Dirac Operators and Spectral Geometry

Written examination

Each problem is worth 5 points, out of a possible total of 30.

1. Let $\mathrm{Cl}_{p,q} := \mathrm{Cl}(\mathbb{R}^{p+q}, g)$ be the real Clifford algebra for the symmetric bilinear form $g(x,x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$, and let Cl^0_{pq} be its even subalgebra. Prove that $\mathrm{Cl}^0_{p+1,q} \simeq \mathrm{Cl}_{q,p}$ as $\mathbb{R}$-algebras.

2. Let $\mathcal{A}$ be the unital $*$-algebra generated by elements x, y, z, and consider $p = \dfrac{1}{2}\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix} \in M_2(\mathcal{A})$. Prove that the relations $p = p^* = p^2$ imply that $\mathcal{A}$ is a polynomial algebra of functions on $\mathbb{S}^2$, by showing that

$$\begin{gathered} x^* = x, \quad y^* = y, \quad z^* = z, \\ [x,y] = [x,z] = [y,z] = 0, \\ x^2 + y^2 + z^2 = 1. \end{gathered}$$

3. Show that the distance function on $\mathbb{S}^1$ determined by the Dirac operator $D\!\!\!/ = -i\, d/d\theta$ is the arc length of the circle.

4. The Dirac operator on the sphere $\mathbb{S}^2$ with its usual round metric has eigenvalues $\pm k$, with respective multiplicities $2k$, for $k = 1, 2, 3, \ldots$. Let $\sigma_N(T)$ denote the sum of the N largest singular values of a compact operator T. If $s > 0$, prove that

$$L = \lim_{N\to\infty} \frac{\sigma_N(|D\!\!\!/|^{-s})}{\log N}$$

exists, with $0 < L < +\infty$, if and only if $s = 2$. Also, find this value of L.

5. A function or distribution u on $\mathbb{R}^n$ is homogeneous of degree λ if rescaling by a factor of t changes u to $t^\lambda u$. Show that the Dirac delta distribution on $\mathbb{R}^n$, defined by $\langle \delta, f\rangle := f(0)$, is homogeneous of degree $-n$.

6. If M is a compact manifold, its space of smooth functions $C^\infty(M)$, with the topology of uniform convergence of all derivatives, is known to be a unital Fréchet $*$-algebra. Prove that $C^\infty(M)$ is a pre-C^*-algebra, i.e., there is a C^*-algebra in which $C^\infty(M)$ is closed under holomorphic functional calculus.

Easy oral questions

E. Describe the real Clifford algebra $\mathrm{Cl}(\mathbb{R}^2)$, where $\mathbb{R}^2$ has its standard Euclidean structure.

E. When are two unital algebras A and B Morita equivalent? If M is an even-dimensional compact manifold, explain why the algebras $C(M)$ and $\Gamma(M, \mathbb{C}\mathrm{l}(M))$ are Morita equivalent.

E. Identify the spectrum of the Dirac operator $\not{D} = -i\, d/d\theta$ on the circle $\mathbb{S}^1$ with its trivial spinor module $\mathcal{S} = C^\infty(\mathbb{S}^1)$.

E. What is a (unital) spectral triple? What are its data, and what conditions must be satisfied by these data?

E. Given a positive elliptic pseudodifferential operator A of order $(-n)$ on a compact manifold of dimension n, why is $\operatorname{Tr}_\omega A$ independent of the choice of Dixmier trace Tr_ω, and how does one compute $\operatorname{Tr}_\omega A$?

E. Give an example of an isospectral deformation of a spectral triple.

More difficult oral questions

D. Let V be a Euclidean space of real dimension n. Explain why the *centre* $Z(\mathbb{C}\mathrm{l}(V))$ of the Clifford algebra $\mathbb{C}\mathrm{l}(V)$ has dimension 1 if n is even, and has dimension 2 if n is odd.

D. Describe briefly the construction of the irreducible representation of the complex Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^{2m})$ on the vector space underlying a complex exterior algebra.

D. Explain why the spinor bundle over the 3-sphere, $S \to \mathbb{S}^3$, is a trivial vector bundle.

D. Explain why the spinor bundle over the 2-sphere, $S \to \mathbb{S}^2$, is a trivial vector bundle.

D. Explain how one can obtain the area of the sphere $\mathbb{S}^2$, knowing the spectrum of the Dirac operator for the round metric on $\mathbb{S}^2$.

D. Suppose that (M, g) is a compact Riemannian spin manifold whose scalar curvature s is positive at each point of M. Outline how the Lichnerowicz formula $D\!\!\!/^2 = \Delta^S + \frac{1}{4}s$ allows one to prove that the Dirac operator has kernel $\{0\}$.

D. Outline the construction of the Dixmier traces as functionals on positive compact operators.

D. A compact operator T on a Hilbert space $\mathcal{H}$ is called *measurable* if it belongs to the Dixmier trace class $\mathcal{L}^{1+}(\mathcal{H})$, *and* if the value of any $\operatorname{Tr}_\omega T$ is independent of which Dixmier trace Tr_ω is used. How can one guarantee that an operator $T \in \mathcal{L}^{1+}(\mathcal{H})$ is measurable? If it is measurable, describe how to compute $\operatorname{Tr}_\omega T$.

D. Describe in detail the definition of the Wodzicki residue, $\operatorname{Wres} P$, of a classical pseudodifferential operator P on a compact manifold M.

D. If $D\!\!\!/$ is the Dirac operator on an n-dimensional compact Riemannian spin manifold (M, g), outline the evaluation of the Wodzicki residue, $\operatorname{Wres} |D\!\!\!/|^{-n}$.

D. Regularity of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is defined using the derivation $[|D|, \cdot]$ instead of the derivation $[D, \cdot]$. Using the standard commutative example where $\mathcal{A} = C^\infty(M)$ and $D = D\!\!\!/$ is a Dirac operator, explain why the operators $[D\!\!\!/, [D\!\!\!/, a]]$ *need not* be bounded for all $a \in C^\infty(M)$.

D. If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple on an infinite-dimensional Hilbert space $\mathcal{H}$, explain why the operator D cannot be bounded.