KK-theory exam questions

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1 Easy questions

1. Define the reduced C^*-algebra $C_r^*(G)$ of a discrete group G.
2. Define the maximal C^* algebra $C_{max}^*(G)$ of a discrete group G.
3. Define the Toeplitz algebra and provide at least one more equivalent description.
4. Define the notion of a Hilbert module over a C^*-algebra A; define the direct sum of a family $\{E_i\}_{i\in I}$ of Hilbert modules over the C^*-algebra A.
5. Let E and F be Hilbert modules over a C^*-algebra A. Provide a definition of the space $\mathcal{L}(E, F)$ of operators from E to F. Provide a definition of the space $\mathcal{K}(E, F)$ of compact operators from E to F. What is $\mathcal{K}(A)$ for a C^*-algebra A?
6. State what it means for two C^*-algebras to be Morita equivalent.
7. Define the notion of a $\mathbb{Z}/2\mathbb{Z}$-graded and ungraded Fredholm module over a an algebra A. Describe Fredholm modules over the algebra $\mathbb{C}$ of complex numbers.
8. Provide the definition of a *relative* Fredholm module over an algebra J.
9. Define K-homology of a C^*-algebra A.

2 More difficult questions

1. Sketch a construction of the Toeplitz extension.
2. Sketch Atiyah's proof of Bott periodicity in K-theory.
3. Sketch Cuntz's proof of Bott periodicity in K-theory.
4. Let A be a C^*-algebra. Sketch the proof of the fact that $\mathcal{L}(A)$ is a maximal essential extension of $\mathcal{K}(A)$.
5. Sketch the proof of the fact that two σ-unital C^*-algebras are Morita equivalent if and only if they are stably isomorphic.

6. Provide a sketch of a construction of the pseudodifferential operator extension.

7. Let $\mathbb{F}_2$ be the nonabelian free group on two generators. Sketch the proof of the Kadison-Kaplansky conjecture: The reduced C^*-algebra $C_r^*(\mathbb{F}_2)$ has no nontrivial idempotents.

8. Let $\alpha = (\rho, H, F)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Fredholm module. Prove that if we ignore the grading and regard this module as an ungraded Fredholm module, then α is homotopic to a degenerate Fredholm module.

9. Let $0 \to J \to A \to A/J \to 0$ be an extension of C^*-algebras. Let $\delta_i : K_i(A/J) \to K_{i+1}(J)$, for $i = 0, 1$, be the connecting homomorphisms in the six term exact sequence in K-theory associated with this algebra extension, and let $\partial^i : K^i(J) \to K^{i+1}(A/J)$ be the corresponding maps in K-homology. Outline the strategy of the proof of the compatibility of these maps with the index pairing.

10. Use the diagram calculus in KK-theory to illustrate the definition of the extension of scalars map
$$\tau_D : KK_i(A, B) \to KK_i(A \otimes D, B \otimes D)$$
where A, B, D are separable C^*-algebras. Use this to provide an illustration of the exterior product in KK-theory:
$$KK_i(A_1, B_1 \otimes D) \times KK_j(D \otimes A_2, B_2) \to KK_{i+j}(A_1 \otimes B_1, A_2 \otimes B_2).$$

3 Written questions

1. Let (A, H, F) be a 1-summable $\mathbb{Z}/2\mathbb{Z}$ graded Fredholm module over an algebra A, and let γ be an involution implementing the grading. Prove that the character
$$\tau(a) = \frac{1}{2}\mathrm{Tr}(\gamma F[F, a])$$
defines a trace on the algebra A.

2. The identity function $u : z \to z$, $z \in S^1$ can be regarded as a unitary element of $C(S^1)$ and a generator of the K-theory group $K_1(C^1)$.

   We define a generator of the K-homology group $K^1(C(S^1))$ as follows. Assume that $C(S^1)$ acts by pointwise multiplication on the Hilbert space $L^2(S^1)$. Let $e_n = e^{2\pi n\theta}$, $\theta \in [0, 1]$ be the standard basis for $L^2(S^1)$ and define an operator F by
$$F(e_n) = \begin{cases} e_n, & n > 0 \\ -e_n, & n < 0 \\ 0, & n = 0 \end{cases}$$
   Let $\alpha = (C(S^1), L^2(S^1), F)$ be the corresponding Fredholm module, which generates $K^1(C(S^1))$. Compute the index pairing
$$\langle [u], [\alpha] \rangle \in \mathbb{Z}$$

3. Let $\mathbb{F}_2$ be the nonabelian free group on two generators, and let T be a tree on which it acts freely and simplicially. Let Δ_0 be the set of vertices and Δ_1 the set of edges of T, and for $x, y \in \Delta_0$ denote by $[x, y]$ the set of vertices on the unique path from x to y. Choose a base point $x_0 \in \Delta_0$ of T. Let $\beta : \Delta_0 \setminus x_0 \to \Delta_1$ be the map defined by sending $x \in \Delta_0 \setminus x_0$ to the unique edge in $\beta(x) \in \Delta_1$ contained in $[x, x_0]$.

   Prove that

   (a) The map β is a bijection.

   (b) For a given $g \in \mathbb{F}_2$, the set of $x \in \Delta_0$ such that $g\beta(g^{-1}x) \neq \beta(x)$ is finite and equals $[x_0, gx_0]$.

   Use this fact to construct a finitely summable Fredholm module over a dense subalgebra $\mathcal{A}$ of $C_r^*(\mathbb{F}_2)$.

4. Let X and Y be contractive self-adjoint operators on a Hilbert space H. We denote by X^b the commutative C^*-subalgebra of $B(H)$ consisting of all elements $\phi(X)$, where $\psi \in C_0(-1, 1)$. The pair (X, Y) is a Schrödinger pair iff

   - Y commutes with X^b modulo compact operators;
   - $X^b \cdot Y^b \subset \mathcal{K}(H)$.

   A strong Schrödinger pair is a Schrödinger pair (X, Y) in which Y commutes with X modulo compacts.

   Let (ρ, H, F) be an ungraded Fredholm module over an algebra J, which we assume to be an ideal in a C^*-algebra A, so that ρ can be extended to A. We assume also that $F^2 = 1$ and F is self-adjoint.

   Let a be a self-adjoint element of A such that $\| \rho(a) \| \leq 1$ and $a^2 - 1 \in J$. Prove that $X = \rho(a)$ and $Y = F$ form a Schrödinger pair.

   Prove also that if the extension of ρ to A makes (ρ, H, F) into a relative Fredholm module, then (X, Y) is a strong Schrödinger pair.

5. Let G be a compact Hausdorff second countable topological group. Let $R(G)$ denote the representation ring of G. Assuming the Peter-Weyl theorem and other basic fundamental facts about the representation theory of compact groups, prove that $KK_G^0(\mathbb{C}, \mathbb{C})$ and $R(G)$ are isomorphic as abelian groups.

   Hint: You may assume that if H is a separable Hilbert space and T_0, T_1 are Fredholm operators on H, then there is a continuous path $\alpha : [0, 1] \to \mathcal{F}(H)$, with $\alpha(0) = T_0$, $\alpha(1) = T_1$ if and only if $Index(T_0) = Index(T_1)$. Here $\mathcal{F}(H)$ is the set of all Fredholm operators on H, topologized by the operator norm.