Cyclic Homology Theory, Part II

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Chapter 1

Homology of Lie algebras of matrices

Theorem 1.1 (Loday-Quillen, Tsygan)**.** *Let k be a characteristic 0 field, A an associative unital k-algebra. Then*

$$\mathrm{H}_\bullet(\mathfrak{gl}(A)) \simeq \Lambda(\mathrm{HC}_{\bullet-1}(A)). \tag{1.1}$$

On the left hand side of isomorphism (1.1) we have matrices of any size, while on the right hand side there are no matrices, and the computations are easier.

There are generalizations of the theorem (1.1) for Lie algebras $\mathfrak{so}(A)$ and $\mathfrak{sp}(A)$. Also, if instad of algebra we take an operad, then on the right hand side will be so called graph homology.

1.1 Leibniz algebras

Definition 1.2. *A **Leibniz (right) algebra** over k is an algebra A with bracket*

$$[-,-]\colon A \otimes A \to A,$$

such that $[-,z]$ is a derivation for each $z \in A$, that is

$$[[x,y],z] = [[x,z],y] + [x,[y,z]].$$

Definition 1.3. *A **Lie algebra** is a Leibniz algebra such that*

$$[x,y] = -[y,x].$$

Under this symmetry property, the Leibniz relation is equivalent to Jacobi relation.

There is a chain complex associated to a Leibniz algebra $\mathfrak{g}$:

$$\mathrm{CL}_\bullet(\mathfrak{g})\colon \ \ldots \to \mathfrak{g}^{\otimes n} \xrightarrow{d} \mathfrak{g}^{\otimes(n-1)} \xrightarrow{d} \ldots \xrightarrow{d} \mathfrak{g} \xrightarrow{0} k$$

$$\mathrm{CL}_n(\mathfrak{g}) = \mathfrak{g}^{\otimes(n-1)}, \quad d\colon \mathrm{CL}_{n+1}(\mathfrak{g}) \to \mathrm{CL}_n(\mathfrak{g})$$

$$d(x_1,\ldots,x_n) := \sum_{1\le i<j\le n} (-1)^j (x_1,\ldots,[x_i,x_j],x_{i+1},\ldots,\widehat{x_j},\ldots,x_n).$$

Lemma 1.4. *The map d is a differential, that is $d^2 = 0$.*

Proof. We will check only the composition

$$\begin{tikzcd}
\mathfrak{g}^{\otimes 3} \arrow[rr, "d"] & & \mathfrak{g}^{\otimes 2} \arrow[rr, "d"] & & \mathfrak{g} \\
& & x \otimes y \arrow[rr, mapsto] & & {[x,y]}
\end{tikzcd}$$

$$x \otimes y \otimes z \longmapsto -[x,z] \otimes y - x \otimes [y,z] + [x,y] \otimes z$$

In this case $d^2 = 0$ is equivalent to Leibniz relation. The general case is analogous. $\square$

Definition 1.5. *The **Leibniz homology** of the Leibniz algebra $\mathfrak{g}$ is*

$$\mathrm{HL}_\bullet(\mathfrak{g}) := \mathrm{H}_\bullet(\mathrm{CL}_\bullet(\mathfrak{g}), d).$$

When $\mathfrak{g}$ is a Lie algebra, then one can pass to the quotient by the action of symmetric group (with signature)

$$\mathfrak{g}^{\otimes n} \twoheadrightarrow \Lambda^n \mathfrak{g}.$$

Then d also passes to the quotient and one obtains a Chevalley-Eilenberg chain complex of $\mathfrak{g}$:

$$C_\bullet(\mathfrak{g}) : \ldots \to \Lambda^n \mathfrak{g} \xrightarrow{d} \Lambda^{n-1} \mathfrak{g} \xrightarrow{d} \ldots \xrightarrow{d} \mathfrak{g} \xrightarrow{0} k$$
$$C_n(\mathfrak{g}) = \Lambda^{n-1}\mathfrak{g}, \quad d\colon C_{n+1}(\mathfrak{g}) \to C_n(\mathfrak{g})$$
$$d(x_1, \ldots, x_n) := \sum_{1 \le i < j \le n} (-1)^{i+j-1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n.$$

For a Lie algebra $\mathfrak{g}$ we define a **Lie algebra homology**:

$$\mathrm{H}_\bullet(\mathfrak{g}) := \mathrm{H}_\bullet(C_\bullet(\mathfrak{g}), d).$$

There is a map $\mathrm{HL}_\bullet(\mathfrak{g}) \to \mathrm{H}_\bullet(\mathfrak{g})$, which is not an isomorphism in general. For example if $\mathfrak{g}$ is abelian, then the boundary in the Leibniz and Chevalley-Eilenberg complex is 0, so we have

$$\mathrm{HL}_n(\mathfrak{g}) = \mathfrak{g}^{\otimes n}$$
$$\mathrm{H}_n(\mathfrak{g}) = \Lambda^n \mathfrak{g}$$

Also if $\mathfrak{g}$ is a simple Lie algebra, then $\mathrm{HL}_n(\mathfrak{g}) = 0$, for $n \ge 1$, but $\mathrm{H}_n(\mathfrak{g})$ does not have to be 0 for $n \ne 1$.

Let $\mathfrak{g}$ be a Lie algebra, and $g \in \mathfrak{g}$. Then g acts on $\mathfrak{g}^{\otimes n}$

$$[g_1 \otimes \cdots \otimes g_n, g] = \sum_{i=1}^{n} g_1 \otimes \cdots \otimes [g_i, g] \otimes \cdots \otimes g_n.$$

Proposition 1.6. *This action is compatible with the boundary map d and it is zero on $\mathrm{H}_\bullet(\mathfrak{g})$.*

Proof. The first part is easy. For the second part we construct for $y \in \mathfrak{g}$ a map

$$\sigma(y)\colon \Lambda^n \mathfrak{g} \to \Lambda^{n+1} \mathfrak{g}$$

$$\alpha \mapsto (-1)^n \alpha \wedge y.$$

Then $\sigma(y)$ is a homotopy from conjugation to zero map, that is

$$d\sigma(y) + \sigma(y) d = [-, y].$$

$\square$

Proposition 1.7. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ be a reductive sub-Lie algebra of $\mathfrak{g}$. Then the surjective map*

$$(\Lambda^n \mathfrak{g}) \twoheadrightarrow (\Lambda^n \mathfrak{g})_{\mathfrak{h}}$$

induces an isomorphism on homology

$$\mathrm{H}_\bullet(\mathfrak{g}) \simeq \mathrm{H}_\bullet((\Lambda^n \mathfrak{g})_{\mathfrak{h}}, d).$$

1.2 Computation of Lie algebra homology $\mathrm{H}_\bullet(\mathfrak{gl}(A))$

Let k be a field, A an associative unital algebra over k. Denote by $M_r(A)$ the algebra of $r \times r$ matrices with coefficients in A, and by $\mathfrak{gl}_r(A)$ the same space, but with its Lie algebra structure given by $[\alpha, \beta] = \alpha\beta - \beta\alpha$. There is an inclusion of Lie algebras $\mathfrak{gl}_r(A) \hookrightarrow \mathfrak{gl}_{r+1}(A)$,

$$\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix},$$

but it does not preserve the identity in $M_r(A)$. With respect to these inclusions we define $\mathfrak{gl}(A) := \bigcup_r \mathfrak{gl}_r(A)$, the Lie algebra of matrices over A. Our aim is to compute $\mathrm{H}_\bullet(\mathfrak{gl}(A)) = \varinjlim_r \mathrm{H}_\bullet(\mathfrak{gl}_r(A))$, but unfortunately we cannot compute $\mathrm{H}_\bullet(\mathfrak{gl}_r(A))$.

Proof. (of the theorem (1.1)) The strategy of the proof of theorem (1.1) can be summarized in the following four steps:

1. Koszul trick.
2. Coinvariant theory.
3. Hopf-Borel (type) theorem.
4. Computation of primitive part.

The idea is to prove that the composition of the following maps is an quasi-isomorphism

$$\begin{tikzcd}
(\mathfrak{gl}(A)^{\otimes n})_{S_n} \arrow[d] \\
((\mathfrak{gl}(A)^{\otimes n})_{S_n})_{\mathfrak{sl}(k)} \arrow[d, equal] \\
((\mathfrak{gl}(k)^{\otimes n} \otimes A^{\otimes n})_{\mathfrak{sl}(k)})_{S_n} \arrow[d, equal] \\
((\mathfrak{gl}(k)^{\otimes n})_{\mathfrak{sl}(k)} \otimes A^{\otimes n})_{S_n} \arrow[d, equal] \\
(k[S_n] \otimes A^{\otimes n})_{S_n} \arrow[d] \\
\Lambda((k[U_n] \otimes A^{\otimes n})_{S_n}) \arrow[d] \\
\Lambda(A^{\otimes n}/(1-t))
\end{tikzcd}$$

where U_n denotes the set of permutations with only one cycle, and $A^{\otimes\bullet}/1-t$ is the Connes complex computing cyclic homology.

1. The algebra $\mathfrak{sl}_r(k)$ is reductive, $(\mathfrak{gl}_r(A)^{\otimes n})_{S_n}$ is an $\mathfrak{sl}_r(k)$-module, and we can consider the projection on the component corresponding to the trivial representation

   $$K \rightarrowtail (\mathfrak{gl}_r(A)^{\otimes n})_{S_n} \twoheadrightarrow ((\mathfrak{gl}_r(A)^{\otimes n})_{S_n})_{\mathfrak{sl}_r(k)}.$$

   The kernel K has trivial homology, so the projection is a quasi-isomorphism.

2. There is an isomorphism $\mathfrak{gl}_r(A) \simeq \mathfrak{gl}_r(k) \otimes A$ which can be proved by decomposing a matrix with entries in A into the elementary matrices E^a_{ij}, having one nonzero entry a in the place (i,j), that is

   $$\sum_{i,j=1}^{r} E^{a_{ij}}_{ij} = \sum_{i,j=1}^{r} E^1_{ij} \otimes a_{ij}.$$

   From this we derive

   $$(\mathfrak{gl}(A)^{\otimes n})_{\mathfrak{sl}_r(k)} = (\mathfrak{gl}_r(k) \otimes A^{\otimes n})_{\mathfrak{sl}_r(k)} = (\mathfrak{gl}_r(k)^{\otimes n})_{\mathfrak{sl}_r(k)} \otimes A^{\otimes n}.$$

   Now we use

   Theorem 1.8. *When k is a characteristic 0 field there is an isomorphism of S_n-modules*

   $$(\mathfrak{gl}_r(k)^{\otimes n})_{\mathfrak{sl}_r(k)} \simeq k[S_n].$$

   Proof. The isomorphism is given by

   $$\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \sum_{\sigma \in S_n} \underbrace{T(\sigma)(\alpha)}_{\in k} \sigma,$$

   where for $sg \in S_n$ which is decomposed into cycles $(i_1 \dots i_k)(j_1 \dots j_l)(\dots)\dots$ we define a map $T(\sigma)\colon (\mathfrak{gl}_r)^{\otimes n} \to k$ by

   $$T(\sigma)(\alpha) = \mathrm{Tr}(\alpha_{i_1} \dots \alpha_{i_k})\,\mathrm{Tr}(\alpha_{j_1} \dots \alpha_{j_l})\,\mathrm{Tr}(\dots)\dots,$$

   which is a product of finite number of elements in k. From the trace property $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ we know that $T(\sigma)$ is well defined.

   Observe that

   $$E^1_{i_1 i_2} \otimes E^1_{i_2 i_3} \otimes \cdots \otimes E^1_{i_n i_1} \mapsto (12\dots n).$$

   The action of S_n on $k[S_n] \otimes A^{\otimes n}$ is conjugation in $k[S_n]$, place permutation on $A^{\otimes n}$ and multiplication by sign. $\square$

3. The diagonal map $\mathfrak{g} \xrightarrow{\Delta} \mathfrak{g} \times \mathfrak{g}$ induces a graded cocommutative coproduct on homology $\mathrm{H}_\bullet(\mathfrak{g}) \to \mathrm{H}_\bullet(\mathfrak{g} \times \mathfrak{g}) = \mathrm{H}_\bullet \otimes \mathrm{H}_\bullet(\mathfrak{g})$. For $\mathfrak{g} = \mathfrak{gl}(A)$ there is a map

   $$\mathfrak{gl}(A) \times \mathfrak{gl}(A) \xrightarrow{\oplus} \mathfrak{gl}(A),$$

   which we can schematically describe as

   $$\left(\begin{pmatrix} * & * & * & \dots \\ * & * & * & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \begin{pmatrix} \star & \star & \star & \dots \\ \star & \star & \star & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \right) \mapsto \begin{pmatrix} * & 0 & * & 0 & * & 0 & \dots \\ 0 & \star & 0 & \star & 0 & \star & \dots \\ * & 0 & * & 0 & * & 0 & \dots \\ 0 & \star & 0 & \star & 0 & \star & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

It induces a graded cocommutative product.

$$\mathrm{H}_\bullet(\mathfrak{gl}(A)) \otimes \mathrm{H}_\bullet(\mathfrak{gl}(A)) \xrightarrow{\mu=\oplus_*} \mathrm{H}_\bullet(\mathfrak{gl}(A))$$

Theorem 1.9. $\mathrm{H}_\bullet(\mathfrak{gl}(A))$ *is a commutative cocommutative bialgebra.*

For any coalgebra $\mathcal{H}$ there is a filtration $F_\bullet\mathcal{H}$ such that , $F_0 = k$ and

$$F_r := \{x \in \mathcal{H} \mid \overline{\Delta}(x) \in F_{r-1} \otimes F_{r-1}\},$$

where $\overline{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$. We say that a coalgebra $\mathcal{H}$ is connected if $\mathcal{H} = \sum_{r \geq 0} F_r\mathcal{H}$. Now we recall the Hopf-Borel theorem:

Theorem 1.10. *If k is a characteristic 0 field, and $\mathcal{H}$ is a connected graded commutative cocommutative bialgebra, then*

$$\mathcal{H} \simeq \Lambda(\mathrm{Prim}(\mathcal{H})).$$

The main point now is that $C_\bullet(\mathfrak{gl}(A))$ is a graded commutative cocommutative bialgebra.

Proposition 1.11. $\bigoplus_n (k[S_n] \otimes A^{\otimes n})_{S_n}$ *is a graded commutative cocommutative bialgebra.*

4. The last step in the proof of theorem (1.1) is determining the primitive part of $(k[S_n] \otimes A^{\otimes n})_{S_n}$. Let U_n denote the permutations with only one cycle. Then

Proposition 1.12.

$$\mathrm{Prim}((k[S_n] \otimes A^{\otimes n})_{S_n}) = (k[U_n] \otimes A^{\otimes n})_{S_n}.$$

Proof. Assume that σ can be decomposed into more than one cycle, $\sigma = (i_1 \dots i_k)(j_1 \dots j_l)$. Then the coproduct gives

$$\Delta((i_1 \dots i_k)(j_1 \dots j_l)) = \sigma \otimes 1 + 1 \otimes \sigma + (i_1 \dots i_k) \otimes (j_1 \dots j_l) \pm (j_1 \dots j_l)(i_1 \dots i_k).$$

We see that σ is primitive if and only if σ has only one cycle. □

Now

$$\mathrm{H}_\bullet(\mathfrak{gl}(A)) = \Lambda(\mathrm{H}_\bullet((k[U_\bullet] \otimes A^{\otimes \bullet})_{S_\bullet})).$$

The symmetric group S_n is acting by conjugation in $k[S_n]$ and $k[U_n]$. As an S_n-representations

$$k[U_n] = \mathrm{Ind}_{C_n}^{S_n} k$$

and the dimension of $k[U_n]$ is $(n-1)!$. Furthermore

$$(\mathrm{Ind}_{C_n}^{S_n} k \otimes A^{\otimes n})_{S_n} \simeq (A^{\otimes n})_{C_n} = A^{\otimes n}/(1-t) = C_n^\lambda(A)$$

and for $a_1 \otimes \cdots \otimes a_n \in C_n^\lambda(A)$ we have by tracing all the steps in the proof

$$
\begin{array}{ccc}
a_1 \otimes \cdots \otimes a_n \in C_n^\lambda(A) & \xrightarrow{\quad b \quad} & b(a_1 \otimes \cdots \otimes a_n) \\
\uparrow & & \| \\
(12\dots n) \otimes (a_1, \dots, a_n) \in k[S_n] \otimes A^{\otimes n} & \longrightarrow & \begin{array}{c} a_1a_2 \otimes a_3 \otimes \cdots \otimes a_n \\ -a_1 \otimes a_2a_3 \otimes \cdots \otimes a_n \\ + \dots + (-1)^n a_na_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \end{array} \\
\uparrow & & \uparrow \\
E_{12}^{a_1} \otimes E_{23}^{a_2} \otimes \cdots \otimes E_{n1}^{a_n} & \xrightarrow{\quad d \quad} & \begin{array}{c} E_{12}^{a_1a_2} \otimes E_{34}^{a_3} \otimes \cdots \otimes E_{n1}^{a_n} \\ +E_{12}^{a_1} \otimes E_{23}^{a_2a_3} \otimes \cdots \otimes E_{n1}^{a_n} \\ + \dots + E_{12}^{a_1} \otimes E_{23}^{a_2} \otimes \cdots \otimes E_{n1}^{a_na_1} \end{array}
\end{array}
$$

We proved that $(k[U_n] \otimes A^{\otimes n})_{S_n}$ is the Connes complex, and thus

$$\Lambda(\mathrm{HC}_{\bullet-1}(A)) \simeq \mathrm{H}_\bullet(\mathfrak{gl}(A)).$$

$\square$

Example 1.13. If $A = k$ we know that

$$\begin{cases} \mathrm{HC}_{2n}(k) & = k \\ \mathrm{HC}_{2n-1}(k) & = 0 \end{cases}$$

From this we can derive

$$\begin{aligned} \mathrm{H}_\bullet(\mathfrak{gl}(k)) &= \Lambda(x_1, x_3, \dots, x_{2n+1}, \dots), \\ \mathrm{H}_\bullet(\mathfrak{gl}_n(k)) &= \Lambda(x_1, x_3, \dots, x_{2n-1}), \\ \mathrm{H}_\bullet(\mathfrak{sl}_2(k)) &= \Lambda(x_3). \end{aligned}$$

1.3 Computation of Leibniz homology $\mathrm{HL}_\bullet(\mathfrak{gl}(A))$

Our aim now is to compute $\mathrm{HL}_\bullet(\mathfrak{gl}(A))$. Recall the steps in the proof of theorem (1.1).

$$
\begin{array}{c}
\mathfrak{gl}(A)^{\otimes n} \\
\Big\downarrow \text{quasi-isomorphism} \\
(\mathfrak{gl}(A)^{\otimes n})_{\mathfrak{sl}(k)} \\
\| \\
(\mathfrak{gl}(k)^{\otimes n} \otimes A^{\otimes n})_{\mathfrak{sl}(k)} \\
\| \\
(\mathfrak{gl}(k)^{\otimes n})_{\mathfrak{sl}(k)} \otimes A^{\otimes n} \\
\| \\
(k[S_n] \otimes A^{\otimes n})_{S_n}
\end{array}
$$

We have to modify the third step, because Leibniz homology is not a Hopf algebra. If $\mathfrak{g}$ is a Leibniz algebra, then $\mathrm{HL}_\bullet(\mathfrak{g})$ is a graded Zinbiel coalgebra which definition we give below.

Definition 1.14. *A **Zinbiel algebra** A is an algebra such that its multiplication satisfies the following identity*

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z + (x \cdot z) \cdot y.$$

Lemma 1.15. *If we define a new product $xy := x \cdot y + y \cdot x$, then it will be associative.*

Zinbiel algebras play the same role to the commutative algebras as the associative algebras to Lie algebras.

Definition 1.16. *A **graded Zinbiel algebra** A is an algebra such that its multiplication satisfies the following identity*

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z + (-1)^{|z||y|}(x \cdot z) \cdot y.$$

Definition 1.17. *A **Zinbiel coalgebra** C is a coalgebra such that its comultiplication $\Delta\colon C \to C \otimes C$ satisfies the following identity*

$$(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta + (\mathrm{Id} \otimes \tau) \circ (\Delta \otimes \mathrm{Id}) \circ \Delta,$$

where $\tau\colon C \otimes C \to C \otimes C$ is given by

$$\tau(x \otimes y) = \begin{cases} y \otimes x & \text{in the non graded case,} \\ (-1)^{|y||x|} y \otimes x & \text{in the graded case.} \end{cases}$$

Proposition 1.18. *The Leibniz homology $\mathrm{HL}_\bullet(\mathfrak{gl}(A))$ is a graded Zinbiel as coalgebra and associative as algebra.*

In short we say that $\mathrm{HL}_\bullet(\mathfrak{gl}(A))$ is a graded $Zinb^c$-As-bialgebra. It means that the Zinbiel coalgebra coproduct and the associative algebra product satisfy some compatibility relation. If one compares the product with the symmetric coproduct, then one obtatins the Hopf formula.

There is a following structure theorem for $Zinb^c$-As-bialgebras.

Theorem 1.19. *If a $Zinb^c$-As-bialgebra $\mathcal{H}$ is connected, then it is free and cofree over its primitive part.*

Corollary 1.20.

$$\mathrm{HL}_\bullet(\mathfrak{gl}(A)) \simeq T(\mathrm{Prim}(\bigoplus_{n \geq 0} k[S_n] \otimes A^{\otimes n})) = T(\bigoplus_{n \geq 0} k[U_n] \otimes A^{\otimes n}).$$

Our aim now is to compute $\mathrm{H}_\bullet(\bigoplus_{n \geq 0} k[U_n] \otimes A^{\otimes n})$.

Theorem 1.21 (Cuvier)**.** *There is a quasi-isomorphism of complexes*

$$\begin{tikzcd} \cdots \arrow[r] & k[U_n] \otimes A^{\otimes n} \arrow[r, "d"] \arrow[d] & k[U_{n-1}] \otimes A^{\otimes (n-1)} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & A^{\otimes n} \arrow[r, "b"] & A^{\otimes (n-1)} \arrow[r] & \cdots \end{tikzcd}$$

The map is given by

$$g(12\dots n)g^{-1} \otimes \alpha \mapsto g(\alpha),$$

where $g \in S_n$, $g(1) = 1$ is chosen in such way that $g(12\dots n)g^{-1}$ is teh cycle which we want to sendo to $A^{\otimes n}$. The map in the opposite direction is

$$\alpha \mapsto (12\dots n) \otimes \alpha.$$

and the one composition is identity on $A^\bullet$ and the second one is homotopic to the identity.

Corollary 1.22.

$$\begin{tikzcd}
\mathrm{HL}_\bullet(\mathfrak{gl}(A)) \arrow[r, "\simeq"] \arrow[d] & T(\mathrm{HH}_{\bullet-1}(A)) \arrow[d] \\
\mathrm{H}_\bullet(\mathfrak{gl}(A)) \arrow[r, "\simeq"] & \Lambda(\mathrm{HC}_{\bullet-1}(A)).
\end{tikzcd}$$

Now we can make a digression on some algebraic topology theorems. Suppose there is a fibration $F \to E \to B$ of H-spaces. There is a spectral sequence

$$E^2_{pq} = \mathrm{H}_p(B, \mathrm{H}_q(F)) \implies \mathrm{H}_{p+q}(E).$$

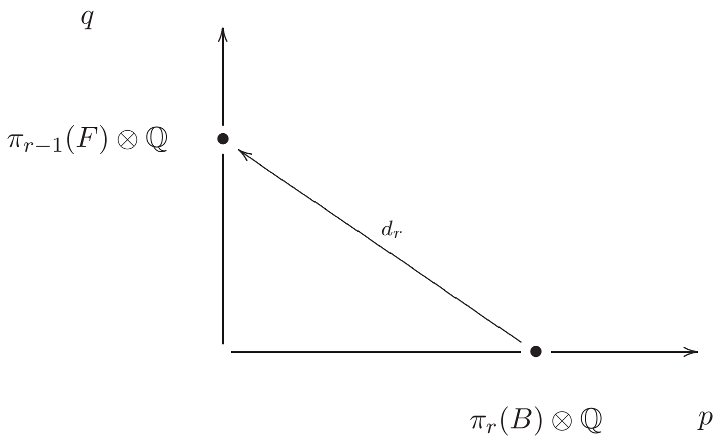
Furthermore for any H-space X

$$\mathrm{Prim}(\mathrm{H}_\bullet(X;\mathbb{Q})) = \pi_{\bullet(X)} \otimes \mathbb{Q},$$

so

$$\mathrm{H}_\bullet(X;\mathbb{Q}) = \Lambda(\pi_\bullet(X) \otimes \mathbb{Q}).$$

If we take a primitive parts on second term of this spectral sequence, then it can be proved that the only nonzero terms will be on the row $q = 0$ and column $p = 0$, and it will be isomorphic to rational homotopy of the base and the fiber.



From this spectral sequence we obtain the long exact sequence of homotopy groups.

The motivation for computing $\mathrm{H}_\bullet(\mathfrak{gl}_r(A))$ for fixed r comes from Macdonald conjecture, which is some identity with sum on the left hand side and product on the right. To prove it, it is sufficient to compute $\mathrm{H}_n(\mathfrak{gl}_r(k[t]/t^k))$. On one side there will be an Euler-Poincae characteristic of the complex, and on the other the Euler-Poincaré characteristic of the homology, which are equal.

Theorem 1.23. *If k is a characteristic 0 field, A is an associative unital algebra, then*

$$\mathrm{H}_n(\mathfrak{gl}_n(A)) \simeq \mathrm{H}_n(\mathfrak{gl}_{n+1}(A)) \simeq \ldots \simeq \mathrm{H}_n(\mathfrak{gl}(A)).$$

Furthermore for commutative A the following sequence is exact

$$\mathrm{H}_n(\mathfrak{gl}_{n-1}(A)) \to \mathrm{H}_n(\mathfrak{gl}_n(A)) \twoheadrightarrow \Omega_A^{n-1}/d\Omega_A^{n-2}.$$

Theorem 1.24. *If k is a characteristic 0 field, A is an associative unital algebra, then*

$$\mathrm{H}_n(\mathrm{GL}_n(F)) \simeq \mathrm{H}_n(\mathrm{GL}_{n+1}(F)) \simeq \ldots \simeq \mathrm{H}_n(\mathrm{GL}(A)),$$

and the following sequence is exact

$$\mathrm{H}_n(\mathrm{GL}_{n-1}(F)) \to \mathrm{H}_n(\mathrm{GL}_n(F)) \to \mathrm{K}_n^M(F) \otimes \mathbb{Q}.$$

where K^M is the Milnor's K-theory.

Theorem 1.25. *If k is a characteristic 0 field, A is an associative unital algebra, then*

$$\mathrm{HL}_n(\mathfrak{gl}_n(A)) \simeq \mathrm{HL}_n(\mathfrak{gl}_{n+1}(A)) \simeq \ldots \simeq \mathrm{HL}_n(\mathfrak{gl}(A)).$$

Furthermore for commutative A there is an exact sequence

$$\mathrm{HL}_n(\mathfrak{gl}_{n-1}(A)) \to \mathrm{HL}_n(\mathfrak{gl}_n(A)) \twoheadrightarrow \Omega_A^{n-1}.$$

Chapter 2

Algebraic operads

2.1 Schur functors and operads

Definition 2.1. *An **algebraic operad** is a functor $\mathcal{P}\colon \mathbf{Vect} \to \mathbf{Vect}$, together with a natural transformationa of functors $\iota\colon \mathrm{Id} \to \mathcal{P}$, $\gamma\colon \mathcal{P}\circ\mathcal{P} \to \mathcal{P}$. They are supposed to satisfy the following relations*

- *γ is associative,*

$$\begin{tikzcd} \mathcal{P}\circ\mathcal{P}\circ\mathcal{P} \arrow[r, "\mathrm{id}\circ\gamma"] \arrow[d, "\gamma\circ\mathrm{id}"'] & \mathcal{P}\circ\mathcal{P} \arrow[d, "\gamma"] \\ \mathcal{P}\circ\mathcal{P} \arrow[r, "\gamma"'] & \mathcal{P} \end{tikzcd}$$

- *ι is a unit for γ.*

If X is a set, then the structure is just inclusion $\{*\} \to X$ and if $X\times X \to X$ is an operation, then we have the notion of **set operad** $\mathcal{P}\colon \mathbf{Sets} \to \mathbf{Sets}$. In analagous way we can define topological operad, chain complex operad etc.

In the sequel, we suppose that $\mathcal{P}$ is Schur functor, which definition we give below.

Definition 2.2. *A **Schur functor** is defined from an S-module $\mathcal{P}$, which is a collection of right S_n-modules, and*

$$\mathcal{P}(V) := \bigoplus_{n\geq 0} \mathcal{P}(n)\otimes_{S_n} V^{\otimes n}.$$

We can as well write

$$\mathcal{P}(V) := \bigoplus_{n\geq 0} (\mathcal{P}(n)\otimes V^{\otimes n})_{S_n}.$$

In characteristic 0 we can take the invariants as well as coinvariants.

In these notes we restrict the study of algebraic operads to Schur functors.

The natural transformation γ gives us for each vector space V a linear map

$$\gamma\colon \mathcal{P}(\mathcal{P}(V)) = \mathcal{P}\left(\bigoplus_{n\geq 0} \mathcal{P}(n)\otimes_{S_n} V^{\otimes n}\right) \to \mathcal{P}(V).$$

and so for each $n\geq 0$, a map

$$\gamma_{i_1\dots i_n} : \mathcal{P}(n)\otimes\mathcal{P}(i_1)\otimes\ldots\otimes\mathcal{P}(i_n) \to \mathcal{P}(i_1+\ldots+i_n), \quad n\geq 0.$$

Starting with $\gamma_{i_1\dots i_n}$, in order to reconstruct the operad we need to assume that it is compatible with the action of the symmetric group

$$S_n \times (S_{i_1} \times \ldots \times S_{i_n}) \to S_{i_1+\ldots+i_n},$$

and that it satisfies a certain associativity property, namely the associativity of $\gamma\colon \mathcal{P}\circ\mathcal{P} \to \mathcal{P}$.

Definition 2.3. *An **algebra over the operad** $\mathcal{P}$ (or $\mathcal{P}$-**algebra**) is a vector space A equipped with a linear map $\gamma_A\colon \mathcal{P}(A) \to A$ such that the following diagrams commute*

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(A)) \arrow[r, "\mathcal{P}(\gamma_A)"] \arrow[d, "\gamma(A)"'] & \mathcal{P}(A) \arrow[d, "\gamma_A"] \\ \mathcal{P}(A) \arrow[r, "\gamma_A"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\iota(A)"] \arrow[dr, "="'] & \mathcal{P}(A) \arrow[d, "\gamma_A"] \\ & A \end{tikzcd}$$

For an algebra over the operad $\mathcal{P}$ and each $n \geq 0$ there is a map $\gamma_n\colon \mathcal{P}(n) \otimes_{S_n} A^{\otimes n} \to A$ and we write

$$(\mu; a_1, \ldots, a_n) \mapsto \gamma(\mu \otimes (a_1, \ldots, a_n)) =: \mu(a_1, \ldots, a_n).$$

We call $\mathcal{P}(n)$ the space of n-ary operations.

Let V be a vector space, and $\mathcal{P}$ an operad. Suppose that we have a type of algebras (for example associative, Leibniz, Lie). We name it $\mathcal{P}$-algebras, where $\mathcal{P}$ denotes the given type. Then we define

Definition 2.4. *The $\mathcal{P}$-algebra A_0 is **free** over V if for any map $V \to A$ to a $\mathcal{P}$-algebra A there is a unique map of $\mathcal{P}$-algebras $A_0 \to A$ such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & A_0 \arrow[d, dashed] \\ & A \end{tikzcd}$$

Let $V = kx_1 \oplus \ldots \oplus kx_n$ be an n-dimensional vector space over k, and $\mathcal{P}(V)$ denote be the free algebra of a given type over V. The multilinear part of $\mathcal{P}(V)$ of degree n (linear in each variable) is a subspace which we denote $\mathcal{P}(n)$ and it inherits an S_n-action. Thus it allows us to construct an operad $\mathcal{P}$ as a Schur functor. If $k \supseteq \mathbb{Q}$ then

$$\mathcal{P}(V) = \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}.$$

2.2 Free operads

The notion of free $\mathcal{P}$- algebra over a vector space V gives rise to a functor from the category **Vect** of vector spaces to a category $\mathcal{P}$-algebras, which is a left adjoint to the forgetful functor

$$\mathrm{Hom}_{\mathcal{P}-\mathbf{alg}}(\mathcal{P}(V), A) \simeq \mathrm{Hom}_{\mathbf{Vect}}(V, A).$$

There is a forgetful functor from the category **Op** of operads to the category of Schur functors. It has a left adjoint $\mathcal{F}$ giving rise to free operads.

Example 2.5. An S-module is given by the family $M(n)$ of S_n-modules. Suppose $M(n)=0$ except $M(2)=k[S_2]\mu$. What is $\mathcal{F}(M)$? We have $\mathrm{id}\in\mathcal{F}(M)(1)$, $\mu\in\mathcal{F}(M)(2)$ and the following two operations in $\mathcal{F}(M)(3)$

$$\mu\circ(\mu,\mathrm{id}),\quad \mu\circ(\mathrm{id},\mu).$$

Thus

$$\mathcal{F}(M)(3)=k[S_3]\mu\circ(\mu,\mathrm{id})\oplus k[S_3]\mu\circ(\mathrm{id},\mu).$$

Proposition 2.6. *The free operad $\mathcal{F}(M)$, where M is binary and free over S_2, has $\mathcal{F}(M)(n)=k[Y_{n-1}]\otimes k[S_n]$, where Y_{n-1} is the set of planar binary trees with n leaves.*

Exercise 2.7. *What is the free operad on N, where $N(n)=0$ except $N(2)=k$ - the trivial representation?*

2.3 Operadic ideals

Definition 2.8. *For a given operad $\mathcal{P}$ and a family of operations in $\mathcal{P}$, the **operadic ideal** generated by this family is a sub-Schur functor I (that is $I(n)\subseteq\mathcal{P}(n)$) linearly generated by all the compositions where at least one of the operations is in the family.*

In another words whenever one of the operations is in I, then the image by γ is also in I.

$$\gamma\colon\mathcal{P}(n)\otimes(\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_n))\to\mathcal{P}(i_1+\ldots+i_n)$$

Proposition 2.9. *The quotient $\mathcal{P}/I$ defined as $(\mathcal{P}/I)(n):=\mathcal{P}(n)/I(n)$ is an operad. It is called the **quotient operad**.*

A type of algebras consists of generating operations and relations (multilinear). Let $\mathcal{F}(M)$ be the free operad over an S-module M such that $M(n)$ is defined by the n-ary operations. If we take the ideal I generated by the relators in $\mathcal{F}(M)(-)$, then the we have an operad associated to a given type of algebras represented as a quotient operad $\mathcal{P}=\mathcal{F}(M)/I$. The space $\mathcal{P}(V)$ is exactly the free algebra over V for the given type.

2.4 Examples

Example 2.10. Associative algebras over k with binary associative operation $\mu\colon A\otimes A\to A$, $\mu(xy)=:xy$. The corresponding operad **As** has

$$\mathbf{As}(n)=k[S_n],$$

$$\mathbf{As}(n)\otimes_{S_n}V^{\otimes n}=V^{\otimes n},$$

$$\gamma\colon\mathbf{As}(n)\otimes\mathbf{As}(i_1)\otimes\cdots\otimes\mathbf{As}(i_n)\to\mathbf{As}(i_1+\ldots+i_n),$$

$$k[S_n]\otimes k[S_{i_1}]\otimes\cdots\otimes k[S_{i_n}]\to k[S_{i_1+\ldots i_n}]$$

$$(\sigma;\omega_1,\ldots,\omega_n)\mapsto\sigma(\omega_1,\ldots,\omega_n)=(\omega_{\sigma(1)}\times\cdots\times\omega_{\sigma(n)}).$$

Example 2.11. Commutative algebras over k with binary commutative operation $\mu\colon A\otimes A\to A$, $\mu(xy) =: xy$. The corresponding operad **Com** has

$$\mathbf{Com}(n) = k,$$

$$\mathbf{Com}(n)\otimes_{S_n} V^{\otimes n} = S^n V,$$
$$\gamma\colon \mathbf{Com}(n)\otimes\mathbf{Com}(i_1)\otimes\cdots\otimes\mathbf{Com}(i_n)\to\mathbf{Com}(i_1+\ldots+i_n),$$
$$\gamma\colon k^{\otimes(n+1)}\xrightarrow{\simeq} k$$

The general construction of the operad associated to algebras of given type uses the following data:

- generating operations μ_n with symmetries which define a right S_n-module $M(n)$, $n\geq 0$
- multilinear relations in $M(n)$, $n\geq 0$.

From the generating operations we can construct a free operad $\mathcal{F}(M)$, and then quotient by the ideal I generated by the relations which gives us $\mathcal{P}=\mathcal{F}(M)/I$.

For example if we have one binary operation μ and one relator $\mu\circ(\mu\otimes\mathrm{id})-\mu\circ(\mathrm{id}\otimes\mu)$, then we can construct an operad **As** for associative algebras.

2.5 Koszul duality of algebras

Definition 2.12. *A* ***quadratic data*** *is a pair (V,R), where V is a vector space and $R\subset V^{\otimes 2}$.*

Definition 2.13. *An* ***quadratic algebra*** *associated with quadratic data (V,R) is a quotient algebra of a tensor algebra $A(V,R) := T(V)/R$.*

The algebra $A := A(V,R)$ has universal property

$$\begin{tikzcd} R \arrow[r, tail] \arrow[rr, bend left, "0"] & T(V) \arrow[r, two heads] \arrow[dr] & T(V)/R \arrow[d, dashed] \\ & & A' \end{tikzcd}$$

Let $T^c(V)$ be the tensor module with the deconcatenation operation $\Delta\colon T^c(V)\to T^c(V)\otimes T^c(V)$,

$$\Delta(v_1,\ldots,v_n)=\sum_{i=1}^{n} v_1\ldots v_i\otimes v_{i+1}\ldots v_n.$$

Definition 2.14. *A* ***quadratic coalgebra*** *associated with quadratic data (V,R) is a subcoalgebra of a cotensor algebra $C := C(V,R)$ having the following universal property*

$$\begin{tikzcd} C(V,R) \arrow[r, tail] \arrow[rr, bend left, "0"] & T^c(V) \arrow[r, two heads] & V^{\otimes 2}/R \\ C' \arrow[u, dashed] \arrow[ur] & & \end{tikzcd}$$

We can write explicitly

$$A = k \oplus V \oplus V^{\otimes 2}/R \oplus V^{\otimes 3}/(V \otimes R + R \otimes V) \oplus \ldots \oplus V^{\otimes n}/(\sum_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}) \oplus \ldots$$

$$C = k \oplus V \oplus R \oplus (V \otimes R \cap R \otimes V) \oplus \ldots \oplus \bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \oplus \ldots$$

$$\alpha: \quad \begin{tikzcd} C \arrow[rr] \arrow[dr] & & A \\ & V \arrow[ur] & \end{tikzcd}$$

We can define a map $d_\alpha \colon C \otimes A \to C \otimes A$ by the composition

$$\begin{tikzcd} C \otimes A \arrow[r, "\Delta\otimes\mathrm{id}"] \arrow[rrr, bend right, "d_\alpha"'] & C \otimes C \otimes A \arrow[r, "\mathrm{id}\otimes\alpha\otimes\mathrm{id}"] & C \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\cdot"] & C \otimes A \end{tikzcd}$$

Lemma 2.15. *The map d_α is a differential, that is $d_\alpha \circ d_\alpha = 0$.*

Proof. This immediately follows from $\alpha * \alpha = 0$ and $d_\alpha \circ d_\alpha = d_{\alpha*\alpha}$. □

Definition 2.16. *A **Koszul complex** of the quadratic algebra $A(V, R)$ is the complex $(C \otimes A, d_\alpha)$.*

Definition 2.17. *A quadratic algebra $A(V, R)$ is said to be **Koszul algebra** if the Koszul complex is acyclic.*

Definition 2.18. *The **Koszul dual algebra** to an algebra $A = A(V, R)$ is defined as*

$$A^! := C^* := C(V, R)^*.$$

We have $A^! = T(V^*)/(R^\perp)$, where $R^\perp$ is defined as the kernel

$$R^\perp \rightarrowtail V^{*\otimes 2} \to R^*.$$

If $\dim V < \infty$, then $V^{**} = V$ and there is an epimorphism $V^{*\otimes 2} \twoheadrightarrow R^*$.

2.6 Bar and cobar constructions

Recall that for associative algebras there is so called bar construction

$$B \colon \mathbf{As-alg} \to \mathbf{DGA-coalg},$$

and for coalgebras there is a dual cobar construction

$$\Omega \colon \mathbf{As-coalg} \to \mathbf{DGA-alg}.$$

Theorem 2.19. *Let (V, R) be a quadratic data. Then the following are equivalent*

1. *$A(V, R)$ is Koszul.*
2. *$C \rightarrowtail B(A)$ is a quasi-isomorphism.*

3. $\Omega(C) \twoheadrightarrow A$ *is a quasi-isomorphism.*

The last two conditions mean that

2. $C \simeq \mathrm{H}^0(B(A))$, $\mathrm{H}^n(B(A)) = 0$ for $n \neq 0$.

3. $A \simeq \mathrm{H}_0(\Omega(C))$, $\mathrm{H}_n(\Omega(C)) = 0$ for $n \neq 0$.

Analogous constructions we can perform for quadratic operads. Starting from generating operations E and relators $R \subset \mathcal{F}(E)(3)$ we can construct an operad $\mathcal{P}(E,R)$ and a cooperad $\mathcal{C}(E,R)$. The cooperads are constructed on the same pattern but using comonoids instead of monoids, that is they are Schur functors with the comonoid structure with comultiplication $\gamma\colon \mathcal{P} \to \mathcal{P} \circ \mathcal{P}$ and counit $\eta\colon \mathcal{P} \to \mathrm{Id}$.

There are a bar and cobar constructions

$$\begin{aligned} B &\colon \mathbf{Op} \to \mathbf{DGA} - \mathbf{coOp} \\ \Omega &\colon \mathbf{coOp} \to \mathbf{DGA} - \mathbf{Op} \end{aligned}$$

Along the same lines we can construct a Koszul complex as $(\mathcal{C} \circ \mathcal{P}, d)$, and if it is acyclic then $\mathcal{P}$ is called a Koszul operad.

Define dual cooperad $\mathcal{P}^! := \mathcal{C}^\vee := \mathcal{C}^* \otimes \mathrm{sgn}$, where sgn is the signature representation of S_n. For any $\mathcal{P}$-algebra A we define a chain complex $\mathcal{P}^{!\vee}(A) := C^{\mathcal{P}}_\bullet(A)$. The Koszul complex $(\mathcal{P}^{!\vee} \circ \mathcal{P}(V))$ is a particular case of this construction.

Definition 2.20. *The homology* $\mathrm{H}^{\mathcal{P}}_\bullet(A) := \mathrm{H}_\bullet(C^{\mathcal{P}}_\bullet(A), d)$ *is called an* ***operadic homology*** *of a* $\mathcal{P}$*-algebra* A*.*

Proposition 2.21. *The Koszul complex is acyclic if and only if* $\mathrm{H}^{\mathcal{P}}_n(\mathcal{P}(V)) = 0$ *for* $n > 1$*, and* $\mathrm{H}^{\mathcal{P}}_1(\mathcal{P}(V)) = V$*.*

Example 2.22. If $\mathcal{P} = \mathbf{Lie}$, $\mathbf{Lie}(n) == \mathrm{Ind}^{S_n}_{C_n} = k[U_n]$ then $\mathcal{P}^! = \mathbf{Com}$, $\mathbf{Com}(n) = k$ for $n \geq 0$. If $\mathfrak{g}$ is a Lie algebra, then $C^{\mathbf{Lie}}_n = \Lambda^n \mathfrak{g}$ is the Chevalley-Eilenberg complex.

Example 2.23. If $\mathcal{P} = \mathbf{Leib}$, $\mathbf{Leib}(n) = k[S_n]$ then $\mathcal{P}^! = \mathbf{Zinb}$, $\mathbf{Zinb}(n) = k[S_n]$ for $n \geq 0$. If A is a Leibniz algebra, hen $C^{\mathbf{Leib}}_n = \mathfrak{g}^{\otimes n}$ is the Leibniz complex.

2.7 Bialgebras and props

Recall that the operad is a Schur functor $\mathcal{P}\colon \mathbf{Vect} \to \mathbf{Vect}$ together with natural transformations $\iota\colon \mathrm{Id} \to \mathcal{P}$, $\gamma\colon \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ definining a monoid structure. An algebra of given type $\mathcal{P}$ gives rise to an operad by the construction of free algebra $\mathcal{P}(V)$ over V. The free algebra is an adjoint functor to the forgetful functor $\mathbf{Alg} \to \mathbf{Vect}$. For bialgebras however, the left adjoint functor to the forgetful functor does not exist. Recall that for an operad $\mathcal{P}$ we have a family of S_n-modules $\mathcal{P}(n)$ such that $\mathcal{P}(V) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$. The module $\mathcal{P}(n)$ is called a space of n-ary operations. For bialgebras one replaces these modules by $\mathcal{P}(n,m)$ for $n, m \geq 0$, because we can do operations and cooperations.

Definition 2.24. *A* ***symmetric monoidal category*** $\mathbb{S}$ *is a category with distinguished object* $0 \in \mathrm{Ob}(\mathcal{S})$ *and an associative product* $\Box\colon \mathbb{S} \times \mathbb{S} \to \mathbb{S}$*. We say that the category is* ***strict*** *if the associativity relation is equality, not only an isomorphism.*

Definition 2.25. *A **prop** is a strict symmetric monoidal category* $\mathbb{S}$ *such that*

$$\mathrm{Ob}(\mathbb{S}) = \{\underline{0}, \underline{1}, \ldots, \underline{n}, \ldots\} \simeq \mathbb{Z}$$

$$\underline{n}\square\underline{m} := \underline{n+m},$$

and the morphisms $\mathrm{Mor}_{\mathcal{S}}(\underline{n}, \underline{m})$ *is a vector space over* k*.*

Definition 2.26. *A **algebra over the prop** $\mathcal{P}$ **(gebra)** is a functor between symmetric monoidal categories*

$$(\mathcal{P}, \square) \to (\mathbf{Vect}, \otimes)$$

$$\underline{1} \mapsto A, \quad \underline{n} \mapsto A^{\otimes n}.$$

Example 2.27. Consider the skeleton category of the cateogory of finite sets, denoted by **Fin**. Objects are $\underline{n} = \{1, 2, \ldots, n\}$ and the morphisms are the set-theoretic maps.

The category of gebras over **Fin** is the category of unital commutative algebras.

$$\begin{tikzcd}
\underline{3} \arrow[d, shift right] \arrow[d, shift left] & \{1,2,3\} \arrow[d, shift right] \arrow[d, shift left] & A^{\otimes 3} \arrow[d, shift right, "\mathrm{id}\otimes\mu"'] \arrow[d, shift left, "\mu\otimes\mathrm{id}"] \\
\underline{2} \arrow[d] & \{1,2\} \arrow[d] & A^{\otimes 2} \arrow[d, "\mu"] \\
\underline{1} & \{1\} & A \\
\underline{0} \arrow[u] & \emptyset \arrow[u] & k \arrow[u]
\end{tikzcd}$$

What if we would like to have a prop corresponding to unital associative algebras? Then the answer is a the category of noncommutative sets ΔS. Its skeleton category **NFin** has the same objects $\{\underline{n}\}$ as **Fin**, but the morphism $f\colon \underline{n} \to \underline{m}$ is a set map together with a total order on each fiber $f^{-1}(i)$. For example we have one map $\underline{2} = \{1, 2\} \to \{1\} = \underline{1}$, but two morphisms $\{1 < 2\} \to \{1\}$ and $\{2 < 1\} \to \{1\}$, which correspond to the two maps $A^{\otimes 2} \to A$ given by $a \otimes b \mapsto ab$ and $a \otimes b \mapsto ba$.

For the Hochschild complex $C_n(A, M) = M \otimes A^{\otimes n}$ there are idempotents $e_n^{(i)}$ which commute with the Hochschild boundary b

$$\begin{tikzcd}
C_n(A,M) \arrow[r, equal] \arrow[d, "b"'] & M \otimes A^{\otimes n} \arrow[loop above, "e_n^{(i)}"] \\
C_{n-1}(A,M) \arrow[r, equal] & M \otimes A^{\otimes (n-1)} \arrow[loop below, "e_n^{(i)}"]
\end{tikzcd}$$

$$be_n^{(i)} = e_{n-1}^{(i)} b.$$

When $A = M$ and there is a B-map we also have

$$Be_n^{(i)} = e_{n-1}^{(i)} B.$$

All these formulas live in $k\mathbf{Fin}$, where $\mathrm{Mor}_{k\mathbf{Fin}}(\underline{n}, \underline{m}) = k[\mathrm{Mor}_{\mathbf{Fin}}(\underline{n}, \underline{m})]$. If A is commutative, then we have

$$b\colon \underline{n} \to \underline{n-1}, \quad b = \sum_{i=1}^{n} d_i$$

$$e_n^{(i)} : \underline{n} \to \underline{n}, \qquad \begin{tikzcd} \underline{n} \arrow[r, "e_n^{(i)}"] \arrow[d, "b"'] & \underline{n} \arrow[d, "b"] \\ \underline{n-1} \arrow[r, "e_n^{(i)}"] & \underline{n-1} \end{tikzcd}$$

in $k[\mathrm{Mor}_{\mathbf{Fin}}(\underline{n}, \underline{m})]$. There is a functor $k\mathbf{Fin} \to M \otimes A^{\otimes n}$ such that

$$\begin{tikzcd} \underline{n} \arrow[r, mapsto] \arrow[d, "f"'] & M \otimes A^{\otimes n} \arrow[d, "f_*"] \\ \underline{m} \arrow[r, mapsto] & M \otimes A^{\otimes m} \end{tikzcd}$$

$$f_*(m, a_1, \ldots, a_n) = (m, b_1, \ldots, b_m), \quad b_i = \prod_{j,\, f(j)=i} a_j.$$

When A is not commutative we take $\mathbf{NFin}$.

Any operad $\mathcal{P}$ gives rise to a prop. One defines

$$\mathcal{P}(n, 1) := \mathcal{P}(n)$$

$$\mathcal{P}(n, m) := ...$$

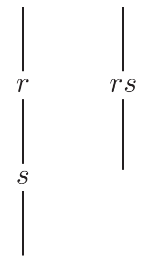
We can say that operads and free operads correspond to abstract rooted trees with data what happens when we contract an edge. If one considers planar rooted trees, then what one gets is so called non-symmetric operad (with no action of symmetric group). If any leaf could be a root, that is we consider abstract trees, then what we get is so called cyclic operads (there is a cyclic action of $\mathbb{Z}/(n+1)\mathbb{Z}$) on $\mathcal{P}(n)$.

Let $\tau = \tau_n \in \mathbb{Z}/(n+1)\mathbb{Z}$ denote the generator. For every n-ary operation $\mu \in \mathcal{P}(n)$ we have $\tau(\mu) \in \mathcal{P}(n)$.

Definition 2.28. *A **cyclic operad** is an operad such that $\mathcal{P}(n)$ has a $\mathbb{Z}/(n+1)\mathbb{Z}$-action. This action together with the S_n-action makes it an S_{n+1}-module.*

There is a relation between cyclic action and composition.

Example 2.29. Let $\mathcal{P}(1) = R$ an associative algebra, and $\mathcal{P}(n) = 0$ for $n \geq 0$.



If there is a cyclic action $r \mapsto \bar{r}$, $\bar{\bar{r}} = r$ on R, then $\overline{rs} = \bar{s}\bar{r}$. The cyclic operad here correspond to cyclic algebra with involution.

Fact 2.30. **As**, **Lie**, **Com**, **Poiss** *are cyclic operad, but* **Leib** *is not.*

Let $\mathcal{P}$ be a cyclic operas. Then we can construct three homology theories $\mathrm{HA}_\bullet$, $\mathrm{HB}_\bullet$, $\mathrm{HC}_\bullet$ which fit into an exact sequence

$$\ldots \to \mathrm{HA}_n(A) \to \mathrm{HB}_n(A) \to \mathrm{HC}_n(A) \to \mathrm{HA}_{n-1}(A) \to \ldots,$$

where A is a $\mathcal{P}$-algebra, and $\mathrm{HB}_n(A) = \mathrm{H}_n^{\mathcal{P}}(A)$ is an operadic homology of A. If $\mathcal{P} = \mathbf{As}$, then HC_n is cyclic homology, $\mathrm{HA}_n = \mathrm{HC}_{n-1}$, $\mathrm{HB}_n = \mathrm{HH}_n$. If $\mathcal{P} = \mathbf{Lie}$, then $\mathrm{HB}_n = \mathrm{H}_n^{\mathrm{Lie}}(\mathfrak{g})$, $\mathrm{HA}_n = \mathrm{H}^{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra and a $\mathfrak{g}$-module via the adjoint representation.

$$\begin{tikzcd}
\mathfrak{g} \otimes \Lambda^n \mathfrak{g} \arrow[r] \arrow[d, "d^{\mathrm{CE}}"'] & \Lambda^{n+1} \mathfrak{g} \arrow[d, "d^{\mathrm{CE}}"'] \\
\mathfrak{g} \otimes \Lambda^{n-1} \mathfrak{g} \arrow[r] & \Lambda^n \mathfrak{g}
\end{tikzcd}$$

$$\mathrm{H}_n^{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}) \qquad \mathrm{H}_{n+1}^{\mathrm{Lie}}(\mathfrak{g})$$

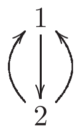
2.8 Graph complex

In the degree n of graph complex the space of n-chains is generated by connected graphs without loops with n edges and the valence of each vertex is ≥ 3. The differential

$$d(\gamma) = \sum_{e \text{ edges}} \pm \gamma / e, \quad d \colon C_n \to C_{n-1}$$

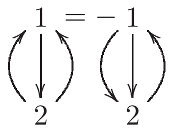
(when there is a loop we send the graph to 0).

To describe it precisely we assume that the graphs are oriented and the set of vertices is labelled by $1, \ldots, K$.

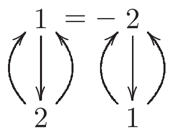


The equivalence relation:

1. changing the orientation of one edge, $\sim_1$



2. permutation of indices, $\sim_2$



The differential in the complex $\widetilde{C_n}$ is given by

$$d(\widetilde{\gamma}) = \sum_{i \xrightarrow{e} j} (-1)^j \widetilde{\gamma}/e$$

and is compatible with the equivalence relation $\sim$. It is also compatible with the first part $\sim_1$ of the equivalence relation. The complex $\widetilde{C_n}$ quotient by $\sim$ gives a graph complex C_n. We denote

$$\mathcal{G}_n := \widetilde{C_n}/\sim_1$$
$$(\mathcal{G}_s)_n := \widetilde{C_n}/\sim_1,\sim_2$$

The complex $(\mathcal{G}_s)_\bullet$ is called the **Kontsevich graph complex**.

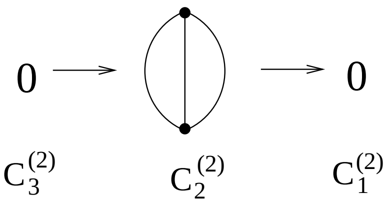


Figure 2.1: $H_2^{(2)} = \mathbb{Q}$

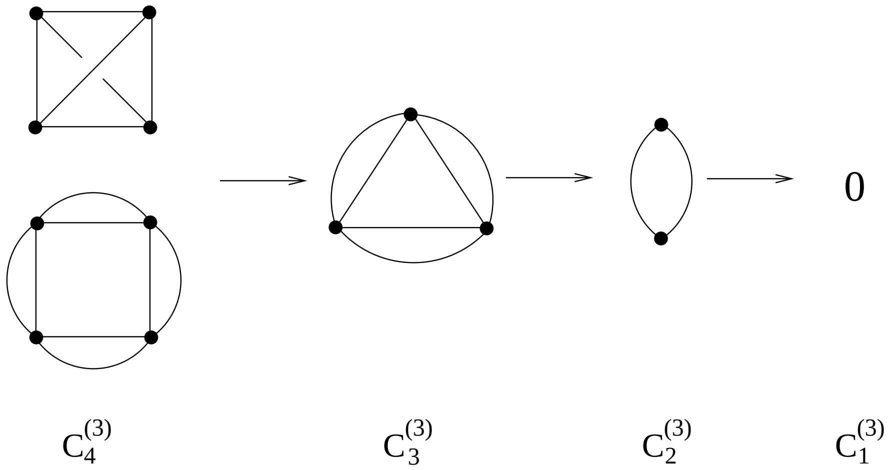


Figure 2.2: $H_4^{(3)} = \mathbb{Q}$

2.9 Symplectic Lie algebra of the commutative operad

We consider the polynomial algebra in $p_1, \dots, p_n, q_1, \dots, q_m$, that is $k[p_1, \dots, q_m] = S(V)$. The Poisson bracket, defined as

$$\{F, G\} := \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}$$

is a Lie bracket, and so, $S(V)$ is a Lie algebra. We denote it by $\mathfrak{sp}_{2m}(\mathbf{Com})$.

Theorem 2.31 (Kontsevich)**.** *If $\mathcal{G}_\bullet$ is the graph complex, then*

$$\mathrm{H}_\bullet(\mathfrak{sp}(\mathbf{Com})) \simeq \Lambda(\mathrm{H}_\bullet(\mathcal{G}_\bullet)).$$

The proof follows the same pattern as before:

1. Koszul trick.
2. Coinvariant theory.
3. Hopf-Borel type theorem, primitives.
4. Making the primitive complex smaller by dividing out acyclic complexes.

The key point in step 2 is the following result.

Theorem 2.32. *Let A_r^- be a linear subspace of degree r in $k[\varphi_{ij}]_{\substack{1\leq i\leq r\\ 1\leq i\leq r}}/\sim$, $\varphi_{ij}\sim\varphi_{ji}$. Then*

$$(V^{\otimes 2m})_{\mathfrak{sp}(V)} \simeq A_r^-.$$

Kontsevich's idea is to compute

$$\bigoplus_{k_1,\dots,k_n} (A_r^-)_{S_{k_1}\times\cdots\times S_{k_n}}$$

which appears to be a vector space generated by graphs divided by the relation $\sim_1$.

In step 3 one proves that the subcomplex of primitives is connected.

In step 4 we get rid of loops and graphs which have vertices of valence 2.

Appendix A

Cyclic duality and Hopf cyclic homology

Lecture given by Piotr Hajac.

A.1 Cyclic duality

Definition A.1. *A* ***cyclic (cocyclic) object*** *is a contravariant (covariant) functor from the cyclic category ΔC to an abelian category $\mathcal{A}$.*

Cyclic objects are used to define homology

$$
\begin{tikzcd}
\cdots & \mathcal{A}_{n-2} & \mathcal{A}_{n-1} \arrow[l, "d_i"] \arrow[r, shift left, "s_i"] \arrow[loop below, "t_{n-1}"] & \mathcal{A}_n \arrow[l, shift left, "d_j"] \arrow[r, shift left, "s_j"] \arrow[loop below, "t_n"] & \mathcal{A}_{n+1} \arrow[l, shift left, "d_k"] \arrow[r, "s_k"] \arrow[loop below, "t_{n+1}"] & \mathcal{A}_{n+2} & \cdots
\end{tikzcd}
$$

where $0 \leq i \leq n-1$, $0 \leq j \leq n$, $0 \leq k \leq n+1$, $n \in N$, and the faces d and degeneracies s satisfy the ΔC^{op} relations.

Cocyclic objects are used to define cohomology

$$
\begin{tikzcd}
\cdots & \mathcal{A}^{n-2} \arrow[r, "\delta_i"'] & \mathcal{A}^{n-1} \arrow[r, shift right, "\delta_j"'] \arrow[loop below, "\tau_{n-1}"] & \mathcal{A}^n \arrow[l, shift right, "\sigma_i"'] \arrow[r, shift right, "\delta_k"'] \arrow[loop below, "\tau_n"] & \mathcal{A}^{n+1} \arrow[l, shift right, "\sigma_j"'] \arrow[loop below, "\tau_{n+1}"] & \mathcal{A}^{n+2} \arrow[l, "\sigma_k"'] & \cdots
\end{tikzcd}
$$

where $0 \leq i \leq n-1$, $0 \leq j \leq n$, $0 \leq k \leq n+1$, $n \in N$, and the faces δ and degeneracies σ satisfy the ΔC relations.

Having a cyclic object, we can construct a cocyclic one, and vice versa.

$$
\begin{tikzcd}
\Delta C^{op} \arrow[r, shift left, "\hat{}"] & \Delta C \arrow[l, shift left, "\check{}"]
\end{tikzcd}
$$

$$
\begin{aligned}
\hat{A}^n &:= \mathcal{A}_n & \check{A}_n &:= \mathcal{A}^n \\
\hat{\delta}_0 &:= t_n s_{n-1} & \check{d}_0 &:= \sigma_{n-1}\tau_n \\
\hat{\delta}_{j+1} &:= s_j,\ 0 \leq j \leq n-1 & \check{d}_{j+1} &:= \sigma_j,\ 0 \leq j \leq n-1 \\
\hat{\sigma}_j &:= d_j & \check{s}_j &:= \delta_j \\
\hat{\tau}_n &:= t_n^{-1} & \check{t}_n &:= \tau_n^{-1}
\end{aligned}
$$

The cyclic category ΔC is isomorphic with its opposite ΔC^{op}. The compositions $\hat{\ }\circ\check{\ }$ and $\check{\ }\circ\hat{\ }$ are inner automorphisms implemented by $\tau_\bullet$ and $t_\bullet$ respectively.

A.2 Cyclic homology of algebra extensions

Let B be a subalgebra of A and M an A-bimodule. Define

$$\mathcal{B}_n := B \otimes_{B\otimes B^{op}} (M \otimes_B \underbrace{A \otimes_B \ldots \otimes_B A}_{n \text{ times}}).$$

Then

$$\mathcal{B}_0 = B \otimes_{B\otimes B^{op}} M = M/[B, M]$$

and B_n can be written in a circle.

The simplicial structure is given by

$$\begin{aligned}
d_0(m, a_1, \ldots, a_n) &:= (ma_1, a_2, \ldots, a_n)\\
d_j(m, a_1, \ldots, a_n) &:= (m, a_1, \ldots, a_j a_{j+1}, \ldots, a_n),\ 1 \leq j \leq n-1\\
d_n(m, a_1, \ldots, a_n) &:= (a_n m, a_1, \ldots, a_{n-1})\\
s_j(m, a_1, \ldots, a_n) &:= (m, a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n).
\end{aligned}$$

Lemma A.2. *The collection $\{B_n\}_{n\in\mathbb{N}}$ is a simplicial module. If $M = A$, then adjoining the morphisms $t_n \colon B_n \to B_n$*

$$t_n(a_0, a_1, \ldots, a_n) := (a_n, a_0, \ldots, a_{n-1})$$

makes $\{B_n\}_{n\in N}$ a cyclic module.

A.3 Hopf Galois extensions

Let H be a Hopf algebra with compultiplication $\Delta \colon H \to H \otimes H$, $\Delta(h) := h^{(1)} \otimes h^{(2)}$. Let M be a raight H-comodule with coaction $\Delta_R \colon M \to M \otimes H$, $\Delta_R(m) := m^{(0)} \otimes m^{(1)}$.

Let A be a right H-comodule algebra via $\Delta_R \colon A \to A \otimes H$ (G-space). Denote

$$B := \{a \in A \mid \Delta_R(a) = a \otimes 1\}$$

(functions on quotient).

Definition A.3. *The extension of algebras $B \subset A$ is called* ***Hopf-Galois extension*** *if*

$$g \colon A \otimes_B A \to A \otimes H$$
$$g(a \otimes_B a') := (a \otimes 1)\Delta_R(a') = a(a')^{(0)} \otimes (a')^{(1)}$$

(canonical map) is an isomorphism.

Since g is a B-module map, we can extend it to B_n

$$\tilde{g}(a_0, \ldots, a_{n-1}, a_n) := (a_0, \ldots, a_{n-1}a_n^{(0)}) \otimes a_n^{(1)}.$$

Using $\tilde{g}$ we have

$$B \otimes_{B\otimes B^{op}} (\underbrace{A \otimes_B \ldots \otimes_B A}_{n \text{ times}}) \xrightarrow{\tilde{g}} B \otimes_{B\otimes B^{op}} (\underbrace{A \otimes_B \ldots \otimes_B A}_{n-1 \text{ times}}) \otimes H$$

After n iterations we land in $B \otimes_{B\otimes B^{op}} A \otimes H^{\otimes n} = A/[A, B] \otimes H^{\otimes n}$. The key idea is to transport the cyclic structure via $\tilde{g}_*$.

A.4 Hopf cyclic homology with coefficients

Definition A.4. *Let M be a left H-module and right H-comodule. It is called **anti-Yetter-Drinfeld module** if*

$$\Delta_R(hm) = h^{(2)}m^{(0)} \otimes h^{(3)}m^{(1)}S(h^{(1)}) \in M \otimes H.$$

*M is **stable** if $m^{(1)}m^{(0)} = m$.*

Denote $\mathcal{H}_n := (H^{\otimes(n+1)}) \otimes_H M$ with diagonal action on $H^{\otimes(n+1)}$.

Theorem A.5 (Jara-Stefan, Hajac-Khalkhali-Rangipour-Sommerhaus)**.** *The following formulas define a cyclic module structure on $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$.*

$$\begin{aligned}
d_j(h_0 \otimes \cdots \otimes h_n) \otimes_H m &:= (h_0 \otimes \cdots \otimes \varepsilon(h_j) \otimes \cdots \otimes h_n) \otimes_H m \\
s_j(h_0 \otimes \cdots \otimes h_n) \otimes_H m &:= (h_0 \otimes \cdots \otimes \Delta h_j \otimes \cdots \otimes h_n) \otimes_H m \\
t_n(h_0 \otimes \cdots \otimes h_n) \otimes_H m &:= (h_n m^{(1)} \otimes h_0 \otimes \cdots \otimes h_{n-1}) \otimes_H m^{(0)}.
\end{aligned}$$

It is well defined if and only if M is stable anti-Yetter-Drinfeld module.

Let $B \subset A$ be a Hopf-Galois extension for a Hopf algebra H. The map $g\colon A\otimes H \to A\otimes_B A$ allows to define the translation map

$$T\colon H \to A \otimes_B A, \quad T(h) := g^{-1}(1 \otimes h) = h^{[1]} \otimes_B h^{[2]}.$$

Lemma A.6 (Mijaschita-Ulbrich, Jara-Stefan, Hajac-Khalkhali-Rangipour-Sommerhaus)**.** *The formula*

$$H \otimes A/[B,A] \xrightarrow{\triangleright} A/[B,A], \quad h \triangleright \bar{a} \mapsto \overline{h^{[2]}ah^{[1]}}$$

defines a left action. Moreover this action satisfies the stable anti-Yetter-Drinfeld module compatibility condition for the induced coaction on $A/[B,A]$.

Example A.7. If $A = H$ then $h \triangleright k = h^{(2)}kS(h^{(1)})$.

Theorem A.8 (Jara-Stefan)**.** *The cyclic modules $\{B \otimes_{B\otimes B^{op}} A^{\otimes_B n}\}_{n\in\mathbb{N}}$ and $\{H^{\otimes(n+1)} \otimes_H A/[B,A]\}_{n\in\mathbb{N}}$ are isomorphic.*

Theorem A.9. *Let A be a left H-module algebra with respect to $H \otimes A \to A$, $h(ab) = h^{(1)}(a)h^{(2)}(b)$, $h(1) = \varepsilon(h)$. Let $M \otimes A^{\otimes(n+1)}$ be a right H-module via $(m \otimes \widetilde{a})h := mh^{(1)} \otimes S(h^{(2)})\widetilde{a}$ and k be a right H-module via ε. Then $\{\mathrm{Hom}_H(M \otimes A^{\otimes(n+1)}, k)\}_{n\in\mathbb{N}}$ is a cocyclic module with the cocyclic structure given by*

$$\begin{aligned}
(\delta_j f)(m \otimes a_0 \otimes \cdots \otimes a_n) &:= f(m \otimes a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n) \\
(\delta_n f)(m \otimes a_0 \otimes \cdots \otimes a_n) &:= f(m^{(0)} \otimes (S^{-1}(m^{-1})a_n)a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \\
(\sigma_j f)(m \otimes a_0 \otimes \cdots \otimes a_n) &:= f(m \otimes a_0 \otimes \cdots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_n) \\
(\tau_n f)(m \otimes a_0 \otimes \cdots \otimes a_n) &:= f(m^{(0)} \otimes S^{-1}(m^{-1})a_n \otimes a_0 \otimes \cdots \otimes a_n).
\end{aligned}$$

Special cases:

1. $H = k = M$ - the standard cyclic homology.

2. $H = k[\sigma, \sigma^{-1}]$, $M = {}^{\sigma}k_{\varepsilon}$ - the twisted cyclic homology.

Appendix B

Twisted homology and Koszul

Lecture given by Ulrich Kraehmer

B.1 Quantum plane

$k = \mathbb{C}$

$xy = qyx$, $q \in k^\times$ not root of unity quadratic algebra A

$e_{ij} = x^i y^j$, $i, j \geq 0$ vector space basis ($\Rightarrow$ PBW algebra)

Automorphisms $\sigma(x) = \lambda x, \sigma(y) = \mu y$, $\lambda, \mu \in k^\times$

Aim: Compute $HC^\sigma_\bullet(A)$

B.1.1 General strategy

Compute the simplicial theory underlying the paracyclic object as $H_\bullet(A, M) = \mathrm{Tor}^{A^e}_\bullet(M, A)$ (in our case $M = {}_\sigma A$) using a nice resolution of A as A^e-module or other techniques

There is a morphisms to the simplicial underlying the cyclic object (in our case this simplicial will be denoted by $\mathrm{HH}^\sigma_\bullet(A)$). Try to compute this.

Do the spectral sequence arising from the (b, B)-bicomplex to obtain as its limit the cyclic theory (here $HC^\sigma_\bullet(A)$)

B.1.2 Step 1 a

By bare hands.

Look for free resolution, i.e., exact complex

$$\ldots \to (A^e)^{b_1} \to (A^e)^{b_0} \to A \to 0$$

of A^e-modules. Should be as small as possible.

Thus $b_0 = 1$, augmentation = multiplication

Lemma: A generated by x_i, then $\ker \mu$ generated as A^e-module by $1 \otimes x_i - x_i \otimes 1$.

Proof: $\sum a_i b_i = 0$ then

$$\sum a_i \otimes b_i = \sum a_i \otimes b_i - a_i b_i \otimes 1 = \sum (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1)$$

hence it is generated by elements of the form $1 \otimes f - f \otimes 1$. But $f \mapsto (1 \otimes f - f \otimes 1)$ satisfies Leibniz:

$$1 \otimes fg - fg \otimes 1 = (f \otimes 1)(1 \otimes g - g \otimes 1) + (1 \otimes g)(1 \otimes f - f \otimes 1).$$

Claim follows

B.1.3 Step 1 b

Thus here: $b_0 = 1, b_1 = 2$:

$$\ldots \to (A^e)^2 \to A^e \to A \to 0$$

Now determine kernel of the first map

$$\begin{aligned} & (a \otimes b, c \otimes d) \\ \mapsto \quad & (a \otimes b)(1 \otimes x - x \otimes 1) + (c \otimes d)(1 \otimes y - y \otimes 1) \\ = \quad & a \otimes xb - ax \otimes b + c \otimes yd - cy \otimes d. \end{aligned}$$

Thus

$$(e_{ij} \otimes e_{kl}, 0) \mapsto e_{ij} \otimes e_{k+1l} - q^{-j} e_{i+1j} \otimes e_{kl}$$

and

$$(0, e_{ij} \otimes e_{kl}) \mapsto q^{-k} e_{ij} \otimes e_{kl+1} - e_{ij+1} \otimes e_{kl}$$

B.1.4 Step 1 c

Playing a bit with grading arguments gives that the kernel is generated as A^e-module by a single element

$$\omega := (1 \otimes y - qy \otimes 1, -q \otimes x + x \otimes 1)$$

Hence we can take $b_2 = 1$. Since A is a domain (again grading argument), the new map

$$A^e \to (A^e)^2, (a \otimes b) \mapsto (a \otimes b)\omega$$

is injective, so thats it: We found a free resolution

$$0 \to A^e \to (A^e)^2 \to A^e \to 0$$

of A as left A^e-module.

B.1.5 Step 2 a

Tensor resolution with ${}_\sigma A$ and take homology

As vector space our complex is

$$0 \to A \to A^2 \to A \to 0$$

The two morphisms are

$$f \mapsto (\sigma(y)f - qfy, -q\sigma(x)f + fx)$$

and

$$(f, g) \mapsto \sigma(x)f - fx + \sigma(y)g - gy.$$

Thus on bases:

$$e_{ij} \mapsto ((\mu q^{-i} - q)e_{ij+1}, (-q\lambda + q^{-j})e_{i+1j})$$

and

$$(e_{ij}, 0) \mapsto (\lambda - q^{-j})e_{i+1j}$$

and

$$(0, e_{kl}) \mapsto (\mu q^{-k} - 1)e_{kl+1}$$

B.1.6 Step 2b

Generators of HH_2:

$$e_{ij}, \quad \lambda = q^{-j-1}, \mu = q^{i+1}$$

Generators of HH_0: Always

$$e_{00}$$

plus

$$e_{0l+1} \quad \mu = 1$$

plus

$$e_{i+10} \quad \lambda = 1,$$

plus

$$e_{i+1j+1} \quad \lambda = q^{-j-1}, \mu = q^{i+1}$$

image contains e_{i+1j} except when $\lambda = q^{-j}$ and e_{kl+1} except when $\mu = q^k$.

B.1.7 Step 2c

HH_1: Kernel generated by

$$(e_{ij,0}) \quad \lambda = q^{-j} \quad plus \quad (0, e_{kl}) \quad \mu = q^k$$

plus

$$((1 - \mu q^{-i-1})e_{ij+1}, (\lambda - q^{-j-1})e_{i+1j})$$

The latter are always trivial in homology.

$$(e_{ij+1}, 0), \quad \lambda = q^{-j-1}$$

is trivial except when also $\mu = q^{i+1}$.

$$(e_{i0}, 0), \quad \lambda = 1$$

are always nontrivial. Similarly,

$$(0, e_{k+1l}) \quad \mu = q^{k+1}$$

is trivial except when $\lambda = q^{-l-1}$ and always nontrivial is

$$(0, e_{0l}), \quad \mu = 1$$

B.1.8 Step 2d

From now on for simplicity

$$\lambda = q^{-1} \quad \mu = q$$

Then

$$\mathrm{HH}_2 : 1$$

$$\mathrm{HH}_1 : (y, 0), (0, x)$$

$$\mathrm{HH}_0 : 1, xy$$

In origina Hochschild:

$$\mathrm{HH}_2 : 1 \otimes x \otimes y - \alpha \otimes y \otimes x$$

is boundary only when $\lambda = q^{-1}, \mu = \alpha = q$

In degree one:

$$x \otimes y, y \otimes x$$

B.2 How to compute $HH^s_\bullet(A)$

• The (b, B)-bicomplex with is not a bicomplex, since

$$b \circ B + B \circ b = \mathrm{id} - T.$$

But the columns form a complex. It computes the Hochschild homology $H_*(A, sA)$ with coefficients in the bimodule $sA = A$
• Define $C^0_n = \ker(\mathrm{id} - T)$. Then we have:
If $C_n = C^0_n \oplus C^1_n$, then $H_*(A, sA) \cong HH^s_*(A)$.
Proof: Since $[b, \mathrm{id} - T] = 0$, $C_* = C^0_* \oplus C^1_*$ as complexes, and we have $HH^s_*(A) = H_*(C^0_*, b)$ and $H_*(A, sA) = HH^s_*(A) \oplus H_*(C^1_*, b)$. But $(\mathrm{id} - T)|_{C^1_*}$ is a bijection, and on C^1_* we have

$$b \circ (\mathrm{id} - T)^{-1} \circ B + (\mathrm{id} - T)^{-1} \circ B \circ b = \mathrm{id}.$$

Hence $(\mathrm{id} - T)^{-1} \circ B$ is a contracting homotopy.
• This applies for example when σ is diagonalizable.

B.3 Cyclic homology

B in normalised:

$$B : f \mapsto 1 \otimes f$$

$$f \otimes g \mapsto 1 \otimes f \otimes g - 1 \otimes \sigma(g) \otimes f$$

$$f \otimes g \otimes h \mapsto 1 \otimes f \otimes g \otimes h + 1 \otimes \sigma(g) \otimes \sigma(h) \otimes f + 1 \otimes \sigma(h) \otimes f \otimes g.$$

On our gens:

$$1 \mapsto 1 \otimes 1, \quad xy \mapsto 1 \otimes xy$$

$$x \otimes y \mapsto 1 \otimes x \otimes y - q \otimes y \otimes x$$

$$y \otimes x \mapsto 1 \otimes y \otimes x - q^{-1} \otimes x \otimes y$$

We have (consider $b(1 \otimes x \otimes y)$)

$$[1 \otimes xy] = [x \otimes y] + q[y \otimes x]$$

Page 2 of spectral sequence:
Nothing in degree 2, the generator of HH_2 is in $\mathrm{im}\, B$
The kernel of B_1 is spanned by $\omega := [x \otimes y] + q[y \otimes x]$ which is in the image of B_0.
The kernel of B_0 is spanned by $[1]$.
Here the spectral sequence stabilises. So periodically:

$$HP_{even} = k[1], HP_{odd} = 0.$$

and not periodically we have to correct

$$HC_0 = k[1] \oplus k \cdot [xy].$$

That is, the quantum plane has the same cyclic theory as the classical one.

B.4 On Koszul

$A = A(V, I)$ quadratic, $A^! = A(V^*, I^\perp)$ Koszul dual, x_i, x^i dual bases in V, V^* Original Koszul complex:

$$K = (A^!)^* \otimes_k A$$

differential multiplication from the right by $e := x^i \otimes x_i$. Here $A^!$ acts on dual space from the right.

Why $d^2 = 0$? Identification:

$$\mathrm{Hom}_k(V^{\otimes 2}, V^{\otimes 2}) \simeq V^* \otimes V \otimes V^* \otimes V \simeq (A_1^! \otimes A_1)^{\otimes 2}$$

and

$$\begin{aligned} & A_2^! \otimes_k A_2 \simeq (V^* \otimes V^*/I^\perp) \otimes (V \otimes V/I) \\ \simeq \quad & I^* \otimes (V \otimes V/I) \simeq \mathrm{Hom}_k(I, V \otimes V/I) \end{aligned}$$

Then μ just become the canical map

$$\mathrm{Hom}_k(V^{\otimes 2}, V^{\otimes 2}) \to \mathrm{Hom}_k(I, V \otimes V/I).$$

Under the dientification $e \otimes e$ becomes id. Hence $\mu(e \otimes e)$ is zero.

B.5 Bimodule complex

This is

$$A \otimes (A^!)^* \otimes A$$

built as total complex of a bicomplex:

$$b_L : r \otimes f \otimes s \mapsto r x_i \otimes f x^i \otimes s$$

$$b_R : r \otimes f \otimes s \mapsto r \otimes x^i f \otimes x_i s$$

These commute and square to zero. Spectral sequence arguemnt: One complex acyclic iff the other is. One resolutoin of k, one resolution of A.

Bibliography

[Con] A. Connes, *Noncommutative Differential Geometry*, Inst. Hautes tudes Sci. Publ. Math. No. 62 (1985), 257–360.

[Lod1] Loday, Jean-Louis *Cyclic homology*, Appendix E by Mara O. Ronco. Second edition. Chapter 13 by the author in collaboration with Teimuraz Pirashvili. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 301. Springer-Verlag, Berlin, 1998. xx+513 pp.

[Lod2] Loday, Jean-Louis; *Generalized bialgebras and triples of operads* Preprint ArXiv:math.QA/0611885

[LodVall] Loday, Jean-Louis; Vallette, Bruno, *Algeraic operads* (book in preparation).

[LodQuill] Loday, Jean-Louis; Quillen, Daniel; *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. 59 (1984), no. 4, 569–591.

[FeiTsy] Feĭgin, B. L.; Tsygan, B. L.; *Additive K-theory* K-theory, arithmetic and geometry (Moscow, 1984–1986), 67–209, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[Kon] Kontsevich, Maxim; *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992), 97–121, Progr. Math., 120, Birkhuser, Basel, 1994.

[ConVog] Conant, James; Vogtmann, Karen; *On a theorem of Kontsevich*, Algebr. Geom. Topol. 3 (2003), 1167–1224 (electronic).