Lectures on Homology of Symbols

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December 2006

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Chapter 1

The algebra of classical symbols

1.1 Local definition of the algebra of symbols

Let X be a C^∞-manifold (not necessarily compact), and E a vector bundle on X. Consider a coordinate patch

$$f_U \colon U \to X, \quad U \subset \mathbb{R}^n.$$

The cotangent bundle $T^*X \to X$ pulls back to U

$$\begin{tikzcd} T_0^*U \arrow[r] \arrow[d, "\pi"'] & T^*U \arrow[r] \arrow[d] & T^*X \arrow[d] \\ U \arrow[r, equal] & U \arrow[r, "f_U"] & X \end{tikzcd}$$

The bundle T_0^*U is defined as $T^*U \setminus U$. There is an isomorphism

$$\begin{tikzcd} T_0^*U \arrow[r, "\simeq"] \arrow[d, "\pi"'] & U \times \mathbb{R}_0^n \subset \mathbb{R}^n \times \mathbb{R}_0^n \\ U & \end{tikzcd}$$

Using it we can denote the coordinates on T_0^*U by (u, ξ), where $u = (u_1, \dots, u_n) \in \mathbb{R}_0^n$, and $\xi \in (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

To each open set U we associate a section $a^U := \sum_{j=0}^\infty a_j^U$, where each a_j^U is a section of the bundle $\mathrm{End}(\pi^* f_U^* E)$, where

$$\begin{tikzcd} \pi^* f_U^* E \arrow[r] \arrow[d] & f_U^\bullet E \arrow[r] \arrow[d] & E \arrow[d] \\ T_0^*U \arrow[r, "\pi"] & U \arrow[r, "f_U"] & X \end{tikzcd}$$

More precisely by a_{m-j}^U we denote the homogeneous part of degree $m - j$

$$a_{m-j}^U \in C^\infty(T_0^*U, \mathrm{End}(\pi^* f_U^* E))(m-j).$$

There is a natural action of $\mathbb{R}_+^*$ on T_0^*X given by $t \cdot (u, \xi) := (u, t\xi)$. The infinitesimal action is provided by the Euler field

$$\Xi = \sum_{i=1}^n \xi_i \partial_{\xi_i}.$$

The homogenity condition for a^U_{m-j} is given by $a^U_{m-j}(u, t\xi) = t^{m-j} a^U_{m-j}(u, \xi)$.

The section a^U belongs to the product

$$\prod_{j=0}^{\infty} C^\infty(T^*_0 U, \mathrm{End}(\pi^* f^*_U E))(m-j)$$

which has a natural structure of Frechet space. With the norm

$$|\xi| := \sqrt{\xi_1^2 + \ldots \xi_n^2}$$

we can write

$$|\xi|^{j-m} a^U_{m-j} \in C^\infty(T^*_0 U, \mathrm{End}(\pi^* f^*_U E))(0) \simeq C^\infty(S^* U, \mathrm{End}(\pi^* f^*_U E)),$$

where S^*U is the cosphere bundle $T^*_0 U/\mathbb{R}^*_+ \xrightarrow{\pi} U$. The cotangent bundle $T^*X \to X$ is canonically oriented and S^*X is canonically oriented (even though we do not have the orientation on X). Now S^*U is a canonically oriented $(2n-1)$-manifold and $S^*U \simeq U \times S^{n-1}$.

The sections a^U are given locally, so we need a compatibility condition. We need a composition law such that it will depend on all jets, not only on 1-jets as usual composition.

$$a^U \circ_u b^U \colon \sum_\alpha \delta^\alpha_\xi a^U D^{[\alpha]}_u b^U$$

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_i \in \mathbb{N}$$

$$D_{u_i} := \frac{1}{i}\partial_{u_i}, \quad D^{[\alpha]}_u = \frac{1}{\alpha!} D^\alpha_u = \frac{1}{\alpha! i^{|\alpha|}} \partial^\alpha_u.$$

If a^U is of order m, b^U of order m' using the notation for classical symbols

$$\mathrm{CS}^m_U(U, E) := \prod_{j=0}^{\infty} (T^*_0 U, \mathrm{End}(\pi^* f^*_U E))(m-j)$$

we can write

$$\circ_u \colon \mathrm{CS}^m_U(U, E) \times \mathrm{CS}^{m'}_U(U, E) \to \mathrm{CS}^{m+m'}_U(U, E), \quad m, m' \in \mathbb{C}.$$

Now suppose we have two open sets $U, V \in \mathbb{R}^n$ such that the images of charts $f_U \colon U \to X$, $f_V \colon V \to X$ have nonempty intersection $f(U) \cap f(V)$. Denote

$$U' := f_U^{-1}(f(U) \cap f(V)), \quad V' := f_V^{-1}(f(U) \cap f(V)),$$

$$f_{UV} := f_U^{-1} \circ f_V \colon V' \to U'.$$

For a smooth map $f \colon X \to Y$ there are induced maps

$$Tf \colon TC \to TY, \quad (Tf)_x \colon (TX)_x \to (TY)_{f(x)},$$
$$Tf^* \colon T^*X \to T^*Y, \quad (Tf)^*_x \colon (T^*X)_x \leftarrow (T^*Y)_{f(x)}.$$

Assume that Tf is invertible

$$((Tf)^*_x)^{-1} \colon (TX)^*_x \to (TY)^*_{f(x)}$$

Define a maps

$$X \times TX \to Y \times TY, \quad (x, v) \mapsto (f(x), (Tf)_x(v)),$$
$$X \times T^*X \to Y \times T^*Y, \quad (x, \xi) \mapsto (f(x), ((Tf)^*)_x^{-1}(\xi)).$$

Now comes the question, to what extend a^V and $(T^*f)^*a^U$ agree? We have

$$\begin{aligned} a^V &= (T^*f)^*(a^U + (\text{arbitrary high order correction terms})) \\ &= (T^*f_{UV})^*(\sum_\alpha \psi_\alpha \partial_\xi^\alpha a^U), \end{aligned}$$

where

$$\psi_\alpha(u, \xi) = D_z^{[\alpha]} e^{i\langle j_u^{>1}(z), (Tf_{UV})_v^*(\xi)\rangle}\big|_{z=u,\, v=(f_V^{-1}\circ f_U)(u)},$$
$$j_u^{>1}(z) = f_V^{-1} \circ f_U - j_U^1(f_V^{-1} \circ f_U),$$

so $j_u^{>1}$ vanishes up to second order at point $u \in U$. The $\psi_\alpha(u, \xi)$ are scalar valued functions on coordinate charts. They do not depend on symbols, only on manifold.

In the whole notes we will be using a projective tensor product of topological vector spaces desribed in the appendix (A).

The product

$$\mathrm{CS}^m(X, E) \times \mathrm{CS}^{m'}(X, E) \to \mathrm{CS}^{m+m'}(X, E)$$

of Frechet spaces is associative. Define the algebra of symbols as

$$\mathrm{CS}(X, E) \colon = \bigcup_{m \in \mathbb{Z}} \mathrm{CS}^m(X, E).$$

Let $a := \{a^U\}_{f_U \colon U \to X}$. The topology on $\mathrm{CS}(X, E)$ is defined as follows. We say that the net $\{a_\lambda\}$ converges to a symbol a if for any $m \in \mathbb{C}$ there exists λ_0 such that $a_\lambda - a \in \mathrm{CS}^m(X, E)$ for all $\lambda \geq \lambda_0$.

The subalgebra $\mathrm{CS}^0(X, E)$ is a Frechet algebra, and $\mathrm{CS}^{-j}(X, E)$, $j \in \mathbb{Z}_+$ is a two sided ideal in $\mathrm{CS}^0(X, E)$.

Remark 1.1. The multiplication

$$\mathrm{CS}^m(X, E) \otimes \mathrm{CS}(X, E) \to \mathrm{CS}(X, E)$$

is not continuous in both arguments.

1.2 Classical pseudodifferentials operators

Let $A \colon C_c^\infty(X, E) \to C^\infty(X, E)$ be a pseudo differential operator. For a chart $f_U \colon U \to X$ there is an operator

$$f_U^\# A \colon C_c^\infty(U, f_U^* E) \to C^\infty(U, f_U^* E)$$

We can define it for $\varphi \in C_c^\infty(U, f_U^* E)$ as follows. First take $(\varphi \circ f_U^{-1})|_{f(\operatorname{supp} \varphi)}$, and then extend by 0, apply A and pullback, as in the following diagram

$$\begin{tikzcd} C_c^\infty(X, E) \arrow[r, "A"] & C^\infty(X, E) \arrow[d, "f_U^*"] \\ C_c^\infty(U, f_U^* E) \arrow[u, "(f_U)_!"] \arrow[r, "f_U^\# A"'] & C^\infty(U, f_U^* E) \end{tikzcd}$$

Explicitly

$$(f_U^{\#} A)\varphi(u) = \int_{\mathbb{R}^n_\xi} \int_U e^{i\langle u-u',\xi\rangle} \beta(u,u',\xi)\varphi(u')du'\bar{d}\xi + (T\varphi)(u),$$

where $\beta \in C^\infty(U \times T^*U, \mathrm{End}(\pi^* f_U^* E))$ is called an amplitude,

$$\beta(u,u',\xi) \sim \sum_{j=0}^{\infty} \beta_{m-j}(u,u',\xi),$$

$$\beta_{m-j}(u,u',t\xi) = t^{m-j}\beta(u,u',\xi),$$

T is a smoothing operator

$$(T\phi)(u) = \int_U K(u,u')\varphi(u')|du'|,$$

and

$$|du| = |du_1 \wedge \cdots \wedge du_n|, \quad \bar{d}\xi = \frac{1}{(2\pi)^n}|d\xi_1 \wedge \cdots \wedge d\xi_n|.$$

By $\mathrm{CL}^m(X,E)$ we denote the space of classical pseudo differential operators, and by $\mathrm{CL}^m_{prop}(X,E)$ the subset of operators which take functions with compact support into functions with compact support. For $A \in \mathrm{CL}^m(X,E)$ there is a decomposition $A = A_{prop} + S$ into a proper part A_{prop} and non proper smoothing part S. Define a Frechet space of arbitrary low order operators by

$$L^{-\infty}(X,E) := \bigcap_{m\in\mathbb{Z}} \mathrm{CL}^m(X,E).$$

There is an isomorphism

$$\mathrm{CL}^m(X,E)/L^{-\infty}(X,E) \xrightarrow{\simeq} \mathrm{CS}^m(X,E).$$

Classical symbols have a product

$$\mathrm{CL}^m_{prop}(X,E) \times \mathrm{CL}^{m'}_{prop}(X,E) \to \mathrm{CL}^{m+m'-1}_{prop}(X,E), \quad m,m' \in \mathbb{C}.$$

We define the algebra of classical symbols as

$$\mathrm{CL}(X,E) := \bigcup_{m\in\mathbb{Z}} \mathrm{CL}^m(X,E).$$

The space of smoothing operators $\mathcal{L}^\infty(X,E)$ is defined as a kernel

$$\mathcal{L}^\infty(X,E) \rightarrowtail \mathrm{CL}(X,E) \twoheadrightarrow \mathrm{CS}(X,E)$$

and if X is closed it is isomorphic (non canonically) to the space of rapidly decaying matrices

$$L^{-\infty} = \{(a_{ij})_{i,j=1\ldots,\infty} \mid |a_{ij}|(i+j)^N \to 0, \text{ as } i+j \to \infty\}.$$

This is the noncommutative orientation class of a closed manifold and index theorem is the way to state that. Index measures to what extend this sequence is not split.

The map

$$\mathrm{CL}(X,E)/L^{-\infty}(X,E) \to \mathrm{CS}(X,E)$$

is defined as follows. For a classical pseudo differential operator

$$A\colon C_c^\infty(X,E) \to C^\infty(X,E)$$

we take the amplitude

$$\beta^U(u,u',\xi) \sim \sum_{j=0}^{\infty} \beta^U_{m-j}(u,u',\xi)$$

and then define $a^U \in \mathrm{CS}(X,E)$ by

$$a^U := \left(e^{\sum_{i=1}^n \partial_{\xi_i} D_{u_i}\beta^U}\right)|_{u=u'}.$$

1.3 Statement of results

The main goal is to compute the Hochschild and cyclic homology of the algebra of symbols $\mathrm{CS}(X)$. Let $T_0^*X = T^*X \setminus X$ and Y^c be the $\mathbb{C}^*$-bundle over the cosphere bundle S^*X defined as

$$\begin{array}{c} Y^c := T_0^*X \times_{\mathbb{R}_+} \mathbb{C}^* \\ \big\downarrow{\scriptstyle \mathbb{C}^*} \\ S^*X \end{array}$$

Theorem 1.2. *There is a canonical isomorphism*

$$\mathrm{HH}_q(\mathrm{CS}(X)) \simeq \mathrm{H}^{2n-q}_{\mathrm{dR}}(Y^c).$$

Regarding cyclic homology, consider on $\mathrm{HC}_q^{\mathrm{cont}}(\mathrm{CS}(X))$ the filtration by the kernels of the iterated S-map:

$$\{0\} = \mathcal{S}_{q0} \subset \mathcal{S}_{q1} \subset \ldots \subset \mathcal{S}_{qt} = \mathrm{HC}_q(\mathrm{CS}(X)),$$

where $t = \left[\frac{q}{2}\right]$ and $\mathcal{S}_{qr} := \ker S_*^{1+r} \cap \mathrm{HC}_q(\mathrm{CS}(X))$.

Theorem 1.3. *The canonical map*

$$I\colon\ \mathrm{HH}_\bullet(\mathrm{CS}(X)) \to \mathrm{HC}_\bullet(\mathrm{CS}(X))$$

is injective. In particular

$$\mathrm{HC}_{qr}(\mathrm{CS}(X)) = \mathrm{gr}_r^S \,\mathrm{HC}_q(\mathrm{CS}(X)) := \mathcal{S}_{qr}/\mathcal{S}_{q,r-1}$$

is canonically isomorphic with

$$\mathrm{H}^{2n-q+2r}_{\mathrm{dR}}(Y^c), \quad r = 0,1,\ldots.$$

1.4 Derivations of the de Rham algebra

Let $\mathcal{O}$ be a commutative k-algebra with unit, and k any commutative ring of coefficients. We define

$$\Omega^\bullet_{\mathcal{O}/k} := \Lambda^\bullet_{\mathcal{O}}\Omega^1_{\mathcal{O}/k},$$

where $\Omega^1_{\mathcal{O}/k}$ can be defined in a three ways:

- Serre's picture
$$\Omega^1_{\mathcal{O}/k} := I_\Delta / I_\Delta^2,$$
where $I_\Delta := \ker(\mathcal{O}^{\otimes 2} \to \mathcal{O})$.
- Hochschild picture
$$\Omega^1_{\mathcal{O}/k} := \mathcal{O}^{\otimes 2} / b\mathcal{O}^{\otimes 3}.$$
- Leibniz picture
$$\Omega^1_{\mathcal{O}/k} := \frac{\mathcal{O}\langle df \mid f \in \mathcal{O}\rangle}{\mathcal{O}\langle d(f+g) - df - dg,\ dc = 0\ (c \in k),\ d(fg) - fdg - gdf\rangle}.$$

The differential $d\colon \mathcal{O} \to \Omega^1_{\mathcal{O}/k}$ is defined in those three pictures as follows

- $f \mapsto d_\Delta f \mod I_\Delta^2 = (1 \otimes f - f \otimes 1) \mod I_\Delta^2$ (Serre's picture),
- $f \mapsto d_\Delta f \mod b\mathcal{O}^{\otimes 3} = (1 \otimes f - f \otimes 1) \mod b\mathcal{O}^{\otimes 3}$ (Hochschild picture),
- $f \mapsto df$ (Leibniz picture).

The derivation $d_\Delta\colon \mathcal{O} \to I_\Delta \subset \mathcal{O} \otimes \mathcal{O}$ is universal in the sense that if we have an $\mathcal{O}$-bimodule M and a derivation $\delta\colon \mathcal{O} \to M$, then there exists a unique $\mathcal{O}$-bimodule map $\widetilde{\delta}$ such that the following diagram commutes

$$\begin{tikzcd}
 & M \\
\mathcal{O} \arrow[ur, "\delta"] \arrow[r, "d"] \arrow[dr, "d_\Delta"'] & I_\Delta / I_\Delta^2 \arrow[u, dashed, "\widetilde{\delta}"'] \\
 & I_\Delta \arrow[u]
\end{tikzcd}$$

Let $\mathrm{Der}^m(\Omega^\bullet) = \mathrm{Der}^m_k(\Omega^\bullet)$ denote the algebra of degree m derivations, and
$$\mathrm{Der}^\bullet(\Omega^\bullet) := \bigoplus_{m \in \mathbb{Z}} \mathrm{Der}^m(\Omega^\bullet).$$
If η is of degree p and ζ of degree q, then for $\delta \in \mathrm{Der}^m(\Omega^\bullet)$ we have
$$\delta(\eta \wedge \zeta) = \delta(\eta) \wedge \zeta + (-1)^{pm} \eta \wedge \delta(\zeta).$$
$$\delta\colon \Omega^p \to \Omega^{p+m}.$$
Furthermore $\mathrm{Der}^\bullet(\Omega^\bullet)$ is a super Lie algebra, that is the commutators satisfy the super Jacobi identity
$$0 = [[a,b],c] + (-1)^{|a|(|b|+|c|)}[[b,c],a] + (-1)^{|c|(|a|+|b|)}[[c,a],b].$$
Denote $\delta_p := \delta|_{\Omega_p}$.

Proposition 1.4. *The set* $\mathrm{Der}^m(\Omega^\bullet)$ *is naturally identified with the set of pairs* (δ_0, δ_1), *where*
$$\delta_0\colon \mathcal{O} \to \Omega^m$$
is a k-linear derivation of $\mathcal{O}$ *with values in* Ω^m,
$$\delta^1\colon \Omega^1 \to \Omega^{m+1}$$

is a k-linear map such that

$$\delta_1(f\alpha) = \delta_0(f) \wedge \alpha + f\delta_1(\alpha).$$

and

$$\delta_1(\alpha)\alpha - (-1)^{m+1}\alpha\delta_1(\alpha) = 0,$$

that is the super commutator $[\delta_1(\alpha), \alpha] = 0$*.*

Any derivation of degree m is uniquely determined by δ_0 and δ_1. Thus $\mathrm{Der}^m(\Omega^\bullet) = 0$ for $m < -1$.

For $\delta_0 = 0$ we have

$$\delta(f\alpha_1 \wedge \cdots \wedge \alpha_p) = \sum_{i=1}^{p} (-1)^{m(i-1)} f\alpha_1 \wedge \cdots \wedge \delta_1(\alpha_i) \wedge \cdots \wedge \alpha_p.$$

Similarly for any $\phi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^{m+1})$ there exists a corresponding derivation

$$\delta_\phi(f\alpha_1 \wedge \cdots \wedge \alpha_p) := \sum_{i=1}^{p} (-1)^{m(i-1)} f\alpha_1 \wedge \cdots \wedge \phi(\alpha_i) \wedge \cdots \wedge \alpha_p.$$

Example 1.5. (The de Rham derivation) Let $d_0 = d\colon \mathcal{O} \to \Omega^1$. Now we will give a construction of $d_1\colon \Omega^1 \to \Omega^2$. Consider a k-linear pairing

$$\mathcal{O} \times \mathcal{O} \to \Omega^2, \quad (f, g) \mapsto df \wedge dg$$

$$\begin{tikzcd} \mathcal{O} \times \mathcal{O} \arrow[r] \arrow[d] & \Omega^2 \\ \mathcal{O} \otimes_k \mathcal{O} \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O} \times \mathcal{O} \arrow[r] \arrow[d] & \Omega^2 \\ (\mathcal{O} \otimes_k \mathcal{O})/I_\Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

Now we can take a restriction to $I_\Delta/I_\Delta^2 \subset (\mathcal{O} \otimes_k \mathcal{O})/I_\Delta^2$. Recall that I_Δ consists of sums of terms of the form

$$\begin{aligned} f_0 d_\Delta f_1 &= f_0(1 \otimes f_1 - f_1 \otimes 1) \\ &= f_0 \otimes f_1 - f_0 f_1 \otimes 1. \end{aligned}$$

Similarly I_Δ^2 consists of sums of terms of the form

$$\begin{aligned} f_0 d_\Delta f_1 d_\Delta f_2 &= f_0(1 \otimes f_1 - f_1 \otimes 1)(1 \otimes f_2 - f_2 \otimes 1) \\ &= f_0(1 \otimes f_1 f_2 + f_1 f_2 \otimes 1 - f_1 \otimes f_2 - f_2 \otimes f_1) \\ &= f_0 \otimes f_1 f_2 + f_0 f_1 f_2 \otimes 1 - f_0 f_1 \otimes f_2 - f_0 f_2 \otimes f_1) \end{aligned}$$

The last expression maps to

$$\begin{aligned} & df_0 \wedge d(f_1 f_2) + d(f_0 f_1 f_2) \wedge d1 - d(f_0 f_1) \wedge df_2 - d(f_0 f_2) \wedge df_1 \\ & = df_0 \wedge ((df_1) f_2 + f_1 df_2) - ((df_0) f_1 + f_0 df_1) \wedge df_2 - ((df_0) f_2 + f_0 df_2) \wedge df_1 \\ & = f_2 df_0 \wedge df_1 + f_1 df_0 \wedge df_2 - f_1 df_0 \wedge df_2 - f_0 df_1 \wedge df_2 - f_2 df_0 \wedge df_1 - f_0 df_2 \wedge df_1 \\ & = -f_0 df_1 \wedge df_2 - f_0 df_2 \wedge df_1 \\ & = 0. \end{aligned}$$

Proposition 1.6. *Any derivation $\delta \in \mathrm{Der}_k^m(\Omega^\bullet)$ can be uniquelly expressed as*

$$[\delta_\phi, d] + \delta_\psi$$

for $\phi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^m)$, $\psi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^{m+1})$.

Example 1.7. ($\mathcal{O}$-linear derivation) For $m = -1$ $\mathrm{Der}_k^{-1}(\Omega^\bullet) = \mathrm{Der}_{\mathcal{O}}^{-1}(\Omega^\bullet)$ and by restriction to Ω^1

$$\mathrm{Der}_{\mathcal{O}}^{-1}(\Omega^\bullet) = \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^\bullet).$$

If $\mathcal{O} = \mathcal{O}(X)$, then $\mathrm{Der}_k \mathcal{O} = \mathcal{T}X$.

$$\begin{tikzcd}
\Omega^1 \arrow[r] & \Omega^\bullet \\
\mathcal{O} \arrow[u, "d"] \arrow[ur] &
\end{tikzcd}$$

Suppose that $\delta, \delta' \in \mathrm{Der}_k^m(\Omega^\bullet)$ are such that

$$\delta_0 = \delta|_{\mathcal{O}} = \delta'|_{\mathcal{O}} = \delta_0'.$$

Then

$$\delta - \delta' \in \mathrm{Der}_{\mathcal{O}}^m(\Omega^\bullet) \quad \mathcal{O} - \text{linear}.$$

Suppose that we have a derivation $D \in \mathrm{Der}_k^1(\Omega^\bullet)$. Then for any $\phi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^m)$ there is a $\delta_\phi \in \mathrm{Der}_{\mathcal{O}}^{m-1}(\Omega^\bullet)$ and

$$[\delta_\phi, D] \in \mathrm{Der}^m(\Omega^\bullet)$$

$$[\delta_\phi, D]_0 = \delta_\phi D = \phi \circ D = d(\text{the de Rham derivation})$$

If there exists $d_1 \colon \Omega^1 \to \Omega^2$, k-linear and satisfying

$$d_1(f\alpha) = df \wedge \alpha + f d\alpha,$$

then there exists a derivation $d \in \mathrm{Der}_k^1(\Omega^\bullet)$.

There is a natural identification between $\mathcal{O}$-modules

$$\begin{tikzcd}
\mathrm{Der}(\mathcal{O}, \Omega^m) \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^m) \arrow[d, leftrightarrow] \\
& \mathrm{Der}_{\mathcal{O}}^{m-1}(\Omega^\bullet)
\end{tikzcd}$$

Let $\eta = \phi \circ d$ which on Ω^0 is $= [\delta_\phi, d]$. Then $\iota_\eta = \delta_\phi$ is the interior product with derivation η. If $m = -1$ this is the classical product of differential forms with a given vector field. Define a Lie derivative with respect to η

$$\mathcal{L}_\eta := [\delta_\phi, d] = [\iota_\eta, d].$$

Then

$$[\mathcal{L}_\eta, d] = [[\iota_{\eta,d}], d] = (-1)^{m-1} d\iota_\eta d - (-1)^m d\iota_\eta d = 0.$$

Any derivation δ is of the form $\delta = \mathcal{L}_\eta + \iota_\zeta$ where $\zeta = \psi \circ d$ for some $\psi \in \mathrm{Hom}_{\mathcal{O}}(\Omega^1, \Omega^{m+1})$. Consider a special $\phi \colon \Omega^1 \to \Omega^m$

$$\phi(\alpha) = \varphi \wedge \alpha$$

for some $\varphi \in \Omega^{m-1}$. Then

$$\begin{aligned}[\delta_\phi, d](\omega) &= \varphi \wedge d\omega - (-1)^{m-1} p d\varphi \wedge \omega \\ &= \varphi \wedge d\omega - (-1)^{m-1} d\varphi \wedge \deg .\end{aligned}$$

A degree map deg is a derivation $\deg = \delta_{\mathrm{id}}$, $\mathrm{id}\colon \Omega^1 \to \Omega^1$, $[\delta_{\mathrm{id}}, d] = d$.

Remark 1.8. To prove identities like $\delta = \delta'$, where δ, δ' are $\mathcal{O}$-linear derivations on $\Omega^\bullet$, it is enough to prove it on $d\mathcal{O} \subset \Omega^1$. For example, for vector fields there is an identity

$$[\mathcal{L}_\eta, \iota_\zeta] = [\iota_\eta, \mathcal{L}_\zeta] = \iota_{[\eta,\zeta]}.$$

The expressions are $\mathcal{O}$-linear, so we can check the equalities by evaluating on df, $f \in \mathcal{O}$.

For $\omega \in \Omega^p$ we have the formula

$$[\delta_{\varphi\wedge -}, d]^2(\omega) = \begin{cases} 0 & m = 1 \\ \frac{1-m}{2} d(\varphi \wedge \varphi) \wedge d\omega & \text{if } m \text{ is odd } \neq 1 \\ (m+p) p d\varphi \wedge d\varphi \wedge \omega & \text{if } m \text{ is even.} \end{cases}$$

For example if $m = 1$ φ is the contact 1-form on $\mathbb{A}^1$, that is $\sum_{i=1}^n \xi_i dx_i$.

$$\omega = \mathcal{L}_\Xi \omega = d\iota_\Xi \omega.$$

In case $m = 0$, for any function $f \in \mathcal{O}$ let $f \cdot -$ denote the multiplication by the function f

$$[\delta_{f\cdot -}, d] = fd - df \wedge \deg, \quad [\delta_{1\cdot -}, d] = d_{\mathrm{dR}}.$$

Let $\eta_1, \ldots, \eta_p \in \mathrm{Der}_k(\mathcal{O})$ (vector fields if $\mathcal{O} = \mathcal{O}(X)$). Then there is a formula

$$[d, \iota_{\eta_1} \ldots \iota_{\eta_p}] = \sum_{1\le i\le p} (-1)^{i-1} \iota_{\eta_1} \ldots \widehat{\iota_{\eta_i}} \ldots \iota_{\eta_p} \mathcal{L}_{\eta_i} + \tag{1.1}$$

$$+ \sum_{1\le i<j\le p} (-1)^{i+j-1} \iota_{[\eta_i,\eta_j]} \iota_{\eta_1} \ldots \widehat{\iota_{\eta_i}} \ldots \widehat{\iota_{\eta_j}} \ldots \iota_{\eta_p}.$$

where $\deg \iota_{\eta_i} = -1$ for all $i = 1, \ldots, p$. Similarly

$$[\iota_{\eta_p} \ldots \iota_{\eta_1}, d] = \sum_{1\le i\le p} (-1)^{i-1} \mathcal{L}_{\eta_i} \iota_{\eta_p} \ldots \widehat{\iota_{\eta_i}} \ldots \iota_{\eta_1} + \tag{1.2}$$

$$+ \sum_{1\le i<j\le p} (-1)^{i+j} \iota_{\eta_p} \ldots \widehat{\iota_{\eta_j}} \ldots \widehat{\iota_{\eta_i}} \ldots \iota_{\eta_1} \iota_{[\eta_i,\eta_j]}.$$

This is in analogy to the Cartan formula for $\omega \in \Omega^{p-1}$

$$(d\omega)(\eta_1, \ldots, \eta_p) = \sum_{1\le i\le p} (-1)^{i-1} \mathcal{L}_{\eta_i} \omega(\eta_1, \ldots, \widehat{\eta}_i, \ldots, \eta_p) + \tag{1.3}$$

$$+ \sum_{1\le i<j\le j} (-1)^{i+j} \omega([\eta_i, \eta_j], \eta_1, \ldots, \widehat{\eta}_i, \ldots, \widehat{\eta}_j, \ldots, \eta_p).$$

1.5 Koszul-Chevalley complex

Let $\mathfrak{m}$ be a $\mathfrak{g}$-module, where $\mathfrak{g}$ is a Lie k-algebra This means that $[,]\colon \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ satisfies Jacobi identity, each $g \in \mathfrak{g}$ acts as an endomorphism of a k-module $\mathfrak{m}$, and the map

$$\mathfrak{g} \to \mathfrak{gl}_k(\mathfrak{m}) = \mathrm{Lie}(\mathrm{End}_k(\mathfrak{m})), \quad g \mapsto \rho_g \text{ - action of } g \text{ on } \mathfrak{m}$$

is a homomorphism of Lie-k-algebras. We have

$$\rho_{[g_1,g_2]} = [\rho_{g_1}, \rho_{g_2}]$$

and $\mathfrak{gl}_k(\mathfrak{m})$ has the right $\mathfrak{g}$-module structure

$$\widetilde{\rho}_g(m) := mg,$$

$$mg_1g_2 - mg_2g_1 = (\widetilde{\rho}_{g_2}\widetilde{\rho}_{g_1} - \widetilde{\rho}_{g_1}\widetilde{\rho}_{g_2})(m) = [\widetilde{\rho}_{g_2}, \widetilde{\rho}_{g_1}] = m[g_2, g_1].$$

This shows that $\widetilde{\rho}\mathfrak{g} \to \mathfrak{gl}(m)$ is an antihomomorphism of Lie algebras (it corresponds to the fact that the inverse $G \to G$, $g \mapsto g^{-1}$ corresponds to $g \mapsto -g$ on $\mathfrak{g}$).

Definition 1.9. ***Koszul-Chevalley complex*** *of a Lie k-algebra* $\mathfrak{g}$ *with coefficients in* $\mathfrak{m}$

$$C_\bullet(\mathfrak{g}, \mathfrak{m}) := \mathfrak{m} \otimes \Lambda_k^\bullet \mathfrak{g}, \quad \partial\colon C_p(\mathfrak{g}, \mathfrak{m}) \to C_{p+1}(\mathfrak{g}, \mathfrak{m}),$$

where

$$\partial(m \otimes g_1 \wedge \cdots \wedge g_p) := \sum_{1 \le i \le p} (-1)^{i-1} g_i m \otimes g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge g_p +$$

$$+ \sum_{1 \le i < j \le p} (-1)^{i+j-1} m \otimes [g_i, g_j] \wedge g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge \widehat{g_j} \wedge \cdots \wedge g_p.$$

$$C^\bullet(\mathfrak{g}, \mathfrak{m}) :== \mathrm{Alt}^\bullet(\mathfrak{g} \times \cdots \times \mathfrak{g}, \mathfrak{m}), \quad \delta\colon C_{p-1}(\mathfrak{g}, \mathfrak{m}) \to C_p(\mathfrak{g}, \mathfrak{m}),$$

where for $\gamma \in \mathrm{Alt}^{p-1}(\mathfrak{g} \times \cdots \times \mathfrak{g}, \mathfrak{m})$ *we define* $\delta(\gamma) \in \mathrm{Alt}^p(\mathfrak{g} \times \cdots \times \mathfrak{g}, \mathfrak{m})$ *by*

$$\delta(\gamma)(g_1, \ldots, g_p) := \sum_{1 \le i \le p} (-1)^{i-1} g_i \gamma(g_1, \ldots, \widehat{g_i}, \ldots, g_p) +$$

$$+ \sum_{1 \le i < j \le p} (-1)^{i+j-1} \gamma([g_i, g_j], g_1, \ldots, \widehat{g_i}, \ldots, \widehat{g_j}, \ldots, g_p).$$

In the next definition we use a relative Tor and Ext groups, which are the derived functorsin the sense of relative homological algebra ([**?**], [**?**]).

Definition 1.10. ***Lie algebra homology and cohomology*** *with coefficients in a* $\mathfrak{g}$*-module* $\mathfrak{m}$

$$\mathrm{H}_\bullet(\mathfrak{g}; \mathfrak{m}) := \mathrm{H}(C_\bullet(\mathfrak{g}, \mathfrak{m}), \partial) \simeq \mathrm{Tor}_\bullet^{(\mathcal{U}(\mathfrak{g}),k)}(k, \mathfrak{m}),$$

$$\mathrm{H}_\bullet(\mathfrak{g}; \mathfrak{m}) := \mathrm{H}(C^\bullet(\mathfrak{g}, \mathfrak{m}), \delta) \simeq \mathrm{Ext}^\bullet_{(\mathcal{U}(\mathfrak{g}),k)}(k, \mathfrak{m}).$$

1.6 A relation between Hochschild and Lie algebra homology

Consider the following situation: A is an associative k-algebra with unit, M an A-bimodule. Let $\mathrm{Lie}(A) = A$ as a k-module with commutator bracket $[a,b] := ab - ba$. Let $a \in A$ act on $m \in M$ by $m \mapsto am - ma$. Consider $d_\Delta \colon A \to A \otimes A^{op}$, $a \mapsto 1 \otimes a^{op} - a \otimes 1$,

$$\begin{aligned}
[d_\Delta a, d_\Delta b] &= -\underbrace{[1 \otimes a^{op}, b \otimes 1]}_{=0} - \underbrace{[a \otimes 1, 1 \otimes b^{op}]}_{=0} + [1 \otimes a^{op}, 1 \otimes b^{op}] + [a \otimes 1, b \otimes 1] \\
&\text{(because } A \otimes 1 \text{ and } 1 \otimes A^{op} \text{ commute in } A\text{)} \\
&= 1 \otimes [a^{op}, b^{op}] + [a,b] \otimes 1 \\
&= 1 \otimes [b,a]^{op} - [b,a] \otimes 1 \\
&= -d_\Delta [a,b].
\end{aligned}$$

Universal derivation is an antihomomorphism, so

$$-d_\Delta \colon \mathrm{Lie}(A) \to \mathrm{Lie}(A \otimes A^{op})$$

is a homomorphism of Lie algebras.

In what follows we will use many arguments based on spectral sequences, and the necessary basics of the theory is presented in appendix (B).

Let $R = \mathcal{U}(\mathrm{Lie}(A))$, $S = A \otimes A^{op}$. Any bimodule N can be viewed as a left $A \otimes A^{op}$-module. The base change spectral sequence takes the form

$$E^2_{pq} = \mathrm{Tor}_p^{A \otimes A^{op}}(\mathrm{Tor}_q^{\mathcal{U}(\mathrm{Lie}(A))}(k, A \otimes A^{op}), N)$$

$$a \cdot (b \otimes c^{op}) = ab \otimes c^{op} - b \otimes a^{op} c^{op} = ab \otimes c^{op} - b \otimes (ca)^{op}$$

Assume that $\mathcal{U}(\mathrm{Lie}(A))$ is flat over k. Then

$$\mathrm{Tor}_p^{A \otimes A^{op}}(\mathrm{Tor}_q^{\mathcal{U}(\mathrm{Lie}(A))}(k, A \otimes A^{op}), N) \simeq \mathrm{Tor}_p^{A \otimes A^{op}}(\mathrm{H}_q(\mathrm{Lie}(A); A \otimes A^{op}), N).$$

In our base change spectral sequence we get an edge homomorphism

$$\mathrm{H}_p(\mathrm{Lie}(A); N) \to \mathrm{Tor}_p^{A \otimes A^{op}}(\mathrm{H}_0(\mathrm{Lie}(A); A \otimes A^{op}), N).$$

In general if $\mathfrak{g}$ is a Lie algebra, and M a $\mathfrak{g}$-module, then $\mathrm{H}_0(\mathfrak{g}; M) = M_\mathfrak{g}$ - the coinvariants of the $\mathfrak{g}$-action. Thus we have a map from Lie algebra homology to Hochschild homology

$$\mathrm{H}_p(\mathrm{Lie}(A); N) \to \mathrm{Tor}_p^{A \otimes A^{op}}(\underbrace{\mathrm{H}_0(\mathrm{Lie}(A); A \otimes A^{op})}_{A}, N) = \mathrm{Tor}_p^{A \otimes A^{op}}(A, N) = \mathrm{H}_p(A; N).$$

When k is of characteristic 0, that map, up to a sign, is induced by inclusion

$$\eta \colon \mathrm{C}_\bullet(\mathrm{Lie}(A); N) \to \mathrm{C}_\bullet(A; N)$$

$$n \otimes a_1 \wedge \cdots \wedge a_p \mapsto \sum_{l_1, \ldots, l_p} (-1)^{\overline{l_1 \ldots l_p}} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_p},$$

where on the right hand side we have a sum over all permutations of the set $\{1, \ldots, p\}$, and $\overline{l_1 \ldots l_p}$ denotes the sign of a permutation.

Proposition 1.11. *The map η is a map of complexes, that is*

$$b\eta = -\eta\partial,$$

where b is the Hochschild boundary, and ∂ the boundary of the Koszul-Chevalley complex.

Proof. On the left hand side we have:

$$\begin{aligned}
b\eta(n \otimes a_1 \wedge \cdots \wedge a_p) &= \sum_{l_1,\ldots,l_p} (-1)^{\overline{l_1\ldots l_p}} n a_{l_1} \otimes \cdots \otimes a_{l_p} \\
&\quad + \sum_{1\le m\le p-1} \sum_{l_1,\ldots,l_p} (-1)^{\overline{l_1\ldots l_p}+m} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p} \\
&\quad + \sum_{l_1,\ldots,l_p} (-1)^{\overline{l_1\ldots l_p}+p} a_{l_p} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_{p-1}} \\
&= \sum_{1\le i\le p} (-1)^{i-1} \sum_{\substack{l_1,\ldots,l_p \\ l_1=i}} (-1)^{\overline{l_2\ldots l_p}} n a_i \otimes a_{l_2} \otimes \cdots \otimes a_{l_p} \\
&\qquad\qquad (\text{because } \overline{i l_2 \ldots l_p} = \overline{l_2 \ldots l_p} \cdot (-1)^{i-1}) \\
&\quad - \sum_{1\le i\le p} (-1)^{i-1} \sum_{\substack{l_1,\ldots,l_p \\ l_p=i}} (-1)^{\overline{l_1\ldots l_{p-1} i}} a_i n \otimes a_{l_1} \otimes \cdots \otimes a_{l_{p-1}} \\
&\qquad\qquad (\text{because } \overline{l_1 \ldots l_{p-1} i} = \overline{l_1 \ldots l_{p-1}} \cdot (-1)^{p-i}) \\
&\quad + \sum_{1\le m\le p-1} \sum_{1\le i<j\le p} \sum_{\substack{l_1\ldots l_p \\ l_m=i,\ l_{m+1}=j}} (-1)^{\overline{l_1\ldots l_p}+m} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p} \\
&\quad + \sum_{1\le m\le p-1} \sum_{1\le j<i\le p} \sum_{\substack{l_1\ldots l_p \\ l_m=j,\ l_{m+1}=i}} (-1)^{\overline{l_1\ldots l_p}+m} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p} \\
&= \sum_{1\le i\le p} (-1)^i [a_i, n] \otimes a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_p \\
&\qquad\qquad (\text{because } \overline{l_1 \ldots l_p} \cdot (-1)^m = \overline{l_1 \ldots l_{m-1} l_{m+2} \ldots l_p} \cdot (-1)^{(i-1)+(j-1)}) \\
&\quad + \sum_{1\le m\le p-1} \sum_{1\le i<j\le p} (-1)^{(i-1)+(j-1)} \sum_{\substack{l_1\ldots l_p \\ l_m=i,\ l_{m+1}=j}} (-1)^{\overline{l_1\ldots l_{m-1} l_{m+2}\ldots l_p}} \\
&\qquad\qquad n \otimes a_{l_1} \otimes \cdots \otimes \underbrace{a_{l_m} a_{l_{m+1}}}_{a_i a_j} \otimes \cdots \otimes a_{l_p} \\
&\quad + \sum_{1\le m\le p-1} \sum_{1\le j<i\le p} (-1)^{(i-1)+(j-1)} \sum_{\substack{l_1\ldots l_p \\ l_m=j,\ l_{m+1}=i}} (-1)^{\overline{l_1\ldots l_{m-1} l_{m+2}\ldots l_p}} \\
&\qquad\qquad n \otimes a_{l_1} \otimes \cdots \otimes \underbrace{a_{l_m} a_{l_{m+1}}}_{a_j a_i} \otimes \cdots \otimes a_{l_p} \\
&= \eta\Bigg(\sum_{1\le i\le p} (-1)^i [a_i, n] \otimes a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_p \\
&\quad + \sum_{1\le i<j\le p} (-1)^{i+j+1} [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge \widehat{a_j} \wedge \cdots \wedge a_p \Bigg) \\
&= -\eta\partial(n \otimes a_1 \wedge \cdots \wedge a_p).
\end{aligned}$$

∎

1.7 Poisson trace

Consider the Lie algebra of derivations $\operatorname{Der}\mathcal{O} = \operatorname{Der}_k \mathcal{O}$. The algebra $\mathcal{O}$ is always a $\operatorname{Der}\mathcal{O}$-module via the natural representation. Let $\varphi \in \Omega^p_{\mathcal{O}/k}$. Then it defines an alternating $\mathcal{O}$-p-linear map

$$\underbrace{\operatorname{Der}\mathcal{O} \times \cdots \times \operatorname{Der}\mathcal{O}}_{p} \to \mathcal{O}$$

$$(\eta_1, \dots, \eta_p) \mapsto \varphi(\eta_1, \dots, \eta_p) := \iota_{\eta_p} \dots \iota_{\eta_1} \varphi \in \Omega^0 = \mathcal{O}.$$

There is an $\mathcal{O}$-linear map,

$$\Omega^\bullet \to \operatorname{Alt}^\bullet_{\mathcal{O}}(\operatorname{Der}\mathcal{O}, \mathcal{O}) \hookrightarrow \operatorname{Alt}^\bullet_k(\operatorname{Der}\mathcal{O}, \mathcal{O})$$

which transforms the de Rham differential d into δ

$$d\varphi \mapsto \delta(\iota_{\eta_p} \dots \iota_{\eta_1} \varphi).$$

(Cartan's picture of de Rham complex).

Let $\Omega^{vol} = \Omega^n$, where n is such that $\Omega^n \neq 0$, $d\colon \Omega^n \to \Omega^{n+1}$ identically 0. Then

$$C_\bullet(\operatorname{Der}\mathcal{O}; \Omega^{vol}) = \Omega^{vol} \otimes_k \Lambda^\bullet_k \operatorname{Der}_k \mathcal{O} \twoheadrightarrow \Omega^{vol} \otimes_{\mathcal{O}} \Lambda^\bullet_{\mathcal{O}} \operatorname{Der}_k \mathcal{O}$$

where the last epimorphism is $\mathcal{O}$-linearization and is an isomorphism if $\mathcal{O}$ is smooth algebra of dim n.

Claim 1.12. *The kernel of $\mathcal{O}$-linearization is a subcomplex of $C_\bullet(\operatorname{Der}\mathcal{O}; \Omega^{vol})$.*

For $\nu \in \Omega^{vol} = \Omega^n$

$$\nu \otimes \eta_1 \wedge \cdots \wedge \eta_p \mapsto \iota_{\eta_1} \dots \iota_{\eta_p} \nu \in \Omega^{n-p} =: \Omega_p.$$

The composition

$$C_\bullet(\operatorname{Der}\mathcal{O}; \Omega^{vol}) \to \Omega^{vol} \otimes_{\mathcal{O}} \Lambda^\bullet_{\mathcal{O}} \operatorname{Der}_k \mathcal{O}$$

is the map of complexes. It suffices to apply the formula for $[d, \iota_{\eta_1} \dots \iota_{\eta_p}]$ only to n-forms.

$$(C_\bullet(\operatorname{Der}_k \mathcal{O}, \Omega^{vol}), \partial) \twoheadrightarrow (\Omega_\bullet, d)$$

(Spencer's picture of de Rham complex).

Now we fix the volume form ν, and denote

$$\operatorname{Der}_k \mathcal{O}_\nu := \{\text{derivations annihilating } \nu\}.$$

There is an $\mathcal{O}$-module morphism

$$\mathcal{O} \to \Omega^{vol}, \quad f \mapsto f\nu,$$

$$C_\bullet(\operatorname{Der}_k \mathcal{O}_\nu, \mathcal{O}) \to C_\bullet(\operatorname{Der}\mathcal{O}, \Omega^{vol}) \to \Omega_\bullet$$

("Divergentless vector fields").

Suppose that $\mathcal{O} = \mathcal{O}(X)$, where X is a symplectic manifold of dimension $2n$, $\omega \in \Omega^2$ is closed and nondegenerate.

$$\omega\colon \operatorname{Der}\mathcal{O} \to \Omega^1, \quad \eta \mapsto \iota_\eta \omega$$

is injective. Furthermore $\omega^n \in \Omega^{vol}$ and we can take $\nu = \omega^n$.

Define $\mathrm{Ham}(X, \omega) \subset \mathrm{Der}_k \mathcal{O}_{\omega^n}$ as

$$\mathrm{Ham}(X, \omega) := \{\eta \in \mathrm{Der}\, \mathcal{O}_{\omega^n} \mid \mathcal{L}_\eta \omega = 0\}.$$

Define $\mathrm{Poiss}(X, \omega)$ as an algebra $\mathcal{O}$ with the Poisson bracket

$$\{f, g\} := \mathcal{L}_{H_f} g = \omega(H_f, H_g) = \iota_{H_g} \iota_{H_f} \omega,$$

where H_f is the vector field characterized by

$$\iota_{H_f} \omega = -df.$$

There is a homomorphism of Lie algebras

$$\mathrm{Poiss}(X, \omega) \to \mathrm{Ham}(X, \omega),$$

and an $\mathcal{O}$-linear map of complexes

$$C_\bullet(\mathrm{Poiss}(X, \omega), \mathrm{ad}) \to C_\bullet(\mathrm{Ham}(X, \omega), \omega^n).$$

$$f_0 \otimes f_1 \wedge \cdots \wedge f_p \mapsto f_0 \omega^n \otimes f_1 \wedge \cdots \wedge f_p.$$

There is also a map

$$C_\bullet(\mathrm{Ham}(X, \omega), \omega^n) \to \Omega_\bullet$$

$$f_0 \omega^n \otimes f_1 \wedge \cdots \wedge f_p \mapsto f_0 \iota_{H_{f_1}} \dots \iota_{H_{f_p}} \omega^n.$$

We have

$$\mathcal{L}_{H_f} = [d, \iota_{H_f}] \omega = 0.$$

Proposition 1.13. *For any $f, g \in \mathcal{O}$*

$$H_{f,g} = [H_f, H_g].$$

Proof. It is sufficient to prove the corresponding identity for contractions

$$\iota_{[H_f, H_g]} = \iota_{H_{\{f,g\}}}.$$

We have

$$\begin{aligned}
\iota_{[H_f, H_g]} \omega &= [\mathcal{L}_{H_f}, \iota_{H_g}] \\
&= \mathcal{L}_H (\iota_{H_g} \omega) - \iota_{H_g} \underbrace{\mathcal{L}_{H_f} \omega}_{0} \\
&= -\mathcal{L}_{H_f}(dg) \\
&= -d(\mathcal{L}_{H_f} g) \\
&= -d\{f, g\} \\
&= -\iota_{H_{\{f,g\}}}.
\end{aligned}$$

$\square$

There is a well defined map, called a **Poisson trace**

$$\mathrm{ptr}_\bullet\colon (C_\bullet(\mathrm{Poiss}(X,\omega);\mathrm{ad}),\partial) \twoheadrightarrow (\Omega_\bullet, d).$$

Let Y be a symplectic manifold, $\dim Y = 2n$, with a symplectic 2-form ω. Then we have a canonical morphism of chain complexes

$$\mathrm{ptr}\colon C_\bullet(\mathrm{Poiss}(Y,\omega);\mathrm{ad}) \twoheadrightarrow \Omega_\bullet(Y),$$

where $\Omega_q(Y) = \Omega^{\dim Y - q}(Y)$, given by

$$f_0 \otimes f_1 \wedge \cdots \wedge f_q \mapsto f_0 \iota_{H_{f_1}} \ldots \iota_{H_{f_q}} \omega^n.$$

An important special case is when Y is a symplectic cone, i.e. Y is acted upon by $\mathbb{R}_+^*$. Let Ξ be the corresponding Euler field (the image of $t\frac{d}{dt}$). We have $t^*\omega = t\omega$ or equivalently

$$\mathcal{L}_\Xi \omega = \omega.$$

1.7.1 Graded Poisson trace

We consider the graded algebra of functions on Y

$$\mathcal{O}_\bullet := \bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m),$$

where

$$\mathcal{O}(m) := \{f \in \mathcal{O} \mid \mathcal{L}_\Xi f = mf\}.$$

Then the Poisson bracket $\{\cdot, \cdot\}$ agrees with the grading in the following way

$$\{\mathcal{O}(l), \mathcal{O}(m)\} \subseteq \mathcal{O}(l+m-1).$$

Let

$$\mathcal{P}_l := \mathcal{O}(l+1), \quad \mathcal{P}_\bullet := \bigoplus_{l\in\mathbb{Z}} \mathcal{P}_l$$

be the graded Lie algebra when equipped with the Poisson bracket $\{\cdot, \cdot\}$. The map $f \mapsto H_f$ is a homomorphism of Lie algebras $\mathcal{O} \to \mathcal{P} = \mathrm{Poiss}(Y,\omega)$, and furthermore

$$\mathcal{L}_\Xi H_f = (\deg(f) - 1)H_f.$$

To check this identity one computes

$$\iota_{[\Xi,H_f]}\omega = [\mathcal{L}_\Xi, \iota_{H_f}]\omega = -\deg(f)df + df = (1-\deg(f))df = (\deg(f)-1)H_f$$

because $\iota_{H_f}\omega = -df$. Thus there is a **graded Poisson trace**

$$\mathrm{ptr}_\bullet\colon C_\bullet(\mathcal{P}_\bullet, \mathrm{ad}) \to \Omega_{\bullet\bullet}(Y)$$

$$\mathrm{ptr}_\bullet\colon \bigoplus_{k\in\mathbb{Z}} C_\bullet^{(k)}(\mathcal{P}_\bullet, \mathrm{ad}) \to \Omega_{\bullet,k+n}(Y),$$

where

$$C_\bullet^{(k)}(\mathcal{P}_\bullet, \mathrm{ad}) = (\mathcal{P}_\bullet \otimes \Lambda^q \mathcal{P}_\bullet)(k+q)$$

and ∂ preserves k. Explicitely we have

$$\begin{aligned}\mathcal{L}_\Xi(f_0\iota_{H_{f_1}} \ldots \iota_{H_{f_q}}\omega^n) &= (l_0 + (l_1 - 1) + \ldots + (l_q - 1) + m) f_0\iota_{H_{f_1}} \ldots \iota_{H_{f_q}}\omega^n \\ &= ((l_0 + \ldots + l_q) + n - q) f_0\iota_{H_{f_1}} \ldots \iota_{H_{f_q}}\omega^n,\end{aligned}$$

$$(\mathcal{P}_\bullet \otimes \Lambda^q \mathcal{P}_\bullet)(l) \to \Omega_q(l-q)$$

1.8 Hochschild homology

Let $C_\bullet(\mathrm{CS}(X))$ be the completed Hochschild complex of $\mathrm{CS}(X)$. Define

$$C_\bullet^{(m)} := C_\bullet(\mathrm{CS}(X))/F_{m-1}C_\bullet(\mathrm{CS}(X)),$$

where $F_{m-1}C_\bullet(\mathrm{CS}(X))$ is the filtration induced by order. Then

$$C_j = \lim_{m\to-\infty} C_j^{(m)}, \quad j \in \mathbb{N}$$

The complexes $C_\bullet^{(m)}$ inherit filtration from $C_\bullet$

$$\{0\} = F_{m-1}C_\bullet^{(m)} \subset F_m C_\bullet^{(m)} \subset \dots$$

where

$$F_p C_\bullet^{(m)} := \begin{cases} F_p C_\bullet(\mathrm{CS}(X))/F_{m-1}C_\bullet(\mathrm{CS}(X)) & \text{for } p \geq m-1, \\ 0 & \text{for } p \leq m-1. \end{cases} \tag{1.4}$$

We have

$$C_j^{(m)} = \lim_{p\to\infty} F_{pj}^{(m)}, \quad m \in \mathbb{Z},\ j \in \mathbb{N}.$$

Let $\mathrm{HH}_\bullet^{(m)}$ denote the homology of $C_\bullet^{(m)}$ and $\mathrm{HH}_\bullet$ the homology of $C_\bullet$. Our first objective will be to find $\mathrm{HH}_\bullet^{(m)}$.

There is a **Milnor short exact sequence**

$$0 \to \lim{}^1 \mathrm{H}_{q+1}(C_\bullet^{(m)}) \to \mathrm{HH}_q(\mathrm{CS}(X)) \to \lim \mathrm{H}_q(C_\bullet^{(m)}) \to 0.$$

If the system $\{\mathrm{H}_{q-1}(C_\bullet^{(m)})\}_{m\to-\infty}$ satisfies the Mittag-Leffler condition, then $\lim^1$ vanishes.

Suppose $\{V_\lambda\}$ is an inverse system of sets (k-modules). It satisfies Mittag-Leffler condition if for all λ the system of subsets $(\mathrm{im}(V_\mu \to V_\lambda))$ for $\mu > \lambda$ stabilizes. The inverse system $\{V_\lambda\}$ can be treated as a sheaf $\widetilde{V}$ over the indexing set Λ with partial order topology. Then

$$\lim{}^p\{V_\lambda\} = \mathrm{H}^p(\Lambda, \widetilde{V}),$$

and in particular $\lim\{V_\lambda\} = \Gamma(\Lambda, \widetilde{V})$.

Theorem 1.14 (Emmanouil). *For $\Lambda = \omega$ - the first infinite ordinal, the inverse system of vector spaces $\{V_\lambda\}$ is Mittag-Leffler if and only if one of the following conditions is satisfied*

$$\lim{}^1\{V_\lambda \otimes_k W\} = 0, \text{ for all vector spaces } W \text{ over } k, \tag{1.5}$$

$$\lim{}^1\{V_\lambda \otimes_k W\} = 0, \text{ for some infinite dimensional vector space } W \text{ over } k. \tag{1.6}$$

Recall that $T_0^*X = T^*X \setminus X$ and Y^c is the $\mathbb{C}^*$-bundle over the cosphere bundle S^*X defined as

$$\begin{array}{c} Y^c := T_0^*X \times_{\mathbb{R}_+} \mathbb{C}^* \\ \Big\downarrow{\scriptstyle \mathbb{C}^*} \\ S^*X \end{array}$$

Consider the eigenspace of the action of the Euler field $\Xi = \sum_{i=1}^{n} \xi_i \partial \xi_i$ on $T_0^* X$

$$\Omega^\bullet(T_0^* X)(m) \subset \Omega^\bullet_{C^\infty}(T_0^* X)$$

$$t^* \eta = t^m \eta$$

Then

$$\Omega^{\bullet\bullet}(T_0^* X) := \bigoplus_{m \in \mathbb{Z}} \Omega^\bullet(T_0^* X)(m)$$

is a bigraded algebra whose cohomology is naturally isomorphic with $\mathrm{H}^\bullet(Y^c)$. We denote it by $\mathrm{H}^\bullet_{\mathrm{dR}}(Y^c)$.

There is a spectral sequence $'E^{(m),r}_{\bullet\bullet}$ converging to $\mathrm{HH}^{(m)}_\bullet$ which is associated with the filtration (1.4) of $\mathrm{C}^{(m)}_\bullet$. Its complete description is provided in the following proposition.

Proposition 1.15. *Assume $m \leq -\dim X = -n$. Then*

a) the second term of a spectral sequence $'E^{(m),r}_{\bullet\bullet}$ which is associated with the filtration on $\mathrm{C}^{(m)}_\bullet$ which is induced by the order filtration as in (1.4) is given by

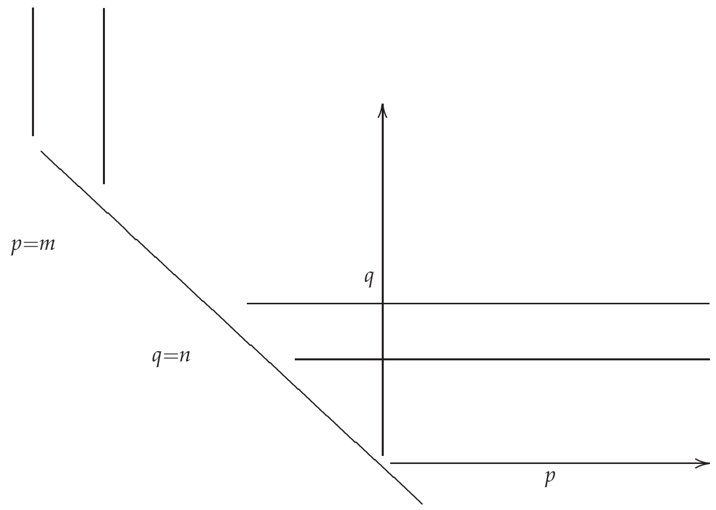
$$'E^{(m),2}_{pq} \simeq \begin{cases} \mathrm{H}^{n-p}_{\mathrm{dR}}(Y^c) & q = n \\ \Omega^{2n-m-q}(n-q)/d\Omega^{2n-1-m-q}(n-q) & p = m \\ 0 & \text{otherwise} \end{cases}$$

b) the spectral sequence $'E^{(m),r}_{\bullet\bullet}$ degenerates at $'E^2$

c) the identification in a) are compatible with the spectral sequence morphisms induced by the canonical spectral sequence projections

$$\mathrm{C}^{(l)}_\bullet \twoheadrightarrow \mathrm{C}^{(m)}_\bullet$$

for $l \leq m$.



Corollary 1.16. *The inverse system of the homology groups* $\{\mathrm{HH}_p^{(m)}\}_{m\in\mathbb{Z}_{<-n}}$ *satisfies Mittag-Leffler condition, in fact*

$$\mathrm{HH}_p^{(l_1,m)} = \mathrm{HH}_p^{(l_2,m)}$$

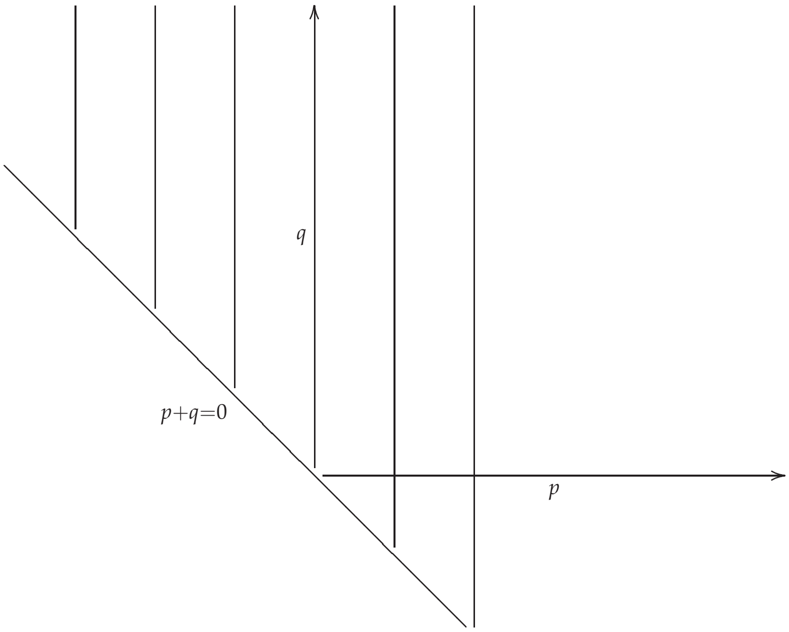
for any $l_1 \le l_2 \le m < -n$*, where* $\mathrm{HH}_p^{(l,m)} := \mathrm{im}(\mathrm{HH}_p^{(l)} \to \mathrm{HH}_p^{(m)})$*.*

Proof. From the proposition (1.15) we obtain a commutative diagrams whose rows are exact.

$$\begin{tikzcd}
\frac{\Omega^{2n-p}(n+m-p)}{d\Omega^{2n-1-p}(n+m-p)} \arrow[r, tail] & \mathrm{H}_p^{(m)} \arrow[r, two heads] & \mathrm{H}_{\mathrm{dR}}^{2n-p}(Y^c) \\
\frac{\Omega^{2n-p}(n+l-p)}{d\Omega^{2n-1-p}(n+l-p)} \arrow[r, tail] \arrow[u, "0"] & \mathrm{H}_p^{(l)} \arrow[r, two heads] \arrow[u] & \mathrm{H}_{\mathrm{dR}}^{2n-p}(Y^c) \arrow[u, equal]
\end{tikzcd}$$

□

Consider a spectral sequence with ${}'E^0_{p\bullet}$ being the p-th component of the graded complex $\mathrm{gr}^F(\mathrm{CS}(X))$.



Taking homology with respect to the differential $d^0_{p\bullet} : {}'E^0_{p\bullet} \to' E^0_{p,\bullet-1}$ we obtain

$${}'E^1_{pq} = \mathrm{HH}_{p+q}(\mathcal{O}_\bullet(X))(p),$$

calculated in terms of differential forms.

If $\mathcal{O}$ is a smooth algebra, there is a map of complexes

$$(C_\bullet, b) \to (\Omega^\bullet, 0)$$

$$f_0 \otimes \cdots \otimes f_q \to f_0 df_1 \wedge \cdots \wedge df_q.$$

But instead of this map we take

$$f_0 \otimes \cdots \otimes f_q \to \frac{(-1)^q}{q!} f_0 \iota_{H_{f_1}} \ldots \iota_{H_{f_q}} \omega^n.$$

We can compose the two maps

$$\begin{tikzcd}
(C_\bullet(\mathrm{Lie}(\mathrm{CS}(X))),\partial) \arrow[r] \arrow[rr, bend right] & (C_\bullet(\mathrm{CS}(X)),b) \arrow[r] & (\Omega_{\bullet\bullet},d).
\end{tikzcd}$$

The first map

$$\eta\colon a_0\otimes a_1\wedge\cdots\wedge a_q \mapsto \sum_{l_1,\dots,l_q}(-1)^{\overline{l_1\dots l_q}}a_0\otimes a_{l_1}\otimes\cdots\otimes a_{l_q},$$

is a map of complexes, while the second one is a map of complexes only if $d=0$. But the composition is still a map of complexes.

We identified ${}'E^{(m),1}_{pq}$ with $\Omega^{2n-p-q}_{\mathcal{O}}(n-q)$ for $p\geq m$ and d^1 with d_{dR}.

To demonstrate that the spectral sequence degenerates at ${}'E_2$ one has to show that the only possibly nontrivial differentials

$$d^{(m),p-m}_{pn}\colon {}'E^{(m),p-m}_{pn}\to{}'E^{(m),p-m}_{m,n+p-m-1}$$

all vanish. This is a consequence of the commutativity of the diagram

$$\begin{tikzcd}
{}'E^{(m),p-m}_{pn} \arrow[r, "d^{(m),p-m}_{pn}"] & {}'E^{(m),p-m}_{m,n+p-m-1} \\
{}'E^{(l),p-m}_{p,n} \arrow[r, "d^{(l),p-m}_{pn}"] \arrow[u, "\simeq"] & {}'E^{(l),p-m}_{m,n+p-m-1} \arrow[u]
\end{tikzcd}$$

for $l<m$.

Now $\mathrm{H}_\bullet=\mathrm{HH}_\bullet(\mathrm{CS}(X))$ is the homology of the projective limit $\lim C^{(m)}_\bullet$. The projective system $C^{(m)}$ satisfies Mittag-Leffler condition. The same holds for the projective systems of homology groups $\{\mathrm{HH}^{(m)}_\bullet\}_{m\in\mathbb{Z}_{<-n}}$ by corollary (1.16). Hence

$$\mathrm{HH}_j=\lim_m \mathrm{HH}^{(m)}_j\simeq \mathrm{H}^{2n-j}_{\mathrm{dR}}(Y^c),$$

and we proved the theorem.

Theorem 1.17. *There is a canonical isomorphism*

$$\mathrm{HH}_q(\mathrm{CS}(X))\simeq \mathrm{H}^{2n-q}_{\mathrm{dR}}(Y^c).$$

1.9 Cyclic homology

We will use the Connes double complex $\mathcal{B}_{\bullet\bullet}(\mathrm{CS}(X))$. The maps I, B, S which involve Hochschild and cyclic homology $\mathrm{HH}_\bullet$, $\mathrm{HC}_\bullet$ are induced by morphism of filtered chain complexes.

$$C_\bullet(\mathrm{CS}(X))\rightarrowtail \mathrm{Tot}(\mathcal{B}_{\bullet\bullet}(\mathrm{CS}(X)))\twoheadrightarrow \mathrm{Tot}(\mathcal{B}_{\bullet\bullet}(\mathrm{CS}(X)))[2]$$

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"'] & {} \arrow[d, "b"'] \\
\mathrm{CS}(X)^{\otimes 3} \arrow[d, "b"'] & \mathrm{CS}(X)^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"'] & \mathrm{CS}(X) \arrow[l, "B"'] \\
\mathrm{CS}(X)^{\otimes 2} \arrow[d, "b"'] & \mathrm{CS}(X) \arrow[l, "B"'] & \\
\mathrm{CS}(X) & &
\end{tikzcd}$$

The first column is a Hochschild complex $C_\bullet(\mathrm{CS}(X))$. The rest is the same complex but shifted diagonally by 1, so the total complex is shifted by 2.

Let us put

$$\mathcal{B}^{(m)}_{\bullet\bullet} := \mathcal{B}_{\bullet\bullet}/F_{m-1}\mathcal{B}_{\bullet\bullet},$$

where $F_p\mathcal{B}_{kl} := F_pC_{l-k}$. Much as we did before we consider the projective system of quotient complexes

$$\operatorname{Tot}\mathcal{B}^{(m)}_{\bullet\bullet} = \operatorname{Tot}\mathcal{B}_{\bullet\bullet}/F_{m-1}\mathcal{B}_{\bullet\bullet}, \quad m \to -\infty.$$

Then we have

$$\mathcal{B}^{(m)}_{kl} = \lim_{p\to\infty} F^{(m)}_{pkl}, \quad m \in \mathbb{Z},\ k,l \geq 0$$

and

$$\mathcal{B}_{kl} = \lim_{m\to-\infty} \mathcal{B}^{(m)}_{kl}, \quad k,l \geq 0,$$

where

$$F^{(m)}_{pkl} := F_p\mathcal{B}_{kl}/F_{m-1}\mathcal{B}_{kl}.$$

Let $\mathrm{HC}^{(m)}_\bullet$ denote the homology of $\operatorname{Tot}\mathcal{B}^{(m)}_{\bullet\bullet}$, and $\mathrm{HC}_{\bullet\bullet}$ the homology of $\operatorname{Tot}\mathcal{B}_{\bullet\bullet}$.

Proposition 1.18. *Assume that $m \leq 0$ and $q \geq 2n+1$. Then there exist isomorphisms*

$$\mathrm{HC}^{(m)}_q \simeq \begin{cases} \mathrm{H}^{ev}_{\mathrm{dR}}(Y^c) & q \text{ even} \\ \mathrm{H}^{odd}_{\mathrm{dR}}(Y^c) & q \text{ odd} \end{cases}$$

compatible with the canonical maps $\mathrm{HC}^{(m')}_q \to \mathrm{HC}^{(m)}_q$ for $m' \leq m$.

In particular, the systems $\{\mathrm{HC}^{(m)}\}_{m\in\mathbb{Z}_{\leq 0}}$ satisfy for $q \geq 2n+1$ the Mittag-Leffler condition. This gives us a corollary.

Corollary 1.19. *There are, for $q \geq 2n+1$, natural isomorphisms*

$$\mathrm{HC}_q \simeq \lim_{m\to-\infty} \mathrm{HC}^{(m)}_q \simeq \begin{cases} \mathrm{H}^{ev}_{\mathrm{dR}}(Y^c) & q \text{ even} \\ \mathrm{H}^{odd}_{\mathrm{dR}}(Y^c) & q \text{ odd} \end{cases}$$

This corollary together with a theorem (1.17) imply the following theorem for cyclic homology of an algebra of symbols if $\dim \mathrm{H}^\bullet_{\mathrm{dR}}(Y^c) < \infty$.

Theorem 1.20. *The canonical map*

$$I\colon\ \mathrm{HH}_\bullet(\mathrm{CS}(X)) \to \mathrm{HC}_\bullet(\mathrm{CS}(X))$$

is injective. In particular

$$\mathrm{HC}_{qr}(\mathrm{CS}(X)) = \mathrm{gr}^S_r \mathrm{HC}_q(\mathrm{CS}(X)) := \mathcal{S}_{qr}/\mathcal{S}_{q,r-1}, \quad \mathcal{S}_{qr} = \ker S^{1+r}_* \cap \mathrm{HC}_q(\mathrm{CS}(X))$$

is canonically isomorphic with

$$\mathrm{H}^{2n-q+2r}_{\mathrm{dR}}(Y^c), \quad r = 0, 1, \ldots.$$

With some more work we can prove the theorem without assumption of finite dimension of $\mathrm{H}^\bullet_{\mathrm{dR}}(Y^c)$. Then one represents X as a union $\bigcup_{j\in\mathbb{N}} X_j$, where each X_j is compact (with smooth or empty boundary) and $X_j \subset \operatorname{Int} X_{j+1}$. Then the restriction maps $\mathrm{CS}(X) \to \mathrm{CS}(X_j)$ induce homomorphisms

$$\theta\colon\ \mathrm{HH}_\bullet(\mathrm{CS}(X)) \to \widehat{\mathrm{HH}}_\bullet := \lim_j \mathrm{HH}_\bullet(\mathrm{CS}(X)), \tag{1.7}$$

$$\eta\colon\ \mathrm{HC}_\bullet(\mathrm{CS}(X)) \to \widehat{\mathrm{HC}}_\bullet := \lim_j \mathrm{HC}_\bullet(\mathrm{CS}(X)). \tag{1.8}$$

For each q there is a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_q(\mathrm{CS}(X)) \arrow[r, "\theta_q"] \arrow[d, "\simeq"'] & \widehat{\mathrm{HH}_q} \arrow[d, "\simeq"] \\
\mathrm{H}^{2n-q}_{\mathrm{dR}} \arrow[r] & \lim_j \mathrm{H}^{2n-q}_{\mathrm{dR}}(Y^c_j)
\end{tikzcd}$$

Notice that also the lower arrow is an isomorphism, since

$$\Omega^\bullet_{\mathcal{O}} = \lim_j \Omega^\bullet_{\mathcal{O}_j},$$

where $\mathcal{O}_j$ denotes the corresponding graded algebra of functions on Y^c_j. Since both projective systems $\{\Omega^\bullet_j\}$ and $\{\mathrm{H}^\bullet_{\mathrm{dR}}(Y^c_j)\}$ satisfy Mittag-Leffler condition, we have that θ in (1.7) is an isomorphism.

The naturality of the Connes exact sequence gives us the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & \widehat{\mathrm{HH}}_q \arrow[r, "\hat{I}"] & \widehat{\mathrm{HC}}_q \arrow[r, "\hat{S}"] & \widehat{\mathrm{HC}}_{q-2} \arrow[r, "0"] & \widehat{\mathrm{HH}}_{q-1} \arrow[r, "\hat{I}"] & \cdots \\
\cdots \arrow[r, "B"] & \mathrm{HH}_q \arrow[r, "I"] \arrow[u, "\theta_q", "\simeq"'] & \mathrm{HC}_q \arrow[r, "S"] \arrow[u, "\eta_q"] & \mathrm{HC}_{q-2} \arrow[r, "B"] \arrow[u, "\eta_{q-2}"] & \mathrm{HH}_{q-1} \arrow[r, "I"] \arrow[u, "\theta_{q-1}", "\simeq"'] & \cdots
\end{tikzcd}$$

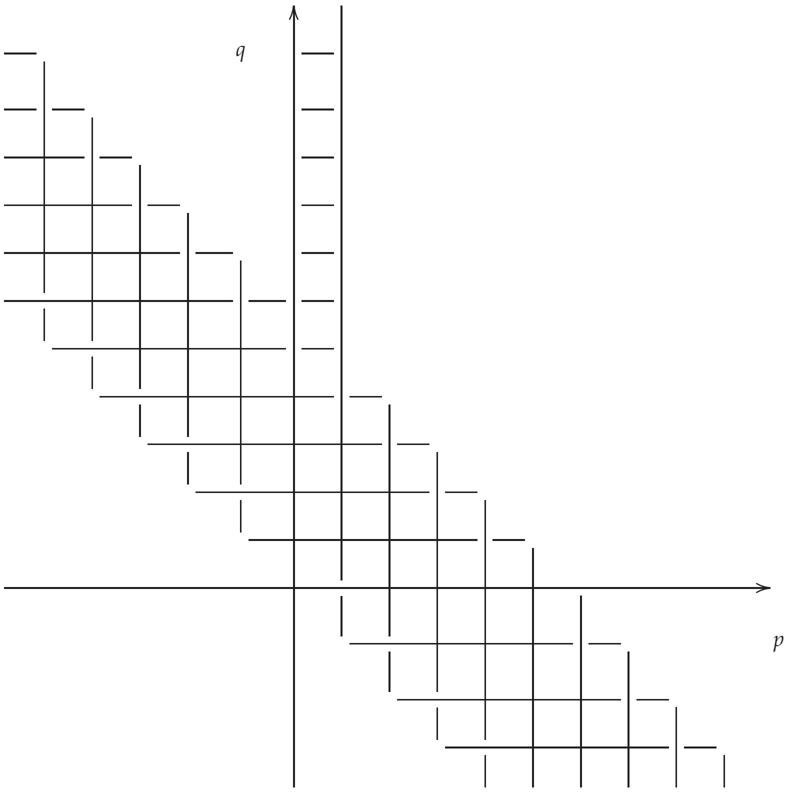
with a priori only the lower sequence being exact. The exactness of the upper sequence follows from

$$\overset{1}{\lim}\, \mathrm{HH}_q(\mathrm{CS}(X_j)) = 0, \text{ for all } q \in \mathbb{N},$$

which is a consequence of the finite-dimensionality of the groups $\mathrm{HH}_q(\mathrm{CS}(X_j)) = \mathrm{H}_{\mathrm{dR}}(Y^c_j)$. Thus the "five lemma" and an easy inductive argument prove that η is an isomorphism and $B = 0$.

Now it remains to prove the proposition (1.18). The filtration $\{F^{(m)}_{p\bullet\bullet} \mid p = m, m+1, \ldots\}$ on $\mathcal{B}^{(m)}_{\bullet\bullet}$ induces a filtration on $\operatorname{Tot}\mathcal{B}^{(m)}_{\bullet\bullet}$. Denote by $E^{(m),r}_{pq}$ the associated spectral sequence which converges to $\mathrm{HC}^{(m)}_\bullet$.

This spectral sequence is a priori located in the region $\{(p,q) \mid p \geq m,\ p+q \geq 0\}$. We shall see that $E^{(m),r}_{pq}$ for $r \geq 1$ vanishes in fact outside the region shown below



i.e. $E^{(m),r}_{pq} = 0$ also if $p + q \geq 2n$ and $p \neq 0$.

Indeed, $E^{(m),1}_{pq}$ is equal, for $p \geq m$, to

$$\mathrm{H}_{p+q}(\mathrm{Tot}\,\mathcal{B}_{\bullet\bullet}(\mathcal{O})(p)) = \mathrm{HC}_{p+q}(\mathcal{O})(p).$$

Actually, the first spectral sequence of the double complex $\mathcal{B}_{\bullet\bullet}(\mathcal{O})(p)$ degenerates at E^2 yielding thus that

$$E^{(m),1}_{pq} \simeq \Omega^{p+q}_{\mathcal{O}}(p)/d\Omega^{p+q-1}_{\mathcal{O}}(p), \quad p \geq m,\ p \neq 0,$$

and

$$E^{(m),1}_{0q} \simeq \mathrm{H}^{\tilde{q}}_{\mathrm{dR}}(Y^c), \quad q \geq 2n,$$

where $\tilde{q}$ is the parity of q and $\mathrm{H}^{\bullet}_{\mathrm{dR}} = \mathrm{H}^{(0)}_{\mathrm{dR}}(Y^c) \oplus \mathrm{H}^{(1)}_{\mathrm{dR}}(Y^c)$. This implies the required location of non-vanishing $E^{(m),r}_{pq}$ and as a corollary gives

$$\mathrm{HC}^{(m)}_q \simeq E^{(m),1}_{0q} \simeq \mathrm{H}^{(\tilde{q})}_{\mathrm{dR}}(Y^c)$$

for $q \geq 2n+1$. The isomorphisms are also compatible with the canonical mappings $\mathrm{HC}^{(m')}_q \to \mathrm{HC}^{(m)}_q$.

1.9.1 Further analysis of spectral sequence

We will use the notation ${}'E_{pq}^{(m),r}$ for the earlier spectral sequence converging to Hochschild homology $\mathrm{HH}^{(m)}$.

First, let us consider the morphism of spectral sequences induced by S

$$\begin{tikzcd} {}'E_{pq}^{(m),r} \arrow[d, "S_{pq}^{(m),r}"] \\ {}'E_{p,q-2}^{(m),r} \end{tikzcd}$$

For r=1 we have

$$E_{pq}^{(m),1} = \begin{cases} \mathrm{HC}_{p+q}(\mathcal{O})(p), \quad \mathcal{O} = \mathrm{gr}(\mathrm{CS}(\mathrm{X})) = \bigoplus_{p\in\mathbb{Z}} \mathcal{O}(p) & p \geq m \\ 0 & p < m \end{cases}$$

Then

$$\begin{tikzcd} E_{pq}^{(m),1} \arrow[d, "S_{pq}^{(m),1}"] \\ E_{p,q-2}^{(m),1} \end{tikzcd}$$

is the corresponding component of the S-map on cyclic homology of graded algebra $\mathcal{O}$.

If $p = 0$

$$\mathrm{HC}_{p+q}(\mathcal{O}) = \overline{\Omega}^q \oplus \mathrm{H}_{\mathrm{dR}}^{q-2} \oplus \mathrm{H}_{\mathrm{dR}}^{q-4} \oplus \dots,$$

where

$$\Omega^\bullet := \Omega^\bullet_{\mathcal{O}}, \quad \mathrm{H}^\bullet_{\mathrm{dR}} := \mathrm{H}^\bullet(\Omega^\bullet).$$

$$\overline{\Omega}^k(p) := \Omega^k(p)/d\Omega^{k-1}(p)$$

For $p \neq 0$

$$\mathrm{HC}_{p+q}(\mathcal{O})(p) = \begin{cases} \overline{\Omega}^{p+q}(p) & p \geq m \\ 0 & p < m \end{cases}$$

$$\begin{tikzcd}[column sep=small]
p=-2 & p=-1 & p=0 & p=1 & p=2 \\
\overline{\Omega}^{q-2}(-2) \arrow[d, "0"] & \overline{\Omega}^{q-1}(-1) \arrow[l, "d^1"'] \arrow[d, "0"] & \overline{\Omega}^{q} \oplus \mathrm{H}_{\mathrm{dR}}^{q-2} \oplus \mathrm{H}_{\mathrm{dR}}^{q-4} \oplus \dots \arrow[l, "d^1"'] \arrow[d] & \overline{\Omega}^{q+1}(1) \arrow[l, "d^1"'] \arrow[d] & \overline{\Omega}^{q+2}(2) \arrow[l, "d^1"'] \arrow[d] \\
\overline{\Omega}^{q-2}(-4) & \overline{\Omega}^{q-1}(-3) \arrow[l, "d^1"'] & \overline{\Omega}^{q-2} \oplus \mathrm{H}_{\mathrm{dR}}^{q-4} \oplus \mathrm{H}_{\mathrm{dR}}^{q-4} \oplus \dots \arrow[l, "d^1"'] & \overline{\Omega}^{q-1}(1) \arrow[l, "d^1"'] & \overline{\Omega}^{q}(2) \arrow[l, "d^1"']
\end{tikzcd}$$

where for $p = 0$ we have

$$\begin{tikzcd}[column sep=small]
\overline{\Omega}^q \arrow[d] & \oplus & \mathrm{H}_{\mathrm{dR}}^{q-2} \arrow[d] & \oplus & \mathrm{H}_{\mathrm{dR}}^{q-4} \arrow[d, equal] & \oplus & \cdots \\
0 & \oplus & \overline{\Omega}^{q-2} & \oplus & \mathrm{H}_{\mathrm{dR}}^{q-4} & \oplus & \cdots
\end{tikzcd}$$

Denote

$$\overline{E}^{(m),1}_{pq} := \begin{cases} \overline{\Omega}^{p+q}(p) & p \geq 0 \\ 0 & p < 0 \end{cases}$$

Corollary 1.21. *There is an isomorphism of chain complexes*

$$(E^{(m),1}_{\bullet,q}, d^1_{\bullet,q}) \simeq (\overline{E}^{(m),1}_{\bullet,q} \oplus (\mathrm{H}^{q-2}_{\mathrm{dR}} \oplus \mathrm{H}^{q-4}_{\mathrm{dR}} \oplus \dots)[0], d)$$

and there is an exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[d] \\ (\mathrm{H}^{q-1}_{\mathrm{dR}}[0], 0) \arrow[d, hook] \\ (\overline{E}^{(m),1}_{\bullet,q-1}, d^1) \arrow[d, "B"] \\ ({}'E^{(m),1}_{\bullet,q}, d^1) \arrow[d, two heads] \\ (\overline{E}^{(m),1}_{\bullet,q}, d^1) \arrow[d] \\ 0 \end{tikzcd}$$

Consider the second spectral sequence of the double complex but arranged according to conventions of Cartan-Eilenberg's book. Denote it by ${}_q\mathcal{E}^r_{\bullet\bullet}$, although it depends also on m.

The ${}_q\mathcal{E}^2_{\bullet\bullet}$ looks as follows.

$$\begin{tikzcd}[column sep=small] & & \mathrm{H}^{q-2}_{\mathrm{dR}} & & & \\ \overline{E}^{(m),2}_{-2,q-1} \arrow[ddr] & \overline{E}^{(m),2}_{-1,q-1} \arrow[ddr] & \overline{E}^{(m),2}_{0,q-1} \arrow[ddr] & \overline{E}^{(m),2}_{1,q-1} \arrow[ddr] & \overline{E}^{(m),2}_{2,q-1} \arrow[ddr] & \overline{E}^{(m),2}_{3,q-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{E}^{(m),2}_{-2,q-1} & \overline{E}^{(m),2}_{-1,q-1} & \overline{E}^{(m),2}_{0,q-1} & \overline{E}^{(m),2}_{1,q-1} & \overline{E}^{(m),2}_{2,q-1} & \overline{E}^{(m),2}_{3,q-1} \end{tikzcd}$$

There is an isomorphism

$$\overline{E}^{(m),2}_{pq} \xrightarrow{\simeq} \overline{E}^{(m),2}_{p+1,q+1}$$

except $(p,q) = (0,q)$, $(1,q-1)$, $(1,q)$, $(2,q)$.

The term $\overline{E}^{(m),2}_{pq}$ appears twice, in ${}_q\mathcal{E}^r_{\bullet\bullet}$ and ${}_{q+1}\mathcal{E}^r_{\bullet\bullet}$.

There are two cases:

$q < n$ then for $l = \left[\frac{q}{2}\right] + 1$

$$\overline{E}^{(m),2}_{0} \xleftarrow{\simeq} \overline{E}^{(m),2}_{-1,q-1} \xleftarrow{\simeq} \overline{E}^{(m),2}_{-2,q-2} \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \overline{E}^{(m),2}_{-l,q-l} \subseteq \mathrm{HC}_{q-2l}(\mathcal{O})(-l) = 0$$

because $q - 2l < 0$.

The $\mathcal{E}^1$-term is the same as the $\mathcal{E}^2$-term:

$$
\begin{tikzcd}
 & & & \mathrm{H}_{\mathrm{dR}}^{q-1} \arrow[dddrr] & & \\
0 & 0 & 0 & \overline{E}_{0,q-1}^{(m),2} = 0 & \overline{E}_{1,q-1}^{(m),2} = 0 & \overline{E}_{2,q-1}^{(m),2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \overline{E}_{1,q}^{(m),2} & \overline{E}_{2,q}^{(m),2}
\end{tikzcd}
$$

In $\mathcal{E}^3$ there are only two terms and the spectral sequence collapses at $\mathcal{E}^4$.

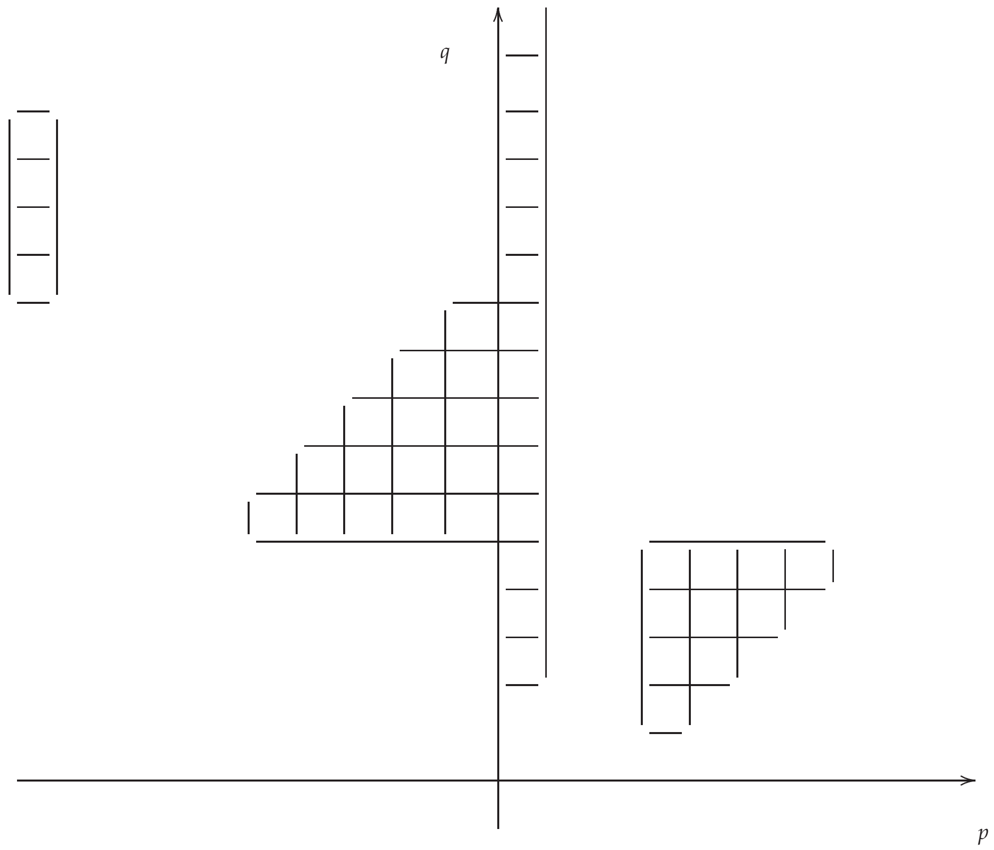
$$
\begin{tikzcd}
 & & & \mathrm{H}_{\mathrm{dR}}^{q-1} \arrow[dddrr] & & & \\
0 & 0 & 0 & 0 & 0 & \mathrm{H}_{\mathrm{dR}}^{q-2} & \mathrm{H}_{\mathrm{dR}}^{q-3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{H}_{\mathrm{dR}}^{q-1} & \mathrm{H}_{\mathrm{dR}}^{q-2}
\end{tikzcd}
$$

$q-1 \geq n$ then for $l = n - \left[\frac{q}{2}\right]$

$$
\overline{E}_{2,q-1}^{(m),2} \xrightarrow{\simeq} \overline{E}_{3,q}^{(m),2} \xrightarrow{\simeq} \overline{E}_{4,q+1}^{(m),2} \xrightarrow{\simeq} \ldots \xrightarrow{\simeq} \overline{E}_{2+l,q+l-1}^{(m),2} \simeq \overline{\Omega}^{2l+q-1}(2+l) = 0
$$

because $2l + q - 1 > 2n$.

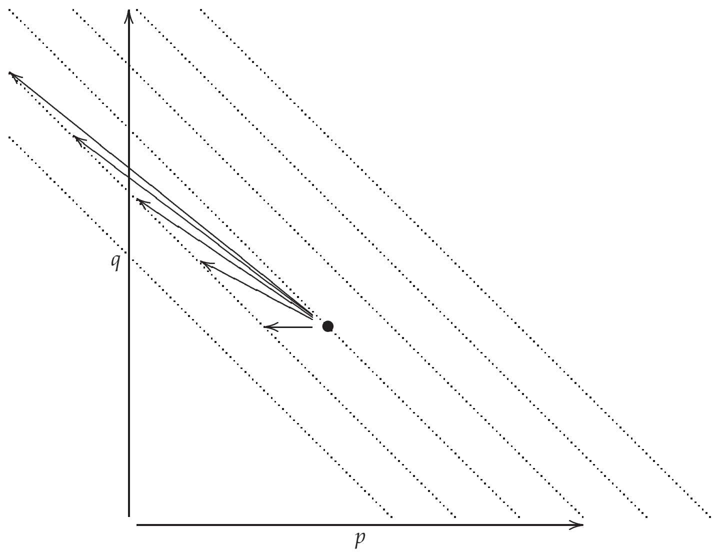
$$
\begin{array}{ccccccc}
 & & & \mathrm{H}_{\mathrm{dR}}^{n-1} & & & \\
0 & 0 & 0 & 0 & 0 & \overline{E}_{2,n-1}^{(m),2} & \overline{E}_{3,n-1}^{(m),2} \\
\mathrm{H}_{\mathrm{dR}}^{n+3} & \mathrm{H}_{\mathrm{dR}}^{n+2} & \mathrm{H}_{\mathrm{dR}}^{n+1} & \mathrm{H}_{\mathrm{dR}}^{n} & \mathrm{H}_{\mathrm{dR}}^{n-1} & \mathrm{H}_{\mathrm{dR}}^{n-2} & \mathrm{H}_{\mathrm{dR}}^{n-3} \\
0 & \overline{E}_{-2,n}^{(m),2} & \overline{E}_{-1,n}^{(m),2} & \overline{E}_{0,n}^{(m),2} & 0 & 0 & 0
\end{array}
$$


q
p

						$\cdots$	0	0	0	$H^{2n+2} \oplus H^{2n+1}$	0	$\cdots$								
						$\cdots$	0	0	0	$H^{2n+1} \oplus H^{2n}$	0	$\cdots$								
						$\cdots$	0	0	0	$H^{2n} \oplus H^{2n-1}$	0	$\cdots$								
						$\cdots$	0	0	H^{2n}	$H^{2n-1} \oplus H^{2n-3}$	0	$\cdots$								
						$\cdots$	0	H^{2n}	H^{2n-1}	$H^{2n-2} \oplus H^{2n-4}$	0	$\cdots$								
						$\cdots$	H^{2n}	H^{2n-1}	H^{2n-2}	$H^{2n-3} \oplus H^{2n-5}$	0	$\cdots$								
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$n+2$	0	0	0	H^{2n}	H^{2n-1}	$\cdots$	H^{n+5}	H^{n+4}	H^{n+3}	$H^{n+2} \oplus H^{n}$	0	0	0	0	0	$\cdots$	0	0	0	0
$n+1$	0	0	H^{2n}	H^{2n-1}	H^{2n-2}	$\cdots$	H^{n+3}	H^{n+3}	H^{n+2}	$H^{n+1} \oplus H^{n-1}$	0	0	0	0	0	$\cdots$	0	0	0	0
n	0	H^{2n}	H^{2n-1}	H^{2n-2}	H^{2n-3}	$\cdots$	H^{n+3}	H^{n+2}	H^{n+1}	$H^{n} \oplus H^{n-2}$	0	0	0	0	0	$\cdots$	0	0	0	0
$n-1$	0	0	0	0	0	$\cdots$	0	0	0	$H^{n-3} \oplus H^{n-5}$	0	H^{n-2}	H^{n-3}	H^{n-4}	H^{n-5}	$\cdots$	H^{3}	H^{2}	H^{1}	H^{0}
$n-2$	0	0	0	0	0	$\cdots$	0	0	0	$H^{n-4} \oplus H^{n-6}$	0	H^{n-3}	H^{n-4}	H^{n-5}	H^{n-6}	$\cdots$	H^{2}	H^{1}	H^{0}	0
	0	0	0	0	0	$\cdots$	0	0	0	$H^{n-5} \oplus H^{n-7}$	0	H^{n-4}	H^{n-5}	H^{n-6}	H^{n-7}	$\cdots$	H^{1}	H^{0}	0	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
						$\cdots$	0	0	0	$H^{2} \oplus H^{0}$	0	H^{3}	H^{2}	H^{1}	H^{0}	$\cdots$				
						$\cdots$	0	0	0	H^{1}	0	H^{2}	H^{1}	H^{0}	0	$\cdots$				
						$\cdots$	0	0	0	H^{0}	0	H^{1}	H^{0}	0	0	$\cdots$				
						$\cdots$	0	0	0	0	0	H^{0}	0	0	0	$\cdots$				
						$\cdots$	0	0	0	0	0	0	0	0	0	$\cdots$				
$\cdots$	$-n-1$	$-n$		$\cdots$		-4	-3	-2	-1	0	1	2	3	4		$\cdots$		$n-1$	n	$\cdots$

1.9.2 Higher differentials

For $r = 1, 2, \ldots$ the differentials in the spectral sequence are as follows



Let E^r_{pq} be a spectral sequence such that each E^r_{pq} (for $r > r_0$) is a finite dimensional vector space. Let R be a region in the (p, q)-plane which contains finitely many boxes. Then

$$\sum_{(p,q)\in R} \dim E^r_{pq} \geq \sum_{(p,q)\in R} \dim E^{r+1}_{pq} \geq \ldots \geq \sum_{(p,q)\in R} \dim E^\infty_{pq}.$$

The equality holds if and only if there is no nontrivial differential originating or leaving R, that is the equality

$$\sum_{(p,q)\in R} \dim E^{r'}_{pq} = \sum_{(p,q)\in R} \dim E^\infty_{pq}$$

is another way of saying that the spectral sequence in region R degenerates at $E^{r'}$.

In our spectral sequence

$$E^{(m),2}_{pq} \implies \mathrm{H}_{p+q}(\mathrm{Tot}\,\mathcal{B}_{\bullet\bullet}(\mathrm{CS}(X))/F_{m-1}\,\mathrm{Tot}\,\mathcal{B}_{\bullet\bullet}(\mathrm{CS}(X)))$$

We claim that the only nonvanishing differentials d^r_{pq} for $r \geq 2$ are

$$d^p_{pq} \colon E^{(m),p}_{pq} \to E^{(m),2}_{0,p+q-1}$$

which inject $E^{(m),2}_{pq} = E^{(m),2}_{pq} \simeq \mathrm{H}^{q-2}_{\mathrm{dR}}$ into $E^{(m),p}_{0,p+q-1}$.

We can define two regions R, R' as follows.

[PICTURE]

Then

$$\sum_{(p,q)\in R} \dim E^{r'}_{pq} = \sum_{(p,q)\in R} \dim E^\infty_{pq}.$$

Suppose that ther is no nontrivial differential originating from R' or nontrivial differential hitting R and originating outside. Then

$$\sum_{(p,q)\in R'} \dim E^r_{pq} - \sum_{(p,q)\in R} \dim E^r_{pq} \geq \sum_{(p,q)\in R'} \dim E^{r+1}_{pq} - \sum_{(p,q)\in R} \dim E^{r+1}_{pq}$$

Equality holds if and only if all d^r inside R are zero, and then for all $r > r_0$ for some r_0

$$\sum_{(p,q)\in R'} \dim E^r_{pq} - \sum_{(p,q)\in R} \dim E^r_{pq} = \sum_{(p,q)\in R'} \dim E^\infty_{pq} - \sum_{(p,q)\in R} \dim E^\infty_{pq}.$$

We can write

$$\sum_{0\le q\le n} \dim E^{(m),2}_{0q} - \sum_{0\le q\le n} \dim E^{(m),\infty}_{0q} = \sum_{p>0} \dim E^{(m),2}_{pq}.$$

For $r \ge 2$ let us introduce the following statements:

$(A)_r$ The natural maps

$$E^{(m),r}_{pq} \to E^{(m),r}_{pq}\langle Y^{n-1}\rangle$$

are isomorphisms for $p > 0$, r fixed.

$(B)_r$ The differentials

$$d^r_{pq}\colon E^{(m),r}_{rq} \to E^{(m),r}_{0,q+r-1}$$

are injective.

$(C)_r$ The differentials

$$d^r_{pq}\colon E^{(m),r}_{pq} \to E^{(m),r}_{p-r,q+r-1}$$

are zero for $p \ge r$.

We prove them by induction on r, simultaneously

$$\begin{tikzcd}
 & (B)_2 & & & (B)_3 \\
(A)_2 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (B)_2 \wedge (C)_2 \arrow[r, Rightarrow] & (A)_3 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
 & (C_2) & & & (C)_3
\end{tikzcd}$$

and so on. Furthermore let us introduce two more sequences of statements:

$(D)_r$ For $p > m$

$$d^r_{pq} = \lim_j d^r_{pq,j}.$$

$(E)_r$ For $p > m$

$$E^{(m),r}_{pq} = \lim_j E^{(m),r}_{pq}\langle Y_j\rangle.$$

These are also proved by induction on r in the following way. The $(E)_r$ implies $(D)_r$ and $(E)_r$ and $(D)_r$ together with the condition that $\{E^{(m),r}_{pq}\langle Y^j\rangle\}$, $\{E^{(m),r+1}_{pq}\langle Y^j\rangle\}$ satisfy Mittag-Leffler condition, imply $(E)_{r+1}$.

The $(A)_2$ statement follows from the following remark. Suppose $\mathrm{H}^k_{\mathrm{dR}}(Y^c) = 0$ for $k > n$ and that $\dim \mathrm{H}^\bullet_{dR}(Y^c) < \infty$. Then

$$\sum_{j=0}^{2n-2} \dim E^{(m),2}_{0j} - \sum_{p>0,q} \dim E^{(m),2}_{pq} = \sum_{j=0}^{2n-2} \dim \mathrm{HC}_j(\mathrm{CS}_Y).$$

The maps

$$\mathrm{H}^j_{\mathrm{dR}}(Y^c) \to \mathrm{H}^j_{\mathrm{dR}}((Y^k)^c)$$

are isomorphisms for $j < k$, monomorphism for $j = k$, zero for $j > k+1$.

Appendix A

Topological tensor products

Let $(E, \{p_\alpha\}_{\alpha\in A})$, $(F, \{q_\beta\}_{\beta\in B})$ be vector spaces with the sytems of seminorms $\{p_\alpha\}_{\alpha\in A}$, $\{q_\beta\}_{\beta\in B}$ respectively. Define a system of seminorms on $E \otimes F$ by

$$(p_\alpha \otimes q_\beta)(\tau) := \inf \sum_{i\in I} p_\alpha(e_i) q_\beta(f_i), \tag{A.1}$$

where infimum is taken over all representations $\tau = \sum_{i\in I} e_i \otimes f_i$, in which I is a finite set.

Definition A.1. *A locally convex space $E \otimes F$ with topology induced by the system of seminorms $\{p_\alpha \otimes q_\beta\}_{(\alpha,\beta)\in A\times B}$ is calles a* ***projective tensor product*** *and denoted by $E \otimes_\pi F$. Its completion is denoted by $E\widehat{\otimes}_\pi F$.*

A bilinear map

$$\phi\colon E \times F \to E\widehat{\otimes}_\pi F, \quad (e, f) \mapsto e \otimes f,$$

is continuous in both variables and has the following universal property.

Fact A.2. *For every bilinear jointly continuous mapping $f\colon E \times F \to W$ into locally convex space W there exists unique continuous linear map $L_\phi\colon E\widehat{\otimes}_\pi F \to W$ such that following diagram commutes.*

$$\begin{tikzcd}
E \times F \arrow[rr, "f"] \arrow[dr, "\phi"'] & & W \\
& E\widehat{\otimes}_\pi F \arrow[ur, dashed, "L_\phi"'] &
\end{tikzcd}$$

Remark A.3. There are also different tensor products on topological vector spaces, like injective and inductive tensor products, but we will not describe them here.

Suppose that $E' = \bigcup_{m\in\mathbb{Z}} E'_m$, where

$$\ldots \subseteq E'_{m-1} \subseteq E'_m \subseteq \ldots$$

is a $\mathbb{Z}$-filtration of E' by locally convex closed vector subspaces of E', and analogously for the space E''. Then define

$$E'\widetilde{\otimes}E'' := \lim_{(l_1,l_2)\in\mathbb{Z}\times\mathbb{Z}} E'_{l_1}\widehat{\otimes}_\pi E''_{l_2}.$$

If for any m there is a continuous projections $E'_m \to E'_{m-1}$, $E''_m \to E''_{m-1}$, then the space $E'_{l_1}\widehat{\otimes}_\pi E''_{l_2}$ is a closed subspace in $E'_{m_1}\widehat{\otimes}_\pi E''_{m_2}$ for any $m_1 \geq l_1$, $m_2 \geq l_2$.

Define a $\mathbb{Z}$-filtration on $E'\widetilde{\otimes}E''$

$$(E'\widetilde{\otimes}E'')_m := \bigcup_{\substack{(l_1,l_2)\in\mathbb{Z}\times\mathbb{Z}\\ l_1+l_2\leq m}} E'_{l_1}\widehat{\otimes}_\pi E''_{l_2}.$$

In similar way we define $E^{(1)}\widetilde{\otimes}\ldots\widetilde{\otimes}E^{(p)}$ with $\mathbb{Z}$-filtration

$$(E^{(1)}\widetilde{\otimes}\ldots\widetilde{\otimes}E^{(p)})_m := \bigcup_{\substack{(l_1,\ldots,l_p)\in\mathbb{Z}^p\\ l_1+\ldots+l_p\leq m}} E^{(1)}_{l_1}\widehat{\otimes}_\pi\ldots\widehat{\otimes}_\pi E^{(p)}_{l_p}.$$

Appendix B

Spectral sequences

Lecture given by prof. Wodzicki on October 2004 in Warsaw,
with remarks added in November 2006.

B.1 Spectral sequence of a filtered complex

Let $(C_\bullet, F, \partial)$ be a filtered chain complex, that is

$$\ldots \subseteq F_p C_\bullet \subseteq F_{p+1} C_\bullet \subseteq \ldots \subseteq C_\bullet.$$

We say that the filtration is

1. **separable** if $\bigcap_p F_p C_n = \{0\}$,
2. **complete** if $C_n \xrightarrow{\simeq} \lim_p C_n / F_p C_n$,
3. **cocomplete** if $\bigcup_p F_p C_n \xrightarrow{\simeq} C_n$,

for all $n \in \mathbb{Z}$.

We define $E^0_{\bullet\bullet} := \mathrm{gr}^F_\bullet C_\bullet$ (the associated graded complex), where $E^0_{pq} := F_p C_{p+q} / F_{p-1} C_{p+q}$, and $d^0_{\bullet\bullet}$ is the boundary operator induced by ∂, $d^0_{pq} \colon E^0_{pq} \to E^0_{p,q-1}$. Thus $(E^0_{\bullet\bullet}, d^0_{\bullet\bullet})$ is the direct sum of complexes

$$(E^0_{\bullet\bullet}, d^0_{\bullet\bullet}) = \bigoplus_{p \in \mathbb{Z}} (E^0_{p\bullet}, d^0_{p\bullet}).$$

Next we define

$$\begin{aligned} E^1_{pq} &:= \mathrm{H}_q(E^0_{p\bullet}, d^0_{p\bullet}) \\ &= \frac{\{c \in F_p C_{p+q} \mid \partial c \in F_{p-1} C_{p+q-1}\}}{\{c \in F_p C_{p+q} \mid c = \partial b \text{ for some } b \in F_p C_{p+q+1}\}} \quad \mod F_{p-1} C_{p+q} \\ &=: \frac{Z^1_{pq} + F_{p-1} C_{p+q}}{B^1_{pq} + F_{p-1} C_{p+q}}. \end{aligned}$$

On E^1_{pq} the boundary operator ∂ induces a boundary operator $d^1_{pq} \colon E^1_{pq} \to E^1_{p-1,q}$ and so on...

Define for $r = 1, 2, \ldots$

$$
\begin{aligned}
E^r_{pq} &= \frac{\{c \in F_pC_{p+q} \mid \partial c \in F_{p-r}C_{p+q-1}\}}{\{c \in F_pC_{p+q} \mid c = \partial b \text{ for some } b \in F_{p+r-1}C_{p+q+1}\}} \quad \bmod F_{p-1}C_{p+q} \\
&=: \frac{Z^r_{pq} + F_{p-1}C_{p+q}}{B^r_{pq} + F_{p-1}C_{p+q}}.
\end{aligned}
$$

$$
\begin{tikzcd}
\cdots & F_{p-r}C_{p+q-1} & F_{p-r}C_{p+q} & F_{p-r}C_{p+q+1} & \cdots \\
\cdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & F_{p-1}C_{p+q-1} & F_{p-1}C_{p+q} & F_{p-1}C_{p+q+1} & \cdots \\
\cdots & F_pC_{p+q-1} & F_pC_{p+q} \arrow[l, "\partial^0"'] \arrow[ul, "\partial^1"] \arrow[uuul, "\partial^r"] & F_pC_{p+q+1} \arrow[l, "\partial^0"'] \arrow[ul, "\partial^1"] \arrow[uuul, "\partial^r"] & \cdots \\
\cdots & F_{p+1}C_{p+q-1} & F_{p+1}C_{p+q} \arrow[l, "\partial^0"'] & F_{p+1}C_{p+q+1} \arrow[l, "\partial^0"'] & \cdots \\
\cdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & F_{p+r}C_{p+q-1} & F_{p+r}C_{p+q} \arrow[l, "\partial^0"'] \arrow[uuul, "\partial^r"] & F_{p+r}C_{p+q+1} \arrow[l, "\partial^0"'] \arrow[uuul, "\partial^r"] & \cdots \\
\cdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & C_{p+q-1} & C_{p+q} \arrow[l, "\partial"'] & C_{p+q+1} \arrow[l, "\partial"'] & \cdots
\end{tikzcd}
$$

Now $E^r_{\bullet\bullet}$ equipped with the boundary operator induced by ∂ becomes a direct sum of complexes

$$
\cdots \leftarrow E^r_{p-r,q+r-1} \xleftarrow{d^r_{pq}} E^r_{pq} \xleftarrow{d^r_{p+r,q-r+1}} E^r_{p+r,q-r+1} \leftarrow \cdots,
$$

$$
\begin{tikzcd}
E^r_{p-r,q+r-1} & E^r_{p-r+1,q+r-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & E^r_{pq} \arrow[ulll] & E^r_{p+1,q} \arrow[ulll] & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & E^r_{p+r,q-r+1} \arrow[ulll] & E^r_{p+r+1,q-r+1} \arrow[ulll]
\end{tikzcd}
$$

which we can denote by $(E^r_{p+\bullet r,q-\bullet(r-1)}, d^r_{p+\bullet r,q-\bullet(r-1)})$. Now E^{r+1}_{pq} is canonically isomorphic to the homology of the complex $(E^r_{p+\bullet r,q-\bullet(r-1)}, d^r_{p+\bullet r,q-\bullet(r-1)})$ at the E^r_{pq}.

For each (p, q) we defined a system of subobjects of $F_p C_{p+q}$:

$$\{0\} = B^0_{pq} \subseteq B^1_{pq} \subseteq \ldots \subseteq B^r_{pq} \subseteq \ldots$$
$$\subseteq \bigcup_r B^r_{pq} =: B^\infty_{pq} \subseteq Z^\infty_{pq} := \bigcap_r Z^r_{pq} \subseteq$$
$$\ldots \subseteq Z^r_{pq} \subseteq \ldots \subseteq Z^1_{pq} \subseteq Z^0_{pq} = F_p C_{p+q},$$

such that

$$E^r_{pq} = Z^r_{pq} / B^r_{pq} \mod F_{p-1} C_{p+q}.$$

Morphism $\varphi\colon (C_\bullet, F, \partial) \to ('C_\bullet, 'F, '\partial)$ of filtered complexes induces a morphism

$$E^r_{\bullet\bullet}(\varphi)\colon E^r_{\bullet\bullet} \to' E^r_{\bullet\bullet}, r \geq 0,$$

of corresponding spectral sequences.

Theorem B.1 (Eilenberg-Moore). *If $E^r_{\bullet\bullet}(\varphi)$ is an isomorphism for some r and both filtrations are complete and cocomplete, then φ is a quasi-isomorphism.*

We say that the spectral sequence $E^r_{\bullet\bullet}$ **converges** to filtered module M if

$$E^\infty_{pq} \simeq F_p M_{p+q} / F_{p-1} M_{p+q}, \quad p, q \in \mathbb{Z}.$$

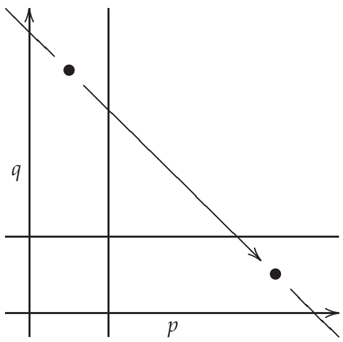
We write then $E^r_{pq} \implies M_{p+q}$.

If the filtration is locally bounded from below (i.e. $F_p C_n = \{0\}$ for $p \ll 0$) and cocomplete, then $E^r_{\bullet\bullet}$ converges to $\mathrm{H}_*(C_\bullet, \partial)$. The homology of a complex $(C_\bullet, \partial)$ is equipped with canonical filtration

$$F_p \, \mathrm{H}_*(C_\bullet, \partial) := \mathrm{im}(\mathrm{H}_*(F_p C_\bullet, \partial) \to \mathrm{H}_*(C_\bullet, \partial)).$$

We say that the spectral sequence $E^r_{\bullet\bullet}$ **degenerates** (or **collapses**) at E^s if $E^s_{\bullet\bullet} \simeq E^\infty_{\bullet\bullet}$.

Consider the r-th term E_r of the spectral sequence.



The source term E^r_{pq} is mapped to the rightmost one $E^r_{p'q'}$. There is a sequence of maps

$$E^r_{pq} \twoheadrightarrow E^{r+1}_{pq} \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{pq} \to \mathrm{H}_{p+q}(C),$$

and similarly

$$\mathrm{H}_{p'+q'}(C) \to E^\infty_{p'q'} \rightarrowtail \cdots \rightarrowtail E^{r+1}_{p'q'} \rightarrowtail E^r_{p'q'}.$$

These maps are called the **edge homomorphisms**. For the first quadrant spectral sequence they correspond to maps from leftmost column $p = 0$

$$E^r_{0q} \to \mathrm{H}_q(C),$$

and to bottom row $q = 0$

$$\mathrm{H}_p(C) \to E^r_{p0}.$$

B.2 Examples

Example B.2. Two spectral sequences associated with the double complex $(C_{\bullet\bullet}, \partial', \partial'')$.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d, "\partial''"'] & C_{p,q+1} \arrow[l, "\partial'"'] \arrow[d, "\partial''"'] & C_{p+1,q+1} \arrow[l, "\partial'"'] \arrow[d, "\partial''"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "\partial''"'] & C_{pq} \arrow[l, "\partial'"'] \arrow[d, "\partial''"'] & C_{p+1,q} \arrow[l, "\partial'"'] \arrow[d, "\partial''"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "\partial'"'] \arrow[d] & C_{p+1,q-1} \arrow[l, "\partial'"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Recall that

$$\partial'^2 = \partial''^2 = 0, \quad [\partial', \partial''] = \partial'\partial'' + \partial''\partial' = 0,$$

and the total complex is defined by

$$(\operatorname{Tot} C)_n := \prod_{p=-\infty}^{-1} C_{p,n-p} \oplus \bigoplus C_{p,n-p}, \quad \partial := \partial' + \partial''.$$

$$
\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & C_{0,n} & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & C_{1,n-1} & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & C_{2,n-2} & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & C_{n-1,1} & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & C_{n0} & \cdots \\
 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots &
\end{array}
$$

There are two filtrations on $\operatorname{Tot} C$:

1. filtration by columns

$$'F_p(\operatorname{Tot} C)_n := \prod_{r \le p} C_{r,n-r}$$

$$
\begin{array}{cccccccccccc}
 & \vdots & \vdots & \vdots & \cdots & \vdots & p & \vdots & \cdots & \vdots & \vdots & \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \vert & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & C_{0,n} & \bullet & \bullet & \cdots & \bullet & \vert & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & C_{1,n-1} & \bullet & \cdots & \bullet & \vert & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & C_{2,n-2} & \cdots & \bullet & \vert & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & C_{p,n-p} & \vert & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \vert & C_{p+1,n-p-1} & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \vert & \bullet & \cdots & C_{n-1,1} & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \vert & \bullet & \cdots & \bullet & C_{n0} & \cdots \\
 & \vdots & \vdots & \vdots & \cdots & \vdots & p & \vdots & \cdots & \vdots & \vdots &
\end{array}
$$

2. filtration by rows

$$''F_p(\operatorname{Tot} C)_n := \bigoplus_{p \le s} C_{n-s,s}$$

$$
\begin{array}{ccccccccccc}
 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & C_{0,n} & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & C_{1,n-1} & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & C_{2,n-2} & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & C_{n-p,p} & \bullet & \cdots & \bullet & \bullet & \cdots \\
p & \multicolumn{9}{c}{\hrulefill} & p \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & C_{n-p-1,p+1} & \cdots & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & C_{n-1,1} & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & C_{n0} & \cdots \\
 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots &
\end{array}
$$

Filtration by rows is complete and cocomplete only if for all $n \in \mathbb{Z}$ $C_{pq} \neq 0$ for only finite number of p, q such that $p + q = n$. Filtration by columns is always complete and cocomplete.

There are two spectral sequences associated to double complex $(C_{\bullet\bullet}, \partial', \partial'')$.

1. First spectral sequence associated to the filtration by columns

$$'E^1_{pq} = \mathrm{H}_q(C_{p\bullet}, \partial'').$$

   It converges to $\mathrm{H}_{p+q}(\mathrm{C}_{\bullet\bullet}) := \mathrm{H}_{p+q}(\mathrm{Tot}(\mathrm{C}_{\bullet\bullet}))$ if $C_{p,n-p} = 0$ for $p \ll 0$ ($n \in \mathbb{Z}$).

2. Second spectral sequence associated to the filtration by rows

$$''E^1_{pq} = \mathrm{H}_q(C_{\bullet p}, \partial').$$

   It converges to $\mathrm{H}_{p+q}(\mathrm{C}_{\bullet\bullet})$ if $C_{p,n-p} = 0$ for $p \ll 0$ and $p \gg 0$ ($n \in \mathbb{Z}$).

Example B.3. Double complex $\mathcal{B}(A)_{\bullet\bullet}$ (Connes double complex). Let A be the associative algebra with unit.

$$\mathcal{B}(A)_{pq} := \begin{cases} A^{\otimes(q-p+1)} & \text{if } q \ge p \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"'] & A \arrow[l, "B"'] & \\
A^{\otimes 2} \arrow[d, "b"'] & A \arrow[l, "B"'] & & \\
A & & &
\end{tikzcd}
$$

Here b is the Hochschild boundary operator and B is defined as

$$B := (1 - t)sN,$$

where

$$
\begin{aligned}
s(a_0 \otimes \cdots \otimes a_n) &:= 1 \otimes a_0 \otimes \cdots \otimes a_n \\
t(a_0 \otimes \cdots \otimes a_n) &:= (-1)^n \otimes a_0 \otimes \cdots \otimes a_{n-1} \\
N(a_0 \otimes \cdots \otimes a_n) &:= (\mathrm{id} + t + \ldots + t^n)(a_0 \otimes \cdots \otimes a_n)
\end{aligned}
$$

Example B.4. Double complex $\mathcal{D}(A)_{\bullet\bullet}$. Here A is commutative k-algebra with unit.

$$\mathcal{D}(A)_{pq} := \begin{cases} \Omega^{q-p}_{A/k} & \text{if } q \geq p \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
\Omega^2_{A/k} \arrow[d, "0"'] & \Omega^1_{A/k} \arrow[l, "d"'] \arrow[d, "0"'] & A \arrow[l, "d"'] & \\
\Omega^1_{A/k} \arrow[d, "0"'] & A \arrow[l, "d"'] & & \\
A & & &
\end{tikzcd}
$$

If $A \xrightarrow{\simeq} A \otimes_{\mathbb{Z}} \mathbb{Q}$ (i.e. the additive group $(A, +)$ is uniquely divisible), then the formula

$$\mu(a_0 \otimes \cdots \otimes a_n) := \frac{1}{n!} a_0 da_0 \wedge \cdots \wedge da_n$$

induces a morphism of double complexes $\mu \colon \mathcal{B}(A)_{\bullet\bullet} \to \mathcal{D}(A)_{\bullet\bullet}$.

On the level of spectral sequences associated with the filtration by columns we obtain surjective maps

$$E^1(pq)(\mu) \colon A^{\otimes(q-p+1)} \twoheadrightarrow \Omega^{q-p}_{A/k}.$$

These maps are isomorphisms if A is a function algebra on the smooth algebraic variety over a perfect field (i.e. of characteristic 0 or such that $k^p = k$ if $\mathrm{char}(k) = p$), or iductive limit of such (for example $A = \mathbb{C}$ as $\mathbb{Q}$-algebra).

The first spectral sequence of a double complex $(\mathcal{D}(A)_{\bullet\bullet}, 0, d) = \bigoplus_{q\geq 0}(\Omega^q_{A/k} \overset{d}{\leftarrow} \ldots \overset{d}{\leftarrow} A)$ degenerates at the term E^2:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
\Omega^2_{A/k}/d\Omega^1_{A/k} \arrow[d, "0"'] & \mathrm{H_{dR}}^1(A) \arrow[l, "d"'] \arrow[d, "0"'] & \mathrm{H_{dR}}^0(A) \arrow[l, "d"'] & \\
\Omega^1_{A/k}/dA \arrow[d, "0"'] & \mathrm{H_{dR}}^0(A) \arrow[l, "d"'] & & \\
A & & &
\end{tikzcd}$$

Thus the first spectral sequence of the double complex $(\mathcal{B}(A)_{\bullet\bullet}, b, B)$ also degenerates at the term E^2, and we get an isomorphism

$$\mathrm{HC}_n(A) := \mathrm{H}_n(\mathcal{B}(A)_{\bullet\bullet}) = \Omega^n_{A/k}/d\Omega^{n-1}_{A/k} \oplus \mathrm{H_{dR}}^{n-2}(A) \oplus \mathrm{H_{dR}}^{n-4}(A) \oplus \ldots.$$

Example B.5. Let $P_\bullet$ be a projective resolution of a right R-module M, and $Q_\bullet$ a projective resolution of a left R-module N. Consider the double complex $P_\bullet \otimes_R Q_\bullet$. Then

$$'E^2_{pq} = \begin{cases} \mathrm{H}_p(P_\bullet \otimes_R N) & q = 0, \\ 0 & q \neq 0 \end{cases}$$

$$''E^2_{pq} = \begin{cases} \mathrm{H}_p(M \otimes_R Q_\bullet) & q = 0, \\ 0 & q \neq 0 \end{cases}$$

Both spectral sequences converge to $\mathrm{H}_{p+q}(P_\bullet \otimes_R Q_\bullet) =: \mathrm{Tor}^R_{p+q}(M, N)$, so we get an important canonical isomorphisms

$$\mathrm{H}_p(P_\bullet \otimes_R N) \simeq \mathrm{Tor}^R_p(M, N) \simeq \mathrm{H}_p(M \otimes_R Q_\bullet).$$

They express the fact that the bifunctor $\otimes_R \colon \mathbf{Mod} - R \times R - \mathbf{Mod} \to \mathbf{Ab}$ is balanced.

Example B.6. Two hiperhomology spectral sequences. A Cartan-Eilenberg resolution of a complex $(C_\bullet, \partial)$ is a double complex $(P_{\bullet\bullet}, \partial', \partial'')$ with augmentation $\eta \colon P_{\bullet 0} \to C_\bullet$ satisfying the following conditions:

1. for all p, q the modules P_{pq}, $\operatorname{im}\partial'_{pq}$, $\ker\partial'_{pq}$, $\mathrm{H}_p(P_{\bullet q}, \partial')$ are projective,

2. the augmented complexes

$$\begin{tikzcd}
P_{p\bullet} \arrow[d, "\eta"'] & \operatorname{im}\partial'_{p\bullet} \arrow[d, "\eta"'] & \ker\partial'_{p\bullet} \arrow[d, "\eta"'] & \mathrm{H}_p(P_{\bullet q}, \partial') \arrow[d, "\eta"'] \\
C_p & \operatorname{im}\partial_p & \ker\partial_p & \mathrm{H}_p(C_\bullet, \partial)
\end{tikzcd}$$

are projective resolutions.

$$
\begin{tikzcd}
\cdots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \cdots \\
\cdots & P_{p-1,q} \arrow[l] \arrow[d, "\partial''_q"'] & P_{p,q} \arrow[l, "\partial'_p"'] \arrow[d, "\partial''_q"'] & P_{p+1,q} \arrow[l, "\partial'_{p+1}"'] \arrow[d, "\partial''_q"'] & \cdots \arrow[l] \\
\cdots & P_{p-1,q-1} \arrow[l] \arrow[d] & P_{p,q-1} \arrow[l, "\partial'_p"'] \arrow[d] & P_{p+1,q-1} \arrow[l, "\partial'_{p+1}"'] \arrow[d] & \cdots \arrow[l] \\
\cdots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \cdots \\
\cdots & P_{p-1,1} \arrow[l] \arrow[d, "\partial''_1"'] & P_{p,1} \arrow[l, "\partial'_p"'] \arrow[d, "\partial''_1"'] & P_{p+1,1} \arrow[l, "\partial'_{p+1}"'] \arrow[d, "\partial''_1"'] & \cdots \arrow[l] \\
\cdots & P_{p-1,0} \arrow[l] \arrow[d, "\eta"'] & P_{p,0} \arrow[l, "\partial'_p"'] \arrow[d, "\eta"'] & P_{p+1,0} \arrow[l, "\partial'_{p+1}"'] \arrow[d, "\eta"'] & \cdots \arrow[l] \\
\cdots & C_{p-1} \arrow[l] & C_p \arrow[l, "\partial_p"'] & C_{p+1} \arrow[l, "\partial_{p+1}"'] & \cdots \arrow[l]
\end{tikzcd}
$$

Such resolution can be obtained from the arbitrary projective resolutions of $\mathrm{H}_p(C_\bullet, \partial)$ and $\operatorname{im}\partial_{p-1}$ by gluing them.

$$
\begin{tikzcd}
P^H_{p\bullet} \arrow[d] & P^Z_{p\bullet} \arrow[l, two heads, dashed] \arrow[d, dashed] & P^B_{p-1,\bullet} \arrow[l, tail, dashed] \arrow[d] \\
\mathrm{H}_p(C_p, \partial) & \ker\partial_p \arrow[l, two heads] & \operatorname{im}\partial_{p-1} \arrow[l, tail]
\end{tikzcd}
\qquad
\begin{tikzcd}
P^B_{p\bullet} \arrow[d] & P_{p\bullet} \arrow[l, two heads, dashed] \arrow[d, dashed] & P^Z_{p\bullet} \arrow[l, tail, dashed] \arrow[d] \\
\operatorname{im}\partial_p & C_p \arrow[l, two heads] & \ker\partial_p \arrow[l, tail]
\end{tikzcd}
$$

For an additive functor F the hiperhomology spectral sequences are the first and second spectral sequences of a double complex $(F(P_{\bullet\bullet}), F(\partial'), F(\partial''))$

$$
\begin{aligned}
{}'E^1_{pq} &= (L_q F)(C_p), \\
{}''E^2_{pq} &= F(P^H_{pq}),
\end{aligned}
$$

and

$$
\begin{aligned}
{}'E^2_{pq} &= \mathrm{H}_p((L_q F)(C_\bullet)), \\
{}''E^2_{pq} &= (L_p F)(\mathrm{H}_q(C_\bullet)).
\end{aligned}
$$

Both spectral sequences converge to

$$\mathbb{L}_{p+q}F(C_\bullet) := \mathrm{H}_{p+q}(F(P_{\bullet\bullet})).$$

if $C_\bullet$ is bounded from below, that is $C_n = 0$ for $n \ll 0$.

Assume that $C_n = 0$ for $n < 0$, $C_\bullet$ is F-acyclic, that is $(L_0 F)(C_n) \xrightarrow{\simeq} C_n$, $(L_p F)(C_n) = 0$ for $p > 0$, and that

$$\mathrm{H}_n(C_\bullet) = \begin{cases} M & n = 0, \\ 0 & n > 0. \end{cases}$$

Such complex is called an F-acyclic resolution of the module M. In that case

$$'E^2_{pq} \simeq \begin{cases} \mathrm{H}_p(F(C_\bullet)) & q = 0, \\ 0 & q \neq 0, \end{cases}$$
$$''E^2_{pq} \simeq \begin{cases} L_pF(M) & p = 0, \\ 0 & p \neq 0. \end{cases}$$

Thus we obtain an isomorphism

$$\mathrm{H}_p(F(C_\bullet)) \simeq (L_pF)(M).$$

We proved a very important fact, that to compute $(L_pF)(M)$ it is enough to use an arbitrary F-acyclic resolution of M.

Example B.7. Flat module is an F-acyclic module for $F = (-) \otimes_R N$, where N is an arbitrary left R-module. For $R = \mathbb{Z}$ flat modules are the torsion free abelian groups. Thus

$$0 \leftarrow \mathbb{Q}/\mathbb{Z} \leftarrow \mathbb{Q} \leftarrow \mathbb{Z} \leftarrow 0$$

is a flat resolution of the group $\mathbb{Q}/\mathbb{Z}$ (injective cogenerator of a category of abelian groups **Ab**). From this we obtain

$$\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Q}/\mathbb{Z}, A) = \ker(A \to A \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{Torsion}(A).$$

Example B.8. Consider two composable additive functors

$$\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C},$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories. Let M be an object in $\mathcal{A}$, $P_\bullet$ its projective resolution. In the hiperhomology spectral sequence we put $C_\bullet = G(P_\bullet)$. Then if G sends projective objects into F-acyclic objects

$$'E^2_{pq} = \mathrm{H}_p((L_qF)(G(P_\bullet))) \simeq \begin{cases} \mathrm{H}_p((F \circ G)(P_\bullet)) = (L_p(F \circ G))(M) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

$$''E^2_{pq} = (L_pF \circ L_qG)(M)$$

In this case we obtain that

$$''E^2_{pq} = (L_pF \circ L_qG)(M) \implies (L_{p+q}(F \circ G))(M).$$

$$'E^2_{pq} = E^\infty_{pq} = \begin{array}{c|cccc} & \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & \cdots & 0 \\ q & 0 & 0 & \cdots & 0 \\ & (L_0(F \circ G))(M) & (L_1(F \circ G))(M) & \cdots & (L_p(F \circ G))(M) \\ \hline & & p & & \end{array}$$

$$''E^2_{pq} = \begin{array}{c|cccc} & \cdots & \cdots & \cdots & \cdots \\ & (L_0F \circ L_qG)(M) & (L_1F \circ L_qG)(M) & \cdots & (L_pF \circ L_qG)(M) \\ & \cdots & \cdots & \cdots & \cdots \\ q & (L_0F \circ L_1G)(M) & (L_1F \circ L_1G)(M) & \cdots & (L_pF \circ L_1G)(M) \\ & (L_0F \circ L_0G)(M) & (L_1F \circ L_0G)(M) & \cdots & (L_pF \circ L_0G)(M) \\ \hline & & p & & \end{array}$$

This spectral sequence is called a spectral sequence of a composition of functors.

Example B.9. Let $\varphi\colon R \to S$ be a homomorphism of unital rings, M a right R-module, N a left S-module. Consider a composition

$$\mathbf{Mod} - R \xrightarrow{G=(-)\otimes_R S} \mathbf{Mod} - S \xrightarrow{F=(-)\otimes_R N} \mathbf{Ab}$$

The spectral sequence of a composition of these two functors (G sends projective R-modules into projective S-modules) in looks as follows:

$$E^2_{pq} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(M,S),N) \implies \mathrm{Tor}^R_{p+q}(M,N)$$

and it is called a base change spectral sequence.

Suppose that $R \to S$ is a homomorphism of k-algebras, M_R, ${}_SN$ are respectively right R-module and left S-module. Their tensor product $M \otimes RN$ gives rise to a sequence of derived functors $\mathrm{Tor}^R_\bullet(M,N)$.

Suppose that $P_\bullet \twoheadrightarrow M$ is a projective R-module resolution of M, and $Q_\bullet \twoheadrightarrow N$ a projective S-module resolution for N.

$$M \otimes_R N \leftarrow P_\bullet \otimes_R Q_\bullet \simeq (P_\bullet \otimes_R S) \otimes_S Q_\bullet$$

Suppose $F(\cdot,\cdot)$ is a functor with both covariant arguments.

$$\begin{tikzcd}
L_qF(\cdot,\cdot) \arrow[r, equal] & L_q^{\{1,2\}}F(\cdot,\cdot) \arrow[dl] \arrow[dr] & \\
L_q^{\{1\}}F(\cdot,\cdot) \arrow[dr] & & L_q^{\{2\}}F(\cdot,\cdot) \arrow[dl] \\
& L_q^{\varnothing}F(\cdot,\cdot) \arrow[d, equal] & \\
& \begin{cases} F & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} &
\end{tikzcd}$$

We say that it is **left balanced** if there are isomorphisms $L_q^{\{1\}} \simeq L_q^{\{1,2\}} \simeq L_q^{\{2\}}$.

$$
\begin{tikzcd}
R^qF(\cdot,\cdot) \arrow[r, equal] & R^q_{\{1,2\}}F(\cdot,\cdot) & \\
R^q_{\{1\}}F(\cdot,\cdot) \arrow[ur] & & R^q_{\{2\}}F(\cdot,\cdot) \arrow[ul] \\
& R^q_{\emptyset}F(\cdot,\cdot) \arrow[ul] \arrow[ur] \arrow[d, equal] & \\
& \begin{cases} F & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} &
\end{tikzcd}
$$

We say that it is **right balanced** if there are isomorphisms $R^q_{\{1\}} \simeq R^q_{\{1,2\}} \simeq R^q_{\{2\}}$.

There is an isomorphism

$$P_\bullet \otimes_R N \xleftarrow{\simeq} P_\bullet \otimes_R Q_\bullet \simeq (P_\bullet \otimes_R S) \otimes_S Q_\bullet$$

$$\operatorname{Tor}_q^R(M, S \otimes_S Q_\bullet) \xrightarrow{\simeq} \operatorname{Tor}_q^R(M, S) \otimes_S Q_\bullet.$$

Taking homology we get

$$\mathrm{H}_p(\operatorname{Tor}_q^R(M, S) \otimes Q_\bullet) \simeq \operatorname{Tor}_p^S(\operatorname{Tor}_q^R(M, S), N),$$

and a base change spectral sequence

$$E^2_{pq} = \operatorname{Tor}_p^S(\operatorname{Tor}_q^R(M, S), N) \implies \operatorname{Tor}^R_{p+q}(M, N).$$

The boundary maps (transgressions) of this spectral sequences are as follows:

$$E^2_{0n} = \operatorname{Tor}_n^R(M, S) \otimes_S N \to \operatorname{Tor}_n^R(M, N)$$

$$\operatorname{Tor}_n^R(M, N) \to E^2_{n0} = \operatorname{Tor}_n^S(M \otimes S, N)$$

Example B.10. For an unital k-algebra A let $\mathrm{Lie}(A)$ denote the associated Lie algebra with bracket $[a, a'] = aa' - a'a$. The universal derivation

$$d_\Delta \colon A \to A \otimes_k A^{op}, \quad d_\Delta(a) = 1 \otimes a^{op} - a \otimes 1$$

is a homomorphism of Lie algebras $\mathrm{Lie}(A) \to \mathrm{Lie}(A \otimes_k A^{op})$, so it induces a homomorphism of associative algebras $R := U(\mathrm{Lie}(A)) \to A \otimes_k A^{op} =: S$. Let $M = k$ (trivial representation of a Lie algebra $\mathrm{Lie}(A)$). The base change spectral sequence has the form

$$E^2_{pq} = \operatorname{Tor}_p^{A \otimes_k A^{op}}(\operatorname{Tor}_q^{U(\mathrm{Lie}(A))}(k, A \otimes_k A^{op}), N) \implies \operatorname{Tor}_{p+q}^{U(\mathrm{Lie}(A))}(k, N),$$

that is if A is flat over k then

$$E^2_{pq} = \operatorname{Tor}_p^{A \otimes_k A^{op}}(\mathrm{H}_q^{\mathrm{Lie}}(A; A \otimes_k A^{op}), N) \implies \mathrm{H}^{\mathrm{Lie}}_{p+q}(k, N).$$

Because $k \otimes_{U(\mathrm{Lie}(A))} (A \otimes A^{op}) \simeq A$ as a right $A \otimes A^{op}$-module, we have that the second boundary map gives a canonical homomorphism

$$\mathrm{H}_n^{\mathrm{Lie}}(A; N) \to \mathrm{H}_n(A; N) \simeq E^2_{n0}.$$

There is a homomorphism of standard chain complexes

$$(C_\bullet(\mathrm{Lie}(A);N),\partial) \to (C_\bullet(A,N),b)$$

where

$$\partial(n\otimes a_1\wedge\cdots\wedge a_n) := \sum_{i=1}^{n}(-1)^i \underbrace{(a_i n - n a_i)}_{-(d_\Delta a)n} \otimes a_1\wedge\cdots\wedge\widehat{a_i}\wedge\cdots\wedge a_n$$
$$+ \sum_{1\le i<j\le n}(-1)^{i+j} n\otimes[a_i,a_j]\wedge a_1\wedge\cdots\wedge\widehat{a_i}\wedge\cdots\wedge\widehat{a_j}\wedge\cdots\wedge a_n$$

In the special case $N=A$ we obtain canonical homomorphism

$$\mathrm{H}_n^{\mathrm{Lie}}(A;\mathrm{ad}) \to \mathrm{HH}_n(A)$$

Example B.11. Hiper-Tor spectral sequences and Künneth spectral sequence. For a right R-module M and a complex of left modules $C_\bullet$ we define

$$\mathbf{Tor}_n^R(M,C_\bullet) := \mathrm{H}_n(P_\bullet\otimes_R C_\bullet)$$

where $P_\bullet\to M$ is a projective resolution of M. Then the first and second spectral sequence of a bicomplex $P_\bullet\otimes_R C_\bullet$ are as follows:

$$\begin{aligned} {}'E^1_{pq} &= P_p\otimes_R \mathrm{H}_q(C)\\ {}'E^2_{pq} &= \mathrm{Tor}_p^R(M,\mathrm{H}_q(C)) \implies \mathbf{Tor}^R_{p+q}(M,C_\bullet)\end{aligned}$$

and

$$\begin{aligned} {}''E^1_{pq} &= \mathrm{Tor}_q^R(M,C_p)\\ {}''E^2_{pq} &= \mathrm{H}_p(\mathrm{Tor}_q^R(M,C_\bullet)) \simeq \begin{cases}\mathrm{H}_p(M\otimes_R C_\bullet) & q=0\\ 0 & q\neq 0\end{cases}\end{aligned}$$

where the isomorphism for E^2_{pq} holds if the complexes $\mathrm{Tor}_q^R(M,C_\bullet)$ are acyclic for $q>0$, for example if C_n are flat. Then we obtain a Künneth spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^R(M,\mathrm{H}_q(C)) \implies \mathrm{H}_{p+q}(M\otimes_R C_\bullet)$$

if $C_n=0$ for $n\ll 0$.

Example B.12. If a group G acts on semigroup S and its representation V, then G acts on Bar-complex $(B_\bullet(S;V),b')$, where $B_q(S;V) = (kS)^{\otimes_k q}\otimes_k V$, and b' is a standard boundary operator. Then

$$\mathbf{Tor}_n^{k[G]}(G,B_\bullet(S;V)) =: \mathrm{H}_n^G(S;V)$$

are the equivariant homology of a semigroup S with coefficients in representation V. In an analogous way one can define equivariant homology of a Lie algebra, Hochschild homology, singular homology of a topological space etc.