A) \to \mathrm{K}_n(A') \to \mathrm{KSC}_{n+1}(A).$$

but this is not necessary for our purposes.

We can give another interpretation of the homotopy fiber of

$$\mathbb{K}(A') \xrightarrow{1-t} \mathbb{K}(A').$$

In a more general framework it is the homotopy fiber Y of two maps $f, g\colon X \to Z$

$$\begin{tikzcd}
Y \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}$$

$$Y = \{(x,s) \mid x \in X,\ s\colon [0,1] \to Z,\ s(0) = f(x),\ s(1) = g(x)\}.$$

$$\begin{tikzcd}
\Omega X \arrow[r] & \Omega Z \arrow[r] & Y \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_{n+1}(X) \arrow[r] & \pi_{n+1}(Z) \arrow[r] & \pi_n(Y) \arrow[r] & \pi_n(X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \pi_n(Z)
\end{tikzcd}$$

In our situation $Y = \mathbb{K}\mathrm{SC}(A)$, $X = Z = \mathbb{K}(A')$, $f = \mathrm{Id}$, $g = t$.

A famous example is the following (Quillen):

$$\mathrm{K}^{alg}(\mathbb{F}_q) \to \mathrm{K}^{top}(\mathbb{C}) \xrightarrow{1-\psi^q} \mathrm{K}^{top}(\mathbb{C}),$$

where $\mathrm{K}^{alg}(\mathbb{F}_q)$ is the algebraic K-theory of Quillen. From this we deduce that

$$\mathrm{K}_{2n-1}(\mathbb{F}_q) = \mathbb{Z}/(q^n - 1)\mathbb{Z},$$
$$\mathrm{K}_{2n}(\mathbb{F}_q) = 0.$$

We note in passing the following theorem

Theorem 2.26. *If A is a stable C^*-algebra ($A \cong \mathcal{K} \otimes A$)*

$$\mathbb{K}^{alg}(A) \sim \mathbb{K}^{top}(A).$$

(Karoubi's conjecture, proved by Suslin and Wodzicki).

Theorem 2.27 (Quillen)**.** *There is a (non canonical) homotopy equivalence*

$$\mathbb{K}^{alg}(A) \sim \mathrm{K}_0(A) \times \mathrm{B}\,\mathrm{GL}(A)^+.$$

The construction of Y^+:

$$\bigvee S^1 \xrightarrow{f_n} Y \to Y_1 \to \bigvee S^2 \to SY$$

$$\bigvee S^2_\alpha \to Y_1 \to Y^+ \to \bigvee S^3_\alpha \to SY_1$$

$$\mathrm{H}_*(\mathrm{GL}(A);\mathbb{Q}) = S(\mathrm{K}^{alg}_*(A))$$

Theorem 2.28 (Borel)**.**

$$\mathrm{H}_*(\mathrm{SL}(\mathbb{Q})) = \Lambda[x_5, x_9, \dots]$$

with rational coefficients.

It implies that $\mathrm{K}_n(\mathbb{Z})$ is finite if $n \neq 5 \mod 8$, and $\mathrm{K}_n(\mathbb{Z}) = \mathbb{Z} \oplus (\text{finite})$ if $n = 5 \mod 8$. Also $\mathrm{K}_3(\mathbb{Z}(i)) = \mathbb{Z} \oplus (\text{finite})$.

2.5.5 Clifford algebras and K-theory

Clifford algebra
$$C^{p,q} := \mathbb{R}[e_1, \dots, e_p, \varepsilon_1, \dots, \varepsilon_q],$$
with relations
$$(e_i)^2 = -1, \quad \varepsilon_j^2 = 1,$$
$$e_i e_j = -e_j e_i,\ i \neq j \text{ etc.}$$
Define
$$\Gamma^{p,q} := \operatorname{coker}(\mathrm{K}(C^{p,q+2}) \to \mathrm{K}(C^{p,q+1})).$$

Theorem 2.29 (Atiyah–Bott–Shapiro)**.** *There is an isomorphism*

$$\Gamma^{p,q} \xrightarrow{\theta} \mathrm{KR}(\mathbb{R}^{p,q}).$$

The idea is to give another definition of higher K-theory using Clifford algebras. Let $\mathcal{E}^{p,q}(X)$ be the category of vector bundles with a $C^{p,q}$-module structure. Then by $\mathrm{K}^{p,q}(X)$ we denote the Grothendieck group of the functor $\mathcal{E}^{p,q+1}(X) \to \mathcal{E}^{p,q}(X)$.

Theorem 2.30.
$$\mathrm{K}^{p,q}(X) = \mathrm{KR}(X \times \mathbb{R}^{p,q}).$$

Let $E = E^0 \oplus E^1$ be a $C^{p,q}$-graded module.

$$\pi \colon X \times B^{p,q} \xrightarrow{\pi} X$$

$$\pi^* E^0 \otimes \mathbb{C} \xrightarrow{v_0 + i v_1} \pi^* E^1 \otimes \mathbb{C}$$

gives an element in the relative group $\mathrm{KR}(X \times B^{p,q}, X \times S^{p,q}) = \mathrm{KR}^{p,q}(X)$.

Let
$$\Gamma^{p,q} := \operatorname{coker}(\mathrm{K}(\mathcal{E}^{p,q+2}(X)) \to \mathrm{K}(\mathcal{E}^{p,q+1}(X))).$$

Corollary 2.31. *For a space X the Atiyah–Bott–Shapiro map*

$$\Gamma^{p,q}(X) \to \mathrm{KR}^{p,q}(X)$$

is always injective.

For a Banach algebra A

$$\Gamma^{p,q}(A) := \mathrm{coker}(\mathrm{K}(C^{p,q+2} \otimes A) \to \mathrm{K}(C^{p,q+1} \otimes A)).$$

Theorem 2.32. $\Gamma^{p,q}(A)$ *injects in* $\mathrm{K}_{q-p}(A)$.

Remark 2.33. More details may be found in [kc08].

Chapter 3

Kasparov theory as a triangulated category

3.1 Additional structure on Kasparov theory

Triangulated categories formalise some additional structure on the stable homotopy category and derived categories that allows to study homology theories in this context. Equivariant Kasparov theory, when viewed as a category, is a triangulated category as well. This section introduces the additional structure on KK that is involved here. In the following sections, we will verify the axioms of a triangulated category for KK and explain what they mean.

To begin with, the category KK is additive. This means that morphism sets form Abelian groups and that there exist finite products and coproducts and that the latter are equal.

A triangulated category is an additive category $\mathcal{T}$ with an automorphism $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and a class $\mathcal{E} \subseteq \text{Triangles}(\mathcal{T})$ of exact triangles, satisfying axioms we will discuss later.

Example 3.1 (The homotopy category of chain complexes over an additive category $\mathcal{A}$). The objects of this category are chain complexes with entries in $\mathcal{A}$, the morphisms are homotopy classes of chain maps. This is an additive category. The suspension automorphism is defined by $\Sigma(C_n, d_n) := (C_{n-1}, -d_{n-1})$ on objects and $\Sigma(f_n) = f_{n-1}$ on chain maps.

Let

$$\begin{tikzcd} I \arrow[r, tail, "i"] & E \arrow[r, two heads, "p"'] & Q, \arrow[l, bend right, "s"'] \end{tikzcd}$$

be a semi-split extension of chain complexes, that is, I, E, Q are chain complexes, i, p are chain maps, and s is a grading preserving map that need not commute with the differential. Hence the chain map

$$\delta_s\colon Q \to I[1], \quad \delta_s = d^E \circ s - s \circ d^Q$$

need not be zero. We call the diagram

$$I \xrightarrow{i} E \xrightarrow{p} Q \xrightarrow{\delta_s} I[1]$$

an *extension triangle*. A triangle is called *exact* if it is isomorphic (in the homotopy category of chain complexes) to such an extension triangle.

Alternatively, we may use mapping cones of chain maps to define the exact triangles, compare the equivalent description of exact triangles in KK using extensions or mapping cones of *-homomorphisms below. For chain complexes, this comparison is quite trivial: the mapping cone of the map δ_s is naturally isomorphic to the extension E.

The triangulated category structure on KK^G is defined as follows:

- The *translation functor* or *suspension functor* in KK^G is defined by $A[-n] := C_0(\mathbb{R}^n) \otimes A$ for $n \geq 0$. This is an automorphism by Bott periodicity.
- A diagram $C[-1] \xrightarrow{w} A \xrightarrow{u} B \xrightarrow{v} C$ in KK^G is called an *exact triangle* if there are an equivariantly semi-split extension $A' \overset{i}{\rightarrowtail} B' \overset{p}{\twoheadrightarrow} C'$ and KK^G-equivalences α, β, γ such that the following diagram commutes:

$$\begin{tikzcd} C[-1] \arrow[r, "w"] \arrow[d, "\gamma[-1]"', "\cong"] & A \arrow[r, "u"] \arrow[d, "\alpha"', "\cong"] & B \arrow[r, "v"] \arrow[d, "\beta"', "\cong"] & C \arrow[d, "\gamma"', "\cong"] \\ C'[-1] \arrow[r, "\delta"'] & A' \arrow[r, "{[i]}"'] & B' \arrow[r, "{[p]}"'] & C'; \end{tikzcd}$$

  here $\delta = \delta_{B'}$ is the class of the extension in $\mathrm{KK}_1^G(C, A) \cong \mathrm{KK}_0^G(C[-1], A)$.

The exact triangles are the correct analogue of extensions in KK^G. Merely knowing the KK^G-classes of i and p in a C*-algebra extension

$$I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$$

does not yet determine the boundary maps. This is why we add the class of the extension in $\mathrm{KK}_1(Q, I)$ as an additional datum. Once this is done, boundary maps and various other constructions with extensions become natural.

Theorem 3.2. *With the translation automorphism and exact triangles defined above,* KK_G *becomes a triangulated category.*

This amounts to checking that the axioms (TR0)–(TR4) of a triangulated category are satisfied. We will formulate and verify these axioms in the following sections. Since the equivariant case is mostly identical to the non-equivariant one, we restrict attention to KK.

We must warn the reader about a notational problem due to the contravariance of the functor **Spaces** $\to$ $\mathbf{C^*{-}alg}$, $X \mapsto C(X)$. This necessarily creates confusion at some points because the notion of a triangulated category is developed for spaces, not for C*-algebras. Fortunately, the opposite of a triangulated category carries a canonical triangulated category structure as well, so that reversing the arrows does not matter much. But this involves inverting the suspension automorphism, and rotating certain exact triangles. For instance, the usual notion of cone in a triangulated category refers to a suspension of the cone in KK due to this reversal of arrows.

3.2 Puppe sequences

First we describe the exact triangles in a different fashion using mapping cones instead of extensions. This is helpful for checking the axioms.

Let $f\colon A \to B$ be a *-homomorphism. We define its cone as in Section 1.2.3:

$$C_f := \{(a,b) \in A \oplus C_0((0,1]) \otimes B \mid f(a) = b(1)\}.$$

The maps in the exact sequence $SB \rightarrowtail C_f \twoheadrightarrow A$ together with the given map f provide a triangle $B[-1] \to C_f \to A \to B$ called a *mapping cone triangle*. On the level of pointed spaces, if $f\colon X \to Y$ is a pointed map, then

$$C_f = X \times [0,1] \sqcup Y \;/\; (x,0) \sim (*,0) \sim (*,t),\ (x,1) \sim f(x)$$

Definition 3.3. *A* mapping cone triangle *is a triangle that is isomorphic to*

$$SB \to C_f \to A \xrightarrow{f} B$$

for some f in KK^G*.*

Theorem 3.4. *A triangle in* KK^G *is exact* (*isomorphic to an extension triangle*) *if and only if it is isomorphic to a mapping cone triangle.*

Proof. First we check that the mapping cone triangle of a map f is isomorphic to an extension triangle. This uses the mapping cylinder and the extension $C_f \rightarrowtail Z_f \twoheadrightarrow B$. Recall the canonical homotopy equivalence $A \simeq Z_f$ from Section 1.2.3. Thus the extension triangle for the mapping cylinder extension is of the form $B[-1] \to C_f \to A \to B$. It can be checked that the class of the extension in $\mathrm{KK}_1(B, C_f) = \mathrm{KK}_0(SB, C_f)$ is the class of the embedding $SB \to C_f$ and that the maps $C_f \to Z_f \to B$ correspond to the maps $C_f \to A \to B$ in the mapping cone triangle. Hence the mapping cone triangle is isomorphic to the extension triangle of the mapping cylinder extension.

Conversely, consider a semi-split extension $I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$. Then the canonical map $I \to C_p$ is a KK-equivalence (compare Section 1.2.3 once again). Hence the extension triangle is isomorphic to a triangle $SQ \to C_p \to E \to Q$. It can be checked that the maps in the latter triangle are the ones in the mapping cone triangle. Hence any extension triangle is isomorphic to a mapping cone triangle. $\square$

As a result, if F is a stable, semi-split exact functor, then we get a *Puppe exact sequence*

$$\cdots \to F(SC_f) \to F(SA) \to F(SB) \to F(C_f) \to F(A) \xrightarrow{F(f)} F(B)$$

Thus we may view $F(C_f)$ as the relative version of F for the map f. Actually, Puppe sequences tend to be easier to establish than long exact sequences for extensions (see [**?**]).

3.3 The first axioms of a triangulated categories

A triangulated category is an additive category with a suspension automorphism and a class of exact triangles, subject to the axioms (TR0)–(TR4). Here we discuss the more elementary axioms (TR0)–(TR3). Roughly speaking, they formalise the steps needed to establish that KK has long exact sequences in both variables.

Axiom 3.5 (TR0)**.** *If a triangle is isomorphic to an exact triangle, then it is exact. Triangles of the form* $0 \to A \xrightarrow{\mathrm{id}} A \to 0$ *are exact.*

Axiom 3.6 (TR1)**.** *Any morphism* $f\colon A \to B$ *can be embedded in an exact triangle*

$$A \xrightarrow{f} B \to C \to A[1].$$

We will see in Proposition 3.13 that this exact triangle is unique up to isomorphism and call C (or $C[-1]$—there are different conventions) a *cone* for f.

Proof. We verify that KK satisfies (TR1) using extension triangles. Let $f \in \mathrm{KK}_0(A, B) \cong \mathrm{KK}_1(SA, B)$ and recall that elements of the latter group are equivalence classes of C*-algebra extensions. Thus f generates a semi-split C*-algebra extension $B \otimes \mathcal{K} \rightarrowtail E \twoheadrightarrow SA$, which yields an extension triangle of the required form:

$$\begin{tikzcd} S^2A \arrow[r] \arrow[d, "\cong"', "\mathrm{Bott}"] & B \otimes \mathcal{K} \arrow[r] \arrow[d, "\cong"', "\mathrm{stability}"] & E \arrow[r] & SA \\ A \arrow[r, "f"'] & B, \end{tikzcd}$$

where we use Bott periodicity and C*-stability of KK. □

Axiom 3.7 (TR2)**.** *The triangle*

$$B[-1] \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B$$

is exact if and only if the triangle

$$A[-1] \xrightarrow{-w[-1]} B[-1] \xrightarrow{-u} C \xrightarrow{-v} A$$

is exact.

We can get rid of an even number of signs because of the isomorphism

$$\begin{tikzcd} A[-1] \arrow[r, "-w{[-1]}"] \arrow[d, "\mathrm{id}"'] & B[-1] \arrow[r, "-u"] \arrow[d, "\mathrm{id}"'] & C \arrow[r, "-v"] \arrow[d, "-\mathrm{id}"'] & A \arrow[d, "-\mathrm{id}"'] \\ A[-1] \arrow[r, "-w{[-1]}"'] & B[-1] \arrow[r, "u"'] & C \arrow[r, "v"'] & A \end{tikzcd}$$

Applying (TR2) three times, we get that

$$B[-2] \xrightarrow{-u[-1]} C[-1] \xrightarrow{-v[-1]} A[-1] \xrightarrow{-w[-1]} B[-1]$$

is exact if and only if $B[-1] \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B$ is exact.

The reason for the signs is that the suspension of a mapping cone triangle for f is the mapping cone triangle for Sf, but this involves a coordinate flip on $\mathbb{R}^2$ on $B[-2] = C_0(\mathbb{R}^2, B)$, which generates a sign.

Proof. Now we verify that (TR2) holds for KK. Let

$$B[-1] \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B$$

be exact, that is, isomorphic to an extension triangle. We may assume that the triangle itself is already an extension triangle of some C*-algebra extension $C \rightarrowtail A \twoheadrightarrow B$. The mapping cone of w fits in an extension $SB \rightarrowtail C_w \twoheadrightarrow A$, which yields an extension triangle $SB \to C_w \to A \to B$. Now recall that C_w is canonically KK-equivalent to C. Hence the extension triangle of the extension $SB \rightarrowtail C_w \twoheadrightarrow A$ is isomorphic to $A[-1] \xrightarrow{-w[-1]} B[-1] \xrightarrow{-u} C \xrightarrow{-v} A$ which is therefore exact. Using Bott periodicity, we also get the converse implication. □

Definition 3.8. *A functor F from a triangulated category to an Abelian category is called* homological *if*

$$F(C) \to F(A) \to F(B)$$

is exact for any exact triangle

$$B[-1] \to C \to A \to B.$$

Proposition 3.9. *If F is homological, then any exact triangle yields a long exact sequence*

$$\cdots \to F_n(C) \to F_n(A) \to F_n(B) \to F_{n-1}(C) \to \cdots$$

where $F_n(A) := F(A[-n])$, $n \in \mathbb{Z}$.

Proof. Use axiom (TR2) repeatedly to get exactness everywhere. □

If F is a semi-split exact, split-exact, C*-stable functor on $\mathbf{C^*{-}alg}$, then the induced functor on KK is homological and hence has long exact sequences for extensions that extend in both directions. This automatic extension of long exact sequences in both directions is the main point of axiom (TR2).

Axiom 3.10 (TR3)**.** *Consider a commuting solid arrow diagram with exact rows*

$$\begin{tikzcd}
B[-1] \arrow[r] \arrow[d, "{\beta[-1]}"'] & C \arrow[r] \arrow[d, dotted, "\exists\gamma"] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
B'[-1] \arrow[r] & C' \arrow[r] & A' \arrow[r] & B'
\end{tikzcd}$$

There exists $\gamma\colon C \to C'$ making the diagram commutative (*but it is not unique*)*.*

Proof. We verify (TR3) for KK. We may assume that the rows are mapping cone triangles:

$$\begin{tikzcd}
SB \arrow[r] \arrow[d, "S\beta"'] & C_f \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
SB' \arrow[r] & C_{f'} \arrow[r] & A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

We know that α is a KK-cycle for $A \to A'$, β is a KK-cycle for $B \to B'$, and there is a homotopy H from $\beta \circ f$ to $f' \circ \alpha$ (because $[\beta \circ f] = [f' \circ \alpha]$ in KK).

We glue these three data together to a KK-morphism from C_f to $C_{f'}$ according to the following picture:

More precisely, we are given the following Kasparov cycles:

$$\alpha = (\mathcal{H}^\alpha_A, \varphi^\alpha, F^\alpha \in B(\mathcal{H}^\alpha)),$$

$$\beta = (\mathcal{H}^\beta_B, \varphi^\beta, F^\beta \in B(\mathcal{H}^\beta)),$$
$$H = (\mathcal{H}^H_{C_0((0,1],B')}, \varphi^H, F^H \in B(\mathcal{H}^H)).$$

These satisfy
$$H|_0 = \beta \circ f = (\mathcal{H}^\beta, \varphi^\beta \circ f, F^\beta),$$
$$H|_1 = f' \circ \alpha = (\mathcal{H}^\alpha \otimes_{f'} B', \varphi^\alpha \otimes \mathrm{id}_{B'}, F^\alpha \otimes \mathrm{id}_{B'}).$$

Then
$$\mathcal{H}^\beta \otimes C_0((0,\tfrac{1}{2}]) \oplus_{\mathcal{H}^\beta \text{ at } \frac{1}{2}} \mathcal{H}^H \oplus_{\mathcal{H}^\alpha \otimes_{f'} B'} \mathcal{H}^\alpha$$

is a Hilbert module over the mapping cone of f'; here we reparametrise $\mathcal{H}^H$ over the interval $[\frac{1}{2}, 1]$. Now $\varphi^\beta \otimes C_0((0,\frac{1}{2}])$, φ^H, φ^α glue to a *-homomorphism $\varphi^\gamma\colon A \to B(\mathcal{H}^\gamma)$. We define the operator F similarly. This yields the desired Kasparov cycle for $\mathrm{KK}_0(C_f, C_{f'})$. □

The following proposition is the main point of axiom (TR3):

Proposition 3.11. *Let D be an object of a category $\mathcal{T}$. Then the functor $A \to \mathcal{T}(D, A)$ is homological. Dually, $A \mapsto \mathcal{T}(A, B)$ is cohomological for every object B in $\mathcal{T}$.*

Proof. Let
$$B[-1] \to C \to A \to B$$
be an exact triangle in $\mathcal{T}$. We have to verify the exactness of

$$\mathcal{T}(D, C) \to \mathcal{T}(D, A) \to \mathcal{T}(D, B).$$

The composite map $\mathcal{T}(D, C) \to \mathcal{T}(D, A) \to \mathcal{T}(D, B)$ vanishes because already the composition $C \to A \to B$ vanishes (exercise). Exactness at $\mathcal{T}(D, A)$ follows from axiom (TR3):

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "0"'] & D \arrow[r, equal] \arrow[d, dotted, "\hat{f}"] & D \arrow[r] \arrow[d, "f"] & 0 \arrow[d, "0"] \\
B[-1] \arrow[r] & C \arrow[r, "u"] & A \arrow[r, "v"] & B.
\end{tikzcd}$$

If $v \circ f = 0$, then we get $\hat{f}\colon D \to C$ with $f = \hat{f} \circ u$. □

In particular, this shows that $\mathrm{KK}^G(-, D)$ is homological and $\mathrm{KK}^G(D, -)$ is cohomological.

Since many general results on triangulated categories only use the axioms (TR0)–(TR3), we postpone the discussion of the last axiom (TR4).

Lemma 3.12 (Five Lemma). *Consider a morphism of exact triangles*

$$\begin{tikzcd} * \arrow[r] \arrow[d, "{\beta[-1]}"'] & * \arrow[r] \arrow[d, "\gamma"'] & * \arrow[r] \arrow[d, "\alpha"'] & * \arrow[d, "\beta"'] \\ * \arrow[r] & * \arrow[r] & * \arrow[r] & * \end{tikzcd}$$

If two of α, β, γ are invertible, then so is the third.

Proof. Assume α, β are invertible. Then $\mathcal{T}(D,\alpha)$, $\mathcal{T}(D,\beta)$, and $\mathcal{T}(D,\alpha[-1])$, $\mathcal{T}(D,\beta[-1])$ are invertible. The exact sequences from Proposition 3.11 yield a diagram

$$\begin{tikzcd} \mathcal{T}(D,A[-1]) \arrow[r] \arrow[d, "{\mathcal{T}(D,\alpha[-1])}"', "\cong"] & \mathcal{T}(D,B[-1]) \arrow[r] \arrow[d, "{\mathcal{T}(D,\beta[-1])}"', "\cong"] & \mathcal{T}(D,C) \arrow[r] \arrow[d, "{\mathcal{T}(D,\gamma)}"'] & \mathcal{T}(D,A) \arrow[r] \arrow[d, "{\mathcal{T}(D,\alpha)}"', "\cong"] & \mathcal{T}(D,B) \arrow[d, "{\mathcal{T}(D,\beta)}"', "\cong"] \\ \mathcal{T}(D,A'[-1]) \arrow[r] & \mathcal{T}(D,B'[-1]) \arrow[r] & \mathcal{T}(D,C') \arrow[r] & \mathcal{T}(D,A') \arrow[r] & \mathcal{T}(D,B') \end{tikzcd}$$

Since the rows are exact, the Five Lemma yields that $\mathcal{T}(D,\gamma)$ is invertible. □

Proposition 3.13. *Let $f\colon A \to B$ be a morphism. There is up to isomorphism a unique exact triangle*

$$B[-1] \to C \to A \xrightarrow{f} B$$

Proof. Existence follows from (TR1). Given two such exact triangles, Axiom (TR3) yields γ in the following diagram:

$$\begin{tikzcd} B[-1] \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, dotted, "\gamma"] & A \arrow[r] \arrow[d, equal] & B \arrow[d, equal] \\ B[-1] \arrow[r] & C' \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

The Five Lemma 3.12 shows that γ is invertible, which gives the asserted uniqueness. □

Hence the object C in an exact triangle $B[-1] \to C \to A \xrightarrow{f} B$ is unique up to isomorphism.

The next lemma completely classifies triangles that are trivial in the sense that either one of the objects or one of the maps in the triangle vanishes.

Lemma 3.14. *Let*

$$B[-1] \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B$$

be an exact triangle. Then

1. *$B = 0$ if and only if v is invertible.*
2. *$u = 0$ if and only if w is an epimorphism, if and only if w is a split epimorphism, if and only if $C \to A \to B$ is a split extension ($A \cong C \oplus B$).*

Proof.

1. If v is invertible, then
$$0 \to C \xrightarrow{v} A \to 0$$
is an exact triangle: use (TR0) and the isomorphism of triangles
$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & C \arrow[r, "v"] \arrow[d, "\cong"'] & A \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\ 0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0 \end{tikzcd}$$
Conversely, assume $B = 0$. The long exact sequence for $\mathcal{T}(D, -)$ shows that $\mathcal{T}(D, B) = 0$ if and only if $\mathcal{T}(D, v)$ is invertible. By the Yoneda Lemma, $\mathcal{T}(D, B) = 0$ for all D if and only if $B = 0$, and $\mathcal{T}(D, v)$ is invertible for all D if and only if v is invertible.

2. If the triangle splits, then w is a split epimorphism and, *a fortiori*, an epimorphism. If w is an epimorphism, then $u = 0$ because $u[-1] \circ w = 0$. It remains to show that the triangle splits if $u = 0$. The exactness of
$$\mathcal{T}(B, A) \xrightarrow{w_*} \mathcal{T}(B, B) \xrightarrow{u_*=0} \mathcal{T}(B, C[1])$$
shows that there is $s\colon B \to A$ with $w \circ s = \mathrm{id}_B$. For any D, the exactness of
$$\cdots \xrightarrow{0} \mathcal{T}(D, C) \to \mathcal{T}(D, A) \to \mathcal{T}(D, B) \xrightarrow{0} \cdots$$
implies that $\mathcal{T}(D, v)$ and $\mathcal{T}(D, s)$ give isomorphism
$$\mathcal{T}(D, C) \oplus \mathcal{T}(D, B) \to \mathcal{T}(D, A).$$
By the Yoneda Lemma, (s, v) induce an isomorphism $C \oplus B \xrightarrow{\cong} A$. □

Furthermore, we claim that any split triangle is exact. Given B, C embed the coordinate projection $w\colon B \oplus C \to B$ in an exact triangle
$$B[-1] \xrightarrow{u} D \to B \oplus C \xrightarrow{w} B$$
using axiom (TR1). Since $B \oplus C \xrightarrow{w} B$ is an epimorphism we have $u = 0$. The long exact sequence
$$\ldots \xrightarrow{0} \mathcal{T}(X, D) \to \mathcal{T}(X, B \oplus C) \to \mathcal{T}(X, B) \xrightarrow{0} \ldots$$
implies $\mathcal{T}(X, D) \cong \mathcal{T}(X, C)$ for all $X \in \mathcal{T}$. Hence $D \cong C$ by the Yoneda Lemma. A similar argument yields the following additivity property:

Lemma 3.15. *If*
$$B_i[-1] \to C_i \to A_i \to B_i$$
are exact triangles for all $i \in I$, and direct sums exist, then
$$\bigoplus_{i\in I} B_i[-1] \to \bigoplus_{i\in I} C_i \to \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} B_i$$
is exact. The same holds for products.

What are the morphisms between triangulated categories? These should be additive functors that are compatible with suspensions and preserve exact triangles. But there is a small issue to keep in mind here. A *stable* functor $\mathcal{T} \to \mathcal{T}'$ between two triangulated categories is a pair consisting of a functor $F\colon \mathcal{T} \to \mathcal{T}'$ and natural isomorphisms $F(A[1]) \cong (FA)[1]$ for all objects A of $\mathcal{T}$. A stable functor is called *exact* if it maps any exact triangle $B[-1] \to C \to A \to B$ to an exact triangle $(FB)[-1] \to FC \to FA \to FB$; the latter involves the map $(FB)[-1] \cong F(B[-1]) \to F(A)$.

For instance, the restriction and induction functors provide exact functors $\mathrm{KK}^G \leftrightarrows \mathrm{KK}^H$ for a closed subgroup $H \subseteq G$, and the descent functor is an exact functor $\mathrm{KK}^G \to \mathrm{KK}$. If $F\colon \mathcal{A} \to \mathcal{A}'$ is additive functor between two additive categories, then the induced functor between the homotopy categories of chain complexes

$$\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{A}')$$

is exact. In these cases, the natural isomorphism that compares the suspensions is the obvious one—this happens in most cases of interest. The following example is an exception where the natural transformation is crucial:

Example 3.16. Let $\Sigma\colon \mathcal{T} \to \mathcal{T}$ be the suspension functor, and

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

an exact triangle. The triangle

$$A[1] \xrightarrow{u[1]} B[1] \xrightarrow{v[1]} C[1] \xrightarrow{w[1]} A[2]$$

could be non-exact: we have to add signs. But when we view Σ as a stable functor with the natural isomorphism

$$\Sigma(A[1]) \xrightarrow{-\mathrm{id}} (\Sigma A)[1]$$

then it becomes an exact functor.

3.4 Cartesian squares and colimits

Recall that a homological functor on KK is essentially the same as a semi-split exact, split-exact C*-stable functor on the category of (separable) C*-algebras. Besides the assertion that such functors descend to functors on KK, this asserts that they have long exact sequences for semi-split extensions. We have already seen that this is equivalent to the existence of Puppe sequences for mapping cones. In this section, we examine some other basic exact sequences associated to pullbacks, pushouts, and inductive limits, following [n-a01, mn06].

Definition 3.17. *A square*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y \arrow[d, "\beta'"] \\ X' \arrow[r, "\alpha'"] & Y' \end{tikzcd}$$

is called homotopy Cartesian *with* differential $\gamma\colon Y'[-1] \to X$ *if*

$$Y'[-1] \xrightarrow{\gamma} X \xrightarrow{\binom{\alpha}{\beta}} Y \oplus X' \xrightarrow{(\beta', -\alpha')} Y' \tag{3.1}$$

is exact.

We can embed any given pair of maps α, β with the same source in a homotopy Cartesian square using axiom (TR1) for the map $\binom{\alpha}{\beta}$ (*homotopy pushout*). Similarly, we may embed a given pair of maps α', β' with the same range into a homotopy Cartesian square (*homotopy pullback*).

If F is a homological functor, then the long exact sequence for the exact triangle (3.1) is a *Mayer–Vietoris exact sequence* for the original homotopy Cartesian square.

Definition 3.18. *Let $(A_n, \alpha_n^{n+1}\colon A_n \to A_{n+1})_{n\in\mathbb{N}}$ be an inductive system in a triangulated category with direct sums. We define its* homotopy colimit

$$\underrightarrow{\operatorname{holim}}(A_n, \alpha_n^{n+1}\colon A_n \to A_{n+1})_{n\in\mathbb{N}}$$

as the cone of the map

$$\bigoplus_{n\in\mathbb{N}} A_n \xrightarrow{\mathrm{id}-S} \bigoplus_{n\in\mathbb{N}} A_n,$$

$$S|_{A_n} = \alpha_n^{n+1}\colon A_n \to A_{n+1},$$

That is, we require an exact triangle

$$\bigoplus_{n\in\mathbb{N}} A_n \xrightarrow{\mathrm{id}-S} \bigoplus_{n\in\mathbb{N}} A_n \longrightarrow \underrightarrow{\operatorname{holim}}(A_n, \alpha_n^{n+1}) \longrightarrow \bigoplus_{n\in\mathbb{N}} A_n[1].$$

Since the cone of a morphism is unique up to isomorphism, homotopy pullbacks, homotopy pushouts, and homotopy colimits are unique up to isomorphism. But since this isomorphism is not canonical—it merely exists—these constructions are not functorial. This lack of functoriality of various constructions sometimes creates problems when working with triangulated categories.

Proposition 3.19. *If $F\colon \mathcal{T} \to \mathbf{Ab}$ is homological and commutes with direct sums, then*

$$F(\underrightarrow{\operatorname{holim}} A_n) = \varinjlim F(A_n).$$

If $\widetilde{F}\colon \mathcal{T} \to \mathbf{Ab}^{\mathrm{op}}$ is co*homological and $\widetilde{F}(\bigoplus A_n) = \prod \widetilde{F}(A_n)$, then there is an exact sequence*

$$\varprojlim{}^1 \widetilde{F}(A_n) \rightarrowtail \widetilde{F}(\underrightarrow{\operatorname{holim}} A_n) \twoheadrightarrow \varprojlim \widetilde{F}(A_n),$$

called Milnor $\varprojlim^1$-sequence.

Proof. Apply F to the exact triangle defining $\underrightarrow{\operatorname{holim}}$. Since F is homological and commutes with direct sums, we get an exact sequence

$$\bigoplus F_n(A_m) \xrightarrow{\mathrm{id}-S} \bigoplus F_n(A_m) \to F_n(\underrightarrow{\operatorname{holim}} A_n) \to \bigoplus F_{n-1}(A_m) \rightarrowtail \bigoplus F_{n-1}(A_m).$$

Now use that a sequence $A \to B \to C \to D \to E$ is exact if and only if

$$\operatorname{coker}(A \to B) \to C \to \ker(D \to E)$$

is an extension. Since the inductive limit functor for Abelian groups is exact, we have

$$\operatorname{coker}(\mathrm{id} - S) = \varinjlim F_n(A_m), \qquad \ker(\mathrm{id} - S) = 0.$$

The assertion in the homological case follows. In the cohomological case, the argument is similar, but with arrows reversed and sums replaced by products. By definition,

$$\ker(\mathrm{id} - S) = \varprojlim F_n(A_m), \qquad \operatorname{coker}(\mathrm{id} - S) = \varprojlim{}^1 F_n(A_m),$$

so that we get the asserted exact sequence. □

Example 3.20. Let $e\colon A \to A$ be an idempotent morphism. The homotopy colimit of the constant inductive system $A \xrightarrow{e} A \xrightarrow{e} A \xrightarrow{e} \cdots$ is a range object for e. Therefore, any triangulated category with countable direct sums is idempotent complete.

The homotopy constructions above satisfy Mayer–Vietoris sequences and Milnor $\varprojlim^1$-sequences by definition. If our triangulated category is the homotopy category of some underlying (model) category, then we may ask whether pullbacks, pushouts, and colimits in this underlying category become homotopy pullbacks, pushouts, and colimits in the triangulated category, respectively. For KK, this leads to the following questions:

1. Let

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y \arrow[d, "\beta'"] \\ X' \arrow[r, "\alpha'"] & Y' \end{tikzcd}$$

   be a pullback diagram of C*-algebras, that is,

$$X = \{(x', y) \in X' \times Y \mid \alpha'(x') = \beta'(y)\}.$$

   When is its image in KK homotopy Cartesian?

2. Let (A_n, α_n) be an inductive system of C*-algebras. Is its inductive limit $\varinjlim(A_n, \alpha_n)$ also a homotopy colimit?

The analogous question for C*-algebra pushouts, that is, free products of C*-algebras with amalgamation, has been studied by Germain and Thomsen.

The second question is answered by the following theorem:

Theorem 3.21 ([mn06])**.** *If all A_n are nuclear, then $\varinjlim(A_n, \alpha_n)$ is a homotopy colimit.*

In general, the colimit is a homotopy colimit if a certain extension built from the data is semi-split. In the nuclear case, this is automatically the case.

Now we discuss the answer to the first question in some detail. First we compare the pullback X to a *homotopy pullback*

$$H = \{(x', y', y) \in X' \times C(I, Y') \times Y \mid \alpha'(x') = y'(0),\ \beta'(y) = y'(1)\}.$$

This really is a homotopy pullback in the sense of our definition because it fits into a semi-split extension

$$SY' \rightarrowtail H \twoheadrightarrow X' \oplus Y,$$

whose class in $\mathrm{KK}_1(X' \oplus Y, SY) \cong \mathrm{KK}_0(X' \oplus Y, Y')$ is $(\beta', -\alpha')$. This last claim follows from the morphisms of exact sequences

$$\begin{tikzcd}
SY' \arrow[r, tail] & H \arrow[r, two heads] & X' \oplus Y \\
SY' \arrow[u, equal] \arrow[r, tail] & \tilde{C}_{\alpha'} \arrow[u] \arrow[r, two heads] & X' \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
SY' \arrow[r, tail] & H \arrow[r, two heads] & X' \oplus Y \\
SY' \arrow[u, equal] \arrow[r, tail] & C_{\beta'} \arrow[u] \arrow[r, two heads] & Y \arrow[u]
\end{tikzcd}$$

and the naturality of the boundary maps for C*-algebra extensions. Here $\tilde{C}_\alpha$ denotes the reflected mapping cone; the reflection is responsible for the sign.

Definition 3.22. *The pullback square is called* admissible *if the canonical map* $X \to H$ *is a* KK*-equivalence.*

Since H is a homotopy pullback, the pullback X is a homotopy pullback as well if and only if the pullback square is admissible.

Proposition 3.23. *If α' is a semi-split epimorphism, then so is α, and the pullback square is admissible. Thus the pullback is a homotopy pullback in this case and we get a Mayer–Vietoris exact sequence*

$$\cdots \to F_n(X) \to F_n(X') \oplus F_n(Y) \to F_n(Y') \to F_{n-1}(X) \to \cdots$$

if F is semi-split exact, split-exact, and C-stable.*

Proof. If α' is a semi-split epimorphism, then so is α, and both have the same kernel. Denote this kernel by K. The canonical map $H \to Y$ is a semi-split epimorphism, and its kernel is naturally isomorphic to the mapping cone of α'. The Five Lemma applied to the morphism of extensions

$$\begin{tikzcd}
C_{\alpha'} \arrow[r, tail, "\alpha"] & H \arrow[r, two heads] & Y \\
K \arrow[u] \arrow[r, tail] & X \arrow[u] \arrow[r, two heads] & Y \arrow[u, equal]
\end{tikzcd}$$

shows that the embedding $X \to H$ is a KK-equivalence if and only if the embedding $K \to C_{\alpha'}$ is a KK-equivalence. But the latter is always the case because K is also the kernel of α' and α' is a semi-split epimorphism (compare the argument after (1.8)). □

3.5 Versions of the octahedral axiom

The fourth axiom of a triangulated category is called octahedral axiom because its original formulation could be drawn on the surface of an octahedron. This statement is equivalent to the following:

Axiom 3.24 (TR4)**.** *Given the solid arrows in the diagram*

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, equal] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & Z \arrow[r, "\alpha_3"] \arrow[d, dotted, "\delta_1"] & X[1] \arrow[d, equal] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\gamma_3"] \arrow[d, dotted, "\delta_2"] & X[1] \arrow[d, "\alpha_1[1]"] \\
 & W \arrow[r, equal] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, dotted, "\delta_3"] & Y[1] \\
 & Y[1] \arrow[r, "\alpha_2[1]"] & Z[1] &
\end{tikzcd}$$

such that $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ are exact triangles, there is an exact triangle $(\delta_1, \delta_2, \delta_3)$ so that the whole diagram commutes.

Since any map can be embedded in an exact triangle, which is unique up to isomorphism, we can recover the solid arrows in the diagram above given only the two composable maps α_1 and β_1. The assertion is that the cones of these maps and of their composition $\beta_1 \circ \alpha_1 = \gamma_1$ are connected by an exact triangle whose maps make the whole diagram commute.

There are several equivalent reformulations of axiom (TR4). We mention a particularly simple one due to Dlab, Parshall and Scott, following [**?**].

Axiom 3.25 (TR4')**.** *Every pair of maps*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \\
X' &
\end{tikzcd}$$

can be completed to a morphism of exact triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, dotted] & Z \arrow[r, "w"] \arrow[d, equal] & X[1] \arrow[d] \\
X' \arrow[r] & Y' \arrow[r, "v'"] & Z \arrow[r] & X'[1]
\end{tikzcd}$$

such that the first square is homotopy Cartesian with differential $w \circ v' \colon Y' \to X[1]$.

Axiom 3.26 (TR4")**.** *Any homotopy Cartesian square*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

with differential $\delta\colon Y' \to X[1]$ *may be completed to a morphism of exact triangles*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, dotted] & Z \arrow[r, "w"] \arrow[d, equal] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r, "v'"] & Z \arrow[r] & X'[1] \end{tikzcd}$$

with $\delta = w \circ v'$.

Proposition 3.27 ([?]). *Axioms* (TR4), (TR4'), (TR4") *are equivalent.*

Each of these equivalent formulations of the octahedral axiom can be checked with moderate effort for KK (see [mn06]).

3.6 Localisation of triangulated categories

Roughly speaking, localisation enlarges a ring (or a category) by adding inverses of certain ring elements (or morphisms). In general, strange things can happen here due to non-commutativity. But in all examples we are going to study, the localisation has a rather simple description.

The motivating example is the derived category of an Abelian category, which is defined as a localisation of its homotopy category of chain complexes. For any additive category $\mathcal{A}$, the homotopy category of chain complexes in $\mathcal{A}$ is a triangulated category (see Example 3.1). This is a purely formal construction and not yet homological algebra. Only if $\mathcal{A}$ carries further structure, say, $\mathcal{A}$ is Abelian, can we talk about things like homology of chain complexes and exact chain complexes. On the categorical level, this allows us to pass to the derived category, in which exact chain complexes become zero and quasi-isomorphisms become invertible.

Definition 3.28. *The localisation of a category* $\mathcal{C}$ *at a family of morphisms* S *is a category* $\mathcal{C}[S^{-1}]$ *together with a functor* $F\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ *such that*

1. $F(s)$ *is invertible for all* $s \in S$;
2. F *is universal among functors with this property, that is, if* $G\colon \mathcal{C} \to \mathcal{C}'$ *is another functor with* $G(s)$ *invertible for all* $s \in S$*, then there is a unique factorisation*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{C}[S^{-1}] \arrow[dl, dotted, "\exists!"] \\ & \mathcal{C}' & \end{tikzcd}$$

In good cases, we have a calculus of fractions that allows us to describe the morphisms in the localisation more concretely. The conditions needed for this generalise the Ore condition for localisation of rings. Mainly, we need a "commutation relation" that turns fractions of the form $f \circ s^{-1}$ into fractions of the form $t^{-1} \circ g$. More precisely, for all $f \in \mathcal{C}$ and $s \in S$, there are $g \in \mathcal{C}$ and $t \in S$ with $tf = gs$, that is, $fs^{-1} = t^{-1}g$. We also assume that S is closed under composition and that if $f_1 s = f_2 s$ for some $s \in S$, then there is $t \in S$ with $tf_1 = tf_2$. Under these assumptions, any morphism in the localisation is of the form $s^{-1} \circ f$ for some $s \in S$, $f \in \mathcal{C}$.

Example 3.29. Let $\mathcal{A}$ be an Abelian category, let $\mathcal{C}$ be the homotopy category of chain complexes in $\mathcal{A}$, and let S be the class of all quasi-isomorphisms, that is, all chain maps that induce an isomorphism on homology. This satisfies the above conditions.

In good cases, we expect a localisation of a triangulated category to be again triangulated. Recall that a morphism in a triangulated category is invertible if and only if its cone is zero. Therefore, when we want to localise a triangulated category we may equally well specify which objects should become zero instead of which maps should be inverted.

Definition 3.30. *A class $\mathcal{N}$ of objects in a triangulated category $\mathcal{T}$ is called* thick *if it satisfies the following conditions:*

1. $0 \in \mathcal{N}$;
2. *if* $A \oplus B \in \mathcal{N}$ *then* $A, B \in \mathcal{N}$;
3. *if the triangle* $A \to B \to C \to A[1]$ *is exact and* $A, B \in \mathcal{N}$, *then* $C \in \mathcal{N}$.

Example 3.31. The object-kernel $\{A \in \mathcal{T} \mid G(A) \cong 0\}$ of an exact functor is thick.

Definition 3.32. *Given a thick subcategory $\mathcal{N} \in \mathcal{T}$, an $\mathcal{N}$-*equivalence *is a morphism in $\mathcal{T}$ whose cone belongs to $\mathcal{N}$.*

The localisation of $\mathcal{T}$ at $\mathcal{N}$ is defined as the localisation at the $\mathcal{N}$-equivalences:

$$\mathcal{T}/\mathcal{N} := \mathcal{T}[(\mathcal{N} - \text{equivalences})^{-1}].$$

Theorem 3.33 ([n-a01]). *Given a thick subcategory $\mathcal{N}$ in a (small) triangulated category $\mathcal{T}$, the $\mathcal{N}$-equivalences have a calculus of fractions, $\mathcal{T}/\mathcal{N}$ is again a triangulated category, and the functor $\mathcal{T} \to \mathcal{T}/\mathcal{N}$ is exact. Furthermore, the kernel of the functor $\mathcal{T} \to \mathcal{T}/\mathcal{N}$ is exactly $\mathcal{N}$, and a morphism in $\mathcal{T}$ becomes invertible in $\mathcal{T}/\mathcal{N}$ if and only if it is an $\mathcal{N}$-equivalence.*

3.7 Complementary subcategories and localisation

Definition 3.34. *The* left orthogonal complement *of a class of objects $\mathcal{N}$ in $\mathcal{T}$ is*

$$\mathcal{N}^{\vdash} := \{L \in \mathcal{T} \mid \mathcal{T}(L, N) = 0 \textit{ for all } N \in \mathcal{N}\}.$$

Definition 3.35. *A pair $(\mathcal{L}, \mathcal{N})$ of thick subcategories in a triangulated category $\mathcal{T}$ is called* complementary *if $\mathcal{L} \subseteq \mathcal{N}^{\vdash}$ and for each $A \in \mathcal{T}$, there is an exact triangle $L \to A \to N \to L[1]$ with $L \in \mathcal{L}$ and $N \in \mathcal{N}$.*

Theorem 3.36. *Let $(\mathcal{L}, \mathcal{N})$ be complementary.*

(1) $\mathcal{L} = \mathcal{N}^{\vdash}$ *and* $\mathcal{N} = \mathcal{L}^{\dashv} = \{N \in \mathcal{T} \mid \mathcal{T}(L, N) = 0 \textit{ for all } N \in \mathcal{N}\}$.

(2) *The exact triangle $L \to A \to N \to L[1]$ with $L \in \mathcal{L}$, $N \in \mathcal{N}$ is unique up to canonical isomorphism and functorial in $\mathcal{A}$.*

(3) *The functors $\mathcal{T} \to \mathcal{L}$, $A \mapsto L$ and $\mathcal{T} \to \mathcal{N}$, $A \mapsto N$ are exact.*

(4) *$\mathcal{L} \to \mathcal{T} \to \mathcal{T}/\mathcal{N}$ and $\mathcal{N} \to \mathcal{T} \to \mathcal{T}/\mathcal{L}$ are equivalences of categories.*

Example 3.37. Let $\mathcal{T}$ be the homotopy category $\mathrm{Ho}(\mathcal{A})$ of chain complexes over an Abelian category $\mathcal{A}$ and let $\mathcal{N}$ be the subcategory of exact chain complexes, which is thick. Let $P \in \mathcal{A}$ be projective and view it as a chain complex concentrated in degree zero. Then homotopy classes of chain maps $P \to C_*$ for a chain complex $C_\bullet$ are in bijection with maps $P \to \mathrm{Ho}(C_*)$: a map

$$\begin{tikzcd}
\cdots \arrow[r] & C_1 \arrow[r, "d_1"] & C_0 \arrow[r, "d_0"] & C_{-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & P \arrow[r] \arrow[u, "f"'] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

is a chain map if and only if $f(P) \subseteq \ker(d_0)$, and it is a boundary if and only if $f(P) \subseteq d_1(C_1)$ because P is projective; moreover, a map $\ker(d_0)/d_1(C_1)$ lifts to a map to $\ker(d_0)$ because P is projective. As a consequence, $P \in \mathcal{N}^\vdash$.

Notice that $\mathcal{N}^\vdash$ is always thick and closed under direct sums. Subcategories with both properties are called *localising*.

Definition 3.38. *The smallest localising subcategory containing a class of objects $\mathcal{P}$ is called the* localising subcategory generated by $\mathcal{P}$ *and denoted by $\langle \mathcal{P} \rangle$.*

We return to Example 3.37. Let P_0, P_1 be projective in $\mathcal{A}$, and let f be a map $P_1 \to P_0$, viewed as a chain map. Its cone is the chain complex

$$C_f := (\cdots \to 0 \to P_1 \xrightarrow{f} P_0 \to 0 \to \cdots),$$

Iterating this construction, we can get any chain complex of projective modules of finite length. All these chain complexes therefore belong to the localising subcategory generated by the projective objects of $\mathcal{A}$.

Theorem 3.39 (Spaltenstein, see also [?])**.** *Let $\mathcal{A}$ be an Abelian category with enough projectives and countable direct sums. Let $\mathcal{N} \subseteq \mathrm{Ho}(\mathcal{A})$ be the full subcategory of exact chain complexes, and let $\mathcal{L}$ be the localising subcategory of $\mathrm{Ho}(\mathcal{A})$ generated by the projective objects of $\mathcal{A}$. Then $(\mathcal{L}, \mathcal{N})$ is complementary.*

The functor $L\colon \mathrm{Ho}(\mathcal{A}) \to \mathcal{L}$ replaces a module M by a projective resolution

$$L(M) = (\cdots \to P_2 \to P_1 \to P_0 \to 0 \to \cdots).$$

The natural map $L(M) \to M$ is the augmentation map $P_0 \to M$ of the resolution. Its cone is the augmented chain complex

$$N(M) = (\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \to \cdots),$$

which belongs to $\mathcal{N}$. Spaltenstein's Theorem generalises this construction to the case where M is a chain complex.

Example 3.40. Let $\mathcal{T} = \mathrm{KK}$, $\mathcal{N} = \{A \in \mathrm{KK} \mid \mathrm{K}_*(A) = 0\}$. Then $\mathbb{C} \in \mathcal{N}^\vdash$ because $\mathrm{KK}_*(\mathbb{C}, A) = \mathrm{K}_*(A) = 0$ for $A \in \mathcal{N}$. Let $\mathcal{B}$ be the localising subcategory generated by $\mathbb{C}$; this subcategory is called the *bootstrap category*. It is exactly the class of all C*-algebras A with the property that the Universal Coefficient Theorem computes $\mathrm{KK}_*(A, B)$ for all B.

Theorem 3.41. *$(\mathcal{B}, \mathcal{N})$ are complementary subcategories of* KK.

The functor $L\colon \mathrm{KK} \to \mathcal{B}$ replaces a separable C*-algebra by one in the bootstrap class with the same K-theory. This is determined uniquely by the Universal Coefficient Theorem.

We will prove Theorems 3.39 and 3.41 at the end of Section 3.8 after developing some more machinery needed for the proof.

Let $\mathcal{T}$ be triangulated and monoidal with tensor unit $\mathbf{1}$, and let $\mathcal{L}$ and $\mathcal{N}$ be thick subcategories with $\mathcal{L} \otimes \mathcal{T} \subseteq \mathcal{L}$ and $\mathcal{N} \otimes \mathcal{T} \subseteq \mathcal{N}$. If $\mathcal{L} \subseteq \mathcal{N}^{\vdash}$ and there is an exact triangle

$$L \to \mathbf{1} \to N \to L[1]$$

with $L \in \mathcal{L}$ and $N \in \mathcal{N}$, then $(\mathcal{L}, \mathcal{N})$ is complementary because the triangle

$$L \otimes A \to \mathbf{1} \otimes A \to N \otimes A \to L \otimes A[1]$$

is exact and has $L \otimes A \in \mathcal{L}$ and $N \otimes A \in \mathcal{N}$ for any $A \in \mathcal{T}$. This trick reduces the study of localisation to the study of a single exact triangle. This is the heart of the Dirac–dual Dirac method for proving injectivity or bijectivity of the Baum–Connes assembly map, see [mn06]. This is why some exterior tensor product structure on $\mathrm{KK}^{\mathcal{G}}$ for a quantum group $\mathcal{G}$ is highly desirable.

3.7.1 Proof of Theorem 3.36

Let $(\mathcal{L}, \mathcal{N})$ be complementary subcategories.

1. $\mathcal{L} = \mathcal{N}^{\vdash}$. The assumptions include $\mathcal{L} \subseteq \mathcal{N}^{\vdash}$. Take $A \in \mathcal{N}^{\vdash}$ and embed it in an exact triangle $L \to A \to N \to L[1]$ with $L \in \mathcal{L}$ and $N \in \mathcal{N}$. The map $A \to N$ vanishes because $A \in \mathcal{N}^{\vdash}$. Lemma 3.14 shows that A is a direct summand of L, hence $A \in \mathcal{L}$ because $\mathcal{L}$ is thick.

2. Consider a diagram

$$\begin{tikzcd} L \arrow[r] & A \arrow[r] \arrow[d, "f"] & N \arrow[r] & L[1] \\ L' \arrow[r] & A' \arrow[r] & N' \arrow[r] & L'[1] \end{tikzcd}$$

   with exact rows, $L, L' \in \mathcal{L}$, and $N, N' \in \mathcal{N}$.

   In the long exact sequence

$$\cdots \to \mathcal{T}_1(L, N') \to \mathcal{T}_0(L, L') \to \mathcal{T}_0(L, A') \to \mathcal{T}_0(L, N') \to \cdots,$$

   the map $\mathcal{T}_0(L, L') \to \mathcal{T}_0(L, A')$ is an isomorphism because $\mathcal{T}_*(L, N') = 0$. Hence there is a *unique* map $L \xrightarrow{L_f} L'$ that makes the solid arrow diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "L_f"] & A \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, dotted, "N_f"] & L[1] \arrow[d, "L_f[1]"] \\ L' \arrow[r] & A' \arrow[r] & N' \arrow[r] & L'[1] \end{tikzcd}$$

commute. A dual argument shows that there is a *unique* map $N_f\colon N \to N'$ that makes the square $AA'NN'$ commute. At the same time, axiom (TR3) yields a map N_f that extends (f, L_f) to a morphism of exact triangles. This can only be the unique map found above. As a result, there is a unique morphism of exact triangles extending f. It follows from this that the exact triangle $L \to A \to N \to L[1]$ is unique up to a canonical isomorphism of exact triangles and functorial.

The argument also shows that $\mathcal{T}_*(\Lambda, A) \cong \mathcal{T}_*(\Lambda, L)$ for all $\Lambda \in \mathcal{L}$, that is, the functor $\mathcal{T} \to \mathcal{L}$, $A \mapsto L$ is right adjoint to the embedding $\mathcal{L} \to \mathcal{T}$. Similarly, the functor $N\colon \mathcal{T} \to \mathcal{N}$ is left adjoint to the embedding $\mathcal{N} \to \mathcal{T}$.

3. Next we check that these functors L and N are exact.

   For an exact triangle
$$L \xrightarrow{u} A \xrightarrow{v} N \xrightarrow{w} L[1]$$
   the triangle
$$L[1] \xrightarrow{u} A[1] \xrightarrow{v} N[1] \xrightarrow{-w} L[2]$$
   is exact. This shows that L and N are stable.

   Let $A \to B \to C \to A[1]$ be an exact triangle in $\mathcal{T}$. Axiom (TR1) provides an exact triangle
$$LA \to LB \to X \to LA[1],$$
   and axiom (TR3) a map $X \xrightarrow{f} C$ so that
$$\begin{tikzcd} LA \arrow[r] \arrow[d, "\pi_A"] & LB \arrow[r] \arrow[d, "\pi_B"] & X \arrow[r] \arrow[d, "f"] & LA[1] \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$
   is a morphism of exact triangles. Since $\mathcal{L}$ is thick, $X \in \mathcal{L}$. Let $\Lambda \in \mathcal{L}$, then the maps π_A and π_B induce isomorphisms on $\mathcal{T}_*(\Lambda, -)$. By the Five Lemma, the map f also induces an isomorphism $\mathcal{T}_*(\Lambda, X) \cong \mathcal{T}_*(\Lambda, C)$. Thus the cone of f belongs to $\mathcal{N} = \mathcal{L}^{\dashv}$. It follows that $X \cong LC$. The maps $LB \to X \to LA[1]$ can only be the unique maps that lift the maps $B \to C \to A[1]$. Hence the exact triangle we have constructed is the L-image of $A \to B \to C \to A[1]$. Thus the functor L is exact. A similar argument shows that N is exact.

4. Let $\mathcal{T}'$ be the category with the same objects as $\mathcal{T}$ and $\mathcal{T}'(A, B) := \mathcal{T}(LA, LB)$. Let $F\colon \mathcal{T} \to \mathcal{T}'$ be the functor that is the identity on objects and L on morphisms. The canonical map $LA \to A$ is an $\mathcal{N}$-equivalence. The functor L maps $\mathcal{N}$-equivalences to isomorphisms because L is exact and $L(A) = 0$ for $A \in \mathcal{N}$. If another functor G maps $\mathcal{N}$-equivalences to isomorphisms, we get
$$\begin{tikzcd} G(LA) \arrow[r] \arrow[d, "\cong"] & G(LB) \arrow[d, "\cong"] \\ G(A) \arrow[r, dotted] & G(B) \end{tikzcd}$$

so that elements of $\mathcal{T}'(A,B)$ give maps $G(A) \to G(B)$. This construction shows that $\mathcal{T}'$ with the functor $F\colon \mathcal{T} \to \mathcal{T}'$satisfies the universal property of $\mathcal{T}[(\mathcal{N} - \text{equivalences})^{-1}]$.

Since the canonical map $LA \to A$ is invertible in $\mathcal{T}$ if $A \in \mathcal{L}$, the map $L^2A \to LA$ is always invertible, so that $LA \to A$ becomes an isomorphism in $\mathcal{T}'$. Hence the restriction of F to $\mathcal{L}$ is essentially surjective and fully faithful. Therefore, $\mathcal{T}'$ is equivalent to $\mathcal{L}$ as claimed. The equivalence $\mathcal{T}' \to \mathcal{L}$ is the functor F^* that is L on objects and the identity on morphisms. There is a unique triangulated category structure on $\mathcal{T}'$ for which the equivalence of categories F^* is an exact functor. Since the functor L is exact, the functor $F\colon \mathcal{T} \to \mathcal{T}'$ is exact for this triangulated category structure as well. Furthermore, if $G\colon \mathcal{T} \to \mathcal{T}''$ is an exact functor vanishing on $\mathcal{N}$, then the induced functor $\overline{G}\colon \mathcal{T}' \to \mathcal{T}''$ is exact. Therefore, $\mathcal{T}'$ is also a localisation of $\mathcal{T}$ in the category of triangulated categories.

3.8 Homological algebra in triangulated categories

It is usually hard to establish that a given pair of subcategories is complementary. Even the rather classical case of Spaltenstein's Theorem 3.39 is non-trivial. In this section, we develop machinery that will, eventually, help us to do so; but it will take a while until we get to that point. The following theory is developed in [**?**] and [**?**], following earlier work by Daniel Christensen ([**?**]) and Apostolos Beligiannis ([**?**]).

We want to use some homological functor $F : \mathcal{T} \to \mathcal{A}$ as a probe to study a triangulated category $\mathcal{T}$; here $\mathcal{A}$ is some Abelian category.

Examples 3.42. In the following, we will consider the following examples.

- $\mathcal{T} = \mathrm{Ho}(\mathcal{A})$ for an Abelian category $\mathcal{A}$, and F is the homology functor $\mathrm{Ho}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$
- $\mathcal{T} = \mathrm{KK}$ and $F = \mathrm{K}_*\colon\ \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$
- $\mathcal{T} = \mathrm{KK}^{(C,\Delta)}$ and $F = \mathrm{K}_*\colon\ \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$, where (C,Δ) is a compact quantum group

In these examples, the target category has its own translation (suspension) automorphism, and F is a stable functor, that is, it intertwines the translation functors (up to a natural isomorphism as in the definition of an exact functor).

Although we will use the functor F in our definitions, it is checked in [**?**] that they only depend on its *morphism-kernel*

$$\mathcal{I}(A,B) = (\ker F)(A,B) := \{\varphi\colon A \to B \mid F(\varphi) = 0\}.$$

This is a finer invariant than the *object-kernel* $\{A \mid F(A) \cong 0\}$. The morphism-kernel $\ker F$ is an ideal in $\mathcal{T}$ invariant under the translation automorphism. Not all such ideals are possible.

Definition 3.43. *An ideal $\mathcal{I}$ in a triangulated category is called* homological *if it is the morphism-kernel of a stable homological functor.*

An intrinsic characterisation of homological ideals is possible but somewhat complicated (Beligiannis develops this in a different notation in [**?**]). A homological ideal allows us to carry over various notions from homological algebra to our category $\mathcal{T}$. The starting point is a good notion of exactness for chain complexes—all other notions follow from that.

Definition 3.44. *Let* $\mathcal{I} = \ker F$. *We call a chain complex* (C_n, d_n) *in* $\mathcal{T}$ $\ker F$-exact *in degree* $n \in \mathbb{Z}$ *if*

$$F(C_{n+1}) \to F(C_n) \to F(C_{n-1})$$

is exact at $F(C_n)$.

See [**?**] for a characterisation of $\ker F$-exact chain complexes that manifestly only depends on $\ker F$.

Definition 3.45. *An object* $A \in \mathcal{T}$ *is* $\mathcal{I}$*-projective if the functor* $\mathcal{T}(A, -)$ *maps* $\mathcal{I}$*-exact chain complexes in* $\mathcal{T}$ *to exact chain complexes.*

Lemma 3.46. *The following statements are equivalent for* $A \in \mathcal{T}$*:*

1. *A is $\mathcal{I}$-projective;*
2. *the map $\mathcal{T}(A, B) \xrightarrow{f_*} \mathcal{T}(A, C)$ vanishes for all $f \in \mathcal{I}(B, C)$;*
3. *$\mathcal{I}(A, C) = 0$ for all $C \in \mathcal{T}$.*

Definition 3.47. *An* $\mathcal{I}$-projective *resolution of* $A \in \mathcal{T}$ *is an* $\mathcal{I}$*-exact chain complex*

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0 \to \cdots$$

with $\mathcal{I}$*-projective* P_i.

Projective resolutions, if they exist, may be used to define derived functors. The issue is whether there are enough projective objects and how to construct them. *Enough projective objects* means, of course, that any object has an $\mathcal{I}$-projective resolution.

We use (partially defined) left adjoints to construct projective objects. In good cases, this yields all $\mathcal{I}$-projective objects and shows that there are enough of them. Let $F\colon \mathcal{T} \to \mathcal{A}$ be a stable homological functor with $\ker F = \mathcal{I}$. Its left adjoint $F^{\vdash}$ is defined on $B \in \mathcal{A}$ if there is $B' := F^{\vdash}(B)$ with a natural isomorphism $\mathcal{T}(B', D) \cong \mathcal{A}(B, F(D))$ for all $D \in \mathcal{T}$. This defines a functor on a subcategory of $\mathcal{A}$.

The functor $\mathcal{T}(F^{\vdash}(B), -)$ factors as follows:

$$\mathcal{T} \xrightarrow{F} \mathcal{A} \xrightarrow{\mathcal{A}(B,-)} \mathbf{Ab}$$

$$D \mapsto F(D) \mapsto \mathcal{A}(B, F(D))$$

and therefore vanishes on $\mathcal{I} = \ker F$. This means that all objects of the form $F^{\vdash}(B)$ are $\mathcal{I}$-projective. Now we examine what happens in our model examples (Examples 3.42).

1. Let $\mathcal{T} = \mathrm{Ho}(\mathcal{A})$, $F = \mathrm{H}_*\colon \mathrm{Ho}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$. Assume that the Abelian category $\mathcal{A}$ has enough projective objects. Recall that if $P \in \mathcal{A}$ is projective, then $\mathcal{T}(P, C_*) = \mathcal{A}(P, \mathrm{H}_*(C_*))$ (see Example 3.37). Thus $\mathrm{H}_*^{\vdash}$ is defined on projective objects of $\mathcal{A}$. Even more, it is defined on all projective objects of $\mathcal{A}^{\mathbb{Z}}$, mapping them to the corresponding chain complex with vanishing boundary map. The general theory shows that, up to isomorphism, that is, chain homotopy equivalence, all projective objects of $\mathrm{Ho}(\mathcal{A})$ are of this form.

2. Let $\mathcal{T} = \mathrm{KK}$, $F = \mathrm{K}_*\colon \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$. By definition, we have
$$\mathrm{KK}(\mathbb{C}, A) = \mathrm{K}_0(A) = \mathrm{Hom}(\mathbb{Z}^{\mathrm{even}}, \mathrm{K}_*(A)).$$
This means that $\mathrm{K}_*^{\vdash}(\mathbb{Z}^{\mathrm{even}}) = \mathbb{C}$. Similarly, $\mathrm{K}_*^{\vdash}(\mathbb{Z}^{\mathrm{odd}}) = \mathbb{C}[1] = C_0(\mathbb{R})$. Since left adjoints commute with direct sums, $\mathrm{K}_*^{\vdash}$ is defined on all free $\mathbb{Z}/2$ graded Abelian groups.

3. Let $\mathcal{T} = \mathrm{KK}^G$ be the equivariant KK-theory for some discrete group G, and let $F\colon \mathrm{KK}^G \to \mathbf{Ab}^{\mathbb{Z}/2}$, $F(A, \alpha) = \mathrm{K}_*(A)$. If $A \in \mathrm{KK}$, $B \in \mathrm{KK}^G$ then
$$\mathrm{KK}^G(C_0(G) \otimes A, B) = \mathrm{KK}(A, B);$$
this is a special case of the adjointness between induction and restriction from open subgroups. It follows that
$$F^{\vdash}(\mathbb{Z}^{\mathrm{even}}) = C_0(G).$$
As above, this implies that $F^{\vdash}$ is defined on all free $\mathbb{Z}/2$-graded Abelian groups.

Proposition 3.48. *Let $F\colon \mathcal{T} \to \mathcal{A}$ be a stable homological functor whose left adjoint is defined on all projective objects of an Abelian category $\mathcal{A}$. If $\mathcal{A}$ has enough projectives, then there are enough* $\ker F$*-projective objects in $\mathcal{T}$, and any* $\ker F$*-projective object is a retract of $F^{\vdash}(B)$ for some projective object $B \in \mathcal{A}$.*

Proof. Let $D \in \mathcal{T}$, we need a projective object $B \in \mathcal{A}$ and a morphism $\pi \in \mathcal{T}(F^{\vdash}(B), D)$ such that $F(\pi)$ is an epimorphism. This is the beginning of a recursive construction of a projective resolution. Let $\rho\colon B \to F(D)$ be an epimorphism with projective B We have
$$\begin{gathered}\mathcal{T}(F^{\vdash}(B), D) \cong \mathcal{A}(B, F(D))\\ \rho^* \leftarrow \rho\end{gathered}$$
We claim that $F(\rho^*)$ is an epimorphism. This follows from the commuting diagram
$$\begin{tikzcd} FF^{\vdash}(B) \arrow[rr, "F(\rho^*)"] & & F(D), \\ & B \arrow[ul, "\varepsilon_B"] \arrow[ur, two heads, "\rho"'] & \end{tikzcd}$$
where $\varepsilon\colon \mathrm{Id} \to FF^{\vdash}$ is the unit of adjointness. $\square$

Once we have an $\mathcal{I}$-projective resolution, we get $\mathcal{I}$- derived functors and, in particular, extension groups $\operatorname{Ext}^*_{\mathcal{T},\mathcal{I}}(A,B)$ for two objects A and B of $\mathcal{T}$:

Definition 3.49. *Let $A \in \mathcal{T}$ and let (P_n, ∂_n) be an $\mathcal{I}$-projective resolution of A. Then $\operatorname{Ext}^n_{\mathcal{T},\mathcal{I}}(A,B)$ is the nth cohomology of the cochain complex*

$$\cdots \leftarrow \mathcal{T}(P_n, B) \leftarrow \mathcal{T}(P_{n-1}, B) \leftarrow \cdots \leftarrow \mathcal{T}(P_0, B) \leftarrow 0.$$

For example
$$\operatorname{Ext}^0_{\mathcal{T},\mathcal{I}} = \ker(\mathcal{T}(P_0, B) \to \mathcal{T}(P_1, B)).$$

The diagram

$$\begin{tikzcd} P_1 \arrow[r] & P_0 \arrow[r] \arrow[dr] & A \arrow[r] \arrow[d] & 0 \\ & & B & \end{tikzcd}$$

provides a natural map $\mathcal{T}(A,B) \to \operatorname{Ext}^0_{\mathcal{T},\mathcal{I}}(A,B)$. But this map is almost never invertible. For one thing, it is easy to see that its kernel is exactly $\mathcal{I}(A,B)$, so that we get $\mathcal{T}(A,B) \bigm/ \mathcal{I}(A,B) \subseteq \operatorname{Ext}^0_{\mathcal{T},\mathcal{I}}(A,B)$. The cokernel of this map is described in [**?**]. Furthermore, there is a natural injective map $\mathcal{I}(A[1],B) \bigm/ \mathcal{I}^2(A[1],B) \subseteq \operatorname{Ext}^1_{\mathcal{T},\mathcal{I}}(A,B)$. These two maps generalise the maps

$$\gamma\colon \operatorname{KK}_*(A,B) \to \operatorname{Hom}\bigl(\mathrm{K}_*(A), \mathrm{K}_*(B)\bigr), \qquad \ker\gamma \to \operatorname{Ext}^1\bigl(\mathrm{K}_{*+1}(A), \mathrm{K}_*(B)\bigr)$$

that appear in the Universal Coefficient Theorem—we will establish that the target groups here are $\operatorname{Ext}^j_{\mathcal{T},\mathcal{I}}(A[j],B)$ for $j = 0,1$. Unless the $\mathcal{I}$-projective resolution of A has length 1, so that all higher extension groups vanish, we cannot expect an exact sequence as in the Universal Coefficient Theorem for Kasparov theory. Instead, we merely get a spectral sequence (see [**?**]). But we will not discuss this in greater detail here.

In some of our examples, the computation of the derived functors reduces to one in the Abelian category $\mathcal{A}$, the target category of the functor $F\colon \mathcal{T} \to \mathcal{A}$ defining our homological ideal. The crucial additional condition for this is that $F \circ F^{\vdash}(B) \cong B$ for all projective objects B in $\mathcal{A}$.

Example 3.50. Let $\mathcal{I} = \ker \mathrm{K}_*$. For $A \in \mathrm{KK}$, there is a resolution of its K-theory

$$\cdots \to 0 \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathrm{K}_*(A) \to 0$$

by countable free Abelian groups. It is easy to see in this case that

$$\mathrm{KK}(\mathrm{K}^{\vdash}_*(P_1), \mathrm{K}^{\vdash}_*(P_0)) = \operatorname{Hom}_{\mathbf{Ab}^{\mathbb{Z}/2}}(P_1, P_0),$$

so that the boundary map d_1 lifts to a map $\widehat{d_1}\colon \mathrm{K}^{\vdash}_*(P_1) \to \mathrm{K}^{\vdash}_*(P_0)$. Furthermore, the adjointness relation $\mathrm{KK}(\mathrm{K}^{\vdash}_*(P_0), A) \cong \mathrm{KK}(P_0, \mathrm{K}_*(A))$ allows us to lift d_0 to $\widehat{d_0} \in \mathrm{KK}(\mathrm{K}^{\vdash}_*(P_0), A)$. Then

$$0 \to \mathrm{K}^{\vdash}_*(P_1) \to \mathrm{K}^{\vdash}_*(P_0) \to A \to 0$$

is an $\mathcal{I}$-projective resolution for $\mathcal{I} = \ker(\mathrm{K}_*)$. Both $\mathrm{K}^{\vdash}_*(P_0)$ and $\mathrm{K}^{\vdash}_*(P_1)$ are direct sums of $\mathbb{C}$ and $C_0(\mathbb{R})$, and $\mathrm{K}_*(\mathrm{K}^{\vdash}_*(P_j)) = P_j$. Hence we have lifted a projective resolution in $\mathbf{Ab}^{\mathbb{Z}/2}$ to an $\mathcal{I}$-projective resolution in KK.

By definition, if $B \in \mathrm{KK}$ then $\mathrm{Ext}^n_{\mathrm{KK},\ker(\mathrm{K}_*)}(A,B)$ is obtained by applying $\mathrm{KK}_*(-,B)$ to the projective resolution constructed above. Using the adjointness property of $\mathrm{K}^{\vdash}_*$, we see that this yields the cochain complex

$$\cdots \leftarrow 0 \leftarrow \mathrm{Hom}\big(P_1, \mathrm{K}_*(B)\big) \leftarrow \mathrm{Hom}\big(P_0, \mathrm{K}_*(B)\big) \leftarrow 0 \leftarrow \cdots,$$

which computes the extension groups of $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$. Thus

$$\mathrm{Ext}^n_{\mathrm{KK},\mathcal{I}}(A,B) = \begin{cases} 0 & \text{for } n \geq 2, \\ \mathrm{Ext}(\mathrm{K}_*(A), \mathrm{K}_*(B)) & \text{for } n = 1, \\ \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) & \text{for } n = 0. \end{cases}$$

The same things happen in general:

Theorem 3.51. *Let $\mathcal{T}$ be an idempotent complete triangulated category and let $\mathcal{A}$ be a stable Abelian category with enough projective objects. Let $F\colon \mathcal{T} \to \mathcal{A}$ be a stable homological functor whose left adjoint $F^{\vdash}$ is defined on all projective objects of $\mathcal{A}$. Assume, moreover, that the counit of adjunction yields isomorphisms $F \circ F^{\vdash}(B) \cong B$ for all projective objects B of $\mathcal{A}$. Then the following holds:*

(1) *F and $F^{\vdash}$ restrict to equivalences of categories between the full subcategories $\mathcal{P}_{\mathcal{I}}\mathcal{T}$ of $\mathcal{I}$-projective objects in $\mathcal{T}$ and $\mathcal{PA}$ of projective objects in $\mathcal{A}$.*

(2) *For any A in $\mathcal{T}$, a projective resolution of $F(A)$ in $\mathcal{A}$ lifts to an $\mathcal{I}$-projective resolution of A in $\mathcal{T}$, and this lifting is unique up to isomorphism.*

(3) $\mathrm{Ext}^*_{\mathcal{T},\mathcal{I}}(A,B) \cong \mathrm{Ext}^*_{\mathcal{A}}(F(A),F(B))$ *for all objects A and B of $\mathcal{T}$.*

(4) *Let $H\colon \mathcal{T} \to \mathcal{C}$ be a homological functor. Then there is, up to natural isomorphism, a unique right-exact functor $\overline{H}\colon \mathcal{A} \to \mathcal{C}$ with $\overline{H} \circ F(P) \cong H(P)$ for all $\mathcal{I}$-projective objects P of $\mathcal{T}$.*

    Let $\mathbb{L}_p\overline{H}\colon \mathcal{A} \to \mathcal{C}$ be the pth left derived functor of $\overline{H}$ and let $\mathbb{L}_pH\colon \mathcal{T} \to \mathcal{C}$ be the pth left derived functor of H, which is defined by applying H to a projective resolution and taking homology. Then $\mathbb{L}^{\mathcal{I}}_pH(A) \cong \mathbb{L}_p\overline{H}\big(F(A)\big)$ for all A in $\mathcal{T}$.

(5) *If $H\colon \mathcal{T} \to \mathcal{C}$ is a homological functor with $\mathcal{I} \subseteq \ker H$, then there is, up to natural isomorphism, a unique exact functor $\overline{H}\colon \mathcal{A} \to \mathcal{C}$ with $\overline{H} \circ F = H$.*

    That is, F is the universal *$\mathcal{I}$-exact homological functor.*

We refer to [**?**] for the proof of this theorem. The first two points say that the description of $\mathcal{I}$-projective objects and $\mathcal{I}$-projective resolutions in $\mathcal{T}$ is equivalent to the study of projective objects and resolutions in $\mathcal{A}$. This explains why the computation of $\mathcal{I}$- derived functors on $\mathcal{T}$ should reduce to the computation of derived functors on $\mathcal{A}$.

The last assertion in the theorem is particularly remarkable because the universal property uniquely characterises the functor F. It can be shown that for any homological ideal $\mathcal{I}$ with enough projective objects there is a universal $\mathcal{I}$-exact stable homological functor (this is due to Beligiannis [**?**]). But this is a mere existence result. Our theorem provides a sufficient condition to detect whether a given functor is the universal one. In fact, this condition is also necessary:

Theorem 3.52. *Let $\mathcal{I} \subseteq \mathcal{T}$ be a homological ideal with enough $\mathcal{I}$-projective objects, and let $F\colon \mathcal{T} \to \mathcal{A}$ be a stable homological functor with* $\ker F = \mathcal{I}$. *This functor is a universal $\mathcal{I}$-exact homological functor if and only if $\mathcal{A}$ has enough projective objects, the left adjoint $F^{\vdash}$ is defined on all projective objects of $\mathcal{A}$, and $F \circ F^{\vdash}(P) \cong P$ for all projctive objects P of $\mathcal{A}$.*

If the defining functor F is not yet universal, then we may replace it by the universal functor with the same morphism-kernel. Typically, this amounts to noticing that the functor F has some internal symmetries and refining its target category accordingly. The following examples illustrate this.

Example 3.53. Let $\mathcal{T} = \mathrm{KK}^G$ for a discrete group G and consider the functor $F\colon \mathrm{KK}^G \to \mathbf{Ab}^{\mathbb{Z}/2}$ that maps a C*-algebra A with G-action α to $\mathrm{K}_*(A)$. We have seen above that the left adjoint of this functor is defined on all free Abelian groups, so that $\mathcal{I} := \ker F$ has enough projective objects. But $F \circ F^{\vdash}(\mathbb{Z}^{\mathrm{even}}) = F(C_0(G)) = \mathbb{Z}[G]^{\mathrm{even}} \neq \mathbb{Z}^{\mathrm{even}}$ unless G is trivial. Thus F is not yet universal.

This is not surprising because derived functors on the target category of F do not really see the group action at all. To remedy the situation, we notice that the G-action on A induces an action of G on $\mathrm{K}_*(A)$, so that $\mathrm{K}_*(A)$ becomes a module over $\mathbb{Z}[G]$. Furthermore, this module is countable. Let $\mathcal{A}$ be the category of all countable $\mathbb{Z}/2$-graded $\mathbb{Z}[G]$-modules and let $\widetilde{F}\colon \mathrm{KK}^G \to \mathcal{A}$ be F enriched appropriately. Then $\ker \widetilde{F} = \ker F$. We claim that $\widetilde{F}$ is universal. Since

$$\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathrm{K}_*(B)) \cong \mathrm{K}_*(B) \cong \mathrm{KK}^G_*(C_0(G), B),$$

we get $\widetilde{F}^{\vdash}(\mathbb{Z}[G]) = C_0(G)$. This implies that $\widetilde{F}^{\vdash}$ is defined on all projective objects of $\mathcal{A}$. Now $\widetilde{F} \circ \widetilde{F}^{\vdash}(\mathbb{Z}[G]) = \mathbb{Z}[G]$, which is the desired answer. Thus $\widetilde{F}$ is universal.

Example 3.54. Let $\mathcal{T}$ be the homotopy category of chain complexes over the Abelian category of R-modules for some ring R, and let $\mathcal{I}$ be the kernel of the homology functor. This is also the kernel of the stable homological functor

$$F = \mathrm{H}_*\colon \mathrm{Ho}(R - \mathbf{Mod}) \to \mathbf{Ab}^{\mathbb{Z}},$$

where we forget the R-module structure on the homology. The latter is not universal, of course. It is easy to check that the associated universal $\mathcal{I}$-exact homological functor is $\mathrm{H}_*\colon \mathrm{Ho}(R - \mathbf{Mod}) \to (R - \mathbf{Mod})^{\mathbb{Z}}$. Hence the passage to the universal functor recovers the category of $\mathbb{Z}$-graded R-modules and the homology functor with values in this category from F.

Example 3.55. Let (C, Δ) be a discrete quantum group; for instance, we could consider $C = C_0(G)$ for a discrete group G or $C = C^*(G)$ for a compact group G. The first case is already discussed in Example 3.53. Let $\mathcal{T} = \mathrm{KK}^{(C,\Delta)}$ be the associated bivariant Kasparov category, and let $F(A, \Delta_A) = \mathrm{K}_*(A)$ for a separable C*-algebra with coaction $\Delta_A\colon A \to \mathcal{M}(A \otimes C)$. We get $F^{\vdash}(\mathbb{Z}^{\mathrm{even}}) = C$ with coaction Δ because

$$\mathrm{KK}^{(C,\Delta)}(C, B) \cong \mathrm{KK}(\mathbb{C}, B) \cong \mathrm{K}_0(B).$$

Hence the left adjoint $F^{\vdash}$ is defined on all free Abelian groups, so that $\mathcal{I} = \ker F$ has enough projective objects. As in Example 3.53, $F \circ F^{\vdash}(\mathbb{Z}^{\mathrm{even}}) = \mathrm{K}_*(C) \neq \mathbb{Z}^{\mathrm{even}}$, so that F is not yet universal. What additional structure on $\mathrm{K}_*(B)$ does a coaction of (C, Δ) entail?

Clearly, $\mathrm{KK}_*^{(C,\Delta)}(C,B)$ is a module over the $\mathbb{Z}/2$-graded ring $R(C) := \mathrm{KK}_*^{(C,\Delta)}(C,C)$ by Kasparov composition product. Let $\mathcal{A}$ be the category of countable $R(C)$-modules. A purely formal computation shows that $\tilde{F}\colon \mathrm{KK}^{(C,\Delta)} \to \mathcal{A}$, $B \mapsto \mathrm{KK}^{(C,\Delta)}(C,B)$ is universal: this works whenever F is representable and commutes with direct sums.

It remains to understand the ring $R(C)$. Since discrete and compact quantum groups are strongly regular, Baaj–Skandalis duality implies

$$\mathrm{KK}_*^{(C,\Delta)}(C,C) \cong \mathrm{KK}_*^{(\widehat{C}^c,\widehat{\Delta}^c)}(C \rtimes \widehat{C}^c, C \rtimes \widehat{C}^c) \cong \mathrm{KK}_*^{(\widehat{C}^c,\widehat{\Delta}^c)}(\mathbb{C},\mathbb{C}) \cong \mathrm{K}_*(C),$$

where we equip $\mathbb{C}$ with the trivial coaction of $\widehat{C}^c$ and, in the last step, use the Green–Julg Theorem for the compact quantum group $\widehat{C}^c$. The resulting ring is the *representation ring* of $\widehat{C}^c$, that is, the ring of finite-dimensional representation with tensor product as multiplication. If $C = C_0(G)$ for a discrete group G, this is just the group ring $\mathbb{Z}[G]$; if $C = C^*(G)$ for a compact group G, this is the representation ring of G.

3.9 From homological ideals to complementary pairs of subcategories

The main goal of this section is to understand the following theorem.

Theorem 3.56. *Let $\mathcal{T}$ be a triangulated category with direct sums and let $F\colon \mathcal{T} \to \mathcal{A}$ be a stable homological functor into some Abelian category that commutes with direct sums. Assume also that the left adjoint of F is defined on all projecive objects of $\mathcal{A}$, so that the ideal $\mathcal{I} = \ker F$ has enough projective objects. Let $\mathcal{P} \subseteq \mathcal{T}$ be the class of projective objects and let $\mathcal{N}$ be the object-kernel of F; let $\langle\mathcal{P}\rangle$ be the localising subcategory generated by $\mathcal{P}$. Then the pair of subcategories $(\langle\mathcal{P}\rangle, \mathcal{N})$ is complementary.*

This theorem applies in numerous cases:

- Let $\mathcal{A}$ be an Abelian category with enough projective objects and exact direct sums. Let $\mathcal{T} = \mathrm{Ho}(\widetilde{\mathcal{A}})$ and let $F\colon \mathcal{T} \to \widetilde{\mathcal{A}}^{\mathbb{Z}}$ be the homology functor. Then $\mathcal{P}$ consists of chain complexes with vanishing boundary map and projective entries, and $\mathcal{N}$ consists of the exact chain complexes. Theorem 3.56 specialises to Spaltenstein's Theorem in this case.
- Let $\mathcal{T} = \mathrm{KK}$ and let F be the K-theory functor. Again, Theorem 3.56 applies. Here $\mathcal{P}$ consists of direct sums of suspensions of $\mathbb{C}$, so that $\langle\mathcal{P}\rangle$ is the bootstrap category, and $\mathcal{N}$ consists of those separable C*-algebras with vanishing K-theory.
- Let $\mathcal{T} = \mathrm{KK}^{(C,\Delta)}$ for a discrete quantum group (C,Δ) and define $F\colon \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$ by $F(B,\beta) = \mathrm{K}_*(B)$. In this case, $\langle\mathcal{P}\rangle$ is the localising subcategory generated by $F^{\vdash}(\mathbb{Z}^{\mathrm{even}}) = (C,\Delta)$, and $\mathcal{N}$ contains all coactions of (C,Δ) on separable C*-algebras with vanishing K-theory.

If $P \in \mathcal{P}$ is $\mathcal{I}$-projective and $N \in \mathcal{N}$, that is, $\mathrm{id}_N \in \mathcal{I}$, then $\mathcal{T}(P,N) = 0$ because $\mathcal{I}$ acts by 0 on $\mathcal{T}(P,-)$. Since the left orthogonal complement $\mathcal{N}^{\vdash}$ of $\mathcal{N}$ is localising, this implies $\langle\mathcal{P}\rangle \subseteq \mathcal{N}^{\vdash}$. To prove Theorem 3.56, it remains to show that for $A \in \mathcal{T}$ can be embedded in an exact triangle $L \to A \to N \to L[1]$ with $L \in \mathcal{L} := \langle \mathcal{P} \rangle$ and $N \in \mathcal{N}$. This is established in [mn08]. Here we only prove a weaker result, namely, that $(\mathcal{N}^{\vdash}, \mathcal{N})$ is complementary. Some more work is necessary to identify $\mathcal{N}^{\vdash} = \mathcal{L}$; in particular, this is the only point in this lecture where the octahedral axiom is used.

The main ingredient in the proof is the phantom tower (it has this name because maps in $\ker F$ are also called *phantom maps*).

Definition 3.57. *Let $B \in \mathcal{T}$. The* phantom tower *is a diagram of the form*

$$\begin{tikzcd}
B \arrow[r, equal] & N_0 \arrow[rr, "\iota_0^1"] & & N_1 \arrow[rr, "\iota_1^2"] \arrow[dl, "\circ" marking] & & N_2 \arrow[rr, "\iota_2^3"] \arrow[dl, "\circ" marking] & & \cdots \arrow[dl, "\circ" marking] \\
& & P_0 \arrow[ul] & & P_1 \arrow[ul] \arrow[ll, "\circ" marking] & & P_2 \arrow[ul] \arrow[ll, "\circ" marking] &
\end{tikzcd} \tag{3.2}$$

where the objects P_n are $\mathcal{I}$-projective, $\iota_n^{n+1} \in \mathcal{I}$, the triangles

$$\begin{tikzcd}
N_n \arrow[rr, "\iota_n^{n+1}"] & & N_{n+1} \arrow[dl, "\circ" marking] \\
& P_n \arrow[ul] &
\end{tikzcd}$$

are exact, and the remaining triangles commute. The circled arrows denote maps of degree 1*, that is, they actually map $N_1 \to P_0[1]$, and so on.*

The bottom row

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \cdots$$

in a phantom tower is a chain complex with differential of degree 1.

Proposition 3.58. *Given a phantom tower as in (3.2), the complex*

$$B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \cdots$$

is an $\mathcal{I}$-projective resolution. Conversely, any projective resolution embeds in a phantom tower, which is unique up to isomorphism.

Proof. First we must check that the sequence $B \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ is $\mathcal{I}$-exact. We know that

$$F_{*+1}(N_{j+1}) \rightarrowtail F_*(P_j) \twoheadrightarrow F_*(N_j)$$

is a short exact sequence because $F_*(\iota_j^{j+1}) = 0$. The Yoneda product of these extensions is an exact chain complex of the form

$$F_*(B) \leftarrow F_*(P_0) \leftarrow F_*(P_1) \leftarrow \cdots .$$

Thus the bottom row in the phantom tower is an $\mathcal{I}$-projective resolution.

Conversely, take a projective resolution

$$B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \cdots .$$

We recursively construct N_j starting with $N_0 = B$, and such that the maps $P_j \to N_j$ are $\mathcal{I}$-epimorphisms. In each step, we embed $N_j \leftarrow P_j$ in an exact triangle

$$P_j \to N_j \xrightarrow{\iota_j^{j+1}} N_{j+1} \to P_j[1].$$

The induction assumption yields that the map $P_j \to N_j$ is an $\mathcal{I}$-epimorphism, so that $\iota_j^{j+1} \in \mathcal{I}$. To complete the induction step, we must lift the boundary map $P_{j+1} \to P_j[1]$ to a map $P_{j+1} \to N_{j+1}$ and check that this lifting is an $\mathcal{I}$-epimorphism.

The first map in the exact sequence

$$\mathcal{T}(P_{j+1}, N_j) \to \mathcal{T}(P_{j+1}, N_{j+1}) \to \mathcal{T}(P_{j+1}, P_j[1]) \to \mathcal{T}(P_{j+1}, N_j[1])$$

vanishes because P_{j+1} is $\mathcal{I}$-projective and $\iota_j^{i+1} \in \mathcal{I}$; hence the second map is injective. This shows that our lifting is unique once it exists. Since we started with a chain complex, the composition $P_{j+1} \to P_j[1] \to P_{j-1}[2]$ vanishes. Since the map $N_j \to P_{j-1}[1]$ from the previous step is an $\mathcal{I}$-monomorphism, $\mathcal{T}(P_{j+1}, N_j[1]) \subseteq \mathcal{T}(P_{j+1}, P_{j-1}[2])$, so that there is no obstruction to lifting the boundary map. Finally, it is routine to check that the unique lifting is indeed an $\mathcal{I}$-epimorphism. $\square$

Now we check that $(\mathcal{N}^{\vdash}, \mathcal{N})$ is complementary. Equivalently, for each $A \in \mathcal{T}$ there is $N \in \mathcal{N}$ and a map $f\colon A \to N$ that induces isomorphisms $\mathcal{T}_*(N, M) \to \mathcal{T}_*(A, M)$ for all $M \in \mathcal{N}$. This means that $A \mapsto N$ is a functor $\mathcal{T} \to \mathcal{N}$ that is left adjoint to the embedding functor $\mathcal{N} \to \mathcal{T}$.

We let N be the homotopy colimit of the phantom tower. Recall that this is defined by an exact triangle

$$\bigoplus_j N_j \xrightarrow{\mathrm{id}-S} \bigoplus_j N_j \to \underrightarrow{\mathrm{holim}}\, N_j \to \bigoplus_j N_j[1], \qquad S = \bigoplus_j \iota_j^{j+1}.$$

Since F commutes with direct sums and $\iota_j^{j+1} \in \ker F$, we get $F(S) = 0$. Therefore, $F(\mathrm{id} - S) = F(\mathrm{id})$ is invertible. Hence $F(\underrightarrow{\mathrm{holim}}\, N_j) = 0$. this means that $N := \underrightarrow{\mathrm{holim}}\, N_j \in \mathcal{N}$.

Let $M \in \mathcal{N}$. Then $\mathcal{T}_*(P_j, M) = 0$ because P_j is $\ker F$-projective. Therefore ι_j^{j+1} induces an isomorphism $\mathcal{T}_*(N_{j+1}, M) \xrightarrow{\cong} \mathcal{T}_*(N_j, M)$. Now we recall the Milnor sequence

$$\underleftarrow{\lim}^1 \mathcal{T}_{*-1}(N_j, M) \rightarrowtail \mathcal{T}_*(\underrightarrow{\mathrm{holim}}\, N_j, M) \twoheadrightarrow \underleftarrow{\lim}\, \mathcal{T}_*(N_j, M).$$

Since the maps in the relevant projective systems are all invertible, $\underleftarrow{\lim}^1 \mathcal{T}_{*-1}(N_j, M) = 0$ and $\underleftarrow{\lim}\, \mathcal{T}_{*-1}(N_j, M) \cong \mathcal{T}_*(A, M)$. Hence N has the required properties.

3.10 Localisation of functors

Let $\mathcal{T}$ be a triangulated category and let $(\mathcal{L}, \mathcal{N})$ be a complementary pair of thick subcategories as in the previous section. Let $F\colon \mathcal{T} \to \mathcal{A}$ be a homological functor with values in some Abelian category $\mathcal{D}$.

For instance, we could take $\mathcal{T} = \mathrm{KK}$, $\mathcal{L}$ the bootstrap category, $\mathcal{N}$ the class of all separable C*-algebras with vanishing K-theory, and $F(A) := F(A \otimes B)$ for some fixed C*-algebra B.

Recall that there are functors

$$L\colon \mathcal{T} \to \mathcal{L}, \quad N\colon \mathcal{T} \to \mathcal{N}$$

and natural exact triangles

$$L(A) \to A \to N(A) \to L(A)[1]$$

Definition 3.59. *The* localisation of the functor F *at* $\mathcal{N}$*, denoted* $\mathbb{L}F$*, is the functor*

$$\mathbb{L}F := F \circ L\colon \mathcal{T} \to \mathcal{A}$$

We may also view this as a functor on $\mathcal{T}/\mathcal{N}$ because L descends to a functor $\mathcal{T}/\mathcal{N} \to \mathcal{L}$. The natural map $L(A) \to A$ induces a natural transformation $\mathbb{L}F \to F$.

Proposition 3.60. *$\mathbb{L}F \to F$ is universal among natural transformations $G \to F$ with G homological and $G|_{\mathcal{N}} = 0$. That is, any natural transformation from such a functor to F factors uniquely through $\mathbb{L}F \to F$:*

$$\begin{tikzcd} G \arrow[r] \arrow[dr, dotted, "\exists!"'] & F \\ & \mathbb{L}F \arrow[u] \end{tikzcd}$$

Proof. To construct the natural transformation, combine the inverse of the isomorphism

$$G(P(B)) \xrightarrow{\cong} G(B)$$

with the natural map

$$G(P(B)) \to F(P(B)) = \mathbb{L}F(B).$$

It is easy to see that this is the only natural transformation $G \to \mathbb{L}F$ that yields the desired factorisation. □

This universal property is used as a definition if $\mathcal{N}$ is not part of a complementary pair. Roughly speaking, the localisation $\mathbb{L}F$ is the best approximation to F that vanishes on $\mathcal{N}$.

Corollary 3.61. *The transformation $\mathbb{L}F \to F$ is invertible if and only if $F|_{\mathcal{N}} = 0$.*

In practice, $\mathcal{L}$ is usually the localising subcategory $\langle \mathcal{P} \rangle$ generated by a smaller class $\mathcal{P}$ of objects.

Proposition 3.62. *Let $(\langle \mathcal{P} \rangle, \mathcal{N})$ be a complementary pair of thick subcategories, let G and F be homological functors on $\mathcal{T}$ that commute with direct sums, and let $\Phi\colon G \to F$ be a natural transformation. Assume that $G|_{\mathcal{N}} = 0$ and that $\Phi_B\colon G(B) \to F(B)$ is invertible for all $B \in \mathcal{P}$. Then Φ descends to a natural isomorphism $G \cong \mathbb{L}F$.*

Proof. The previous proposition yields a natural transformation $\Psi\colon G \to \mathbb{L}F$. This is invertible on $\mathcal{P}$ because $\mathbb{L}F(B) \cong F(B)$ for $B \in \mathcal{P}$. Since G and $\mathbb{L}F$ are homological and commute with direct sums, the class of objects where Ψ is invertible is localising. Hence it contains $\mathcal{L} = \langle \mathcal{P} \rangle$. It also contains $\mathcal{N}$ because both G and $\mathbb{L}F$ vanish on $\mathcal{N}$. Thus it contains $\mathcal{T}$. $\square$

Example 3.63 (for Proposition 3.62). Let G be a discrete torsion-free group, $\mathcal{T} = \mathrm{KK}^G$,

$$\mathcal{N} = \{A \in \mathrm{KK}^G \mid A \cong 0 \text{ in } \mathrm{KK}\,(\text{non-equivariantly})\},$$
$$\mathcal{P} = \{C_0(G, A) \text{ with the free } G\text{-action of C*-algebra}\}.$$

Take $F(A) := \mathrm{K}_*(G \ltimes_r A)$ and a natural transformation $\Phi\colon G \implies F$. Then if $G|_{\mathcal{N}} = 0$ and $\Phi|_{\mathcal{P}}$ is invertible, then $G \cong \mathbf{L}F$, and Φ is equivalent to the Baum-Connes assembly map.

3.11 The Baum–Connes conjecture

Usually, we do not expect the map $\mathbb{L}F \to F$ that compares a functor to its localisation to be an isomorphism. In noncommutative topology, this sometimes happens for rather deep reasons. Here we explain how this is related to the Baum–Connes conjecture (see [mn06] for more details).

Let $\mathcal{T} = \mathrm{KK}^G$ for a locally compact group G. In order to get started, we need a homological ideal $\mathcal{I}$ in $\mathcal{T}$. We let $f \in \mathcal{I}$ if $\mathrm{Res}_G^H(f) = 0$ for all compact subgroups $H \subseteq G$, where $\mathrm{Res}_G^H\colon \mathrm{KK}^G(A,B) \to \mathrm{KK}^H(A,B)$ denotes the restriction functor. The ideal $\mathcal{I}$ is, by definition, the kernel of an exact functor, not of a stable homological functor. But it can be shown that morphism-kernels of exact functors are homological ideals as well (see [**?**]). In order to find enough projective objects for this ideal, we would like the restriction functor Res_G^H to have a left adjoint. This works out nicely if $H \subseteq G$ is open, so that G/H is discrete: in that case, the induction functor $\mathrm{Ind}_H^G\colon \mathrm{KK}^H \to \mathrm{KK}^G$ is left adjoint to Res_G^H. More generally, if G/H is a smooth manifold, then the left adjoint is still defined on sufficiently many objects of KK^H and sufficiently close to the induction functor for the following arguments to go through. Since any locally compact group contains sufficiently many subgroups H for which G/H is a smooth manifold, the following discussion can be carried over to general locally compact groups. However, we only prove assertions for discrete groups to simplify the discussion.

The localising subcategory $\langle \mathcal{P} \rangle$ generated by the $\mathcal{I}$-projective objects is the localising subcategory generated by *compactly induced* actions, that is, objects of the form $\mathrm{Ind}_H^G(A)$ for a compact subgroup $H \subseteq G$ and $A \in \mathrm{KK}^H$; the proof uses that such objects are $\mathcal{I}$-projective by the adjointness of induction and restriction, and that any object A admits an $\mathcal{I}$-epimorphism from $\bigoplus \mathrm{Ind}_H^G \mathrm{Res}_G^H A$ onto it, where H runs through a suitable set of compact subgroups of G. The object-kernel of our homological ideal is the subcategory $\mathcal{N}$ of all G-C*-algebras N with $\mathrm{Res}_G^H(N) \cong 0$ in KK^H for all compact subgroups H of G. Theorem 3.56 shows that $(\langle \mathcal{P} \rangle, \mathcal{N})$ is complementary. In many examples, $\langle \mathcal{P} \rangle$ consists of all G-C*-algebras that are KK^G-equivalent to one on which G acts properly (in fact, this is probably always true, but the proof is not written down in complete generality).

Both $\mathcal{P}$ and $\mathcal{N}$ are closed under tensor products with arbitrary G-C*-algebras. Hence it suffices to study a single exact triangle of the form $L \to \mathbb{C} \to N \to L[1]$ with $L \in \langle \mathcal{P} \rangle$ and $N \in \mathcal{N}$: tensoring with A yields such a triangle for any A. The map $L \to \mathbb{C}$ is called a *Dirac morphism* in [mn06].

Next we compare the Baum–Connes assembly map with the localisation at $\mathcal{N}$. This is the map from

$$\mathrm{K}_*^{\mathrm{top}}(G,B) := \varinjlim_X \mathrm{KK}^G(C_0(X),B), \qquad X \ G\text{-compact in } \underline{E}G,$$

to $\mathrm{K}_*(C_\mathrm{r}^*G)$ defined as follows:

$$\mathrm{KK}_*^G(C_0(X),B) \to \mathrm{KK}_*(G \ltimes_\mathrm{r} C_0(X), G \ltimes_\mathrm{r} B) \to \mathrm{K}_*(G \ltimes_\mathrm{r} B),$$

where the first map is the descent functor and the second map is induced by a canonical class in $\mathrm{K}_0(G \ltimes_\mathrm{r} C_0(X))$ for any proper cocompact G-space called *Mishchenko line bundle*.

See also Theorem 2.18 in Chapter 2.

Theorem 3.64. *Let $\mathcal{T} = \mathrm{KK}^G$ for a locally compact group G, let $\mathcal{N}$ be as above, and let $F(B) = \mathrm{K}_*(G \ltimes_\mathrm{r} B)$. Let $\mathbb{L}F$ be the localisation of F at $\mathcal{N}$. Then there is a natural isomorphism $\mathbb{L}F(B) \cong \mathrm{K}_*^{\mathrm{top}}(G,B)$ that intertwines the canonical map $\mathbb{L}F \to F$ and the Baum–Connes assembly map with coefficients in B. That is, the Baum–Connes assembly map is equivalent to the natural transformation $\mathbb{L}F(B) \to F(B)$.*

Proof. The proof is based on some (non-trivial) formal properties of the Baum–Connes assembly map. First, $\mathrm{K}_*^{\mathrm{top}}(G,B)$ vanishes for $B \in \mathcal{N}$; even more, $\mathrm{K}_*^{\mathrm{top}}(G,B) = 0$ if $\mathrm{K}_*(H \ltimes B) = 0$ for all compact subgroups $H \subseteq G$ because there is a spectral sequence computing $\mathrm{K}_*^{\mathrm{top}}(G,B)$ whose first page consists of groups of the form $\mathrm{K}_*(H \ltimes B)$. Secondly, the Baum–Connes assembly map with coefficients B is invertible if G acts properly on B; in particular, it is invertible if B is of the form $\mathrm{Ind}_H^G(B_0)$ for some compact subgroup H. These two facts plugged into Proposition 3.62 yield the assertion. □

This description of the Baum–Connes assembly map with coefficients yields several reformulations of the Baum–Connes property with coefficients:

Corollary 3.65. *Let G be a locally compact group. The following assertions are equivalent:*

(1) *the Baum–Connes assembly map is an isomorphism for all coefficient algebras B;*

(2) $\mathrm{K}_*(G \ltimes_\mathrm{r} B) = 0$ *if B is H-equivariantly contractible for any compact subgroup $H \subseteq G$;*

(3) *if a G-equivariant *-homomorphism $f\colon A \to B$ is an H-homotopy equivalence for all compact subgroups (that is, it has an H-equivariant inverse up to H-equivariant homotopies), then it induces an isomorphism* $\mathrm{K}_*(G \ltimes_\mathrm{r} A) \cong \mathrm{K}_*(G \ltimes_\mathrm{r} B)$;

(4) $\mathrm{K}_*(G \ltimes_\mathrm{r} B) = 0$ *for all $B \in \mathcal{N}$;*

(5) *if* $f \in \mathrm{KK}_0^G(A,B)$ *becomes invertible in* KK^H *for all compact subgroups* $H \subseteq G$*, then it induces an isomorphism* $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A) \cong \mathrm{K}_*(G \ltimes_{\mathrm{r}} B)$*;*

(6) $\mathrm{K}_*(G \ltimes_{\mathrm{r}} B) = 0$ *if* $\mathrm{K}_*(H \ltimes B) = 0$ *for all compact subgroups* $H \subseteq G$*;*

(7) *if* $f \in \mathrm{KK}_0^G(A,B)$ *induces an isomorphism* $\mathrm{K}_*(H \ltimes A) \cong \mathrm{K}_*(H \ltimes B)$ *for all compact subgroups* $H \subseteq G$*, then it induces an isomorphism* $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A) \cong \mathrm{K}_*(G \ltimes_{\mathrm{r}} B)$*.*

Proof. The equivalence of (1) and (4) is Corollary 3.61. Any object of $\mathcal{N}$ is KK^G-equivalent to a G-C*-algebra as in (2). Therefore, (2) and (4) are equivalent. It is clear that (6) implies (4). Conversely, (1) implies (6) because $\mathrm{K}_*^{\mathrm{top}}(G,B)$ vanishes if B is as in (6). Therefore, (1), (2), (4), and (6) are equivalent. Trivially, (7)$\Longrightarrow$(5)$\Longrightarrow$(3). Moreover, (3) implies (2) because if B is as in (2) then we may apply (3) to the zero map $0 \to B$.

Thus it only remains to check that (6) implies (7). If f is as in (7), then we may embed it in an exact triangle $A \to B \to C \to A[1]$ using Axiom (TR1). The long exact sequence for the homological functor $\mathrm{K}_*(H \ltimes -)$ shows that $\mathrm{K}_*(H \ltimes C) = 0$ for all compact subgroups H of G. Now (6) yields $\mathrm{K}_*(G \ltimes_{\mathrm{r}} C) = 0$. Finally, the long exact sequence for the homological functor $\mathrm{K}_*(G \ltimes_{\mathrm{r}} -)$ shows that $\mathrm{K}_*(G \ltimes_{\mathrm{r}} f)$ is invertible as asserted in (7). □

Let us consider the case where G is torsion-free, that is, G has no compact subgroups besides the trivial group. Then in all statements above H can only be the trivial group. Thus the Baum–Connes conjecture is equivalent to the following rigidity property: if $f\colon A \to B$ is a G-equivariant *-homomorphism that is also a (non-equivariant) homotopy equivalence, then f induces an isomorphism $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A) \cong \mathrm{K}_*(G \ltimes_{\mathrm{r}} B)$.

Theorem 3.66 (Higson–Kasparov)**.** *The Baum–Connes conjecture with coefficients holds for all amenable groups. Even more, if* G *is amenable, then* $\mathcal{N} = 0$*, that is,* $A \cong 0$ *if* $\mathrm{Res}_G^H(A) \cong 0$ *for all compact subgroups* $H \subseteq G$*.*

In particular, this theorem applies to all Abelian groups such as $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. The stronger statement in the second statement of the theorem is equivalent, in the terminology of the dual Dirac method, to the statement that the γ-element of the group G exists and is equal to 1. This is known to be false for groups with property (T) such as higher rank Lie groups and lattices in such groups.

Example 3.67. Let $\mathbb{Z}$ act by translation on $\mathbb{R}$; we extend this action to the half-open interval $(-\infty,\infty]$ by fixing $+\infty$. Although it is has no equivariant linear section, the extension

$$C_0(\mathbb{R}) \to C_0((-\infty,\infty]) \to \mathbb{C}$$

has a class in $\mathrm{KK}_1^{\mathbb{Z}}(C_0(\mathbb{R}),\mathbb{C})$ because this extension acquires an equivariant completely positive contractive section after we stabilise it by $\mathcal{K}(\ell^2\mathbb{Z})$ (see [**?**] for a proof using Baaj–Skandalis duality). The C*-algebra $C_0((-\infty,\infty])$ in the middle is non-equivariantly contractible and therefore belongs to $\mathcal{N}$. The Higson–Kasparov Theorem in this case predicts that $C_0((-\infty,\infty]) \cong 0$ in $\mathrm{KK}^{\mathbb{Z}}$. Thus the boundary map of the extension provides an invertible element in $\mathrm{KK}_1(\mathbb{C}, C_0(\mathbb{R}))$. Since the exterior tensor product defines a functor on $\mathrm{KK}^{\mathbb{Z}}$, we get an invertible element in $\mathrm{KK}_1(A, C_0(\mathbb{R}) \otimes A)$ for any $\mathbb{Z}$-C*-algebra A. Thus any $\mathbb{Z}$-C*-algebra is $\mathrm{KK}^{\mathbb{Z}}$-equivalent to one with a proper action of $\mathbb{Z}$, namely, $C_0(\mathbb{R}) \otimes SA$.

Since the action of $\mathbb{Z}$ on $\mathbb{R}$ is free and proper, the crossed product $\mathbb{Z} \ltimes (C_0(\mathbb{R}) \otimes A) = \mathbb{Z} \ltimes_{\mathrm{r}} (C_0(\mathbb{R}) \otimes A)$ is Morita equivalent to the generalised fixed point algebra

$$T_\alpha := \{f \in C_b(\mathbb{R}, A) \mid f(x+1) = \alpha f(x)\} \cong \{f \in C([0,1], A) \mid f(1) = \alpha(f(0))\},$$

where $\alpha \in \mathrm{Aut}(A)$ describes the $\mathbb{Z}$-action. This C*-algebra is called the *mapping torus* of α. It is commutative if A is commutative. The Higson–Kasparov Theorem implies that the mapping torus is KK-equivalent to the suspension of the crossed product $\mathbb{Z} \ltimes A$, so that the K-theories agree up to a dimension shift. From this, it is easy to deduce the Pimsner–Voiculescu exact sequence for crossed products by $\mathbb{Z}$ for any homological invariant for C*-algebras.

The only step in the above reasoning that is non-trivial is that $C_0((-\infty, \infty]) \cong 0$ in $\mathrm{KK}^{\mathbb{Z}}$ or, equivalently, that the class η of the resulting extension in $\mathrm{KK}_1^{\mathbb{Z}}(\mathbb{C}, C_0(\mathbb{R}))$ is invertible. This latter fact can be checked by hand by writing down a candidate $D \in \mathrm{KK}_1^{\mathbb{Z}}(C_0(\mathbb{R}), \mathbb{C})$ for the inverse and checking that the two resulting Kasparov products are homotopic to the identity in $\mathrm{KK}_0^{\mathbb{Z}}(\mathbb{C}, \mathbb{C})$ and $\mathrm{KK}_0^{\mathbb{Z}}(C_0(\mathbb{R}), C_0(\mathbb{R}))$. We only remark here that the inverse is the equivariant K-homology class of the Dirac operator on $\mathbb{R}$. Up to a dimension shift, this is the Dirac morphism of $\mathbb{Z}$, and η is the dual Dirac morphism. If we forget the $\mathbb{Z}$-action, then D and η are exactly the classes that generate the Bott periodicity isomorphisms; the fact that they are inverse to each other is a well-known fact of index theory. It is possible to prove the Higson–Kasparov Theorem for $\mathbb{Z}$ by going through this proof of Bott periodicity and checking that all the relevant constructions are sufficiently $\mathbb{Z}$-equivariant to carry them over from KK to $\mathrm{KK}^{\mathbb{Z}}$.

3.12 Towards an analogue of the Baum–Connes conjecture for quantum groups

The approach to the Baum–Connes assembly map outlined above is particularly suitable to extend the Baum–Connes conjecture to other objects than groups such as quantum groups. For this purpose, the main achievement is that our definitions no longer use proper actions—whose meaning for quantum groups is unclear—but only compact subgroups. Nevertheless, the correct analogue of the family of compact subgroups for a quantum group is still unclear because quantum groups may have too few subgroups, so that more general objects than subgroups have to be allowed.

Before we discuss this issue, we restrict attention to the case of a torsion-free discrete quantum group (C, Δ). We let $\mathcal{I}$ be the kernel of the restriction functor $\mathrm{KK}^C \to \mathrm{KK}$ (we drop the coactions from our notation to avoid clutter). Since (C, Δ) is assumed discrete, this restriction functor has a left adjoint, which maps $A \in \mathrm{KK}$ to $A \otimes C$ with the coaction $\mathrm{id}_A \otimes \Delta$; that is, $\mathrm{KK}_*^C(A \otimes C, B) \cong \mathrm{KK}_*(A, B)$ for any separable C*-algebra B and any coaction of (C, Δ) on B. We may argue exactly as in the group case now. Let $\mathcal{P}$ be the class of objects of KK^C of the form $A \otimes C$, and let $\mathcal{N}$ consist of all objects A of KK^C with $A \cong 0$ in KK, that is, after forgetting the coaction. Then $(\langle \mathcal{P} \rangle, \mathcal{N})$ is a complementary pair of localising subcategories of KK^C. The Baum–Connes assembly map for (C, Δ) with coefficients in B is the natural map $\mathbb{L}F(B) \to F(B)$, where $F(B)$ is the K-theory of the reduced crossed product, $F(B) := \mathrm{K}_*(B \rtimes_{\mathrm{r}} \widehat{C}^c)$. This is the unique natural transformation $G \to F$ that is invertible on objects of $\mathcal{P}$ for a functor G that vanishes on $\mathcal{N}$. Thus, to check that some other construction of an assembly map $G \to F$ agrees with this one, it suffices to check that $G(N) = 0$ for $N \in \mathcal{N}$ and that $G(B) \to F(B)$ is invertible for $B \in \mathcal{P}$.

The above assembly map can be constructed for any discrete quantum group, torsion-free or not. But for quantum groups with torsion, we do not expect it to be invertible. In the group case, this map is the classical assembly map $\mathrm{K}_*(BG) \to \mathrm{K}_*(C *_{\mathrm{r}} G)$, where BG is the usual . This is never an isomorphism unless G is torsion-free. Nevertheless, there are several interesting quantum groups where it is reasonable to conjecture that the assembly map above is invertible or, equivalently, that $\mathrm{K}_*(B \rtimes_{\mathrm{r}} \widehat{C}^c) = 0$ for $B \in \mathcal{N}$.

Here we only consider the case of duals of compact groups, following [**?**]. The quantum group $C^*(G)$ for a group G is discrete if and only if G is compact. Since any open normal subgroup H in G yields a finite quantum subgroup $C^*(G/H)$ in $C^*(G)$, the quantum group $C^*(G)$ certainly has torsion unless G is connected. Somewhat surprisingly, this assumption is not yet enough: we also need that G should have no non-trivial projective representations. This is equivalent to the assumption that G have torsion-free fundamental group. Thus we assume now that G is a connected compact group with torsion-free fundamental group. We also assume G to be a Lie group to simplify the discussion, although this assumption may be removed. We can use, for example, $G = \mathrm{SU}(n)$ or $G = \mathbb{T}^n$. Since $C^*(G)$ is amenable, it makes no difference whether we use reduced or full crossed products, and the Higson–Kasparov Theorem for groups leads us to expect the following theorem:

Theorem 3.68 ([**?**]). *Let G be a connected Lie group with torsion-free fundamental group and let A be a C^*-algebra with a coaction $\delta \in \mathrm{Mor}(A, A \otimes C^*(G))$.*
If $\mathrm{K}_*(A) = 0$, then $\mathrm{K}_*(A \rtimes C_0(G)) = 0$.
If $A \cong 0$ in KK, then $A \cong 0$ in $\mathrm{KK}^{C^*(G)}$.

The first statement follows, in fact, from the second one and an inspection of $\mathcal{I}$-projective resolutions. Let $L\colon \mathrm{KK}^{C^*(G)} \to \langle \mathcal{P} \rangle$ be the localisation functor. If (P_n, d_n) is an $\mathcal{I}$-projective resolution of A, then the general theory shows that L maps A into the localising subcategory of $\mathrm{KK}^{C^*(G)}$ generated by the objects P_n. Now if $\mathrm{K}_*(A) = 0$, then we can choose these such that $\mathrm{K}_*(P_n \rtimes C_0(G)) = 0$, so that $\mathrm{K}_*(L(A) \rtimes C_0(G)) = 0$. The second statement in the theorem simply means that $L(A) \cong A$, so that we get $\mathrm{K}_*(A \rtimes C_0(G)) = 0$ if $\mathrm{K}_*(A) = 0$.

We may use Baaj–Skandalis duality to turn coactions of $C^*(G)$ into coactions of the dual $C_0(G)$, that is, group actions of G. This turns Theorem 3.68 into a statement about equivariant bivariant K-theory for compact groups. The duality functor $\mathrm{KK}^G \to \mathrm{KK}^{C^*(G)}$ maps a G-C*-algebra B to $B \rtimes G$ with the canonical coaction. Hence the assumption of the first statement in Theorem 3.68 is $\mathrm{K}_*^G(B) := \mathrm{K}_*(B \rtimes G) = 0$, and the conclusion is $\mathrm{K}_*(B) = 0$ because $B \rtimes G \rtimes C_0(G) \cong B \otimes \mathcal{K}(L^2 G)$ is Morita equivalent to B. Reformulating the second statement similarly, we arrive at the following equivalent reformulation of Theorem 3.68:

Theorem 3.69 ([?]). *Let G be a connected Lie group with torsion-free fundamental group and let A be a C^*-algebra with an action of G.*

If $\mathrm{K}_*^G(A) = 0$, then $\mathrm{K}_*(A) = 0$.

If $A \rtimes G \cong 0$ in KK, then $A \cong 0$ in KK^G.

The first statement is already known for some time, see [?], so that the Baum–Connes conjecture for the dual of G reduces to a known statement. The second statement is not contained in [?]; it may be used to improve the description of the equivariant bootstrap class in [?] and to formulate a variant of the Universal Coefficient Theorem that appears in [?] that reduces the computation of $\mathrm{KK}^G(A, B)$ not to $\mathrm{K}_*^G(A)$ and $\mathrm{K}_*^G(B)$, but to $\mathrm{KK}(A \rtimes G, B \rtimes G)$; this variant has the advantage that it converges for arbitrary A and B without any bootstrap class assumptions (see also [?, ?]).

Theorem 3.69 certainly becomes false if G is a finite group, say, $G = \mathbb{Z}/2$: there exists a $\mathbb{Z}/2$-action on a contractible C*-algebra A such that $\mathrm{K}_*(A \rtimes \mathbb{Z}/2) \neq 0$; then the dual action of $\mathbb{Z}/2$ on $A \rtimes \mathbb{Z}/2$ provides a counterexample.

We have already described the universal homological functor for the homological ideal $\mathcal{I}$ in Example 3.55: it is the K-theory functor $\mathrm{KK}^{C^*(G)} \to \mathcal{A}$, where $\mathcal{A}$ is the category of all countable $\mathbb{Z}/2$-graded $\mathrm{R}(G)$-modules, where $\mathrm{R}(G)$ denotes the representation ring of G. We view this as a functor $\mathrm{KK}^G \to \mathcal{A}$, $B \mapsto \mathrm{K}_*^G(B)$, via Baaj–Skandalis duality; the $\mathrm{R}(G)$-module structure agrees with the one by exterior product.

The following is an application of Theorem 3.69:

Proposition 3.70. *Let A be a C^*-algebra with $G \ltimes A$ in the bootstrap class. Assume that $\mathrm{K}_*^G(A)$ is free as an $\mathrm{R}(G)$-module, say, $\mathrm{KK}_0^G(\mathbb{C}, A) \cong \mathrm{R}(G)^N$ and $\mathrm{KK}_1^G(\mathbb{C}, A) \cong 0$ as $\mathrm{R}(G)$-modules. Then $A \cong \mathbb{C}^N$ in KK^G.*

Proof. Recall that $\mathrm{KK}_0^G(\mathbb{C}, \mathbb{C}) \cong \mathrm{R}(G)$ and $\mathrm{KK}_1^G(\mathbb{C}, \mathbb{C}) \cong 0$. By assumption, we have an isomorphism $\mathrm{R}(G)^N \cong \mathrm{KK}_0^G(\mathbb{C}, \mathbb{C}^N) \cong \mathrm{KK}_0^G(\mathbb{C}, A)$. This isomorphism corresponds to choosing N basis vectors in $\mathrm{KK}_0^G(\mathbb{C}, A)$, which we may combine to a single element of $\mathrm{KK}_0^G(\mathbb{C}^N, A) \cong \mathrm{KK}_0^G(\mathbb{C}, A)^N$. This yields some $f \in \mathrm{KK}_0^G(\mathbb{C}^N, A)$ such that the induced map $f_*\colon \mathrm{KK}_0^G(\mathbb{C}, \mathbb{C}^N) \to \mathrm{KK}_0^G(\mathbb{C}, A)$ is the given isomorphism. Since $\mathbb{C}^N \rtimes G$ and $A \rtimes G$ belong to the bootstrap class and $f \rtimes G$ acts by an invertible map on K-theory, $f \rtimes G$ is a KK-equivalence. Now we use the second part of Theorem 3.69 to conclude that f is invertible in KK^G. Here we used once again the trick that invertibility of a morphism in a triangulated category is equivalent to vanishing of its cone. $\square$

Now let $T \leq G$ be a maximal torus and let $A = C(G/T)$. Then

$$\mathrm{K}_*^G(C(G/T)) \cong \mathrm{K}_*^T(\mathrm{pt}) \cong \mathrm{R}(T).$$

Let W be the Weyl group of G. Basic results of representation theory assert that $\mathrm{R}(G) = \mathrm{R}(T)^W$ and that $\mathrm{R}(T) \cong \mathrm{R}(G)^N$ for some $N \in \mathbb{N}$. Therefore, the last proposition predicts that $C(G/T)$ is KK^G-equivalent to $\mathbb{C}^N$. In fact, the proof of Theorem 3.69 begins by establishing this fact directly:

Lemma 3.71. *There is a KK^G-equivalence $\mathbb{C}^N \cong C(G/T)$.*

Proof. As in the proof of the proposition above, we may lift the isomorphism $\mathrm{K}_*^G(\mathbb{C}^N) \cong \mathrm{K}_*^G(C(G/T))$ to a class $f \in \mathrm{KK}^G(\mathbb{C}^N, C(G/T))$. We must show that it is invertible. Since the induced map

$$f_*\colon \mathrm{KK}^G_*(\mathbb{C}^N, \mathbb{C}^N) \to \mathrm{KK}^G_*(\mathbb{C}^N, C(G/T))$$

is an isomorphism by construction, it suffices to establish that f_* induces an isomorphism

$$f_*\colon \mathrm{KK}^G_*(C(G/T), \mathbb{C}^N) \to \mathrm{KK}^G_*(C(G/T), C(G/T))$$

as well and then to invoke the Yoneda Lemma.

This statement may be simplified using Poincaré duality in KK^G. Our assumptions on G ensure that G/T has a G-equivariant spin structure. Hence there is a natural isomorphism

$$\mathrm{KK}^G_*(C(G/T), B) \cong \mathrm{KK}^G_*(\mathbb{C}, C(G/T) \otimes B) \cong \mathrm{K}^G_*(C(G/T) \otimes B) \cong \mathrm{K}^T_*(B),$$

where the last step uses the Morita equivalence $G \ltimes C(G/T, B) \sim_M T \ltimes B$. Thus we are reduced to proving that $f_*\colon \mathrm{K}^T_*(\mathbb{C}^N) \to rK^T_*(C(G/T))$ is invertible. The domain is simply $\mathrm{K}^T_*(\mathbb{C}^N) \cong \mathrm{R}(T)^N$. Thus we are reduced to computing the T-equivariant K-theory of the homogeneous space G/T. This has already been done a long time ago, and the result confirms the expectation that $f_*\colon \mathrm{K}^T_*(\mathbb{C}^N) \to rK^T_*(C(G/T))$ is invertible. □

With this lemma, we can reduce the assertion of Theorem 3.69 to the corresponding statement for the subgroup T. For all $A \in \mathrm{KK}^G$, we get $A \otimes \mathbb{C}^N = A^N \cong A \otimes C(G/T)$ in KK^G. As a consequence, $A \rtimes G \cong 0$ in KK if and only if $A^N \rtimes G \cong 0$ in KK, if and only if $A \rtimes T \sim_M (C(G/T) \otimes A) \rtimes G \cong 0$ in KK; and $A \cong 0$ in KK^G if and only if $A^N \cong 0$ in KK^G if and only if $C(G/T) \otimes A \cong 0$ in KK^G. The latter can be rewritten as $C(G/T) \otimes A = \mathrm{Ind}_T^G \mathrm{Res}_G^T(A)$. If we assume that Theorem 3.69 holds for T, then vanishing of $A \rtimes G$ implies, via the vanishing of $A \rtimes T$ and hence of $\mathrm{Res}_G^T(A)$ in KK^T, the vanishing of A in KK^G. Thus Theorem 3.69 holds for G once it holds for T.

Finally, we return to the equivalent formulation of the problem in Theorem 3.68. Since T is Abelian, $C^*(T) \cong C_0(\mathbb{Z}^n)$ for some $n \in \mathbb{N}$. Thus Theorem 3.68 is equivalent to the Higson–Kasparov Theorem for the group $\mathbb{Z}^n$. Hence Theorem 3.69 holds for T, and the proof is finished.

Which ingredients did this proof use? First, we needed some elementary K-theory computations to get the KK^G-equivalence $\mathbb{C}^N \cong C(G/T)$; then we used the exterior tensor product in KK^G to get a KK^G-equivalence $A^N \cong \mathrm{Ind}_T^G \mathrm{Res}_G^T(A)$ for an arbitrary G-C*-algebra A. This reduced the problem from G to the maximal torus T, where we already know the statement because its dual is an Abelian *group*, for which the Baum–Connes conjecture is known.

Christian Voigt has extended each of these steps to the compact quantum group $\mathrm{SU}_q(2)$. I expect that this argument works in much greater generality for all quantum deformations of simply connected compact simple Lie groups. The main obstacle are the elementary K-theory computations. There is always an undeformed maximal torus, and the representation ring $\mathrm{R}(G_q)$ of the compact quantum group and its maximal torus are related as in the classical case. We also know that $\mathrm{KK}^{G_q}(B, C(G_q/T)) \cong \mathrm{KK}^T(B, \mathbb{C})$. But it is not so easy to compute $\mathrm{KK}^{G_q}(C(G_q/T), C(G_q/T)) \cong \mathrm{KK}^T(C(G_q/T), \mathbb{C})$: this involves constructing equivariant K-homology classes on $C(G_q/T)$.

Finally, we discuss what could be the replacement for the family of compact subgroups in a quantum group. The most obvious choice uses the following notion of subgroup:

Definition 3.72. *A* closed quantum subgroup *of* (A, Δ) *is a quotient* A/I *to which* Δ *descends.*

Example 3.73. Closed quantum subgroups of $C_0(G)$ are $C_0(H)$ for closed subgroups $H \leq G$, as it should be. But there are too few closed quantum subgroups of $C^*_{\mathrm{r}}(G)$: while we would expect to see all closed subgroups of G, we only get $C^*_{\mathrm{r}}(G/N)$ if $N \leq G$ is a closed, amenable, normal subgroup; this is compact if and only if G/N is discrete, that is, N is open. Many locally compact groups such as $\mathrm{GL}_2(\mathbb{Q}_p)$ have lots of compact open subgroups, but no open normal subgroup. Such groups have no compact quantum subgroups.

Definition 3.74. *A* proper quantum homogeneous space *for* (A, Δ) *is a* C^**-subalgebra* B *of* A *that is a left* Δ*-coideal, that is,* $\Delta(B) \subseteq M(B \otimes A)$*. (Here "proper" means that the coaction on them is proper.)*

Example 3.75. If G is a group and $H \subseteq G$ is a compact subgroup, then $B = C_0(G/H) \subseteq C_0(G)$ is a proper quantum homogeneous space; these are the only proper quantum homogeneous spaces in this case. $C^*_{\mathrm{r}}(H) \subseteq C^*_{\mathrm{r}}(G)$ is a proper quantum homogeneous space for any open subgroup $H \subseteq G$; again, these are all the examples.

Let G be a compact Lie group, so that $C^*(G)$ is a discrete quantum group. Our description of proper quantum homogeneous spaces shows that there are no non-trivial ones if and only if G is connected. For instance, $\mathrm{SO}(3)$ has no proper quantum homogeneous spaces. This is a problem because the assembly map constructed above is not always an isomorphism for $\mathrm{SO}(3)$. Since we expect the Baum–Connes assembly map to be invertible in this case, we have to further modify our notion of torsion. The problem for $\mathrm{SO}(3)$ is related to its projective representations.

One way to approach the problem is to ask for particularly simple actions of a quantum group. If $H \subseteq G$ is compact, then the crossed product $C_0(G/H) \rtimes G$ is Morita equivalent to $C^*(H)$, which is isomorphic to a direct sum of matrix algebras. In fact, the same is true for any proper quantum homogeneous space of a regular quantum group: always, $B \rtimes \widehat{A}^c \subseteq A \rtimes \widehat{A}^c \cong \mathcal{K}(L^2 A)$ is a direct sum of matrix algebras or the algebra of compact operators. For brevity, we call such C*-algebras *elementary*. Elementary C*-algebras are those for which the computation of their K-theory may be considered a trivial combinatorial problem. Therefore, it is reasonable to consider the localising subcategory they generate. By Baaj–Skandalis duality, coactions of A with elementary crossed products are in bijection with coactions of the dual quantum group $\widehat{A}^c$ on elementary C*-algebras.

Definition 3.76. *Let* (A, Δ) *be a strongly regular locally compact quantum group. Let* $\mathcal{R}$ *be the set of all coactions of* $\widehat{A}^c$ *on separable elementary* C^**-algebras.*

The category $\mathcal{R}$ *is countably additive because countable direct sums of elementary* C^**-algebras are again elementary. Let* $\mathcal{R}_0$ *be the subset of all indecomposable objects of* $\mathcal{R}$*. Furthermore,* $\mathcal{R}$ *and* $\mathcal{R}_0$ *are closed under equivariant*

Morita equivalence. Let $\mathcal{R}_1$ be the set of equivariant Morita equivalence classes of objects in $\mathcal{R}_0$.

We call (A,Δ) torsion-free *if $\mathcal{R}_1$ has only one element; equivalently, any coaction of $\widehat{A}^c$ on an elementary C^*-algebra is a direct sum of coactions on $\mathcal{K}(\mathcal{H})$ induced by a coaction on $\mathcal{H}$.*

By design, if A is torsion-free then $\widehat{A}^c$ has no non-trivial projective representations. It can be checked that $C_0(G)$ for a discrete group G is torsion-free in this sense if and only if G is torsion-free and that $C^*(G)$ for a compact group G is torsion-free if and only if G is connected and has no non-trivial projective unitary representations. Christian Voigt has shown that deformations of $C^*(G)$ for simply connected simple compact Lie groups remain torsion-free; an example of this is $C^*(\mathrm{SU}_q(n))$ for all $n \in \mathbb{N}_{\geq 2}$.

Currently, the best proposal for the Baum–Connes assembly map for a regular quantum group I know is the following. Let $\mathcal{P}$ be the collection of objects of $\mathrm{KK}^{(A,\Delta)}$ that are Baaj–Skandalis dual to objects of $\mathcal{R}$ (or $\mathcal{R}_1$, this makes essentially no difference). Let $\mathcal{L}$ be the localising subcategory they generate. Let $\mathcal{N} := \mathcal{L}^{\dashv}$ be its right orthogonal complement and assume that $(\mathcal{L},\mathcal{N})$ is complementary. Then the Baum–Connes assembly map should be the canonical map $\mathbb{L}F(B) \to F(B)$, where $\mathbb{L}$ denotes the localisation at $\mathcal{N}$ and $F(B) := \mathrm{K}_*(B \rtimes \widehat{A}^c)$. For groups, this agrees with the usual Baum–Connes assembly map. But so far we have not studied enough examples to be certain that this is the final formulation of the assembly map.

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