Cyclic Homology Theory

Written Exam

1. (10pt) Let $\mathcal{C}$ be a small category (i.e., the objects form a set). Define $\mathcal{C}_n$ as the set of n composable morphisms:

$$C_0 \xrightarrow{f_1} C_1 \longrightarrow \cdots \xrightarrow{f_n} C_n \ .$$

(Thus $\mathcal{C}_0 = Obj\ \mathcal{C}$, $\mathcal{C}_1 = Mor\ \mathcal{C}$, etc.) Construct faces and degeneracies making it into a simplicial set (called the *nerve* of the category). Prove all the simplicial relations.

2. (10pt) Compute $HH_*(A)$ and $HC_*(A)$ of the truncated polynomial ring $A = K[t]/(t^2)$.

3. (15pt) Consider the cyclic bicomplex $CC_{**}(A)$ of the algebra A whose horizontal boundary maps are alternatively $(1-t)$ and N, and whose vertical boundary maps are alternatively b and $-b'$. Compute the E^2 term of the spectral sequence, where $E^2_{pq} = H^h_p H^v_q(CC_{**}(A))$.

4. (10pt) Let H be a Hopf algebra, $H_\bullet \otimes H_\bullet$ a right H module vie the diagonal action $(x \otimes y)h = xh^{(1)} \otimes yh^{(2)}$, and ${}_\bullet H$ a left H module via the adjoint action $h \triangleright k = h^{(2)} k S(h^{(1)})$. Remembering that $h \otimes k \mapsto hS(k^{(1)}) \otimes k^{(2)}$ is an H-linear isomorphism from $H_\bullet \otimes H_\bullet$ to $H \otimes H$, prove that the map $\beta : H \otimes H \to (H_\bullet \otimes H_\bullet) \otimes_H {}_\bullet H$, $\beta(h \otimes k) := k^{(2)} \otimes 1 \otimes_H hk^{(1)}$ is an isomorphism. Show that it intertwines the flip map with the cyclic operator $\tau(h_0 \otimes h_1 \otimes_H k) = h_1 k^{(2)} \otimes h_0 \otimes_H k^{(1)}$, i.e., $\tau(\beta(h \otimes k)) = \beta(k \otimes h)$.

5. (5pt) Let $\mathfrak{g}$ be any Lie algebra. Define $\widetilde{\mathfrak{g}}$ as the super-Lie algebra spanned by the symbols d and ι_η, $\mathcal{L}_\eta$ (linear in $\eta \in \mathfrak{g}$) of degrees $(1, -1, 0)$ subject to the relations

$$[d,d] = 0, \quad [\iota_{\eta_1}, \iota_{\eta_2}] = 0, \quad [d, \iota_\eta] = \mathcal{L}_\eta, \quad [d, \mathcal{L}_\eta] = 0, \quad [\iota_{\eta_1}, \mathcal{L}_{\eta_2}] = -\iota_{[\eta_1,\eta_2]}, \quad [\mathcal{L}_{\eta_1}, \mathcal{L}_{\eta_2}] = \mathcal{L}_{[\eta_1,\eta_2]}.$$

In the enveloping algebra of $\widetilde{\mathfrak{g}}$, prove the formula

$$[d, \iota_{\eta_1} \dots \iota_{\eta_p}] = \sum_{1 \le i \le p} (-1)^{i-1} \iota_{\eta_1} \dots \widehat{\iota_{\eta_i}} \dots \iota_{\eta_p} \mathcal{L}_{\eta_i} + \sum_{1 \le i < j \le p} (-1)^{i+j-1} \iota_{[\eta_i,\eta_j]} \iota_{\eta_1} \dots \widehat{\iota_{\eta_i}} \dots \widehat{\iota_{\eta_j}} \dots \iota_{\eta_p}$$

D1. Show that, in the simplicial set $S^1_\bullet$ representing the circle, there are $n+1$ elements of degree n. Show that there is a natural bijection with the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$.

D2. Let $C_{*,*}$ be a complex of modules in the first quadrant. We suppose that whenever p is odd, the vertical complex $C_{p,*}$ has a homotopy which makes it acyclic. Show that $Tot(C_{*,*})$ is quasi-isomorphic to a complex made out of the modules $C_{p,q}$ for p even.

D3. Let $\mathcal{B}(A)$ be the (b,B)-bicomplex of the algebra A, made with the Hochschild boundary b and the Connes-Rinehart differential B. Give a strategy to prove that the total complex $Tot\ \mathcal{B}(A)$ is quasi-isomorphic to Connes' cyclic complex $C^\lambda(A)$, where $C^\lambda_n(A) = A^{\otimes n+1}/(1-t)$.

D4. Let A be an associative algebra and let $\mathcal{M}_r(A)$ be the algebra of $r \times r$-matrices with entries in A. Outline a proof that the trace map $tr : \mathcal{M}_r(A) \to A$ can be extended to a morphism of complexes $Tr : C_*(\mathcal{M}_r(A)) \to C_*(A)$ which is a quasi-isomorphism. Here $C_*(A)$ is the Hochschild complex with boundary b.

D5. Let $D : A \to A$ be a derivation of the algebra A (i.e., $D(ab) = aD(b) + D(a)b$). Show that D can be extended to a chain map $L_D : C_*(A) \to C_*(A)$. Show that this map is trivial on homology when the derivation is internal (i.e., $D = \mathrm{ad}x$).

D6. Outline a proof of the equivalence of Serre's, Hochschild's and Leibniz's definitions of $\Omega^1_{\mathcal{O}/k}$.

D7. Describe $E^2_{p,q}$ for the spectral sequence abutting to the cyclic homology of the algebra of symbols.

D8. Argue that for any exhaustion $X = \bigcup_{i\in I} X_i$ of a manifold X by compact submanifolds X_i with boundary, $X_i \Subset X_{i+1}$,

$$\lim_{i\in I} \frac{\Omega^q(X_i)}{d\Omega^{q-1}(X_i)} = \frac{\Omega^q(X)}{d\Omega^{q-1}(X)}.$$

D9. Show that the differential d on $\Omega^\bullet_{\mathcal{O}/k}$ is well defined by the formula

$$d(f_0 df_1 \wedge \cdots \wedge df_n) = df_0 \wedge df_1 \wedge \cdots \wedge df_n.$$

D10. Compute $H_*(\mathfrak{gl}(k))$, where k is a field.

E1. What is the relationship between Hochschild homology $HH_*(A)$ and differential forms $\Omega^*_{A/k}$?

E2. Define the multiplication of symbols. Compute products of symbols
a) $\xi_2 \circ \log(1 + x_1)$,
b) $\xi_1 \circ e^{x_1}$.

E3. Explain the concept of cyclic duality.

E4. Let $\mathcal{A}s$ be the operad of nonunital associative algebras.
a) What is the S_n-representation $\mathcal{A}s(n)$?
b) Describe explicitly the composition map

$$\mathcal{A}s(n) \otimes \mathcal{A}s(i_1) \otimes \cdots \otimes \mathcal{A}s(i_n) \to \mathcal{A}s(i_1 + \cdots + i_n).$$

E5. At what term does the spectral sequence abutting to the
a) cyclic homology,
b) Hochschild homology,
of the algebra of symbols degenerate?