Dirac Operators and Spectral Geometry

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Introduction and Overview

Noncommutative geometry asks: "What is the geometry of the Quantum World?"

Quantum field theory considers aggregates of "particles", which are of two general species, "bosons" and "fermions". These are described by solutions of (relativistic) wave equations:

- Bosons: Klein–Gordon equation, $(\Box + m^2)\phi(x) = \rho_b(x)$ – "source term";
- Fermions: Dirac equation, $(i\partial\!\!\!/ - m)\psi(x) = \rho_f(x)$ – "source term";

where $x = (t, \vec{x}) = (x^0, x^1, x^2, x^3)$; $\Box = -\partial^2/\partial t^2 + \partial^2/\partial\vec{x}^2$; and $\partial\!\!\!/ = \sum_{\mu=0}^{3} \gamma^\mu \, \partial/\partial x^\mu$. In order that $\partial\!\!\!/$ be a "square root of $\Box$", we need $(\gamma^0)^2 = -1$, $(\gamma^j)^2 = +1$ for $j = 1, 2, 3$ and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$. Thus, the γ^μ must be matrices; in fact there are four (4×4) matrices satisfying these relations.

Point-like measurements are often ruled out by quantum mechanics; thus we replace *points* $x \in M$ by *coordinates* $f \in C(M)$. The metric distance on a Riemannian manifold (M, g) can be computed in two ways:

$$\begin{aligned} d_g(p, q) &:= \inf\{\, \mathrm{length}(\gamma \colon [0,1] \to M) : \gamma(0) = p,\ \gamma(1) = q \,\} \\ &= \sup\{\, |f(p) - f(q)| : f \in C(M), \|D\!\!\!\!/, f\| \leq 1 \,\}, \end{aligned}$$

where $D\!\!\!\!/$ is a Dirac operator with positive-definite signature (all $(\gamma^\mu)^2 = +1$) if it exists, so the Dirac operator specifies the metric. $D\!\!\!\!/$ is an (unbounded) operator on a Hilbert space $\mathcal{H} = L^2(M, S)$ of "square-integrable spinors" and $C^\infty(M)$ also acts on $\mathcal{H}$ by multiplication operators with $\|[D\!\!\!\!/, f]\| = \|\,\mathrm{grad}\, f\|_\infty$.

Noncommutative geometry generalizes $(C^\infty(M), L^2(M, S), D\!\!\!\!/)$ to a **spectral triple** of the form $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a "smooth" algebra acting on a Hilbert space $\mathcal{H}$, D is an (unbounded) selfadjoint operator on $\mathcal{H}$, subject to certain conditions: in particular that $[D, a]$ be a bounded operator for each $a \in \mathcal{A}$. The tasks of the geometer are then:

1. To describe (metric) differential geometry in an operator language.
2. To reconstruct (ordinary) geometry in the operator framework.
3. To develop new geometries with noncommutative coordinate algebras.

The long-term goal is to geometrize quantum physics at very high energy scales, but we are still a long way from there.

The general program of these lectures is as follows.

(A) The classical theory of spinors and Dirac operators in the Riemannian case.

(B) The operational toolkit for noncommutative generalization.

(C) Reconstruction: how to recover differential geometry from the operator framework.

(D) Examples of spectral triples with noncommutative coordinate algebras.

Chapter 1

Clifford algebras and spinor representations

Here are a few general references on Clifford algebras, in reverse chronological order: [GVF, 2001], [Fri, 1997/2000], [BGV, 1992], [LM, 1989] and [ABS, 1964]. (See the bibliography for details.)

1.1 Clifford algebras

We start with (V, g), where $V \simeq \mathbb{R}^n$ and g is a **nondegenerate symmetric bilinear form**. If $q(v) = g(v, v)$, then $2g(u, v) = q(u + v) - q(u) - q(v)$. Thus g is determined by the corresponding "quadratic form" q.

Definition 1.1. *The* ***Clifford algebra*** $\mathrm{Cl}(V, g)$ *is an algebra (over* $\mathbb{R}$*) generated by the vectors* $v \in V$ *subject to the relations* $uv + vu = 2g(u, v)1$ *for* $u, v \in V$*.*

The existence of this algebra can be seen in two ways. First of all, let $\mathcal{T}(V)$ be the tensor algebra on V, that is, $\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} V^{\otimes n}$. Then

$$\mathrm{Cl}(V, g) := \mathcal{T}(V)/\,\mathrm{Ideal}\langle u \otimes v + v \otimes u - 2g(u, v)\,1 : u, v \in V\rangle. \tag{1.1}$$

Since the relations are *not* homogeneous, the $\mathbb{Z}$-grading of $\mathcal{T}(V)$ is lost, only a $\mathbb{Z}_2$-grading remains:

$$\mathrm{Cl}(V, g) = \mathrm{Cl}^0(V, g) \oplus \mathrm{Cl}^1(V, g).$$

The second option is to define $\mathrm{Cl}(V, g)$ as a subalgebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$ generated by all expressions $c(v) = \varepsilon(v) + \iota(v)$ for $v \in V$, where

$$\begin{aligned} \varepsilon(v)\colon u_1 \wedge \cdots \wedge u_k &\mapsto v \wedge u_1 \wedge \cdots \wedge u_k \\ \iota(v)\colon u_1 \wedge \cdots \wedge u_k &\mapsto \sum_{j=1}^{k} (-1)^{j-1} g(v, u_j) u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_k. \end{aligned}$$

Note that $\varepsilon(v)^2 = 0$, $\iota(v)^2 = 0$, and $\varepsilon(v)\iota(u) + \iota(u)\varepsilon(v) = g(v, u)\,1$. Thus

$$\begin{aligned} c(v)^2 = g(v, v)\,1 \quad &\text{for all} \quad v \in V, \\ c(u)c(v) + c(v)c(u) = 2g(u, v)\,1 \quad &\text{for all} \quad u, v \in V. \end{aligned}$$

Thus these operators on $\Lambda^\bullet V$ do provide a representation of the algebra (1.1).

Dimension count: suppose $\{e_1, \dots, e_n\}$ is an orthonormal basis for (V, g), i.e., $g(e_k, e_k) = \pm 1$ and $g(e_j, e_k) = 0$ for $j \neq k$. Then the $c(e_j)$ anticommute and thus a basis for $\mathrm{Cl}(V, g)$ is $\{c(e_{k_1}) \dots c(e_{k_r}) : 1 \leq k_1 < \cdots < k_r \leq n\}$, labelled by $K = \{k_1, \dots, k_r\} \subseteq \{1, \dots, n\}$. Indeed,

$$c(e_{k_1}) \dots c(e_{k_r}) \colon 1 \mapsto e_{k_1} \wedge \cdots \wedge e_{k_r} \equiv e_K \in \Lambda^\bullet V$$

and these are linearly independent. Thus the dimension of the subalgebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$ generated by all $c(v)$ is just $\dim \Lambda^\bullet V = 2^n$. Now, a moment's thought shows that in the abstract presentation (1.1), the algebra $\mathrm{Cl}(V, g)$ is generated as a vector space by the 2^n products $e_{k_1} e_{k_2} \dots e_{k_r}$, and these are linearly independent since the operators $c(e_{k_1}) \dots c(e_{k_r})$ are linearly independent in $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$. Therefore, this representation of $\mathrm{Cl}(V, g)$ is faithful, and $\dim \mathrm{Cl}(V, g) = 2^n$.

The so-called "symbol map":

$$\sigma : a \mapsto a(1) : \mathrm{Cl}(V, g) \to \Lambda^\bullet V$$

is inverted by a "quantization map":

$$Q : u_1 \wedge u_2 \wedge \cdots \wedge u_r \longmapsto \frac{1}{r!} \sum_{\tau \in S_r} (-1)^\tau \, c(u_{\tau(1)}) c(u_{\tau(2)}) \dots c(u_{\tau(r)}). \tag{1.2}$$

To see that it is an inverse to σ, one only needs to check it on the products of elements of an orthonormal basis of (V, g).

From now, we write uv instead of $c(u)c(v)$, etc., in $\mathrm{Cl}(V, g)$.

1.2 The universality property

Chevalley [Che] has pointed out the usefulness of the following property of Clifford algebras, which is an immediate consequence of their definition.

Lemma 1.2. *Any $\mathbb{R}$-linear map $f \colon V \to A$ (an $\mathbb{R}$-algebra) that satisfies*

$$f(v)^2 = g(v, v)\, 1_A \quad \textit{for all} \quad v \in V$$

extends to an unique unital $\mathbb{R}$-algebra homomorphism $\tilde{f} \colon \mathrm{Cl}(V, g) \to A$.

Proof. There is really nothing to prove: $\tilde{f}(v_1 v_2 \dots v_r) := f(v_1) f(v_2) \dots f(v_r)$ gives the uniqueness, provided only that this recipe is well-defined. But observe that

$$\begin{aligned} \tilde{f}(uv + vu - 2g(u, v)1) &= \tilde{f}((u+v)^2 - u^2 - v^2) - 2g(u, v)\, \tilde{f}(1) \\ &= [q(u+v) - q(u) - q(v) - 2g(u, v)]\, 1_A = 0. \end{aligned}$$

□

Here are a few applications of universality that yield several useful operations on the Clifford algebra.

1. *Grading*: take $A = \mathrm{Cl}(V, g)$ itself; the linear map $v \mapsto -v$ on V extends to an *automorphism* $\chi \in \mathrm{Aut}(\mathrm{Cl}(V, g))$ satisfying $\chi^2 = \mathrm{id}_A$, given by

$$\chi(v_1 \dots v_r) := (-1)^r \, v_1 \dots v_r.$$

This operator gives the $\mathbb{Z}_2$-grading

$$\mathrm{Cl}(V, g) =: \mathrm{Cl}^0(V, g) \oplus \mathrm{Cl}^1(V, g).$$

2. *Reversal*: take $A = \mathrm{Cl}(V,g)^\circ$, the opposite algebra. Then the map $v \mapsto v$, considered as the inclusion $V \hookrightarrow A$, extends to an *antiautomorphism* $a \mapsto a^!$ of $\mathrm{Cl}(V,g)$, given by $(v_1 v_2 \dots v_r)^! := v_r \dots v_2 v_1$.

3. *Complex conjugation*: the **complexification** of $\mathrm{Cl}(V,g)$ is $\mathrm{Cl}(V,g) \otimes_{\mathbb{R}} \mathbb{C}$, which is isomorphic to $\mathrm{Cl}(V^{\mathbb{C}}, g^{\mathbb{C}})$ as a $\mathbb{C}$-algebra. Now take A to be $\mathrm{Cl}(V,g) \otimes_{\mathbb{R}} \mathbb{C}$ and define $f\colon v \mapsto \bar{v} : V^{\mathbb{C}} \to V^{\mathbb{C}} \hookrightarrow A$ (a real-linear map). It extends to an *antilinear automorphism* of A. Note that Lemma 1.2 guarantees $\mathbb{R}$-linearity, but not $\mathbb{C}$-linearity, of the extension even when A is a $\mathbb{C}$-algebra.

4. *Adjoint*: Also, $a^* := (\bar{a})^!$ is an *antilinear involution* on $\mathrm{Cl}(V,g) \otimes_{\mathbb{R}} \mathbb{C}$.

5. *Charge conjugation*: $\kappa(a) := \chi(\bar{a}) : v_1 \dots v_r \mapsto (-1)^r \bar{v}_1 \dots \bar{v}_r$ is an antilinear automorphism of $\mathrm{Cl}(V,g) \otimes_{\mathbb{R}} \mathbb{C}$.

Notation. We write $\mathbb{C}\mathrm{l}(V) := \mathrm{Cl}(V,g) \otimes_{\mathbb{R}} \mathbb{C}$ to denote the complexified Clifford algebra. Up to isomorphism, this is independent of the signature of the symmetric bilinear form g, because all complex nondegenerate bilinear forms are congruent.

1.3 The trace

Proposition 1.3. *There is an unique trace $\tau\colon \mathbb{C}\mathrm{l}(V) \to \mathbb{C}$ such that $\tau(1) = 1$ and $\tau(a) = 0$ for a odd.*

Proof. If $\{e_1, \dots, e_n\}$ is an orthonormal basis for (V,g), then

$$\tau(e_{k_1} \dots e_{k_{2r}}) = \tau(-e_{k_2} \dots e_{k_{2r}} e_{k_1}) = -\tau(e_{k_1} \dots e_{k_{2r}}) = 0.$$

(Here we have moved e_{k_1} to the right by anticommutation, and returned it to the left with the trace property.) Thus, if $a = \sum_{K \text{ even}} a_{k_1 \dots k_{2r}}\, e_{k_1} \dots e_{k_{2r}}$ lies in $\mathbb{C}\mathrm{l}^0(V)$, then $\tau(a) = a_\emptyset$. We will check that $a_\emptyset$ does not depend on the orthonormal basis used. Suppose $e'_j = \sum_{k=1}^n h_{kj}\, e_j$, with $H^t H = 1_n$, is another orthonormal basis. Then

$$e'_i e'_j = (\vec{h}_i \cdot \vec{h}_j)\, 1 + \sum_{k<l} c_{ij}^{kl}\, e_k e_l,$$

but $\vec{h}_i \cdot \vec{h}_j = [H^t H]_{ij} = 0$ for $i \neq j$. Next, the matrix of $e_k e_l \mapsto e'_i e'_j$ is $H \wedge H$, of size $\binom{n}{2}$, that is also orthogonal, so $e'_i e'_j e'_r e'_s$ has zero scalar part in the $e_k e_l e_p e_q$-expansion; and so on: the same is true for expressions $e'_{j_1} \dots e'_{j_{2r}}$ by induction. Thus $\tau(a) = a_\emptyset$ does not depend on $\{e_1, \dots, e_n\}$. □

Remark 1.4. At this point, it was remarked that for existence of the trace, one could use the restriction of the (normalized) trace on $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\Lambda^\bullet V^{\mathbb{C}})$, in which $\mathbb{C}\mathrm{l}(V)$ is embedded. True enough: although one must see why odd elements must have trace zero. For that, it is enough to note that if $a \in \mathbb{C}\mathrm{l}^1(V)$, then $c(a)$ takes even [respectively, odd] elements of the $\mathbb{Z}$-graded algebra $\mathrm{End}_{\mathbb{C}}(V^{\mathbb{C}})$ to odd [respectively, even] elements; thus, in any basis, the matrix of $c(a)$ will have only zeroes on the diagonal, so that $\mathrm{tr}(c(a)) = 0$. Nonetheless, Proposition 1.3 is useful in that it establishes the uniqueness of the trace.

Now $\mathbb{C}\mathrm{l}(V)$ is a *Hilbert space* with scalar product

$$\langle a \mid b \rangle := \tau(a^* b).$$

1.4 Periodicity

Write $\mathrm{Cl}_{pq} := \mathrm{Cl}(\mathbb{R}^{p+q}, g)$, where g has signature (p,q), and the orthonormal basis is written as $\{e_1, \dots, e_p, \varepsilon_1, \dots, \varepsilon_q\}$, where $e_1^2 = \cdots = e_p^2 = 1$ and $\varepsilon_1^2 = \cdots = \varepsilon_q^2 = -1$. For example,

$$\begin{aligned}
&\mathrm{Cl}_{10} = \mathbb{R} \oplus \mathbb{R};\\
&\mathrm{Cl}_{01} = \mathbb{C}, \quad \text{with} \quad \varepsilon_1 = i;\\
&\mathrm{Cl}_{20} = M_2(\mathbb{R}), \quad \text{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\\
&\mathrm{Cl}_{02} = \mathbb{H}, \quad \text{with} \quad \varepsilon_1 = i, \ \varepsilon_2 = j, \ \varepsilon_1\varepsilon_2 = k.
\end{aligned}$$

Lemma 1.5 ("(1,1)-periodicity"). $\mathrm{Cl}_{p+1,q+1} \simeq \mathrm{Cl}_{pq} \otimes M_2(\mathbb{R})$.

Proof. Take $V = \mathbb{R}^{p+q+2}$, $A = \mathrm{Cl}_{pq} \otimes M_2(\mathbb{R})$. Define $f\colon V \to A$ on basic vectors by

$$\begin{aligned}
f(e_r) &:= e_r \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad r = 1, \dots, p,\\
f(\varepsilon_s) &:= \varepsilon_s \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad s = 1, \dots, q,\\
f(e_{p+1}) &:= 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\\
f(\varepsilon_{q+1}) &:= 1 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{1.3}$$

Thus $f(e_k)^2 = +1$, $f(\varepsilon_l)^2 = -1$ in all cases, and all $f(e_k)$, $f(\varepsilon_l)$ anticommute. This entails that f extends by linearity to a linear map satisfying $f(v)^2 = g(v,v)\,1$ for all $v \in V$. Hence there exists a homomorphism $\tilde{f}\colon \mathrm{Cl}_{p+1,q+1} \to A$, which is surjective since the right hand sides of (1.3) generate A as an $\mathbb{R}$-algebra. It is an isomorphism, because the dimensions over $\mathbb{R}$ are equal. ∎

Lemma 1.6. $\mathrm{Cl}^0_{p+1,q} \simeq \mathrm{Cl}_{qp}$.

Proof. Define $f\colon \mathbb{R}^{q+p} \to \mathrm{Cl}^0_{p+1,q}$ on basic vectors by

$$\begin{aligned}
f(e_r) &:= e_r e_{p+1}, \qquad r = 1, \dots, q,\\
f(\varepsilon_s) &:= \varepsilon_s e_{p+1}, \qquad s = 1, \dots, p.
\end{aligned}$$

Then

$$\begin{aligned}
f(e_r)^2 &= e_r e_{p+1} e_r e_{p+1} = -e_r^2 e_{p+1}^2 = -e_r^2 = -1,\\
f(\varepsilon_s)^2 &= \varepsilon_s e_{p+1} \varepsilon_s e_{p+1} = -\varepsilon_s^2 e_{p+1}^2 = -\varepsilon_s^2 = +1,
\end{aligned}$$

and all $f(e_r)$, $f(\varepsilon_s)$ anticommute. The rest of the proof is like that of the previous Lemma. ∎

Lemma 1.7. $\mathrm{Cl}_{p+4,q} \simeq \mathrm{Cl}_{pq} \otimes M_2(\mathbb{H}) \simeq \mathrm{Cl}_{p,q+4}$.

Proof. We will prove the first isomorphism. Take $A = \mathrm{Cl}_{pq} \otimes M_2(\mathbb{H})$; define $f\colon \mathbb{R}^{p+4+q} \to A$ by

$$\begin{aligned}
f(e_r) &:= e_r \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \qquad r = 1, \dots, p,\\
f(\varepsilon_s) &:= \varepsilon_s \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \qquad s = 1, \dots, q,
\end{aligned}$$

and on the remaining four basic vectors, define

$$f(e_{p+1}) := 1 \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad f(e_{p+2}) := 1 \otimes \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix},$$
$$f(e_{p+3}) := 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad f(e_{p+4}) := 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \square$$

Corollary 1.8 ("(+8)-periodicity"). $\mathrm{Cl}_{p+8,q} \simeq \mathrm{Cl}_{pq} \otimes M_{16}(\mathbb{R}) \simeq \mathrm{Cl}_{p,q+8}$.

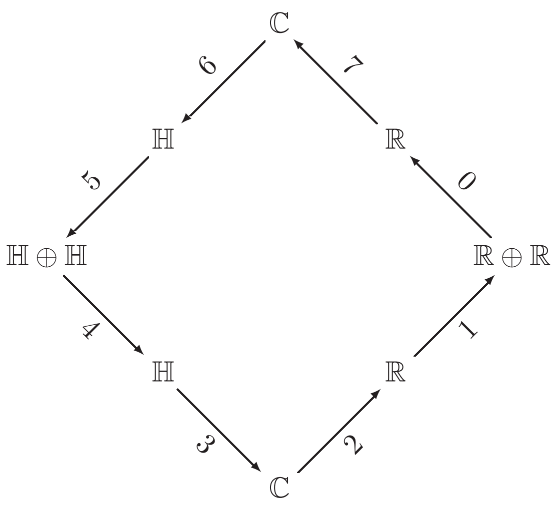
Proof. This reduces to $M_2(\mathbb{H}) \otimes_{\mathbb{R}} M_2(\mathbb{H}) \simeq M_{16}(\mathbb{R})$, that in turn reduces to $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq M_4(\mathbb{R})$, which is left as an exercise. $\square$

All Cl_{pq} are given, up to $M_N(\mathbb{R})$ tensor factors, by Cl_{p0} for $p = 1, \dots, 8$:

$$\begin{aligned}
\mathrm{Cl}_{10} &= \mathbb{R} \oplus \mathbb{R} \\
\mathrm{Cl}_{20} &= M_2(\mathbb{R}) \\
\mathrm{Cl}_{30} &= M_2(\mathbb{C}) \\
\mathrm{Cl}_{40} &= M_2(\mathbb{H}) \\
\mathrm{Cl}_{50} &= M_2(\mathbb{H}) \oplus M_2(\mathbb{H}) \\
\mathrm{Cl}_{60} &= M_4(\mathbb{H}) \\
\mathrm{Cl}_{70} &= M_8(\mathbb{C}) \\
\mathrm{Cl}_{80} &= M_{16}(\mathbb{R})
\end{aligned} \tag{1.4}$$

Two algebras Cl_{10} and Cl_{50} are direct sums of simple algebras, and the others are simple. We could also define $\mathrm{Cl}_{00} = \mathbb{R}$ (the base field), so that Corollary 1.8 holds even when $p = q = 0$.

Those eight algebras Cl_{p0} can be arranged on a "spinorial clock", which is taken from Budinich and Trautman's book [BT].



If $p - q \equiv m \bmod 8$, then Cl_{pq} is of the form $A \otimes M_N(\mathbb{R})$, where A is the diagram entry at the head of the arrow labelled m. Moreover, Lemma 1.6 says that the even subalgebra Cl^0_{pq} is of the same kind, where A is now the diagram entry at the tail of the arrow labelled m. The matrix size N is easily determined from the real dimension, in each case. In this way, the spinorial clock displays the full classification of real Clifford algebras.

1.5 Chirality

From now on, $n = 2m$ for n even, $n = 2m+1$ for n odd. We take $\mathbb{C}\mathrm{l}(V) \simeq \mathrm{Cl}(V,g) \otimes_{\mathbb{R}} \mathbb{C}$ with g always positive definite.

Suppose $\{e_1, \dots, e_n\}$ is an *oriented* orthonormal basis for (V,g). If $e'_k = \sum_{j=1}^n h_{jk}\, e_j$ with $H^t H = 1_n$, then $e'_1 \dots e'_n = (\det H)\, e_1 \dots e_n$, and $\det H = \pm 1$. We restrict to the oriented case $\det H = +1$, so the expression $e_1 e_2 \dots e_n$ is independent of $\{e_1, e_2, \dots, e_n\}$. Thus

$$\gamma := (-i)^m e_1 e_2 \dots e_n$$

is well-defined in $\mathbb{C}\mathrm{l}(V)$. Now

$$\gamma^* = i^m e_n \dots e_2 e_1 = (-i)^m (-1)^m (-1)^{n(n-1)/2} e_1 e_2 \dots e_n = (-1)^m (-1)^{n(n-1)/2} \gamma,$$

and

$$\frac{n(n-1)}{2} = \left\{ \begin{matrix} m(2m-1), & n \text{ even} \\ (2m+1)m, & n \text{ odd} \end{matrix} \right\} \equiv m \bmod 2,$$

so $\gamma^* = \gamma$. But also $\gamma^*\gamma = (e_n \dots e_2 e_1)(e_1 e_2 \dots e_n) = (+1)^n = 1$, so γ is "unitary". Hence $\gamma^2 = 1$, so $\frac{1+\gamma}{2}$, $\frac{1-\gamma}{2}$ are "orthogonal projectors" in $\mathbb{C}\mathrm{l}(V)$.

Since $\gamma e_j = (-1)^{n-1} e_j \gamma$, we get that if n is odd, then γ is *central* in $\mathbb{C}\mathrm{l}(V)$; and for n even, γ anticommutes with V, but is central in the even subalgebra $\mathbb{C}\mathrm{l}^0(V)$. Moreover, when n is even and $v \in V$, then $\gamma v \gamma = -v$, so that $\gamma(\cdot)\gamma = \chi \in \mathrm{Aut}(\mathbb{C}\mathrm{l}(V))$.

Proposition 1.9. *The centre of* $\mathbb{C}\mathrm{l}(V)$ *is* $\mathbb{C}1$ *if* n *is even; and it is* $\mathbb{C}1 \oplus \mathbb{C}\gamma$ *if* n *is odd.*

Proof. Denote this centre by $Z(\mathbb{C}\mathrm{l}(V))$.

Even case: $a \in Z(\mathbb{C}\mathrm{l}(V))$ implies $\gamma a \gamma = a\gamma^2 = a$, so a lies in $\mathbb{C}\mathrm{l}^0(V)$.

If $a = \sum_{K \text{ even}} a_K\, e_{k_1} \dots e_{k_r}$, then $0 = a - e_j a e_j = \sum_{K \text{ even},\, j \in K} 2a_K e_{k_1} \dots e_{k_r}$, so $a_K = 0$ if $j \in K$. Since this holds for any j, we conclude that $a = a_\emptyset 1 = \tau(a)\, 1$. Therefore $Z(\mathbb{C}\mathrm{l}(V)) \simeq \mathbb{C}1$ when n is even.

Odd case: If $a = a_0 + a_1$ (even + odd) lies in $Z(\mathbb{C}\mathrm{l}(V))$, then

$$0 = [a,v] = \underbrace{[a_1,v]}_{\text{even}} + \underbrace{[a_0,v]}_{\text{odd}} \quad \text{for all} \quad v \in V,$$

so $[a_0,v] = [a_1,v] = 0$ for all $v \in V$. In particular, $a_0 \in Z(\mathbb{C}\mathrm{l}^0(V)) \simeq \mathbb{C}1$, and thus $a_0 = \tau(a)\,1$.

Also, $a_1\gamma$ is even and central, so $a_1\gamma = \tau(a\gamma)\,1$ and $a_1 = \tau(a\gamma)\,\gamma$. Thus $Z(\mathbb{C}\mathrm{l}(V)) = \mathbb{C}1 \oplus \mathbb{C}\gamma$ when n is odd. □

1.6 Spin^c and Spin groups

Let v be a unit vector, $g(v,v) = 1$. Then $v^2 = 1$ in $\mathrm{Cl}(V,g)$, so $v = v^*$ and $v^*v = vv^* = 1$ in $\mathbb{C}\mathrm{l}(V)$. If $w = \lambda v \in V^{\mathbb{C}}$ with $|\lambda| = 1$, then $ww^* = w^*w = 1$ in $\mathbb{C}\mathrm{l}(V)$ also. Now

$$\begin{aligned} \langle wa \mid wb \rangle &= \tau(a^* w^* w b) = \tau(a^* b) = \langle a \mid b \rangle, \\ \langle aw \mid bw \rangle &= \tau(w^* a^* b w) = \tau(w w^* a^* b) = \tau(a^* b) = \langle a \mid b \rangle, \end{aligned}$$

so $a \mapsto wa$, $a \mapsto aw$ are unitary operators in $\mathcal{L}(\mathbb{C}\mathrm{l}(V))$.

⟦Exercise: Conversely, if $u \in \mathbb{C}\mathrm{l}(V)$ and $a \mapsto ua$ and $a \mapsto au$ are both unitary, then $u^*u = uu^* = 1$.⟧

If $v, x \in V$, with $g(v,v) = 1$, then

$$-vxv^{-1} = -vxv = (xv - 2g(v,x))v = x - 2g(v,x)v \ \underline{\in V}.$$

This is a reflection of x in the hyperplane orthogonal to v. For $w = \lambda v$, $|\lambda| = 1$ we also get $-wxw^{-1} = -\lambda\bar{\lambda}vxv^{-1} = -vxv^{-1}$, which is the same as above. If $a = w_1 \dots w_r$ is a product of unit vectors in $V^{\mathbb{C}}$, then

$$\chi(a)xa^{-1} = (-1)^r w_1 \dots w_r x w_r^{-1} \dots w_1^{-1}$$

is a product of r reflections of $x \in V$. If $r = 2k$ is *even*, and $a = w_1 \dots w_{2k}$, then $axa^{-1} \in V$ after k rotations. Thus $\phi(a) \colon x \mapsto axa^{-1}$ lies in $\mathrm{SO}(V) = \mathrm{SO}(V,g)$.

Definition 1.10. *The set of all even products of unitary vectors,*

$$\mathrm{Spin}^{\mathrm{c}}(V) := \{\, u = w_1 \dots w_{2k} : w_j \in V^{\mathbb{C}},\ w_j^* w_j = 1,\ k = 0, 1, \dots, m \,\},$$

is a group *included in* $\mathrm{Cl}^0(V)$*, and* $\phi \colon \mathrm{Spin}^{\mathrm{c}}(V) \to \mathrm{SO}(V)$ *is a group homomorphism.*

The inverse of $u = w_1 \dots w_{2k}$ is $u^{-1} = u^* = \bar{w}_{2k} \dots \bar{w}_1$.

Suppose $u \in \ker\phi$, which means that $uxu^{-1} = x$ for all $x \in V$. Thus $\ker\phi \subset Z(\mathbb{C}\mathrm{l}(V))$ for n even, and $\ker\phi \subset Z(\mathrm{Cl}^0(V))$ for n odd; in both cases, u lies in $\mathbb{C}1$. It follows that $\ker\phi \simeq \{\, \lambda \in \mathbb{C} : |\lambda| = 1 \,\} = \mathbb{T} = \mathrm{U}(1)$. Therefore, there is a *short exact sequence* (SES) of groups:

$$1 \to \mathbb{T} \to \mathrm{Spin}^{\mathrm{c}}(V) \xrightarrow{\phi} \mathrm{SO}(V) \to 1. \tag{1.5}$$

If $u = w_1 \dots w_{2k} \in \mathrm{Spin}^{\mathrm{c}}(V)$ with $w_j = \lambda_j v_j$ where $\lambda_j \in \mathbb{T}$ and $v_j \in V$, then $u^! = w_{2k} \dots w_1$, and $u^! u = \lambda_1^2 \lambda_2^2 \dots \lambda_n^2 \in \mathbb{T}$. Thus, $u^! u$ is central, so $(u_1u_2)^! u_1 u_2 = u_2^! u_1^! u_1 u_2 = u_1^! u_1 u_2^! u_2$, so that $u \mapsto u^! u$ is a *homomorphism* $\nu \colon \mathrm{Spin}^{\mathrm{c}}(V) \to \mathbb{T}$, which restricts to $\mathbb{T} \subset \mathrm{Spin}^{\mathrm{c}}(V)$ as $\lambda \mapsto \lambda^2$. The combined $(\phi, \nu) \colon \mathrm{Spin}^{\mathrm{c}}(V) \to \mathrm{SO}(V) \times \mathbb{T}$ is a homomorphism with kernel $\{\pm 1\}$.

Definition 1.11. $\mathrm{Spin}(V) := \ker\nu \leq \mathrm{Spin}^{\mathrm{c}}(V)$.

Indeed $\mathrm{Spin}(V)$ is included in (the even part of) the real Clifford algebra $\mathrm{Cl}^0(V,g)$:

$$u^*u = 1,\ u^! u = 1 \implies u^* = u^! \implies \bar{u} = u \implies u \in \mathrm{Cl}^0(V,g).$$

The SES (1.5) now becomes

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \xrightarrow{\phi} \mathrm{SO}(V) \to 1, \tag{1.6}$$

so that ϕ is a double covering of $\mathrm{SO}(V)$. Furthermore, $\mathrm{Spin}^{\mathrm{c}}(V) \simeq \mathrm{Spin}(V) \times_{\mathbb{Z}_2} \mathbb{T}$.

Example 1.12. Case $n = 2$: We write $\mathrm{Spin}(n) \equiv \mathrm{Spin}(\mathbb{R}^n)$. It is easy to check that

$$\begin{aligned}
\mathrm{Spin}(V) &= \{\, \alpha + \beta e_1 e_2 : \alpha, \beta \in \mathbb{R},\ \alpha^2 + \beta^2 = 1 \,\} \\
&= \{\, u = \cos\frac{\psi}{2} + \sin\frac{\psi}{2} e_1 e_2 : -2\pi < \psi \leq 2\pi \,\} \simeq \mathbb{T}.
\end{aligned}$$

We compute

$$\begin{aligned}
ue_1u^{-1} &= \Big(\cos\frac{\psi}{2} + \sin\frac{\psi}{2} e_1e_2\Big) e_1 \Big(\cos\frac{\psi}{2} - \sin\frac{\psi}{2} e_1e_2\Big) = (\cos\psi)\, e_1 - (\sin\psi)\, e_2, \\
ue_2u^{-1} &= \Big(\cos\frac{\psi}{2} + \sin\frac{\psi}{2} e_1e_2\Big) e_2 \Big(\cos\frac{\psi}{2} - \sin\frac{\psi}{2} e_1e_2\Big) = (\sin\psi)\, e_1 + (\cos\psi)\, e_2,
\end{aligned}$$

so that

$$\phi(u) = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \in \mathrm{SO}(2),$$

which is (nontrivial) double covering of the circle.

Example 1.13. $\mathrm{Spin}(3) \simeq \mathrm{SU}(2) = \{\text{unit quaternions}\}$ in $\mathrm{Cl}^0_{30} \simeq \mathrm{Cl}_{02} \simeq \mathbb{H}$, and $\phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is the adjoint representation of $\mathrm{SU}(2)$.

Example 1.14. $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ in $\mathrm{Cl}^0_{40} \simeq \mathrm{Cl}_{03} \simeq \mathbb{H} \oplus \mathbb{H}$. If $u = (q, p)$ with $q, p \in \mathrm{SU}(2)$, then $\phi(u)$ becomes $x \mapsto qxp^{-1}$ for $x \in \mathbb{H} \simeq \mathbb{R}^4$, and this map lies in $\mathrm{SO}(4)$. If $\phi(u) = 1_{\mathbb{H}}$, then $1 \mapsto qp^{-1}$, so $p = q$, and $x \mapsto qxq^{-1} = x$, so q is central; hence $p = q = \pm 1$ and ϕ is indeed a double covering of $\mathrm{SO}(4)$.

1.7 The Lie algebra of $\mathrm{Spin}(V)$

Recall the linear isomorphism $Q\colon \Lambda^\bullet V \to \mathrm{Cl}(V, g)$, inverse to $\sigma\colon a \mapsto c(a)1$. Write

$$b = Q(u \wedge v) = \tfrac{1}{2}(uv - vu) = uv + g(u, v)\,1 \in \mathrm{Cl}^0(V, g).$$

Note in passing that $b^! = \frac{1}{2}(vu - uv) = -b$.

Although the algebra $\mathrm{Cl}(V, g)$ is not $\mathbb{Z}$-graded, it is *$\mathbb{Z}$-filtered*: we may write $\mathrm{Cl}^{\leq k}(V, g)$ to denote the vector subspace generated by products of *at most* k vectors from V. With that notation, the subspace $Q(\Lambda^2 V)$ may also be described as the set of all even elements $b \in \mathrm{Cl}^{\leq 2}(V, g)$ with $\tau(b) = 0$.

For $x \in V$, we compute

$$[b, x] = [uv, x] = uvx + uxv - uxv - xuv = 2g(v, x)u - 2g(u, x)v \ \underline{\in V},$$

so $\operatorname{ad} b\colon V \to V$. Also

$$[b, b'] = \tfrac{1}{2}[b, u'v' - v'u'] = \tfrac{1}{2}[b, u']v' + \tfrac{1}{2}u'[b, v'] - \tfrac{1}{2}[b, v']u' - \tfrac{1}{2}v'[b, u']$$

so that $[b, b'] \in \mathrm{Cl}^{\leq 2}(V, g)$ with $\tau([b, b']) = 0$. Hence $[b, b'] \in Q(\Lambda^2 V)$, and this is a *Lie algebra.* Next,

$$g(y, [b, x]) = 2g(v, x)g(y, u) - 2g(u, x)g(y, v) = -g([b, y], x),$$

so that $\operatorname{ad} b$ is *skewsymmetric*: thus $\operatorname{ad} b \in \mathfrak{so}(V)$. By the Jacobi identity,

$$[[b, b'], x] = [b, [b', x]] - [b', [b, x]] \quad \text{for all} \quad x \in V,$$

and so $\operatorname{ad}([b, b']) = [\operatorname{ad} b, \operatorname{ad} b']$. Thus, $\operatorname{ad}\colon Q(\Lambda^2 V) \to \mathfrak{so}(V)$ is a Lie algebra homomorphism.

If $\operatorname{ad} b = 0$, so that $[b, x] = 0$ for all $x \in V$, then $b \in Z(\mathrm{Cl}^0(V)) \simeq \mathbb{C}1$. But $\tau(b) = 0$ then implies $b = 0$, so ad is injective. Since $\dim \Lambda^2 V = n(n-1)/2 = \dim \mathfrak{so}(V)$, we see that $\operatorname{ad}\colon Q(\Lambda^2 V) \to \mathfrak{so}(V)$ is a *Lie algebra isomorphism.*

There is an important formula for the inverse of ad. For $A \in \mathfrak{so}(V)$, define

$$\dot{\mu}(A) = \frac{1}{4} \sum_{j,k=1}^{n} g(e_j, Ae_k)\, e_j e_k = \frac{1}{2} \sum_{j<k} g(e_j, Ae_k)\, e_j e_k. \tag{1.7}$$

Since $\tau(\dot\mu(A)) = 0$, we get $\dot\mu(A) \in Q(\Lambda^2 V)$. Also

$$
\begin{aligned}
[\dot\mu(A), e_r] &= \frac14 \sum_{j,k} g(e_j, Ae_k)\bigl(e_j \underbrace{\{e_k, e_r\}}_{\delta_{kr}} - \underbrace{\{e_j, e_r\}}_{\delta_{jr}} e_k\bigr) \\
&= \frac12 \sum_j g(e_j, Ae_r)\, e_j - \frac12 \sum_k g(e_r, Ae_k)\, e_k = \sum_j g(e_j, Ae_r)\, e_j \\
&= Ae_r,
\end{aligned}
$$

where we have used the anticommutator notation $\{X, Y\} := XY + YX$. Hence $\operatorname{ad}(\dot\mu(A)) = A \in \mathfrak{so}(V)$.

Now consider $u = \exp b := 1 + \sum_{k\geq 1} \frac{1}{k!} b^k \in \mathrm{Cl}^0(V, g)$ for $b \in Q(\Lambda^2 V)$. Then $u^* u = u^! u = \exp(-b)\exp b = 1$ since $b^! = -b$. Also, u is unitary and even, and if $x \in V$ then

$$
\begin{aligned}
uxu^{-1} &= \sum_{k,l\geq 0} \frac{1}{k!\,l!}\, b^k x(-b)^l \\
&= \sum_{r\geq 0} \frac{1}{r!} \sum_{k=0}^{r} \binom{r}{k} b^k x(-b)^{r-k} \\
&= \sum_{r\geq 0} \frac{1}{r!} (\operatorname{ad} b)^r(x) \ \underline{\in V},
\end{aligned}
$$

and thus $u = \exp(b)$ lies in $\mathrm{Spin}(V)$. When $b = \dot\mu(A)$, we get $\phi(\exp(b)) = \exp(\operatorname{ad} b) = \exp(A)$, and it is known that $\exp\colon \mathfrak{so}(V) \to \mathrm{SO}(V)$ is surjective (a property of compact connected matrix groups).

Now $\exp(Q(\Lambda^2 V))$ is a subset of $\mathrm{Spin}(V)$ covering all of $\mathrm{SO}(V)$. If we can show that $-1 = \exp c$ for some c, then $-\exp b = (\exp b)(\exp c) = \exp(b + c)$, provided that c, b *commute*. If $b = \dot\mu(A)$, we can express the skewsymmetric matrix A as a direct sum of 2×2 skewsymmetric blocks in a suitable orthonormal basis:

$$
A = \begin{pmatrix}
0 & * & & & & & & \\
* & 0 & & & & & & \\
& & 0 & * & & & & \\
& & * & 0 & & & & \\
& & & & \ddots & & & \\
& & & & & 0 & * & \\
& & & & & * & 0 & \\
& & & & & & & \ddots
\end{pmatrix}.
$$

That is, we can choose the (oriented) orthonormal basis $\{e_1, \dots, e_n\}$ so that

$$
b = \tfrac12 g(e_1, Ae_2)\, e_1e_2 + \tfrac12 g(e_3, Ae_4)\, e_3e_4 + \cdots + \tfrac12 g(e_{2r-1}, Ae_{2r})\, e_{2r-1}e_{2r}
$$

with $r \leq m$. Now this particular e_1e_2 commutes with b: $(e_1e_2)b = b(e_1e_2)$; take $c := \pi e_1e_2$. Then $\exp c = \exp(\pi e_1e_2) = \cos\pi + \sin\pi\, e_1e_2 = -1$. We have shown that $\exp\colon Q(\Lambda^2 V) \to \mathrm{Spin}(V)$ is surjective.

Note that $t \mapsto \exp(te_1e_2)$, for $0 \leq t \leq \pi$, is a path in $\mathrm{Spin}(V)$ from $+1$ to -1. Since $\pi_1(\mathrm{SO}(V)) \simeq \mathbb{Z}_2$ for $n \geq 3$, the double covering $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ is nontrivial. We get an important consequence.

Corollary 1.15. $\mathrm{Spin}(n)$ *is simply connected, for* $n \geq 3$. $\square$

1.8 Orthogonal complex structures

Suppose that $n = 2m$ is even, $V \simeq \mathbb{R}^{2m}$. Then V can be identified with $\mathbb{C}^m$, *but not canonically.*

Definition 1.16. *An operator $J \in \operatorname{End}_{\mathbb{R}} V$ is called an* ***orthogonal complex structure****, written $J \in \mathcal{J}(V, g)$, if*

(a) $J^2 = -1$ *in* $\operatorname{End}_{\mathbb{R}} V$*;*

(b) $g(Ju, Jv) = g(u, v)$ *for all* $u, v \in V$*.*

Then also $g(Ju, v) = -g(Ju, J^2 v) = -g(u, Jv)$, so that J is skewsymmetric with respect to g: $J^t = -J$. Note that (b) says that $J^t J = 1$.

We can now make V a $\mathbb{C}$-module by setting $iv := Jv$, that is,

$$(\alpha + i\beta)v := \alpha v + \beta J v \quad \text{for all} \quad \alpha, \beta \in \mathbb{R}.$$

We define a hermitian scalar product on V by

$$\langle u \mid v \rangle_J := g(u, v) + i\, g(Ju, v)$$

Note that $\langle Ju \,|\, v\rangle_J = -i\langle u \,|\, v\rangle_J$ and $\langle u \,|\, Jv\rangle_J = +i\langle u \,|\, v\rangle_J$ (check it!). We denote the resulting m-dimensional complex Hilbert space by V_J.

If $\{u_1, \dots, u_m\}$ is an orthonormal basis for $V_J := (V, \langle\cdot|\cdot\rangle_J)$, then $\{u_1, Ju_1, \dots, u_m, Ju_m\}$ is an orthonormal oriented basis for V (over $\mathbb{R}$). The orientation may or may not be compatible with the given one on V.

Exercise 1.17. *If $2m = 4$, show that all such J can be parametrized by two disjoint copies of $\mathbb{S}^2$, one for each orientation.*

If $J^2 = -1$, $J^t J = 1$ and if $h \in \mathrm{O}(n) = \mathrm{O}(V, g)$ is an orthogonal linear transformation, then $K := hJh^{-1}$ is also an orthogonal complex structure. In that case,

$$\begin{aligned}\langle hu \mid hv \rangle_K &= g(hu, hv) + i\, g((hJh^{-1})hu, hv) \\ &= g(u, v) + i\, g(Ju, v) = \langle u \mid v \rangle_J,\end{aligned}$$

so that $h \colon V_J \to V_K$ is unitary. Thus $hJh^{-1} = J$ if and only if $h \in \mathrm{U}(V_J) \simeq \mathrm{U}(m)$. In short: $\mathrm{O}(n) = \mathrm{O}(2m)$ acts transitively on $\mathcal{J}(V, g)$ with isotropy subgroups isomorphic to $\mathrm{U}(m)$. Hence, as a manifold,

$$\mathcal{J}(V, g) \approx \mathrm{O}(2m)/\,\mathrm{U}(m).$$

Those J which are compatible with orientation on V form one component (of two), homeomorphic to $\mathrm{SO}(2m)/\,\mathrm{U}(m)$.

We may complexify V to get $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$. Take

$$W_J := \{\, v - iJv \in V^{\mathbb{C}} : v \in V \,\} = \tfrac{1}{2}(1 - iJ)V = P_J V.$$

This is an isotropic subspace for the symmetric billinear form $g^{\mathbb{C}}$ on $V^{\mathbb{C}}$.

$$g(u - iJu, v - iJv) = g(u, v) - ig(u, Jv) - ig(Ju, v) - g(Ju, Jv) = 0.$$

The conjugate subspace

$$\overline{W}_J = \{\, v + iJv : v \in V \,\} = \tfrac{1}{2}(1 + iJ)V = P_J V$$

satisfies $W_J \oplus \overline{W}_J \simeq V^{\mathbb{C}}$, an orthogonal direct sum for the hermitian scalar product

$$\langle\langle w \mid z \rangle\rangle := 2\, g(\bar{w}, z) \quad \text{for } \ w, z \in V^{\mathbb{C}}.$$

Note that $P_J^2 = P_J$ and $P_J = P_J^*$ with respect to this product. We say that W_J is a **polarization** of $V^{\mathbb{C}}$. Also $P_J \colon V_J \to W_J$ is an unitary isomorphism.

Conversely: given a splitting $V = W \oplus \overline{W}$, orthogonal with respect to $\langle\langle \cdot \mid \cdot \rangle\rangle$, write $w =: u - iv$ for $w \in W$, with $u, v \in V$; then $J_W \colon u \mapsto v$ lies in $\mathcal{J}(V, g)$, and $W_{J_W} = W$ (exercise). Thus the correspondence $J \leftrightarrow W_J$ is bijective.

1.9 Irreducible representations of $\mathbb{C}\mathrm{l}(V)$

We continue to suppose that $n = 2m$ is even.

Definition 1.18. *The (fermionic)* ***Fock space*** *corresponding to $J \in \mathcal{J}(V, g)$ is defined as*

$$\mathcal{F}_J(V) := \Lambda^{\bullet} W_J,$$

with hermitian scalar product given by

$$\langle\langle w_1 \wedge \cdots \wedge w_k \mid z_1 \wedge \cdots \wedge z_l \rangle\rangle := \delta_{kl} \det\bigl[\langle\langle w_i \mid z_j \rangle\rangle\bigr]. \tag{1.8}$$

This is a complex Hilbert space of dimension 2^m. Choose and fix a unit vector $\Omega \in \Lambda^0 W_J$: it is unique up to a factor $\lambda \in \mathbb{T}$. For $w \in W_J$ (so that $\bar{w} \in \overline{W}_J = W_J^{\perp}$), we write

$$\begin{aligned}
\varepsilon(w) &: z_1 \wedge \cdots \wedge z_k \mapsto w \wedge z_1 \wedge \cdots \wedge z_k, \\
\iota(\bar{w}) &: z_1 \wedge \cdots \wedge z_k \mapsto \sum_{j=1}^{k} (-1)^{k-1} \langle\langle w \mid z_j \rangle\rangle\, z_1 \wedge \cdots \wedge \widehat{z_j} \wedge \cdots \wedge z_k.
\end{aligned}$$

For $v \in V$, write $w = \frac{1}{2}(v - iJv) = P_J v \in W_J$ and define

$$c_J(v) := \varepsilon(w) + \iota(\bar{w}) = \varepsilon(P_J v) + \iota(P_{-J} v).$$

Then

$$c_J^2(v) := \langle\langle w \mid w \rangle\rangle\, 1 = \langle v \mid v \rangle_J\, 1 = g(v, v)\, 1,$$

so that $c_J \colon V \to \mathrm{End}_{\mathbb{C}}(\mathcal{F}_J V) \equiv \mathcal{L}(\mathcal{F}_J V)$. That is to say, c_J is a *representation* of $\mathbb{C}\mathrm{l}(V)$ on the Hilbert space $\mathcal{F}_J V$.

Note that we complexify the representation of $\mathrm{Cl}(V, g)$, given by universality. One can check that

$$c_J(w) = \varepsilon(w) \quad \text{if } \ w \in W_j; \quad c_J(\bar{z}) = \iota(\bar{z}) \quad \text{if } \ \bar{z} \in \overline{W}_J.$$

From (1.8) and the properties of determinants, it is easy to check that the operators $\varepsilon(w)$ and $\iota(\bar{w})$ are adjoint to one another, that is, $\varepsilon(w)^\dagger = \iota(\bar{w})$ for $w \in W_J$; in particular, $c_J(v)^\dagger = c_J(v)$ for $v \in V$. (This is a consequence of our choice of g to have positive definite signature: were we to have taken g to be negative definite, as in done in many books, then the operators $c_J(v)$ would have been skewadjoint.) More generally, we get $c_J(a)^\dagger = c_J(a^*)$ for $a \in \mathbb{C}\mathrm{l}(V)$: we say that c_J is a *selfadjoint representation* of the $*$-algebra $\mathbb{C}\mathrm{l}(V)$ on the Fock space $\mathcal{F}_J(V)$.

Now, if $T \in \mathcal{L}(\mathcal{F}_J(V))$ *commutes* with $c_J(V^{\mathbb{C}})$, then in particular $\iota(\bar{z}) T\Omega = T\iota(\bar{z})\Omega = T(0) = 0$ for $\bar{z} \in \overline{W}_J$. Therefore $T\Omega \in \Lambda^0 W_J$, i.e., $T\Omega = t\Omega$ for some $t \in \mathbb{C}$. Now

$$T(w_1 \wedge \cdots \wedge w_k) = T\varepsilon(w_1) \ldots \varepsilon(w_k)\Omega = \varepsilon(w_1) \ldots \varepsilon(w_k) T\Omega = t\, w_1 \wedge \cdots \wedge w_k$$

for $w_1, \dots, w_k \in W_J$. Thus $T = t\,1 \in \mathcal{L}(\Lambda^\bullet W_J)$. By Schur's lemma, the representation c_J is *irreducible*.

Suppose $K \in \mathcal{J}(V, g)$ with $K = hJh^{-1}$ for $h \in \mathrm{O}(2m)$. Then $hJ = Kh$, $hP_{\pm J} = P_{\pm K}h$, and so $c_K(hv) = (\Lambda^\bullet h)\, c_J(v)$. By universality again, we get $c_K \circ \Lambda^\bullet h = \Lambda^\bullet h \circ c_J$, so that the irreducible representations c_K and c_J are *equivalent*.

The Fock space is $\mathbb{Z}_2$-*graded* as $\Lambda^{\mathrm{even}} W_J \oplus \Lambda^{\mathrm{odd}} W_J$. ¿What operator determines its $\mathbb{Z}_2$-grading? In fact, this operator is $c_J(\gamma)$. To see that, write $\gamma = (-1)^m e_1 e_2 \dots e_{2m}$, where $e_{2j} = Je_{2j-1}$ for $j = 1, \dots, m$. If $z_1 := P_J e_1 = \frac{1}{2}(e_1 - ie_2)$ we get

$$\bar{z}_1 z_1 - z_1 \bar{z}_1 = \tfrac{1}{4}(e_1 + ie_2)(e_1 - ie_2) - \tfrac{1}{4}(e_1 - ie_2)(e_1 + ie_2) = -e_1 e_2.$$

With $z_j := P_J e_{2j-1} = \frac{1}{2}(e_{2j-1} - ie_{2j})$, this gives

$$\gamma = (\bar{z}_1 z_1 - z_1 \bar{z}_1) \dots (\bar{z}_m z_m - z_m \bar{z}_m) \quad \text{in} \quad \mathrm{Cl}^0(V).$$

Now $c_J(\bar{z}_j z_j - z_j \bar{z}_j) = \iota(\bar{z}_j)\varepsilon(z_j) - \varepsilon(z_j)\iota(\bar{z}_j)$ is the operator

$$z_{k_1} \wedge \dots \wedge z_{k_r} \longmapsto \begin{cases} -z_{k_1} \wedge \dots \wedge z_{k_r} & \text{if } j \in K \\ +z_{k_1} \wedge \dots \wedge z_{k_r} & \text{if } j \notin K. \end{cases}$$

Thus $c_J(\gamma)$ acts as $(-1)^k$ on $\Lambda^k W_J$: this is indeed the $\mathbb{Z}_2$-grading operator.

Finally, the *odd case* is treated as follows. Let $U := \mathbb{R}\text{-}\mathrm{span}\{e_1, \dots, e_{2m}\} \leq V$. Then $\mathrm{Cl}(U) \simeq \mathrm{Cl}^0(V)$ via $u \mapsto i\,ue_{em+1}$, extended to $\mathrm{Cl}(U)$. Now $\mathcal{F}_J(U)$ is an irreducible $\mathrm{Cl}^0(V)$-module, while $Z(\mathrm{Cl}(V)) = \mathbb{C}1 \oplus \mathbb{C}\gamma$. Since $\gamma^2 = 1$, we can extend the action of $\mathrm{Cl}^0(V)$ on $\mathcal{F}_J(U)$ to the full $\mathrm{Cl}(V)$ by setting either $c_J(\gamma) := +1$ or $c'_J(\gamma) := -1$ on $\mathcal{F}_J(U)$.

These representations c_J, c'_J are *inequivalent*, since $T(1) = (-1)T$ is not possible unless $T = 0$, using Schur's lemma again. Thus $\mathrm{Cl}(V)$ has two irreducible Fock representations of dimension 2^m in the odd case.

Proposition 1.19. *The Fock representations yield all irreducible representations of* $\mathrm{Cl}(V)$. *If* $\dim_{\mathbb{R}} V = 2m$, *the irreducible representation is unique up to equivalence; if* $\dim_{\mathbb{R}} V = 2m+1$, *there are exactly two such representations.*

Proof. We have already described and classified the Fock representations. It remains to show that this list is complete.

We have seen that up to tensoring with a matrix algebra $M_N(\mathbb{R})$, the real Clifford algebras occur in eight species. The periodicity of *complex* Clifford algebras is much simpler, and may be obtained from (1.4) by complexifying each algebra found there. Since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$, we obtain directly that

$$\mathrm{Cl}(\mathbb{R}^{2m}) \simeq M_{2^m}(\mathbb{C}) \quad \text{and} \quad \mathrm{Cl}(\mathbb{R}^{2m+1}) \simeq M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}). \tag{1.9}$$

From this it is clear that, when $\dim V$ is even, $\mathrm{Cl}(V)$ is a simple matrix algebra and therefore all irreducible representations are equivalent and arise from matrix multiplication on a minimal left ideal, whose dimension is 2^m. Similar arguments in the odd case show that there are at most two inequivalent representations of $\mathrm{Cl}(V)$. Thus the Fock representations we have constructed account for all of them: there are no others. $\square$

1.10 Representations of $\mathrm{Spin}^c(V)$

We obtain representations of the *group* $\mathrm{Spin}^c(V)$ by restriction of the irreducible representations of $\mathbb{C}l(V)$.

$$\mathrm{Spin}^c(V) = \{\, w_1 w_2 \dots w_{2k} : w_i \in V^{\mathbb{C}},\ w_i^* w_i = 1 \,\}.$$

We have to check whether these restrictions are irreducible or not.

Even case, $n = 2m$: γ belongs to $\mathrm{Spin}^c(V)$ and is central there, so $c_J(\gamma)$ commutes with $c_J(\mathrm{Spin}^c(V))$. Thus the group representation reduces over $\Lambda^\bullet W_J = \Lambda^{\mathrm{even}} W_J \oplus \Lambda^{\mathrm{odd}} W_J$: there are two subrepresentations. Since

$$w_1 \wedge \cdots \wedge w_{2k} = \varepsilon(w_1) \dots \varepsilon(w_{2k}) = c_J(w_1) \dots c_J(w_{2k})\, \Omega,$$

we get at once that $c_J(\mathrm{Spin}^c(V))\, \Omega = \Lambda^{\mathrm{even}} W_J$: the "even" subrepresentation is irreducible.

If $w_1, w_2 \in W_J$ are unit vectors, then

$$c_J(w_2 \bar{w}_1)\, w_1 = \varepsilon(w_2)\iota(\bar{w}_1)\, w_1 = \varepsilon(w_2)\, \Omega = w_2.$$

From there we soon conclude that $c_J(\mathrm{Spin}^c(V))\, w_1 = \Lambda^{\mathrm{odd}} W_J$: the "odd" subrepresentation is also irreducible.

¿Are these subrepresentations equivalent? *No*: for suppose $R\colon \Lambda^{\mathrm{even}} W_J \to \Lambda^{\mathrm{odd}} W_J$ intertwines both subrepresentations. Then in particular $Rc_J(\gamma) = c_J(\gamma)R$ means that $R(+1) = (-1)R : \Lambda^{\mathrm{even}} W_J \to \Lambda^{\mathrm{odd}} W_J$, so that $R = 0$.

Conclusion: The algebra representation c_J of $\mathbb{C}l(V)$ restricts to a group representation c_J of $\mathrm{Spin}^c(V)$ which is the direct sum of two inequivalent irreducible subrepresentations, if $\dim V$ is even.

Odd case, $n = 2m + 1$: There are two irreducible representations c_J and c'_J of $\mathbb{C}l(V)$ on $\mathcal{F}_J(U)$, but they *coincide* on $\mathbb{C}l^0(V)$: in this case, γ is odd. Declaring $c_J(\gamma)$ to be, say, $+1$ on $\mathcal{F}(U)$, we get for $w_1, \dots, w_{2k+1} \in W_J$:

$$\begin{aligned} w_1 \wedge \cdots \wedge w_{2k} &= c_J(w_1 \dots w_{2k})\, \Omega, \\ w_1 \wedge \cdots \wedge w_{2k+1} &= c_J(w_1 \dots w_{2k+1}\gamma)\, \Omega, \end{aligned}$$

so that in this case, $\Lambda^\bullet W_J$ is an irreducible representation if $\dim V$ is odd.

Conclusion: The two algebra representations c_J and c'_J of $\mathbb{C}l(V)$ restrict to the same group representation c_J of $\mathrm{Spin}^c(V)$ which is already irreducible, if $\dim V$ is odd.

Chapter 2

Spinor modules over compact Riemannian manifolds

2.1 Remarks on Riemannian geometry

Let M be a *compact* C^∞ manifold *without boundary*, of dimension n. Compactness is not crucial for some of our arguments (although it may be for others), but is very convenient, since it means that the algebras $C(M)$ and $C^\infty(M)$ are *unital*: the unit is the constant function 1. For convenience we use the function algebra $A = C(M)$ —a commutative C^*-algebra— at the beginning. We will change to $\mathcal{A} = C^\infty(M)$ later, when the differential structure becomes important.

Any A-module (or more precisely, a "symmetric A-bimodule") which is *finitely generated and projective* is of the form $\mathcal{E} = \Gamma(M, E)$ for $E \to M$ a (complex) vector bundle. Two important cases are

$$\begin{aligned}\mathfrak{X}(M) = \Gamma(M, T_{\mathbb{C}}M) &= \text{ (continuous) vector fields on } M;\\ \mathcal{A}^1(M) = \Gamma(M, T^*_{\mathbb{C}}M) &= \text{ (continuous) 1-forms on } M.\end{aligned}$$

These are *dual* to each other: $\mathcal{A}^1(M) \simeq \operatorname{Hom}_A(\mathfrak{X}(M), A)$, where Hom_A means "A-module maps" commuting with the action of A (by multiplication).

Definition 2.1. *A **Riemannian metric** on M is a symmetric bilinear form*

$$g \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C(M)$$

such that:

1. *$g(X, Y)$ is a real function if X, Y are real vector fields;*
2. *g is $C(M)$-*bilinear*: $g(fX, Y) = g(X, fY) = f\,g(X, Y)$, if $f \in C(M)$;*
3. *$g(X, X) \geq$ for X real, with $g(X, X) = 0 \implies X = 0$ in $\mathfrak{X}(M)$.*

The second condition entails that g is given by a continuous family of symmetric bilinear maps $g_x \colon T_x^{\mathbb{C}}M \times T_x^{\mathbb{C}}M \to \mathbb{C}$ or $g_x \colon T_xM \times T_xM \to \mathbb{R}$; the latter version is positive definite.

Fact 2.2. *Riemannian metrics always exist (in abundance).*

Since each g_x is positive definite, there are "musical isomorphisms" between $\mathfrak{X}(M)$ and $\mathcal{A}^1(M)$, as A-modules

$$\mathfrak{X}(M) \underset{\alpha^\sharp \leftarrow \alpha}{\overset{X \to X^\flat}{\rightleftarrows}} \mathcal{A}^1(M), \quad \text{given by} \quad \begin{cases} X^\flat(Y) & := g(X,Y) \\ \alpha(Y) & =: g(\alpha^\sharp, Y). \end{cases}$$

They are mutually inverse, of course. In fact, they can be used to transfer the metric form $\mathfrak{X}(M)$ to $\mathcal{A}^1(M)$:

$$g(\alpha,\beta) := g(\alpha^\sharp,\beta^\sharp), \quad \text{for } \alpha,\beta \in \mathcal{A}^1(M).$$

One should perhaps write $g^{-1}(\alpha,\beta)$ —as is done in [GVF]— since in local coordinates $g_{ij} := g(\partial/\partial x^i, \partial/\partial x^j)$ and $g^{rs} := g(dx^r, dx^s)$ have inverse matrices: $[g^{rs}] = [g_{ij}]^{-1}$.

If $f \in C^1(M)$, the *gradient* of f is $\operatorname{grad} f := (df)^\sharp$, so that

$$g(\operatorname{grad} f, Y) = df(Y) := Yf.$$

2.2 Clifford algebra bundles

More generally a real vector bundle $E \to M$ is a **Euclidean bundle** if, with $\mathcal{E} = \Gamma(M, E^{\mathbb{C}})$, there is a symmetric A-bilinear form $g\colon \mathcal{E} \times \mathcal{E} \to A = C(M)$ such that

1. $g(s,t) \in C(M;\mathbb{R})$ when s,t lie in $\Gamma(M,E)$ —the real sections;
2. $g(s,s) \geq 0$ for $s \in \Gamma(M,E)$, with $g(s,s) = 0 \implies s = 0$.

By defining $(s \mid t) := g(s^*,t)$, we get a **hermitian pairing** with values in A:

- $(s \mid t)$ is A-linear in t;
- $(t \mid s) = \overline{(s \mid t)} \in A$;
- $(s \mid s) \geq 0$, with $(s \mid s) = 0 \implies s = 0$ in $\mathcal{E}$;
- $(s \mid ta) = (s \mid t)\, a$ for all $s,t \in \mathcal{E}$ and $a \in A$.

These properties make $\mathcal{E}$ a (right) C^*-module over A, with C^*-norm given by

$$\|s\|_{\mathcal{E}} := \sqrt{\|(s \mid s)\|_A} \quad \text{for } s \in \mathcal{E}.$$

For each $x \in M$, we can form $\mathrm{Cl}(E_x) := \mathrm{Cl}(E_x, g_x) \otimes_{\mathbb{R}} \mathbb{C}$. Using the linear isomorphisms $\sigma_x\colon \mathrm{Cl}(E_x) \to (\Lambda^\bullet E_x)^{\mathbb{C}}$, we see that these are fibres of a vector bundle $\mathrm{Cl}(E) \to M$, isomorphic to $(\Lambda^\bullet E)^{\mathbb{C}} \to M$ as $\mathbb{C}$-vector bundles (but not as algebras!). Under $(\kappa\lambda)(x) := \kappa(x)\lambda(x)$, the sections of $\mathrm{Cl}(E)$ also form an algebra $\Gamma(M, \mathrm{Cl}(E))$. It has an A-valued pairing

$$(\kappa \mid \lambda)\colon x \mapsto \tau(\kappa(x)^*\lambda(x)).$$

By defining $\|\kappa\| := \sup_{x\in M} \|\kappa(x)\|_{\mathrm{Cl}(E_x)}$, this becomes a C^*-algebra.

Lemma 2.3. *If g,h are two different "metrics" on $\mathcal{E} = \mathcal{A}^1(M)$, the corresponding C^*-algebras*

$$B_g := \Gamma(M, \mathrm{Cl}(T^*M, g) \otimes_{\mathbb{R}} \mathbb{C}) \quad \textit{and} \quad B_h := \Gamma(M, \mathrm{Cl}(T^*M, h) \otimes_{\mathbb{R}} \mathbb{C})$$

are isomorphic.

Proof. We compose $\alpha \mapsto \alpha^{\sharp_g} \colon \mathcal{A}^1(M) \to \mathfrak{X}(M)$ and $X \mapsto X^{\flat_h} \colon \mathfrak{X}(M) \to \mathcal{A}^1(M)$ to get an A-linear isomorphism $\rho\colon \mathcal{A}^1(M) \to \mathcal{A}^1(M)$. Now

$$h(\bar{\alpha}, \rho(\alpha)) = \bar{\alpha}(\alpha^{\sharp_g}) = g(\bar{\alpha}, \alpha) \geq 0, \quad \text{for all} \quad \alpha \in \mathcal{A}^1(M). \tag{2.1}$$

At each $x \in M$, the $\mathbb{C}$-vector space $T_x^{\mathbb{C}}(M)$ may be regarded as a Hilbert space with scalar product $\langle \alpha_x \mid \beta_x \rangle_h := h_x(\bar{\alpha}_x, \beta_x)$, and now (2.1) says that each $\rho_x \in \operatorname{End}_{\mathbb{C}}(T_x^{\mathbb{C}} M)$ is a positive operator with a positive square root σ_x: we thereby obtain an A-linear isomorphism $\sigma\colon \mathcal{A}^1(M) \to \mathcal{A}^1(M)$ such that $\rho = \sigma^2$. We may regard σ as an injective A-linear map from $\mathcal{A}^1(M)$ into the algebra B_h; when $\alpha \in \mathcal{A}^1(M)$ is real, we get

$$\sigma(\alpha)^2 = h(\sigma(\alpha), \sigma(\alpha))\, 1 = h(\alpha, \rho(\alpha))\, 1 = g(\alpha, \alpha)\, 1.$$

By a now familiar argument, applied to each σ_x separately, we may extend σ to an A-linear unital *-algebra homomorphism $\tilde{\sigma}\colon B_g \to B_h$. Exchanging g and h gives an inverse homomorphism $\tilde{\sigma}^{-1}\colon B_h \to B_g$. Since any unital $*$-homomorphism between C^*-algebras is automatically norm-decreasing and thus continuous, we conclude that $\tilde{\sigma}\colon B_g \to B_h$ is an isomorphism of C^*-algebras. ☐

Definition 2.4. *A **Clifford module** over (M, g) is a finitely generated projective A-module, with $A = C(M)$, of the form $\mathcal{E} = \Gamma(M, E)$ for E a (complexified) Euclidean bundle, together with an A-linear homomorphism $c\colon B \to \Gamma(M, \operatorname{End} E)$, where $B := \Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$ is the Clifford algebra bundle generated by $\mathcal{A}^1(M)$, such that*

$$(s \mid c(\kappa)t) = (c(\kappa^*)s \mid t) \quad \textit{for all} \quad s, t \in \mathcal{E},\ \kappa \in B.$$

Example 2.5. Take $\mathcal{E} = \mathcal{A}^\bullet(M) = \Gamma(M, (\Lambda^\bullet T^*M)^{\mathbb{C}})$ —all differential forms on M— with $c(\alpha)\colon \omega \mapsto \varepsilon(\alpha)\omega + \iota(\alpha^\sharp)\omega$ for $\omega \in \mathcal{A}^\bullet(M)$ and $\alpha \in \mathcal{A}^1(M)$ real. Then $\mathcal{E}$ is indeed a Clifford module, but it is a rather large one: it may have nontrivial submodules. The goal of the next subsection is top explore how some minimal submodules may be constructed.

2.3 The existence of Spin^c structures

Suppose $n = 2m + 1 = \dim M$ is odd. Then the fibres of B are semisimple but not simple: $\mathbb{C}\mathrm{l}(T_x^*M) \simeq M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$. We shall restrict to the even subalgebras, $\mathbb{C}\mathrm{l}^0(T_x^*M) \simeq M_{2^m}(\mathbb{C})$, by demanding *that $c(\gamma)$ act as the identity in all cases.* Then we may adopt the convention that

$$c(\kappa) := c(\kappa\gamma) \quad \text{when } \kappa \text{ is odd}.$$

Notice here that $\kappa\gamma$ is even; and $c(\gamma) = c(\gamma^2) = +1$ is required for consistency of this rule.

We take $A = C(M)$, but for B we now take

$$B := \begin{cases} \Gamma(M, \mathbb{C}\mathrm{l}(T^*M)), & \text{if } \dim M \text{ is even}, \\ \Gamma(M, \mathbb{C}\mathrm{l}^0(T^*M)), & \text{if } \dim M \text{ is odd}. \end{cases} \tag{2.2}$$

The fibres of these bundles are *central simple algebras* of finite dimension 2^{2m} in all cases.

We classify the algebras B as follows. Taking

$$\underline{B} := \begin{cases} \{\, B_x = \mathbb{C}\mathrm{l}(T_x^*M) : x \in M \,\}, & \text{if } \dim M \text{ is even} \\ \{\, B_x = \mathbb{C}\mathrm{l}^0(T_x^*M) : x \in M \,\}, & \text{if } \dim M \text{ is odd} \end{cases}$$

to be the collection of fibres, we can say that $\underline{B}$ is a "continuous field of simple matrix algebras", which moreover is *locally trivial.* There is an invariant

$$\delta(\underline{B}) \in \mathrm{H}^3(M;\mathbb{Z})$$

for such fields, found by Karrer [Kar] and in more generality —allowing the compact operators $\mathcal{K}$ as an infinite-dimensional simple matrix algebra— by Dixmier and Douady [Dix].

Here is a (rather pedestrian) sketch of how $\delta(\underline{B})$ is constructed:

If $x \in M$, take $p_x \in B_x$ to be a *projector of rank one*, that is,

$$p_x = p_x^* = p_x^2 \quad \text{and} \quad \operatorname{tr} p_x = 1.$$

On the left ideal $S_x := B_x p_x$, we introduce a hermitian scalar product

$$\langle a_x p_x \mid b_x p_x \rangle := \operatorname{tr}(p_x a_x^* b_x p_x). \tag{2.3}$$

Notice that the recipe

$$|a_x p_x\rangle\langle b_x p_x| : c_x p_x \mapsto (a_x p_x)(b_x p_x)^*(c_x p_x) = (a_x p_x b_x^*)(c_x p_x)$$

identifies $\mathcal{L}(S_x)$ —or $\mathcal{K}(S_x)$ in the infinite-dimensional case— with B_x, since the two-sided ideal $\operatorname{span}\{\, a_x p_x b_x^* : a_x, b_x \in B_x \,\}$ equals B_x by simplicity.

By local triviality, this can be done locally with varying x. If $\{U_i\}$ is a "good" open cover[1] of M, we get local fields $\underline{S}_i = \{\, S_{i,x} : x \in U_i \,\}$ with isomorphisms $\underline{\theta}_i \colon \mathcal{L}(\underline{S}_i) \to \underline{B}\big|_{U_i}$ of fields of simple C^*-algebras. On nonempty intersections $U_{ij} := U_i \cap U_j$, we get $*$-algebra isomorphisms $\underline{\theta}_i^{-1}\underline{\theta}_j \colon \mathcal{L}(\underline{S}_j) \to \mathcal{L}(\underline{S}_i)$, so there are fields of unitary maps $\underline{u}_{ij} \colon \underline{S}_j \to \underline{S}_i$ such that $\underline{\theta}_i^{-1}\underline{\theta}_j = \underline{u}_{ij}(\cdot)\underline{u}_{ij}^{-1}$.

On $U_{ijk} := U_i \cap U_j \cap U_k$, we see that $(\operatorname{Ad}\underline{u}_{ij})(\operatorname{Ad}\underline{u}_{jk}) = \operatorname{Ad}\underline{u}_{ik}$, and so

$$\underline{u}_{ij}\underline{u}_{jk} = \underline{\lambda}_{ijk}\underline{u}_{ik},$$

where $\underline{\lambda}_{ijk} \colon U_{ijk} \to \mathbb{T}$ are *scalar* maps. We may now check that $\underline{\lambda}_{jkl}\underline{\lambda}_{ikl}^{-1}\underline{\lambda}_{ijl} = \underline{\lambda}_{ijk}$ on U_{ijkl}. Thus $\underline{\lambda}$ is a *Čech 2-cocycle*, and its Čech cohomology class lies in $\check{\mathrm{H}}^2(M;\mathbb{T}) \simeq \mathrm{H}^3(M;\mathbb{Z})$. We may go one more step in order to exhibit this isomorphism: if we write $\underline{\lambda}_{ijk} = \exp(2\pi i \underline{f}_{ijk})$ —we can take logarithms since U_{ijk} is simply connected— then

$$a_{ijkl} := \underline{f}_{ijk} - \underline{f}_{ijl} + \underline{f}_{ikl} - \underline{f}_{jkl}$$

takes values in $\mathbb{Z}$ (and since each U_{ijkl} is connected, these will be constant functions); thus, these a_{ijkl} form a $\mathbb{Z}$-valued 3-cocycle, a. Finally, one may check that its class $[a] \in \mathrm{H}^3(M;\mathbb{Z})$ is independent of all choices made so far. We define $\delta(\underline{B}) := [a]$, which is called the *Dixmier–Douady class* of $\underline{B}$.

Suppose now that the Hilbert spaces $S_x \simeq \mathbb{C}^{2^m}$ can be chosen *globally* for $x \in M$ —not just locally for $x \in U_i$— that is, they are fibres of a vector bundle $S \to M$ (that may b gifted with a Hermitian metric) such that $\mathcal{L}(S_x) \simeq B_x$, for $x \in M$, via a single field of isomorphisms $\underline{\theta} \colon \underline{\mathcal{L}(S)} \to \underline{B}$ such that $\underline{\theta}_i = \underline{\theta}\big|_{U_i}$ for each U_i. Then $\underline{u}_{ij} = \underline{\theta}_i^{-1}\underline{\theta}_j = \mathrm{id}$ over U_{ij}, and so $\underline{\lambda}_{ijk} = 1$ over U_{ijk}, and $a_{ijkl} = 0$ over each U_{ijkl}; hence $\delta(\underline{B}) = [a] = 0$ in $\mathrm{H}^3(M;\mathbb{Z})$.

[1]The word *good* has a precise technical meaning: namely, that all nonempty finite intersections of open sets of the cover are both connected and simply connected. On Riemannian manifolds, good open covers may always be formed using geodesically convex balls.

Conversely if $\delta(\underline{B}) = 0$, so that $[\underline{\lambda}]$ is trivial in $\check{\mathrm{H}}^2(M;\mathbb{T})$, i.e., $\underline{\lambda}$ is a 2-coboundary, then there are maps $\underline{\nu}_{ij} \colon U_{ij} \to \mathbb{T}$ such that $\underline{\lambda}_{ijk} = \underline{\nu}_{ij}\underline{\nu}_{ik}^{-1}\underline{\nu}_{jk}$ on U_{ijk}. Setting $\underline{v}_{ij} := \underline{\nu}_{ij}^{-1}\underline{u}_{ij}$, we get local fields of unitaries such that $\underline{v}_{ij}\underline{v}_{jk} = \underline{v}_{ik}$ on each U_{ijk}. These $\underline{v}_{ij} \colon \underline{S}_j \to \underline{S}_i$ are therefore *transition functions* for a (Hermitian) vector bundle $S \to M$ such that $\underline{S}\big|_{U_i} \simeq \underline{S}_i$ for each U_i. Let $\mathcal{S} := \Gamma(M, S)$ denote the A-module of sections of this bundle. Now the pointwise isomorphisms $B_x \simeq \operatorname{End} S_x$, for each $x \in M$, imply that $B \simeq \operatorname{End}_A S$ as A-modules, and indeed as C^*-algebras. We summarize all this in the following Proposition.

Proposition 2.6. *Let (M, g) be a compact Riemannian manifold. With $A = C(M)$ and B the algebra of Clifford sections given by (2.2), the Dixmier–Douady class vanishes, i.e., $\delta(\underline{B}) = 0$, if and only if there is a finitely generated projective A-module $\mathcal{S}$, carrying a selfadjoint action of B by A-linear operators, such that* $\operatorname{End}_A(S) \simeq B$.

2.4 Morita equivalence for (commutative) unital algebras

Definition 2.7. *If A, B are* unital $\mathbb{C}$*-algebras, we say that they are* ***Morita equivalent*** *if there is a B-A-bimodule $\mathcal{E}$ and an A-B-bimodule $\mathcal{F}$, such that $\mathcal{E} \otimes_A \mathcal{F} \simeq B$ as B-B-bimodules and $\mathcal{F} \otimes_B \mathcal{E} \simeq A$ as A-A-bimodules. We say that such an $\mathcal{E}$ is an "equivalence bimodule".*

In general, we may choose $\mathcal{F} \simeq \mathcal{E}^\sharp := \operatorname{Hom}_A(\mathcal{E}, A)$ to be the "dual" right A-module with a specified action of B. We can then identify $\operatorname{End}_A \mathcal{E} \simeq B$.

Fact 2.8. $\operatorname{End}_A \mathcal{E} \simeq B$ *whenever $\mathcal{E}$ is an equivalence B-A-bimodule.*

Fact 2.9. *Since A, B are unital, each $\mathcal{E}$ is finitely generated and projective (and full).*

Remark 2.10. There is a C*-version, due to Rieffel, whereby all bimodules are provided with compatible A-valued and B-valued Hermitian pairings. This becomes nontrivial in the more general context of nonunital algebras. The full story is told in [RW]. We shall not need this machinery in the unital case: remember that M is taken to be compact.

Notation. For isomorphism classes $[\mathcal{E}]$ of bimodules, we form the set

$$\operatorname{Mrt}(B, A) := \{\, [\mathcal{E}] : \mathcal{E} \text{ is a } B\text{-}A\text{-equivalence bimodule} \,\}.$$

In the case $B = A$, we write $\operatorname{Pic}(A) := \operatorname{Mrt}(A, A)$; this is called the "Picard group" of A.

We call an A-bimodule $\mathcal{E}$ *symmetric* if the left and right actions are the same: $a \triangleright x = x \triangleleft a$ for $x \in \mathcal{E}$ and $a \in A$. When A is commutative, a symmetric A-A-bimodule can be called, more simply, an "A-module" —as we have already been doing. Even when A is commutative, an A-A-equivalence bimodule $\mathcal{E}$ need not be symmetric. Indeed, suppose $\phi, \psi \in \operatorname{Aut}(A)$. Then we define ${}_\phi\mathcal{E}_\psi$ to be the same vector space $\mathcal{E}$, but with the bi-action of A on $\mathcal{E}$ twisted as follows:

$$a_1 \triangleright a_0 \triangleleft a_2 := \phi(a_1)\, a_0\, \psi(a_2)$$

For $\phi = \psi = \mathrm{id}$, this is the original A-A-bimodule (when either $\phi = \mathrm{id}$ or $\psi = \mathrm{id}$, we shall not write that subscript). In particular, we can apply this twisting to $\mathcal{E} = A$ itself.

Lemma 2.11. *If A is a unital algebra, there exists an A-A-bimodule isomorphism $\theta \colon A \to {}_\phi A$ if and only if ϕ is inner.*

Proof. If $\theta \colon A \to {}_\phi A$ is an A-bimodule isomorphism, then

$$\phi(a)\theta(1) = \theta(a) = \theta(1)a$$

so that $\phi(a) = uau^{-1}$, where $u = \theta(1)$ is invertible. ∎

Thus the "outer automorphism group" $\mathrm{Out}(A) := \mathrm{Aut}(A)/\mathrm{Inn}(A)$ classifies the asymmetric A-bimodules. When A is commutative, so that $\mathrm{Inn}(A)$ is trivial, this is just $\mathrm{Aut}(A)$.

Recall that

$$\mathrm{Aut}(C(M)) \simeq \mathrm{Homeo}(M), \qquad \mathrm{Aut}(C^\infty(M)) \simeq \mathrm{Diff}(M),$$

where $\phi(f) \colon x \mapsto f(\phi^{-1}x)$ for $f \in C(M)$. We shall write $\mathrm{Pic}_A(A)$, following [BW], to denote the isomorphism classes of symmetric A-bimodules. (This repairs an oversight in [GVF, Chap. 9], which did not distinguish between $\mathrm{Pic}(A)$ and $\mathrm{Pic}_A(A)$, as was pointed out to me by Henrique Bursztyn.)

Fact 2.12. $\mathrm{Pic}(A) \simeq \mathrm{Pic}_A(A) \rtimes \mathrm{Aut}(A)$ *as a semidirect product of groups, with product given by* $([\mathcal{E}], \phi) \cdot ([\mathcal{F}], \psi) = ([{}_\psi\mathcal{E}_\psi \otimes_A \mathcal{F}], \phi \circ \phi)$.

The proof is not difficult, but we refer to the paper [BW].

Lemma 2.13. *For* $A = C(M)$ *or* $C^\infty(M)$, $\mathrm{Pic}_A(A) \simeq \mathrm{H}^2(M; \mathbb{Z})$.

Proof. Since invertible A-modules $\mathcal{L}$ are given by $\mathcal{L} = \Gamma(M, L)$ —either continuous or smooth sections, respectively— where $L \to M$ are $\mathbb{C}$-line bundles; and these are classified by the first Chern class $c_1(L) \in \mathrm{H}^2(M; \mathbb{Z})$, obtained from $[\underline{\lambda}] = [\underline{\lambda}_{ij}] \in \check{\mathrm{H}}^1(M; \mathbb{T}) \simeq \mathrm{H}^2(M; \mathbb{Z})$. Indeed, here $\mathcal{L}^\sharp = \Gamma(M, L^*)$, where $L^* \to M$ is the dual bundle and $\mathcal{L} \otimes_A \mathcal{L}^\sharp = \Gamma(M, L \otimes L^*) \simeq \Gamma(M, M \times \mathbb{C}) \simeq C(M)$ or $C^\infty(M)$, respectively.

The group operation in $\mathrm{Pic}_A(A)$ is $[\mathcal{L}_1] \cdot [\mathcal{L}_2] = [\mathcal{L}_1 \otimes_A \mathcal{L}_2]$: since $\mathcal{L}_1 \otimes_A \mathcal{L}_2 \simeq \Gamma(M, L_1 \otimes L_2)$, it is again a module of sections for a $\mathbb{C}$-line bundle. $\square$

2.5 Classification of spinor modules

In this section, $A = C(M)$ and as before, $B = \Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$ or $B = \Gamma(M, \mathbb{C}\mathrm{l}^0(T^*M))$, according as the dimension of M is even or odd.

Consider now the set $\mathrm{Mrt}(B, A)$ of isomorphism classes of B-A-bimodules: we have seen that $\delta(B) = 0$ if and only if $\mathrm{Mrt}(B, A)$ is nonempty. We shall assume from now on that indeed $\delta(B) = 0$, so that there exists at least one B-A-bimodule $\mathcal{S} = \Gamma(M, S)$ —continuous sections, for the moment— such that at each $x \in M$, S_x is an irreducible representation of the simple algebra B_x. Therefore, any such $\mathcal{S}$ has a partner $\mathcal{S}^\sharp = \mathrm{Hom}_A(\mathcal{S}, A)$ such that $\mathcal{S} \otimes_A \mathcal{S}^\sharp \simeq B$ and $\mathcal{S}^\sharp \otimes_B \mathcal{S} \simeq A$: in other words, $\mathcal{S}$ is an equivalence B-A-bimodule, and its isomorphism class $[\mathcal{S}]$ is an element of $\mathrm{Mrt}(B, A)$.

Since $\mathcal{S}^\sharp \simeq \Gamma(M, S^*)$ where $S^* \to M$ is the dual vector bundle to $S \to M$, we can write this equivalence fibrewise: $\mathcal{S}_x \otimes_\mathbb{C} \mathcal{S}_x^* = \mathrm{End}_\mathbb{C}(\mathcal{S}_x) \simeq B_x$ and then $\mathcal{S}_x^* \otimes_{B_x} \mathcal{S}_x \simeq \mathbb{C}$, for $x \in M$.

Lemma 2.14. $\mathrm{Mrt}(B, A)$ *is a principal homogeneous space for the group* $\mathrm{Pic}_A(A)$, *when* $\delta(B) = 0$.

Proof. There is a right action of $\mathrm{Pic}_A(A)$ on $\mathrm{Mrt}(B, A)$, given by $[\mathcal{S}] \cdot [\mathcal{L}] := [\mathcal{S} \otimes_A \mathcal{L}]$. We say that the spinor module $\mathcal{S}$ is "twisted" by the invertible A-module $\mathcal{L}$.

If $\mathcal{S} \otimes_A \mathcal{S}^\sharp \simeq B$ and $\mathcal{S}^\sharp \otimes_B \mathcal{S} \simeq A$, then for $\mathcal{S}^1 := \mathcal{S} \otimes_A \mathcal{L}$ we get

$$\begin{aligned}
\mathcal{S}_1 \otimes_A \mathcal{S}_1^\sharp &= \mathcal{S} \otimes_A \mathcal{L} \otimes_A \mathcal{L}^\sharp \otimes_A \mathcal{S}^\sharp \simeq \mathcal{S} \otimes_A A \otimes_A \mathcal{S}^\sharp \simeq \mathcal{S} \otimes_A \mathcal{S}^\sharp \simeq B, \\
\mathcal{S}_1^\sharp \otimes_B \mathcal{S}_1 &= \mathcal{L}^\sharp \otimes_A \mathcal{S}^\sharp \otimes_B \mathcal{S} \otimes_A \mathcal{L} \simeq \mathcal{L}^\sharp \otimes_A A \otimes_A \mathcal{L} \simeq \mathcal{L}^\sharp \otimes_A \mathcal{L} \simeq A.
\end{aligned}$$

Thus $\mathcal{S}_1$ is again an equivalence B-A-bimodule. Moreover, under the natural isomorphism

$$\mathrm{Hom}_B(\mathcal{S}',\mathcal{S}) \otimes_A \mathcal{L} \simeq \mathrm{Hom}_B(\mathcal{S}',\mathcal{S} \otimes_A \mathcal{L}) \ : \ F \otimes l \mapsto [s' \mapsto F(s') \otimes l],$$

we see that $\mathrm{Hom}_B(\mathcal{S},\mathcal{S} \otimes_A \mathcal{L}) \simeq \mathcal{S}^\sharp \otimes_B \mathcal{S} \otimes_A \mathcal{L} \simeq A \otimes_A \mathcal{L} = \mathcal{L}$, so that the right action of $\mathrm{Pic}_A(A)$ is *free*. On the other hand, the identification

$$\mathcal{S} \otimes \mathrm{Hom}_B(\mathcal{S},\mathcal{S}') \simeq \mathrm{Hom}_B(\mathrm{End}_A(\mathcal{S}),\mathcal{S}') \ : \ s \otimes F \mapsto [G \mapsto (F \circ G)(s)]$$

yields, for $\mathcal{L} := \mathrm{Hom}_B(\mathcal{S},\mathcal{S}')$, the isomorphism

$$\mathcal{S} \otimes_A \mathcal{L} \simeq \mathrm{Hom}_B(B,\mathcal{S}') \simeq \mathcal{S}',$$

so that the action of $\mathrm{Pic}_A(A)$ is *transitive*. □

To proceed, we explain how B acts on $\mathcal{S}^\sharp = \mathrm{Hom}_A(\mathcal{S},A)$. The spinor module $\mathcal{S}$ carries an A-valued hermitian pairing (2.3) given by the local scalar products defined in the construction of $\mathcal{S}$, that may be written

$$(\psi \mid \phi) \ : \ x \mapsto \langle \psi_x \mid \phi_x \rangle, \quad \text{for } x \in M. \tag{2.4}$$

We can identify elements of $\mathcal{S}^\sharp$ with "bra-vectors" $\langle\psi|$ using this pairing, namely, we define $\langle\psi|$ to be the map $\phi \mapsto (\psi|\phi) \in A$. Since A is unital, there is a "Riesz theorem" for A-modules showing that all elements of $\mathcal{S}^\sharp$ are of this form. Now the left B-action is defined by

$$b\,\langle\psi| := \langle\psi| \circ \chi(b^!).$$

Recall that $b \mapsto \chi(b!)$ is a linear antiautomorphism of B.

Remark 2.15. In these notes, there are many inner products. As a convention, angle brackets $\langle\cdot\mid\cdot\rangle$ take values in $\mathbb{C}$ —we shall call them *scalar products* to emphasize this— while round brackets $(\cdot\mid\cdot)$ take values in an algebra A —we use the word *pairing* to signal that.

Lemma 2.16. *Let $\mathcal{L}_\mathcal{S} := \mathrm{Hom}_B(\mathcal{S}^\sharp,\mathcal{S})$ be the A-module for which $\mathcal{S}^\sharp \otimes_A \mathcal{L}_\mathcal{S} \simeq \mathcal{S}$. Then $\mathcal{L}_{\mathcal{S}\otimes_A\mathcal{L}} \simeq \mathcal{L}_\mathcal{S} \otimes_A \mathcal{L} \otimes_A \mathcal{L}$, so that the twisting $[\mathcal{S}] \mapsto [\mathcal{S} \otimes_A \mathcal{L}]$ on $\mathrm{Mrt}(B,A)$ induces a translation by $[\mathcal{L} \otimes_A \mathcal{L}] = 2[\mathcal{L}]$ on $\mathrm{Pic}_A(A)$.*

Proof. First observe that, since $\mathcal{L}$ is an invertible A-module, the dual of $\mathcal{S}\otimes_A\mathcal{L}$ is isomorphic to $\mathcal{S}^\sharp \otimes_A \mathcal{L}^\sharp$. The commutativity of the group $\mathrm{Pic}_A(A)$ the shows that

$$\begin{aligned}(\mathcal{S} \otimes_A \mathcal{L})^\sharp \otimes_A (\mathcal{L}_\mathcal{S} \otimes_A \mathcal{L} \otimes_A \mathcal{L}) &\simeq \mathcal{S}^\sharp \otimes_A \mathcal{L}^\sharp \otimes_A (\mathcal{L} \otimes_A \mathcal{L}_\mathcal{S} \otimes_A \mathcal{L}) \\ &\simeq \mathcal{S}^\sharp \otimes_A \mathcal{L}_\mathcal{S} \otimes_A \mathcal{L} \simeq \mathcal{S} \otimes_A \mathcal{L},\end{aligned}$$

and the freeness of the action now implies the result. □

Thus, the "mod 2 reduction" $j_*[\mathcal{L}_\mathcal{S}] \in \mathrm{H}^2(M;\mathbb{Z}_2)$, coming from the short exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_2 \to 0$, is independent of $[\mathcal{S}]$. Indeed, it defines an invariant $\kappa[B] \in \mathrm{H}^2(M;\mathbb{Z})$. This is clear, when one takes into account the corresponding long exact sequence in Čech cohomology and the governing assumption that $\delta(B) = 0$:

$$\cdots \to \mathrm{H}^1(M;\mathbb{Z}_2) \xrightarrow{\partial} \mathrm{H}^2(M;\mathbb{Z}) \xrightarrow{(\times 2)_*} \mathrm{H}^2(M;\mathbb{Z}) \xrightarrow{j_*} \mathrm{H}^2(M;\mathbb{Z}_2) \xrightarrow{\partial} \mathrm{H}^3(M;\mathbb{Z}) \to \cdots \tag{2.5}$$

Remark 2.17. It can be shown that $\kappa(B) = w_2(TM) = w_2(T^*M)$, the familiar second *Stiefel–Whitney class* of the tangent (or cotangent) bundle. See, for instance, the original papers of Karrer [Kar] and Plymen [Ply], and the lecture notes by Schröder [Schd].

¿What is the meaning of the condition $\kappa(B) = 0$? It means that, by replacing any original choice of $\mathcal{S}$ by a suitably twisted $\mathcal{S} \otimes_A \mathcal{L}$, we can arrange that $\mathcal{L}_\mathcal{S}$ is trivial, i.e. $\mathcal{L}_\mathcal{S} \simeq A$, or better yet, that

$$\mathcal{S}^\sharp \simeq \mathcal{S} \quad \text{as } B\text{-}A\text{-bimodules.}$$

We now reformulate this condition in terms of a certain antilinear operator C; later on, in the context of spectral triples, we shall rename it to J.

Proposition 2.18. *There is a B-A-bimodule isomorphism $\mathcal{S}^\sharp \simeq \mathcal{S}$ if and only if there is an antilinear endomorphism C of $\mathcal{S}$ such that*

(a) $C(\psi\, a) = C(\psi)\, \bar a$ *for* $\psi \in \mathcal{S}$, $a \in A$;

(b) $C(b\, \psi) = \chi(\bar b)\, C(\psi)$ *for* $\psi \in \mathcal{S}$, $b \in B$;

(c) C *is* antiunitary *in the sense that* $(C\phi \mid C\psi) = (\psi \mid \phi) \in A$, *for* $\phi, \psi \in \mathcal{S}$;

(d) $C^2 = \pm 1$ *on* $\mathcal{S}$ *whenever* M *is connected.*

Proof. Ad (a): We provisionally define C by $C(\psi) := T\langle\psi|$, where $T\colon \mathcal{S}^\sharp \to S$ is the given B-A-bimodule isomorphism. Now since $\langle\psi a| = \langle\psi|\, \bar a$ because $(\psi\, a \mid \phi) = \bar a\, (\psi \mid \phi) = (\psi \mid \phi)\, \bar a = (\psi \mid \phi\, \bar a)$ —since A is commutative— we get $C(\psi\, a) = T(\langle\psi|\, \bar a) = T\langle\psi|\, \bar a = C(\psi)\, \bar a$.

Ad (b): The formula for the B-action on $\mathcal{S}^\sharp$, and the relation $(b\psi \mid \phi) = (\psi \mid b^*\phi)$ give

$$C(b\, \psi) = T\langle b\, \psi| = T(\langle\psi| \circ b^*) = T(\chi(b^!)\, \langle\psi|) = \chi(\bar b)\, T\langle\psi|.$$

Ad (c): The pairing $(\phi \mid C\psi)$ is antilinear (and bounded) in both ϕ and ψ, and thus of the form $(\psi \mid \chi)$ for some $\chi \in \mathcal{S}$, by the aforementioned "Riesz theorem". Thus we get an adjoint map to C, namely the antilinear map $C^\dagger\colon \phi \mapsto \chi$ —obeying the rule for transposing antilinear operators, i.e., $(\psi \mid C^\dagger\phi) = (\phi \mid C\psi)$. Next, notice that $C^\dagger C$ is an A-linear bijective endomorphism of $\mathcal{S}$, that commutes with each $b \in B$:

$$\begin{aligned}
(\phi \mid C^\dagger C b\psi) &= (Cb\psi \mid C\phi) = (\chi(\bar b) C\psi \mid C\phi) \\
&= (C\psi \mid \chi(b^!) C\phi) = (C\psi \mid Cb^*\phi) \\
&= (b^*\phi \mid C^\dagger C\psi) = (\phi \mid bC^\dagger C\psi).
\end{aligned}$$

Therefore $C^\dagger C \in \operatorname{End}_B(\mathcal{S}) \simeq A$, i.e. there is an invertible $a \in A$ such that $C^\dagger C = a 1_\mathcal{S}$. If $a \neq 1$, we can now *redefine* $C \mapsto a^{-\frac12} C$, keeping (a) and (b), so with the redefinition $C(\psi) := a^{-\frac12} T\langle\psi|$, we get $C^\dagger C = 1$, i.e., $(C\phi \mid C\psi) = (\psi \mid C^\dagger C\phi) = (\psi \mid \phi)$.

Ad (d): Finally, C^2 is A-linear, and $C^2\, b = C\, \chi(\bar b)\, C = b\, C^2$ for $b \in B$, so $C^2 = u 1_\mathcal{S}$ with $u \in A$. From the relations

$$\begin{aligned}
uC = C^3 = Cu &= \bar u C \quad \text{by antilinearity of } C, \\
\bar u u\, 1_\mathcal{S} = (C^\dagger)^2 C^2 &= C^{-2} C^2 = 1_\mathcal{S},
\end{aligned}$$

we get $u = \bar u$ and hence $u^2 = 1$. Thus $u \in A = C(M)$ takes the values ± 1 only, so $u = \pm 1$ when M is connected. (More generally, C^2 lies in $\mathrm{H}^0(M, \mathbb{Z}_2)$.) ∎

The antilinear operator $C: \mathcal{S} \to \mathcal{S}$, which becomes an *antiunitary* operator on a suitable Hilbert-space completion of $\mathcal{S}$, is called the **charge conjugation**. It exists if and only if $\kappa(B) = 0$.

¿What, then, are spin$^{\mathrm{c}}$ and spin structures on M? We choose on M a metric (without losing generality), and also an *orientation* ε, which organizes the action of B, in that a change $\varepsilon \mapsto -\varepsilon$ induces $c(\gamma) \mapsto -c(\gamma)$, which either

(i) reverses the $\mathbb{Z}_2$-grading of $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, in the even case; or

(ii) changes the action on $\mathcal{S}$ of each $c(\alpha)$ to $-c(\alpha)$, for $\alpha \in \mathcal{A}^1(M)$, in the odd case —recall that $c(\alpha) := c(\alpha\gamma)$ in the odd case.

Definition 2.19. *Let (M, ε) be a compact boundaryless orientable manifold, together with a chosen orientation ε. Let $A = C(M)$ and let B be specified as before (in terms of a fixed but arbitrary Riemannian metric on M). If $\delta(B) = 0$ in $\mathrm{H}^3(M; \mathbb{Z})$, a* ***spin$^{\mathrm{c}}$ structure*** *on (M, ε) is an isomorphism class $[\mathcal{S}]$ of equivalence B-A-bimodules.*

If $\delta(B) = 0$ and if $\kappa(B) = 0$ in $\mathrm{H}^2(M; \mathbb{Z}_2)$, a pair $(\mathcal{S}, C)$ give data for a spin structure, when $\mathcal{S}$ is an equivalence B-A-bimodule such that $\mathcal{S}^\sharp \simeq \mathcal{S}$, and C is a charge conjugation operator on $\mathcal{S}$. A ***spin structure*** *on (M, ε) is an isomorphism class of such pairs.*

Remark 2.20. There is an alternative treatment, given in many books, that defines spin$^{\mathrm{c}}$ or spin structures using *principal G-bundles* for $G = \mathrm{Spin}^c(\mathbb{R}^n)$ or $G = \mathrm{Spin}(\mathbb{R}^n)$ respectively. The equivalence of the two approaches is treated in [Ply] and [Schd].

Atiyah, Bott and Shapiro [ABS] called a spin$^{\mathrm{c}}$ structure a "K-orientation", for reasons which may be obvious to K-theorists. At any rate, it is a finer invariant than the orientation class $[\varepsilon]$, provided it exists.

In the long cohomology exact sequence there is a boundary homomorphism

$$\mathrm{H}^2(M; \mathbb{Z}_2) \xrightarrow{\partial} \mathrm{H}^3(M; \mathbb{Z}).$$

By examining the definitions of the various Čech cocyles that we have obtained so far, one can show that $\delta(B) = \partial(\kappa(B))$.

Remark 2.21. It is known that $\delta(B) = 0$ for $\dim M \leq 4$: manifolds of dimensions $1, 2, 3, 4$ always carry spin$^{\mathrm{c}}$ structures. There are 5-dimensional manifolds for which $\delta(B) \neq 0$; the best-known is the homogeneous space $\mathrm{SU}(3)/\mathrm{SO}(3)$. A homotopy-theoretic proof of the obstruction for this example in given in [Fri].

A complex manifold has a natural orientation and a natural spin$^{\mathrm{c}}$ structure coming from its complex structure. Thus $\mathbb{C}P^m$ come with a spin$^{\mathrm{c}}$ structure, for all m. However, it is known that $\mathbb{C}P^m$ admits spin structures if and only if m is odd: therefore, $\mathbb{C}P^2$ is a 4-dimensional manifold without spin structures.

2.6 The spin connection

We now leave the topological level and introduce differential structure. Thus we replace $A = C(M)$ by $\mathcal{A} = C^\infty(M)$, and continuous sections Γ_{cont} by smooth sections Γ_{smooth}. Thus $\mathcal{S} = \Gamma(M, S)$ will henceforth denote the $\mathcal{A}$-module of *smooth* spinors.

Our treatment of Morita equivalence of *unital* algebras passes without change to the smooth level. We can go back with the functor $- \otimes_{C^\infty(M)} C(M)$, if desired.

Definition 2.22. *A* ***connection*** *on a (finitely generated projective) $\mathcal{A}$-module $\mathcal{E} = \Gamma(M, E)$ is a $\mathbb{C}$-linear map $\nabla \colon \mathcal{E} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E} = \Gamma(M, T^*M \otimes E) \equiv \mathcal{A}^1(M, E)$, satisfying the Leibniz rule*

$$\nabla(fs) = df \otimes s + f\,\nabla s.$$

It extends to an odd derivation of degree +1 on $\mathcal{A}^\bullet(M) \otimes_{\mathcal{A}} \mathcal{E} = \Gamma(M, \Lambda^\bullet T^*M \otimes E) \equiv \mathcal{A}^\bullet(M, E)$ with grading inherited from that of $\mathcal{A}^\bullet(M)$, leaving $\mathcal{E}$ trivially graded, so that $\nabla(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^{|\omega|}\omega \wedge \nabla\sigma$ for $\omega \in \mathcal{A}^\bullet(M)$, $\sigma \in \mathcal{A}^\bullet(M, E)$.

Employing the usual contraction of vector fields with forms in $\mathcal{A}^\bullet(M)$, namely,

$$\iota_X \omega(Y_1, \dots, Y_k) := \omega(X, Y_1, \dots, Y_k) \quad \text{for} \quad \omega \in \mathcal{A}^{k+1}(M),$$

extended to $\mathcal{A}^\bullet(M) \otimes_{\mathcal{A}} \mathcal{E}$ as $\iota_X \otimes \mathrm{id}_{\mathcal{E}}$ —but still written ι_X— we get operators ∇_X on $\mathcal{A}^\bullet(M, E)$ of degree 0 by defining

$$\nabla_X := \iota_X \circ \nabla + \nabla \circ \iota_X.$$

This is $\mathcal{A}$-linear in X. Moreover, if $\omega \in \mathcal{A}^\bullet(M)$ and $s \in \mathcal{E}$, one can check that $\nabla_X(\omega \otimes s) = \mathcal{L}_X \omega \otimes s + \omega\,\nabla_X s$, where $\mathcal{L}_X = \iota_X\, d + d\,\iota_X$ is the Lie derivative of forms with respect to X.

Exercise 2.23. *Verify that $\nabla_X(\iota_Y \sigma) = \iota_Y(\nabla_X \sigma) + \iota_{[X,Y]}\sigma$ for $\sigma \in \mathcal{A}^\bullet(M, E)$.*

Exercise 2.24. *If $\mathcal{E} = \mathfrak{X}(M) = \Gamma(M, TM)$, then show that*

$$\nabla_X Y - \nabla_Y X - [X, Y] = \iota_Y \iota_X \nabla\theta,$$

where $\theta \in \mathcal{A}^1(M, TM)$ is the fundamental 1-form *defined by $\iota_X \theta := X$. We say that ∇ is* ***torsionfree*** *if $\nabla\theta = 0$ in $\mathcal{A}^2(M, TM)$.*

Exercise 2.25. *Show that $\iota_Y \iota_X \nabla^2 = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, where the degree +2 operator ∇^2 on $\mathcal{A}^\bullet(M, E)$ is the* ***curvature*** *of ∇.*

We mention two natural constructions for connections, on tensor products of $\mathcal{A}$-modules and on dual $\mathcal{A}$-modules. Firstly, if $\nabla' \colon \mathcal{F} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{F}$ is another connection in another $\mathcal{A}$-module, then

$$\widetilde{\nabla}(s \otimes t) := \nabla s \otimes t + s \otimes \nabla' t$$

(extneded by linearity, as usual) makes $\widetilde{\nabla}$ a connection on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$.

Next, if $\mathcal{E}^\sharp = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$, then the **dual connection** $\nabla^\sharp$ on $\mathcal{E}^\sharp$ is determined by

$$\begin{aligned} d(\zeta(s)) &=: (\nabla^\sharp \zeta)(s) + \zeta(\nabla s) \quad \text{in } \mathcal{A}^1(M); \quad \text{or equivalently} \\ X(\zeta(s)) &=: (\nabla^\sharp_X \zeta)(s) + \zeta(\nabla_X s) \quad \text{in } \mathcal{A}, \quad \text{for } X \in \mathfrak{X}(M), \end{aligned}$$

whenever $\zeta \in \mathcal{E}^\sharp$ and $s \in \mathcal{E}$.

Definition 2.26. *If $\mathcal{E}$ an $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued Hermitian pairing, we say that a* connection *∇ on $\mathcal{E}$ is* Hermitian *if*

$$\begin{aligned} (\nabla s \mid t) + (s \mid \nabla t) &= d\,(s \mid t), \quad \textit{or, in other words,} \\ (\nabla_X s \mid t) + (s \mid \nabla_X t) &= X\,(s \mid t), \quad \textit{for any } \text{real } X \in \mathfrak{X}(M). \end{aligned}$$

If ∇, ∇' are connections on $\mathcal{E}$, then $\nabla' - \nabla$ is an $\mathcal{A}$-module map: $(\nabla' - \nabla)(fs) = f(\nabla' - \nabla)s$, so that *locally*, over $U \subset M$ for which $E\big|_U \to U$ is trivial, we can write

$$\nabla = d + \alpha, \quad \text{where } \alpha \in \mathcal{A}^1(U, \mathrm{End}\, E).$$

Fact 2.27. *On $\mathfrak{X}(M) = \Gamma(M, TM)$ there is, for each Riemannian metric g, a* unique torsion-free connection that is compatible with *g:*

$$g(\nabla X, Y) + g(X, \nabla Y) = d(g(X,Y)) \quad \textit{for } X, Y \in \mathfrak{X}(M), \quad \textit{or}$$

$$g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = Z(g(X,Y)) \quad \textit{for } X, Y, Z \in \mathfrak{X}(M).$$

The explicit formula for this connection is

$$\begin{aligned} 2g(\nabla_X Y, Z) = X(g(Y,Z)) &+ Y(g(X,Z)) - Z(g(X,Y)) \\ &+ g(Y, [Z,X]) + g(Z, [X,Y]) - g(X, [Y,Z]). \end{aligned} \tag{2.6}$$

It is called **Levi-Civita connection** associated to g. (The proof of existence consists in showing that the right hand side of this expression is $\mathcal{A}$-linear in Y and Z, and obeys a Leibniz rule with respect to X, so it gives a connection; and uniqueness is obtained by checking that metric compatibility and torsion freedom make the right hand side automatic.)

The *dual connection* on $\mathcal{A}^1(M)$ will also be called the "Levi-Civita connection". At the risk of some confusion, we shall use the same symbol ∇ for both of these Levi-Civita connections.

Local formulas From now on, we assume that $U \subset M$ is an open chart domain over which the tangent and cotangent bundles are trivial. Local coordinates are functions $x^1, \dots, x^n \in C^\infty(U)$, and we denote $\partial_j := \partial/\partial x^j \in \mathfrak{X}(M)\big|_U$ for the local basis of vector fields; by definition, their Lie brackets vanish: $[\partial_i, \partial_j] = 0$. We define the *Christoffel symbols* $\Gamma^k_{ij} \in C^\infty(U)$ by

$$\nabla_{\partial_i} \partial_j =: \Gamma^k_{ij}\, \partial_k, \quad \text{or} \quad \nabla \partial_j =: \Gamma^k_{ij}\, dx^i \otimes \partial_k.$$

The explicit expression (2.6) for the Levi-Civita connection reduces to a local formula over U, namely

$$\Gamma^k_{ij} := \tfrac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}); \qquad \text{here } [g^{rs}] = [g_{ij}]^{-1}. \tag{2.7}$$

Notice that $\Gamma^k_{ji} = \Gamma^k_{ij}$; this is beacuse of torsion freedom.

Dually, the coefficients of the Levi-Civita connection on 1-forms are $-\Gamma^k_{ij}$ (note the change of sign):

$$\nabla_{\partial_i}(dx^k) = -\Gamma^k_{ij}\, dx^j, \quad \text{or} \quad \nabla(dx^k) = -\Gamma^k_{ij}\, dx^i \otimes dx^j.$$

Since the Riemannian metric gives a concept of (fibrewise) orthogonality on the tangent and cotangent bundles, we can select *local orthonormal bases*:

$$\begin{aligned} \{E_1, \dots, E_n\} \quad &\text{for } \mathfrak{X}(M)\big|_U = \Gamma(U, TM): \quad g(E_\alpha, E_\beta) = \delta_{\alpha\beta}; \\ \{\theta^1, \dots, \theta^n\} \quad &\text{for } \mathcal{A}^1(M)\big|_U = \Gamma(U, T^*M): \quad g(\theta^\alpha, \theta^\beta) = \delta^{\alpha\beta}. \end{aligned}$$

We rewrite the Christoffel symbols in these local bases:

$$\nabla E_\alpha =: \widetilde{\Gamma}^\beta_{i\alpha}\, dx^i \otimes E_\beta, \qquad \nabla \theta^\beta = -\widetilde{\Gamma}^\beta_{i\alpha}\, dx^i \otimes \theta^\alpha.$$

Metric compatibility means that, for each fixed i, the $\widetilde{\Gamma}^{\bullet}_{i\bullet}$ are *skewsymmetric matrices*:

$$\widetilde{\Gamma}^\beta_{i\alpha} + \widetilde{\Gamma}^\alpha_{i\beta} = -g(\nabla_{\partial_i} \theta^\beta, \theta^\alpha) - g(\theta^\beta, \nabla_{\partial_i} \theta^\alpha) = -\partial_i(\delta^{\alpha\beta}) = 0.$$

Thus $\widetilde{\Gamma}$ lies in $\mathcal{A}^1(U, \mathfrak{so}(T^*M)) \simeq \mathcal{A}^1(U) \otimes_{\mathbb{R}} \mathfrak{so}(\mathbb{R}^n)$.

Definition 2.28. *On a spinor module $\mathcal{S} = \Gamma(M, S)$, a **spin^c-connection** is any* Hermitian *connection $\nabla^S \colon \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$ which is compatible with the action of B in the following way:*

$$\nabla^S(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi \quad \textit{for } \alpha \in \mathcal{A}^1(M),\ \psi \in \mathcal{S}; \quad \textit{or}$$
$$\nabla^S_X(c(\alpha)\psi) = c(\nabla_X\alpha)\psi + c(\alpha)\nabla^S_X\psi \quad \textit{for } \alpha \in \mathcal{A}^1(M),\ \psi \in \mathcal{S},\ X \in \mathfrak{X}(M), \tag{2.8}$$

where $\nabla\alpha$ and $\nabla_X\alpha$ refer to the Levi-Civita connection *on $\mathcal{A}^1(M)$.*

*If $(\mathcal{S}, C)$ are data for a spin structure, we say ∇^S is a **spin connection** if, moreover, each $\nabla_X \colon \mathcal{S} \to \mathcal{S}$ commutes with C whenever X is real.*

Before discussing existence, let us look first at local formulas. We thus write "$\nabla = d - \widetilde{\Gamma}$ locally" for the Levi-Civita connection, with an implicit choice of local orthonormal bases of 1-forms. We recall that there are isomorphisms of Lie algebras

$$\dot\mu \colon \mathfrak{so}(T^*_xM) \to Q(\Lambda^2 T^*_xM) \equiv \mathfrak{spin}(T^*_xM)$$

with the property that $\operatorname{ad}(\dot\mu(A)) = A$ for $A \in \mathfrak{so}(T^*_xM)$; in other words, $[\dot\mu(A), v] = Av$ for $v \in T^*_xM$ —this is a commutator for the Clifford product in $\mathrm{Cl}(T^*_xM, g_x)$. On the chart domain U, we can apply $\dot\mu$ to $\widetilde{\Gamma}$ fibrewise; this means that

$$[\dot\mu(\widetilde{\Gamma}), c(\alpha)] = c(\widetilde{\Gamma}\,\alpha)$$

for $\alpha \in \mathcal{A}^1(M)$ with support in U, $\widetilde{\Gamma} \in \Gamma(U, \operatorname{End} T^*M)$ is mapped to $\dot\mu(\widetilde{\Gamma}) \in \Gamma(U, \operatorname{End} S)$, and $c(\alpha)$ again denotes the Clifford action action of α on $\mathcal{S}\big|_U = \Gamma(U, S)$.

In this way we get the local expression of a connection,

$$\nabla^S := d - \dot\mu(\widetilde{\Gamma}), \quad \text{acting on} \quad \mathcal{S}\big|_U. \tag{2.9}$$

Suppose we take $\alpha \in \mathcal{A}^1(M)$ with support in U, and $\psi \in \mathcal{S}\big|_U$. Then

$$\begin{aligned}
\nabla^S(c(\alpha)\psi) &= d(c(\alpha)\psi) - \dot\mu(\widetilde{\Gamma})c(\alpha)\psi \\
&= c(d\alpha)\,\psi + c(\alpha)\,d\psi - \dot\mu(\widetilde{\Gamma})c(\alpha)\psi \\
&= c(\alpha)(d\psi - \dot\mu(\widetilde{\Gamma})\psi) + (c(d\alpha) - [\dot\mu(\widetilde{\Gamma}), c(\alpha)]\psi) \\
&= c(\alpha)\nabla^S\psi + c(d\alpha - \widetilde{\Gamma}\,\alpha)\psi \\
&= c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi.
\end{aligned} \tag{2.10}$$

Thus $\nabla^S := d - \dot\mu(\Gamma)$ provides a local solution to the existence of $\nabla^S \colon \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$ satisfying the Leibniz rule:

$$\nabla^S(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi.$$

Physicists like to write $\gamma^\alpha := c(\theta^\alpha)$ for a given local orthonormal basis of $\mathcal{A}^1(M)$ —so that the γ^α are *fixed* matrices. For convenience, we also write $\gamma_\beta = \delta_{\alpha\beta}\,\gamma^\alpha$ also (in the Euclidean signature, which we are always using here); in other words, $\gamma_\beta = \gamma^\beta$ but with its index lowered for use with the Einstein summation convention. Thus the Clifford relations are just

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta^{\alpha\beta}, \quad \text{for } \alpha, \beta = 1, \dots, n.$$

The formula (1.7) for $\dot\mu$ can now be rewritten as

$$\dot\mu(\widetilde{\Gamma}) = -\tfrac{1}{4}\widetilde{\Gamma}^\beta_{\bullet\alpha}\,\gamma^\alpha\gamma_\beta.$$

A more sensible notation arrives by introducing matrix-valued functions $\omega_1, \dots, \omega_n \in \Gamma(U, \operatorname{End} T^*M)$ as follows:

$$\omega_i := -\tfrac{1}{4}\widetilde{\Gamma}^{\beta}_{i\alpha}\, \gamma^\alpha \gamma_\beta.$$

Let us look at the calculation (2.10) again, after contracting with a vectorfield X. We get

$$\begin{aligned}[\nabla^S_X, c(\alpha)]\, \psi &= [\mathcal{L}_X - X^i \dot{\mu}(\omega_i), c(\alpha)]\, \psi \\ &= \bigl(c(\mathcal{L}_X \alpha) - X^i\, c(\omega_i \alpha)\bigr)\, \psi = c(\nabla_X \alpha)\, \psi.\end{aligned}$$

Thus the local coefficients of ∇^S_X are $-\frac{1}{4}X^i\widetilde{\Gamma}^{\beta}_{i\alpha}\,\gamma^\alpha\gamma_\beta$, for $X \in \mathfrak{X}(M)$.

Now suppose S comes from a *spin* structure on M. Since $C(\psi a) = (C\psi)\bar{a}$ for $a \in \mathcal{A} = C^\infty(M)$, the operator C acts locally (as a field of antilinear conjugations $C_x \colon S_x \to S_x$); and since $C(b) = \chi(\bar{b})C$ for $b \in B$, we get, for $\alpha, \beta = 1, \dots, n$:

$$C\gamma^\alpha\gamma^\beta = Cc(\theta^\alpha)c(\theta^\beta) = c(\theta^\alpha)c(\theta^\beta)C = \gamma^\alpha\gamma^\beta C.$$

Thus $\nabla^S_X C - C\nabla^S_X$ vanishes over U, provided $X\big|_U$ is real.

Suppose that ∇' is another local connection defined on $\mathcal{S}\big|_U$ and satisfying the Leibniz rule (2.10) there. Then

$$\nabla' - \nabla^S = \beta \in \mathcal{A}^1(U, \operatorname{End} S)$$

and $c(\kappa)\beta\psi = \beta c(\kappa)\psi$ for all $\kappa \in \mathcal{B}\big|_U$. Thus β_x is a scalar matrix in $\operatorname{End}(S_x)$, for each $x \in U$. To fix β, we ask that both ∇' and ∇^S be *Hermitian* connections; this entails that each β_x is skew-hermitian:

$$\langle \beta_x\phi_x \mid \psi_x\rangle + \langle \phi_x \mid \beta_x\psi_x\rangle = 0 \quad \text{for } x \in U,$$

so this scalar is purely imaginary. On the other hand, if ∇'_X, like ∇^S_X, commutes with C whenever the coefficients X^i of X are real functions, then this scalar must be purely real. Therefore, $\beta = 0$.

Proposition 2.29. *If $(\mathcal{S}, C)$ are data for a spin structure on M, then there is a unique Hermitian spin connection $\nabla^S \colon \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$, such that*

$$\nabla^S(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi, \quad \textit{for } \alpha \in \mathcal{A}^1(M),\ \psi \in \mathcal{S},$$

and such that $\nabla^S_X C = C\nabla^S_X$ for $X \in \mathfrak{X}(M)$ real.

Proof. We have shown that ∇^S exists locally with the recipe (2.9) on any chart domain. This recipe gives a local Hermitian connection since $\dot{\mu}(\widetilde{\Gamma})$ is skewadjoint —because the representation c is selfadjoint— and it commutes with C. Any other local connection with these properties must coincide with (2.9) over U.

Furthermore, on overlaps $U_1 \cap U_2$ of chart domains, we have shown that $\beta := \nabla^S|_{U_1} - \nabla^S|_{U_2} \in \mathcal{A}^1(U_1 \cap U_2, \operatorname{End} S)$ vanishes. Therefore, the local expressions can be assembled into a globally defined spin connection. ☐

Remark 2.30. If $\mathcal{S}$ is only a spinor module for a spin^c structure, then the uniqueness argument for the local spin connection fails. We can only conclude that $\nabla^S|_{U_1} - \nabla^S|_{U_2} = i(\alpha_1 - \alpha_2) \otimes 1_{\operatorname{End} S}$, where $\alpha_1 \in \mathcal{A}^1(U_1)$ and $\alpha_2 \in \mathcal{A}^1(U_2)$ are real 1-forms. We may be able to patch these "gauge potentials" to get a connection ∇ of a line bundle $\mathcal{L}^\sharp = \Gamma(M, L^*)$. Then one can show that on $\mathcal{S} \otimes \mathcal{L}$, there is a connection $\nabla^{S,\alpha}$ that satisfies the Leibniz rule above, and hermiticity. These are "spin^c connections" for the twisted spin^c structures.

If ∇ is any connection on an $\mathcal{A}$-module $\mathcal{E} = \Gamma(M, E)$, then

$$\nabla^2(fs) = \nabla(df \otimes s + f\,\nabla s) = \big(d(df) \otimes s - df\,\nabla s\big) + \big(df\,\nabla s + f\,\nabla^2 s\big) = f\,\nabla^2 s,$$

for $f \in \mathcal{A}$, so that ∇^2 is tensorial: $\nabla s = R\,s$ for a certain 2-form $R \in \mathcal{A}^2(M, \operatorname{End} E)$, the *curvature* of ∇. For the Levi-Civita connection, a local calculation gives

$$\nabla^2\alpha = (d - \widetilde{\Gamma})(d\alpha - \widetilde{\Gamma}\,\alpha) = -d(\widetilde{\Gamma}\,\alpha) - \widetilde{\Gamma}\,d\alpha + \widetilde{\Gamma}(\widetilde{\Gamma}\,\alpha) = (-d\widetilde{\Gamma} + \widetilde{\Gamma} \wedge \widetilde{\Gamma})\,\alpha,$$

which yields the local expression for the Riemannian curvature tensor:

$$R\big|_U = -d\widetilde{\Gamma} + \widetilde{\Gamma} \wedge \widetilde{\Gamma} \in \mathcal{A}^2(U, \mathfrak{so}(T^*M)).$$

Likewise, the curvature R^S of the *spin connection* is locally given by

$$\dot{\mu}(R) = -d\dot{\mu}(\widetilde{\Gamma}) + \dot{\mu}(\widetilde{\Gamma}) \wedge \dot{\mu}(\widetilde{\Gamma}) \in \mathcal{A}^2(U, \operatorname{End} S).$$

One can check these formulas to get more familiar expressions by computing $R(X,Y) = \iota_Y \iota_X R$ and likewise $R^S(X,Y)$, for $X, Y \in \mathfrak{X}(M)$.

2.7 Epilogue: counting the spin structures

A spin structure on (M, ε) is an equivalence class of pairs $(\mathcal{S}, C)$, but ¿what can be said about the equivalence relation?

First, $\mathcal{S}$ has a class $[\mathcal{S}] \in \operatorname{Mrt}(B, A)$: these are classified by $\mathrm{H}^2(M;\mathbb{Z})$. If $(\mathcal{S}_1, C_1)$ is another spin structure, then $C_1 \colon \mathcal{S}_1^\sharp \to \mathcal{S}_1$ comes from a B-A-bimodule isomorphism $T_1 \colon \mathcal{S}_1 \to \mathcal{S}_1$. But now $\mathcal{S}_1 \simeq \mathcal{S} \otimes \mathcal{L}$ for some $\mathcal{L}$, where $[\mathcal{L}] \in \mathrm{H}^2(M;\mathbb{Z})$ is well defined. Thus we get

$$\begin{tikzcd}
\mathcal{S}^\sharp \arrow[r, "- \otimes_A \mathcal{L}^\sharp"] \arrow[d, "T"'] & \mathcal{S}^\sharp \otimes_A \mathcal{L}^\sharp = (\mathcal{S} \otimes_A \mathcal{L})^\sharp \arrow[d, "T_1"] \\
\mathcal{S} \arrow[r, "- \otimes_A \mathcal{L}"] & \mathcal{S} \otimes_A \mathcal{L}
\end{tikzcd}$$

and therefore $(\mathcal{S}_1, C_1) \sim (\mathcal{S}, C)$ if this diagram commutes. Now

$$\begin{aligned}
\mathcal{S}_1 &\simeq \mathcal{S}_1^\sharp \otimes_A \operatorname{Hom}_B(\mathcal{S}_1^\sharp, \mathcal{S}_1) \simeq \mathcal{S}_1^\sharp \otimes_A \operatorname{Hom}_B(\mathcal{S}^\sharp \otimes_A \mathcal{L}^\sharp, \mathcal{S} \otimes_A \mathcal{L}) \\
&\simeq \mathcal{S}_1^\sharp \otimes_A \mathcal{L} \otimes_A \operatorname{Hom}_B(\mathcal{S}^\sharp, \mathcal{S}) \otimes_A \mathcal{L} \simeq \mathcal{S}_1^\sharp \otimes_A \mathcal{L} \otimes_A \mathcal{L},
\end{aligned}$$

since $\operatorname{Hom}_B(\mathcal{S}^\sharp, \mathcal{S})$ is trivial: the existence of T shows that $[\mathcal{S}^\sharp] = [\mathcal{S}]$ in $\operatorname{Mrt}(B, A)$. The conclusion is that $\mathcal{S}_1 \simeq \mathcal{S}_1^\sharp \otimes_A \mathcal{L} \otimes_A \mathcal{L}$. Thus $\mathcal{S}_1$ is also selfdual if and only if $\mathcal{L} \otimes_A \mathcal{L}$ is trivial: $(\times 2)_*[\mathcal{L}] = 0$ in $\mathrm{H}^2(M;\mathbb{Z})$. But, using the long exact sequence (2.5), we find that $\ker(\times 2)_* = \operatorname{im}\{\partial \colon \mathrm{H}^1(M;\mathbb{Z}_2) \to \mathrm{H}^2(M;\mathbb{Z})\}$.

Conclusion: Those $[\mathcal{S} \otimes_A \mathcal{L}] \in \operatorname{Mrt}(B, A)$ for which $\mathcal{L} \otimes_A \mathcal{L}$ is trivial, but $\mathcal{L}$ is not, i.e., the distinct spin structures on (M, ε), are classified by $\mathrm{H}^1(M, \mathbb{Z}_2)$.

Remark 2.31. The group $\mathrm{H}^1(M, \mathbb{Z}_2)$ is known to classify *real* line bundles over M. If a twist by $\mathcal{L}$ exchanges the spinor modules for two spin structures, there is an antilinear automorphism of $\mathcal{L}$ which matches the two charge conjugation operators, and the part of $\mathcal{L}$ fixed by this automorphism comprises the sections of the corresponding $\mathbb{R}$-line bundle over M.

Chapter 3

Dirac operators

Suppose we are given a compact oriented (boundaryless) Riemannian manifold (M,ε) and a spinor module with charge conjugation $(\mathcal{S},C)$, together with a Riemannian metric g, so that the Clifford action $c\colon \mathcal{B} \to \mathrm{End}_{\mathcal{A}}(\mathcal{S})$ has been specified. We can also write it as $\hat{c} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{S}, \mathcal{S})$ by setting $\hat{c}(\kappa \otimes \psi) := c(\kappa)\,\psi$.

Definition 3.1. *Using the inclusion $\mathcal{A}^1(M) \hookrightarrow \mathcal{B}$ —where in the odd dimensional case this is given by $c(\alpha) := c(\alpha\gamma)$, as before— we can form the composition*

$$D\!\!\!/ := -i\,\hat{c} \circ \nabla^S \tag{3.1}$$

where

$$\mathcal{S} \xrightarrow{\nabla^S} \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S} \xrightarrow{\hat{c}} \mathcal{S},$$

*so that $D\!\!\!/\colon \mathcal{S} \to \mathcal{S}$ is $\mathbb{C}$-linear. This is the **Dirac operator** associated to $(\mathcal{S},C)$ and g.*

The $(-i)$ is included in the definition to make $D\!\!\!/$ symmetric (instead of skewsymmetric) as an operator on a Hilbert space, because we have chosen g to be positive definite, that is, $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = +2\,\delta^{\alpha\beta}$. Historically, $D\!\!\!/$ was introduced as $-i\gamma^\mu\delta_\mu = \gamma^\mu p_\mu$ where the p_μ are components of a 4-momentum, but in the Minkowskian signature.

Using local (coordinate or orthonormal) bases for $\mathfrak{X}(M)$ and $\mathcal{A}^1(M)$, we get nicer formulas:

$$D\!\!\!/\psi = -i\,\hat{c}(\nabla^S\psi) = -i\,c(dx^j)\nabla^S_{\partial_j}\psi = -i\,\gamma^\alpha\nabla^S_{E_\alpha}\psi. \tag{3.2}$$

The essential algebraic property of $D\!\!\!/$ is the *commutation relation*:

$$[D\!\!\!/, a] = -i\,c(da), \quad \text{for all} \quad a \in \mathcal{A} = C^\infty(M). \tag{3.3}$$

Indeed,

$$\begin{aligned}
[D\!\!\!/, a]\,\psi &= -i\,\hat{c}(\nabla^S(a\psi)) + ia\,\hat{c}(\nabla^S\psi) \\
&= -i\,\hat{c}(\nabla^S(a\psi) - a\,\nabla^S\psi) \\
&= -i\,\hat{c}(da \otimes \psi) = -i\,c(da)\,\psi, \quad \text{for } \psi \in \mathcal{S}.
\end{aligned}$$

3.1 The metric distance property

As an operator, we can make sense of $[D\!\!\!/, a]$ by conferring on $\mathcal{S}$ the structure of a Hilbert space: if we write $\det g := \det[g_{ij}]$ for short, then

$$\nu_g := \sqrt{\det g}\,dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \mathcal{A}^n(M)$$

is the Riemannian volume form (for the given orientation ε and metric g). In the notation, we assume that all local charts are consistent with the given orientation, which just means that $\det[g_{ij}] > 0$ in any local chart. The scalar product on $\mathcal{S}$ is then given by

$$\langle \phi \mid \psi \rangle := \int_M (\phi \mid \psi)\, \nu_g \quad \text{for } \phi, \psi \in \mathcal{S}.$$

On completion in the norm $\|\psi\| := \sqrt{\langle \psi \mid \psi \rangle}$, we get the Hilbert space $\mathcal{H} := L^2(M, S)$ of L^2-spinors on M.

Using the gradient $\operatorname{grad} a := (da)^\sharp \in \mathfrak{X}(M)$, we can compute

$$\begin{aligned}
\|[D\!\!\!/, a]\|^2 = \|c(da)\|^2 &= \sup_{x \in M} \|c_x(da(x))\|^2 \\
&= \sup_{x \in M} g_x(d\bar{a}(x), da(x)) \quad \text{with } g_x \text{ on } \ (T_x^*M)^{\mathbb{C}} \\
&= \sup_{x \in M} g_x\big(\operatorname{grad} \bar{a}\big|_x, \operatorname{grad} a\big|_x\big) \quad \text{using the dual } g_x \text{ on } \ (T_xM)^{\mathbb{C}} \\
&=: \sup_{x \in M} \big\|\operatorname{grad} a\big|_x\big\|^2 =: \|\operatorname{grad} a\|_\infty^2.
\end{aligned}$$

Classically, we compute distances on a (connected) Riemannian manifold by the formula

$$d(x, y) := \inf\{\, \operatorname{length}(\gamma) : \gamma \colon [0,1] \to M;\ \gamma(0) = x,\ \gamma(1) = y \,\},$$

with the infimum taken over all piecewise-smooth paths γ in M from x to y. For $a \in C^\infty(M)$, we then get

$$\begin{aligned}
a(y) - a(x) = a(\gamma(1)) - a(\gamma(0)) &= \int_0^1 \frac{d}{dt}[a(\gamma(t))]\, dt \\
&= \int_0^1 \dot\gamma(a)\big|_{\gamma(t)}\, dt = \int_0^1 da(\dot\gamma)\big|_{\gamma(t)}\, dt = \int_0^1 da_{\gamma(t)}(\dot\gamma(t))\, dt \\
&= \int_0^1 g_{\gamma(t)}\big(\operatorname{grad} a\big|_{\gamma(t)}, \dot\gamma(t)\big)\, dt,
\end{aligned}$$

and we can estimate this difference by

$$\begin{aligned}
|a(y) - a(x)| &\le \int_0^1 \big|\operatorname{grad} a\big|_{\gamma(t)}\big|\, |\dot\gamma(t)|\, dt \\
&\le \|\operatorname{grad} a\|_\infty \int_0^1 |\dot\gamma(t)|\, dt = \|\operatorname{grad} a\|_\infty \operatorname{length}(\gamma) \\
&= \|[D\!\!\!/, a]\| \operatorname{length}(\gamma).
\end{aligned}$$

Thus

$$\sup\{\, |a(y) - a(x)| : a \in C(M),\ \|[D\!\!\!/, a]\| \le 1 \,\} \le \inf_\gamma \operatorname{length}(\gamma) =: d(x, y). \tag{3.4}$$

In this supremum, we can use $a \in C(M)$ not necessarily smooth; a need only be continuous with $\operatorname{grad} a$ (ν-essentially) bounded. Since we have obtained $|a(y) - a(x)| \le \|\operatorname{grad} a\|_\infty d(x, y)$, we see that a need only be *Lipschitz* on M —with respect to the distance d— with Lipschitz constant $\le \|\operatorname{grad} a\|_\infty$. In fact, this is the best general Lipschitz constant: fix $x \in M$, and set $a_x(y) := d(x, y)$. This function lies in $C(M)$, and $|a_x(y) - a_x(z)| \le d(y, z)$ by the triangle inequality for d. Since $\|\operatorname{grad} a_x\|_\infty = 1$ by a local geodesic calculation, we see that $a = a_x$ makes the inequality in (3.4) sharp:

$$\begin{aligned} d(x,y) &= \sup\{\, |a(y) - a(x)| : \|\operatorname{grad} a\|_\infty \leq 1 \,\} \\ &= \sup\{\, |a(y) - a(x)| : a \in C(M),\ \|[D\!\!\!/, a]\| \leq 1 \,\}, \end{aligned} \tag{3.5}$$

so that $D\!\!\!/$ determines the Riemannian distance d, which in turn determines the metric g. (The Myers–Steenrod theorem of differential geometry says that g is uniquely determined by its distaqnce function d.)

Example 3.2. Take $M = \mathbb{S}^1$ ($n = 1$, $m = 0$, $2^m = 1$). The trivial line bundle is a spinor bundle, with $\mathcal{S} = C^\infty(\mathbb{S}^1) = \mathcal{A}$, and C is just the complex conjugation K of functions. With the *flat* metric on $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$, we can identify $\mathcal{S}$ with the set of smooth 1-periodic functions on $\mathbb{R}$, so both ∇ and ∇^S are trivial since $\Gamma^1_{11} = 0$. Therefore,

$$D\!\!\!/ = -i \frac{d}{d\theta}$$

is the Dirac operator in this case. Thus $[D\!\!\!/, f] = -if'$ for $f \in \mathcal{A}$, and for $\alpha, \beta \in [0,1]$, we get

$$|f(\beta) - f(\alpha)| = \left| \int_\alpha^\beta f'(\theta)\, d\theta \right| \leq \int_\alpha^\beta |f'(\theta)|\, d\theta \leq |\beta - \alpha| \quad \text{whenever } \|f'\|_\infty \leq 1.$$

Using $f_\alpha(\beta) := |\beta - \alpha|$ for $\alpha - \frac{1}{2} \leq \beta \leq \alpha + \frac{1}{2}$ wrapped around $\mathbb{R}/\mathbb{Z}$, we get a Lipschitz function making the inequality sharp. Thus $d(\alpha, \beta) = |\beta - \alpha|$ provided $|\beta - \alpha| \leq \frac{1}{2}$: this is just the *arc length* on the circle of circumference 1.

More generally, the formula for $d(x,y)$ yields the length of the minimal geodesic from x to y, provided y is closer to x, than the "cut-locus" of x.

3.2 Symmetry of the Dirac operator

We now regard $D\!\!\!/$ as an operator on $L^2(M,S)$, defined initially on the dense domain $\mathcal{S} = \Gamma_{\mathrm{smooth}}(M,S)$.

Proposition 3.3. *$D\!\!\!/$ is* symmetric*: that is, whenever $\phi, \psi \in \mathcal{S}$, the following equality holds:*

$$\langle D\!\!\!/\phi \mid \psi \rangle = \langle \phi \mid D\!\!\!/\psi \rangle.$$

Proof. We compute the pairings $(D\!\!\!/\phi \mid \psi)$ and $(\phi \mid D\!\!\!/\psi)$, which take values in $\mathcal{A} = C^\infty(M)$. We need a formula for the *divergence* of a vector field: $\mathcal{L}_X \nu_g =: (\operatorname{div} X)\, \nu_g$, so that

$$\int_M (\operatorname{div} X)\, \nu_g = \int_M \mathcal{L}_X(\nu_g) = \int_M \iota_X(d\nu_g) + d(\iota_X \nu_g) = \int_M d(\iota_X \nu_g) = 0$$

by Stokes' theorem (remember that M has no boundary). This formula is

$$\operatorname{div} X = \partial_j X^j + \Gamma^j_{jk} X^k = dx^j(\nabla_{\partial_j} X),$$

as can easily be checked; on the right hand side we use the Levi-Civita connection on $\mathfrak{X}(M)$. Now we abbreviate $c^j := c(dx^j) \in \Gamma(U, \operatorname{End} S)$, for $j = 1, \dots, n$. Then we compute the difference of $\mathcal{A}$-valued pairings:

$$\begin{aligned} i(\phi \mid D\!\!\!/\psi) - i(D\!\!\!/\phi \mid \psi) &= (\phi \mid c^j\, \nabla^S_{\partial_j} \psi) + (c^j\, \nabla^S_{\partial_j} \phi \mid \psi) \\ &= (\phi \mid \nabla^S_{\partial_j} c^j\, \psi) - (\phi \mid c(\nabla_{\partial_j} dx^j)\psi) + (\nabla^S_{\partial_j} \phi \mid c^j\, \psi) \\ &= \partial_j (\phi \mid c^j\, \psi) - (\phi \mid c(\nabla_{\partial_j} dx^j)\psi). \end{aligned}$$

Here we have used the Leibniz rule for ∇^S, the selfadjointness of c^j since dx^j is a real local 1-form, and the hermiticity of ∇^S.

By duality, the map $\alpha \mapsto (\phi \mid c(\alpha)\psi)$, which takes 1-forms to functions, defines a vector field $Z_{\phi\psi}$ —because $\mathfrak{X}(M) = \mathrm{End}_{C^\infty(M)}(\mathcal{A}^1(M), C^\infty(M))$— so the right hand side becomes

$$\partial_j(dx^j(Z_{\phi\psi})) - (\nabla_{\partial_j} dx^j)(Z_{\phi\psi}) = dx^j(\nabla_{\partial_j} Z_{\phi\psi}) = \operatorname{div} Z_{\phi\psi},$$

where we have used the Leibniz rule for the dual Levi-Civita connections on $\mathcal{A}^1(M)$ and on $\mathfrak{X}(M)$, respectively. Thus

$$(\phi \mid D\!\!\!/\psi) - (D\!\!\!/\phi \mid \psi) = -i \operatorname{div} Z_{\phi\psi}$$

which has integral zero. □

3.3 Selfadjointness of the Dirac operator

If T is a densely defined operator on a Hilbert space $\mathcal{H}$, its *adjoint* T^* has domain

$$\mathrm{Dom}\, T^* := \{\, \phi \in \mathcal{H} : \exists\, \chi \in \mathcal{H} \text{ with } \langle T\psi \mid \phi\rangle = \langle \psi \mid \chi\rangle \text{ for all } \psi \in \mathrm{Dom}\, T \,\}$$

and then $T^*\phi := \chi$, of course, so that the formula $\langle T\psi|\phi\rangle = \langle \psi|T^*\phi\rangle$ holds. If T is *symmetric*, then clearly $\mathrm{Dom}\, T \subseteq \mathrm{Dom}\, T^*$ with $T^* = T$ on $\mathrm{Dom}\, T$: that is, T^* is an *extension* of T to a larger domain.

The second adjoint $T^{**} =: \overline{T}$ is called the *closure* of T (symmetric operators always have this closure), where the domain of the closure is

$$\begin{aligned}\mathrm{Dom}\, \overline{T} := \{\, \psi \in \mathcal{H} : \exists\, \phi \in \mathcal{H} &\text{ and a sequence } \{\psi_n\} \subset \mathrm{Dom}\, T, \\ &\text{such that } \psi_n \to \psi \text{ and } T\psi_n \to \phi \text{ in } \mathcal{H} \,\}\end{aligned}$$

In other words, the graph of $\overline{T}$ in $\mathcal{H} \oplus \mathcal{H}$ is the closure of the graph of T. And then, of course, we put $\overline{T}\psi := \phi$. When T is symmetric, we get

$$\mathrm{Dom}\, T \subseteq \mathrm{Dom}\, \overline{T} \subseteq \mathrm{Dom}\, T^*.$$

Definition 3.4. *We say that T is* ***selfadjoint*** *if $T = T^*$; thus T is symmetric and closed. Otherwise, we say that T is* ***essentially selfadjoint*** *if it is symmetric and its closure $\overline{T}$ is selfadjoint.*

Remark 3.5. Selfadjoint operators have *real spectra*: $\mathrm{sp}(T) \subseteq \mathbb{R}$. This is crucial: an unbounded operator that is merely symmetric may have non-real elements in its spectrum. Moreover, selfadjoint operators obey the *spectral theorem*: $T = \int_{\mathbb{R}} \lambda\, dE_T(\lambda)$, where E_T is a "projector-valued measure" on Borel subsets of $\mathbb{R}$ with support in $\mathrm{sp}(T)$.

The main result of this chapter is that the Dirac operator on a compact Riemannian spin manifold is essentially selfadjoint. This was proved by Wolf in 1973; he actually showed the result also for noncompact manifolds which are *complete* with respect to the Riemannian distance given by the metric [Wolf]. In his proof, completeness is needed to establish that closed geodesic balls are compact; that proof is also given in the book by Friedrich [Fri]. For simplicity, we deal here only with the compact case.

Theorem 3.6. *Let (M, g) be a compact boundaryless Riemannian spin manifold. The Dirac operator $D\!\!\!/$ is essentially selfadjoint on its original domain $\mathcal{S}$.*

Proof. There is a natural norm on $\operatorname{Dom} D\!\!\!/^*$, given by

$$|||\psi|||^2 := \|\psi\|^2 + \|D\!\!\!/^*\psi\|^2.$$

We claim that $\mathcal{S} = \Gamma_{\text{smooth}}(M, S)$ is dense in $\operatorname{Dom} D\!\!\!/^*$ for this norm. Using a finite partition of unity $f_1 + \cdots + f_r = 1$ with each $f_i \in \mathcal{A}$ supported in a chart domain U_i over which $S|_{U_i} \to U_i$ is trivial, it is enough to show that any $f_i\phi$, with $\phi \in \operatorname{Dom} D\!\!\!/^*$, can be approximated in the $|||\cdot|||$-norm by elements of $\Gamma_{\text{smooth}}(U_i, S)$. Thus we can suppose that $\operatorname{supp}\phi \subset U_i$, and regard $\phi \in L^2(U_i, S)$ as a 2^m-tuple of functions $\phi = \{\phi_k\}$ with each $\phi_k \in L^2(U_i, \nu_g)$.

Previous formulas now show that

$$\begin{aligned}\langle D\!\!\!/^*\phi \mid \psi\rangle = \langle \phi \mid D\!\!\!/\psi\rangle &= \int_M (\phi \mid c^j\, \nabla^S_{\partial_j}\psi) = \int (c^j\phi \mid \nabla^S_{\partial_j}\psi) \\ &= \int \big[\partial_j(c^j\,\phi \mid \psi) - (\nabla^S_{\partial_j} c^j\phi \mid \psi)\big]\,\nu_g \\ &= \int_M \big[-(c^j\phi \mid \psi)\,(\operatorname{div}\partial_j) - (\nabla^S_{\partial_j} c^j\phi \mid \psi)\big]\,\nu_g\end{aligned}$$

after an integration by parts, so that $D\!\!\!/^*$ is given by the formula $D\!\!\!/^* = -(\nabla^S_{\partial_j} + \operatorname{div}\partial_j)\,c(dx^j)$, as a vector-valued distribution on U_i; in particular, it is also a differential operator (the difference $D\!\!\!/^* - D\!\!\!/$ will soon be seen to vanish).

Now, if $\{h_r\}$ is a smooth delta-sequence, then for large enough r we can convolve both ϕ and $D\!\!\!/^*\phi$ with h_r, while remaining supported in U_i —the convolution is defined after pulling back functions on the chart domain U_i to an fixed open subset of $\mathbb{R}^n$. Thus we find that $\phi * h_r \to \phi$ and $D\!\!\!/^*(\phi * h_r) \to D\!\!\!/^*\phi$ in $L^2(U_i, \nu_g)^{2^m}$, so that $|||\phi * h_r - \phi||| \to 0$. But the spinors $\phi * h_r$ are smooth since the h_r are smooth, so we conclude that $\mathcal{S}$ is $|||\cdot|||$-dense in $\operatorname{Dom} D\!\!\!/^*$.

But now $D\!\!\!/^*(\phi * h_r) = D\!\!\!/(\phi * h_r)$ since $\mathcal{S} = \operatorname{Dom} D\!\!\!/$, so we have shown that ϕ lies in $\operatorname{Dom} \overline{D\!\!\!/}$ and that $\overline{D\!\!\!/}\phi = D\!\!\!/^*\phi$. Thus $\operatorname{Dom}\overline{D\!\!\!/} = \operatorname{Dom} D\!\!\!/^*$, and it follows that $\overline{D\!\!\!/} = D\!\!\!/^{**} = D\!\!\!/^*$: which establishes that $\overline{D\!\!\!/}$ is selfadjoint. □

3.4 The Schrödinger–Lichnerowicz formula

If $E \to M$ is any smooth vector bundle with connection ∇^E on $\mathcal{E} = \Gamma(M, E)$, we can consider not only $\nabla^E \colon \mathcal{E} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$, but also the connection $\nabla^{E'} := \nabla \otimes 1 + 1 \otimes \nabla^E$ on the tensor product bundle $\mathcal{E}' = \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$; here ∇ is once again the Levi-Civita connection on $\mathcal{A}^1(M)$. Their composition is an operator $\nabla^{E'} \circ \nabla^E$ from $\mathcal{E}$ to $\mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$; using the metric g on $\mathcal{A}^1(M)$ we can take the trace over the first two factors, ending up with a **Laplacian**:

$$\Delta^E := -\operatorname{Tr}_g(\nabla^{E'} \circ \nabla^E) : \mathcal{E} \to \mathcal{E}. \tag{3.6}$$

The minus sign is a convention to yield a positive operator (instead of a negative one) [BGV]. Locally, this means:

$$\Delta^E = -g^{ij}(\nabla^E_{\partial_i}\nabla^E_{\partial_j} - \Gamma^k_{ij}\,\nabla^E_{\partial_k}).$$

Definition 3.7. *In particular, when $E = M \times \mathbb{C}$ is the trivial line bundle, we get the "scalar Laplacian"*

$$\Delta = -g^{ij}(\partial_i\,\partial_j - \Gamma^k_{ij}\,\partial_k), \tag{3.7}$$

also known as the "Laplace–Beltrami operator" on $\mathcal{A} = C^\infty(M)$. Likewise, when $E = S$, we get the ***spinor Laplacian*** *for a spin manifold.*

Before examining the relation between the Dirac operator and the spinor Laplacian, we collect a few well-known formulas for the *Riemann curvature tensor*, R. These can be found in many places, for instance [BGV]; perhaps the best reference is Milnor's little book [Mil].

The square of the Levi-Civita connection on $\mathfrak{X}(M)$ is $C^\infty(M)$-linear, so it is given by $\nabla^2 X = R(X)$, where $R \in \mathcal{A}^2(M, \operatorname{End} TM)$. In local coordinates, its components are $R_{ijkl} := g(\partial_i, R(\partial_k, \partial_l)\,\partial_j)$.

Taking a trace over the first and third indices, we get the *Ricci tensor*, whose components are $R_{jl} := g^{ik} R_{ijkl}$. The trace of the Ricci tensor is the *scalar curvature* (or "curvature scalar") $s := g^{jl} R_{jl} = g^{jl} g^{ik} R_{ijkl} \in C^\infty(M)$. Under exchange of indices, R has the following skewsymmetry and symmetry relations:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}; \qquad R_{ijkl} = R_{klij}.$$

The (first) *Bianchi identity* says that the cyclic sum over three indices vanishes:

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0.$$

Moreover, the Ricci tensor is symmetric: $R_{jl} = R_{lj}$.

The formula in the next Proposition is generally attributed to Lichnerowicz [Lich, 1963], but was anticipated by Schrödinger in a little-known paper [Sch1, 1932].

Proposition 3.8. *Let (M, g) be a compact Riemannian spin manifold with spinor module $\mathcal{S}$. Then*

$$\not{D}^2 = \Delta^S + \tfrac{1}{4} s \tag{3.8}$$

as an operator on $\mathcal{S}$,

Proof. It is enough to prove the equality when applied to spinors ψ supported in a chart domain, so we may use local coordinate formulas. Since $\not{D} = -i\, c^j\, \nabla^S_{\partial_j}$, we get

$$\begin{aligned}
\not{D}^2 &= -c^i\, \nabla^S_{\partial_i} c^j\, \nabla^S_{\partial_j} = -c^i c^j\, \nabla^S_{\partial_i} \nabla^S_{\partial_j} - c^i c(\nabla_{\partial_i} dx^k)\, \nabla^S_{\partial_k} \\
&= -c^i c^j\, (\nabla^S_{\partial_i} \nabla^S_{\partial_j} - \Gamma^k_{ij}\, \nabla^S_{\partial_k}),
\end{aligned}$$

and from $\Gamma^k_{ij} = \Gamma^k_{ji}$ (torsion freedom) and the Clifford relation $c^i c^j + c^j c^i = 2g^{ij}$, we get

$$\begin{aligned}
\not{D}^2 &= -g^{ij} (\nabla^S_{\partial_i} \nabla^S_{\partial_j} - \Gamma^k_{ij}\, \nabla^S_{\partial_k}) - \tfrac{1}{2} c^i c^j\, [\nabla^S_{\partial_i}, \nabla^S_{\partial_j}] \\
&= \Delta^S - \tfrac{1}{2} c^i c^j\, [\nabla^S_{\partial_i}, \nabla^S_{\partial_j}].
\end{aligned}$$

Since $[\partial_i, \partial_j] = 0$, the commutator $[\nabla^S_{\partial_i}, \nabla^S_{\partial_j}]$ is a spin-curvature term:

$$[\nabla^S_{\partial_k}, \nabla^S_{\partial_l}] = R^S(\partial_k, \partial_l) = \tfrac{1}{4} R_{ijkl}\, c^i c^j,$$

because the curvature R^S of ∇^S is given by $\dot{\mu}(R)$. Hence,

$$\not{D}^2 - \Delta^S = -\tfrac{1}{8} R_{ijkl}\, c^k c^l c^i c^j = \tfrac{1}{8} R_{jikl}\, c^k c^l c^i c^j. \tag{3.9}$$

Since R_{jikl} has cyclic sum zero in the indices i, k, l, we can also skewsymetrize $c^k c^l c^i = c(dx^k)\, c(dx^l)\, c(dx^i)$. It is a simple exercise to check that

$$c^k c^l c^i = Q(dx^k \wedge dx^l \wedge dx^i) + g^{li} c^k - g^{ki} c^l + g^{kl} c^i,$$

If we now skewsymmetrize the right hand side of (3.9) in the indices i, k, l —which does not change its value— the Q-term contributes zero to the result. Also, the term $g^{kl} c^i c^j = g^{lk} c^i c^j$ contributes zero, while $g^{li} c^k c^j$ and $-g^{ki} c^l c^j$ contribute equally. Thus,

$$\begin{aligned} D\!\!\!/^2 - \Delta^S &= \tfrac14 R_{ijkl} g^{ik} c^l c^j = \tfrac14 R_{jl} c^l c^j = \tfrac18 R_{jl} (c^l c^j + c^j c^l) \\ &= \tfrac14 R_{jl} g^{jl} = \tfrac14 s. \end{aligned} \qquad \square$$

One consequence of the formula (3.8) is a famous "vanishing theorem" of Lichnerowicz.

Corollary 3.9. *If $s(x) \geq 0$ for all $x \in M$, and $s(x_0) > 0$ at some point $x_0 \in M$, then* $\ker D\!\!\!/ = \{0\}$.

Proof. Suppose that $\psi \in \mathcal{S}$ satisfies $D\!\!\!/\psi = 0$. Then

$$0 = \|D\!\!\!/\psi\|^2 = \langle \psi \mid D\!\!\!/^2 \psi \rangle = \langle \psi \mid \Delta^S \psi \rangle + \int_M \tfrac14 s \, (\psi \mid \psi) \, \nu_g. \tag{3.10}$$

Now it is easy to check that, after an integration by parts over M and discarding a divergence term,

$$\langle \psi \mid \Delta^S \psi \rangle = g^{ij} \, \langle \nabla^S_{\partial_i} \psi \mid \nabla^S_{\partial_j} \psi \rangle. \tag{3.11}$$

Since the matrix $[g^{ij}]$ is positive definite, this (by the way) shows that Δ^S is a positive operator; and since $s \geq 0$, both terms on the right hand side of (3.10) are nonnegative; so they must both vanish, since their sum is zero.

Moreover, (3.11) shows that $\langle \psi \mid \Delta^S \psi \rangle = 0$ implies $\nabla^S \psi = 0$. This in turn implies that $\partial_j \, (\psi \mid \psi) = (\nabla^S_{\partial_j} \psi \mid \psi) + (\psi \mid \nabla^S_{\partial_j} \psi)$ vanishes for each j, so that $k := (\psi \mid \psi)$ is a constant function. But now (3.10) reduces to $0 = k \int_M s \, \nu_g$, which entails $k = 0$ and then $\psi = 0$. $\square$

We saw by example (Appendix A.2) that on $\mathbb{S}^2$, the Dirac operator for the round metric has spectrum $\operatorname{sp}(D\!\!\!/) = \mathbb{N} \setminus \{0\}$: here $s \equiv 2$ and $\ker D\!\!\!/ = \{0\}$. Thus there are no "harmonic spinors" on $\mathbb{S}^2$.

3.5 The spectral growth of the Dirac operator

Since $D\!\!\!/^2 = \Delta^S + \frac14 s$, and Δ^S is closely related to the Laplacian Δ on the (compact, boundaryless) Riemannian manifold (M, g), the general features of $\operatorname{sp}(D\!\!\!/)$ may be deduced from those of $\operatorname{sp}(\Delta)$.

We require two main properties of Laplacians on compact Riemannian manifolds without boundary. Recall that

$$\Delta = -\operatorname{Tr}_g(\nabla^{T^*M \otimes T^*M} \circ \nabla^{T^*M}) = -g^{ij} \, (\partial_i \, \partial_j - \Gamma^k_{ij} \, \partial_k)$$

is the local expression for the Laplacian (which depends on g through the Levi-Civita connection and g^{ij}). Thus Δ is a second order differential operator on $C^\infty(M)$.

Fact 3.10. *The Laplacian Δ extends to a positive selfadjoint operator on $L^2(M, \nu_g)$ —also denoted by Δ— and $(1 + \Delta)$ has a compact inverse.*

To make Δ selfadjoint, we must complete $C^\infty(M)$ to a larger domain, by defining

$$|\!|\!| f |\!|\!|^2 := \langle f \mid (1 + \Delta) f \rangle = \|f\|^2 + g^{ij} \, \langle \partial_i f \mid \partial_j f \rangle,$$

where $\langle f \mid f \rangle := \int_M |f|^2 \, \nu_g$. Taking $\operatorname{Dom} \Delta := \{ f \in L^2(M, \nu_g) : |||f||| < \infty \}$, Δ becomes selfadjoint and $(1 + \Delta)^{-1} \colon L^2(M, \nu_g) \to (\operatorname{Dom} \Delta, |||\cdot|||)$ is bounded. Then one shows that the inclusion $(\operatorname{Dom} \Delta, |||\cdot|||) \hookrightarrow L^2(M, \nu_g)$ is a compact operator (by Rellich's theorem); and $(1 + \Delta)^{-1}$, as a bounded operator on $L^2(M, \nu_g)$, is then the composition of these two, so it is also compact.

Corollary 3.11. *Δ has discrete (point) spectrum of finite multiplicity.*

Proof. Since $(1+\Delta)^{-1}$ is compact, its spectrum —except for 0— consists only of eigenvalues of finite multiplicity. Therefore, the same is true of $1 + \Delta$, and of Δ itself. Indeed,

$$\operatorname{sp}((1 + \Delta)^{-1}) = \left\{ \frac{1}{1 + \lambda_0}, \frac{1}{1 + \lambda_1}, \frac{1}{1 + \lambda_2}, \dots \right\}$$

with $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ being the list of eigenvalues of Δ in increasing order. These are *counted with multiplicity*: an eigenvalue of multiplicity r appears exactly r times on the list. This $\lambda_k \to \infty$, since $(1 + \lambda_k)^{-1} \to 0$, as $k \to \infty$. □

As a convention, when A is a compact positive operator, we write $\lambda_k(A)$ to denote the k-th eigenvalue of A in decreasing order (with multiplicity): $\lambda_0(A) \geq \lambda_1(A) \geq \cdots$; on the other hand, if A is an unbounded positive selfadjoint operator with compact inverse, we write the eigenvalues in increasing order, as we did for Δ.

Fact 3.12 (Weyl's theorem). *The counting function for* $\operatorname{sp}(\Delta)$ *is*

$$N_\Delta(\lambda) := \#\{ \lambda_k(\Delta) : \lambda_k(\Delta) \leq \lambda \}.$$

For large λ, the following asymptotic estimate holds:

$$N_\Delta(\lambda) \sim C_n \operatorname{Vol}(M) \, \lambda^{n/2} \quad \text{as } \lambda \to \infty,$$

where $n = \dim M$, and $\operatorname{Vol}(M) = \int_M \nu_g$ is the total volume of the manifold M. The constant C_n, that depends only on the dimension n, is

$$C_n = \frac{\Omega_n}{n(2\pi)^n} = \frac{1}{(4\pi)^{n/2} \Gamma(\frac{n}{2} + 1)},$$

where $\Omega_n = \operatorname{Vol}(\mathbb{S}^{n-1}) = 2 \, \pi^{n/2} / \Gamma(\frac{n}{2})$.

We shall not prove Weyl's theorem, in particular why the number of eigenvalues (up to λ) is proportional to $\operatorname{Vol}(M)$, but we shall compute the constant by considering an example. For a simple and clear exposition of the proof, we recommend Higson's ICTP lectures [Hig].

Example 3.13. Take $M = \mathbb{T}^n = \mathbb{R}^n / bZ^n$ to be the n-torus with unit volume. Identify $C^\infty(\mathbb{T}^n)$ with the smooth periodic functions on the unit cube $[0, 1]^n$. For the flat metric on $\mathbb{T}^n$, and local coordinates $t = (t^1, \dots, t^n)$, we get

$$\Delta = -\left(\frac{\partial}{\partial t^1} \right)^2 - \left(\frac{\partial}{\partial t^2} \right)^2 - \cdots - \left(\frac{\partial}{\partial t^n} \right)^2.$$

Thus we find eigenfunctions, labelled by $r = (r_1, \dots, r_n) \in \mathbb{Z}^n$:

$$\phi_r := e^{2\pi i r \cdot t}, \quad \text{for } t \in [0, 1]^n.$$

Since $\{ \phi_r : r \in \mathbb{Z}^n \}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$, these are a complete set of eigenfunctions, and therefore

$$\operatorname{sp}(\Delta) = \{ 4\pi^2 \, |r|^2 : r \in \mathbb{Z}^n \}.$$

If $B(0; R)$ is the ball of radius R, centered at $0 \in \mathbb{R}^n$, then

$$\begin{aligned} N_\Delta(\lambda) &= \#\{ r \in \mathbb{Z}^n : 4\pi^2 |r|^2 \leq \lambda \} \\ &\sim \operatorname{Vol}(B(0; \sqrt{\lambda/4\pi^2})) = \left(\frac{\lambda}{4\pi^2}\right)^{n/2} \operatorname{Vol}(B(0;1)) \\ &= \frac{\lambda^{n/2}}{(2\pi)^n} \frac{\Omega_n}{n} = \frac{\Omega_n}{n(2\pi)^n} \lambda^{n/2}, \quad \text{as } \lambda \to \infty, \end{aligned}$$

using

$$\operatorname{Vol}(B(0;1)) = \int_{B(0;1)} dx^1 \wedge \cdots \wedge dx^n = \int_0^1 \left(\int_{\mathbb{S}^{n-1}} \nu \right) r^{n-1} \, dr = \Omega_n \int_0^1 r^{n-1} \, dr = \frac{\Omega_n}{n}.$$

For the spinor Laplacian Δ^S, a similar estimate holds, but with C_n replaced by $2^m C_n$ (recall that in the flat torus case with untwisted spin structure, $\mathcal{S} \simeq C^\infty(\mathbb{T}^n) \otimes \mathbb{C}^{2^m}$). Now by Lichnerowicz' formula, $D\!\!\!/^2$ differs from Δ^S by a bounded multiplication operator $\frac{1}{4}s$, thus $N_{D\!\!\!/^2}(\lambda) \sim N_{\Delta^S}(\lambda)$ as $\lambda \to \infty$, hence

$$N_{D\!\!\!/^2}(\lambda) \sim \frac{2^m \, \Omega_n}{n(2\pi)^n} \operatorname{Vol}(M) \, \lambda^{n/2}, \quad \text{as } \lambda \to \infty.$$

Consider the positive operator $|D\!\!\!/| := (D\!\!\!/^2)^{1/2}$; remember that μ is an eigenvalue for $|D\!\!\!/|$ if and only if μ^2 is an eigenvalue for $D\!\!\!/^2$ (with the same multiplicity). We arrive at the following estimate.

Corollary 3.14.

$$N_{|D\!\!\!/|}(\lambda) \sim \frac{2^m \, \Omega_n}{n(2\pi)^n} \operatorname{Vol}(M) \, \lambda^n, \quad \textit{as } \lambda \to \infty. \qquad \square$$

Example 3.15. For $M = \mathbb{S}^2$, with $n = 2$, we have seen (in Appendix A.2) that

$$\begin{aligned} \operatorname{sp}(D\!\!\!/) &= \{ \pm(l + \tfrac{1}{2}) : l + \tfrac{1}{2} \in \mathbb{N} + \tfrac{1}{2} \}, \quad \text{with multiplicities } 2l+1 \\ &= \{ \pm k : k = 1, 2, 3, \dots \}, \quad \text{with multiplicities } 2k. \end{aligned}$$

Therefore

$$N_{|D\!\!\!/|}(\lambda) = \sum_{1 \leq k \leq \lambda} 4k = 2\lfloor\lambda\rfloor(\lfloor\lambda\rfloor + 1) \sim 2\lambda(\lambda+1) \sim 2\lambda^2, \quad \text{as } \lambda \to \infty.$$

Now $C_2 = \dfrac{\Omega_2}{2(2\pi)^2} = \dfrac{2\pi}{8\pi^2} = \dfrac{1}{4\pi}$ and $2C_2 = \dfrac{1}{2\pi}$ for spinors. Therefore $2C_2 \operatorname{Area}(\mathbb{S}^2)\, \lambda^2 = 2\lambda^2$, so

$$\operatorname{Area}(\mathbb{S}^2) = \frac{1}{C_2} = 4\pi.$$

In other words, Weyl's theorem allows us to deduce the area of the 2-sphere $\mathbb{S}^2$ from (the knowledge of the circumference of the circle $\Omega_2 = 2\pi$ and) the growth of the spectrum of the Dirac operator on $\mathbb{S}^2$.

Chapter 4

Spectral Growth and Dixmier Traces

4.1 Definition of spectral triples

We start with the definition of the main concept in noncommutative geometry.

Definition 4.1. *A (unital)* **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ *consists of:*

- *an* algebra $\mathcal{A}$ *with an involution* $a \mapsto a^*$*, equipped with a faithful representation on:*
- *a* Hilbert space $\mathcal{H}$*; and also*
- *a* selfadjoint operator D *on* $\mathcal{H}$*, with dense domain* $\operatorname{Dom} D \subset \mathcal{H}$*, such that* $a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ *for all* $a \in \mathcal{A}$*,*

satisfying the following two conditions:

- *the operator* $[D, a]$*, defined initially on* $\operatorname{Dom} D$*, extends to a* bounded operator *on* $\mathcal{H}$*, for each* $a \in \mathcal{A}$*;*
- D *has* compact resolvent*:* $(D - \lambda)^{-1}$ *is compact, when* $\lambda \notin \operatorname{sp}(D)$*.*

For now, and until further notice, all spectral triples will be defined over unital algebras. The compact-resolvent condition must be modified if $\mathcal{A}$ is nonunital: as well as enlarging $\mathcal{A}$ to a unital algebra, we require only that the products $a(D-\lambda)^{-1}$, for $a \in \mathcal{A}$ and $\lambda \notin \operatorname{sp}(D)$, be compact operators.

Example 4.2. Let (M, ε) be an oriented compact boundaryless manifold which is spin, i.e. admits spin structures, and $(\mathcal{S}, C)$ be data for a specific spin structure. Choose a Riemannian metric g on M (which allows us to define ∇ and ∇^S) and let $D\!\!\!\!/ = -i\,\hat{c} \circ \nabla^S$ be the corresponding Dirac operator, extended to be a selfadjoint operator on $L^2(M, S)$. Then $(C^\infty(M), L^2(M, S), D\!\!\!\!/)$ is a spectral triple. Here $[D\!\!\!\!/, a] = -i\,c(da)$ is a bounded operator on spinors, with $\|[D\!\!\!\!/, a]\| = \|\operatorname{grad} a\|_\infty$, for $a \in C^\infty(M)$. We know by now that $(D\!\!\!\!/^2 + 1)^{-1} = (D\!\!\!\!/ - i)^{-1}(D\!\!\!\!/ + i)^{-1}$ is compact, so $(D\!\!\!\!/ \pm i)^{-1}$ is compact. We refer to these spectral triples as "standard commutative examples".

Note that, if $\lambda, \mu \notin \operatorname{sp}(D)$, then $(D-\lambda)^{-1} - (D-\mu)^{-1} = (\lambda - \mu)(D-\lambda)^{-1}(D-\mu)^{-1}$ —this is the famous "resolvent equation"— since

$$(D-\lambda)\big((D-\lambda)^{-1} - (D-\mu)^{-1}\big)(D-\mu) = (D-\mu) - (D-\lambda) = \lambda - \mu.$$

Thus $(D-\lambda)^{-1}$ is compact if and only if $(D-\mu)^{-1}$ is compact, so we need only to check this condition for *one* value of λ. In the same way, we get the following useful result.

Lemma 4.3. *D has compact resolvent if and only if $(D^2+1)^{-1}$ is compact.*

Proof. We may take $\lambda = -i$, since the selfadjointness of D implies that $\pm i \notin \operatorname{sp}(D)$. Thus, D has compact resolvent if and only if $(D+i)^{-1}$ is compact. Let $T = (D+i)^{-1}$; then the proof reduces to the well-known result that a bounded operator T is compact if and only if T^*T is compact. $\square$

By the spectral theorem $(D^2+1)^{1/2} - |D| = f(D)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is the continuous function $f(\lambda) := \sqrt{\lambda^2+1} - |\lambda| = \frac{1}{\sqrt{\lambda^2+1}+|\lambda|}$; and $0 < f(\lambda) \leq 1$ for all $\lambda \in \mathbb{R}$, so that $\|f(D)\| \leq 1$. Or more precisely: the operator $f(D) := (D^2+1)^{1/2} - |D|$, defined initially on $\operatorname{Dom} D$, extends to a bounded operator on $\mathcal{H}$, of norm at most 1.

In many arguments to come, we shall employ $|D|$ and $|D|^{-1}$ as if we knew that $\ker D = \{0\}$. However, even if $\ker D \neq \{0\}$, we can always replace $|D|$ by $(D^2+1)^{1/2}$ and $|D|^{-1}$ by $(D^2+1)^{-1/2}$, at the cost of some extra calculation.

4.2 Logarithmic divergence of spectra

If A is a positive selfadjoint operator with compact resolvent, let $\{\, \lambda_k(A) : k \in \mathbb{N} \,\}$ be its eigenvalues listed in increasing order, $\lambda_0(A) \leq \lambda_1(A) \leq \lambda_2(A) \leq \cdots$ (an eigenvalue of multiplicity r occurs exactly r times in the list). The *counting function* $N_A(\lambda)$, defined for $\lambda > 0$, is the number of eigenvalues not exceeding λ:

$$N_A(\lambda) := \#\{\, k \in \mathbb{N} : \lambda_k(A) \leq \lambda \,\}.$$

If A is invertible (i.e., if $\lambda_0(A) > 0$), we can define the "zeta function"

$$\zeta_A(s) := \operatorname{Tr} A^{-s} = \sum_{k \geq 0} \lambda_k(A)^{-s}, \quad \text{for } s > 0,$$

where we understand that $\zeta_A(s) = +\infty$ when A^{-s} is not traceless. For real s, $\zeta_A(s)$ is a nonnegative decreasing function.

It is actually more useful to consider *finite partial sums.*

Notation. If $T \in \mathcal{K}(\mathcal{H})$ is any compact operator, and if $k \in \mathbb{N}$, let $s_k(T)$, called the k-th **singular value** of T, be the k-th eigenvalue of the compact positive operator $|T| := (T^*T)^{1/2}$, where these are listed in decreasing order, with multiplicity. Thus $s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots$ and each singular value occurs only finitely many times in the list, namely, the finite multiplicity of the that eigenvalue of $|T|$; therefore, $s_k(T) \to 0$ as $k \to \infty$. Note that $s_0(T) = \|T\|$ since $s_0(T)^2$ is the largest eigenvalue of T^*T, so that $s_0(T)^2 = \|T^*T\| = \|T\|^2$. For each $N \in \mathbb{N}$, write

$$\sigma_N(T) := \sum_{k=0}^{N-1} s_k(T).$$

We shall see later that for many spectral triples, the counting function of the positive (unbounded) operator $|D|$ has polynomial growth: for some n, one can verify an asymptotic relation $N_{|D|}(\lambda) \sim C_n' \lambda^n$. In that case we can take $A := |D|^{-n}$, which is compact. Then the number of eigenvalues of A that are $\geq \varepsilon$ equals $N_{|D|}(\lambda)$ for $\lambda = 1/\varepsilon$. This suggests heuristically that for N close to $N_{|D|}(1/\varepsilon)$, the N-th eigenvalue is roughly C/ε for some constant C, so that $\sigma_N(|D|^{-n}) = O(\log N)$. We now check this condition in a few examples.

Example 4.4. We estimate $\sigma_N(|D\!\!\!/|^{-s})$ for $s > 0$, where $D\!\!\!/$ is the Dirac operator on the sphere $\mathbb{S}^2$ with its spin structure and its rotation-invariant metric. We know that the eigenvalues of $|D\!\!\!/|$ are $k = 1, 2, 3, \dots$ with respective multiplicities $2(2k) = 4, 8, 12, \dots$. For $r = 1, 2, 3, \dots$, let

$$N_r := \sum_{k=1}^{r} 4k = 2r(r+1) \sim 2r^2 \quad \text{as } r \to \infty,$$

so that $\log N_r \sim 2 \log r$ as $r \to \infty$. Next,

$$\sigma_{N_r}(|D\!\!\!/|^{-s}) = \sum_{k=1}^{r} 4k(k^{-s}) = 4 \sum_{k=1}^{r} k^{1-s},$$

and thus

$$\frac{\sigma_{N_r}(|D\!\!\!/|^{-s})}{\log N_r} \sim \frac{4}{2\log r} \sum_{k=1}^{r} k^{1-s} \sim \frac{2}{\log r} \int_1^r t^{1-s}\, dt \quad \text{as } r \to \infty,$$

by the "integral test" of elementary calculus. There are three cases to consider:

- If $s < 2$, then $\dfrac{2}{\log r} \displaystyle\int_1^r t^{1-s}\, dt = \dfrac{r^{2-s} - 1}{2 - s}$ diverges as $r \to \infty$;
- if $s > 2$, then $\dfrac{2}{\log r} \displaystyle\int_1^r t^{1-s}\, dt \to 0$ as $r \to \infty$; while
- if $s = 2$, then $\dfrac{\sigma_{N_r}(|D\!\!\!/|^{-s})}{\log N_r} \sim \dfrac{2\log r}{\log_r} \to 2.$

Finally, note that if $N_{r-1} \leq N \leq N_r$, then

$$\frac{\sigma_{N_{r-1}}(|D\!\!\!/|^{-s})}{\log N_r} \leq \frac{\sigma_N(|D\!\!\!/|^{-s})}{\log N} \leq \frac{\sigma_{N_r}(|D\!\!\!/|^{-s})}{\log N_{r-1}},$$

while $\log N \sim \log N_{r-1} \sim \log N_r \sim 2 \log r$ as $r \to \infty$. Thus

$$\lim_{N\to\infty} \frac{\sigma_N(|D\!\!\!/|^{-s})}{\log N} = \lim_{r\to\infty} \frac{\sigma_{N_r}(|D\!\!\!/|^{-s})}{\log N_r} = \begin{cases} +\infty & \text{if } s < 2, \\ 2 & \text{if } s = 2, \\ 0 & \text{if } s > 2. \end{cases}$$

We express this result by saying that for $s = 2$, "the spectrum of $|D\!\!\!/|^{-2}$ diverges logarithmically". There is precisely one exponent, namely $s = 2$, for which this limit is neither zero nor infinite.

Exercise 4.5. *Do the same calculation for $D\!\!\!/$ on the torus $\mathbb{T}^n$, whose spectrum we know: show that the spectrum of $|D\!\!\!/|^{-s}$ diverges logarithmically if and only if $s = n = \dim \mathbb{T}^n$.*

4.3 Some eigenvalue inequalities

Let $\mathcal{H}$ be a separable (infinite-dimensional) Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the ideal of compact operators on $\mathcal{H}$. Each $T \in \mathcal{K}$ has a polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2} \in \mathcal{K}$, and $U \in \mathcal{L}(\mathcal{H})$ is a partial isometry. This factorization is unique if we require that $U = 0$ on $\ker |T|$, since U must map the range of $|T|$ isometrically onto the range of T.

The spectral theorem yields an orthonormal family $\{\psi_k\}$ in $\mathcal{H}$, such that

$$|T| = \sum_{k\geq 0} s_k(T)\,|\psi_k\rangle\langle\psi_k|, \qquad T = \sum_{k\geq 0} s_k(T)\,|U\psi_k\rangle\langle\psi_k|.$$

(If $|T|$ is invertible, this is an orthonormal basis for $\mathcal{H}$. Otherwise, we can adjoin an orthonormal basis for $\ker|T|$ to the family $\{\psi_k\}$.) Since $\phi_k := U\psi_k$ gives another orthonormal family, any $T \in \mathcal{K}$ has an expansion of the form

$$T = \sum_{k\geq 0} s_k(T)\,|\phi_k\rangle\langle\psi_k|, \tag{4.1}$$

for some pair of orthonormal families $\{\phi_k\}$, $\{\psi_k\}$.

If V_1, V_2 are *unitary* operators on $\mathcal{H}$, we can then write

$$V_1 T V_2 = \sum_{k\geq 0} s_k(T)\,|V_1\phi_k\rangle\langle V_2^*\psi_k|,$$

and conclude that $s_k(V_1TV_2) = s_k(T)$ for each k, and hence that

$$\sigma_N(V_1TV_2) = \sigma_N(T).$$

Therefore, *any* norm $|||T|||$ that is built from the sequence $\{\, s_k(T) : k \in \mathbb{N}\,\}$ is unitarily invariant, that is, $|||V_1TV_2||| = |||T|||$ for V_1, V_2 unitary.

Example 4.6. If $\|T\|$ is the usual operator norm on $\mathcal{K}$, then

$$\|T\| = \|T^*T\|^{1/2} = \|\,|T|\,\| = \sup_{k\geq 0} s_k(T) = s_0(T).$$

A compact operator T is called *trace-class*, and we write $T \in \mathcal{L}^1 = \mathcal{L}^1(\mathcal{H})$, if the following series converges:

$$\|T\|_1 := \operatorname{Tr}|T| = \sum_{k\geq 0} s_k(T) = \lim_{N\to\infty} \sigma_N(T).$$

For $1 < p < \infty$, there are *Schatten classes* $\mathcal{L}^p = \mathcal{L}^p(\mathcal{H})$ consisting of operators for which the following norm is finite:

$$\|T\|_p = \Bigl(\sum_{k\geq 0} s_k(T)^p\Bigr)^{1/p}.$$

There are strict inclusions $\mathcal{L}^1 \subset \mathcal{L}^r \subset \mathcal{L}^p \subset \mathcal{K}$ for $1 < r < p < \infty$.

Soon, we shall introduce a "Dixmier trace class" $\mathcal{L}^{1+}(\mathcal{H})$, with yet another norm built from singular values, such that $\mathcal{L}^1 \subset \mathcal{L}^{1+} \subset \mathcal{L}^p$ for $p > 1$.

Much is known about the singular values of compact operators. For instance, the following relation holds, for $T \in \mathcal{K}$:

$$s_k(T) = \inf\{\, \|T(1-P)\| : P = P^2 = P^*,\ \dim P(\mathcal{H}) \leq k\,\}. \tag{4.2}$$

This comes from a well-known minimax principle: see [RS], for instance. The infimum is indeed attained at the projector Q of rank k whose range is $Q(\mathcal{H}) := \operatorname{span}\{\psi_0, \dots, \psi_{k-1}\}$, when T is given by (4.1), since $T(1-Q) = T - TQ = \sum_{j\geq k} s_j(T)|\phi_j\rangle\langle\psi_j|$ is an operator with norm $\|T - TQ\| = s_k(T)$.

Lemma 4.7. *If $T \in \mathcal{K}$, then*

$$\sigma_N(T) = \sup\{\, \|TP\|_1 : P = P^2 = P^*,\ \operatorname{rank} P = N \,\}. \tag{4.3a}$$

If A is a positive *compact operator, then it is also true that*

$$\sigma_N(A) = \sup\{\, \operatorname{Tr}(PAP) : P = P^2 = P^*,\ \operatorname{rank} P = N \,\}. \tag{4.3b}$$

Proof. If P is a projector of finite rank N, then $(TP)^*(TP) = PT^*TP$ and thus $|TP|$ has finite rank $\leq N$, so $\|TP\|_1 = \sum_{k=0}^{N-1} s_k(TP) = \sigma_N(TP)$. From the formula (4.2), it follows that $0 \leq A \leq B$ in $\mathcal{K}$ implies $s_k(A) \leq s_k(B)$ for each $k \in \mathbb{N}$, and in particular, since $0 \leq PT^*TP \leq T^*T$, we get $s_k(TP) \leq s_k(T)$. Thus $\sigma_N(TP) \leq \sigma_N(T)$ also. We conclude that the right hand side of (4.3a) is $\leq \sigma_N(T)$.

If we write T in the form (4.1) and then choose Q, as before, to be the projector with range $\operatorname{span}\{\psi_0, \dots, \psi_{k-1}\}$, then $|TQ| = \sum_{j=0}^{k-1} s_j(T)\, |\psi_j\rangle\langle\psi_j|$ and thus $\|TQ\|_1 = \sigma_N(T)$.

When $A \in \mathcal{K}$ is positive, and P is a projector of rank n, then $\operatorname{Tr}(PAP) = \operatorname{Tr}(AP) \leq \|AP\|_1 \leq \sigma_N(A)$. To see that the supremum in (4.3b) is attained, we can write $A = \sum_{k\geq 0} s_k(A)\, |\psi_k\rangle\langle\psi_k|$ and note that $QA = AQ = QAQ$, so that AQ is also a positive operator. It then follows that $\operatorname{Tr}(QAQ) = \operatorname{Tr}(AQ) = \|AQ\|_1 = \sigma_N(A)$. □

Corollary 4.8. *Each σ_N is a norm on $\mathcal{K}$: $\sigma_N(S+T) \leq \sigma_N(S) + \sigma_N(T)$ for $S, T \in \mathcal{K}$.*

Proof. This follows from $\|SP + TP\|_1 \leq \|SP\|_1 + \|TP\|_1$ for $P = P^2 = P^*$, $\operatorname{rank} P = N$. □

Lemma 4.9. *If $T \in \mathcal{K}$, then*

$$\sigma_N(T) = \inf\{\, \|R\|_1 + N\, \|S\| : R, S \in \mathcal{K} \text{ with } R + S = T \,\}.$$

Proof. If $T = U|T|$, then $|T| = U^*T$ (by the details of polar decomposition, this is true even though U might not be unitary), so $T = R + S$ implies $U^*T = U^*R + U^*S$; thus, we can suppose that $T \geq 0$.

If we now split $T =: R + S$, then $\sigma_N(T) \leq \sigma_N(R) + \sigma_N(S) \leq \|R\|_1 + \sigma_N(S)$, while

$$\sigma_N(S) = \sum_{0\leq k<N} s_k(S) \leq \sum_{0\leq k<N} s_0(S) = N\, \|S\|.$$

For $T = \sum_{k\geq 0} s_k(T)\, |\psi_k\rangle\langle\psi_k|$, we consider the special splitting into positive operators,

$$\widetilde{R} := \sum_{0\leq k<N} (s_k(T) - s_N(T))\, |\psi_k\rangle\langle\psi_k|, \qquad \widetilde{S} := T - \widetilde{R}.$$

Then $\|\widetilde{R}\|_1 = \sigma_N(T) - N\, s_N(T)$, while $\|\widetilde{S}\| = s_N(T)$ by inspection. □

The triangle inequality in Corollary 4.8 is not good enough for our needs: our goal is get an *additive* functional, rather than just a subadditive one. The next step is to extract from (4.3b) a sort of "wrong-way triangle inequality", at least for *positive compact operators*.

Lemma 4.10. *If $A \geq 0$, $B \geq 0$ are positive compact operators, and if $M, N \in \mathbb{N}$, then*

$$\sigma_{M+N}(A+B) \geq \sigma_M(A) + \sigma_N(B).$$

Proof. From (4.3b) we obtain $\sigma_M(A) = \sup\{\operatorname{Tr}(PAP) : P = P^2 = P^*,\ \operatorname{rank} P = M\}$ and $\sigma_N(B) = \sup\{\operatorname{Tr}(P'BP') : P' = P'^2 = P'^*,\ \operatorname{rank} P' = N\}$. Now $\operatorname{rank}(P + P') = \dim(P\mathcal{H} + P'\mathcal{H}) \leq M + N$, so if P'' is any projector of rank $M + N$ whose range includes the subspace $P\mathcal{H} + P'\mathcal{H}$, then $P \leq P''$ and $P' \leq P''$ as operators. Therefore,

$$\operatorname{Tr}(PAP) + \operatorname{Tr}(P'BP') \leq \operatorname{Tr}(P''AP'') + \operatorname{Tr}(P''BP'') = \operatorname{Tr}(P''(A+B)P''),$$

so that $\sigma_M(A)+\sigma_N(B) \leq \sup_{P''} \operatorname{Tr}(P''(A+B)P'') \leq \sigma_{2N}(A+B)$. (Notice how this argument requires additivity of the trace: it would not have worked with $\|\cdot\|_1$ instead of Tr, hence the restriction to the case of positive operators.) $\square$

Corollary 4.11. *If $A, B \in \mathcal{K}$ with $A \geq 0$, $B \geq 0$, then*

$$\sigma_N(A + B) \leq \sigma_N(A) + \sigma_N(B) \leq \sigma_{2N}(A + B). \qquad \square$$

We see that the functional $A \mapsto \sigma_N(A)/\log N$ is not far from being additive functional on the positive cone $\mathcal{K}_+$. But to get a truly additive functional, we must try to take the limit $N \to \infty$, and here things become more interesting.

4.4 Dixmier traces

It is a bit awkward to deal with the index N of $\sigma_N(A)$ as a discrete variable, but we can fix this by a simple linear interpolation.

If $N \leq \lambda \leq N + 1$, so that $\lambda = N + t$ with $0 \leq t \leq 1$, we put

$$\sigma_\lambda(A) := (1 - t)\sigma_N(A) + t\sigma_{N+1}(A).$$

Note that $\sigma_\lambda(A + B) \leq \sigma_\lambda(A) + \sigma_\lambda(B)$ now holds for all $\lambda \geq 0$: every σ_λ is a norm on $\mathcal{K}$.

Exercise 4.12. *Check that $\sigma_\lambda(A + B) \leq \sigma_\lambda(A) + \sigma_\lambda(B) \leq \sigma_{2\lambda}(A + B)$, for $A, B \geq 0$ in $\mathcal{K}$, also holds for all $\lambda \geq 0$.*

Definition 4.13. *The* ***Dixmier ideal*** $\mathcal{L}^{1+} = \mathcal{L}^{1+}(\mathcal{H}) = \mathcal{L}^{1,\infty}(\mathcal{H})$ *is defined to be*

$$\mathcal{L}^{1+} := \left\{ T \in \mathcal{K} : \sup_{\lambda \geq e} \frac{\sigma_\lambda(T)}{\log \lambda} < \infty \right\}.$$

(The e here is by convention: any constant > 1 would do. Also, the notation $\mathcal{L}^{1+}$ is not universally accepted: some authors prefer the clumsier notation $\mathcal{L}^{1,\infty}$, or even $\mathcal{L}^{(1,\infty)}$, which comes from the historical origin of these operator ideals in real interpolation theory: see [Con, IV.C] for that.)

Since each σ_λ is a norm on $\mathcal{K}$, so also is this supremum whenever it is finite. Thus $\mathcal{L}^{1+}$ has a natural (¡unitarily invariant!) norm

$$\|T\|_{1+} := \sup_{\lambda \geq e} \frac{\sigma_\lambda(T)}{\log \lambda} \quad \text{for } T \in \mathcal{L}^{1+}.$$

As stated, the norm depends on the chosen constant e, but the ideal $\mathcal{L}^{1+}(\mathcal{H})$ does not.

Note that $T \in \mathcal{K}$ is *traceclass* if and only if $\sigma_\lambda(T)$ is bounded (by $\|T\|_1$, for instance) without need for the factor $(1/\log \lambda)$. Thus $\mathcal{L}^1(\mathcal{H}) \subset \mathcal{L}^{1+}(\mathcal{H})$.

Remark 4.14. If the bounded function $\sigma_\lambda(T)/\log\lambda$ is actually *convergent* as $\lambda \to \infty$, or equivalently, if $\sigma_N(T)/\log N$ converges as $N \to \infty$, then clearly

$$\lim_{N\to\infty} \frac{\sigma_N(T)}{\log N} = \lim_{\lambda\to\infty} \frac{\sigma_\lambda(T)}{\log\lambda} \leq \|T\|_{1+}.$$

We get an *additive* functional defined on $\mathcal{L}^{1+}$ in three more steps. First, we dampen the oscillations in $\sigma_\lambda(T)/\log\lambda$ by taking a *Cesàro mean* with respect to the logarithmic measure on an interval $[\lambda_0,\infty)$ for some $\lambda_0 > e$. For definiteness, we choose $\lambda_0 = 3$. Our treatment closely follows the appendix of the local-index paper of Connes and Moscovici [CM].

Definition 4.15. *For $\lambda \geq 3$, we set*

$$\tau_\lambda(T) := \frac{1}{\log\lambda}\int_3^\lambda \frac{\sigma_u(T)}{\log u}\,\frac{du}{u}, \quad \textit{for } T \in \mathcal{L}^{1+}(u). \tag{4.4}$$

Exercise 4.16. *Check the triangle inequality $\tau_\lambda(S+T) \leq \tau_\lambda(S) + \tau_\lambda(T)$ for $\lambda \geq 3$.*

Lemma 4.17 (Connes–Moscovici). *If $A \geq 0$, $B \geq 0$ in $\mathcal{L}^{1+}(\mathcal{H})$, then*

$$\tau_\lambda(A) + \tau_\lambda(B) - \tau_\lambda(A+B) = O\left(\frac{\log\log\lambda}{\log\lambda}\right) \quad \textit{as } \lambda \to \infty.$$

Proof. First of all, it is clear that $\dfrac{\sigma_u(A+B)}{\log u} \leq \|A\|_{1+} + \|B\|_{1+}$ for $\lambda \geq e$. Next,

$$\begin{aligned}
\tau_\lambda(A) + \tau_\lambda(B) - \tau_\lambda(A+B) &\leq \frac{1}{\log\lambda}\int_3^\lambda \left(\frac{\sigma_{2u}(A+B)}{\log u} - \frac{\sigma_u(A+B)}{\log u}\right)\frac{du}{u} \\
= \frac{1}{\log\lambda}\int_6^{2\lambda} &\left(\frac{\sigma_u(A+B)}{\log(u/2)} - \frac{\sigma_u(A+B)}{\log u}\right)\frac{du}{u} - \frac{1}{\log\lambda}\left(\int_3^\lambda - \int_6^{2\lambda}\right)\frac{\sigma_u(A+B)}{\log u}\,\frac{du}{u}.
\end{aligned}$$

The second term can be rewritten as

$$\frac{1}{\log\lambda}\left(\int_3^6 - \int_6^{2\lambda}\right)\frac{\sigma_u(A+B)}{\log u}\,\frac{du}{u}.$$

Since $\int_3^6 \frac{du}{u} = \int_\lambda^{2\lambda}\frac{du}{u} = \log 2$, we get an estimate of $\frac{2\log 2}{\log\lambda}\|A+B\|_1$. For the first term, we compute

$$\begin{aligned}
\frac{1}{\log\lambda}\int_6^{2\lambda} \frac{\sigma_u(A+B)}{\log u}\left(\frac{\log u}{\log(u/2)} - 1\right)\frac{du}{u} &\leq \frac{\|A+B\|_{1+}}{\log\lambda}\int_3^\lambda \left(\frac{\log 2u}{\log u} - 1\right)\frac{du}{u} \\
= \frac{\|A+B\|_{1+}}{\log\lambda}\log 2\int_3^\lambda \frac{du}{u\log u} &< \frac{\|A+B\|_{1+}}{\log\lambda}\log 2\,(\log\log\lambda). \qquad \square
\end{aligned}$$

Since the failure of additivity of τ_λ vanishes as $\lambda \to \infty$, the second step is to quotient out by functions vanishing at infinity. For that we consider the "corona" C^*-algebra

$$B_\infty := \frac{C_b([3,\infty))}{C_0([3,\infty))}.$$

The function $\lambda \mapsto \tau_\lambda(A)$, for $A \geq 0$ in $\mathcal{L}^{1+}$, lies in $C_b([3,\infty))$, and its image $\tau(A)$ in B_∞ defines an additive map, that is,

$$\tau(A+B) = \tau(A) + \tau(B) \quad \text{for } A \geq 0,\ B \geq 0 \quad \text{in } \mathcal{L}^{1+}.$$

The final step is to compose this map with a state on B_∞.

Definition 4.18. *For $A \geq 0$ in $\mathcal{L}^{1+}$, let $\tau(A) \in (B_\infty)_+$ denote the image, under the quotient map $C_b([3,\infty)) \to B_\infty$, of the bounded function $\lambda \mapsto \tau_\lambda(A)$. This yields an additive map between positive cones, $\tau\colon (\mathcal{L}^{1+})_+ \to (B_\infty)_+$. Since the "four positive parts" of any operator in $\mathcal{L}^{1+}$ also lie in $\mathcal{L}^{1+}$, as is easily checked, this map extends in the obvious way to a positive linear map $\tau\colon \mathcal{L}^{1+} \to B_\infty$. Moreover, τ is invariant under unitary conjugation, i.e., $\tau(UAU^*) = \tau(A)$ for each unitary $U \in \mathcal{L}(\mathcal{H})$.*

For each state *$\omega\colon B_\infty \to \mathbb{C}$, we can now define a* ***Dixmier trace*** Tr_ω *on $\mathcal{L}^{1+}(\mathcal{H})$ by*

$$\operatorname{Tr}_\omega T := \omega(\tau(T)).$$

Since $\operatorname{Tr}_\omega(UAU^*) = \operatorname{Tr}_\omega(A)$ for positive $A \in \mathcal{L}^{1+}(\mathcal{H})$ and unitary $U \in \mathcal{L}(\mathcal{H})$, each such positive linear functional on $\mathcal{L}^{1+}(\mathcal{H})$ is indeed a trace.

This definition has a *drawback*: since B_∞ is a non-separable C^*-algebra, there is no way to *exhibit* even one such state. However, Dixmier traces are still computable in a special case: if $\lim_{\lambda\to\infty} \tau_\lambda(T)$ exists, then $\tau(T)$ coincides with the image of a constant function in B_∞, and since the state ω is normalized, $\omega(1) = 1$, the value $\omega(\tau(T))$ equals this limit:

$$\operatorname{Tr}_\omega T = \lim_{\lambda\to\infty} \tau_\lambda(T)$$

is independent of ω, provided that the limit exists. Such operators are called *measurable*. When this happens, we shall suppress the label ω and write $\operatorname{Tr}^+ T$ for the common value of all Dixmier traces.

The use of the Cesàro mean (4.4) simplifies the original definition that Dixmier [Dix1] gave of these traces. A detailed analysis of these (and other related) functionals was made recently by Lord, Sedaev and Sukochev [LSS], who called them "Connes–Dixmier traces". As an unexpected consequence of their work, they have shown that a *positive* operator $A \in \mathcal{L}^{1+}(\mathcal{H})$ is measurable if *and only if* the original sequence $\{\, \sigma_N(A)/\log N : N \in \mathbb{N} \,\}$ is already convergent. Thus it is not necessary to compute $\tau_\lambda(A)$, since

$$\operatorname{Tr}^+ A = \lim_{N\to\infty} \frac{\sigma_N(A)}{\log N} \quad \text{for positive, measurable } A \in \mathcal{L}^{1+}.$$

Chapter 5

Symbols and Traces

5.1 Classical pseudodifferential operators

In order to develop a symbol calculus for Dirac operators and their powers, we shall temporarily restrict our attention to a single chart domain $U \subset M$, over which the cotangent bundle is trivial: $T^*M\big|_U \simeq U \times \mathbb{R}^n$. If P is an operator on $C^\infty(M)$, or more generally, on a space of sections $\Gamma(M, E)$ of a vector bundle $E \to M$, and if $\{\phi_1, \dots, \phi_m\}$ is a finite partition of unity in $C^\infty(M)$, then $P(f) = \sum_{i,j=1}^m \phi_i P(\phi_j f)$, so we may as well consider operators which are defined on a single chart domain of M. At some later stage, we must ensure that the important properties of such operators are globally defined, independently of the choice of local coordinates.

We will work, then, in *local coordinates* $(x^1, \dots, x^n)$ over a chart domain U; the local coordinates of the cotangent bundle $T^*M\big|_U$ are

$$(x, \xi) = (x^1, \dots, x^n, \xi_1, \dots, \xi_n), \quad \text{where } \xi \in T_x^*M.$$

Let $E \to M$ be a vector bundle of rank r. We assume (without loss of generality) that the vector bundle E is also trivial over U, so we can identify $\Gamma(U, \operatorname{End} E)$ with $U \times M_r(\mathbb{C})$.

A **differential operator** acting on (smooth) local sections $f \in \Gamma(U, E)$ is an operator P of the form

$$P = \sum_{|\alpha| \le d} a_\alpha(x)\, D^\alpha, \quad \text{with } a_\alpha \in \Gamma(U, \operatorname{End} E),$$

where we use the notation $D^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}$, and $D_j := -i\,\partial/\partial x^j$, the positive integer d is the *order* of P.

The local coordinates allow us to identify U with an open subset of $\mathbb{R}^n$. The coefficients a_α are matrix-valued functions $U \to M_r(\mathbb{C})$.

By a Fourier transformation, we can write, for $f \in C_c^\infty(U, \mathbb{R}^r)$,

$$\begin{aligned} Pf(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi}\, p(x,\xi)\, \hat f(\xi)\, d^n\xi \\ &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi}\, p(x,\xi)\, f(y)\, d^n y\, d^n\xi, \end{aligned} \tag{5.1}$$

where $p(x, \xi)$ is a *polynomial* of order d in the ξ-variable, called the (complete) *symbol* of P. (Clearly, this symbol depends on the choice of local coordinates.) Here

$$K_p(x,y) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}\, p(x,\xi)\, f(y)\, d^n\xi \tag{5.2}$$

is the **kernel** of P, as an integral operator: the inverse Fourier transform of $p(x,\xi)$.

For the Dirac operator $D\!\!\!/$, we can use the local expression of the spin connection to write $D\!\!\!/ = -i\,c(dx^j)\,\nabla^S_{\partial_j} = -i\,c(dx^j)(\partial_j + \omega_j(x))$, so the corresponding symbol is

$$p(x,\xi) = c(dx^j)(\xi_j - i\,\omega_j(x)). \tag{5.3}$$

This is a first-order polynomial in the ξ_j variables, so that $D\!\!\!/$ is a first order differential operator. The *leading term* in $p(x,\xi)$ —the part that is homogeneous in ξ_j of degree one— is $c(dx^j)\xi_j = c(\xi_j\,dx^j) = c(\xi)$, where $\xi = \xi_j\,dx^j$ can be regarded as an element of $\mathcal{A}^1(U)$.

More generally, a *pseudodifferential* operator P is given locally by an integral of the form (5.1), where the symbol $p(x,\xi)$ need no longer be a polynomial. In that case, we must specify certain classes of symbols for which these integrals make sense.

Definition 5.1. *The vector space $S^d(U)$ of (scalar)* symbols of order $\leq d$*, consists of functions $p \in C^\infty(U \times \mathbb{R}^n)$ such that, for any compact $K \subset U$, and any multiindices $\alpha,\beta \in \mathbb{N}^n$, there exists a constant $C_{K\alpha\beta}$ such that*

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{K\alpha\beta}(1+|\xi|^2)^{\frac{1}{2}(d-|\alpha|)} \quad \text{for all } x \in K, \xi \in \mathbb{R}. \tag{5.4}$$

Here D_x^β and D_ξ^α denote derivatives in the x^i variables and in the ξ_j variables, respectively. We use $(1+|\xi|^2)^{\frac{1}{2}}$ instead of $|\xi|$ to avoid problems at $\xi = 0$.

In the same way, we define matrix-valued symbols of order $\leq d$ as smooth functions $p\colon U \times \mathbb{R}^n \to M_r(\mathbb{C})$ satisfying the same norm estimates, but with the absolute value $|\cdot|$ on the left hand side of (5.4) replaced by a matrix norm in $M_r(\mathbb{C})$. By a small abuse of notation, we shall write $p \in S^d(U)$ also in the matrix-valued case.

When $p(x,\xi)$ is a *polynomial* in ξ, of order at most d, we can isolate its homogeneous parts:

$$p(x,\xi) = \sum_{j=0}^{d} p_{d-j}(x,\xi), \quad \text{where} \quad p_{d-j}(x,t\xi) = t^{d-j}\,p_{d-j}(x,\xi) \quad \text{for } t > 0.$$

Definition 5.2. *More generally, an element $p \in S^d(U)$ is called a **classical symbol** if we can find a sequence of terms $p_d(x,\xi)$, $p_{d-1}(x,\xi)$, $p_{d-2}(x,\xi)$,…, with $p_{d-j}(x,t\xi) = t^{d-j}\,p_{d-j}(x,\xi)$ for $t > 0$, such that for each $k = 0,1,2,\dots,$*

$$p - \sum_{j=0}^{k-1} p_{d-j} \in S^{d-k}(U).$$

When this is possible, we write

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_{d-j}(x,\xi), \tag{5.5}$$

and regard this series as an asymptotic development of the symbol p. This expansion does not determine $p(x,\xi)$ uniquely: a symbol in $\bigcap_{k\in\mathbb{N}} S^{d-k}(U)$ is called "smoothing", and smoothing symbols are exactly those symbols whose asymptotic expansion is zero.

Definition 5.3. *A **classical pseudodifferential operator** of order d, over $U \subset \mathbb{R}^n$, is an operator P defined by (5.1), for which $p(x,\xi)$ is a classical symbol in $S^d(U)$ whose leading term $p_d(x,\xi)$ does not vanish. This leading term is called the **principal symbol** of P, and we also denote it by $\sigma^P(x,\xi) := p_d(x,\xi)$.*

We need a formula for the symbol of the composition of two classical pseudodifferential operators ("classical ΨDOs", for short). It is not clear *a priori* when and if two such operators are composable: we remit to [Tay], for instance, for the full story on compositions (and adjoints) of classical pseudodifferential operators, and for the justification of the following formula.

If P is a classical ΨDOs of order d_1 with symbol $p \in S^{d_1}(U)$, and if Q is a classical ΨDO of orders d_2 with symbol $p \in S^{d_2}(U)$, then the symbol $p \circ q$ of the composition PQ lies in $S^{d_1+d_2}(U)$ and its asymptotic development is given by

$$(p \circ q)(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x,\xi)\, D_x^\alpha q(x,\xi). \tag{5.6}$$

To find the terms $(p \circ q)_{d_1+d_2-j}(x,\xi)$ of the symbol expansion, one must substitute (5.5) for both p and q into the right hand side of (5.6) and rearrange a finite number of terms. For the case $j = 0$, one need only use $\alpha = 0$ —since D_ξ^α lowers the order by $|\alpha|$— and in particular, the principal symbols compose easily:

$$(p \circ q)_{d_1+d_2}(x,\xi) = p_{d_1}(x,\xi)\, q_{d_2}(x,\xi).$$

The composition formula is valid for both scalar-valued and matrix-valued symbols, provided the matrix size r is the same for both operators.

Exercise 5.4. *If P and Q are classical ΨDOs with scalar-valued symbols, show that the principal symbol of $[P,Q] = PQ - QP$ is $-i\,\{\sigma^P, \sigma^Q\}$, where $\{\cdot,\cdot\}$ is the* Poisson bracket *of functions:*

$$-i\,\{\sigma^P(x,\xi), \sigma^Q(x,\xi)\} = -i \sum_{j=1}^n \frac{\partial \sigma^P}{\partial \xi_j} \frac{\partial \sigma^Q}{\partial x^j} - \frac{\partial \sigma^Q}{\partial \xi_j} \frac{\partial \sigma^P}{\partial x^j}.$$

Conclude that the order of $[P,Q]$ is $\leq d_1 + d_2 - 1$. ¿What can be said about the order of $[P,Q]$ if P and Q have matrix-valued symbols of size $r > 1$?

Suppose U and V are open subsets of $\mathbb{R}^n$ and that $\phi\colon U \to V$ is a diffeomorphism. If P is a ΨDO over U, then $\phi_* P\colon f \mapsto P(\phi^* f) \circ \phi^{-1}$ is a ΨDO over V, as can be verified by an explicit change-of-variable calculation. If P is classical, then so also is $\phi_* P$. If p^ϕ denotes the symbol of $\phi_* P$, we find that the *principal symbols* are related by

$$p_d(x,\xi) = p_d^\phi(\phi(x), \phi'(x)^{-t}\xi),$$

where $\phi'(x)^{-t}$ is the contragredient matrix (inverse transpose) to $\phi'(x)$.

This is the change-of-variable rule for the cotangent bundle. The conclusion is that, for any *scalar* ΨDO P that we may be able to define over a compact manifold M, the complete symbol $p(x,\xi)$ will depend on the local coordinates for a given chart of M, but the leading term $p_d = \sigma^P$ will make sense as an element of $C^\infty(T^*M)$ —i.e., a function on the total space of the cotangent bundle. (The subleading terms $p_{d-j}(x,\xi)$, for $j \geq 1$, will not be invariant under local coordinate changes.)

When P is defined on sections of a vector bundle $E \to M$ of rank r, the principal symbol σ^P becomes a section of the bundle $\pi^*(\operatorname{End} E) \to T^*M$, i.e., the pullback of $\operatorname{End} E \to M$ via the cotangent projection $\pi\colon T^*M \to M$.

For the *Dirac operator* $D\!\!\!/$, which is a first-order differential operator on $\Gamma(M,S)$, we get $\sigma^{D\!\!\!/} \in \Gamma(T^*M, \pi^*(\operatorname{End} S))$. From (5.3), we get at once

$$\sigma^{D\!\!\!/}(x,\xi) = c(\xi_j\, dx^j) = c(\xi).$$

Since taking the principal symbol is a multiplicative procedure, we also obtain

$$\sigma^{\not{D}^2}(x,\xi) = (\sigma_{\not{D}}(x,\xi))^2 = c(\xi)^2 = g(\xi,\xi)\,1_{2^m}.$$

(Here we use the handy notation 1_r for the $r \times r$ identity matrix.) Notice that the principal symbol of Δ^S is also $g(\xi,\xi)\,1_{2^m}$, since $\not{D}^2 - \Delta^S = \frac{1}{4}s$ is a term of order zero (it is independent of the ξ_j variables), thus $\not{D}^2$ and Δ^S have the same principal symbol.

Note that $\sigma^{\not{D}^2}(x,\xi)$ only vanishes when $\xi = 0$, that is, on the zero section of T^*M.

Definition 5.5. *A ΨDO P is called **elliptic** if $\sigma_P(x,\xi)$ is invertible when $\xi \neq 0$, i.e., off the zero section of T^*M.*

In particular, $\not{D}$, $\not{D}^2$, Δ, Δ^S are all elliptic differential operators.

5.2 Homogeneity of distributions

We now wish to pass from the symbol p of a classical ΨDO, with a given symbol expansion

$$p(x,\xi) = \sum_{j=0}^{N-1} p_{d-j}(x,\xi) + r_N(x,\xi), \quad r_N \in S^d(U).$$

to the operator kernel (5.2), by taking an inverse Fourier transform. However, the terms in this expansion may give divergent integrals when $y = x$. Therefore, we first need to look more closely at the inverse Fourier transforms of negative powers of $|\xi|$.

Assume that $n \geq 2$, for the rest of this section.

Definition 5.6. *Let $\lambda \in \mathbb{R}$. A function $\phi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is **homogeneous of degree $\boldsymbol{\lambda}$**, or "λ-homogeneous", if*

$$\phi(t\,\xi) = t^\lambda\,\phi(\xi) \quad \textit{for all} \quad t > 0,\ \xi \neq 0.$$

Thus if $\xi = r\omega$ with $r = |\xi| > 0$ and $\omega = \xi/|\xi| \in \mathbb{S}^{n-1}$, we can write $\phi(\xi) = r^\lambda\,\psi(\omega)$ for some $\psi\colon \mathbb{S}^{n-1} \to \mathbb{C}$.

We can extend this definition to (tempered) distributions on $\mathbb{R}^n$. Write ϕ_t for the dilation of ϕ by the scale factor t, that is, $\phi_t(\xi) := \phi(t\xi)$, so that the λ-homogeneity condition can be written as $\phi_t = t^\lambda\,\phi$ for $t > 0$.

The change-of variables formula for functions,

$$\int_{\mathbb{R}^n} u(t\xi)\,\phi(\xi)\,d^n\xi = \int_{\mathbb{R}^n} t^{-n}\,u(\eta)\,\phi(\eta/t)\,d^n\eta,$$

suggests the following definition of homogeneity.

Definition 5.7. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a temepered distribution on $\mathbb{R}^n$. For $t > 0$, the* dilation *u_t of u by the scale factor t is defined by*

$$\langle u_t, \phi\rangle := t^{-n}\langle u, \phi_{1/t}\rangle, \quad \textit{for}\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

*We say that u is **homogeneous of degree $\boldsymbol{\lambda}$** if $u_t = t^\lambda\,u$ for all $t > 0$.*

Example 5.8. The Dirac δ is homogeneous of degree $-n$, since for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \delta_t, \phi\rangle = t^{-n}\,\langle \delta, \phi_{1/t}\rangle = t^{-n}\,\phi_{1/t}(0) = t^{-n}\,\phi(0) = t^{-n}\,\langle \delta, \phi\rangle.$$

Suppose now that u is a smooth function on $\mathbb{R}^n \setminus \{0\}$, such that

$$u(\xi) = r^\lambda\,v(\omega), \quad \text{for} \quad \xi = r\omega,\ r = |\xi| > 0,\ \omega \in \mathbb{S}^{n-1}.$$

We would like *to extend it to a (tempered) distribution* on the whole $\mathbb{R}^n$. There are several cases to consider.

Case 1 If $\lambda > 0$, then just put $u(0) := 0$. In this case, u extends to $\mathbb{R}^n$ as a homogeneous *function*.

Case 2 If $-n < \lambda \leq 0$, then $u(0)$ may not exist, but $u(\xi)$ is *locally integrable* near 0, so $\langle u, \phi\rangle$ is defined. Indeed, if $B = B(0;1)$ and 1_B is its indicator function, and if σ denotes the usual volume form on $\mathbb{S}^{n-1}$, then

$$\begin{aligned}\langle u, 1_B\rangle &:= \int_B u(\xi)\, d^n\xi = \int_{\mathbb{S}^{n-1}} v(\omega)\,\sigma \int_0^1 r^\lambda (r^{n-1}\,dr) \\ &= C\int_0^1 r^{\lambda+n-1}\,dr < \infty, \quad \text{since } \lambda + n - 1 > -1.\end{aligned}$$

Case 3 Suppose $\lambda = -n$, and that $\int_{\mathbb{S}^{n-1}} v(\omega)\,\sigma = 0$.

We define a distribution $\mathrm{P}u$ by the following trick. Let $f\colon [0,\infty) \to \mathbb{R}$ be a *cutoff function*, such that:

$$f(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq 1, \end{cases}$$

and f decreases smoothly from 1 to 0 on $[\frac{1}{2}, 1]$. Replace the test function ϕ by $\phi(\xi) - \phi(0)f(|\xi|)$, and put

$$\langle \mathrm{P}u, \phi\rangle := \int_{\mathbb{R}^n} u(\xi)\big(\phi(\xi) - \phi(0)\,f(|\xi|)\big)\,d^n\xi. \tag{5.7}$$

If $g(t)$ is another cutoff function with the same properties, the right hand side of this formula changes by

$$\int_{\mathbb{R}^n} u(\xi)\phi(0)\big(f(r) - g(r)\big)\,d^n\xi = \phi(0)\int_{\mathbb{S}^{n-1}} v(\omega)\,\sigma \int_{1/2}^1 \big(f(r) - g(r)\big)\,\frac{dr}{r} = 0,$$

since $u(\xi)\,d^n\xi = r^{-n}v(\omega)\,\sigma\,r^{n-1}\,dr = v(\omega)\,\sigma\,dr/r$ by homogeneity. Thus $\langle \mathrm{P}u, \phi\rangle$ is independent of the cutoff chosen. Indeed, since

$$\int_{|\xi|>\varepsilon} u(\xi) f(|\xi|)\,d^n\xi = \int_\varepsilon^1 f(r)\,\frac{dr}{r}\int_{\mathbb{S}^{n-1}} v(\omega)\,\sigma = 0,$$

for any $\varepsilon > 0$, we get another formula for $\mathrm{P}u$:

$$\langle \mathrm{P}u, \phi\rangle = \lim_{\varepsilon\downarrow 0}\int_{|\xi|>\varepsilon} u(\xi)\phi(\xi)\,d\xi.$$

Therefore, $\mathrm{P}u$ is just the "Cauchy principal part" of u at $\xi = 0$.

Lemma 5.9. *When u is a $(-n)$-homogeneous function on $\mathbb{R}^n \setminus \{0\}$ whose integral over $\mathbb{S}^{n-1}$ vanishes, its principal-part extension $\mathrm{P}u$ is a homogeneous distribution of degree $(-n)$.*

Proof. For each $t > 0$, we observe that

$$\begin{aligned}\langle (\mathrm{P}u)_t, \phi\rangle &= t^{-n}\langle \mathrm{P}u, \phi_{1/t}\rangle = t^{-n}\lim_{\varepsilon\downarrow 0}\int_{|\xi|>\varepsilon} u(\xi)\,\phi(\xi/t)\,d\xi \\ &= t^{-n}\lim_{\varepsilon\downarrow 0}\int_{|\eta|>\varepsilon/t} u(\eta)\,\phi(\eta)\,d\eta = t^{-n}\langle \mathrm{P}u, \phi\rangle.\end{aligned}$$ ∎

Case 4 Consider the function $u(\xi) := |\xi|^{-n}$ for $\xi \neq 0$. (By averaging $v(\omega)$ over $\mathbb{S}^{n-1}$, one can see that any smooth $(-n)$-homogeneous function on $\mathbb{R}^n \setminus \{0\}$ is a linear combination of $|\xi|^{-n}$ and a function in Case 3.

We can try the cutoff regularization, anyway. Let $R_f u$ be given by the recipe of (5.7):

$$\langle R_f u, \phi \rangle := \int_{\mathbb{R}^n} u(\xi)\bigl(\phi(\xi) - \phi(0)\, f(|\xi|)\bigr)\, d^n\xi. \tag{5.8}$$

However, in the present case, $R_f u$ is *not* homogeneous!

Lemma 5.10. *If* $\delta\colon \phi \mapsto \phi(0)$ *is the Dirac delta, and if* $u(\xi) := |\xi|^{-n}$ *for* $\xi \neq 0$*, then*

$$(R_f u)_t - t^{-n}\, R_f u = (\Omega_n\, t^{-n} \log t)\, \delta. \tag{5.9}$$

Proof. We compute $\langle (R_f u)_t - t^{-n} R_f u, \phi\rangle$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$. Since $u(\xi) := |\xi|^{-n}$ and $f(|\xi|)$ are both rotation-invariant, we can first integrate over $\mathbb{S}^{n-1}$, so we may suppose that ϕ is radial: $\phi(\xi) = \psi(|\xi|)$ for some $\psi\colon [0,\infty) \to \mathbb{C}$. Then

$$\begin{aligned}
\langle R_f u, \phi\rangle &= \int_{\mathbb{R}^n} r^{-n}(\psi(r) - \psi(0)\, f(r))\, \sigma\, r^{n-1}\, dr \\
&= \Omega_n \int_0^\infty \bigl(\psi(r) - \psi(0)\, f(r)\bigr)\, \frac{dr}{r} \\
&= \Omega_n \int_0^\infty \Bigl(\psi\Bigl(\frac{r}{t}\Bigr) - \psi(0)\, f\Bigl(\frac{r}{t}\Bigr)\Bigr)\, \frac{dr}{r}, \quad \text{for any } t > 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle (R_f u)_t - t^{-n} R_f u, \phi\rangle &= t^{-n}\langle R_f u, \phi_{1/t} - \phi\rangle \\
&= \Omega_n \phi(0)\, t^{-n} \int_0^\infty \Bigl(f\Bigl(\frac{r}{t}\Bigr) - f(r)\Bigr)\, \frac{dr}{r} \\
&= \Omega_n \phi(0)\, t^{-n} \int_0^\infty \int_r^{r/t} f'(s)\, ds\, \frac{dr}{r} \\
&= \Omega_n \phi(0)\, t^{-n} \int_0^\infty \int_{st}^{s} \frac{dr}{r}\, f'(s)\, ds \\
&= \Omega_n \phi(0)\, t^{-n}(-\log t) \int_{1/2}^1 f'(s)\, ds = \Omega_n \phi(0)\, t^{-n} \log t.
\end{aligned}$$

The extra $\log t$-term measures the failure of homogeneity of the regularization $R_f u$. ∎

Case 5 $u(\xi) = |\xi|^{-n-j}$ for $j = 1, 2, 3, \dots$.

Any cutoff function f gives a regularization by "Taylor subtraction", as follows:

$$\langle \widetilde{R}_f u, \phi\rangle := \int_{\mathbb{R}^n} |\xi|^{-n-j}\biggl(\phi(\xi) - \sum_{|\alpha| \leq j} \frac{i^\alpha}{\alpha!} D^\alpha \phi(0) \xi^\alpha\, f(|\xi|)\biggr)\, d^n\xi.$$

Again one finds that $\widetilde{R}_f u$ is not homogeneous, by a straightforward calculation along the lines of the previous Lemma. This can be simplified a little by the following observation [GVF]. One can find constants c_α for $|\alpha| \leq j$, such that the modified regularization $R_f u := \widetilde{R}_f u - \sum_{|\alpha|<j} c_\alpha\, D^\alpha \delta$ has a "failure of homogeneity" of the form

$$(R_f u)_t - t^{-n-j}\, R_f u = t^{-n-j} \log t \biggl(\sum_{|\alpha|=j} c_\alpha\, D^\alpha \delta\biggr).$$

That completes our study of the extensions of homogeneous functions to distributions on $\mathbb{R}^n$. We need a remark about their Fourier transforms. Recall that the Fourier transformation $\mathcal{F}$ preserves the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and by duality it also preserves $\mathcal{S}'(\mathbb{R}^n)$. If u is a λ-homogeneous function on $\mathbb{R}^n \setminus \{0\}$, its Fourier transform is $\mathcal{F}u(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi}\, u(x)\, d^n x$, thus

$$(\mathcal{F}u)_t(\xi) = \int_{\mathbb{R}^n} e^{itx\xi}\, u(x)\, d^n x = t^{-n} \int_{\mathbb{R}^n} e^{-iy\xi}\, u(y/t)\, d^n y = t^{-n-\lambda}\, \mathcal{F}u(\xi).$$

It follows that $\mathcal{F}$, and also the inverse transformation $\mathcal{F}^{-1}$, take homogeneous functions (or distributions) of degree λ to homogeneous functions (or distributions) of degree $(-n-\lambda)$.

5.3 The Wodzicki residue

Now we return to the symbol expansion of a classical ΨDO P, of integral order $d \in \mathbb{Z}$, with

$$p(x,\xi) = \sum_{j=0}^{N-1} p_{d-j}(x,\xi) + r_N(x,\xi),$$

where $r_N \in S^{d-N}(U)$, and $p_{d-j}(x,t\xi) = t^{d-j}\, p_{d-j}(x,\xi)$. Now apply $\mathcal{F}_2^{-1}$, the inverse Fourier transform in the second variable, to this sum, to get the integral kernel

$$k_P(x,y) = \sum_{j=0}^{N-1} h_{j-d-n}(x, x-y) + (\mathcal{F}_2^{-1} r_N)(x, x-y).$$

If $N > n+d$, then $r_N \in S^{d-N}(U)$ is integrable in ξ, so the term $\mathcal{F}_2^{-1} r_N(x,z)$ is bounded as $z \to 0$. For the terms $h_{j-d-n}(x,z)$, there are 3 cases, which may give singularities. So *before* applying $\mathcal{F}_2^{-1}$ to $p_{d-j}(x,\xi)$, we must *regularize* $p_{d-j}(x,\xi)$ to $R_f p_{d-j}(x,\xi)$ by using a suitable cutoff:

$$p(x,\xi) = \sum_{j=0}^{N-1} R_f p_{d-j}(x,\xi) + s_N(x,\xi),$$

with s_N integrable. Now take $h_{j-d-n} := \mathcal{F}_2^{-1}(R_f p_{d-j})$.

Case 1 Suppose $d-j > -n$. Then $k := j-d-n < 0$, and $R_f p_{d-j}(x,\xi)$ is homogeneous of degree greater than $-n$, so $h_k(x,z)$ is homogeneous of degree k. These terms have no failure of homogeneity.

Before examining the other two cases, we return to the context of functions on $\mathbb{R}^n \setminus \{0\}$, and look first at $w_0(z) := (2\pi)^n \mathcal{F}^{-1}(R_f |\xi|^{-n})$. Since (5.9) holds with $u(\xi) = |\xi|^{-n}$ for $\xi \neq 0$, and since $(2\pi)^n \mathcal{F}^{-1}(\delta) = 1$, we get

$$t^{-n} w_0(z/t) - t^{-n} w_0(z) = \Omega_n\, t^{-n} \log t \quad \text{for } t > 0,$$

or more simply,

$$w_0(z/t) - w_0(z) = \Omega_n \log t. \tag{5.10}$$

Notice that $C = w_0(z/|z|)$ is a constant, because w_0 is rotation-invariant. Substituting $t := |z|$ in (5.10) gives

$$w_0(z) = C - \Omega_n \log |z|, \tag{5.11}$$

so that w_0 "diverges logarithmically". We can suppress the constant term if we replace $R_f|\xi|^{-n}$ by $R_f|\xi|^{-n} - C\delta$, since we must then subtracting the constant C from the inverse Fourier transform.

For $j = 1, 2, \dots$, we define $w_j(z) := (2\pi)^n \mathcal{F}^{-1}(R_f|\xi|^{-n-j})$. A similar analysis shows that $w_j(z) = q_j(z) - r_j(z) \log|z|$, where both q_j and r_j are homogeneous of degree $j > 0$. In this case, $w_j(z)$ remains bounded as $z \to 0$.

We now return to the examination of the terms h_{j-d-n} in the integral kernel $k(x, y)$.

Case 2 Suppose $d - j < -n$. Then $k = j - d - n > 0$, and we find that $h_{j-d-n}(x, z)$ remains bounded as $z \to 0$.

Case 3 Consider the case $d - j = -n$. Then we get $h_0(x, z) = -u_0(x) \log|z|$, after possibly subtracting a term depending only on x. We have proved the following result.

Proposition 5.11. *If P is a classical pseudodifferential operator of integral order d, then its kernel has the following form near the diagonal:*

$$k_P(x, y) = \sum_{-d-n \leq k < 0} h_k(x, x - y) - u_0(x) \log|x - y| + O(1), \tag{5.12}$$

where each $h_k(x, \cdot)$ is homogeneous of negative degree k, $u_0(x)$ is independent of $x - y$, and $O(1)$ stands for a term which remains bounded as $y \to x$. $\square$

To compute $u_0(x)$, the coefficient of logarithmic divergence, we change coordinates by a local diffeomorphism $\psi(x)$. Note that

$$\log|\psi(x) - \psi(y)| \sim \log|\psi'(x) \cdot (x - y)| \sim \log|x - y| \quad \text{as } y \to x,$$

while $k_P(x, y) \mapsto k_P(\psi(x), \psi(y))\, L(x, y)$, where $L(x, y) \to |\det \psi'(x)|$ as $y \to x$, by the change of variables formula for $|d^n y|$. (We use a 1-density, not an oriented volume form, to do integration; however, if we agree to fix an orientation on M and use only coordinate changes that preserve the orientation, for which $\det \psi'(x) > 0$ at each x, then we need not make this distinction). Thus the log-divergent term transforms as follows:

$$-u_0(x) \log|x - y| \mapsto -u_0(\psi(x)) |\det \psi'(x)| \log|x - y|.$$

For the case of scalar pseudodifferential operators, this is all we need. In the general case of operators acting on sections of a vector bundle $E \to M$, we replace $u_0(x) \in \operatorname{End} E_x$ by its matrix trace $\operatorname{tr} u_0(x) \in \mathbb{C}$. The previous formula then says that *the 1-density* $\operatorname{tr} u_0(x)\, |d^n x|$ *is invariant under local coordinate changes.*

Now, when we regularize $p_{-n}(x, \xi)$ to obtain this 1-density after applying $\mathcal{F}_2^{-1}$, we can first subtract the homogeneous "principal part", at each $x \in U$, since this will not change the coefficient of logarithmic divergence. This subtraction is done by replacing $p_{-n}(x, \xi)$ by its average over the sphere $|\xi| = 1$ in the cotangent space $T_x^* M$. That is to say, we get the same $u_0(x)$ if we replace $p_{-n}(x, \xi)$ by $\Omega_n^{-1} |\xi|^{-n} \int_{|\omega|=1} p_{-n}(x, \omega)\, \sigma$. On applying (5.11) (with $C = 0$) at each x, we conclude that

$$\operatorname{tr} u_0(x) = \int_{|\omega|=1} \operatorname{tr} p_{-n}(x, \omega)\, \sigma.$$

Definition 5.12. *The **Wodzicki residue density** of a classical ΨDO P, acting on sections of a vector bundle $E \to M$, is well defined by the local formula*

$$\operatorname{wres}_x P := \left(\int_{|\omega|=1} \operatorname{tr} p_{-n}(x, \omega)\, \sigma \right) |d^n x|, \quad \text{at } x \in M.$$

*The **Wodzicki residue** of P is the integral of this 1-density:*

$$\operatorname{Wres} P := \int_M \operatorname{wres}_x P = \int_M \left(\int_{|\omega|=1} \operatorname{tr} p_{-n}(x, \omega)\, \sigma \right) |d^n x|.$$

We shall now show that Wres is a *trace* on the algebra of classical pseudodifferential operators on M acting on a given vector bundle.

We begin with another important property of homogneous functions on $\mathbb{R}^n \setminus \{0\}$. We shall make use of the *Euler vector field* on this space:

$$R = \sum_{j=1}^n \xi_j \frac{\partial}{\partial \xi_j} = r \frac{\partial}{\partial r}.$$

Notice that h is λ-homogeneous if and only if $Rh = \lambda h$, since $Rh(r\omega) = r\frac{\partial}{\partial r}(r^\lambda h(\omega)) = \lambda r^\lambda h(\omega) = \lambda h(r\omega)$.

Lemma 5.13. *If $\lambda \neq -n$, any λ-homogeneous function h on $\mathbb{R}^n \setminus \{0\}$ is a finite sum of derivatives.*

Proof. It is enough to notice that

$$\sum_{j=1}^n \frac{\partial}{\partial \xi_j}(\xi_j\, h(\xi)) = nh(\xi) + Rh(\xi) = (n + \lambda)h(\xi),$$

which implies $h = \frac{1}{n+\lambda} \sum_{j=1}^n \frac{\partial}{\partial \xi_j}(\xi_j\, h)$. ◻

Lemma 5.14. *If $h \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is $(-n)$-homogeneous, with $n > 1$, then h is a finite sum of derivatives if and only if $\int_{\mathbb{S}^{n-1}} h\, \sigma = 0$.*

Proof. Suppose first that $\int_{\mathbb{S}^{n-1}} h\, \sigma = 0$. Since $h(\xi) = r^{-n} h(\omega)$, h is determined by its restriction to $\mathbb{S}^{n-1}$, and the hypothesis says that $\langle 1 \mid h \rangle = 0$ in $L^2(\mathbb{S}^{n-1}, \sigma)$. Thus $h \in (\mathbb{C}1)^\perp = (\ker \Delta)^\perp = \operatorname{im} \Delta$, where Δ is the Laplacian on the sphere $\mathbb{S}^{n-1}$ (which is a Fredholm operator on the Hilbert space $L^2(\mathbb{S}^{n-1}, \sigma)$, with closed range). Thus the equation $h = \Delta g$ has a unique (and C^∞, since Δ is elliptic) solution g on $\mathbb{S}^{n-1}$. Extend g to $\mathbb{R}^n \setminus \{0\}$ by setting $g(r\omega) := r^{-n+2} g(\omega)$ for $0 < r < +\infty$. Since the Laplacian on $\mathbb{R}^n$ is

$$\Delta_{\mathbb{R}^n} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta,$$

we get $\Delta_{\mathbb{R}^n} g = h$, and thus $h = \sum_{j=1}^n \frac{\partial}{\partial \xi_j} \left(-\frac{\partial g}{\partial \xi_j} \right)$ is a finite sum of derivatives.

Suppose instead that $h(\xi) = \frac{\partial f}{\partial \xi_1}$ for $\xi \neq 0$, where f is $(-n+1)$-homogeneous. Let σ' be the volume form on $\mathbb{S}^{n-2}$, and notice that

$$\int_{\mathbb{S}^{n-2}} \int_{\mathbb{R}} \frac{\partial f}{\partial \xi_1}(\xi_1, \omega')\, d\xi_1 \wedge \sigma' = \int_{\mathbb{S}^{n-2}} [f(+\infty, \omega') - f(-\infty, \omega')]\, \sigma' = 0$$

since $f(\xi_1, \omega') \to 0$ as $\xi_1 \to \pm\infty$, by homogenity. Thus we must show that

$$\int_{\mathbb{S}^{n-1}} h\,\sigma = \int_{\mathbb{R}\times\mathbb{S}^{n-2}} h\,d\xi_1 \wedge \sigma'.$$

By Stokes' theorem, we must show that the difference is the integral of the zero n-form on the tube T, whose oriented boundary is $(\mathbb{R}\times\mathbb{S}^{n-2}) - \mathbb{S}^{n-1}$. (Picture a ball stuck in a cylinder of radius 1; T is the region inside the cylinder but outside the ball.) Consider the $(n-1)$-form

$$\tilde{\sigma} = \sum_{j=1}^{n} (-1)^{j-1}\xi_j\, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \in \mathcal{A}^{n-1}(\mathbb{R}^n \setminus \{0\}).$$

If $i \colon \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is the inclusion, then $\sigma = i^*\tilde{\sigma}$. Now $\tilde{\sigma} = \iota_R\nu$, where $\nu = d\xi_1 \wedge \cdots \wedge d\xi_n$, and R is the Euler vector field on $\mathbb{R}^n \setminus \{0\}$. Since $Rh = -nh$ by homogenity, we find that

$$\begin{aligned} d(h\tilde{\sigma}) &= dh \wedge \tilde{\sigma} + h\,d\tilde{\sigma} = dh \wedge \tilde{\sigma} + nh\,\nu \\ &= dh \wedge \iota_R\nu - (Rh)\,\nu = dh \wedge \iota_R\nu - \iota_R(dh)\nu \\ &= -\iota_R(dh \wedge \nu) = \iota_R(0) = 0. \end{aligned}$$

Thus $h\tilde{\sigma}$ is a closed form on $\mathbb{R}^n \setminus \{0\}$, that restricts to $h\sigma$ on $\mathbb{S}^{n-1}$ and to $h\,d\xi_1 \wedge \sigma'$ on $\mathbb{R}\times\mathbb{S}^{n-2}$, therefore

$$\int_{\mathbb{S}^{n-1}} h\,\sigma - \int_{\mathbb{R}\times\mathbb{S}^{n-2}} h\,d\xi_1 \wedge \sigma' = \int_{\partial T} h\,\sigma' = \int_T d(h\tilde{\sigma}) = 0,$$

by Stokes' theorem. Thus $\int_{\mathbb{S}^{n-1}} h\,\sigma = 0$, as required. $\square$

Proposition 5.15. Wres *is a trace on the algebra of classical pseudodifferential operators acting on a fixed vector bundle $E \to M$.*

Proof. We must show that $\mathrm{Wres}([P,Q]) = 0$, for all classical pseudodifferential operators P, Q on M. First we consider the scalar case, where $E = M$. Let $p(x,\xi)$, $q(x,\xi)$ be the complete symbols of P, Q respectively, in some coordinate chart of M. Then if $r(x,\xi)$ is the complete symbol of $[P,Q]$, we know that

$$r(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha p\, D_x^\alpha q - D_\xi^\alpha q\, D_x^\alpha p). \tag{5.13}$$

In particular, the *principal* symbol of $R = [P,Q]$ comes from the terms with $|\alpha| = 1$ in this expansion:

$$\sigma_R(x,\xi) = -i\sum_{j=1}^{n}\left(\frac{\partial p}{\partial\xi_j}\frac{\partial q}{\partial x^j} - \frac{\partial q}{\partial\xi_j}\frac{\partial p}{\partial x^j}\right) = -i\sum_{j=1}^{n}\frac{\partial}{\partial\xi_j}\left(p\,\frac{\partial q}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(p\,\frac{\partial q}{\partial\xi_j}\right).$$

In like manner, when $\alpha_j = 2$ and the other $\alpha_i = 0$, we get the terms

$$\begin{aligned} -\frac{1}{2}&\left(\frac{\partial^2 p}{\partial\xi_j^2}\frac{\partial^2 q}{\partial(x^j)^2} - \frac{\partial^2 q}{\partial\xi_j^2}\frac{\partial^2 p}{\partial(x^j)^2}\right) \\ &= -\frac{1}{2}\frac{\partial}{\partial\xi_j}\left(\frac{\partial p}{\partial\xi_j}\frac{\partial^2 q}{\partial(x^j)^2} - p\,\frac{\partial^3 q}{\partial\xi_j\partial(x_j)^2}\right) - \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial p}{\partial x^j}\frac{\partial^2 q}{\partial\xi_j^2} - p\,\frac{\partial^3 q}{\partial\xi_j^2\partial x_j}\right). \end{aligned}$$

By induction, all terms in the expansion (5.13) that contribute to $r_{-n}(x,\xi)$ are finite sums of derivatives.

In the general case, if $p(x,\xi) = [p_{kl}(x,\xi)]$ and $q(x,\xi) = [q_{kl}(x,\xi)]$ are square matrices, the same argument applies to the sums

$$\sum_{k,l} (D^\alpha_\xi p_{kl}\, D^\alpha_x q_{lk} - D^\alpha_\xi q_{lk}\, D^\alpha_x p_{kl}), \quad \text{for each } \alpha \in \mathbb{N}^n,$$

that contribute to the expansion of $\operatorname{tr} r_{-n}(x,\xi)$. Thus, $\operatorname{tr} r_{-n}(x,\xi)$ is a finite sum of derivatives in the variables x^j and ξ_j. Write

$$\operatorname{tr} r_{-n}(x,\xi) = \sum_{j=1}^n \frac{\partial f_j}{\partial x^j} + \frac{\partial g_j}{\partial \xi_j},$$

where $f_j(x,\xi)$, $g_j(x,\xi)$ vanish outside $K \times \mathbb{R}^n$ for some compact subset $K \subset U$ of a coordinate chart of M. (This can be guaranteed by first writing $P = \sum_r \psi_r P$ and $Q = \sum_r \psi_r Q$ for a suitable partition of unity $\{\psi_r\}$ on M.) Then

$$F_j(x) := \int_{|\xi|=1} f_j(x,\xi)\, \sigma_\xi$$

has $\operatorname{supp} F_j \subset K$, so that

$$\int_{|\xi|=1} \frac{\partial f_j}{\partial x^j}\, \sigma = \frac{\partial F_j}{\partial x^j}, \quad \text{and} \quad \int_U \frac{\partial F_j}{\partial x^j}\, |d^n x| = 0.$$

By construction, $\operatorname{tr} r_{-n}(x,\xi)$, and each $\frac{\partial g_j}{\partial \xi_j}(x,\xi)$ also, are $(-n)$-homogeneous in ξ. Lemma 5.14 now implies that

$$\operatorname{Wres}([P,Q]) = \int_M \left(\int_{|\xi|=1} \left(\sum_{j=1}^n \frac{\partial g_j}{\partial \xi_j} \right) \sigma \right) d^n x = 0. \qquad \square$$

To show that the trace is unique (up to constants) when $n > 1$, let T be any trace on the algebra of classical pseudodifferential operators. Again we suppose that all symbols are supported in a coordinate chart $U \subset M$, and we note that the formulas for composition of symbols give the commutation relations

$$[x^j, f] = i\, \frac{\partial f}{\partial \xi_j}, \qquad [\xi_j, f] = -i\, \frac{\partial f}{\partial x^j}.$$

By Lemmas 5.13 and 5.14, $T(P)$ thus depends only on the homogeneous term $\operatorname{tr} p_{-n}(x,\xi)$, of degree $-n$, and moreover $T(P) = 0$ if $\int_{|\xi|=1} \operatorname{tr} p_{-n}(x,\xi)\, \sigma = 0$. We can replace $\operatorname{tr} p_{-n}(x,\xi)$ with $|\xi|^{-n} \int_{|\xi|=1} \operatorname{tr} p_{-n}(x,\xi)\, \sigma$, without changing $T(P)$. Now $f \mapsto T(f(x)\, |\xi|^{-n})$ is a linear functional on $C^\infty_c(U)$ that kills derivatives with respect to each x^j, so it is a multiple of the Lebesgue integral:

$$T(f) = C \int_U f(x)\, |d^n x| \quad \text{for some} \quad C \in \mathbb{C}.$$

Therefore,

$$T(P) = C \int_U \int_{|\xi|=1} \operatorname{tr} p_{-n}(x,\xi)\, \sigma\, |d^n x| = C \operatorname{Wres}(P).$$

Example 5.16. If (M, g) is a compact Riemann spin manifold with Dirac operator $D\!\!\!/$, then

$$\operatorname{Wres} |D\!\!\!/|^{-n} = 2^m \, \Omega_n \operatorname{Vol}(M).$$

Proof. Recall that the principal symbol of $|D\!\!\!/|$ is $\sigma^{D\!\!\!/}(x, \xi) = c(\xi)$, and that of $D\!\!\!/^2$ (or of Δ^S) is $\sigma^{D\!\!\!/^2}(x, \xi) = c(\xi)^2 = g(\xi, \xi) \, 1_{2^m}$. (Recall that 1_{2^m} means the identity matrix of size 2^m, which is the rank of the spinor bundle.) Thus the principal symbol of $|D\!\!\!/|^{-n}$ is $\sigma^{|D\!\!\!/|^{-n}}(x, \xi) = g(\xi, \xi)^{-n/2} \, 1_{2^m}$. This is homogeneous of degree $-n$, so that $p_{-n}(x, \xi)$ is actually the principal symbol when $P = |D\!\!\!/|^{-n}$. Therefore, $\operatorname{tr} p_{-n}(x, \xi) = 2^m \, g(\xi, \xi)^{-n/2}$.

Now $g(\xi, \xi) = g^{ij} \, \xi_i \xi_j$ in local coordinates on T^*M. To compute its integral over the Euclidean sphere $|\xi| = 1$ [rather than over the ellipsoid $g(\xi, \xi) = 1$], we make a change of coordinates $x \mapsto y = \psi(x)$, and we note that $(x, \xi) \mapsto (y, \eta)$ where $\xi = \psi'(x)^t \, \eta$. We can choose ψ such that $\psi'(x) = [g^{ij}(x)]^{1/2}$, a positive-definite $n \times n$ matrix, in which case $g^{ij}(x) \, \xi_i \xi_j = \delta^{kl} \, \eta_k \eta_l = |\eta|^2$. Now $\operatorname{tr} p_{-n}(y, \eta) = 2^m \, |\eta|^{-n}$, so $\int_{|\eta|=1} \operatorname{tr} \sigma_{-n}(y, \eta) = 2^m \, \Omega_n$, and the Wodzicki residue density is

$$\operatorname{wres}_x |D\!\!\!/|^{-n} = 2^m \, \Omega_n \, |d^n y| = 2^m \, \Omega_n \, \det \psi'(x) \, |d^n x|.$$

But $\det \psi'(x)) = \sqrt{\det g(x)}$ by construction, so we arrive at

$$\operatorname{wres}_x |D\!\!\!/|^{-n} = 2^m \, \Omega_n \sqrt{\det(g_x)} \, |d^n x| = 2^m \, \Omega_n \, \nu_g.$$

Integrating this over M gives $\operatorname{Wres} |D\!\!\!/|^{-n} = 2^m \, \Omega_n \operatorname{Vol}(M)$, as claimed. □

What we have gained? We no longer need the full spectrum of the Dirac operator: its *principal symbol* is enough to give the Wodzicki residue.

5.4 Dixmier trace and Wodzicki residue

There is a third method of computing the logarithmic divergence of the spectrum of $D\!\!\!/$, by means of residue calculus applied to powers of pseudodifferential operators. We shall give (only) a brief outline of what is involved.

Suppose that H is an *elliptic* pseudodifferential operator on $\Gamma(M, E)$ that extends to a positive selfadjoint operator (also denoted here by H) on the Hilbert space $L^2(M, E)$, which is defined as the completion of $\{\, s \in \Gamma(M, E) : \int_M (s \mid s) \, \nu_g < \infty \,\}$ in the norm $\|\psi\| := \sqrt{\langle \psi \mid \psi \rangle}$, where

$$\langle \phi \mid \psi \rangle := \int_M (\phi \mid \psi) \, \nu_g$$

is the scalar product introduced in Section 3.1. We have in mind the example $H = |D\!\!\!/| = (D\!\!\!/^2)^{1/2}$ or else $H = (D\!\!\!/^2 + 1)^{1/2}$, in case $\ker D\!\!\!/ \neq \{0\}$.

Since M is compact, the operator H on $L^2(M, E)$ is known to be Fredholm [Tay], thus $\ker H$ is finite dimensional. We can define its powers H^{-s}, for $s \in \mathbb{C}$, by holomorphic functional calculus:

$$H^{-s} := \frac{1}{2\pi i} \oint_\Gamma \lambda^{-s} (\lambda - H)^{-1} \, d\lambda$$

where Γ is a contour that winds once anticlockwise around the spectrum of H, excluding 0 to avoid the branch point of λ^{-s}. (We define $H^{-s}\psi := 0$ for $\psi \in \ker H$.)

By applying the same Cauchy integrals to the complete symbol of H, one can show that H^{-s} is pseudodifferential, and obtain much information about its integral kernel. This was first done by Seeley [See]. He found that the following properties hold.

- If H has order $d > 0$, then for $\Re s > n/d$, H^{-s} is traceless and $\zeta_H(s) := \operatorname{Tr} H^{-s}$ is holomorphic on this open half-plane.
- For $x \neq y$, the function $s \mapsto K_{H^{-s}}(x, y)$ extends from the half-plane $\Re s > n/d$ to all of $\mathbb{C}$, as an entire function.
- For $x = y$, the function $s \mapsto K_{H^{-s}}(x, x)$ can be continued to a meromorphic function on $\mathbb{C}$, with possible poles only at $\{\, s = (n-k)/d : k = 0, 1, 2, \dots \,\}$.
- The residues at these poles are computed by integrating certain symbol terms over the sphere $|\xi| = 1$ in $T_x^* M$.

Later on, Wodzicki [Wodz] made a deep study of the spectral asymptotics of these operators, and in particular found that at $s = n/d$, the operator $H^{-n/d}$ is of order $(-n)$, and the residue at this pole depends only on its principal symbol; in fact,

$$\operatorname*{Res}_{s=n/d} K_{H^{-s}}(x, x)\, |d^n x| = \frac{1}{d(2\pi)^n} \operatorname{wres}_x H^{-n/d}.$$

Corollary 5.17. *If A is a positive elliptic ΨDO of order $(-n) = -\dim M$ on $L^2(M, E)$, then $s \mapsto \operatorname{Tr} A^s$ is convergent and holomorphic on $\{\, s \in \mathbb{C} : \Re s > -1 \,\}$, it continues meromorphically to $\mathbb{C}$ with a (simple) pole at $s = 1$, and*

$$\operatorname*{Res}_{s=1}(\operatorname{Tr} A^s) = \frac{1}{n(2\pi)^n} \operatorname{Wres} A. \qquad \square$$

(For the proof, one applies Seeley's theory to $H = A^{-1}$.)

A basic result in noncommutative geometry is **Connes' trace theorem** of 1988 [Con1], which shows that this residue is actually a Dixmier trace.

Theorem 5.18 (Connes). *If A is a positive elliptic ΨDO of order $(-n) = -\dim M$ on $\mathcal{H} = L^2(M, E)$, the operator A lies in the Dixmier trace class $\mathcal{L}^{1+}(\mathcal{H})$, it is "measurable", i.e., $\operatorname{Tr}_\omega A =: \operatorname{Tr}^+ A$ is independent of ω, and the following equalities hold:*

$$\operatorname{Tr}^+ A = \operatorname*{Res}_{s=1}(\operatorname{Tr} A^s) = \frac{1}{n(2\pi)^n} \operatorname{Wres} A. \qquad \square$$

We omit the proof, but a few comments can be made. In view of what was already said, it is enough to establish the first equality. The elliptic operator $H = A^{-1}$, of order n, has compact resolvent [Tay], so that A itself is compact (we ignore any finite-dimensional kernel). If the eigenvalues of A are $\lambda_k = s_k(A)$ (listed in decreasing order), the first equality reduces to the following known theorem on divergent series:

Proposition 5.19 (Hardy). *Suppose that $\lambda_k \downarrow 0$ as $k \to \infty$, that $\sum_{k=1}^\infty \lambda_k^s < \infty$ for $s > 1$, and that $\lim_{s \downarrow 1} (s-1) \sum_{k=1}^\infty \lambda_k^s = C$ exists. Then $\frac{1}{\log N} \sum_{k=1}^N \lambda_k \to C$ as $N \to \infty$.*

For a proof of the Proposition, see [GVF, pp. 294–295].

Next note that both $\operatorname{Tr}^+ A$ and $\operatorname{Wres} A$ are bilinear in A, so we can weaken the positivity hypothesis when comparing them. (There are other zeta-residue formulas available which are bilinear in A, but we do not go into that here.)

Corollary 5.20. *If A is a linear combination of positive elliptic pseudodifferential operators of order $(-n)$, then $A \in \mathcal{L}^+$, A is measurable, and $\operatorname{Tr}^+ A = \frac{1}{n(2\pi)^n} \operatorname{Wres} A$.*

Chapter 6

Spectral Triples: General Theory

6.1 The Dixmier trace revisited

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, whose algebra $\mathcal{A}$ is unital. We continue to assume, for convenience, that $\ker D = \{0\}$, so that D^{-1} is a compact operator on $\mathcal{H}$. Suppose now that $|D|^{-p} \in \mathcal{L}^{1+}$, for some $p \geq 1$. Then the functional on $\mathcal{A}$ given by $a \mapsto \operatorname{Tr}_\omega(a\,|D|^{-p})$, for some particular ω, is our candidate for a "noncommutative integral". To see why that should be so, we first examine the commutative case.

Proposition 6.1. *If M is a compact boundaryless n-dimensional spin manifold, with Riemannian metric g and Dirac operator $D\!\!\!/$, then for each $a \in C^\infty(M)$ the operator $a\,|D\!\!\!/|^{-n}$ is measurable, and*

$$\operatorname{Tr}^+(a\,|D\!\!\!/|^{-n}) = C_n \int_M a\,\nu_g,$$

where C_n is a constant depending only on n, namely,

$$C_n = \frac{2^m\,\Omega_n}{n(2\pi)^n}, \quad \text{that is,} \quad \begin{cases} C_{2m} = \dfrac{1}{m!(2\pi)^m} & \text{if } n = 2m, \\ C_{2m+1} = \dfrac{1}{(2m+1)!!\,\pi^{m+1}} & \text{if } n = 2m+1. \end{cases}$$

Proof. We know that $|D\!\!\!/|^{-n}$ is a ΨDO with principal symbol $\sigma^{|D\!\!\!/|^{-n}}(x,\xi) = g_x(\xi,\xi)^{-n/2}\,1_{2^m}$, a scalar matrix of size $2^m \times 2^m$. As a multiplication operator on $L^2(M,S)$, a is a ΨDO of order 0, with principal symbol $\sigma^a(x,\xi) = a(x)\,1_{2^m}$. Thus $a|D\!\!\!/|^{-n}$ is of order $-n$, with $\operatorname{tr}\sigma^{a|D\!\!\!/|^{-n}}(x,\xi) = 2^m\,a(x)\,g_x(\xi,\xi)^{-n/2}$.

Now by the trace theorem, we find, using the calculation in Example **??**,

$$\begin{aligned}
\operatorname{Tr}^+(a|D\!\!\!/|^{-n}) &= \frac{1}{n(2\pi)^n}\operatorname{Wres}(a|D\!\!\!/|^{-n}) \\
&= \frac{2^m}{n(2\pi)^n}\int_M a(x)\left(\int_{|\xi|=1} g_x(\xi,\xi)^{-n/2}\,\sigma\right)|d^nx| \\
&= \frac{2^m\,\Omega_n}{n(2\pi)^n}\int_M a(x)\sqrt{\det g_x}\,|d^nx| \\
&= \frac{2^m\,\Omega_n}{n(2\pi)^n}\int_M a(x)\,\nu_g. \qquad \square
\end{aligned}$$

Therefore, the functional $a \mapsto \operatorname{Tr}^+(a\,|D\!\!\!/|^{-n})$ is just the usual integral with respect to the Riemannian volume form, expect for the normalization constant. Therefore, it can be adapted to more general spectral triples as a "noncommutative integral".

However, in the noncommutative case, it is not obvious that $a \mapsto \mathrm{Tr}^+(a\,|\not{D}|^{-n})$ will be itself a trace. ¿Why should $\mathrm{Tr}^+(ab\,|\not{D}|^{-n})$ be equal to $\mathrm{Tr}^+(ba\,|\not{D}|^{-n}) = \mathrm{Tr}^+(a\,|\not{D}|^{-n}\,b)$? To check this tracial property of the noncommutative integral, we need the Hölder inequality for Dixmier traces.

Fact 6.2 (Horn's inequality). *If $T, S \in \mathcal{K}$ and $n \in \mathbb{N}$, then*

$$\sigma_n(TS) \leq \sum_{k=0}^{n-1} s_k(T)\, s_k(s). \tag{6.1}$$

Proposition 6.3. (a) *If $T \in \mathcal{L}^{1+}$ and S is a bounded operator on $\mathcal{H}$, then for any Dixmier trace Tr_ω, the following inequality holds:*

$$\mathrm{Tr}_\omega |TS| \leq (\mathrm{Tr}_\omega |T|)\, \|S\|. \tag{6.2a}$$

(b) *Let $1 < p < \infty$ and $q = p/(p-1)$, so that $\frac{1}{p} + \frac{1}{q} = 1$, and let $T, S \in \mathcal{K}$ be such that $|T|^p, |S|^q \in \mathcal{L}^{1+}$. Then for any Tr_ω, we get*

$$\mathrm{Tr}_\omega |TS| \leq (\mathrm{Tr}_\omega |T|^p)^{1/p}\, (\mathrm{Tr}_\omega |T|^q)^{1/q}. \tag{6.2b}$$

Proof. Ad (a): By the minimax formula (4.2) for singular values, we find, for each $k \in \mathbb{N}$,

$$\begin{aligned} s_k(TS) &= \inf\{\, \|(1-P)TS\| : P = P^2 = P^*,\ \operatorname{rank} P \leq k \,\} \\ &\leq \inf\{\, \|(1-P)T\|\, \|S\| : P = P^2 = P^*,\ \operatorname{rank} P \leq k \,\} = s_k(T)\, \|S\|, \end{aligned}$$

Summing over $k = 0, 1, \dots, n-1$, we get $\sigma_n(TS) \leq \sigma_n(T)\, \|S\|$. Thus

$$\frac{\sigma_n(TS)}{\log n} \leq \frac{\sigma_n(T)}{\log n}\, \|S\| \quad \text{for all} \quad n \geq 2,$$

and linear interpolation gives the same relation with N replaced by and real $\lambda \geq 2$. Using the definition (4.4) of τ_λ and integrating over $\lambda \geq 3$, we get

$$\tau_\lambda(TS) \leq \tau_\lambda(T)\, \|S\| \quad \text{for all} \quad \lambda \geq 3,$$

and therefore $\mathrm{Tr}_\omega |TS| \leq (\mathrm{Tr}_\omega |T|)\|S\|$ for all ω.

Ad (b): From (6.1) and the ordinary Hölder inequality in $\mathbb{R}^n$, we get

$$\sigma_n(TS) \leq \biggl(\sum_{0 \leq k < n} s_k(T)^p \biggr)^{1/p} \biggl(\sum_{0 \leq k < n} s_k(S)^q \biggr)^{1/q} = \sigma_n(|T|^p)^{1/p}\, \sigma_n(|S|^q)^{1/q}.$$

If $n \leq \lambda < n+1$ with $\lambda = n + t$, then, with $a_n := \sigma_n(|T|^p)^{1/p}$ and $b_n := \sigma_n(|S|^q)^{1/q}$,

$$\begin{aligned} \sigma_\lambda(TS) &= (1-t)\sigma_n(TS) + t\sigma_{n+1}(TS) \\ &\leq (1-t)a_n b_n + t a_{n+1} b_{n+1} \\ &\leq \bigl((1-t)a_n^p + t a_{n+1}^p\bigr)^{1/p} \bigl((1-t)a_n^q + t a_{n+1}^q\bigr)^{1/q} \\ &= \sigma_\lambda(|T|^p)^{1/p}\, \sigma_\lambda(|S|^q)^{1/q} \quad \text{for all } \lambda \geq 2, \end{aligned}$$

where we have used the Hölder inequality in $\mathbb{R}^2$. Again we employ (4.4) and use the Hölder inequality for the integral $\frac{1}{\log \lambda} \int_3^\lambda (\cdot)\, \frac{du}{u}$. This gives

$$\tau_\lambda(TS) \leq \tau_\lambda(|T|^p)^{1/p}\, \tau_\lambda(|T|^q)^{1/q}, \quad \text{for } \lambda \geq 3.$$

Thus $\tau(|TS|) \leq \tau(|T|^p)^{1/p}\, \tau(|T|^q)^{1/q}$ as positive elements of the corona C^*-algebra B_∞. Finally, we use the Hölder inequality for the state ω of this commutative C^*-algebra, namely

$$\omega\big(\tau(|T|)^p)^{1/p}\, \tau(|S|^q)^{1/q}\big) \leq \omega(\tau(|T|)^p)^{1/p}\, \omega(\tau(|S|^q)^{1/q},$$

and the result (6.2b) follows at once. $\square$

Proposition 6.4. *Let $(\mathcal{A}, \mathcal{H}, D)$ be any spectral triple whose operator D is invertible, and let $a \in \mathcal{A}$. Then the commutator $[|D|^r, a]$ is a bounded operator for each r such that $0 < r < 1$.*

We postpone the proof of this Proposition until later. It is a crucial property of spectral triple that this bounded commutator property is *not* automatic for the case $r = 1$, that is, the commutators $[|D|, a]$ need not be bounded in general.

Theorem 6.5. *If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple such that $|D|^{-p} \in \mathcal{L}^{1+}(\mathcal{H})$ for some $p \geq 1$, then for each $a \in \mathcal{A}$ and any $T \in \mathcal{L}(\mathcal{H})$, the following tracial property holds:*

$$\operatorname{Tr}_\omega(aT\,|D|^{-p}) = \operatorname{Tr}_\omega(Ta\,|D|^{-p}) \quad \textit{for all} \quad \omega. \tag{6.3}$$

Proof. Note that $aT\,|D|^{-p}$ and $Ta\,|D|^{-p}$ lie in $\mathcal{L}^{1+}$ since $\mathcal{L}^{1+}$ is an ideal in $\mathcal{L}(\mathcal{H})$. Also the Hölder inequality (6.2a) gives

$$\big|\operatorname{Tr}_\omega([a, T]\,|D|^{-p})\big| = \big|\operatorname{Tr}_\omega(T\,[|D|^{-p}, a])\big| \leq \|T\|\ \operatorname{Tr}_\omega\big|[|D|^{-p}, a]\big|,$$

so we must show that $\operatorname{Tr}_\omega\big|[|D|^{-p}, a]\big| = 0$ for all $a \in \mathcal{A}$. We have not supposed that $p \in \mathbb{N}$, so write $p = kr$ with $k \in \mathbb{N}$, $0 < r < 1$, and let $R := |D|^{-r}$, a positive compact operator. Then

$$[|D|^{-p}, a] = [R^k, a] = \sum_{j=1}^{k} R^{j-1}\,[R, a]\,R^{k-j} = -\sum_{j=1}^{k} R^j\,[|D|^r, a]\,R^{k-j+1}.$$

Applying Hölder's inequality to each term, we get

$$\operatorname{Tr}_\omega\big|R^j\,[|D|^r, a]\,R^{k-j+1}\big| \leq \|[|D|^r, a]\|\,(\operatorname{Tr}_\omega R^{jp_j})^{1/p_j}\,(\operatorname{Tr}_\omega R^{(k-j+1)q_j})^{1/q_j},$$

where $q_j = p_j/(p_j - 1)$ and the number $p_j > 1$ must be chosen so that all R^{jp_j} and all $R^{(k-j+1)q_j}$ are trace-class: for that, we need $rjp_j > p$ and $r(k - j + 1)q_j > p$. This will happen if we take

$$p_j := \frac{p}{r(j - \frac{1}{2})}, \qquad q_j := \frac{p}{r(k - j + \frac{1}{2})},$$

and then $\frac{1}{p_j} + \frac{1}{q_j} = 1$, since $rk = p$. Since Tr_ω vanishes on $\mathcal{L}^1(\mathcal{H})$, we need only to check that

$$|D|^{-p} \in \mathcal{L}^{1+}(\mathcal{H}) \implies |D|^{-s} \in \mathcal{L}^1(\mathcal{H}) \quad \text{for all} \ \ s > p. \tag{6.4}$$

This is a consequence of the next lemma.

Lemma 6.6. *If $A \in \mathcal{L}^{1+}(\mathcal{H})$ and $A \geq 0$, then $A^s \in \mathcal{L}^1(\mathcal{H})$ for $s > 1$.*

Proof. We need the following result on sequence spaces. If E is a Banach space, we denote by E^* the dual Banach space of continuous linear forms on E.

Fact 6.7. *If $\mathbf{s} := \{\, (s_0, s_1, \dots) \in \mathbb{C}^\mathbb{N} : (s_0 + \cdots + s_{n-1})/\log n$ is bounded $\}$, if $\mathbf{s}_0$ is the closure of the finite sequences in $\mathbf{s}$, and if $\mathbf{t} := \{\, (t_0, t_1, \dots) \in \mathbb{C}^\mathbb{N} : \sum_{k\geq 0} |t_k|/(k + 1) < \infty \,\}$, then $\mathbf{s}_0$, $\mathbf{s}$ and $\mathbf{t}$ are complete in the obvious norms, and under the standard duality pairing $\langle s, t\rangle := \sum_{k\geq 0} s_k t_k$, there are isometric isomorphisms $\mathbf{s}_0^* \simeq \mathbf{t}$ and $\mathbf{t}^* \simeq \mathbf{s}$.*

Now let $\mathcal{K}^- := \{\, T \in \mathcal{K} : \{s_k(T)\}_{k\geq 0} \in \mathbf{t} \,\}$, and let $\mathcal{L}_0^{1+}$ be the closure of the finite-rank operators in $\mathcal{L}^{1+}$. Then $(\mathcal{L}_0^{1+})^* \simeq \mathcal{K}^-$ and $(\mathcal{K}^-)^* \simeq \mathcal{L}^{1+}$ as Banach spaces.

For $T \in \mathcal{L}^q$ with $1 < q < \infty$, the Hölder inequality for sequences gives

$$\sum_{k\geq 0} \frac{s_k(T)}{k+1} \leq \biggl(\sum_{k\geq 0} s_k(T)^q\biggr)^{1/q} \biggl(\sum_{k\geq 0} \frac{1}{(k+1)^p}\biggr)^{1/p} = \|T\|_q \, \zeta(p)^{1/p} < \infty,$$

so that $\mathcal{L}^q \subset \mathcal{K}^-$ for all $1 < q < \infty$. Since $(\mathcal{L}^q)^* \simeq \mathcal{L}^p$ with $p = q/(q-1)$, we conclude that $\mathcal{L}^{1+} \subset \mathcal{L}^p$ for all $p > 1$. (This is why we employ the notation $\mathcal{L}^{1+}$, of course.)

Now if $A \in \mathcal{L}^{1+}$ with $A \geq 0$, then $A^s \in \mathcal{L}^{p/s}(\mathcal{H})$ whenever $1 < s \leq p$. In particular, when $p = s$, we see that

$$\|A^s\|_1 = \sum_{k\geq 0} s_k(A^s) = \sum_{k\geq 0} \lambda_k(A^s) = \sum_{k\geq 0} \lambda_k(A)^s = (\|A\|_s)^s < +\infty,$$

since $A \in \mathcal{L}^{1+}$ implies $A \in \mathcal{L}^s$. ☐

This establishes (6.4) and concludes the proof of Theorem 6.5. ☐

Corollary 6.8. *If $A \geq 0$ is in $\mathcal{L}^{1+}$, and $\operatorname{Tr}^+ A > 0$, then $\operatorname{Tr}^+ A^s = 0$ for $s > 1$.*

To establish Proposition 6.4, we use the following commutator estimate, due to Helton and Howe [HH].

Lemma 6.9. *Let D be a selfadjoint operator on $\mathcal{H}$, and let $a \in \mathcal{L}(\mathcal{H})$ with $a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ be such that $[D,a]$ extends to a bounded operator on $\mathcal{H}$. Suppose also that $g\colon \mathbb{R} \to \mathbb{R}$ is smooth, with Fourier transform is a function $\hat{g}$ such that $t \mapsto t\,\hat{g}(t)$ is integrable on $\mathbb{R}$. Then $[g(D),a]$ extends to a bounded operator on $\mathcal{H}$, such that*

$$\|[g(D),a]\| \leq \frac{1}{2\pi} \|[D,a]\| \int_{\mathbb{R}} |t\,\hat{g}(t)|\,dt.$$

Proof. We may define

$$g(D) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(t) e^{itD}\,dt$$

for any smooth function g, since $\hat{g}$ has compact support. Now

$$\begin{aligned} \langle \phi \mid [e^{itD}, a]\psi \rangle &= \int_0^1 \frac{d}{ds} \langle \phi \mid e^{istD}\, a\, e^{i(1-s)tD} \psi \rangle \,ds \\ &= it \int_0^1 \frac{d}{ds} \bigl(\langle De^{-istD}\phi \mid ae^{i(1-s)tD}\psi \rangle - \langle \phi \mid e^{istD} aDe^{i(1-s)tD}\psi \rangle \bigr)\,ds \end{aligned}$$

for $\phi, \psi \in \operatorname{Dom} D$, and thus

$$\begin{aligned} &\bigl| \langle g(D)\phi \mid a\psi \rangle - \langle \phi \mid ag(D)\psi \rangle \bigr| \\ &\qquad = \frac{1}{2\pi} \biggl| \int_{\mathbb{R}} t\hat{g}(t) \int_0^1 \frac{d}{ds} \bigl(\langle De^{-istD}\phi \mid ae^{i(1-s)tD}\psi \rangle - \langle \phi \mid e^{istD} aDe^{i(1-s)tD}\psi \rangle \bigr)\,ds\,dt \biggr| \\ &\qquad \leq \frac{\|[D,a]\|}{2\pi} \,\|\phi\|\,\|\psi\| \int_{\mathbb{R}} |t\,\hat{g}(t)|\,dt, \end{aligned}$$

so that $[g(D),a]$ extends to a bounded operator, and the required estimate holds. ☐

Proof of Proposition 6.4. We want to apply Lemma 6.9, using $|x|^r$ instead of $g(x)$, $x \in \mathbb{R}$. But $x \mapsto |x|^r$ is not smooth at $x = 0$ (although it is homogeneous of degree r), so we modify it near $x = 0$ to get a smooth function $g(x)$ such that $g(x) = |x|^r$ for $|x| \geq \delta$, for some $\delta > 0$. Thus $g(x) = |x|^r + h(x)$, where $\operatorname{supp} h \subset [-\delta, \delta]$. We can write its derivative as a sum of two terms, $g'(x) = u(x) + h'(x)$, where $\operatorname{supp} h' \subset [-\delta, \delta]$ and u is homogeneous of negative degree $r - 1$. Taking Fourier transforms on $\mathbb{R}$, we get $it\,\hat{g}(t) = \hat{u}(t) + \hat{h}'(t)$, where $\hat{h}'(t)$ is analytic and $\hat{u}$ is homogeneous of degree $-1 - (r - 1) = -r$, with $-1 < -r < 0$. Thus $t\hat{g}(t)$ is locally integrable near $t = 0$, and $t\hat{g}(t) \to 0$ rapidly for large t, since g' is smooth. We end up with an estimate

$$\|[|D|^r, a]\| \leq C_r \|[D, a]\| + \|[h(D), a]\|,$$

where $C_r := (2\pi)^{-1} \int_{\mathbb{R}} |t\hat{g}(t)|\, dt$ is finite, and $\|[h(D), a]\|$ is finite since $h(D)$ is a bounded operator. $\square$

6.2 Regularity of spectral triples

The arguments of the previous section are not applicable to determine whether $[|D|, a]$ is bounded, in the case $r = 1$. This must be formulated as an assumption. In fact, we shall ask for much more: we want each element $a \in \mathcal{A}$, and each bounded operator $[D, a]$ too, to lie in the *smooth domain* of the following derivation.

Notation. We denote by δ the derivation on $\mathcal{L}(\mathcal{H})$ given by taking the commutator with $|D|$. It is an unbounded derivation, whose domain is

$$\operatorname{Dom} \delta := \{\, T \in \mathcal{L}(\mathcal{H}) : T(\operatorname{Dom} |D|) \subseteq \operatorname{Dom} |D|,\ \ [|D|, T] \text{ is bounded } \}.$$

We write $\delta(T) := [|D|, T]$ for $T \in \operatorname{Dom} \delta$.

Definition 6.10. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called **regular**, if for each $a \in \mathcal{A}$, the operators a and $[D, a]$ lie in $\bigcap_{k \in \mathbb{N}} \operatorname{Dom} \delta^k$.*

The regularity condition does not depend on the invertibility of D (that is, the condition $\ker D = \{0\}$) which we have been assuming, to simplify certain calculations. One can always replace $|D|$ by $\langle D \rangle := (D^2 + 1)^{1/2}$ in the definition, since $f(D) := \langle D \rangle - |D|$ is bounded. If δ' denotes the derivation $\delta'(T) := [\langle D \rangle, T] = \delta(T) + [f(D), T]$, then clearly $\operatorname{Dom} \delta' = \operatorname{Dom} \delta$, and it is easy to show by induction that $\operatorname{Dom} \delta'^k = \operatorname{Dom} \delta^k$ for each $k \in \mathbb{N}$, so one may instead define regularity using δ'. This is the approach taken in the work of Carey et al [CPRS], who use the term "QC^∞" instead of "regular" for this class of spectral triples.

Definition 6.11. *Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, with D invertible. For each $s \geq 0$, consider the operator $|D|^s$ defined by functional calculus. Define also $\mathcal{H}^s := \operatorname{Dom} |D|^s$ for $s \geq 0$, with the Hilbert norm $\|\xi\|_s := \sqrt{\|\xi\|^2 + \||D|^s \xi\|^2}$. Their intersection $\mathcal{H}^\infty = \bigcap_{s \geq 0} \operatorname{Dom} |D|^s = \bigcap_{k=0}^{\infty} \operatorname{Dom} |D|^k$ is the smooth domain of the positive selfadjoint operator $|D|$. Its topology is defined by the seminorms $\| \cdot \|_k$, for $k \in \mathbb{N}$. Each $\mathcal{H}^s$ (and thus also $\mathcal{H}^\infty$) is complete, since the operators $|D|^s$ are closed, thus $\mathcal{H}^\infty$ is a Fréchet space.*

Since $a \in \mathcal{A}$ implies $a \in \operatorname{Dom} \delta$, we see that $a(\mathcal{H}^1) \subseteq \mathcal{H}^1$, and then we can write $a(|D|\xi) = |D|(a\xi) - [|D|, a]\, \xi$ for $\xi \in \mathcal{H}^1$. Also,

$$\begin{aligned}
\|a\xi\|_1^2 &= \|a\xi\|^2 + \||D| a\xi\|^2 \\
&= \|a\xi\|^2 + \|a|D|\xi + \delta(a)\xi\|^2 \\
&\leq \|a\xi\|^2 + 2\|\delta(a)\xi\|^2 + 2\|a|D|\xi\|^2 \\
&\leq \max\{\|a\|^2 + 2\|\delta(a)\|^2, 2\|a\|^2\}\, \|\xi\|_1^2,
\end{aligned}$$

where we have used the parallelogram law $\|\xi+\eta\|^2+\|\xi-\eta\|^2=2\|\xi\|^2+2\|\eta\|^2$. Therefore, a extends to a bounded operator on $\mathcal{H}^1$. If $(\mathcal{A},\mathcal{H},D)$ is regular, then by induction we find that $a(\mathcal{H}^k)\subset\mathcal{H}^k$ continuously for each k, so that $a(\mathcal{H}^\infty)\subset\mathcal{H}^\infty$ continuously, too.

Definition 6.12. *If $r\in\mathbb{Z}$, let Op_D^r be the vector space of linear maps $T\colon\mathcal{H}^\infty\to\mathcal{H}^\infty$ for which there are constants C_k, for $k\in\mathbb{N}$, $k\geq r$, such that*

$$\|T\xi\|_{k-r}\leq C_k\,\|\xi\|_k \quad \text{for all } \xi\in\mathcal{H}^\infty.$$

Every such T extends to a bounded operator from $\mathcal{H}^k$ to $\mathcal{H}^{k-r}$, for each $k\in\mathbb{N}$. Note that $|D|^r\in\mathrm{Op}_D^r$ for each $r\in\mathbb{Z}$. If $T\in\mathrm{Op}_D^r$ and $S\in\mathrm{Op}_D^s$, then $ST\in\mathrm{Op}_D^{r+s}$.

Suppose $(\mathcal{A},\mathcal{H},D)$ is regular. Then $\mathcal{A}\subset\mathrm{Op}_D^0$ and $[D,\mathcal{A}]:=\{\,[D,a]:a\in\mathcal{A}\,\}\subset\mathrm{Op}_D^0$, too. Moreover, if $a\in\mathcal{A}$, then

$$\begin{aligned}
[D^2,a]&=[|D|^2,a]=|D|\,[|D|,a]+[|D|,a]\,|D| \\
&=|D|\delta(a)+\delta(a)|D|=2|D|\delta(a)-[|D|,\delta(a)] \\
&=2|D|\delta(a)-\delta^2(a),
\end{aligned}$$

so that $[D^2,a]\in\mathrm{Op}_D^1$. Also $[D^2,[D,a]]\in\mathrm{Op}_D^1$ in the same way.

If b lies the subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $\mathcal{A}$ and $[D,\mathcal{A}]$, we introduce

$$\begin{aligned}
L(b)&:=|D|^{-1}\,[D^2,b]=2\delta(b)-|D|^{-1}\,\delta^2(b), \\
R(b)&:=[D^2,b]\,|D|^{-1}=2\delta(b)+\delta^2(b)\,|D|^{-1}.
\end{aligned} \tag{6.5}$$

If $b\in\bigcap_{k\geq 0}\mathrm{Dom}\,\delta^k$, then $L(b)$ and $R(b)$ lie in Op_D^0. The operations L and R commute: indeed,

$$L(R(b))=|D|^{-1}\,[D^2,[D^2,b]\,|D|^{-1}]=|D|^{-1}\,[D^2,[D^2,b]\,|D|^{-1}=R(L(b)).$$

Note also that $L^2(b)=|D|^{-2}\,[D^2,[D^2,b]]$.

Proposition 6.13. *If D is invertible, then $\bigcap_{k,l\geq 0}\mathrm{Dom}(L^kR^l)=\bigcap_{m\geq 0}\mathrm{Dom}\,\delta^m\subset\mathcal{L}(\mathcal{H})$.*

Proof. We use the following identity for $|D|^{-1}$, obtained from the spectral theorem:

$$|D|^{-1}=\frac{2}{\pi}\int_0^\infty (D^2+\mu^2)^{-1}\,d\mu, \tag{6.6}$$

in order to compute the commutators. We shall show that $\mathrm{Dom}\,L^2\cap\mathrm{Dom}\,R\subset\mathrm{Dom}\,\delta$.

Indeed, if $b\in\mathrm{Dom}\,L^2\cap\mathrm{Dom}\,R$ implies $b\in\mathrm{Dom}\,\delta$, then $b\in\mathrm{Dom}\,L^4\cap\mathrm{Dom}\,L^2R\cap\mathrm{Dom}\,R^2$ implies $\delta b\in\mathrm{Dom}\,L^2\cap\mathrm{Dom}\,R$, so $b\in\mathrm{Dom}\,\delta^2$. By induction, $\bigcap_{k,l\geq 0}\mathrm{Dom}(L^kR^l)\subset\bigcap_{m\geq 0}\delta^m$. The converse inclusion is clear, from (6.5).

Take $b \in \operatorname{Dom} L^2 \cap \operatorname{Dom} R$, and compute $[|D|, b]$ as follows:

$$\begin{aligned}
[|D|, b] &= [D^2|D|^{-1}, b] = [D^2, b]\,|D|^{-1} + D^2\,[|D|^{-1}, b] \\
&= \frac{2}{\pi}\int_0^\infty \big([D^2, b](D^2+\mu^2)^{-1} + D^2[(D^2+\mu^2)^{-1}, b]\big)\,d\mu \\
&= \frac{2}{\pi}\int_0^\infty \big([D^2, b](D^2+\mu^2)^{-1} - D^2(D^2+\mu^2)^{-1}[D^2+\mu^2, b](D^2+\mu^2)^{-1}\big)\,d\mu \\
&= \frac{2}{\pi}\int_0^\infty (1 - D^2(D^2+\mu^2)^{-1})[D^2, b](D^2+\mu^2)^{-1}\,d\mu \\
&= \frac{2}{\pi}\int_0^\infty \mu^2(D^2+\mu^2)^{-1}[D^2, b](D^2+\mu^2)^{-1}\,d\mu \\
&= \frac{2}{\pi}\int_0^\infty \big([D^2, b](D^2+\mu^2)^{-2} + [(D^2+\mu^2)^{-1}, [D^2, b]](D^2+\mu^2)^{-1}\big)\mu^2\,d\mu \\
&= \frac{2}{\pi}\int_0^\infty \big(R(b)|D|(D^2+\mu^2)^{-2} - (D^2+\mu^2)^{-1}D^2L^2(b)(D^2+\mu^2)^{-2}\big)\mu^2\,d\mu
\end{aligned}$$

Now $R(b)$ and $\frac{D^2}{D^2+\mu^2}L^2(b)$ are bounded, by hypothesis. Also

$$\frac{2}{\pi}\int_0^\infty \frac{x\mu^2\,d\mu}{(x^2+\mu^2)^2} = \frac{2}{\pi}\int_0^\infty \frac{t^2\,dt}{1+t^2} = \frac{2}{\pi}\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{1}{2},$$

while

$$\int_0^\infty (D^2+\mu^2)^{-2}\mu^2\,d\mu = \int_0^1 (D^2+\mu^2)^{-2}\mu^2\,d\mu + \int_1^\infty (D^2+\mu^2)^{-2}\mu^2\,d\mu$$

is bounded by

$$\int_0^1 \|D^{-4}\|\mu^2\,d\mu + \int_0^\infty \mu^{-2}\,d\mu = \frac{1}{3}\,\|D\|^{-4} + 1.$$

Thus $[|D|, b]$ is bounded with the estimate

$$\|[|D|, b]\| \le \frac{1}{2}\|R(b)\| + \left(\frac{1}{3}\|D\|^{-4} + 1\right)\frac{2}{\pi}\,\|L^2(b)\|.$$

Hence $b \in \operatorname{Dom}\delta$, as desired. □

Corollary 6.14. *The standard commutative example $(C^\infty(M), L^2(M, S), D\!\!\!/)$ is a regular spectral triple.*

Proof. We need one more fact from the theory of ΨDOs (see [Tay], for example): over a compact manifold M, with a hermitian vector bundle E, a ΨDO of order zero is bounded as an operator on $L^2(M, E)$. Thus we need only show that, if $b = a$ or $b = [D\!\!\!/, a] = -i\,c(da)$, then $L^k R^l$ is a ΨDO of order ≤ 0, for each $k, l \in \mathbb{N}$.

For $k = l = 0$, note that $\psi \mapsto a\psi$ and $\psi \mapsto [D\!\!\!/, a]\psi = -i\,c(da)\psi$ are bounded multiplication and Clifford-action operators. Their (principal) symbols are

$$\begin{aligned}
\sigma^a(x, \xi) &= a(x)\,1_{2^m}, \\
\sigma^{[D\!\!\!/, a]}(x, \xi) &= -i\sum_{j=1}^n \partial_j a(x)c^j = -i\,\{c(\xi), a(x)\} = -i\,c(da).
\end{aligned}$$

For $k + l > 0$, we use $L^k R^l = |D\!\!\!/|^{-k}(\operatorname{ad} D\!\!\!/^2)^{k+l}(\cdot)|D\!\!\!/|^{-l}$. Now $D\!\!\!/^2$ is a second-order ΨDO, with $\sigma_2^{D\!\!\!/^2}(x, \xi) = g(\xi, \xi)\,1_{2^m}$, so that when P is of order d then $[D\!\!\!/^2, P]$ is of order $\le d + 1$. Hence, if $a \in C^\infty(M)$, then $(\operatorname{ad} D\!\!\!/^2)^{k+l}(a)$ is of order $\leq k+l$, and thus $L^k R^l$ is of order ≤ 0. The same is true if a is replaced by $-i\,c(da)$. Thus $L^k R^l(b)$ is bounded, if $b \in \mathcal{A}$ or $b \in [D\!\!\!/, \mathcal{A}]$. $\square$

This example also shows why regularity is defined using the derivation $\delta = [|D|, \cdot]$ instead of the apparently simpler derivation $[D, \cdot]$. Indeed, we have just seen that for $a \in C^\infty(M)$, the operator $[|D\!\!\!/|, [D\!\!\!/, a]]$ has order zero (and therefore, it lies in $\mathrm{Op}^0_{D\!\!\!/}$. On the other hand, $[D\!\!\!/, [D\!\!\!/, a]]$ is in general a ΨDO of order 1 (and so it lies in $\mathrm{Op}^1_{D\!\!\!/}$). Indeed, the first-order terms in its symbol are

$$[\sigma^{D\!\!\!/}, \sigma^{[D\!\!\!/,a]}](x,\xi) = [c^j \xi_j, -i\, c^k\, \partial_k a(x)] = -i\,[c^j, c^k]\, \xi_j\, \partial_k a(x)$$

which need not vanish since c^j, c^k do not commute. In contrast, the principal symbol of $|D\!\!\!/|$ is a scalar matrix, which commutes with that of $[D\!\!\!/, a]$, and the order of the commutator drops to zero.

6.3 Pre-C*-algebras

If any spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the algebra $\mathcal{A}$ is a (unital) $*$-algebra of bounded operators acting on a Hilbert space $\mathcal{H}$ [or, if one wishes to regard $\mathcal{A}$ abstractly, a faithful representation $\pi \colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is given]. Let A be the norm closure of $\mathcal{A}$ [or of $\pi(\mathcal{A})$] in $\mathcal{L}(\mathcal{H})$: it is a C^*-algebra in which $\mathcal{A}$ is a dense $*$-subalgebra.

A priori, the only functional calculus available for $\mathcal{A}$ is the holomorphic one:

$$f(a) := \frac{1}{2\pi i} \oint_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\, d\lambda, \tag{6.7}$$

where Γ is a contour in $\mathbb{C}$ winding (once positively) around $\mathrm{sp}(a)$, and $\mathrm{sp}(a)$ means the spectrum of a in the C^*-algebra A. To ensure that $a \in \mathcal{A}$ implies $f(a) \in \mathcal{A}$, we need the following property:

If $a \in \mathcal{A}$ has an inverse $a^{-1} \in A$, then in fact a^{-1} lies in $\mathcal{A}$ (briefly: $\mathcal{A} \cap A^\times = \mathcal{A}^\times$, where $\mathcal{A}^\times$ is the group of invertible elements of A). If this condition holds, then $\frac{1}{2\pi i}\oint_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\, d\lambda$ is a limit of Riemann sums lying in $\mathcal{A}$. To ensure convergence in $\mathcal{A}$ (they do converge in A), we need only ask that $\mathcal{A}$ be complete in some topology that is finer than the C^*-norm topology.

Definition 6.15. *A **pre-C^*-algebra** is a subalgebra of $\mathcal{A}$ of a C^*-algebra A, which is stable under the holomorphic functional calculus of A.*

Remark 6.16. This condition appears in Blackadar's book [Bla] under the name "local C^*-algebra". However, one can wonder how such a property could be checked in practice. Consider the two conditions on a $*$-subalgebra $\mathcal{A}$ of a unital C^*-algebra A:

(a) $\mathcal{A}$ is *stable under holomorphic functional calculus*; that is, $a \in \mathcal{A}$ implies $f(a) \in \mathcal{A}$, according to (6.7).

(b) $\mathcal{A}$ is *spectrally invariant* in A [Schw], that is,

$$a \in \mathcal{A} \text{ and } a^{-1} \in A \implies a^{-1} \in \mathcal{A}. \tag{6.8}$$

In particular, (6.8) implies $\mathrm{sp}_{\mathcal{A}}(a) = \mathrm{sp}_A(a)$, for all $a \in \mathcal{A}$.

Question: If $\mathcal{A}$ is known to have a (locally convex) vector space topology under which $\mathcal{A}$ is complete (needed for convergence of the Riemann sums defining the contour integral) and such that the inclusion $\mathcal{A} \hookrightarrow A$ is continuous, ¿are (a) and (b) equivalent?

Ad (a) $\Longrightarrow$ (b): This is clear: if $a \in \mathcal{A}$, $a^{-1} \in A$, use $f(\lambda) := 1/\lambda$ outside $\mathrm{sp}_A(a)$.

Ad (b) $\Longrightarrow$ (a): To prove that the integral converges in $\mathcal{A}$, because $\mathcal{A}$ is complete, we need to show that the integrand is continuous. Note that since the inclusion $i \colon \mathcal{A} \hookrightarrow A$ is continuous, then $\mathcal{A}^\times = \{\, a \in \mathcal{A} : a^{-1} \in \mathcal{A} \,\} = \mathcal{A} \cap A^\times = i^{-1}(A^\times)$ is open in $\mathcal{A}$. But we still need to show that $a \mapsto a^{-1} \colon \mathcal{A}^\times \to \mathcal{A}^\times$ is continuous. This will follow if $\mathcal{A}$ is a *Fréchet* algebra [Schw].

Corollary 6.17 (Schweitzer). *If $\mathcal{A}$ is a unital Fréchet algebra, and if $\|\cdot\|_A$ is continuous in the topology of $\mathcal{A}$, then Conditions (a) and (b) are equivalent.*

If $\mathcal{A}$ is a nonunital algebra, we can always adjoin a unit in the usual way, and work with $\widetilde{\mathcal{A}} := \mathbb{C} \oplus \mathcal{A}$ whose unit is $(1, 0)$, and with its C^*-completion $\widetilde{A} := \mathbb{C} \oplus A$. Since the multiplication rule in $\widetilde{\mathcal{A}}$ is $(\lambda, a)(\mu, b) := (\lambda\mu, \mu a + \lambda b + ab)$, we see that $1 + a := (1, a)$ is invertible in $\widetilde{\mathcal{A}}$, with inverse $(1, b)$, if and only if $a + b + ab = 0$.

Lemma 6.18. *If $\mathcal{A}$ is a unital, Fréchet pre-C^*-algebra, then so also is $M_n(\mathcal{A}) = M_n(\mathbb{C}) \otimes \mathcal{A}$.*

Sketch proof. It is enough to show that $a \in M_n(\mathcal{A})$ is invertible for a close to the identity 1_n in the norm of $M_n(A)$. But for a close to 1_n, the procedure of Gaussian elimination gives matrix factorization $a =: ldu$, where

$$l = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \dots & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & d_n \end{pmatrix}, \qquad u = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

with $d_j \in \mathcal{A}$ such that $\|d_j - 1\|_A < 1$, for $j = 1, \dots, n$. Thus d^{-1} exists, and $a^{-1} = u^{-1} d^{-1} l^{-1} \in M_n(\mathcal{A})$.

For $n = 2$, we get explicitly

$$a = \begin{pmatrix} 1 & 0 \\ a_{21} a_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_1 1 & 0 \\ 0 & a_{22} - a_{21} a_{11}^{-1} a_{12} \end{pmatrix} \begin{pmatrix} 1 & a_{11}^{-1} a_{12} \\ 0 & 1 \end{pmatrix},$$

provided $\|1 - a_{11}\|_A < 1$. For larger n, if $\|1_n - a\|_{M_n(A)} < \delta$ for δ small enough, we can perform $(n-1)$ steps of Gaussian elimination (without any exchanges of rows or columns) and get the factorization $a = ldu$ in $M_n(\mathcal{A})$ with d invertible. $\square$

Lemma 6.19. *The Schwartz algebra $\mathcal{S}(\mathbb{R}^n)$ is a nonunital pre-C^*-algebra.*

Proof. We represent $\mathcal{S}(\mathbb{R}^n)$ by multiplication operators on $L^2(\mathbb{R}^n)$. Its C^*-completion is $C_0(\mathbb{R}^n)$. Note that $\mathbb{C}1 \oplus C_0(\mathbb{R}^n) \simeq C(\mathbb{S}^n)$. Suppose $f \in \mathcal{S}(\mathbb{R}^n)$, and that there exists $g \in C_0(\mathbb{R}^n)$ such that $(1 + f)(1 + g) = 1$. Then $f + g + fg = 0$, and $1 + g = 1/(1 + f)$ in $C(\mathbb{S}^n)$. Now, since f is C^∞, then in particular g is smooth on $\mathbb{R}^n$ and all derivatives $\partial^\alpha g$ are bounded. This entails that $fg \in \mathcal{S}(\mathbb{R}^n)$ also.

Finally, $g = -f - fg$ lies in $\mathcal{S}(\mathbb{R}^n)$, so that $(1 + g) = (1 + f)^{-1}$ lies in $\mathbb{C}1 \oplus \mathcal{S}(\mathbb{R}^n)$, as required. $\square$

Example 6.20. If M is compact boundaryless smooth manifold, then $C^\infty(M)$ is a unital Fréchet pre-C^*-algebra. The topology on $C^\infty(M)$ is that of "uniform convergence of all derivatives":

$$f_k \to f \text{ in } C^\infty(M) \quad \text{if and only if } \|X_1 \dots X_r f_k - X_1 \dots X_r f\|_\infty \to 0 \quad \text{as } k \to \infty,$$

for each finite set of vector fields $\{X_1, \dots, X_r\} \in \mathfrak{X}(M)$. This makes $C^\infty(M)$ a Fréchet space. If $f \in C^\infty(M)$ is invertible in $C(M)$, then $f(x) \neq 0$ for any $x \in X$, and so $1/f$ is also smooth. Thus $C^\infty(M)^\times = C^\infty(M) \cap C(M)^\times$.

We state, without proof, two important facts about Fréchet pre-C^*-algebras.

Fact 6.21. *If $\mathcal{A}$ is a Fréchet pre-C^*-algebra and A is its C^*-completion, then $\mathrm{K}_j(\mathcal{A}) = \mathrm{K}_j(A)$ for $j = 0, 1$. More precisely, if $i\colon \mathcal{A} \to A$ is the (continuous, dense) inclusion, then $i_*\colon \mathrm{K}_j(\mathcal{A}) \to \mathrm{K}_j(A)$ is an surjective isomorphism, for $j = 0$ or 1.*

This invariance of K-theory was proved by Bost [Bost]. For K_0, the spectral invariance plays the main role. For K_1, one must first formulate a topological K_1-theory is a category of "good" locally convex algebras (thus whose invertible elements form an open subset and for which inversion is continuous), and it is known that Fréchet pre-C^*-algebras are "good" in this sense.

Fact 6.22. *If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, we can confer on $\mathcal{A}$ the topology given by the seminorms*

$$q_k(a) := \|\delta^k(a)\|, \quad q'_k(a) := \|\delta^k([D, a])\|, \quad \textit{for each } k \in \mathbb{N}. \tag{6.9}$$

The completion $\mathcal{A}_\delta$ of $\mathcal{A}$ is then a Fréchet pre-C^*-algebra, and $(\mathcal{A}_\delta, \mathcal{H}, D)$ is again a regular spectral triple.

These properties of the completed spectral triple are due to Rennie [Ren]. We now discuss another result of Rennie, namely that such completed algebras of regular spectral triples are endowed with a C^∞ *functional calculus.*

Proposition 6.23. *If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, for which $\mathcal{A}$ is complete in the Fréchet topology determined by the seminorms* (6.9)*, then $\mathcal{A}$ admits a C^∞-functional calculus. Namely, if $a = a^* \in \mathcal{A}$, and if $f\colon \mathbb{R} \to \mathbb{C}$ is a compactly supported smooth function whose support includes a neighbourhood of* $\operatorname{sp}(a)$*, then the following element $f(a)$ lies in $\mathcal{A}$:*

$$f(a) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(s) \exp(isa)\, ds. \tag{6.10}$$

Remark 6.24. One may use the continuous functional calculus in the C^*-algebra A to define the one-parameter unitary group $s \mapsto \exp(isa)$, for $s \in \mathbb{R}$. Then the right hand side of (6.10) coincides with the element $f(a) \in A$ defined by the continuous functional calculus in A.

Proof. The map $\delta = \operatorname{ad} |D|\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a closed derivation [BR] since $|D|$ is a selfadjoint operator. To show that $f(a) \in \operatorname{Dom} \delta$ and that

$$\delta(f(a)) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)\, \delta(\exp(ita))\, dt, \tag{6.11}$$

we need to show that the integral on the right hand side converges. Indeed, by the same token, the formula

$$\delta(\exp(ita)) = it \int_0^1 \exp(ista)\, \delta(a)\, \exp(i(1-s)ta)\, ds$$

shows that $\exp(ita) \in \operatorname{Dom}\delta$ because

$$\left| it \int_0^1 \exp(ista)\,\delta(a)\,\exp(i(1-s)ta)\,ds \right| \leq |t| \int_0^1 \|\delta(a)\|\,ds = |t|\,\|\delta(a)\|,$$

and dominated convergence of the integral follows. Plugging this estimate into (6.11), we get

$$\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(t)|\,\|\delta(\exp(ita))\|\,dt \leq \frac{1}{2\pi}\,\|\delta(a)\| \int_{\mathbb{R}} |t\hat{f}(t)|\,dt < +\infty,$$

since $f \in C_c^\infty(\mathbb{R})$ implies $\hat{f} \in \mathcal{S}(\mathbb{R})$. Thus $f(a) \in \operatorname{Dom}\delta$, and (6.11) holds.

Now let A_m, for $m \in \mathbb{N}$, be the completion of $\mathcal{A}$ in the norm

$$a \mapsto \sum_{k=0}^m q_k(a) + q'_k(a) = \sum_{k=0}^m \|\delta^k(a)\| + \|\delta^k([D,a])\|.$$

For $m = 0$, we get

$$\|f(a)\| + \|[D, f(a)]\| \leq \frac{1}{2\pi} \int_{\mathbb{R}} (|\hat{f}(t)| + \|[D,a]\|\,|t\hat{f}(t)|)\,dt$$

by replacing $|D|$ by D in the previous argument.

Therefore, $f(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(s)\,\exp(isa)\,ds$ is a limit of Riemann sums converging in the norm $q_0 + q'_0$, so that $f(a) \in \mathcal{A}_0$. Next, since δ and $(\operatorname{ad} D)$ are commuting derivations (on $\mathcal{A}$), we get $[D, f(a)] \in \operatorname{Dom}\delta$, with

$$\begin{aligned}
\big\|\delta([D, f(a)])\big\| &\leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(t)|\,\|\delta([D, \exp(ita)])\|\,dt \\
&\leq \frac{1}{2\pi} \int_{\mathbb{R}} \big(|t\hat{f}(t)|\,\|\delta([D,a])\| + |t^2\hat{f}(t)|\,\|\delta(a)\|\,\|[D,a]\|\big)\,dt,
\end{aligned}$$

since $t\hat{f}(t)$ and $t^2\hat{f}(t)$ lie in $\mathcal{S}(\mathbb{R})$. We conclude that δ extends to a closed derivation from A_0 to $\mathcal{L}(\mathcal{H})$.

By an (ugly) induction on m, we find that for $k = 0, 1, \dots, m$ $f(a)$ and $[D, f(a)]$ lie in $\operatorname{Dom}\delta^k$, and that δ extends to a closed derivation from A_m to $\mathcal{L}(\mathcal{H})$, and that $f(a) \in A_m$. By hypothesis, $\mathcal{A} = \bigcap_{m\in\mathbb{N}} A_m$, and thus $f(a) \in \mathcal{A}$. □

Before showing how this smooth functional calculus can yield useful results, we pause for a couple of technical lemmas on approximation of idempotents and projectors, in Fréchet pre-C^*-algebras. The first is an adaptation of a proposition of [Bost].

Lemma 6.25. *Let $\mathcal{A}$ be an unital Fréchet pre-C^*-algebra, with C^*-norm $\|\cdot\|$. Then for each ε with $0 < \varepsilon < \frac{1}{8}$, we can find $\delta \leq \varepsilon$ such that, for each $v \in \mathcal{A}$ with $\|v - v^2\| < \delta$ and $\|1 - 2v\| < 1 + \delta$, there is an idempotent $e = e^2 \in \mathcal{A}$ such that $\|e - v\| < \varepsilon$.*

Proof. Consider the holomorphic function

$$f : \{\, \lambda \in \mathbb{C} : |\lambda| < \tfrac{1}{4} \,\} \to \mathbb{C} \quad \text{defined by} \quad f(\lambda) := \tfrac{1}{2}\big(1 - \sqrt{1 + 4\lambda}\big),$$

where we choose the branch of the square root for which $\sqrt{1} = +1$. Note that $f(0) = 0$, and that $(1 - 2f(\lambda))^2 = 1 + 4\lambda$, so that

$$f(\lambda)^2 - f(\lambda) = \lambda \quad \text{for} \quad |\lambda| < \tfrac{1}{4}.$$

If $x \in A$ with $\|x\| < \frac{1}{8}$, then $(1+4x)^{-1}$ exists since $\|1-(1+4x)\| < \frac{1}{2}$, and

$$\|x(1+4x)^{-1}\| \leq \|x\|\,\|(1+4x)^{-1}\| \leq \|x\| \sum_{k=0}^{\infty} \|(-4x)^k\| \leq \|x\| \sum_{k=0}^{\infty} \|4x\|^k = \frac{\|x\|}{1-4\|x\|} < \frac{1}{4},$$

since $\dfrac{t}{1-4t}$ increases from 0 to $\frac{1}{4}$ for $0 \leq t \leq \frac{1}{8}$.

Now let $x := v^2 - v$, (thus $\|x\| < \frac{1}{8}$), and let $y := -x(1+4x)^{-1} = (v-v^2)(1-2v)^{-2}$, for which $\|y\| < \frac{1}{4}$. Note that $\|x\| \to 0$ implies $\|y\| \to 0$, which in turn implies $\|f(y)\| \to 0$, so that for each $\varepsilon \in (0, \frac{1}{8})$, we can choose $\delta \leq \varepsilon$ such that $\|1-2v\|\|f(y)\| < \varepsilon$ whenever $\|1-2v\| < 1+\delta$ and $\|v - v^2\| = \|x\| < \delta$.

Finally, let $v_t := v + (1-2v)f(ty)$ for $0 \leq t \leq 1$, and take $e := v_1$. Since $f(0) = 0$, we get $v_0 = v$. Our estimates show that $\|e - v\| = \|(1-2v)f(y)\| < \varepsilon$. By holomorphic functional calculus, $v \in \mathcal{A}$ implies that x, y, v_t, e all lie in $\mathcal{A}$, too. We compute

$$\begin{aligned}
v_t^2 - v_t &= (v + (1-2v)f(ty))^2 - (v + (1-2v)f(ty)) \\
&= v^2 - v - (1-2v)^2 f(ty) + (1-2v)^2 f(ty)^2 \\
&= v^2 - v + (1-2v)^2 \big(f(ty)^2 - f(ty)\big) \\
&= v^2 - v + (1-2v)^2 ty = (v^2 - v)(1-t),
\end{aligned}$$

and in particular $e^2 - e = 0$, as required. □

Lemma 6.25 says that in a unital Fréchet pre-C^*-algebra $\mathcal{A}$, an "almost idempotent" $v \in \mathcal{A}$ that is not far from being a projector (since $\|1-2v\|$ is close to 1) can be retracted to a genuine idempotent in $\mathcal{A}$. The next Lemma says that projectors in the C^*-completion of $\mathcal{A}$ can be approximated by projectors lying in $\mathcal{A}$.

Lemma 6.26. *Let $\mathcal{A}$ be an unital Fréchet pre-C^*-algebra, whose C^*-completion is A. If $\tilde{q} = \tilde{q}^2 = \tilde{q}^*$ is a projector in A, then for any $\varepsilon > 0$, we can find a projector $q = q^2 = q^* \in \mathcal{A}$ such that $\|q - \tilde{q}\| < \varepsilon$.*

Proof. For a suitable $\delta \in (0,1)$, to be chosen later, we can find $v \in \mathcal{A}$ such that $v^* = v$ and $\|v - \tilde{q}\| < \delta$, because $\mathcal{A}$ is dense in A. Now

$$\|v^2 - v\| \leq \|v^2 - \tilde{q}^2 + \tilde{q} - v\| \leq (\|v + \tilde{q}\| + 1)\|v - \tilde{q}\| < (3+\delta)\delta < 4\delta,$$

and

$$\|1 - 2v\| \leq \|1 - 2\tilde{q}\| + 2\|\tilde{q} - v\| < 1 + 2\delta.$$

Lemma 6.25 now provides an idempotent $e = e^2 \in \mathcal{A}$ such that $\|e - v\| < \varepsilon/4$, for δ small enough (in particular, we must take $\delta < \varepsilon/4$). To replace e by a projector q, we may use Kaplansky's formula (in the C^*-algebra A: see [GVF, p. 88], for example) to define

$$q := ee^*(ee^* + (1-e^*)(1-e))^{-1}.$$

Indeed, $ee^* + (1-e^*)(1-e) = 1 + (e-e^*)(e^*-e) \geq 1$ in A, so it is invertible in A, and thus also in $\mathcal{A}$ because $\mathcal{A}$ is a pre-C^*-algebra. Thus $q \in \mathcal{A}$. One checks that $q^* = q$ and $q^2 = q$. Note also that $eq = q$.

If $\mathcal{A}$ is represented faithfully on a Hilbert space $\mathcal{H}$, we can decompose $\mathcal{H}$ as $q\mathcal{H} \oplus (1-q)\mathcal{H}$. With respect to this decomposition, we can write

$$q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & T \\ 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} R & V \\ V^* & S \end{pmatrix},$$

where $R = R^* \in \mathcal{L}(q\mathcal{H})$, $S = S^* \in \mathcal{L}((1-q)\mathcal{H})$, and $V, T\colon (1-q)\mathcal{H} \to q\mathcal{H}$ are bounded.

Now $\|e - v\| < \varepsilon/4$, so $\|(v-e)^*(v-e)\| < \varepsilon^2/16$; it follows that

$$\|(R-1)^2 + VV^*\| < \frac{\varepsilon^2}{16}, \qquad \|(V-T)^*(V-T) + S^2\| < \frac{\varepsilon^2}{16}.$$

Thus $\|VV^*\| < \varepsilon^2/16$, i.e., $\|V\| < \varepsilon/4$, and likewise $\|V - T\| < \varepsilon/4$. Therefore, $\|q - e\| = \|T\| < \varepsilon/2$. Finally,

$$\|q - \tilde{q}\| \leq \|q - e\| + \|e - v\| + \|v - \tilde{q}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \delta \leq \varepsilon. \qquad \square$$

Theorem 6.27. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, in which $\mathcal{A}$ is a unital Fréchet pre-C^*-algebra; and assume that $\mathcal{A}$ is* commutative. *Let $X = M(A)$ be the character space of A, a compact Hausdorff space such that $A \simeq C(X)$. Then, for each finite open cover $\{U_1, \dots, U_m\}$ of X, we can choose a subordinate partition of unity $\{\phi_1, \dots, \phi_m\}$:*

$$\phi_k \in C(X), \quad 0 \leq \phi_k \leq 1, \quad \operatorname{supp}\phi_k \subset U_k, \quad \phi_1 + \cdots + \phi_m = 1,$$

in such a way that $\phi_k \in \mathcal{A}$ for $k = 1, 2, \dots, m$.

Remark 6.28. By definition, $X = M(A)$ is the set of all nonzero $*$-homomorphisms $\phi\colon A \to \mathbb{C}$. Note that if $\phi \in M(A)$, then $\phi(1)^2 = \phi(1)$ implies $\phi(1) = 1$. Gelfand's theorem provided a $*$-isomorphism of unital C^*-algebras from A onto $C(X)$, so that we can regard elements of A as continuous functions on X.

Now let $\phi \in M(\mathcal{A})$. Recall that $\mathrm{sp}_{\mathcal{A}}(a) = \mathrm{sp}_A(a)$ for all $a \in \mathcal{A}$, since $\mathcal{A}$ is a pre-C^*-algebra. Because $(a - \lambda 1)b = 1$ in $\mathcal{A}$ implies $(\phi(a) - \lambda)\phi(b) = 1$, we see that $\lambda \notin \mathrm{sp}_{\mathcal{A}}(a) \implies \lambda \neq \phi(a)$; therefore, $\phi(a) \in \mathrm{sp}_A(a)$ for all $\phi \in M(\mathcal{A})$. Thus $|\phi(a)| \leq r(a) \leq \|a\|_A$, where $r(a)$ is the spectral radius of a, so that ϕ extends to A by continuity. This means that $M(\mathcal{A}) = M(A) = X$, so that we can regard X as the character space of the pre-C^*-algebra $\mathcal{A}$.

Proof of Theorem 6.27. We first choose a partition of unity $\{\tilde{\phi}_1, \dots, \tilde{\phi}_m\}$ in $C(X) = A$ subordinate to $\{U_1, \dots, U_m\}$. Let $\tilde{q} \in M_m(A)$ be the matrix whose (j,k)-entry is $\sqrt{\tilde{\phi}_j \tilde{\phi}_k}$. Then $\tilde{q} = \tilde{q}^2 = \tilde{q}^*$. Choose $\varepsilon \in (0, 1/m)$. We apply Lemma 6.26 to the pre-C^*-algebra $M_m(\mathcal{A}) \subset M_m(A)$, to get $q = q^2 = q^* \in M_m(\mathcal{A})$, with $\|q - \tilde{q}\| < \varepsilon$.

Let $\psi_i := q_{ij} \in \mathcal{A}$ for $j = 1, \dots, m$. Then $\psi_1 + \cdots + \psi_m = \operatorname{tr} q$, and $\|\operatorname{tr} q - \operatorname{tr}\tilde{q}\| \leq \sum_{j=1}^m \|q_{jj} - \tilde{q}_{jj}\| \leq m\varepsilon < 1$, so that $\operatorname{tr} q = \operatorname{tr}\tilde{q}$, because $\tilde{q} \mapsto \operatorname{tr}\tilde{q}$ is an integer-valued function on X (and $\operatorname{tr}\tilde{q}$ is the rank of the vector bundle corresponding to the projector $\tilde{q}$). Thus,

$$\psi_1 + \cdots + \psi_m = \operatorname{tr} q = \operatorname{tr}\tilde{q} = \phi_1 + \cdots + \phi_m = 1.$$

Furthermore, $0 \leq \psi_k \leq 1$ since ψ_k is the (k,k)-element of $q = q^*q \in M_m(A)$, and thus $\{\psi_1, \dots, \psi_m\}$ is a partition of unity with elements in $\mathcal{A}$. We can modify $\{\psi_k\}$ to get $\{\phi_k\}$ that will be subordinated to $\{U_k\}$, as follows.

Let $g\colon \mathbb{R} \to [0,1]$ be smooth with $\operatorname{supp} g \subseteq [\varepsilon, 1+\varepsilon]$, and $g(t) > 0$ for $\varepsilon < t \leq 1$. Now $V_k := \{\, x \in X : \psi_k(x) > \varepsilon \,\} \subset U_k$, since $\|\psi_k - \phi_k\| < \varepsilon$. Let $\chi_k := g \circ \psi_k$. By the smooth functional calculus, we find that $\chi_k \in \mathcal{A}$, for $k = 1, \dots, m$, and $\sum_{j=1}^m \chi_j(x) > 0$ for all $x \in X$, since otherwise $\sum_{j=1}^m \psi_j(x) \leq m\varepsilon < 1$, impossible. Therefore, $\chi_1 + \cdots + \chi_m$ is invertible in $C(X) = A$, and thus also in $\mathcal{A}$, so we can take $\phi_k := \chi_k(\chi_1 + \cdots + \chi_m)^{-1} \in \mathcal{A}$, having $\operatorname{supp}\phi_k \subset U_k$. Now $\{\phi_1, \dots, \phi_m\}$ is the desired partition of unity. $\square$

6.4 Real spectral triples

Recall that a spin structure on an oriented compact manifold (M, ε) is represented by a pair $(\mathcal{S}, C)$, where $\mathcal{S}$ is a $\mathcal{B}$-$\mathcal{A}$-bimodule and, according to Proposition 2.18, $C \colon \mathcal{S} \to \mathcal{S}$ is an antilinear map such that $C(\psi\, a) = C(\psi)\, \bar{a}$ for $a \in \mathcal{A}$; $C(b\, \psi) = \chi(\bar{b})\, C(\psi)$ for $b \in \mathcal{B}$; and, by choosing a metric g on M, which determines a Hermitian pairing on $\mathcal{S}$, we can also require that $(C\phi \mid C\psi) = (\psi \mid \phi) \in \mathcal{A}$ for $\phi, \psi \in \mathcal{S}$. $\mathcal{S}$ may be completed to a Hilbert space $\mathcal{H} = L^2(M, S)$, with scalar product $\langle \phi \mid \psi \rangle = \int_M (\phi \mid \psi)\, \nu_g$. It is clear that C extends to a bounded antilinear operator on $\mathcal{H}$ such that $\langle C\phi \mid C\psi \rangle = \langle \psi \mid \phi \rangle$ by integration with respect to ν_g, so that (the extended version of) C is *antiunitary* on $\mathcal{H}$. Moreover, the Dirac operator is $D\!\!\!/ = -i\hat{c} \circ \nabla^S$, where by construction the spin connection ∇^S commutes with C: that is, ∇^S_X commutes with C, for each $X \in \mathfrak{X}(M)$.

The property $C(\psi a) = C(\psi)\, \bar{a}$ shows that, for each $x \in X$, $\psi(x) \mapsto C(\psi)(x)$ is an antilinear operator C_x on the fibre S_x of the spinor bundle, which is a Fock space with $\dim_{\mathbb{C}} S_x = 2^m$. Thus, to determine whether C commutes with $D\!\!\!/$ or not, we can work with the local representation $D\!\!\!/ = -i\gamma^\alpha\, \nabla^S_{E_\alpha}$. Here $\gamma^\alpha = c(\theta^\alpha)$, for $\alpha = 1, \dots, n$, is a local section of the Clifford algebra bundle $\mathbb{C}\mathrm{l}(T^*M) \to M$, and the property $C(b\psi) = \chi(\bar{b})\, C(\psi)$ says that $C(\gamma^\alpha \psi) = -\gamma^\alpha\, C(\psi)$ whenever ψ is supported on a local chart domain.

However, replacing b by $\gamma^\alpha \in \Gamma(U, \mathbb{C}\mathrm{l}^1(T^*M))$ is only allowed when the dimension n is even. In the odd case, $\mathcal{B}$ consists of sections of the bundle $\mathbb{C}\mathrm{l}^0(T^*M)$, and we can only write relations like $C(\gamma^\alpha\gamma^\beta\psi) = \gamma^\alpha\gamma^\beta\, C(\psi)$ for $\psi \in \Gamma(U, S)$. But since C is antilinear, in the even case we get

$$C D\!\!\!/ \psi = C(-i\gamma^\alpha\, \nabla^S_{E_\alpha}\psi) = iC(\gamma^\alpha\, \nabla^S_{E_\alpha}\psi) = -i\gamma^\alpha\, C(\nabla^S_{E_\alpha}\psi) = D\!\!\!/ C\psi.$$

Thus $[D\!\!\!/, C] = 0$ on $\mathcal{H}$, when $n = 2m$ is even.

¿What happens in the odd-dimensional case? Consider what happens on a single fibre S_x, which carries a selfadjoint representation of $B_x = \mathbb{C}\mathrm{l}^0(T^*_xM)$. Recall that we use the convention that $c(\omega) := c(\omega\gamma)$ to extend the action of $\mathcal{B}$ to all of $\Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$, where $\gamma = (-i)^m\theta^1 \dots \theta^{2m+1}$ is the chirality element. For $\omega = \theta^\alpha$, then gives $C\, c(\omega)\, C^{-1} = C\, c(\omega\gamma)\, C^{-1} = c(\chi(\overline{\omega\gamma})) = c(\overline{\omega\gamma})$ since $\omega\gamma$ is even for ω odd, and $\overline{\omega\gamma} = i^m\, \theta^\alpha\theta^1 \dots \theta^{2m+1} = (-1)^m\omega\gamma$. We conclude that $C\, c(\omega)\, C^{-1} = (-1)^m\, c(\omega)$ for $\omega \in \mathcal{A}^1(M)$ real, and therefore $C D\!\!\!/ = (-1)^{m+1}\, D\!\!\!/ C$ by antilinearity of C. We sum up:

$$C D\!\!\!/ = \begin{cases} +D\!\!\!/ C, & \text{if } n \not\equiv 1 \mod 4, \\ -D\!\!\!/ C, & \text{if } n \equiv 1 \mod 4. \end{cases}$$

In the even case, $\mathcal{B} = \Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$ contains the operator $\Gamma = c(\gamma)$ which extends to a selfadjoint unitary operator on $\mathcal{H}$. Recall from Definition 1.18 that $c_J(\gamma)$ is the $\mathbb{Z}_2$-grading operator on the Fock space $\Lambda^\bullet W_J$, the model for S_x. If $\mathcal{H}^\pm = L^2(M, S^\pm)$ denotes the completion of $\mathcal{S}^\pm$ in the norm of $\mathcal{H}$, then $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, with Γ being the $\mathbb{Z}_2$-grading operator. Now γ is even and $\bar{\gamma} = (-1)^m\, \gamma$ as before, so that $C\Gamma = (-1)^m\Gamma C$ whenever $n = 2m$.

When M is a connected manifold, there is a third sign associated with C, since we know that $C^2 = \pm 1$. Once more, the sign can be found by examining the case of a single fibre S_x, so we ask whether an irreducible representation S of $\mathbb{C}\mathrm{l}(V)$ admits an *antiunitary conjugation* $C \colon S \to S$ such that $C\, c_J(v)\, C^{-1} = \pm c_J(v)$ for $v \in V$ (plus sign if $\dim V = 1 \mod 4$) and either $C^2 = +1$ or $C^2 = -1$. By periodicity of the Clifford algebras, the sign depends only on n mod 8, where $n = \dim V$.

Note that if $\{\gamma^1, \dots, \gamma^n\}$ generate $\mathrm{Cl}_{n,0}$, then $\{-i\gamma^1, \dots, -i\gamma^n\}$ generate $\mathrm{Cl}_{0,n} = \mathrm{Cl}(\mathbb{R}^n, g)$ with g negative-definite. Thus one can equally well work with $\mathrm{Cl}_{0,q}$, for $q = 0, 1, \dots, 7$. Since $\mathrm{Cl}_{p,0} \otimes_{\mathbb{R}} M_N(\mathbb{R}) \simeq \mathrm{Cl}_{0,8-p} \otimes_{\mathbb{R}} M_{N'}(\mathbb{R})$ for $p = 0, 1, \dots, 7$ and suitable matrix sizes N, N', we get, from our classification (1.4) of the Clifford algebras $\mathrm{Cl}_{p,0}$:

- for $q \equiv 0, 6, 7 \bmod 8$, $\mathrm{Cl}_{0,q}$ is an algebra over $\mathbb{R}$,
- for $q \equiv 1, 5 \bmod 8$, $\mathrm{Cl}_{0,q}$ is an algebra over $\mathbb{C}$,
- for $q \equiv 2, 3, 4 \bmod 8$, $\mathrm{Cl}_{0,q}$ is an algebra over $\mathbb{H}$.

On a case-by-case basis, using this classification, one finds that $C^2 = -1$ if and only if $n = 2, 3, 4, 5 \bmod 8$.

Exercise 6.29. *Find five matrices* $\varepsilon_1, \dots, \varepsilon_5 \in M_4(\mathbb{C})$*, generating a representation of* Cl_{05}*, and an antiunitary operator* C *on* $\mathbb{C}^4$ *such that* $C\,\varepsilon_j\,C^{-1} = -\varepsilon_j$ *for* $j = 1, \dots, 5$*. Show that* C *is unique up to multiples* $C \mapsto \lambda C$ *with* $\lambda \in \mathbb{C}$ *and* $|\lambda| = 1$*; and that* $C^2 = -1_4$*.*

Summary: There are two tables of signs

$n \bmod 8$	0	2	4	6
$C^2 = \pm 1$	$+$	$-$	$-$	$+$
$C \not{D} = \pm \not{D} C$	$+$	$+$	$+$	$+$
$C\Gamma = \pm \Gamma C$	$+$	$-$	$+$	$-$

$n \bmod 8$	1	3	5	7
$C^2 = \pm 1$	$+$	$-$	$-$	$+$
$C \not{D} = \pm \not{D} C$	$-$	$+$	$-$	$+$

There is a deeper reason why only these signs can occur, and why they depend on $n \bmod 8$: the data set $(\mathcal{A}, \mathcal{H}, \not{D}, C, \Gamma)$ determines a class in the "Real" KR-homology $\mathrm{KR}^\bullet(\mathcal{A})$, and $\mathrm{KR}^{j+8}(\mathcal{A}) \simeq \mathrm{KR}^j(\mathcal{A})$ by Bott periodicity. We leave this story for Prof. Brodzki's course. (But see [GVF, Sec. 9.5] for a pedestrian approach.)

"Real" KR-homology is a theory for algebras with involution: in the *commutative* case, we may just take $a \mapsto a^*$, and we ask that $C\,a\,C^{-1} = a^*$ i.e., that C implement the involution. This is trivial for the manifold case, since $C(\psi a) = C(\psi)\bar{a} =: a^* C(\psi)$, the a^* here being multiplication by $\bar{a}$.

In the *noncommutative* case, the operator Ca^*C^{-1} would generate a second representation of $\mathcal{A}$, in fact an antirepresentation (that is, a representation of the opposite algebra $\mathcal{A}^\circ$) and we should require that this commute with the original representation of $\mathcal{A}$.

Definition 6.30. *A* ***real spectral triple*** *is a spectral triple* $(\mathcal{A}, \mathcal{H}, D)$*, together with an antiunitary operator* $J \colon \mathcal{H} \to \mathcal{H}$ *such that* $J(\mathrm{Dom}\, D) \subset \mathrm{Dom}\, D$*, and* $[a, Jb^*J^{-1}] = 0$ *for all* $a, b \in \mathcal{A}$*.*

Definition 6.31. *A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is* ***even*** *if there is a selfadjoint unitary operator* Γ *on* $\mathcal{H}$ *such that* $a\Gamma = \Gamma a$ *for all* $a \in \mathcal{A}$*,* $\Gamma(\mathrm{Dom}\, D) = \mathrm{Dom}\, D$*, and* $D\Gamma = -\Gamma D$*. If no such* $\mathbb{Z}_2$*-grading operator* Γ *is given, we say that the spectral triple is* ***odd****.*

We have seen that in the standard commutative example, the *even* case arises when the auxiliary algebra $\mathcal{B}$ contains a natural $\mathbb{Z}_2$-grading operator, and this happens exactly when *the manifold dimension is even.* Now, the manifold dimension is determined by the spectral growth of the Dirac operator, and this spectral version of dimension may be used for noncommutative spectral triples, too. To make this more precise, we must look more closely at spectral growth.

6.5 Summability of spectral triples

Definition 6.32. *For $1 < p < \infty$, there is an operator ideal $\mathcal{L}^{p+}(\mathcal{H}) = \mathcal{L}^{p,\infty}(\mathcal{H})$, defined as follows:*

$$\mathcal{L}^{p+}(\mathcal{H}) := \{\, T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(N^{(p-1)/p}) \quad \text{as } N \to \infty \,\},$$

with norm $\|T\|_{p+} := \sup_{N\geq 1} \sigma_N(T)/N^{(p-1)/p}$.

For instance, if $A \geq 0$ with $s_k(A) := \dfrac{1}{(k+1)^{1/p}}$, then $A \in \mathcal{L}^{p+}$ by the integral test:

$$\sigma_N(A) \sim \int_1^N t^{-1/p}\, dt \sim \frac{p}{p-1}\, N^{(p-1)/p}, \quad \text{as } N \to \infty.$$

Indeed, since $p > 1$, $T \in \mathcal{L}^{p+}$ implies $s_k(T) = O((k+1)^{-1/p})$. To see that, recall that $s_0(T)+\cdots+s_k(t) = \sigma_{k+1}(T)$; since $\{s_k(T)\}$ is decreasing, this implies $(k+1)s_k(T) \leq \sigma_{k+1}(T)$, and thus $s_k(T) \leq \frac{1}{k+1}\, \sigma_{k+1}(T) \leq C(k+1)^{-1/p}$ for some constant C.

Therefore, $T \in \mathcal{L}^{p+}$ implies $s_k(T^p) = O(\frac{1}{k+1})$ and then $\sigma_N(T^p) = O(\log N)$, so that $T^p \in \mathcal{L}^{1+}$, which serves to justify the notation $\mathcal{L}^{p+}$. It turns out, however, that there are, for any $p > 1$, positive operators $B \in \mathcal{L}^{1+}$ such that $B^{1/p} \notin \mathcal{L}^{p+}$, so the implication "$T \in \mathcal{L}^{p+} \implies T^p \in \mathcal{L}^{1+}$" is a one-way street. For an example, see [GVF, Sec. 7.C].

Definition 6.33. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is **p$^+$-summable** for some p with $1 \leq p < \infty$ if $(D^2+1)^{-1/2} \in \mathcal{L}^{p+}(\mathcal{H})$. If D is invertible, this is equivalent to requiring $|D|^{-1} \in \mathcal{L}^{p+}(\mathcal{H})$.*

Definition 6.34. *Let $p \in [1, \infty)$. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has **spectral dimension p** if it is p$^+$-summable and moreover*

$$0 < \operatorname{Tr}_\omega((D^2+1)^{-p/2}) < \infty \quad \textit{for any Dixmier trace } \operatorname{Tr}_\omega\,.$$

If D is invertible, this is equivalent to $0 < \operatorname{Tr}_\omega(|D|^{-p}) < \infty$ for any Tr_ω.

For positivity of all Dixmier traces, it suffices that $\liminf_{N\to\infty} \frac{1}{\log N} \sigma_N((D^2+1)^{-p/2}) > 0$. Note that, in view of Corollary 6.8, this can happen for *at most one value* of p.

Proposition 6.35. *If $(\mathcal{A}, \mathcal{H}, D)$ is a p$^+$-summable spectral triple, with D invertible, let*

$$F := D\,|D|^{-1} \tag{6.12}$$

*be the **phase** of the selfadjoint operator D. Then, for each $a \in \mathcal{A}$, the commutator $[F, a]$ lies in $\mathcal{L}^{p+}(\mathcal{H})$.*

Proof. First we show that $[F, a] \in \mathcal{K}(\mathcal{H})$, using the spectral formula (6.6) for $|D|^{-1}$. Indeed,

$$\begin{aligned}
[F, a] &= [D\,|D|^{-1}, a] = [D, a]\,|D|^{-1} + D\,[|D|^{-1}, a] \\
&= \frac{2}{\pi}\int_0^\infty \big([D, a]\,(D^2+\mu)^{-1} + D\,[(D^2+\mu)^{-1}, a]\big)\, d\mu \\
&= \frac{2}{\pi}\int_0^\infty \big([D, a]\,(D^2+\mu)^{-1} - D(D^2+\mu)^{-1}\,[D^2, a]\,(D^2+\mu)^{-1}\big)\, d\mu \qquad (6.13)\\
&= \frac{2}{\pi}\int_0^\infty \big(\mu^2(D^2+\mu)^{-1}\,[D, a]\,(D^2+\mu)^{-1} - D(D^2+\mu)^{-1}\,[D, a]\,D(D^2+\mu)^{-1}\big)\, d\mu.
\end{aligned}$$

In the integrand, $[D, a]$ is bounded by the hypothesis that $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. Next, $(D^2 + \mu)^{-1} = (D - i\mu)^{-1}(D + i\mu)^{-1} \in \mathcal{K}(\mathcal{H})$ and

$$D(D^2 + \mu)^{-1} = \underbrace{D(D^2 + \mu)^{-\frac{1}{2}}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{(D^2 + \mu)^{-\frac{1}{2}}}_{\in \mathcal{K}(\mathcal{H})}$$

is also compact. Thus the integrand lies in $\mathcal{K}(\mathcal{H})$ for each μ, hence $[F, a] \in \mathcal{K}(\mathcal{H})$, that is, the integral converges in the norm of this C^*-algebra.

Next to show that $[F, a] \in \mathcal{L}^{p+}(\mathcal{H})$, we may assume that $a^* = -a$, since

$$i\,[F, a] = [F, \tfrac{i}{2}(a^* + a)] - i\,[F, \tfrac{1}{2}(a^* - a)].$$

Note that this assumption implies that the bounded operators $[F, a]$ and $[D, a]$ are selfadjoint.

If we *replace* the term $[D, a]$ by its norm $\|[D, a]\|$ on the right hand side of (6.13), this integral changes into

$$\begin{aligned}
&\frac{2}{\pi} \int_0^\infty \bigl(\mu^2 (D^2 + \mu)^{-1}\, \|[D, a]\|\, (D^2 + \mu)^{-1} - D(D^2 + \mu)^{-1}\, \|[D, a]\|\, D(D^2 + \mu)^{-1}\bigr)\, d\mu \\
&= \frac{2}{\pi}\, \|[D, a]\| \int_0^\infty \bigl((\mu^2 (D^2 + \mu)^{-2} + D^2 (D^2 + \mu)^{-2})\bigr)\, d\mu \\
&\leq \frac{2}{\pi}\, \|[D, a]\| \int_0^\infty \bigl((\mu^2 (D^2 + \mu)^{-2} + D^2 (D^2 + \mu)^{-2})\bigr)\, d\mu \\
&= \frac{2}{\pi} \|[D, a]\| \int_0^\infty (D^2 + \mu)^{-1}\, d\mu = \|[D, a]\|\, |D|^{-1},
\end{aligned}$$

where these are inequalities among selfadjoint elements of the C^*-algebra $\mathcal{K}(\mathcal{H})$. Therefore, if we plug in the order relation

$$-\|[D, a]\| \leq [D, a] \leq \|[D, a]\|$$

among selfadjoint elements of $\mathcal{L}(\mathcal{H})$ into the right hand side of (6.13), we obtain the operator inequalities

$$-\|[D, a]\|\, |D|^{-1} \leq [F, a] \leq \|[D, a]\|\, |D|^{-1}.$$

Thus the singular values of $[F, a]$ are dominated by those of $|D|^{-1}$. We now conclude that $|D|^{-1} \in \mathcal{L}^{p+}$ implies $[F, a] \in \mathcal{L}^{p+}$, for all $a \in \mathcal{A}$. □

The assumption that D is invertible in the statement of Proposition 6.35 is not essential (though the proof does depend on it, of course). With some extra work, we can modify the proof to show that $(D^2 + 1)^{-1/2} \in \mathcal{L}^{p+}$ implies that all $[F, a] \in \mathcal{L}^{p+}$, where F is redefined to mean $F := D\,(D^2 + 1)^{-1/2}$, in contrast to (6.12). This is proved in [CPRS], in full generality.

Chapter 7

Spectral Triples: Examples

7.1 Geometric conditions on spectral triples

We begin by listing a set of requirements on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, whose algebra $\mathcal{A}$ is unital but not necessarily commutative, such that $(\mathcal{A}, \mathcal{H}, D)$ provides a "spin geometry" generalization of our "standard commutative example" $(C^\infty(M), L^2(M, S), \not{D})$. Again we shall assume, for convenience, that D is invertible.

Condition 1 (Spectral dimension). *There is an* integer $n \in \{1, 2, \dots\}$, *called the spectral dimension of* $(\mathcal{A}, \mathcal{H}, D)$, *such that* $|D|^{-1} \in \mathcal{L}^{n+}(\mathcal{H})$, *and* $0 < \mathrm{Tr}_\omega(|D|^{-n}) < \infty$ *for any Dixmier trace* Tr_ω.

When n *is* even, *the spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is also even: that is, there exists a selfadjoint unitary operator* $\Gamma \in \mathcal{L}(\mathcal{H})$ *such that* $\Gamma(\mathrm{Dom}\, D) = \mathrm{Dom}\, D$, *satisfying* $a\Gamma = \Gamma a$ *for all* $a \in \mathcal{A}$, *and* $D\Gamma = -\Gamma D$.

Remark 7.1. It is useful to allow the case $n = 0$ as a possible spectral dimension. There are two cases to consider:

- $\mathcal{H}$ is an infinite-dimensional Hilbert space, but the spectrum of the operator D has exponential growth, so that $N_{|D|}(\lambda) = O(\lambda^\varepsilon)$ as $\lambda \to \infty$, for any exponent $\varepsilon > 0$. This is what happens in the example by Dąbrowski and Sitarz [DS] of a spectral triple on the standard Podleś sphere $\mathbb{S}^2_q$ with $0 < q < 1$, where the operator D has the same eigenvalue multiplicities as the Dirac operator on $\mathbb{S}^2$ (see Section A.2), but the eigenvalue $\pm(l + \frac{1}{2})$ is replaced by $\pm(q^{-l-\frac{1}{2}} - q^{l+\frac{1}{2}})/(q^{-1} - q)$, for $l = \frac{1}{2}, \frac{3}{2}, \dots$.

- $\mathcal{H}$ is finite-dimensional, $\mathcal{A}$ is a finite-dimensional matrix algebra, and D is a hermitian matrix. In this case, we assign to $(\mathcal{A}, \mathcal{H}, D)$ the spectral dimension $n = 0$, and replace the Dixmier traces Tr_ω by the ordinary matrix trace tr.

Condition 2 (Regularity). *For each* $a \in \mathcal{A}$, *the bounded operators* a *and* $[D, a]$ *lie in the smooth domain* $\bigcap_{k\geq 1} \mathrm{Dom}\, \delta^k$ *of the derivation* $\delta\colon T \mapsto [|D|, T]$.

Moreover, $\mathcal{A}$ *is complete in the topology given by the seminorms* $q_k\colon a \mapsto \|\delta^k(a)\|$ *and* $q'_k\colon a \mapsto \|\delta^k([D, a])\|$. *This ensures that* $\mathcal{A}$ *is a Fréchet pre-*C^**-algebra.*

Condition 3 (Finiteness). *The subspace of smooth vectors* $\mathcal{H}^\infty := \bigcap_{k\in\mathbb{N}} \mathrm{Dom}\, D^k$ *is a* finitely generated projective *left* $\mathcal{A}$*-module.*

This is equivalent to saying that, for some $N \in \mathbb{N}$, *there is a projector* $p = p^2 = p^*$ *in* $M_N(\mathcal{A})$ *such that* $\mathcal{H}^\infty \simeq \mathcal{A}^N p$ *as left* $\mathcal{A}$*-modules.*

Condition 4 (Real structure). *There is an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$ satisfying $J^2 = \pm 1$, $JDJ^{-1} = \pm D$, and $J\Gamma = \pm \Gamma J$ in the even case, where the signs depend only on $n \bmod 8$ (and thus are given by the table of signs for the standard commutative examples). Moreover, $b \mapsto Jb^* J^{-1}$ is an antirepresentation of $\mathcal{A}$ on $\mathcal{H}$ (that is, a representation of the opposite algebra $\mathcal{A}^\circ$), which commutes with the given representation of $\mathcal{A}$:*

$$[a, Jb^* J^{-1}] = 0, \quad \text{for all } a, b \in \mathcal{A}.$$

Condition 5 (First order). *For each $a, b \in \mathcal{A}$, the following relation holds:*

$$[[D, a], Jb^* J^{-1}] = 0, \quad \text{for all } a, b \in \mathcal{A}.$$

This generalizes, to the noncommutative context, the condition that D be a first-order differential operator.

Since

$$[[D, a], Jb^* J^{-1}] = [[D, Jb^* J^{-1}], a] + [D, \underbrace{[a, Jb^* J^{-1}]}_{=0}],$$

this is equivalent to the condition that $[a, [D, Jb^* J^{-1}]] = 0$.

Condition 6 (Orientation). *There is a Hochschild n-cycle*

$$\mathbf{c} = \textstyle\sum_j (a_j^0 \otimes b_j^0) \otimes a_j^1 \otimes \cdots \otimes a_j^n \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ),$$

such that

$$\pi_D(\mathbf{c}) \equiv \textstyle\sum_j a_j^0 (J b_j^{0*} J^{-1}) \, [D, a_j^1] \dots [D, a_j^n] = \begin{cases} \Gamma, & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}. \end{cases} \tag{7.1}$$

In many examples, including the noncommutative examples we shall meet in the next two sections, one can often take $b_j^0 = 1$, so that $\mathbf{c}$ may be replaced, for convenience, by the cycle $\sum_j a_j^0 \otimes a_j^1 \otimes \cdots \otimes a_j^n \in Z_n(\mathcal{A}, \mathcal{A})$. In the commutative case, where $\mathcal{A}^\circ = \mathcal{A}$, this identification may be justified: the product map $m \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a homomorphism.

The data set $(\mathcal{A}, \mathcal{H}, D; \Gamma \text{ or } 1, J, \mathbf{c})$ satisfying these six conditions constitute a "noncommutative spin geometry". In the fundamental paper where these conditions were first laid out [Con2], Connes added one more nondegeneracy condition (Poincaré duality in K-theory) as a requirement. We shall not go into this matter here.

To understand the orientation condition in the standard commutative example, we show that $\mathbf{c}$ arises from a *volume form* on the oriented compact manifold M. Choose a metric g on M and let ν_g be the corresponding Riemannian volume form. Furthermore, let $\{(U_j, a_j)\}$ be a finite atlas of charts on M, where $a_j \colon U_j \to \mathbb{R}^n$, and let $\{f_j\}$ be a partition of unity subordinate to the open cover $\{U_j\}$; then for $r = 1, \dots, n$, each $f_j a_j^r$ lies in $C^\infty(M)$ with $\operatorname{supp}(f_j a_j^r) \subset U_j$. Over each U_j, let $\{\theta_j^1, \dots, \theta_j^n\}$ be a local orthonormal basis of 1-forms (with respect to the metric g). Then

$$\nu_g\big|_{U_j} = \theta_j^1 \wedge \cdots \wedge \theta_j^n = h_j \, da_j^1 \wedge \cdots \wedge da_j^n,$$

for some smooth functions $h_j \colon U_j \to \mathbb{C}$. We write $a_j^0 := (-i)^m f_j h_j \in C^\infty(M)$, where as usual, $n = 2m$ or $n = 2m + 1$. With that notation, we get

$$(-i)^m \nu_g = (-i)^m \textstyle\sum_j f_j \big(\nu_g\big|_{U_j}\big) = \sum_j a_j^0 \, da_j^1 \wedge \cdots \wedge da_j^n.$$

Now we define

$$\mathbf{c} := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sum_j a_j^0 \otimes a_j^{\sigma(1)} \otimes \cdots \otimes a_j^{\sigma(n)}. \tag{7.2}$$

Exercise 7.2. *Show that the Hochschild boundary* $b\mathbf{c}$ *of the chain* (7.2) *is zero because* $\mathcal{A}$ *is commutative.*

Therefore, $\mathbf{c}$ is a Hochschild n-cycle in $Z_n(\mathcal{A},\mathcal{A})$, for $\mathcal{A} = C^\infty(M)$. Its representative as a bounded operator on $\mathcal{H}$ is

$$\begin{aligned}
\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma \sum_j a_j^0\,[D\!\!\!/, a_j^{\sigma(1)}]\dots[D\!\!\!/, a_j^{\sigma(n)}] &= \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma \sum_j a_j^0\, c(da_j^{\sigma(1)})\dots c(da_j^{\sigma(n)}) \\
&= \frac{(-i)^m}{n!}\sum_j f_j \sum_{\sigma\in S_n}(-1)^\sigma c(\theta_j^{\sigma(1)})\dots c(\theta_j^{\sigma(n)}) \\
&= \Big(\sum_j f_j\Big)(-i)^m\, c(\theta_j^1)\dots c(\theta_j^n) \\
&= c(\gamma) = \Gamma \text{ or } 1,
\end{aligned}$$

since $c(\gamma) = \Gamma$ for $n = 2m$, and $c(\gamma) = 1$ for $n = 2m+1$.

This calculation shows that the elements $a_j^1,\dots,a_j^n$ occurring in the cycle c are local coordinate functions for M. An alternative approach would be to embed M in some $\mathbb{R}^N$ and take the a_j^r to be some of the cartesian coordinates of $\mathbb{R}^N$, regarded as functions on M. This is illustrated in the following example.

Example 7.3. By regarding the sphere $\mathbb{S}^2$ as embedded in $\mathbb{R}^3$,

$$\mathbb{S}^2 = \{\,(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 = 1\,\},$$

we can write down its volume form for the rotation-invariant metric g as

$$\nu = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy.$$

The corresponding Hochschild 2-cycle is

$$\mathbf{c} := -\frac{i}{2}\sum_{\text{cyclic}}(x\otimes y\otimes z - x\otimes z\otimes y),$$

summing over cyclic permutations of the letters x, y, z.

If $D\!\!\!/$ is the Dirac operator on $\mathbb{S}^2$ for this "round" metric and the unique spin structure on $\mathbb{S}^2$ compatible with its usual orientation (see Section A.2), then

$$-\frac{i}{2}\sum_{\text{cyclic}}\big(x\,[D\!\!\!/, y]\,[D\!\!\!/, z] - x\,[D\!\!\!/, z]\,[D\!\!\!/, y]\big) = \Gamma = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

on $\mathcal{H} = \mathcal{H}^+\oplus\mathcal{H}^-$, which is the completion of the spinor module $\mathcal{S} = \mathcal{S}^+\oplus\mathcal{S}^-$.

Consider the following element of $M_2(\mathcal{A})$, with $\mathcal{A} = C^\infty(\mathbb{S}^2)$:

$$p := \frac{1}{2}\begin{pmatrix}1+z & x+iy\\ x-iy & 1-z\end{pmatrix}. \tag{7.3}$$

Note that $\operatorname{tr}(p-\frac{1}{2}) = 0$, where $\operatorname{tr}(a) := a_{11}+a_{22}$ means the matrix trace $\operatorname{tr}\colon M_2(\mathcal{A})\to\mathcal{A}$.

Exercise 7.4. *Show that, if* $\mathcal{A}$ *is any* $*$*-algebra and* $p\in M_2(\mathcal{A})$ *is given by* (7.3)*, then the projector relations* $p = p^* = p^2$ *are equivalent to the following relations among* $x, y, z\in\mathcal{A}$*:*

$$\begin{gathered}
x^* = x,\quad y^* = y,\quad z^* = z,\\
[x,y] = [x,z] = [y,z] = 0,\\
x^2+y^2+z^2 = 1.
\end{gathered}$$

Exercise 7.5. *Check that* $\operatorname{tr}(p\,dp \wedge dp) = -\frac{i}{2}\,\nu$.

If we replace $-\frac{i}{2}\,\nu$ by the Hochschild 2-cycle $\mathbf{c}$, the same calculation that solves the previous exercise also shows that $\pi_D(\mathbf{c}) = \Gamma$.

This computation has a deeper significance. One can show that the left $\mathcal{A}$-module $M_2(\mathcal{A})\,p$ is isomorphic to $\mathcal{E}_1$ in our classification of $\mathcal{A}$-modules of sections of line bundles over $\mathbb{S}^2$; and we have seen in Section A.3 that $\mathcal{E}_1 \simeq \Gamma(\mathbb{S}^2, L)$ where $L \to \mathbb{S}^2$ is the tautological line bundle. The first Chern class $c_1(L)$ equals (a standard multiple of) $[\nu] \in H^2_{\mathrm{dR}}(\mathbb{S}^2)$. One can trace a parallel relation between spin^c structures on $\mathbb{S}^2$ defined, via the principal $U(1)$-bundle $SU(2) \to \mathbb{S}^2$, on associated line bundles, and the Chern classes of each such line bundle. For that, we refer to [BHMS].

7.2 Isospectral deformations of commutative spectral triples

To some extent, one can recover the sphere $\mathbb{S}^2$ from spectral triple data alone. Thus, if $\mathcal{A}$ is a $*$-subalgebra of some C^*-algebra containing elements x, y, z, and if the matrix

$$p = \frac{1}{2}\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix} \in M_2(\mathcal{A})$$

is a projector, i.e., $p = p^* = p^2$, then by Exercise 7.4, the elements x, y, z commute, they are selfadjoint, and they satisfy $x^2 + y^2 + z^2 = 1$. Thus, the commutative C^*-algebra A generated by x, y, z is of the form $C(X)$, where $X \subseteq \mathbb{S}^2$ is a closed subset. If $\mathcal{A}$ is now a pre-C^*-subalgebra of A containing x, y, z, and is the algebra of some spectral triple $(\mathcal{A}, \mathcal{H}, D)$, then the extra condition $\pi_D(\mathbf{c}) = \Gamma$ can only hold if X is the support of the measure ν. This means that $X = \mathbb{S}^2$.

A similar argument can be tried, to obtain an "algebraic" description of $\mathbb{S}^4$. What follows is a heuristic motivation, following [CL]. One looks for a projector $p \in M_4(\mathcal{A})$, of the form

$$p = \begin{pmatrix} 1+z & 0 & a & b \\ 0 & 1+z & -b^* & a \\ a^* & -b & 1-z & 0 \\ b^* & a^* & 0 & 1-z \end{pmatrix} = \begin{pmatrix} (1+z)1_2 & q \\ q^* & (1-z)1_2 \end{pmatrix}, \quad \text{where } q = \begin{pmatrix} a & b \\ -b^* & a \end{pmatrix}.$$

Then $p = p^*$ only if $z = z^*$, and then $p^2 = p$ implies that $-1 \leq z \leq 1$ in the C^*-completion A of $\mathcal{A}$, $\left[\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right), q\right] = 0$, and $qq^* = q^*q = \begin{pmatrix} 1-z^2 & 0 \\ 0 & 1-z^2 \end{pmatrix}$. From that, one finds that a, a^*, b, b^*, z commute, and $aa^* + bb^* = 1 - z^2$. Thus $A = C(X)$ with $X \subseteq \mathbb{S}^4$. Again, it can be shown that the equality $X = \mathbb{S}^4$ is reached by some extra conditions, namely,

$$\begin{aligned} \operatorname{tr}(p - \tfrac{1}{2}) &= 0, \\ \operatorname{tr}((p - \tfrac{1}{2})\,dp\,dp) &= 0 \quad \text{in } \Omega^2(\mathcal{A}), \\ \pi_D((p - \tfrac{1}{2})\,dp\,dp\,dp\,dp) &= \Gamma \quad \text{in } \mathcal{L}(\mathcal{H}). \end{aligned}$$

However, if one takes instead $q := \begin{pmatrix} a & b \\ -\bar{\lambda} b^* & a^* \end{pmatrix}$ with $\lambda = e^{2\pi i\theta}$, then there is another, noncommutative, solution [CL]: now A is the C^*-algebra generated by a, b and $z = z^*$, where z is central, and the other relations are

$$\begin{gathered} ab = e^{-2\pi i\theta} ba, \qquad a^*b = e^{2\pi i\theta} ba^*, \\ aa^* = a^*a, \qquad bb^* = b^*b, \qquad aa^* + bb^* = 1 - z^2. \end{gathered} \tag{7.4}$$

To find a solution to these relations, where the central element z is taken to be a scalar multiple of 1, we substitute

$$\begin{aligned} a &= u \sin\psi \cos\phi \\ b &= v \sin\psi \cos\phi \\ z &= (\cos\psi)\, 1 \end{aligned}$$

with $-\pi \leq \psi \leq \pi$ and $-\pi < \phi \leq \pi$, say. In this way, the commutation relations (7.4) reduce to

$$uu^* = u^*u = 1, \qquad vv^* = v^*v = 1, \qquad vu = e^{2\pi i\theta} uv.$$

These are the relations for the unitary generators of a noncommutative 2-torus: see Section A.4. Thus, by fixing values of ϕ, ψ with $\psi \neq \pm\pi$ and $\phi \notin \frac{\pi}{2}\mathbb{Z}$, we get a homomorphism from A to $C(\mathbb{T}^2_\theta)$, the C^*-algebra of the noncommutative 2-torus with parameters $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \in M_2(A)$.

We look for a suitable algebra $\mathcal{A}$, generated by elements satisfying the above relations, by examining a *Moyal deformation* of $C^\infty(\mathbb{S}^4)$. One should first note that $\mathbb{S}^4 \subset \mathbb{R}^5 = \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ carries an obvious action of $\mathbb{T}^2$, namely,

$$(t_1, t_2) \cdot (\alpha, \beta, z) := (t_1\alpha, t_2\beta, z), \quad \text{for } |t_1| = |t_2| = 1,$$

which preserves the defining relation $\alpha\bar\alpha + \beta\bar\beta + z^2 = 1$ of $\mathbb{S}^4$. The action is *not free*: there are two fixed points $(0, 0, \pm 1)$, and for each t with $-1 < t < 1$ there are two circular orbits, namely $\{\, (\alpha, 0, t) : \alpha\bar\alpha = 1 - t^2 \,\}$ and $\{\, (\beta, 0, t) : \beta\bar\beta = 1 - t^2 \,\}$. The remaining orbits are copies of $\mathbb{T}^2$. The construction which follows will produce a "noncommutative space" $\mathbb{S}^4_\theta$ that can be thought of as the sphere $\mathbb{S}^4$ with each principal orbit $\mathbb{T}^2$ replaced by a noncommutative torus $\mathbb{T}^2_\theta$, while the $\mathbb{S}^1$-orbits and the two fixed points remain unchanged.

In quantum mechanics, the **Moyal product** of two functions $f, h \in \mathcal{S}(\mathbb{R}^n)$ is defined as an (oscillatory) integral of the form

$$(f \star h)(x) := (\pi\theta)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + s)\, h(x + t)\, e^{-2is(\Theta^{-1}t)}\, ds\, dt. \tag{7.5}$$

Here $n = 2m$ is even, $\Theta = -\Theta^t \in M_n(\mathbb{R})$ is an invertible skewsymmetric matrix, and $\det\Theta = \theta^n$ with $\theta > 0$. In the next section, we shall interpret this formula in a precise manner (see Definition 7.17 below), and show that $f \star h$ lies in $\mathcal{S}(\mathbb{R}^n)$ also. Formally, at any rate, one can rewrite it as an ordinary Fourier integral:

$$(f \star h)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - \tfrac{1}{2}\Theta u)\, h(x + t)\, e^{-iut}\, du\, dt,$$

with the advantage that now Θ need not be invertible (so that n need no longer be even). It was noticed by Rieffel [Rie] that one can replace the translation action of $\mathbb{R}^n$ on f, h by any (strongly continuous) action α of some $\mathbb{R}^l$ on a C^*-algebra A. Then, given $\Theta = -\Theta^t \in M_l(\mathbb{R})$, one can define

$$a \star b := \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \alpha_{\frac{1}{2}\Theta u}(a)\, \alpha_{-t}(b)\, e^{2\pi i ut}\, du\, dt,$$

provided that the integral makes sense. In particular, if α is *periodic action* of $\mathbb{R}^l$, i.e., $\alpha_{t+r} = \alpha_t$ for $r \in \mathbb{Z}^n$, so that α is effectively an action of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, then one can describe the Moyal deformation as follows.

Definition 7.6. *Let A be a unital C^*-algebra, and suppose that there is an action α of $\mathbb{T}^l$ on A by $*$-automorphisms, which is strongly continuous. For each $r \in \mathbb{Z}^l$, let $A_{(r)}$ be the* spectral subspace

$$A_{(r)} := \{\, a \in A : \alpha_t(a) = t^r a \text{ for all } t \in \mathbb{T}^l \,\}, \quad \text{where } t^r := t_1^{r_1} \dots t_n^{r_n} \in \mathbb{T}.$$

Let $\mathcal{A} := \{\, a \in A : t \mapsto \alpha_t(a) \text{ is smooth} \,\}$ be the "smooth subalgebra" for the action of $\mathbb{T}^l$. It can be shown that $\mathcal{A}$ is a Fréchet pre-C^*-algebra, and each $a \in \mathcal{A}$ can be written as a convergent series $a = \sum_{r \in \mathbb{Z}^l} a_r$, where $a_r \in A_{(r)}$ and $\|a_r\| \to 0$ rapidly as $|r| \to \infty$.

Definition 7.7. *Fix $\Theta = -\Theta^t \in M_l(\mathbb{R})$. The **Moyal product** of two elements $a, b \in \mathcal{A}$, with $a = \sum_r a_r$ and $b = \sum_s b_s$, is defined as $a \star b := \sum_{r,s} a_r \star b_s$, where*

$$a_r \star b_s := \sigma(r,s)\, a_r b_s, \quad \text{with} \quad \sigma(r,s) := \exp\bigl\{-\pi i \textstyle\sum_{j,k=1}^{l} r_j \theta_{jk} s_k\bigr\}. \tag{7.6}$$

For actions of $\mathbb{T}^l$, Rieffel [Rie] showed that the integral formula and the series formula for $a \star b$ are equivalent, when a, b belong to the smooth subalgebra $\mathcal{A}$.

Definition 7.8. *Let M be a compact Riemannian manifold, carrying a continuous action of $\mathbb{T}^l$ by isometries $\{\sigma_t\}_{t \in \mathbb{T}^l}$. Then $\alpha_t(f) := f \circ \sigma_t$ is a strongly continuous action of $\mathbb{T}^l$. Given $\Theta = -\Theta^t \in M_l(\mathbb{R})$, Rieffel's construction provides a Moyal product on $C^\infty(M_\Theta) := (C^\infty(M), \star)$, whose C^*-completion in a suitable norm is $C(M_\Theta) := (C(M), \star)$. In particular, for $M = \mathbb{S}^4$ with the round metric and $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$, these are the algebras $C^\infty(\mathbb{S}^4_\theta)$ and $C(\mathbb{S}^4_\theta)$ introduced by Connes and Landi [CL].*

To deform the spectral triple $(C^\infty(M), L^2(M,S), D\!\!\!/)$, we need a further step. Since each σ_t is an isometry of M, it defines an automorphism of the tangent bundle TM (with $T_xM \to T_{\sigma_t(x)}M$), and of the cotangent bundle T^*M (with $T^*_{\sigma_t(x)}M \to T^*_xM$), preserving the orientation and the metric on each bundle. But the group $\mathrm{SO}(T^*_xM, g_x)$ does not act directly on the fibre S_x of the spinor bundle. Instead, the action of the Clifford algebra $\mathcal{B}$ on $\mathcal{H} = L^2(M,S)$ yields a homomorphism $\mathrm{Spin}(T^*_xM, g_x) \to \mathrm{End}(S_x)$ for each $x \in M$, and we know that there is a double covering $\mathrm{Ad}_x \colon \mathrm{Spin}(T^*_xM, g_x) \to \mathrm{SO}(T^*_xM, g_x)$ by conjugation.

It turns out [CDV] that one can lift the isometric action $\alpha \colon \mathbb{T}^l \to SO(T^*M)$ to an action of another torus $\tau \colon \widetilde{\mathbb{T}}^l \to \mathrm{Aut}(S)$, where there is a covering map $\pi \colon \widetilde{\mathbb{T}}^l \to \mathbb{T}^l$ such that $\pi(\pm 1) = 1$, making the following diagram commute:

$$\begin{tikzcd} \widetilde{\mathbb{T}}^l \arrow[r, "\tau_t"] \arrow[d, "\pi"'] & \mathrm{Aut}(S) \arrow[d, "\mathrm{Ad}"] \\ \mathbb{T}^l \arrow[r, "\alpha_t"] & \mathrm{SO}(T^*M) \end{tikzcd}$$

Fact 7.9. *One can find a covering of $\mathbb{T}^l$ by a torus $\widetilde{\mathbb{T}}^l$, and a representation $\tilde{t} \mapsto \tau_{\tilde{t}}$ of $\widetilde{\mathbb{T}}^l$ on* $\mathrm{Aut}(S)$ *such that* $\mathrm{Ad}(\tau_{\tilde{t}}) = \alpha_t$ *if* $\pi(\tilde{t}) = t \in \mathbb{T}^l$. *For $f \in \mathcal{A} = C^\infty(M)$ and $\phi, \psi \in \mathcal{S} = \Gamma(M,S)$, this implies that*

$$\tau_{\tilde{t}}(f\psi) = \alpha_t(f)\, \tau_{\tilde{t}}\psi, \tag{7.7}$$

$$(\tau_{\tilde{t}}\phi \mid \tau_{\tilde{t}}\psi) = \alpha_t(\phi \mid \psi). \tag{7.8}$$

Integrating over M, and recalling that σ_t is an isometry, we get $\langle \tau_{\tilde{t}}\phi \mid \tau_{\tilde{t}}\psi \rangle = \langle \phi \mid \psi \rangle$, so that τ extends to a unitary representation of $\widetilde{\mathbb{T}}^l$ on $\mathcal{H} = L^2(M,S)$.

We can regard $\mathbb{T}^l$ as $\mathbb{R}^l/(\mathbb{Z}^l + \hat{\mathbb{Z}}^l)$, where $\hat{\mathbb{Z}}^l = \mathbb{Z}^l + (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. With this convention, one can show that the set of commuting selfadjoint operators $P_1, \dots, P_l$ on $\mathcal{H}$ which generate the unitary representation of $\widetilde{\mathbb{T}}^l$, i.e.,

$$\tau_{\tilde{t}} =: \exp(it_1 P_1 + \cdots + it_n P_n),$$

have half-integer spectra: $\operatorname{sp}(P_j) \subseteq \frac{1}{2}\mathbb{Z}$.

Now define a family of unitary operators $\{\, \sigma(r, P) : r \in \mathbb{Z}^l \,\}$ by

$$\sigma(r, P) := \exp\Bigl(-\pi i \textstyle\sum_{j,k} r_j \theta_{jk} \, P_k\Bigr),$$

that is, we substitute s_k by P_k in the cocycle formula $\sigma(r, s)$ of (7.6).

Exercise 7.10. *Show that, since the action α is isometric, the unitary operator $\tau_{\tilde{t}}$ commutes with $D\!\!\!/$ and with the charge conjugation operator C, for each $\tilde{t} \in \widetilde{\mathbb{T}}^l$.*

It follows that each $\sigma(r, P)$ commutes with $D\!\!\!/$, too. However, the operators $\sigma(r, P)$ need *not* commute with the multiplication operators $\psi \mapsto f\psi$, for $f \in C^\infty(M)$. Indeed, (7.7) implies that $\tau_{\tilde{t}} f \tau_{-\tilde{t}} = \alpha_t(f)$ for each $\tilde{t} \in \widetilde{\mathbb{T}}^l$.

Exercise 7.11. *If $h_s \in A_{(s)}$ and $r \in \mathbb{Z}^l$, show that $\sigma(r, P)\, h_s = h_s\, \sigma(r, P + s)$.*

We are now ready to exhibit the isospectral deformation of the standard commutative example for a spin manifold M carrying an isometric action of $\mathbb{T}^l$, with respect to a fixed matrix Θ of deformation parameters. The deformation is called *isospectral* for the simple reason that the operator D of the deformed spectral triple is the *same* Dirac operator of the undeformed case, so it is no surprise that its spectrum does not change. What does change is the algebra: in fact, the underlying vector space of $\mathcal{A}$ is unchanged, but the product operation is deformed, and consequently its representation on $\mathcal{H}$ changes, too.

Theorem 7.12. *If $\mathcal{A}_\Theta = (C^\infty(M), \star)$, $\mathcal{H} = L^2(M, S)$ and $D = D\!\!\!/$, then there is a representation of $\mathcal{A}_\Theta$ by bounded operators on $\mathcal{H}$, such that $(\mathcal{A}_\Theta, \mathcal{H}, D\!\!\!/)$ is a spectral triple with the same Dirac operator as the standard commutative example $(C^\infty(M), \mathcal{H}, D\!\!\!/)$. Moreover, the charge conjugation operator C is a real structure on $(\mathcal{A}_\Theta, \mathcal{H}, D\!\!\!/)$, and the first order property holds.*

Proof. If $f \in \mathcal{A}$, write $f = \sum_{r \in \mathbb{Z}^l} f_r$ as a decomposition into spectral subspaces, where $\alpha_t(f_r) = t^r f_r$ for $t \in \mathbb{T}^l$, $r \in \mathbb{Z}^l$. Define

$$L(f) := \sum_r f_r \, \sigma(r, P) \in \mathcal{L}(\mathcal{H}).$$

Then $f \mapsto L(f)$ is a representation of the algebra $\mathcal{A}_\Theta$:

$$\begin{aligned} L(f)L(h) &= \sum_{r,s} f_r \, \sigma(r, P)\, h_s \, \sigma(s, P) \\ &= \sum_{r,s} f_r h_s \, \sigma(r, P + s)\, \sigma(s, P) \\ &= \sum_{r,s} f_r h_s \, \sigma(r, s)\, \sigma(r + s, P) \\ &= \sum_{r,s} f_r \star h_s \, \sigma(r + s, P) = L(f \star h). \end{aligned}$$

The last equality follows because $f_r \in A_{(r)}$, $h_s \in A_{(s)}$ imply that both $f_r h_s$ and $f_r \star h_s$ lie in $A_{(r+s)}$ —these products differ only by the phase factor $\sigma(r, s)$— and therefore $(f \star h)_p = \sum_{r+s=p} f_r \star h_s$.

Since $\alpha_t(f_r^*) = \alpha_t(f_r)^* = t^{-r} f_r^*$, we see that $(f^*)_s = (f_{-s})^*$ for $s \in \mathbb{Z}^l$. Thus $L(f)^* = \sum_r f_r^* \, \sigma(-r, P) = \sum_r (f^*)_{-r} \, \sigma(-r, P) = L(f^*)$, so that L is actually a $*$-representation.

Since $D\!\!\!/$ commutes with each $\sigma(r, P)$, we get

$$[D\!\!\!/, L(f)] = \sum_r [D\!\!\!/, f_r] \, \sigma(r, P) =: L([D\!\!\!/, f]), \tag{7.9}$$

where we remark that $\tau_{\tilde{t}} \, [D\!\!\!/, f_r] \, \tau_{-\tilde{t}} = [D\!\!\!/, \tau_{\tilde{t}} f_r \tau_{-\tilde{t}}] = t^r \, [D\!\!\!/, f_r]$, so that the operators $[D\!\!\!/, f]$, for $f \in \mathcal{A}$, decompose into spectral subspaces under the action $t \mapsto \mathrm{Ad}(\tau_{\tilde{t}})$ by automorphisms of $\mathcal{L}(\mathcal{H})$, which extends $t \mapsto \alpha_t$ by automorphisms of $\mathcal{A}$.

Next, since the antilinear operator C commutes with all unitaries $\sigma(r, P)$, we deduce that $C P_j C^{-1} = -P_j$ for $j = 1, 2, \dots, l$. Therefore, we can define an antirepresentation of $\mathcal{A}_\Theta$ on $\mathcal{H}$ by

$$R(f) := C \, L(f)^* \, C^{-1} = \sum_r \sigma(r, P)^* \, C f_r C^{-1} = \sum_r \sigma(-r, P) \, f_r = \sum_r f_r \, \sigma(-r, P).$$

Notice that $\sigma(r, P)^* = \sigma(-r, P)$ commutes with f_r in view of Exercise 7.11 and the relation $\sigma(-r, r) = 1$.

The left and right multiplication operators commute, since

$$\begin{aligned}
[L(f), R(h)] &:= \sum_{r,s} [\sigma(r, P) \, f_r, h_s \, \sigma(-s, P)] \\
&= \sum_{r,s} [f_r, h_s] \, \sigma(r, s) \, \sigma(r - s, P) = 0,
\end{aligned}$$

where $[f_r, h_s] = 0$ because, with its original product. $\mathcal{A}$ is commutative. The same calculation shows also that

$$[[D\!\!\!/, L(f)], R(h)] = \big[L([D\!\!\!/, f]), R(h)\big] = \sum_{r,s} [[D\!\!\!/, f_r], h_s] \, \sigma(r, s) \, \sigma(r - s, P) = 0,$$

since $([D\!\!\!/, f])_r = [D\!\!\!/, f_r]$ for each r and $[[D\!\!\!/, f_r], h_s] = 0$ by the first-order property of the undeformed spectral triple $(C^\infty(M), \mathcal{H}, D\!\!\!/)$. □

When $\dim M$ is even, and Γ is the $\mathbb{Z}_2$-grading operator Γ on the spinor space $\mathcal{H}$, we should note that the orientation condition $\pi_{D\!\!\!/}(\mathbf{c}) = \Gamma$ says, among other things, that Γ appears in the algebra generated by the operators f and $[D\!\!\!/, f]$, for $f \in \mathcal{A}$. The representation L of $\mathcal{A}_\Theta$ extends to this algebra of operators by using (7.9) as a definition of $L([D\!\!\!/, f])$. In the formula (7.1) for $\pi_{D\!\!\!/}(\mathbf{c})$, if we replace all terms a_j^r by $L(a_j^r)$, then we obtain $L(\pi_{D\!\!\!/}(\mathbf{c})) = L(\Gamma) = \Gamma$. Thus $\mathbf{c}$ may also be regarded as a Hochschild n-cycle over $\mathcal{A}_\Theta$, and *the orientation condition* $\pi_{D\!\!\!/}(\mathbf{c}) = \Gamma$ *is unchanged by the deformation*. In odd dimensions, the same is true, with Γ replaced by 1.

In conclusion: the isospectral deformation procedure of Connes and Landi yields a family of noncommutative spectral triples that satisfy all of our stated conditions for a noncommutative spin geometry. (Moreover [CL], Poincaré duality in K-theory is stable under deformation, too.)

7.3 The Moyal plane as a nonunital spectral triple

In order to extend the notion of spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to include the case where the algebra $\mathcal{A}$ may be nonunital, we modify Definition 4.1 as follows.

Definition 7.13. *A* ***nonunital spectral triple*** $(\mathcal{A}, \mathcal{H}, D)$ *consists of a nonunital $*$-algebra $\mathcal{A}$, equipped with a faithful representation on a Hilbert space $\mathcal{H}$, and a selfadjoint operator D on $\mathcal{H}$ with $a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ for all $a \in \mathcal{A}$, such that*

- $[D, a]$ *extends to a bounded operator on $\mathcal{H}$, for each $a \in \mathcal{A}$;*
- $a\,(D^2 + 1)^{-1/2}$ *is a compact operator, for each $a \in \mathcal{A}$.*

In general, D may have continuous spectrum, so that the operator $(D^2 + 1)^{-1/2}$ will usually not be compact. But it is enough to ask that it become compact when mollified by any multiplication operator in $\mathcal{A}$. An equivalent condition is that $a(D-\lambda)^{-1}$ be compact, for all $\lambda \notin \operatorname{sp}(D)$. In the nonunital case, there is no advantage in supposing that D be invertible, so it is better to work directly with $(D^2 + 1)^{1/2}$ instead of $|D|$.

Remark 7.14. The simplest commutative example of a nonunital spectral triple is given by

$$\mathcal{A} = C_0^\infty(\mathbb{R}^n), \qquad \mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^m}, \qquad \not{D} = -i\,\gamma^j\,\frac{\partial}{\partial x^j},$$

describing the noncompact manifold $\mathbb{R}^n$ with trivial spinor bundle $\mathbb{R}^n \times \mathbb{C}^{2^m} \to \mathbb{R}^n$ and flat metric: as always, $n = 2m$ or $n = 2m+1$. Here $C_0^\infty(\mathbb{R}^n)$ is the space of smooth functions that vanish at infinity together with all derivatives: it is a $*$-algebra under pointwise multiplication and complex conjugation of functions. Here $\operatorname{sp}(\not{D}) = \mathbb{R}$ and $(\not{D}^2 + 1)^{-1/2}$ is not compact. However, it is known [Sim] that if $f \in L^p(\mathbb{R}^n)$ with $p > n$, then $f\,(\not{D}^2 + 1)^{-1/2} \in \mathcal{L}^p(\mathcal{H})$.

The simplest *noncommutative, nonunital* example is an isospectral deformation of this commutative case, where we use the same Dirac operator $\not{D} = -i\,\gamma^j\,\partial/\partial x^j$ on the same spinor space $\mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^m}$, but we change the algebra by replacing the ordinary product of functions by a Moyal product.

Before giving the details, we summarize the effect of this nonunital isospectral deformation on the conditions given in Section 7.1 to define a "noncommutative spin geometry".

- The *reality* and *first-order* conditions are unchanged: we use the same charge conjugation operator C as in the undeformed case.
- The *regularity* condition is essentially unchanged: all that is needed is to replace the derivation $\delta\colon T \mapsto [|D|, T]$ by the derivation $\delta_1\colon T \mapsto [(\not{D}^2+1)^{1/2}, T]$, because $\operatorname{Dom}\delta_1^k = \operatorname{Dom}\delta^k$ for each $k \in \mathbb{N}$ since $(\not{D}^2 + 1)^{1/2} - |D|$ is a bounded operator.
- For the *orientation condition*, the Hochschild n-cycle will not lie in $Z_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ but rather in $Z_p(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^\circ)$, where $\widetilde{\mathcal{A}}$ is a *unitization* of $\mathcal{A}$, that is, a unital $*$-algebra in which $\mathcal{A}$ is included as an essential ideal.
- For the *finiteness* condition, we ask that $\mathcal{H}^\infty = \mathcal{A}^N p$, for some projector $p = p^2 = p^*$ lying in $M_N(\widetilde{\mathcal{A}})$. Thus $\mathcal{H}^\infty$ can be regarded as the pullback, via the inclusion $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}}$, of the finitely generated projective left $\widetilde{\mathcal{A}}$-module $\widetilde{\mathcal{A}}^N p$.
- To define the integer n as the *spectral dimension*, we would like to be able to assert that $a\,(D^2 + 1)^{-1/2}$ lies in $\mathcal{L}^{n+}(\mathcal{H})$ for each $a \in \mathcal{A}$, and that $0 < \operatorname{Tr}_\omega(a\,(D^2 + 1)^{-n/2}) < \infty$ whenever a is positive and nonzero. It turns out that we can only verify this for a belonging to a certain *dense subalgebra* of $\mathcal{A}$, in the Moyal-plane example: see below.

Exercise 7.15. *Check the assertion on regularity: show that* $\operatorname{Dom}\delta_1 = \operatorname{Dom}\delta$ *and that* $\operatorname{Dom}\delta_1^k = \operatorname{Dom}\delta^k$ *for each* $k \in \mathbb{N}$*, by induction on* k.

Exercise 7.16. *Show that Proposition 6.13 holds without the assumption that* D *is invertible. Namely, if* $L_1(b) := (D^2+1)^{-1/2}\,[D^2, b]$ *and* $R_1(b) := [D^2, b]\,(D^2+1)^{-1/2}$*, show that* $\bigcap_{k,l\geq 0} \operatorname{Dom}(L_1^k R_1^l) = \bigcap_{m\geq 0} \operatorname{Dom}\delta_1^m$ *by adapting the proof of Proposition 6.13.*

In what follows, we will sketch the main features of the Moyal-plane spectral triple. A complete treatment can be found in Gayral *et al* [GGISV], on which this outline is based. Our main concern here is to identify the "correct" algebra $\mathcal{A}$ and its unitization $\widetilde{\mathcal{A}}$ so that the modified spin-geometry conditions will hold.

We now recall the Moyal product over $\mathbb{R}^n$, discussed in the previous Section. It depends on a real skewsymmetric matrix $\Theta \in M_n(\mathbb{R})$ of "deformation parameters". For $n = 2$, such a matrix is of the form $\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ for some $\theta \in \mathbb{R}$; and for $n = 2m$ or $n = 2m+1$, Θ is similar to a direct sum of m such matrices with possibly different values of θ (so Θ cannot be invertible if n is odd). For convenience, we now take n to be even, and we shall suppose that all values of θ are the same. (In applications to quantum mechanics, where the Moyal product originated [Moy], $\theta = \hbar$ is the Planck constant.) Thus, we choose

$$\Theta := \theta\, S \in M_{2m}(\mathbb{R}), \quad \text{with } S := \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}, \quad \theta > 0. \tag{7.10}$$

Note that $\det\Theta = \theta^{2m} > 0$.

Definition 7.17. *Let* $n = 2m$ *be even, let* $\theta > 0$*, and let* $f, h \in \mathcal{S}(\mathbb{R}^n)$*. Their* ***Moyal product*** $f \star_\theta h \in \mathcal{S}(\mathbb{R}^n)$ *is defined as follows:*

$$\begin{aligned} (f \star_\theta h)(x) &:= (\pi\theta)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x+s)\, h(x+t)\, e^{2is(St)/\theta}\, ds\, dt \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x - \tfrac12\theta Su)\, h(x+t)\, e^{-iut}\, du\, dt \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} f(x - \tfrac12\theta Su)\, \hat{h}(u)\, e^{iux}\, du. \end{aligned} \tag{7.11}$$

Here $\hat{h}(u) := \int_{\mathbb{R}^n} h(t)\, e^{-iut}\, dt$ *is the Fourier transform. Since* $h \mapsto \hat{h}$ *preserves the Schwartz space* $\mathcal{S}(\mathbb{R}^n)$*, the third integral is a twisted version of the usual convolution of* f *and* $\hat{h}$*, and one can check that this integral converges to an element of* $\mathcal{S}(\mathbb{R}^n)$.

The first or second integral in (7.11) can also be regarded as defining the Moyal product $f \star_\theta h$, where f and h need not be Schwartz functions, provided that the integrals are understood in some generalized sense. Thus Rieffel [Rie], for instance, considers them as oscillatory integrals. Here we shall extend the Moyal product by duality, as follows. It is easy to see that $\|f \star_\theta h\|_\infty \leq (\pi\theta)^{-n}\|f\|_1\,\|h\|_1$, from the first integral in (7.11). By applying similar estimates to the functions $x^\alpha\,\partial^\beta\,(f \star_\theta h)$, for $\alpha, \beta \in \mathbb{N}^n$, one can verify that the product $(f, h) \mapsto f \star_\theta h$ is a jointly continuous bilinear map from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Here are some elementary properties of the Moyal product that are easy to check formally; they can be verified rigorously by some work with oscillatory integrals: see [Rie].

1. The Moyal product is associative: $(f \star_\theta g) \star_\theta h = f \star_\theta (g \star_\theta h)$.

2. The Leibniz rule holds: $\partial_j(f \star_\theta h) = \partial_j f \star_\theta h + f \star_\theta \partial_j h$ for $j = 1, \dots, n$.

3. Complex conjugation is an involution: $\overline{f \star_\theta h} = \bar{h} \star_\theta \bar{f}$.

4. Integration over $\mathbb{R}^n$ is a trace for the Moyal product:

$$\int_{\mathbb{R}^n} (f \star_\theta h)(x)\, dx = \int_{\mathbb{R}^n} (h \star_\theta f)(x)\, dx = \int_{\mathbb{R}^n} f(x)\, h(x)\, dx. \tag{7.12}$$

We denote $\mathcal{S}_\theta := (\mathcal{S}(\mathbb{R}^n), \star_\theta)$. It is a Fréchet $*$-algebra, with the usual topology of $\mathcal{S}(\mathbb{R}^n)$.

The trace property gives us a (suitably normalized) bilinear pairing:

$$\langle f, h \rangle := (\pi\theta)^{-m} \int_{\mathbb{R}^n} (f \star_\theta h)(x)\, dx.$$

Together with associativity, this gives the relation

$$\langle f \star_\theta g, h \rangle = \langle f, g \star_\theta h \rangle = (\pi\theta)^{-m} \int_{\mathbb{R}^n} (f \star_\theta g \star_\theta h)(x)\, dx,$$

valid for $f, g, h \in \mathcal{S}(\mathbb{R}^n)$. Now, if $T \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, and if $f \in \mathcal{S}(\mathbb{R}^n)$, we can define $T \star_\theta f$, $f \star_\theta T \in \mathcal{S}'(\mathbb{R}^n)$ by the continuity of the Moyal product:

$$\langle T \star_\theta f, h \rangle := \langle T, f \star_\theta h \rangle, \qquad \langle f \star_\theta T, h \rangle := \langle T, h \star_\theta f \rangle.$$

In this way, $\mathcal{S}'(\mathbb{R}^n)$ becomes a bimodule over $\mathcal{S}_\theta$. Inside this bimodule, we can identify a multiplier algebra in the obvious way.

Definition 7.18. *The **Moyal algebra** $\mathcal{M}_\theta = \mathcal{M}_\theta(\mathbb{R}^n)$ is defined as the set of (left and right) multipliers for $\mathcal{S}(\mathbb{R}^n)$ within $\mathcal{S}'(\mathbb{R}^n)$:*

$$\mathcal{M}_\theta := \{\, R \in \mathcal{S}'(\mathbb{R}^n) : R \star_\theta f \in \mathcal{S}(\mathbb{R}^n),\ f \star_\theta R \in \mathcal{S}(\mathbb{R}^n) \text{ for all } f \in \mathcal{S}(\mathbb{R}^n) \,\}.$$

This is a $*$-algebra, and $\mathcal{S}'(\mathbb{R}^n)$ is an $\mathcal{M}_\theta$-bimodule, under the operations

$$\langle T \star_\theta R, f \rangle := \langle T, R \star_\theta f \rangle, \qquad \langle R \star_\theta T, f \rangle := \langle T, f \star_\theta R \rangle.$$

This Moyal algebra is very large: for instance, it contains all polynomials on $\mathbb{R}^n$. However, because it contains many unbounded elements, it cannot serve as a coordinate algebra for a spectral triple. Even so, it is a starting point for a second approach, developed in [GV]. Consider the quadratic polynomials $H_r := \frac{1}{2}(x_r^2 + x_{m+r}^2)$ for $r = 1, \dots, m$. In the quantum-mechanical interpretation, these are Hamiltonians for a set of m independent harmonic oscillators; but for now, it is enough to know that they belong to $\mathcal{M}_\theta$. It turns out that the left and right Moyal multiplications by these H_r have a set of joint eigenfunctions $\{\, f_{kl} : k, l \in \mathbb{N}^m \,\}$ belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, with the following properties:

- The eigenvalues are half-integer multiples of θ, namely,

$$H_r \star_\theta f_{kl} = \theta(k_r + \tfrac{1}{2}) f_{kl}, \qquad f_{kl} \star_\theta H_r = \theta(l_r + \tfrac{1}{2}) f_{kl}.$$

- The eigenfunctions form a set of matrix units for the Moyal product: $f_{kl} \star_\theta f_{rs} = \delta_{lr} f_{ks}$ and $\bar{f}_{kl} = f_{lk}$ for all $k, l, r, s \in \mathbb{N}^m$.

- Any $f \in \mathcal{S}(\mathbb{R}^n)$ is given by a series $f = (2\pi\theta)^{-m/2} \sum_{kl} \alpha_{kl}\, f_{kl}$, converging in the topology of $\mathcal{S}(\mathbb{R}^n)$, such that $\alpha_{kl} \to 0$ rapidly.

- The subset $\{ (2\pi\theta)^{-m/2} f_{kl} : k, l \in \mathbb{N}^m \}$ of $\mathcal{S}(\mathbb{R}^n)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

For example, when $n = 2$ and $k = l \in \mathbb{N}$, f_{kk} is given by

$$f_{kk}(x_1, x_2) := 2\,(-1)^k\, e^{-(x_1^2+x_2^2)/\theta}\, L_k^0(\tfrac{2}{\theta}(x_1^2 + x_2^2)),$$

where L_k^0 is the Laguerre polynomial of order k.

Because of these properties, we can extend the Moyal product to pairs of functions in $L^2(\mathbb{R}^n)$. If $f = (2\pi\theta)^{-m/2} \sum_{k,l} \alpha_{kl} f_{kl}$ and $h = (2\pi\theta)^{-m/2} \sum_{k,l} \beta_{kl} f_{kl}$, we define

$$f \star_\theta h := (2\pi\theta)^{-m} \sum_{k,r,l} \alpha_{kr}\beta_{rl}\, f_{kl}. \tag{7.13}$$

The Schwarz inequality for sequences shows that

$$\begin{aligned} \|f \star_\theta h\|_2^2 &= (2\pi\theta)^{-2m} \Bigl\| \sum_{k,r,l} \alpha_{kr}\beta_{rl}\, f_{kl} \Bigr\|_2^2 = (2\pi\theta)^{-m} \sum_{k,l} \Bigl| \sum_r \alpha_{kr}\beta_{rl} \Bigr|^2 \\ &\leq (2\pi\theta)^{-m} \sum_{k,r} |\alpha_{kr}|^2 \sum_{r,l} |\beta_{rl}|^2 = (2\pi\theta)^{-m} \|f\|_2^2\, \|h\|_2^2. \end{aligned} \tag{7.14}$$

This calculation guarantees that the series (7.13) converges whenever $f, h \in L^2(\mathbb{R}^n)$; and that the operator $L(f) \colon h \mapsto f \star_\theta h$ extends to a bounded operator in $\mathcal{L}(L^2(\mathbb{R}^n))$ whenever $f \in L^2(\mathbb{R}^n)$. Moreover, it gives a bound on the operator norm:

$$\|L(f)\| \leq (2\pi\theta)^{-m/2}\, \|f\|_2.$$

Now the Schwartz-multiplier algebra $\mathcal{M}_\theta$ can be replaced by an L^2-multiplier algebra. By duality in sequence spaces, any $T \in \mathcal{S}'(\mathbb{R}^n)$ can be given an expansion in terms of the $\{f_{kl}\}$ basis, and in this way one can define an algebra

$$A_\theta := \{ R \in \mathcal{S}'(\mathbb{R}^n) : R \star_\theta f \in L^2(\mathbb{R}^n) \text{ for all } f \in L^2(\mathbb{R}^n) \}.$$

This is actually a C^*-algebra, with operator norm $\|L(R)\| := \sup\{ \|R \star_\theta f\|_2 / \|f\|_2 : f \neq 0 \}$.

There is a unitary isomorphism $W \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^m)$ (tensor product of Hilbert spaces), such that $W\, L(f)\, W^{-1} = \sigma(f) \otimes 1$, where σ is the (irreducible) Schrödinger representation; that is to say, $f \mapsto L(f)$ is equivalent to the Schrödinger representation with infinite multiplicity. One can show that $A_\theta = W^{-1}\, \mathcal{L}(L^2(\mathbb{R}^m))\, W$, whereas the norm closure of the $*$-algebra $(\mathcal{S}(\mathbb{R}^n), \star_\theta)$ is $W^{-1}\, \mathcal{K}(L^2(\mathbb{R}^m))\, W$. For the details, consult [VG] and [GGISV].

The analogue of Lemma 6.19 holds, too: *$\mathcal{S}_\theta$ is a nonunital pre-C^*-algebra.* As in the proof of Lemma 6.19, if $f \in \mathcal{S}_\theta$, suppose the equation $(1 + f) \star_\theta (1 + g) = 1$ has a solution g in the unital C^*-algebra A_θ. We may also write

$$f + g + f \star_\theta g = 0 \quad \text{and} \quad f + g + g \star_\theta f = 0, \tag{7.15}$$

and we wish to show that $g \in \mathcal{S}_\theta$. Since $g = -f - f \star_\theta g$, it is enough to show that $f \star_\theta g \in \mathcal{S}_\theta$. Now, left-multiplying the second equation in (7.15) by f gives $f \star_\theta f + f \star_\theta g + f \star_\theta g \star_\theta f = 0$, so it is enough to check that $f \star_\theta g \star_\theta f \in \mathcal{S}(\mathbb{R}^n)$ whenever $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in A_\theta$. This turns out to be true: the necessary norm estimates are given in [VG].

However, the algebra $\mathcal{S}_\theta$ is not the best candidate for the coordinate algebra of the Moyal spectral triple. We now introduce a better algebra.

Definition 7.19. *Consider the following space of smooth functions on* $\mathbb{R}^n$*:*

$$\mathcal{D}_{L^2}(\mathbb{R}^n) := \{\, f \in C^\infty(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}^n \,\},$$

introduced by Laurent Schwartz in his book on distributions [Schz]. It is a Fréchet space, under the norms $p_r(f) := \sum_{|\alpha|\le r} \|\partial^\alpha f\|_2$*, for* $r \in \mathbb{N}$*. The Leibniz rule for the Moyal product and the inequality* (7.14) *show that if* $f, h \in \mathcal{D}_{L^2}(\mathbb{R}^n)$*, then*

$$\begin{aligned} \|\partial^\alpha (f \star_\theta h)\|_2 &\le \sum_{0\le\beta\le\alpha} \|\partial^\beta f \star_\theta \partial^{\alpha-\beta} h)\|_2 \\ &\le (2\pi\theta)^{-m/2} \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} \|\partial^\beta f\|_2 \, \|\partial^{\alpha-\beta} h\|_2^2, \end{aligned}$$

so that $\mathcal{D}_{L^2}(\mathbb{R}^n)$ *is actually an* algebra *under the Moyal product; and that this product is continuous for the given Fréchet topology. Moreover, since complex conjugation is an isometry for each norm* p_r*, it is a* $*$*-algebra with a continuous involution. We write* $\mathcal{A}_\theta := (\mathcal{D}_{L^2}(\mathbb{R}^n), \star_\theta)$ *to denote this Fréchet* $*$*-algebra.*

It does not matter whether these derivatives $\partial^\alpha f$ are taken to be distributional derivatives only, since arguments based on Sobolev's Lemma show that if f and all its distributional derivatives are square-integrable, then f is actually a smooth function.

The algebra $\mathcal{A}_\theta$ is nonunital. Next, we introduce the preferred *unitization* of $\mathcal{A}_\theta$.

Definition 7.20. *Another space of smooth functions on* $\mathbb{R}^n$ *is found also in [Schz]:*

$$\mathcal{B}(\mathbb{R}^n) := \{\, f \in C^\infty(\mathbb{R}^n) : \partial^\alpha f \text{ is bounded on } \mathbb{R}^n, \text{ for all } \alpha \in \mathbb{N}^n \,\}.$$

It is also a Fréchet space, under the norms $q_r(f) := \max_{|\alpha|\le r} \|\partial^\alpha f\|_\infty$*, for* $r \in \mathbb{N}$*.*

We shall soon prove that $\mathcal{B}(\mathbb{R}^n)$ *is also a* $*$*-algebra under the Moyal product; we denote it by* $\widetilde{\mathcal{A}}_\theta := (\mathcal{D}_{L^2}(\mathbb{R}^n), \star_\theta)$*.*

It is proved in Schwartz' book that $\mathcal{D}_{L^2}(\mathbb{R}^n) \subset \mathcal{B}(\mathbb{R}^n)$, and that the inclusion is continuous for the given topologies. (This is not as obvious as it seems, because in general square-integrable functions on $\mathbb{R}^n$ need not be bounded.) Combining this with knowledge of the Moyal multiplier algebras, we end up with the following inclusions [GGISV]:

$$\mathcal{S}_\theta \subset \mathcal{A}_\theta \subset \widetilde{\mathcal{A}}_\theta \subset A_\theta \cap \mathcal{M}_\theta.$$

The inclusion $\widetilde{\mathcal{A}}_\theta \subset A_\theta$ is a consequence of the Calderón–Vaillancourt theorem, which says that a pseudodifferential operator of order zero on $\mathbb{R}^n$, whose symbol is differentiable to a high enough order, gives a bounded operator on $L^2(\mathbb{R}^n)$; we may notice that the second integral in (7.11) says that $L(f)$ is pseudodifferential, with symbol $p(x,\xi) = f(x - \frac{1}{2}\theta S\xi)$.

Proposition 7.21. $\mathcal{B}(\mathbb{R}^n)$ *is a Fréchet* $*$*-algebra under the Moyal product.*

Proof. If $f, h \in \mathcal{B}(\mathbb{R}^n)$ and if $s \in \mathbb{N}$, we shall find estimates of the form

$$q_s(f \star_\theta h) \le C_{rs}\, q_r(f)\, q_r(h) \quad \text{whenever } r \ge s + n + 2. \tag{7.16}$$

This shows that $f \star_\theta h$ lies in $\mathcal{B}(\mathbb{R}^n)$ whenever $f, h \in \mathcal{B}(\mathbb{R}^n)$, and that $(f,h) \mapsto f \star_\theta h$ is a jointly continuous bilinear operation on $\mathcal{B}(\mathbb{R}^n)$. Since complex conjugation is clearly isometric for each q_r, the involution is continuous, too.

To justify the estimates (7.16), we first notice that, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
(\partial^\beta f \star_\theta \partial^\gamma h)(x) &= (\pi\theta)^{-n} \iint \frac{\partial^\beta f(x+y)}{(1+|y|^2)^k} \, \frac{\partial^\gamma h(x+z)}{(1+|z|^2)^k} \, (1+|y|^2)^k (1+|z|^2)^k \, e^{2iy(Sz)/\theta} \, dy \, dz \\
&= (\pi\theta)^{-n} \iint \frac{\partial^\beta f(x+y)}{(1+|y|^2)^k} \, \frac{\partial^\gamma h(x+z)}{(1+|z|^2)^k} \, P_k(\partial_y, \partial_z) \big[e^{2iy(Sz)/\theta} \big] \, dy \, dz \\
&= (\pi\theta)^{-n} \iint e^{2iy(Sz)/\theta} \, P_k(-\partial_y, -\partial_z) \left[\frac{\partial^\beta f(x+y)}{(1+|y|^2)^k} \, \frac{\partial^\gamma h(x+z)}{(1+|z|^2)^k} \right] dy \, dz,
\end{aligned}
$$

where P_k is a certain polynomial of degree $2k$ in both y_j and z_j variables, and for the third line we integrate by parts. It is not hard to find constants such that $|\partial^\alpha((1+|x|^2)^{-k})| \leq C'_{\alpha k}(1+|x|^2)^{-k}$ for each $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. Thus, we get estimates of the form

$$
\begin{aligned}
|(\partial^\beta f \star_\theta \partial^\gamma h)(x)| &\leq \sum_{|\mu|,|\nu| \leq 2k} C''_{\mu\nu} \iint \frac{|\partial^{\beta+\mu} f(x+y)| \, |\partial^{\gamma+\nu} h(x+z)|}{(1+|y|^2)^k \, (1+|z|^2)^k} \, dy \, dz \\
&\leq C'''_{kr} \, q_r(f) \, q_r(h) \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^k} \int_{\mathbb{R}^n} \frac{dz}{(1+|z|^2)^k},
\end{aligned}
$$

provided $r \geq |\beta| + |\gamma| + 2k$; and we need $k > n/2$ so that the right hand side is finite. For $|\beta| + |\gamma| \leq s$, we only need to choose k so that $n < 2k \leq r - s$, and this is always possible for $r \geq s + n + 2$. □

Rieffel, in [Rie], showed that $\widetilde{\mathcal{A}}_\theta$ is the space of smooth vectors for the action of $\mathbb{R}^n$ (by translations) on its C^*-completion; this entails that $\widetilde{\mathcal{A}}_\theta$ is a *pre-C^*-algebra*.

Now, the inclusion $\widetilde{\mathcal{A}}_\theta \subset \mathcal{A}_\theta$ means that $\|\partial^\alpha f \star_\theta \partial^\beta h\|_2$ is finite, whenever $f \in \widetilde{\mathcal{A}}_\theta$ and $h \in \mathcal{A}_\theta$; therefore, $f \star_\theta h$ lies in $\mathcal{A}_\theta$ also. A similar argument shows that $h \star_\theta f$ lies in $\mathcal{A}_\theta$. Thus, $\mathcal{A}_\theta$ is an *ideal* in $\widetilde{\mathcal{A}}_\theta$. (In fact, it is an essential ideal; that is to say, if $f \star_\theta h = 0$ for all $h \in \mathcal{A}_\theta$, then $f = 0$; this can be seen by taking $h = f_{kl}$ for any $k, l \in \mathbb{N}^n$ and checking that f must vanish.)

Lemma 7.22. *$\mathcal{A}_\theta$ is a nonunital pre-C^*-algebra.*

Proof. Since $\mathcal{A}_\theta$ is Fréchet, we only need to show that it is spectrally invariant. In the nonunital case, this means that if $f \in \mathcal{A}_\theta$, and the equations $f + g + f \star_\theta g = f + g + g \star_\theta f = 0$ have a solution g in the C^*-completion of $\mathcal{A}_\theta$, then g lies in $\mathcal{A}_\theta$. Now since $f \in \widetilde{\mathcal{A}}_\theta$ and $\widetilde{\mathcal{A}}_\theta$ is already a pre-C^*-algebra, we see that $g \in \widetilde{\mathcal{A}}_\theta$. But $\mathcal{A}_\theta$ is an ideal in $\widetilde{\mathcal{A}}_\theta$, and thus $f \star_\theta g \in \mathcal{A}_\theta$. This implies that $g = -f - f \star_\theta g$ lies in $\mathcal{A}_\theta$, too. □

An important family of elements in $\widetilde{\mathcal{A}}_\theta$ that do not belong to $\mathcal{A}_\theta$ are the *plane waves*:

$$
u_k(x) := e^{2\pi i k x}, \quad \text{for each } \ k \in \mathbb{R}^n.
$$

It is immediate from the formulas (7.11) that

$$
u_k \star_\theta u_l = e^{-\pi i \theta k(Sl)} \, u_{k+l}, \quad \text{for all } \ k, l \in \mathbb{R}^n.
$$

In particular, by taking $k, l \in \mathbb{Z}^n$ to be *integral* vectors, we get an inclusion $C^\infty(\mathbb{T}^n_{\theta S}) \hookrightarrow \widetilde{\mathcal{A}}_\theta$: the smooth algebra of the noncommutative n-torus, for $\Theta = \theta S$, can be identified with a subalgebra of *periodic functions* in $\widetilde{\mathcal{A}}_\theta$.

In particular, the Hochschild n-cycle $\mathbf{c}$ representing the orientation of this noncommutative torus can also be regarded as an n-cycle over $\widetilde{\mathcal{A}}_\theta$. We can write $u_k = v_1^{k_1} \star_\theta \cdots \star_\theta v_n^{k_n}$ where $v_j = u_{e_j}$ for the standard orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. The expression for $\mathbf{c}$ is

$$\mathbf{c} = \frac{1}{n!\,(2\pi i)^n} \sum_{\sigma \in S_n} (-1)^\sigma \big(v_{\sigma(1)} v_{\sigma(2)} \dots v_{\sigma(n)}\big)^{-1} \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

(When $\theta = 0$, we can write $v_j = e^{2\pi i t_j}$, and the right hand side reduces to $dt_1 \wedge \cdots \wedge dt_n$, the usual volume form for either $\mathbb{R}^n$ or the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.)

We refer to [GGISV] for the discussion of the spectral dimension properties of the triple $(\mathcal{A}_\theta, L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^m}, D\!\!\!\!/)$. Briefly, the facts are these. If $\pi(f) := L(f) \otimes 1_{2^m}$ denotes the representation of $\mathcal{A}_\theta$ on the spinor space $\mathcal{H}$ by componentwise left Moyal multiplication, then one can show that, for any $f \in \mathcal{A}_\theta$, we get

$$\pi(f)\,(D\!\!\!\!/^2 + 1)^{-1/2} \in \mathcal{L}^p(\mathcal{H}), \quad \text{for all } \ p > n.$$

In particular, these operators are compact, so this triple is indeed a nonunital spectral triple. However, this is not quite enough to guarantee that

$$\pi(f)\,(D\!\!\!\!/^2 + 1)^{-1/2} \in \mathcal{L}^{n+}(\mathcal{H}), \tag{7.17}$$

for *every* $f \in \mathcal{A}_\theta$. Instead, what is found in [GGISV] is that (7.17) holds for f lying in the (dense) subalgebra $\mathcal{S}_\theta$. The key lemma which makes the proof work is a "strong factorization" property of $\mathcal{S}_\theta$, proved in [GV]: namely, that any $f \in \mathcal{S}_\theta$ can be expressed (without taking finite sums) as a product $f = g \star_\theta h$, with $g, h \in \mathcal{S}_\theta$. This factorization property fails for the full algebra $\mathcal{A}_\theta$.

Once (7.17) has been established, one can proceed to compute its Dixmier trace. It turns out that $\operatorname{Tr}_\omega(\pi(f)\,(D\!\!\!\!/^2 + 1)^{-1/2})$ is unchanged from its value when $\theta = 0$, namely $(2^m\,\Omega_n/n\,(2\pi)^n) \int_{\mathbb{R}^n} f(x)\,dx$. The end result is that the spectral dimension condition for nonunital spectral triples is the expected one, but that Dixmier-traceability as in (7.17) should only be required for a dense subalgebra of the original algebra.

7.4 A geometric spectral triple over $SU_q(2)$

(The notes for this part will appear later. In the meantime, have a look at [DLSSV].)

Appendix A

Exercises

A.1 Examples of Dirac operators

A.1.1 The circle

Let $M := \mathbb{S}^1$, regarded as $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$; that is to say, we parametrize the circle by the half-open interval $[0,1)$ rather than $[0,2\pi)$, say. Then $\mathcal{A} = C^\infty(\mathbb{S}^1)$ can be identified with periodic smooth functions on $\mathbb{R}$ with period 1:

$$\mathcal{A} \simeq \{\, f \in C^\infty(\mathbb{R}) : f(t+1) \equiv f(t) \,\}.$$

Since $\mathrm{Cl}(\mathbb{R}) = \mathbb{C}1 \oplus \mathbb{C}e_1$ as a $\mathbb{Z}_2$-graded algebra, we see that $\mathcal{B} = \mathcal{A}$ in this case; and since $n = 1$, $m = 0$ and $2^m = 1$, there is a "trivial" spin structure given by $\mathcal{S} := \mathcal{A}$ itself. The charge conjugation is just $C = K$, where K means *complex conjugation* of functions. With the flat metric on the circle, the Dirac operator is just

$$D\!\!\!/ := -i\,\frac{d}{dt}.$$

Exercise A.1. *Show that its spectrum is*

$$\mathrm{sp}(D\!\!\!/) = 2\pi\mathbb{Z} = \{\, 2\pi k : k \in \mathbb{Z} \,\},$$

by first checking that the eigenfunctions $\psi_k(t) := e^{2\pi i k t}$ form an orthonormal basis for the Hilbert-space completion $\mathcal{H}$ of $\mathcal{S}$ – using Fourier series theory.

The point is that the closed span of these eigenvectors is all of $\mathcal{H}$, so that $\mathrm{sp}(D\!\!\!/)$ contains no more than the corresponding eigenvalues.

Next, consider

$$\mathcal{S}' := \{\, \phi \in C^\infty(\mathbb{R}) : \phi(t+1) \equiv -\phi(t) \,\},$$

which can be thought of as the space of smooth functions on the interval $[0,1]$ "with antiperiodic boundary conditions".

Exercise A.2. *Explain in detail how $\mathcal{S}'$ can be regarded as a $\mathcal{B}$-$\mathcal{A}$-bimodule, and how $C = K$ acts on it as a charge-conjugation operator. Taking $D\!\!\!/ := -i\,d/dt$ again, but now as an operator with domain $\mathcal{S}'$ on the Hilbert-space completion of $\mathcal{S}'$, show that its spectrum is now*

$$\mathrm{sp}(D\!\!\!/) = 2\pi(\mathbb{Z} + \tfrac{1}{2}) = \{\, \pi(2k+1) : k \in \mathbb{Z} \,\},$$

by checking that $\phi_k(t) := e^{\pi i(2k+1)t}$ are a complete set of eigenfunctions.

The circle $\mathbb{S}^1$ thus carries two inequivalent spin structures: their inequivalence is most clearly manifest in the different spectra of the Dirac operators. Notice that $0 \in \operatorname{sp}(D\!\!\!/)$ for the "untwisted" spin structure where $\mathcal{S} = \mathcal{A}$, while $0 \notin \operatorname{sp}(D\!\!\!/)$ for the "twisted" spin structure whose spinor module is $\mathcal{S}'$. There are no more spin structures to be found, since $H^1(\mathbb{S}^1, \mathbb{Z}_2) = \mathbb{Z}_2$.

A.1.2 The (flat) torus

On the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, we use the Riemannian metric coming from the usual flat metric on $\mathbb{R}^2$. Thus, if we regard $\mathcal{A} = C^\infty(\mathbb{T}^2)$ as the smooth periodic functions on $\mathbb{R}^2$ with $f(t^1,t^2) \equiv f(t^1+1,t^2) \equiv f(t^1,t^2+1)$, then (t^1,t^2) define local coordinates on $\mathbb{T}^2$, with respect to which all Christoffel symbols are zero, namely $\Gamma^k_{ij} = 0$, and thus $\nabla = d$ represents the Levi-Civita connection on 1-forms.

In this case, $n = 2$, $m = 1$ and $2^m = 2$, so we use "two-component" spinors; that is, the spinor bundles $S \to \mathbb{T}^2$ are of rank two. There is the "untwisted" one, where S is the trivial rank-two $\mathbb{C}$-vector bundle, and $\mathcal{S} \simeq \mathcal{A}^2$. The Clifford algebra in this case is just $\mathcal{B} = M_2(\mathcal{A})$. Using the standard Pauli matrices:

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can write the charge conjugation operator as

$$C = -i\,\sigma^2\,K$$

where K again denotes (componentwise) complex conjugation.

Exercise A.3. *Find three more spinor structures on $\mathbb{T}^2$, exhibiting each spinor module as a $\mathcal{B}$-$\mathcal{A}$-bimodule, with the appropriate action of C. (Use $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.)*

Exercise A.4. *Check that*

$$D\!\!\!/ = -i\,(\sigma^1\,\partial_1 + \sigma^2\,\partial_2) = \begin{pmatrix} 0 & -\partial_2 - i\partial_1 \\ \partial_2 - i\partial_1 & 0 \end{pmatrix}$$

where $\partial_1 = \partial/\partial t^1$ and $\partial_2 = \partial/\partial t^2$, is indeed the Dirac operator on the untwisted spinor module $\mathcal{S} = \mathcal{A}^2$. Compute $\operatorname{sp}(D\!\!\!/^2)$ *by finding a complete set of eigenvectors. Then show that*

$$\operatorname{sp}(D\!\!\!/) = \{\, \pm 2\pi\sqrt{r_1^2 + r_2^2} : (r_1, r_2) \in \mathbb{Z} \,\}$$

by finding the eigenspinors for each of these eigenvalues. ¿What can be said of the multiplicities *of these eigenvalues? and what is the dimension of* $\ker D\!\!\!/$*?*

Notice that σ^3 does not appear in the formula for $D\!\!\!/$; its role here is to give the $\mathbb{Z}_2$-grading operator: $c(\gamma) = \sigma^3$ —regarded as a constant function with values in $M_2(\mathbb{C})$— in view of the relation $\sigma^3 = -i\,\sigma^1\sigma^2$ among Pauli matrices.

On the 3-torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$, where now $n = 3$, $m = 1$ and again $2^m = 2$, we get two-component spinors. Again we may use a flat metric and an untwisted spin structure with $\mathcal{S} = \mathcal{A}^2$. The charge conjugation is still $C = -i\,\sigma^2\,K$ on $\mathcal{S}$, so that $C^2 = -1$ also in this 3-dimensional case. The Dirac operator is now

$$D\!\!\!/ = -i\,(\sigma^1\,\partial_1 + \sigma^2\,\partial_2 + \sigma^3\,\partial_3) = \begin{pmatrix} -i\,\partial_3 & -\partial_2 - i\partial_1 \\ \partial_2 - i\partial_1 & i\,\partial_3 \end{pmatrix}.$$

Exercise A.5. *Compute* $\operatorname{sp}(D\!\!\!/^2)$ *and* $\operatorname{sp}(D\!\!\!/)$ *for this Dirac operator on $\mathbb{T}^3$.*

A.1.3 The Hodge–Dirac operator on $\mathbb{S}^2$

If M is a compact, oriented Riemannian manifold that has no spin^c structures, ¿can one define Dirac-like operators on an $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{E}$ that is not pointwise irreducible under the action of $\mathcal{B}$? It turns out that one can do so, if $\mathcal{E}$ carries a "Clifford connection", that is, a connection $\nabla^{\mathcal{E}}$ such that

$$\nabla^{\mathcal{E}}(c(\alpha)s) = c(\nabla\alpha)\, s + c(\alpha)\, \nabla^{\mathcal{E}} s,$$

for $\alpha \in \mathcal{A}^1(M)$, $s \in \mathcal{E}$, and which is Hermitian with respect to a suitable $\mathcal{A}$-valued sesquilinear pairing on $\mathcal{E}$. For instance, we may take $\mathcal{E} = \mathcal{A}^\bullet(M)$, the full algebra of differential forms on M, which we know to be a left $\mathcal{B}$-module under the action generated by $c(\alpha) = \varepsilon(\alpha) + \iota(\alpha^\sharp)$. The Clifford connection is just the Levi-Civita connection on all forms, obtaining by extending the one on $\mathcal{A}^1(M)$ with the Leibniz rule (and setting $\nabla f := df$ on functions). The pairing $(\alpha \mid \beta) := g(\bar\alpha, \beta)$ extends to a pairing on $\mathcal{A}^\bullet(M)$; by integrating the result over M with respect to the volume form ν_g, we get a scalar product on forms, and we can then complete $\mathcal{A}^\bullet(M)$ to a Hilbert space.

If $\{E_1, \dots, E_n\}$ and $\{\theta^1, \dots, \theta^n\}$ are local orthonormal sections for $\mathcal{X}(M)$ and $\mathcal{A}^1(M)$ respectively, compatible with the given orientation, so that $c(\theta^j) = \varepsilon(\theta^j) + \iota(E_j)$ locally, then

$$\star := c(\gamma) = (-i)^m\, c(\theta^1)\, c(\theta^2) \dots c(\theta^n)$$

is globally well-defined as an $\mathcal{A}$-*linear operator* taking $\mathcal{A}^\bullet(M)$ onto itself, such that $\star^2 = 1$. This is the *Hodge star* operator, and it exchanges forms of high and low degree.

Exercise A.6. *If $\{1, \dots, n\} = \{i_1, \dots, i_k\} \uplus \{j_1, \dots, j_{n-k}\}$, show that locally,*

$$\star(\theta^{i_1} \wedge \dots \wedge \theta^{i_k}) = \pm i^m\, \theta^{j_1} \wedge \dots \wedge \theta^{j_{n-k}},$$

where the sign depends on $i_1, \dots, i_k$. Conclude that $\star$ maps $\mathcal{A}^k(M)$ onto $\mathcal{A}^{n-k}(M)$, for each $k = 0, 1, \dots, n$.

(Actually, our sign conventions differ from the usual ones in differential geometry books, that do not include the factor $(-i)^m$. With the standard conventions, $\star^2 = \pm 1$ on each $\mathcal{A}^k(M)$, with a sign depending on the degree k.)

The *codifferential* δ on $\mathcal{A}^\bullet(M)$ is defined by

$$\delta := -\star d \star.$$

This operation *lowers* the form degree by 1. The **Hodge–Dirac** operator is defined to be $-i(d + \delta)$ on $\mathcal{A}^\bullet(M)$. One can show that, on the Hilbert-space completion, the operators d and $-\delta$ are adjoint to one another, so that $-i(d+\delta)$ extends to a selfadjoint operator. (With the more usual sign conventions, d and $+\delta$ are adjoint, so that the Hodge–Dirac operator is written simply $d + \delta$.)

Now we take $M = \mathbb{S}^2$, the 2-sphere of radius 1. The round (i.e., rotation-invariant) metric on $\mathbb{S}^2$ is written $g = d\theta^2 + \sin^2\theta\, d\phi^2$ in the usual spherical coordinates, which means that $\{d\theta, \sin\theta\, d\phi\}$ is a local orthonormal basis of 1-forms on $\mathbb{S}^2$. The area form is $\nu = \sin\theta\, d\theta \wedge d\phi$. The Hodge star is specified by defining it on 1 and on $d\theta$:

$$\star(1) := -i\,\nu, \qquad \star(d\theta) := i \sin\theta\, d\phi.$$

To find the eigenforms of the Hodge–Dirac operator, it is convenient to use another set of coordinates, obtained form the Cartesian relation $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ by setting $\zeta := x^1 + ix^2 = e^{i\phi}\cos\theta$, along with $x^3 = \cos\theta$; the pair (ζ, x^3) can serve as coordinates for $\mathbb{S}^2$, subject to the relation $\zeta\bar\zeta + (x^3)^2 = 1$. (The extra variable $\bar\zeta$ gives a third coordinate, extending $\mathbb{S}^2$ to $\mathbb{R}^3$.)

Exercise A.7. *Check that in the* (ζ, x^3) *coordinates, the Hodge star is given by*

$$\star(\zeta) = -i\, d\zeta \wedge dx^3, \qquad \star(d\zeta) = x^3\, d\zeta - \zeta\, dx^3.$$

Exercise A.8. *Consider the (complex) vectorfields on* $\mathbb{R}^3$ *given by*

$$L_+ := 2ix^3 \frac{\partial}{\partial \bar\zeta} + i\zeta \frac{\partial}{\partial x^3}, \quad L_- := 2ix^3 \frac{\partial}{\partial \zeta} - i\bar\zeta \frac{\partial}{\partial x^3}, \quad L_3 := i\bar\zeta \frac{\partial}{\partial \bar\zeta} - i\zeta \frac{\partial}{\partial \zeta}.$$

Verify the commutation relations $[L_+, L_-] = -2i\, L_3$, $[L_3, L_-] = i\, L_-$ *and* $[L_3, L_+] = -i\, L_+$.

These commutation relations show that if $L_\pm =: L_1 \pm i\, L_2$, then L_1, L_2, L_3 generate a representation of the Lie algebra of the rotation group SO(3). One obtains representation spaces of $SO(3)$ by finding functions f_0 ("highest weight vectors") such that $L_3 f_0$ is a multiple of f_0, $L_+ f_0 = 0$, and $\{\, (L_-)^r f_0 : r \in \mathbb{N} \,\}$ spans a space of finite dimension. To get spaces of differential forms with these properties, one extends each vector field L_j to an operator on $\mathcal{A}^\bullet(\mathbb{S}^2)$, namely its *Lie derivative* $\mathcal{L}_j$, just by requiring that $\mathcal{L}_j\, d = d\, \mathcal{L}_j$. Since the Hodge star operator is unchanged by applying a rotation to an orthonormal basis of 1-forms, one can also show that $\mathcal{L}_j \star = \star \mathcal{L}_j$, so that the Hodge–Dirac operator $-i(d+\delta)$ commutes with each $\mathcal{L}_j$. This gives a method of finding subspaces of joint eigenforms for each eigenvalue of the Hodge–Dirac operator.

We introduce the following families of forms:

$$\begin{aligned} \phi_l^+ &:= i\zeta^l(1 - i\nu), \quad l = 0,1,2,3,\dots; \qquad & \psi_l^+ &:= \zeta^{l-1}(d\zeta + \star(d\zeta)), \quad l = 1,2,3,\dots; \\ \phi_l^- &:= i\zeta^l(1 + i\nu), \quad l = 0,1,2,3,\dots; \qquad & \psi_l^- &:= \zeta^{l-1}(d\zeta - \star(d\zeta)), \quad l = 1,2,3,\dots. \end{aligned}$$

Clearly, $\star(\phi_l^\pm) = \pm\phi_l^\pm$ and $\star(\psi_l^\pm) = \pm\psi_l^\pm$. Thus ϕ_l^+ and ψ_l^+ are even, while ϕ_l^- and ψ_l^- are odd, with respecting to the $\mathbb{Z}_2$-grading on forms given by $\mathcal{A}^\bullet(\mathbb{S}^2) = \mathcal{A}^+(\mathbb{S}^2) \oplus \mathcal{A}^-(\mathbb{S}^2)$, where $\mathcal{A}^\pm(\mathbb{S}^2) := \frac{1}{2}(1 \pm \star)\, \mathcal{A}^\bullet(\mathbb{S}^2)$.

Exercise A.9. *Show that*

$$\begin{aligned} -i(d+\delta)\phi_l^\pm &= l\psi^\mp, \quad \textit{for } l = 0,1,2,\dots \\ -i(d+\delta)\psi_l^\pm &= (l+1)\phi^\mp, \quad \textit{for } l = 1,2,3,\dots \end{aligned}$$

and conclude that each of ϕ_l^+, ϕ_l^-, ψ_l^+ *and* ψ_l^- *is an eigenvector for* $(-i(d+\delta))^2 = -(d\delta + \delta d)$ *with eigenvalue* $l(l+1)$. *Find corresponding eigenspinors for* $-i(d+\delta)$ *with eigenvalues* $\pm\sqrt{l(l+1)}$.

Exercise A.10. *Show that* $L_3(\zeta^l) = -il\, \zeta^l$, $L_+ \zeta^l = 0$, *and that* $(L_-)^k(\zeta^l)$ *is a linear combination of terms* $(x^3)^{k-2r}\bar\zeta^r\zeta^{l-k+r}$ *that does not vanish for* $k = 0,1,\dots,2l$, *and that* $(L_-)^{2l+1}(\zeta^l) = 0$. *Check that* $L_+(L_-)^k(\zeta^l)$ *is a multiple of* $(L_-)^{k-1}(\zeta^l)$, *for* $k = 1,\dots,2l$.

Exercise A.11. *Show that*

$$\mathcal{L}_3\phi_l^\pm = -il\, \phi_l^\pm, \quad \mathcal{L}_+\phi_l^\pm = 0; \qquad \mathcal{L}_3\psi_l^\pm = -il\, \psi_l^\pm, \quad \mathcal{L}_+\psi_l^\pm = 0;$$

for each possible value of l. *Conclude that the forms* $\mathcal{L}_-^k(\phi_l^\pm)$ *and* $\mathcal{L}_-^k(\psi_l^\pm)$ *vanish if and only if* $k \geq 2l+1$. *¿What can now be said about the multiplicities of the eigenvalues of* $-i(d+\delta)$*?*

With some more works, it can be shown that all these eigenforms span a dense subspace of the Hilbert-space completion of $\mathcal{A}^\bullet(\mathbb{S}^2)$, so that these eigenvalues in fact give the full spectrum of the Hodge–Dirac operator.

A.2 The Dirac operator on the sphere $\mathbb{S}^2$

A.2.1 The spinor bundle S on $\mathbb{S}^2$

Consider the 2-dimensional sphere $\mathbb{S}^2$, with its usual orientation, $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\} \simeq \mathbb{C}P^1$. The usual spherical coordinates on $\mathbb{S}^2$ are

$$p = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{S}^2.$$

The poles are $N = (0,0,1)$ and $S = (0,0,-1)$. Let $U_N = \mathbb{S}^2 \setminus \{N\}$, $U_S = \mathbb{S}^2 \setminus \{S\}$ be the two charts on $\mathbb{S}^2$. Consider the stereographic projections $p \mapsto z : U_N \to \mathbb{C}$, $p \mapsto \zeta : U_S \to \mathbb{C}$ given by

$$z := e^{-i\phi} \cot\frac{\theta}{2}, \qquad \zeta := e^{+i\phi} \tan\frac{\theta}{2},$$

so that $\zeta = 1/z$ on $U_N \cap U_S$. Write

$$q := 1 + z\bar{z} = \frac{2}{1-\cos\theta}, \quad \text{and} \quad q' := 1 + \zeta\bar{\zeta} = \frac{q}{z\bar{z}}.$$

The sphere $\mathbb{S}^2$ has only the "trivial" spin structure $\mathcal{S} = \Gamma(\mathbb{S}^2, S)$, where $S \to \mathbb{S}^2$ has rank two. Now $S = S^+ \oplus S^-$, where $S^\pm \to \mathbb{S}^2$ are complex *line bundles*, and these may be (and are) nontrivial. We argue that $S^+ \to \mathbb{S}^2$ is the "tautological" line bundle coming from $\mathbb{S}^2 \simeq \mathbb{C}P^1$. We know already that

$$\mathcal{S}^\sharp \simeq \mathcal{S} \iff S^* \simeq S \impliedby (S^+)^* \simeq S^-$$

and the converse $S^* \simeq S \implies (S^+)^* \simeq S^-$ will hold provided we can show that $S^\pm \to \mathbb{S}^2$ are nontrivial line bundles. (Otherwise, S^+ and S^- would each be selfdual, but we know that the only selfdual line bundle on $\mathbb{S}^2$ is the trivial one, since $H^2(\mathbb{S}^2, \mathbb{Z}) \simeq \mathbb{Z}$.)

Consider now the (tautological) line bundle $L \to \mathbb{S}^2$, where

$$L_z := \{ (\lambda z_0, \lambda z_1) \in \mathbb{C}^2 : \lambda \in \ bC \}, \quad \text{if } z = \frac{z_1}{z_0}, \qquad L_\infty := \{ (0,\lambda) \in \mathbb{C}^2 : \lambda \in \mathbb{C} \}.$$

In other words, L_z is the complex line through the point $(1,z)$, for $z \in \mathbb{C}$. A particular *local* section of L, defined over U_N, is $\sigma_N(z) := (q^{-\frac12}, zq^{-\frac12})$, which is normalized so that $(\sigma_N \mid \sigma_N) = q^{-1}(1 + \bar{z}z) = 1$ on U_N: this hermitian pairing on $\Gamma(\mathbb{S}^2, L)$ comes from the standard scalar product on $\mathbb{C}^2$ —each L_z is a line in $\mathbb{C}^2$.

Let also $\sigma_S(\zeta) := (\zeta q'^{-\frac12}, q'^{-\frac12})$, normalized so that $(\sigma_S \mid \sigma_S) = 1$ on U_S. Now if $z \neq 0$, then

$$\sigma_S(z^{-1}) = \left(\frac{1}{z\sqrt{q'}}, \frac{1}{\sqrt{q'}} \right) = (\bar{z}/z)^{1/2} \left(\frac{1}{\sqrt{q}}, \frac{z}{\sqrt{q}} \right) = (\bar{z}/z)^{1/2}\, \sigma_N(z).$$

To avoid ambiguity, we state that $(\bar{z}/z)^{1/2}$ means $e^{-i\phi}$, and also $(z/\bar{z})^{1/2}$ will mean $e^{+i\phi}$.

A smooth section of L is given by two functions $\psi_N^+(z,\bar{z})$ and $\psi_S^+(\zeta,\bar{\zeta})$ satisfying the relation $\psi_N^+(z,\bar{z})\sigma_N(z) = \psi_S^+(\zeta,\bar{\zeta})\sigma_S(\zeta)$ on $U_N \cap U_S$. Thus we argue that

$$\psi_N^+(z,\bar{z}) = (\bar{z}/z)^{1/2} \psi_S^+(z^{-1}, \bar{z}^{-1}) \quad \text{for } z \neq 0,$$

and ψ_N^+, ψ_S^+ are regular at $z = 0$ or $\zeta = 0$ respectively. Likewise, a pair of smooth functions ψ_N^-, ψ_S^- on $\mathbb{C}$ is a section of the dual line bundle $L^* \to \mathbb{S}^2$ if and only if

$$\psi_N^-(z,\bar{z}) = (z/\bar{z})^{1/2} \psi_S^-(z^{-1}, \bar{z}^{-1}) \quad \text{for } z \neq 0.$$

We claim now that we can identify $S^+ \simeq L$ and $S^- \simeq L^* = L^{-1}$ —here the notation L^{-1} means that $[L^{-1}]$ is the inverse of $[L]$ in the Picard group $H^2(\mathbb{S}^2, \mathbb{Z})$ that classifies $\mathbb{C}$-line bundles— so that a *spinor* in $\mathbb{S} = \Gamma(\mathbb{S}^2, S)$ is given precisely by two pairs of smooth functions

$$\begin{pmatrix} \psi_N^+(z,\bar{z}) \\ \psi_N^-(z,\bar{z}) \end{pmatrix} \quad \text{on } U_N, \qquad \begin{pmatrix} \psi_S^+(\zeta,\bar{\zeta}) \\ \psi_S^-(\zeta,\bar{\zeta}) \end{pmatrix} \quad \text{on } U_S,$$

satisfying the above transformation rules. (The nontrivial thing is that the spinor components must both be regular at the south pole $z = 0$ and the north pole $\zeta = 0$, respectively.)

Since $\mathcal{S} \otimes_{\mathbb{A}} \mathcal{S}^* \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{S}) \simeq \mathcal{B} \simeq \mathcal{A}^\bullet(\mathbb{S}^2)$ as $\mathcal{A}$-module isomorphisms (we know that $\mathcal{B} \simeq \mathcal{A}^\bullet(\mathbb{S}^2)$ as sections of *vector* bundles), it is enough to show that, as vector bundles,

$$\mathcal{A}^\bullet(\mathbb{S}^2) \simeq L^0 \oplus L^2 \oplus L^{-2} \oplus L^0,$$

where $L^2 = L \otimes L$, $L^{-2} = L^* \otimes L^*$, and $L^0 = \mathbb{S}^2 \times \mathbb{C}$ is the trivial line bundle. It is clear that $\mathcal{A}^0(\mathbb{S}^2) = C^\infty(\mathbb{S}^2) = \mathcal{A} = \Gamma(\mathbb{S}^2, L^0)$; and furthermore, $\mathcal{A}^2(\mathbb{S}^2) \simeq \mathcal{A} = \Gamma(\mathbb{S}^2, L^0)$ since $\Lambda^2 T^*\mathbb{S}^2$ has a nonvanishing global section, namely the volume form $\nu = \sin\theta d\theta \wedge d\phi$.

With respect to the "round" metric on $\mathbb{S}^2$, namely,

$$g := d\theta^2 + \sin^2\theta \, d\phi^2 = \frac{4}{q^2}\,(dx^1 \otimes dx^1 + dx^2 \otimes dx^2),$$

the pairs of 1-forms $\left\{ \dfrac{dz}{q}, \dfrac{d\bar{z}}{q} \right\}$ and $\left\{ -\dfrac{d\zeta}{q'}, -\dfrac{d\bar{\zeta}}{q'} \right\}$ are local bases for $\mathcal{A}^1(\mathbb{S}^2)$, over U_N and U_S respectively.

Exercise A.12. *Write, for $\alpha \in \mathcal{A}^1(\mathbb{S}^2)$,*

$$\begin{aligned} \alpha &=: f_N(z,\bar{z})\,\frac{dz}{q} + g_N(z,\bar{z})\,\frac{d\bar{z}}{q} \quad \textit{on } U_N, \\ &=: -f_S(\zeta,\bar{\zeta})\,\frac{d\zeta}{q'} - g_S(\zeta,\bar{\zeta})\,\frac{d\bar{\zeta}}{q'} \quad \textit{on } U_S. \end{aligned}$$

Show that

$$\begin{aligned} f_N(z,\bar{z}) &= (\bar{z}/z)\, f_S(z^{-1}, \bar{z}^{-1}) \\ g_N(z,\bar{z}) &= (z/\bar{z})\, g_S(z^{-1}, \bar{z}^{-1}) \end{aligned}$$

on $U_N \cap U_S$, and conclude that $\mathcal{A}^1(\mathbb{S}^2) \simeq \Gamma(\mathbb{S}^2, L^2 \oplus L^{-2})$.

Note that the last exercise now justifies the claim that the half-spin bundles were indeed $S^+ \oplus S^- \simeq L \oplus L^*$.

A.2.2 The spin connection ∇^S over $\mathbb{S}^2$

Given any local orthonormal basis of 1-forms $\{E_1, \dots, E_n\}$, we can compute Christoffel symbols with all three indices taken from this basis, by setting $\widehat{\Gamma}^\beta_{\mu\alpha} := (E_\mu)^i \widetilde{\Gamma}^\beta_{i\alpha}$, or equivalently, by requiring that

$$\nabla_{E_\mu} E_\alpha =: \widehat{\Gamma}^\beta_{\mu\alpha}\, E_\beta$$

for $\mu, \alpha, \beta = 1, 2, \dots, n$. (This works because the first index is tensorial).

Exercise A.13. *On U_N, take $z =: x^1 + ix^2$. Compute the ordinary Christoffel symbols Γ^k_{ij} in the (x^1, x^2) coordinates for the round metric $g = (4/q^2)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$, and then show that*

$$\widehat{\Gamma}^\beta_{\mu\alpha} = \delta_{\mu\alpha} x^\beta - \delta_{\mu\beta} x^\alpha \quad \textit{for } \mu, \alpha, \beta = 1, 2.$$

This yields the local orthonormal bases $E_1 := \frac{1}{2} q\, \partial/\partial x^1$, $E_2 := \frac{1}{2} q\, \partial/\partial x^2$ for vector fields, and dually $\theta^1 = (2/q)\, dx^1$, $\theta^2 = (2/q)\, dx^2$ for 1-forms. However, since $\mathbb{S}^2 = \mathbb{C}P^1$ is a complex manifold, it is convenient to pass to "isotropic" bases, as follows. We introduce

$$\begin{aligned} E_+ &:= E_1 - iE_2 = q\frac{\partial}{\partial z}, & \theta^+ &:= \tfrac{1}{2}(\theta^1 + i\theta^2) = \frac{dz}{q}, \\ E_- &:= E_1 + iE_2 = q\frac{\partial}{\partial \bar z}, & \theta^- &:= \tfrac{1}{2}(\theta^1 - i\theta^2) = \frac{d\bar z}{q}. \end{aligned}$$

Exercise A.14. *Verify that the Levi-Civita connection on $\mathcal{A}^1(\mathbb{S}^2)$ is given, in these isotropic local bases, by*

$$\begin{aligned} \nabla_{E_+}\left(\frac{dz}{q}\right) &= \bar z\, \frac{dz}{q}, & \nabla_{E_-}\left(\frac{dz}{q}\right) &= -z\, \frac{dz}{q}, \\ \nabla_{E_+}\left(\frac{d\bar z}{q}\right) &= -\bar z\, \frac{d\bar z}{q}, & \nabla_{E_-}\left(\frac{d\bar z}{q}\right) &= z\, \frac{d\bar z}{q}. \end{aligned}$$

The Clifford action on spinors is given (over U_N, say) by $\gamma^1 := \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^2 := \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The $\mathbb{Z}_2$-grading operator is given by

$$\chi := (-i)\, \sigma^1 \sigma^2 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin connection is now specified by

$$\nabla^S_{E_\pm} := E_\pm - \tfrac{1}{4} \widehat{\Gamma}^\beta_{\pm\alpha}\, \gamma^\alpha \gamma_\beta.$$

Exercise A.15. *Verify that, over U_N, ∇^S is determined by*

$$\nabla^S_{E_+} = q\frac{\partial}{\partial z} + \tfrac{1}{2} \bar z\, \chi, \qquad \nabla^S_{E_-} = q\frac{\partial}{\partial \bar z} - \tfrac{1}{2} z\, \chi.$$

Conclude that the Dirac operator $D\!\!\!\!/ = -i\sigma^1\, \nabla^S_{E_1} - i\sigma^2\, \nabla^S_{E_2}$ is given, over U_N, by

$$D\!\!\!\!/ = -i \begin{pmatrix} 0 & q\frac{\partial}{\partial z} - \frac{1}{2}\bar z \\ q\frac{\partial}{\partial \bar z} - \frac{1}{2} z & 0 \end{pmatrix}.$$

A similar expression is valid over U_S, by replacing $z, \bar z, q$ by $\zeta, \bar\zeta, q'$ respectively, and by changing the overall $(-i)$ factor to $(+i)$. This formal change of sign is brought about by the local coordinate transformation formulas induced by $\zeta = 1/z$. (Here is an instance of the "unique continuation property" of $D\!\!\!\!/$: the local expression for the Dirac operator on any one chart determines its expressions on any overlapping chart, and then by induction, on the whole manifold.)

Exercise A.16. *By integrating spinor pairings with the volume form $\nu = \sin\theta d\theta \wedge d\phi = 2iq^{-2}\, dz \wedge d\bar{z}$, check that $D\!\!\!/$ is indeed symmetric as an operator on $L^2(\mathbb{S}^2, S)$ with domain $\mathcal{S}$.*

Exercise A.17. *Show that the spinor Laplacian Δ^S is given in the isotropic basis by*

$$\Delta^s = -\tfrac{1}{2}\big(\nabla^S_{E_+}\nabla^S_{E_-} + \nabla^S_{E_-}\nabla^S_{E_+} - z\nabla^S_{E_+} - \bar{z}\nabla^S_{E_-}\big),$$

and compute directly *that $D\!\!\!/^2 = \Delta^S + \frac{1}{2}$. This is consistent with the value $s \equiv 2$ of the scalar curvature of $\mathbb{S}^2$, taking into account how the metric g is normalized.*

A.2.3 Spinor harmonics and the Dirac operator spectrum

Newman and Penrose (1966) introduced a family of special functions on $\mathbb{S}^2$ that yield an orthonormal basis of spinors, in the same way that the conventional spherical harmonics Y_{lm} yield an orthonormal basis of L^2-functions. For functions, l and m are integers, but the spinors are labelled by "half-odd-integers" in $\mathbb{Z}+\frac{1}{2}$. When expressed in our coordinates $(z, \bar{z})$, they are given as follows.

For $l \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\} = \mathbb{N} + \frac{1}{2}$, and $m \in \{-l, -l+1, \dots, l-1, l\}$, write

$$Y^+_{lm}(z, \bar{z}) := C_{lm}\, q^{-l} \sum_{r-s=m-\frac{1}{2}} \binom{l-\frac{1}{2}}{r}\binom{l+\frac{1}{2}}{s} z^r(-\bar{z})^s,$$

$$Y^-_{lm}(z, \bar{z}) := C_{lm}\, q^{-l} \sum_{r-s=m+\frac{1}{2}} \binom{l+\frac{1}{2}}{r}\binom{l-\frac{1}{2}}{s} z^r(-\bar{z})^s,$$

where r, s are integers with $0 \le r \le l \mp \frac{1}{2}$ and $0 \le s \le l \pm \frac{1}{2}$ respectively; and

$$C_{lm} = (-1)^{l-m}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!(l-m)!}{(l+\frac{1}{2})!(l-\frac{1}{2})!}}.$$

Exercise A.18. *Show that $Y^\pm_{lm}$ are half-spinors in $S^\pm$, by applying the transformation laws under $z \mapsto z^{-1}$ and checking the regularity at the poles.*

Then define pairs of full spinors by

$$Y'_{lm} := \frac{1}{\sqrt{2}}\begin{pmatrix} Y^+_{lm} \\ iY^-_{lm}\end{pmatrix}, \qquad Y''_{lm} := \frac{1}{\sqrt{2}}\begin{pmatrix} -Y^+_{lm} \\ iY^-_{lm}\end{pmatrix}.$$

These turn out to be eigenspinors for the Dirac operator.

Exercise A.19. *Verify the following eigenvalue relations:*

$$D\!\!\!/ Y'_{lm} = (l + \tfrac{1}{2})\, Y'_{lm}, \qquad D\!\!\!/ Y''_{lm} = -(l + \tfrac{1}{2})\, Y''_{lm}.$$

Goldberg *et al* (1967) showed that these half-spinors are special cases of matrix elements $\mathcal{D}^l_{nm}$ of the irreducible group representations for $SU(2)$, namely,

$$Y^\pm_{lm}(z, \bar{z}) = \sqrt{\frac{2l+1}{4\pi}}\mathcal{D}^l_{\mp\frac{1}{2},m}(-\phi, \theta, -\phi),$$

By setting $h^\pm_{lm}(\theta, \phi, \psi) := e^{\mp\frac{1}{2}(\phi+\psi)}Y^\pm_{lm}(z, \bar{z})$, we get an orthonormal set of elements of $L^2(\mathrm{SU}(2))$, such that $\int_{\mathrm{SU}(2)} |h^\pm_{lm}(g)|^2\, dg = (1/4\pi)\int_{\mathbb{S}^2} |Y^\pm_{lm}|^2\, \nu$. The Plancherel formula for

SU(2) can then be used to show that these are a complete set of eigenvalues for $D\!\!\!\!/$. Thus we have obtained the spectrum:

$$\mathrm{sp}(D\!\!\!\!/) = \{\, \pm(l+\tfrac{1}{2}) : l \in \mathbb{N}+\tfrac{1}{2} \,\} = \{\pm 1, \pm 2, \pm 3, \dots\} = \mathbb{N} \setminus \{0\},$$

with respectively multiplicities $(2l+1)$ in each case, since the index m in $Y^{\pm}_{lm}$ takes $(2l+1)$ distinct values.

Postscript: Since $s \equiv 2$ and $D\!\!\!\!/^2 = \Delta^S + \frac{1}{2}$, we also get

$$\mathrm{sp}(\Delta^S) = \{\, (l+\tfrac{1}{2})^2 - \tfrac{1}{2} = l^2 + l - \tfrac{1}{4} : l \in \mathbb{N}+\tfrac{1}{2} \,\}$$

with multiplicities $2(2l+1)$ in each case. Note that

$$\mathrm{sp}(D\!\!\!\!/^2) = \{\, (l+\tfrac{1}{2})^2 = l^2 + l + \tfrac{1}{4} : l \in \mathbb{N}+\tfrac{1}{2} \,\}.$$

The operator $\underline{C}$ given by $\underline{C} := \Delta^S + \frac{1}{4} = D\!\!\!\!/ - \frac{1}{4}$ has spectrum

$$\mathrm{sp}(\underline{C}) = \{\, l(l+1) : l \in \mathbb{N}+\tfrac{1}{2} \,\},$$

with multiplicities $2(2l+1)$ again. This $\underline{C}$ comes from the *Casimir element* in the centre of $\mathcal{U}(\mathfrak{su}(2))$, represented on $\mathcal{H} = L^2(\mathbb{S}^2, S)$ via the rotation action of SU(2) on the sphere $\mathbb{S}^2$. There is a general result for compact symmetric spaces $M = G/K$ with a G-invariant spin structure, namely that $D\!\!\!\!/ = \underline{C}_G + \frac{1}{8}s$, or $\Delta^S = \underline{C}_G - \frac{1}{8}s$. This is a nice companion result, albeit only for homogeneous spaces, to the Schrödinger–Lichnerowicz formula. Details are given in Section 3.5 of Friedrich's book.

A.3 Spin^c Dirac operators on the 2-sphere

We know that finitely generated projective modules over the C^*-algebra $A = C(\mathbb{S}^2)$ are of the form $p\,A^k$, where $p = [p_{ij}]$ is an $k \times k$ matrix with elements in A, such that p $(= p^2 = p^*)$ is an orthogonal projector, whose rank is $\operatorname{tr} p = p_{11} + \cdots + p_{kk}$. To get modules of sections of *line bundles*, we impose the condition that $\operatorname{tr} p = 1$, so that $p\,A^k$ is an A-module "of rank one". It turns out that it is enough to consider the case $k = 2$ of 2×2 matrices.

Exercise A.20. *Check that any projector $p \in M_2(C(\mathbb{S}^2))$ is of the form*

$$p = \frac{1}{2}\begin{pmatrix} 1+n_3 & n_1 - in_2 \\ n_1 + in_2 & 1 - n_3 \end{pmatrix},$$

where $n_1^2 + n_2^2 + n_3^2 = 1$, so that $\vec{n} = (n_1, n_2, n_3)$ is a continuous function from $\mathbb{S}^2$ to $\mathbb{S}^2$.

After stereographic projection, we can replace $\vec{n}$ by $f(z) := \dfrac{n_1 - in_2}{1 - n_3}$. where $z = e^{-i\phi}\cot\frac{\theta}{2}$ is allowed to take the value $z = \infty$ at the north pole. Then f is a continuous map from the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$ into itself. If two projectors p and q are homotopic —there is a continuous path of projectors $\{\, p_t : 0 \leq t \leq 1 \,\}$ with $p_0 = p$ and $p_1 = q$— then they give the same class $[p] = [q]$ in $K^0(\mathbb{S}^2)$; and this happens if and only if the corresponding maps $\vec{n}$, or functions $f(z)$, are homotopic.

Exercise A.21. *Consider, for each $m = 1, 2, 3, \dots$, the maps*

$$z \mapsto f_m(z) := z^m \quad \textit{and} \quad z \mapsto f_{-m}(z) := \bar{z}^m$$

of the Riemann sphere into itself. ¿Can you describe the corresponding maps $\vec{n}$ of $\mathbb{S}^2$ into itself? Can you show that any two of these maps are not homotopic?

Let $\mathcal{E}_{(m)} = p_m\, A^2$ and $\mathcal{E}_{(-m)} = p_{-m}\, A^2$, where

$$p_m(z) = \frac{1}{1 + z^m \bar{z}^m} \begin{pmatrix} z^m \bar{z}^m & z^m \\ \bar{z}^m & 1 \end{pmatrix}, \qquad p_{-m}(z) = \frac{1}{1 + z^m \bar{z}^m} \begin{pmatrix} z^m \bar{z}^m & \bar{z}^m \\ z^m & 1 \end{pmatrix},$$

with the obvious definition (¿what is it?) for $z = \infty$.

Exercise A.22. *Show that $\mathcal{E}_{(1)}$ is isomorphic to the space of sections of the tautological line bundle $L \to \mathbb{C}P^1$ [hint: apply p_1 to any element of $\mathcal{A}^2$ and examine the result]. Show also that $\mathcal{E}_{(-1)}$ gives the space of sections of the dual line bundle $L^* \to \mathbb{C}P^1$.*

Exercise A.23. *For $m = 2, 3, \dots$, show that $\mathcal{E}_{(m)} \simeq \mathcal{E}_{(1)} \otimes_A \cdots \otimes_A \mathcal{E}_{(1)}$ (m times) by examining the components of elements of $p_m A^2$. ¿What is the analogous result for $\mathcal{E}_{(-m)}$?*

For $m \in \mathbb{Z}$, $m \neq 0$, we redefine $\mathcal{E}_{(m)} := p_m \mathcal{A}^2$ with $\mathcal{A} = C^\infty(\mathbb{S}^2)$; so that $\mathcal{E}_{(m)}$ now denotes *smooth* sections over a nontrivial line bundle on $\mathbb{S}^2$. We can identify each element of $\mathcal{E}_{(m)}$ with a smooth function $f_N \colon U_N \to \mathbb{C}$ for which there is another smooth function $f_S \colon U_S \to \mathbb{C}$, such that

$$f_N(z) = (\bar{z}/z)^{m/2}\, f_S(z^{-1}) \quad \text{for all } z \neq 0. \qquad (\underline{m})$$

Here, as before, $(\bar{z}/z)$ means $e^{i\phi}$ in polar coordinates.

Exercise A.24. *Writing $E_+ := q\,\partial/\partial z$ and $E_- := q\,\partial/\partial\bar{z}$ as before, where $q = 1 + z\bar{z}$, show that when the operators*

$$\nabla^{(m)}_{E_+} = q\,\frac{\partial}{\partial z} + \tfrac{1}{2} m\bar{z}, \qquad \nabla^{(m)}_{E_-} = q\,\frac{\partial}{\partial \bar{z}} - \tfrac{1}{2} m z,$$

are applied to functions f_N that satisfy $(\underline{m})$, the image also satisfies $(\underline{m})$. Thus they are components of a connection $\nabla^{(m)}$ on $\mathcal{E}_{(m)}$.

To get all the spin^c structures on $\mathbb{S}^2$, we twist the spinor module $\mathcal{S}$ for the spin structure, namely $\mathcal{S} = \mathcal{E}_{(1)} \oplus \mathcal{E}_{(-1)}$, by the rank-one module $\mathcal{E}_{(m)}$. On the tensor product $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)}$ we use the connection

$$\nabla^{S,m} := \nabla^S \otimes 1_{\mathcal{E}_{(m)}} + 1_{\mathcal{S}} \otimes \nabla^{(m)}.$$

Exercise A.25. *Show that the Dirac operator $D\!\!\!/_m := -i\,\hat{c} \circ \nabla^{S,m}$, that acts on $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)}$, is given by*

$$D\!\!\!/_m \equiv \begin{pmatrix} 0 & D\!\!\!/_m^- \\ D\!\!\!/_m^+ & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & q\frac{\partial}{\partial z} + \frac{1}{2}(m-1)\,\bar{z} \\ q\frac{\partial}{\partial \bar{z}} - \frac{1}{2}(m+1)\,z & 0 \end{pmatrix}.$$

Check also that

$$D\!\!\!/_m^+ = -i\, q^{(m+3)/2}\,\frac{\partial}{\partial \bar{z}}\, q^{-(m+1)/2} \quad \textit{and} \quad D\!\!\!/_m^- = -i\, q^{-(m-3)/2}\,\frac{\partial}{\partial z}\, q^{(m-1)/2},$$

where these powers of q are multiplication operators on suitable spaces on functions on U_N.

Exercise A.26. *If $m < 0$, show that any element of* $\ker \not{D}_m^+$ *is of the form* $a(z)\, q^{(m+1)/2}$ *where* $a(z)$ *is a holomorphic polynomial of degree* $< |m|$*. Also, if* $m \geq 0$*, show that* $\ker \not{D}_m^+ = 0$*.*

Exercise A.27. *If $m > 0$, show that any element of* $\ker \not{D}_m^-$ *is of the form* $b(\bar{z})\, q^{-(m-1)/2}$ *where* $b(\bar{z})$ *is an* anti*holomorphic polynomial of degree* $< m$*. Also, if* $m \leq 0$*, show that* $\ker \not{D}_m^- = 0$*. Conclude that the index of* $\not{D}_m$ *equals* $-m$ *in all cases.*

The *sign* of a selfadjoint operator D on a Hilbert space is given by the relation $D =: F\,|D| = F\,(D^2)^{1/2}$, where we put $F := 0$ on $\ker D$. Thus F is a bounded selfadjoint operator such that $1 - F^2$ is the orthogonal projector whose range is $\ker D$. When $\ker D$ is finite-dimensional, $1 - F^2$ has finite rank, so it is a compact operator.

An *even Fredholm module* over an algebra $\mathcal{A}$ is given by:

1. a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$;
2. a representation $a \mapsto \pi(a) = \begin{pmatrix} \pi^0(a) & 0 \\ 0 & \pi^1(a) \end{pmatrix}$ of $\mathcal{A}$ on $\mathcal{H}$ by bounded operators that commute with the $\mathbb{Z}_2$-grading;
3. a selfadjoint operator $F = \begin{pmatrix} 0 & F^- \\ F^+ & 0 \end{pmatrix}$ on $\mathcal{H}$ that anticommutes with the $\mathbb{Z}_2$-grading, such that $F^2 - 1$ and $[F, \pi(a)]$ are compact operators on $\mathcal{H}$, for each $a \in \mathcal{A}$.

We can extend the twisted Dirac operator $\not{D}_m$ to a selfadjoint operator on $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, where $\mathcal{H}^0$ and $\mathcal{H}^1$ are two copies of the Hilbert space $L^2(\mathbb{S}^2, \nu)$ where $\nu = 2i\, q^{-2}\, dz\, d\bar{z}$. We define $\pi^0(a) = \pi^1(a)$ to be the usual multiplication operator of a function $a \in C^\infty(\mathbb{S}^2)$ on this L^2-space.

Exercise A.28. *Show that* $\not{D}_m$*, given by the above formulas on its original domain, is a symmetric operator on* $\mathcal{H}$*.*

Exercise A.29. *Check that the sign* F_m *of the twisted Dirac operator* $\not{D}_m$ *determines a Fredholm module over* $C^\infty(\mathbb{S}^2)$*.*

A.4 A spectral triple on the noncommutative torus

To define a spectral triple over a noncommutative algebra, we introduce the so-called *noncommutative torus*. In fact, there are many such tori, labelled by a dimension n and by a family of parameters θ_{ij} forming a real skewsymmetric matrix $\Theta = -\Theta^t \in M_n(\mathbb{R})$.

Fix an integer $n \in \{2, 3, 4, \dots\}$. In the algebra $A_0 := C(\mathbb{T}^n)$, one can write down Fourier-series expansions:

$$f(\phi_1, \dots, \phi_n) \longleftrightarrow \sum_{r \in \mathbb{Z}^n} c_r\, e^{2\pi i r\cdot\phi}, \qquad c_r := \int_{[0,1]^n} e^{-2\pi i r\cdot\phi}\, f(\phi)\, d^n\phi \in \mathbb{C},$$

where $r \cdot \phi := r_1\phi_1 + \cdots + r_n\phi_n$, as usual. To ensure that this series converges uniformly and represents $f(\phi)$, we retreat to the dense subalgebra $\mathcal{A}_0 := C^\infty(\mathbb{T}^n)$, in which the coefficients c_r decrease rapidly to zero as $|r| \to \infty$. On the space of multisequences $\mathbf{c} := \{c_r\}_{r\in\mathbb{Z}^n}$, we introduce the seminorms

$$p_k(\mathbf{c}) := \Bigl(\sum_{r\in\mathbb{Z}^n} (1 + r\cdot r)^k |c_r|^2 \Bigr)^{1/2}, \quad \text{for all } k \in \mathbb{N}.$$

We say that *"$c_r \to 0$ rapidly"* if $p_k(\mathbf{c}) < \infty$ for every k. Notice that $p_{k+1}(\mathbf{c}) \geq p_k(\mathbf{c})$ for each k; these seminorms induce, on rapidly decreasing sequences, the topology of a Fréchet space, which indeed coincides with the usual Fréchet topology on $C^\infty(\mathbb{T}^n)$, i.e., the topology of uniform convergence of the functions and of all their derivatives.

We can think of A_0 as the C^*-algebra generated by n commuting unitary elements, namely the functions u_j defined by $u_j(\phi_1, \dots, \phi_n) := e^{2\pi i \phi_j}$, for $j = 1, \dots, n$.

Noncommutativity appears when we choose a real skewsymmetric matrix $\Theta \in M_n(\mathbb{R})$, and introduce the (universal) C^*-algebra A_Θ generated by *unitary* elements $u_1, \dots, u_n$ which no longer commute: instead, they satisfy the *commutation relations*

$$u_k\, u_j = e^{2\pi i \theta_{jk}}\, u_j\, u_k, \quad \text{for } j,k = 1, \dots, n.$$

(In quantum mechanics, these are called "Weyl's form of the canonical commutation relations".) To form polynomials with these generators, we introduce a *Weyl system* of unitary elements $\{\, u^r : r \in \mathbb{Z}^n \,\}$ in A_Θ, by defining

$$u^r := \exp\bigl\{\pi i \textstyle\sum_{j<k} r_j \theta_{jk} r_k\bigr\}\, u_1^{r_1} u_2^{r_2} \dots u_n^{r_n}.$$

Exercise A.30. *Show that $(u^r)^* = u^{-r}$ for $r \in \mathbb{Z}^n$, and that*

$$u^r\, u^s = \sigma(r,s)\, u^{r+s}, \quad \textit{where} \quad \sigma(r,s) := \exp\bigl\{-\pi i \textstyle\sum_{j,k} r_j \theta_{jk} s_k\bigr\}.$$

Verify directly that

$$\sigma(r, s+t)\, \sigma(s,t) = \sigma(r,s)\, \sigma(r+s, t), \quad \textit{for } r,s,t \in \mathbb{Z}^n.$$

Notice that $\sigma(r, \pm r) = 1$ by skewsymmetry of Θ.

We now define $\mathcal{A}_\Theta =: C^\infty(\mathbb{T}^n_\Theta)$ to be the *dense $*$-subalgebra* of A_Θ consisting of elements of the form

$$a = \sum_{r \in \mathbb{Z}^n} a_r\, u^r$$

where $a_r \in \mathbb{C}$ for each r, and $a_r \to 0$ *rapidly.*

Exercise A.31. *Check that this series converges in the norm of $\mathcal{A}^\Theta$, by considering the series $\sum_r (1 + r \cdot r)^{-k}$ for large enough k.*

There is an *action* of the abelian Lie group $\mathbb{T}^n$ by $*$-automorphisms on the C^*-algebra A_Θ, given by

$$z \cdot u^r := z_1^{r_1} z_2^{r_2} \dots z_n^{r_n}\, u^r \quad \text{for } r \in \mathbb{Z}^n,$$

or, more simply, $z \cdot u_j = z_j\, u_j$, where $z = (z_1, \dots, z_n) \in \mathbb{T}^n$. This action is generated by a set of n commuting *derivations* $\delta_1, \dots, \delta_n$, namely,

$$\delta_j(a) := \frac{d}{dt}\bigg|_{t=0} e^{2\pi i t \phi_j} \cdot a,$$

whose domain is the set of all $a \in A$ for which the map $t \mapsto e^{2\pi i t \phi_j} \cdot a$ is differentiable.

Exercise A.32. *Show that $u_r \in \operatorname{Dom} \delta_j$, and that $\delta_j(u^r) = 2\pi i r_j\, u^r$ for all $r \in \mathbb{Z}^n$ and $j = 1, \dots, n$. Conclude that the common smooth domain $\bigcap_{m \in \mathbb{Z}^n} \operatorname{Dom}(\delta_1^{m_1} \dots \delta_n^{m_n})$ is equal to the subalgebra $\mathcal{A}_\Theta$.*

The result of the previous exercise shows that $\mathcal{A}_\Theta$ is just the "smooth subalgebra" of the C^*-algebra A_Θ with respect to the action of $\mathbb{T}^n$. It is known that any such smooth subalgebra, under a continuous action of a compact Lie group on a C^*-algebra, is actually a *pre-C^*-algebra*.

Exercise A.33. *Define a linear operator $E\colon A_\Theta \to A_\Theta$ by averaging over the orbits of this $\mathbb{T}^n$-action:*

$$E(a) := \int_{[0,1]^n} (e^{-2\pi i\phi_1}, \dots, e^{-2\pi i\phi_n}) \cdot a \; d\phi_1 \dots d\phi_n.$$

Check that $E(1) = 1$, that $E(a^*) = E(a)^*$, that $E(a^*a) \geq 0$ and $\|E(a)\| \leq \|a\|$ for all $a \in A_\Theta$; where "$x \geq 0$" means that x is a positive *element of A_Θ. Then show the "conditional expectation" property:*

$$E\big(E(a)\, b\, E(c)\big) = E(a)\, E(b)\, E(c) \quad \textit{for all } a, b, c \in A_\Theta.$$

By considering $b = a - E(a)$, show also that $E(a^*a) \geq E(a)^* E(a)$ for $a \in A_\Theta$.

Exercise A.34. *If $a = \sum_r a_r\, u^r \in \mathcal{A}_\Theta$, check that $E(a) = a_0\, 1$. Conclude that the range of E is the $*$-subalgebra $\mathbb{C}\, 1$, and that*

$$\tau(a)\, 1 := E(a)$$

defines a trace *on A_Θ; by continuity, it is enough to check the trace property on the dense subalgebra $\mathcal{A}_\Theta$.*

Exercise A.35. *If instead we only consider the action of a subgroup $\mathbb{T}^k$ of $\mathbb{T}^n$, we can define a conditional expectation*

$$E_k(a) := \int_{[0,1]^k} (e^{-2\pi i\phi_1}, \dots, e^{-2\pi i\phi_k}, 1, \dots, 1) \cdot a \; d\phi_1 \dots d\phi_k.$$

In this case the range of E_k will be isomorphic to a C^*-algebra A_Φ where Φ is a certain real skewsymmetric matrix in $M_{n-k}(\mathbb{R})$. Compute the matrix Φ in terms of the matrix Θ. In particular, ¿what is the range of E_k for the case $k = n-1$?

We now define $\mathcal{H}_\tau$ to be the *completion* of A_Θ in the norm

$$\|a\|_2 := \sqrt{\tau(a^*a)}.$$

We remark that $\|a\|_2 \leq \|a\|$ for all a, so that the inclusion map $\eta_\tau\colon A_\Theta \to \mathcal{H}_\tau$ is continuous. It is convenient to write $\underline{a} := \eta_\tau(a)$ to denote the element $a \in A_\Theta$ regarded as a vector in $\mathcal{H}_\tau$. It turns out that the trace τ is faithful, so that $\mathcal{H}_\tau$ is just the Hilbert space of the "GNS representation" π_τ of A_Θ. This representation is defined —first on $\eta_\tau(A_\Theta)$, then extended by continuity— by

$$\pi_\tau(a) : \underline{b} \mapsto \underline{ab} : \mathcal{H}_\tau \to \mathcal{H}_\tau, \quad \text{for each } a \in A_\Theta.$$

Exercise A.36. *Define an* antilinear *operator $J_0\colon \mathcal{H}_\tau \to \mathcal{H}_\tau$ by setting*

$$J_0(\underline{a}) := \underline{a^*}, \quad \textit{for } a \in \eta_\tau(A_\Theta).$$

Show that J_0 is an isometry on this domain, so that it extends to all of $\mathcal{H}_\tau$; and show that the extended J_0 is an antiunitary *operator on $\mathcal{H}_\tau$. For $b \in A_\Theta$, consider the operator*

$$\pi'_\tau(b) := J_0 \pi_\tau(b^*) J_0.$$

Check that $\pi'_\tau(b) : \underline{c} \mapsto \underline{cb}$ for $c \in A_\Theta$. Conclude that $[\pi_\tau(a), \pi'_\tau(b)] = 0$ for all $a, b \in A_\Theta$.

The analogue of the L^2-spinor space for the noncommutative torus is just the tensor product $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$, where as usual, $n = 2m$ or $n = 2m+1$ according as n is even or odd. (In the commutative case $\Theta = 0$, this means that we are using the spinor module for the untwisted spin structure on $\mathbb{T}^n$.) Recall that we can regard $\mathbb{C}^{2^m}$ as a Fock space $\Lambda^\bullet\mathbb{C}^m$, carrying an irreducible represenation of the matrix algebra $B = \mathbb{C}\mathrm{l}(\mathbb{R}^n)$ if n is even, or $B = \mathbb{C}\mathrm{l}^0(\mathbb{R}^n)$ if n is odd. In the even case, there is a $\mathbb{Z}_2$-grading operator $\Gamma := 1_{\mathcal{H}_\tau} \otimes c(\gamma)$, satisfying $\Gamma^2 = 1$ and $\Gamma^* = \Gamma$.

The charge conjugation on B, that we have written $b \mapsto \chi(\bar{b})$, is implemented by an antiunitary operator on $\mathbb{C}^{2^m}$ of the form C_0K, where K is complex conjugation and C_0 is a certain $2^m \times 2^m$ matrix: this means that $(C_0K)\, b\, (C_0K)^{-1} = \chi(\bar{b})$ as operators on $\mathbb{C}^{2^m}$.

For instance, if $n = 2$ or 3, then $C_0 = i\,\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now let $J := J_0 \otimes C_0$. This is an antiunitary operator on $\mathcal{H}$, such that $J^2 = \pm 1$ according as $C_0^2 = \pm 1$.

Exercise A.37. *Show that $\delta_j(a^*) = (\delta_j(a))^*$ and that $\tau(\delta_j(a)) = 0$ for all $a \in \mathcal{A}_\Theta$. Conclude that the densely defined operator $\underline{\delta}_j : \underline{a} \mapsto \underline{\delta_j(a)}$, with domain $\eta_\tau(\mathcal{A}_\Theta)$, is* skewsymmetric *in the sense that*

$$\langle \underline{\delta}_j(\underline{a}) \mid \underline{b} \rangle = -\langle \underline{a} \mid \underline{\delta}_j(\underline{b}) \rangle, \quad \textit{for all } \underline{a}, \underline{b} \in \operatorname{Dom} \underline{\delta}_j.$$

The closure of this operator, still denoted by $\underline{\delta}_j$, is then an unbounded skewadjoint operator on $\mathcal{H}$.

Let $\gamma^1, \dots, \gamma^n$ be the generators of the action of the Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ on $\mathbb{C}^{2^m}$: they are a set of $2^m \times 2^m$ matrices such that $\gamma^j\gamma^k + \gamma^k\gamma^j = 2\delta^{jk}$ for $j, k = 1, \dots, n$. The operator C_0K is determined by the relations

$$(C_0K)\, \gamma^j\, (C_0K)^{-1} = -\gamma^j \quad \text{for } j = 1, \dots, n.$$

We can now define the *Dirac operator* on $\mathcal{H}$ by

$$D := -i \sum_{j=1}^n \underline{\delta}_j \otimes \gamma^j.$$

Exercise A.38. *Show that $JDJ^{-1} = \pm D$ on the domain $\mathcal{A}_\Theta$.*

Exercise A.39. *If $\{ s_\alpha : \alpha = 1, \dots, 2^m \}$ is an orthonormal basis of $\mathbb{C}^{2^m}$, define $\psi_{r\alpha} := \underline{u^r} \otimes s_\alpha \in \mathcal{H}$. Show that $\{ \psi_{r\alpha} : r \in \mathbb{Z}^n,\ \alpha = 1, \dots, 2^m \}$ is an orthonormal basis of $\mathcal{H}$ that diagonalizes D^2, by checking that*

$$D^2\psi_{r\alpha} = 4\pi^2(r \cdot r)\, \psi_{r\alpha} \quad \textit{for each } r, \alpha.$$

¿What is the spectrum (with its multiplicities) of $|D|$? What is the spectrum of D itself?

Exercise A.40. *We can invert D on the orthogonal complement of the finite-dimensional space* $\ker D = \operatorname{span}\{ \psi_{0\alpha} : \alpha = 1, \dots, 2^m \}$*. Show that, for each $s > 0$, the expression*

$$\operatorname{Tr}^+ |D|^{-s} := \lim_{N \to \infty} \frac{\sigma_N(|D|^{-s})}{\log N}$$

either exists as a finite limit, or diverges to $+\infty$. (Show that we may use a subsequence where $N = N_R := \#\{ r \in \mathbb{Z}^n : r \cdot r \leq R^2 \}$ for some $R > 0$.) Verify that the $0 < \operatorname{Tr}^+ |D|^{-s} < +\infty$ *if and only if $s = n$; and compute the value of* $\operatorname{Tr}^+ |D|^{-n}$.

Exercise A.41. *If $a \in \mathcal{A}^\Theta$, show that both a and $[D, a]$, considered as bounded operators on $\mathcal{H}$, lie in the smooth domain of the operator $T \mapsto [|D|, T]$.*

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