Foliations, C*-algebras and index theory
Part III

Paul Baum

notes taken by:
Paweł Witkowski

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Chapter 1

Bott periodicity and index theorem

1.1 Bott periodicity

In one of it's forms Bott periodicity theorem can be stated as

$$\pi_j(\mathrm{GL}(n,\mathbb{C})) = \begin{cases} 0 & j \text{ even}, \\ \mathbb{Z} & j \text{ odd} \end{cases}$$

for $j = 0, 1, 2, \ldots, 2n-1$. This is the original form of Bott's theorem. It has reformulation, for example in the language of topological K-theory or the K-theory of C^*-algebras where it gives an isomorphism

$$\mathrm{K}_j(A) \simeq \mathrm{K}_j(A \otimes C_0(\mathbb{R}^2)).$$

The homotopy groups are constructed as follows. We take maps from the sphere

$$S^j \xrightarrow{f} X$$

preserving base points $p_0 \in S^j$, $x_0 \in X$, i.e. $f(p_0) = x_0$. On the set of homotopy classes of such maps we give a group structure by composing with the map contracting the equator of the sphere S^j making it a wedge of two copies of S^j.

$$S^j \to S^j \vee S^j \xrightarrow{f \vee g} X.$$

The sphere S^n can be described as

$$S^n = \{(t_1, t_2, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid t_1^2 + \ldots + t_{n+1}^2 = 1\},$$

however if n is odd we can embed it in the complex space $\mathbb{C}^n$

$$S^{2r-1} = \{(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{C}^r \mid \sum_{j=1}^{r} \overline{\lambda_j}\lambda_j = 1\}.$$

For a map $f\colon S^n \to S^n$ define degree $\deg(f) \in \mathbb{Z}$ as follows. On homology f induces a map

$$f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}) \to \mathrm{H}_n(S^n;\mathbb{Z}) \simeq \mathbb{Z}$$

Then for each $u \in \mathrm{H}_n(S^n;\mathbb{Z})$

$$f_*(u) = \deg(f)u.$$

We shall define for $j = 1, 3, 5, \dots, 2n-1$ a homomorphism of abelian groups

$$\beta\colon \pi_j(\mathrm{GL}(n,\mathbb{C})) \to \mathbb{Z}.$$

First consider $j = 2n-1$ and a map

$$f\colon S^{2n-1} \to \mathrm{GL}(n,\mathbb{C}).$$

For $p \in S^{2n-1}$

$$f(p) = \begin{bmatrix} \lambda_{11}(p) & \lambda_{12}(p) & \dots & \lambda_{1n}(p) \\ \lambda_{21}(p) & \lambda_{22}(p) & \dots & \lambda_{2n}(p) \\ \vdots & \vdots & & \vdots \\ \lambda_{n1}(p) & \lambda_{n2}(p) & \dots & \lambda_{nn}(p) \end{bmatrix}$$

We take first column and divide it by the norm i.e.

$$f_1(p) := (\lambda_{11}(p), \lambda_{21}(p), \dots, \lambda_{n1}(p))/(\sum_{j=1}^{n} \overline{\lambda_{ji}(p)}\lambda_{ji}(p))^{\frac{1}{2}}.$$

This gives a map $f_1\colon S^{2n-1} \to S^{2n-1}$. Now define

$$\beta(f) := \frac{\deg(f_1)}{(n-1)!} \in \mathbb{Z}.$$

It os a part of Bott's theorem that this number actually is an integer.

For all n the unitary subgroup $U(n) \in \mathrm{GL}(n,\mathbb{C})$ is a maximal compact subgroup, and the inclusion induces a homotopy equivalence. From the fibering

$$U(n-1) \to U(n) \to S^{2n-1}$$

and the homotopy exact sequence we get that for $0 < j < 2n-1$ the homotopy

$$\pi_j(U(n-1)) \simeq \pi_j(U(n)).$$

Lemma 1.1. *If $j = 2r-1$ with $1 \leqslant r < n$, then*

$$f\colon S^{2r-1} \to GL(n,\mathbb{C})$$

is homotopic to a map

$$\tilde{f}\colon S^{2r-1} \to \mathrm{GL}(n,\mathbb{C})$$

of the form

$$\tilde{f}(p) = \begin{bmatrix} \lambda_{11}(p) & \lambda_{12}(p) & \dots & \lambda_{1r}(p) & 0 & 0 & \dots & 0 \\ \lambda_{21}(p) & \lambda_{22}(p) & \dots & \lambda_{2r}(p) & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_{r1}(p) & \lambda_{r2}(p) & \dots & \lambda_{rr}(p) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Define

$$\tilde{f}_1 \colon S^{2r-1} \to S^{2r-1}$$

$$\tilde{f}_1(p) := (\lambda_{11}(p), \lambda_{21}(p), \ldots, \lambda_{r1}(p))/(\sum_{j=1}^{r} \overline{\lambda_{ji}(p)}\lambda_{ji}(p))^{\frac{1}{2}}.$$

and as before

$$\beta(f) := \frac{\deg(\tilde{f}_1)}{(r-1)!} \in \mathbb{Z}.$$

This number is an integer, which is part of the

Theorem 1.2. *For $j = 1, 3, 5, \ldots, 2n-1$*

$$\beta \colon \pi_j(\mathrm{GL}(n, \mathbb{C})) \to \mathbb{Z}$$

is an isomorphism.

1.2 Elliptic operators

Let X be a C^∞-manifold (Hausdorff, second countable, finite dimensional, without boundary), and $E \to X$ a complex C^∞-vector bundle. For each $p \in X$, E_p is a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} E_p < \infty$.

By $C^\infty(X, E)$ we denote a $\mathbb{C}$-vector space of all C^∞ sections of E,

$$\dim_{\mathbb{C}} C^\infty(X, E) = \infty.$$

If E^0, E^1 are two vector bundles on X, then an **elliptic differential operator** (or elliptic $\psi D0$) is a $\mathbb{C}$-linear map

$$D \colon C^\infty(X, E^0) \mapsto C^\infty(X, E^1)$$

which is differential operator (or an $\psi D0$) and which satisfies a condition called **ellipticity**.

Example 1.3. Laplacian on $X = \mathbb{R}^n$, $E^0 = E^1 = X \times \mathbb{C}$

$$D = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

This operator is elliptic of order 2, because polynomial

$$\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$$

has no real zeroes except $(0, 0, \ldots, 0)$.

Example 1.4. Cauchy-Riemann operator on $X = \mathbb{R}^2$, $E^0 = E^1 = X \times \mathbb{C}$

$$D = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}.$$

This operator is elliptic of order 1, because polynomial $\xi_1 + i\xi_2$ has no real zeroes except $(0, 0)$. If we denote

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

then $\bar{\partial} f = 0$ iff. f is holomorphic.

Definition of ellipticity

$$D\colon C^\infty(X, E^0) \to C^\infty(X, E^1)$$

differential operator of order r. To each $p \in X$ and $\xi \in T_p^*X = \mathrm{Hom}_{\mathbb{R}}(T_pX, \mathbb{R})$ we shall associate a map of $\mathbb{C}$-vector spaces

$$\sigma(\xi, D)\colon E_p^0 \to E_p^1.$$

To do this, given $v \in E_p^0$ and $\xi \in T_p^*X$, choose

1. Section $s \in C^\infty(X, E^0)$ with $s(p) = v$,
2. C^∞-function $f\colon X \to \mathbb{R}$ with $f(p) = 0$ and $(df)(p) = \xi$.

Lemma 1.5. *$D(f^r s)(p)$ depends only on D, ξ, v; and does not depend on the choice of s and f.*

Set

$$\sigma(\xi, D)v := D(f^r s)(p),$$
$$\sigma(\xi, D)\colon E_p^0 \to E_p^1.$$

Definition 1.6. *A differential operator of order r*

$$D\colon C^\infty(X, E^0) \to C^\infty(X, E^1)$$

is **elliptic** *if whenever $p \in X$ and $0 \neq \xi \in T_p^*X$, then $\sigma(\xi, D)\colon E_p^0 \to E_p^1$ is an isomorphism of $\mathbb{C}$-vector spaces.*

Example 1.7. Let X be a manifold. We consider complex valued differential forms on X i.e. elements of

$$\Lambda^j(T_{\mathbb{C}}^*X), \quad T_{\mathbb{C}}^*X := T^*X \otimes_{\mathbb{R}} \mathbb{C}, \quad j = 0, 1, 2, \ldots.$$

The de Rham operator

$$C^\infty(X, \Lambda^j(T_{\mathbb{C}}^*X)) \xrightarrow{d} C^\infty(X, \Lambda^{j+1}(T_{\mathbb{C}}^*X))$$

is a differential operator of order 1. For $p \in X$, $\xi \in T_p^*X$, $v \in (\Lambda^j(T_{\mathbb{C}}^*X))_p$ choose a form ω such that

$$\omega(p) = v.$$

Then choose a function $f\colon X \to \mathbb{R}$ such that

$$(df)(p) = \xi.$$

We have

$$d(f\omega) = df \wedge \omega + f d\omega$$
$$d(f\omega)_p = (df \wedge \omega)_p \text{ because } f(p) = 0,$$

and thus the map $\sigma(\xi, d)$ is given by

$$v \mapsto \xi \wedge v \text{ because } (df)(p) = \xi.$$

This operator is not elliptic. However if we take

$$\bigoplus_j C^\infty(X, \Lambda^{2j}(T_{\mathbb{C}}^*X)) \xrightarrow{d+d^*} \bigoplus_j C^\infty(X, \Lambda^{2j-1}(T_{\mathbb{C}}^*X)),$$

where $d^*\colon \Lambda^j(T^*_{\mathbb{C}}X) \to \Lambda^{j-1}(T^*_{\mathbb{C}}X)$ is formal adjoint to d, then $d+d^*$ is elliptic, and $\sigma(\xi, d+d^*)$ is given by

$$v \mapsto \xi \wedge v + \iota(\xi)v,$$

where $\iota(\xi)$ is a contraction of form by ξ.

Lemma 1.8. *If X is compact and D is elliptic, then*

$$\dim_{\mathbb{C}}(\ker D) < \infty, \ \textit{and}$$

$$\dim_{\mathbb{C}}(\operatorname{coker} D) < \infty.$$

Definition 1.9. *If X is compact and D is elliptic, then*

$$\operatorname{Index}(D) := \dim_{\mathbb{C}}(\ker D) - \dim_{\mathbb{C}}(\operatorname{coker} D).$$

Theorem 1.10 (Atiyah-Singer). *If X is compact and D is elliptic, then*

$$\operatorname{Index}(D) = (\textit{topological formula}).$$

Example 1.11. Toeplitz operator

$$X = S^1 = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^2 + a_2^2 = 1\}$$

Take a trivial bundles $E^0 = E^1 = S^1 \times \mathbb{C}$. Sections of these are just smooth functions on S^1

$$C^\infty(S^1, E^0) = C^\infty(S^1, E^0) = C^\infty(S^1).$$

Any $u \in C^\infty(S^1)$, $u\colon S^1 \to \mathbb{C}$, has a Fourrier series

$$u = \sum_{n=-\infty}^{n=\infty} a_n e^{in\theta}, \ \ a_n \in \mathbb{C}.$$

We have a decomposition

$$C^\infty(S^1) = C^\infty_+(S^1) \oplus C^\infty_-(S^1)$$
$$C^\infty_+(S^1) = \{u \in C^\infty(S^1) \mid a_n = 0 \ \forall\, n < 0\}$$
$$C^\infty_-(S^1) = \{u \in C^\infty(S^1) \mid a_n = 0 \ \forall\, n \geqslant 0\}$$

Denote the projection

$$P\colon C^\infty(S^1) \to C^\infty_+(S^1),$$
$$P\left(\sum_{n=-\infty}^{n=\infty} a_n e^{in\theta}\right) = \sum_{n=0}^{n=\infty} a_n e^{in\theta}.$$

Fix a C^∞ function $\alpha\colon S^1 \to \mathbb{C}$ and define

$$T_\alpha(u) = P(\alpha u), \ \ \alpha(u)(\lambda) = (\alpha\lambda)u(\lambda), \ \ u \in C^\infty(S^1), \ \ \lambda \in S^1.$$

Define operator

$$D_\alpha\colon C^\infty(S^1) \to C^\infty(S^1)$$

using decomposition $C^\infty(S^1) = C^\infty_+(S^1) \oplus C^\infty_-(S^1)$

$$D_\alpha = \begin{bmatrix} T_\alpha & 0 \\ 0 & I \end{bmatrix}$$

$$D_\alpha u = \begin{cases} T_\alpha u & u \in C^\infty_+(S^1) \\ u & u \in C^\infty_-(S^1) \end{cases}$$

Proposition 1.12. *1.* $D_\alpha\colon C^\infty(S^1)\to C^\infty(S^1)$ *is a pseudo-differential operator* ($\psi D0$),

2. D_α *is elliptic iff.* $\alpha(\lambda)\neq 0$ *for all* $\lambda\in S^1$,

3. *if* $\alpha(\lambda)\neq 0$ *for all* $\lambda\in S^1$, *then*

$$\mathrm{Index}(D_\alpha) = -(\textit{winding number})(\alpha) = -\frac{1}{2\pi i}\int_{S^1}\frac{d\alpha}{\alpha}.$$

Remark 1.13. Winding number is also present in our Bott periodicity theorem. Indeed

$$\pi_1(\mathrm{GL}(n,\mathbb{C}))\simeq\mathbb{Z}$$

and the isomorphism is given by winding number of

$$S^1\mapsto \mathrm{GL}(n,\mathbb{C})\xrightarrow{\det}\mathbb{C}^*.$$

Example 1.14. Classical Riemann-Roch. Let X be connected Riemann surface, i.e compact connected complex analitic manifold with $\dim_{\mathbb{C}} X = 1$. The genus of X is a number of holes which is equal to

$$g=\frac{1}{2}\operatorname{rank}\mathrm{H}_1(X;\mathbb{Z}).$$

Assume we are given a complex analitic line bundle L on X. For each $p\in X$, L_p is a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} L_p = 1$. The degree $\deg(L)$ of this bundle can be defined as follows. Choose any meromorphic section u of L. Then the order of u at $p\in X$ is defined as

$$\mathrm{ord}_p(u)=\begin{cases}0 & \text{if } p\in X \text{ is neither a zero nor a pole of } u,\\ n & \text{if } p\in X \text{ is a zero of order } n \text{ of } u,\\ n & \text{if } p\in X \text{ is a pole of order } n \text{ of } u.\end{cases}$$

Then

$$\deg(L):=\sum_{p\in X}\mathrm{ord}_p(u).$$

Lemma 1.15. $\deg(L)$ *does not depend on the choice of meromorphic section* u.

Remark 1.16. Another way to describe the degree is to evaluate first Chern class of bundle L on the fundamental class of the base X

$$\deg(L)=\langle c_1(L),[X]\rangle\in\mathbb{Z}.$$

Consider operator

$$\bar\partial\colon C^\infty(X,L)\to C^\infty(X,L\otimes\Lambda^{0,1}T^*_{\mathbb{C}}X)$$

given for $s=f\alpha$ by

$$\bar\partial s=\frac{\partial f}{\partial\bar z}\otimes d\bar z$$

$$z=x+iy,\ \ d\bar z=dx-idy,\ \ \frac{\partial f}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right),$$

and $C^\infty(X,L\otimes\Lambda^{0,1}T^*_{\mathbb{C}}X)$ are complex valued 1-forms of type $(0,1)$. Then $u\in C^\infty(X,L)$ is holomorphic iff. $\bar\partial u=0$.

Theorem 1.17 *(Riemann-Roch).*

$$\mathrm{Index}(\bar\partial)=\deg(L)-g+1.$$

1.2.1 Pseudodifferenital operators

When we consider non compact manifolds the Atiyah-Singer index theorem must be stated using elliptic pseudodifferential operators.

Let $U \in \mathbb{R}^n$ be an open subset, $m \in \mathbb{Z}$. Define a subspace

$$S^m(U) \subset C^\infty(U \times \mathbb{R}^n)$$

$$\phi \in S^m(U), \quad \phi\colon U \times \mathbb{R}^n \to \mathbb{C}, \quad (x,\xi) \mapsto \phi(x,\xi),$$

by the condition

Function $\phi \in S^m(U)$ if and only if for every compact subset $\Delta \subset U$ and for all multiindices α, β there exists constant $C_{\phi,\alpha,\beta,\Delta}$ with

$$\left| D_x^\beta D_\xi^\alpha \phi(x,\xi) \leqslant C_{\phi,\alpha,\beta,\Delta}(1+|\xi|)^{m-|\alpha|}. \right|$$

Constant $C_{\phi,\alpha,\beta,\Delta}$ depends on $\phi, \alpha, \beta, \Delta$,

$$D_\xi^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \left(\frac{1}{i}\frac{\partial}{\partial \xi_2}\right)^{\alpha_2} \cdots \left(\frac{1}{i}\frac{\partial}{\partial \xi_n}\right)^{\alpha_n},$$

$$|\alpha| = \sum_{j=1}^n \alpha_j.$$

Now define a subspace

$$S_0^m(U) \subset S^m(U)$$

Function $\phi \in S_0^m(U)$ if and only if

$$\lim_{\lambda \to \infty} \frac{\phi(x,\lambda,\xi)}{\lambda^m}$$

exists.

For $\phi \in S^m(U)$ set

$$\sigma_\phi(x,\xi) = \lim_{\lambda \to \infty} \frac{\phi(x,\lambda,\xi)}{\lambda^m}.$$

Then σ_ϕ is a C^∞ function defined on $U \times (\mathbb{R}^n \times \{0\})$

$$\sigma_\phi\colon U \times (\mathbb{R}^n - \{0\}) \to \mathbb{C}$$

and σ_ϕ is homogeneous of degree m in ξ. We take it as a symbol of the following operator

$$P_\phi\colon C_c^\infty(U) \to C_c^\infty(U)$$

$$P_\phi(x) = \frac{1}{2\pi} \int \phi(x,\xi)\hat{f}(\xi)e^{\langle x,\xi\rangle} d\xi$$

1.3 Topological formula of Atiyah-Singer

Let X be compact hausdorff topological space, E $\mathbb{C}$-vector bundle on X. To describe the topological index formula one has to introduce Chern character $\mathrm{ch}(E)$ and the Todd class $\mathrm{Td}(E)$, both being elements of $\bigoplus_j \mathrm{H}^j(X;\mathbb{Q})$.

For line bundle $L \to X$

$$\mathrm{ch}(E) = e^{c_1(L)} = 1 + c_1(L) + \frac{c_1^2(L)}{2} + \ldots$$

For a sum of a line bundles $E = L_1 \oplus L_2 \oplus \ldots \oplus L_n$

$$\mathrm{ch}(L_1 \oplus L_2 \oplus \ldots \oplus L_n) = e^{c_1(L_1)} + e^{c_1(L_2)} + \ldots + e^{c_1(L_n)} =$$
$$= \mathrm{ch}(L_1) + \mathrm{ch}(L_2) + \ldots + \mathrm{ch}(L_n).$$

General formula can be obtained using above and splitting principle. Just as Chern character is based on a function e^x, the Todd class is based on a function

$$\frac{x}{1-e^{-x}} = \frac{x}{1 - [1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \ldots]} =$$
$$= \frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \ldots.$$

Now for a line bundle $L \to X$ we have

$$\mathrm{Td}(L) = \frac{c_1(L)}{1 - e^{-c_1(L)}}$$

and for a sum of line bundles

$$\mathrm{Td}(L_1 \oplus L_2 \oplus \ldots \oplus L_n) = \mathrm{Td}(L_1) \cup \mathrm{Td}(L_2) \cup \ldots \cup \mathrm{Td}(L_n).$$

Let $E^0, E^1 \to X$ be vector bundles on X, $\psi \colon E^0 \to E^1$ a vector bundle map.

$$\mathrm{Support}(\psi) := \{\, p \in X \mid \psi \colon E^0_p \to E^1_p \text{ is not an isomorphism of } \mathbb{C}\text{-vector spaces}\}$$

Assume $\mathrm{Support}(\psi)$ is compact. Then

$$\mathrm{ch}(\psi) \in \mathrm{H}^{2j}_c(X;\mathbb{Q}).$$

Example 1.18. Let M be a compact C^∞-manifold with no boundary. For a pair of vector bundles

$$\begin{tikzcd} E^0 \arrow[rr] \arrow[dr] & & E^1 \arrow[dl] \\ & M & \end{tikzcd}$$

let

$$D \colon C^\infty(M, E^0) \to C^\infty(M, E^1)$$

be an elliptic operator. The cotangent bundle

$$\pi \colon T^*M \to M, \quad \pi(T^*_pM) = p$$

induces pullback bundles on T^*M

$$\begin{tikzcd} \pi^*E^0,\ \pi^*E^1 \arrow[r] \arrow[d] & E^0,\ E^1 \arrow[d] \\ T^*M \arrow[r, "\pi"] & M \end{tikzcd}$$

The symbol of D is a mapping

$$\sigma \colon \pi^*E^0 \to \pi^*E^1.$$

Then

$$\mathrm{ch}(\sigma) \in \mathrm{H}^{2j}_c(T^*M;\mathbb{Q})$$

and

$$\mathrm{Index}(D) = (ch(\sigma) \cup \pi^* \mathrm{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}))[T^*M].$$

In the proof of index theorem one uses

Lemma 1.19. *Let A be an abelian group. Let $\varphi\colon A \to \mathbb{Z}$ and $\tau\colon A \to \mathbb{Z}$ be homomorphisms. Assume that φ is an isomorphism. Assume also that there exists $a \in A$, with $a \neq 0$ and $\varphi(a) = \tau(a)$. Then $\varphi = \tau$.*

Now we shall describe appriopriate abelian group A.

Definition 1.20. *A* **symbol datum** *is a 4-tuple (M, F^0, F^1, σ) such that*

1. *M is a C^∞-manifold, finite dimensional, Hausdorff, second countable, with $\pi_0(M)$ finite, and with no boundary,*
2. *F^0, F^1 are complex vector bundles on T^*M,*
3. *σ is a vector bundle map $F^0 \to F^1$ with* $\mathrm{Support}(\sigma)$ *compact.*

On a set of such 4-tuples we will define an equivalence relation $\sim$, and then put

$$A := \{(M, F^0, F^1, \sigma)\}/\sim$$

$$(M, F^0, F^1, \sigma) + (W, E^0, E^1, \theta) = (M \cup W, F^0 \cup E^0, F^1 \cup E^1, \sigma \cup \theta).$$

Now the two homomorphisma which are mentioned in the lemma (1.19) are as follows

$$\varphi\colon A \to \mathbb{Z}$$

$$\varphi(M, F^0, F^1, \sigma) := (ch(\sigma) \cup \pi^* \operatorname{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}))[T^*M],$$

$$\tau\colon A \to \mathbb{Z}$$

$$\tau(M, F^0, F^1, \sigma) = \operatorname{Index}(D),$$

where D is any elliptic operator on M whose symbol datum is (M, F^0, F^1, σ).

Remark 1.21. If M is non-compact then D will be an elliptic pseudodifferential operator (ΨDO) on M which is trivial at infinity.

The equivalence relation $\sim$ betwen symbol data is described in five steps

1. isomorphism,
2. homotopy of σ,
3. direct sum - disjoint union,
4. excision,
5. vector bundle modification.

1. **Isomorphism**.

   4-tuples (M, F^0, F^1, σ) and (W, E^0, E^1, θ) are isomorphic

   $$(M, F^0, F^1, \sigma) \simeq (W, E^0, E^1, \theta)$$

   if and only if there exists a diffeomorphism

   $$h\colon M \to W$$

such that one can assign in a continuous way, to each $\xi \in T^*W$ an isomorphisms of vector spaces

$$\eta^0_\xi \colon E^0_\xi \xrightarrow{\simeq} F^0_{h'\xi},$$

$$\eta^1_\xi \colon E^1_\xi \xrightarrow{\simeq} F^1_{h'\xi},$$

with commutativity in the diagram

$$\begin{tikzcd} E^0_\xi \arrow[r, "\eta^0_\xi"] \arrow[d, "\theta"'] & F^0_{h'\xi} \arrow[d, "\sigma"] \\ E^1_\xi \arrow[r, "\eta^1_\xi"] & F^1_{h'\xi} \end{tikzcd}$$

where

$$h' \colon T^*W \to T^*M$$

is the map of cotangent bundles induced by $h \colon M \to W$.

2. **Homotopy of σ.**

   We consider homotopies between symbol data (M, F^0, F^1, σ) such that M, F^0, F^1 is fixed, and for $0 \leqslant t \leqslant 1$ we have family of symbols σ_t. Then

   $$(M, F^0, F^1, \sigma_0) \sim (M, F^0, F^1, \sigma_1).$$

   Furthermore the set of $(\sigma, t) \in T^*M \times [0,1]$ such that

   $$\sigma_t(\xi) \colon F^0_\xi \to F^1_\xi$$

   is not an isomorphism of $\mathbb{C}$-vector spaces, is compact.

3. **Direct sum - disjoint union**.

   Let (M, F^0, F^1, σ) and (M, E^0, E^1, θ) be two symbol data with the same M. Then

   $$(M, F^0, F^1, \sigma) \cup (M, F^0, F^1, \theta) \sim (M, F^0 \oplus E^0, F^1 \oplus E^1, \sigma \oplus \theta).$$

4. **Excision**.

   Let (M, F^0, F^1, σ) be a symbol data. Recall that

   $$\mathrm{Support}(\sigma) = \{\ \xi \in T^*M \bigm| \sigma(\xi) \colon F^0_\xi \to F^1_\xi \text{ is not an isomorphism of } \mathbb{C} \text{ vector spaces}\}.$$

   Denote by π the projection $T^*M \to M$. Let $U \subset M$ be an open set with

   $$\pi(\mathrm{Support}(\sigma)) \subset U.$$

   Then

   $$(M, F^0, F^1, \sigma) \sim (U, F^0|_{T^*U}, F^1|_{T^*U}, \sigma|_{T^*U}).$$

5. **Vector bundle modification**.

   Let (M, F^0, F^1, σ) be a symbol data and E any C^∞-vector bundle on M. Then we describe another 4-tuples $(E, -, -, -)$ which will be equivalent to the given one. First we give a basic example of symbol datum.

Example 1.22. For each $n = 1, 3, 5, \ldots$ w shall define a symbol datum $\mathbb{R}^{n\wedge}$. For $n = 1$ we take

$$\mathbb{R}^{1\wedge} = (\mathbb{R}, (T^*\mathbb{R}) \times \mathbb{C}, (T^*\mathbb{R}) \times \mathbb{C}, \cdot)$$

One has

$$T^*\mathbb{R} = \mathbb{R} \times \mathbb{R} = \mathbb{C}$$

$$(t_1, t_2 dx) \leftrightarrow t_1 + it_2, \quad t_1, t_2 \in \mathbb{R}$$

and $\cdot$ denotes multiplication on the second coordinate

$$(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_1 \cdot \lambda_2).$$

For $n > 1$ we put

$$\mathbb{R}^{n\wedge} := \mathbb{R}^{1\wedge} \times \mathbb{R}^{1\wedge} \times \ldots \times \mathbb{R}^{1\wedge}.$$

More explicitly

$$\mathbb{R}^{n\wedge} = (\mathbb{R}^n, (T^*\mathbb{R}^n) \times \Lambda^{ev}\mathbb{C}^n, (T^*\mathbb{R}^n) \times \Lambda^{odd}\mathbb{C}^n, \wedge + \iota),$$

where

$$T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n,$$

$$\Lambda^{ev}\mathbb{C}^n = \bigoplus_j \Lambda^{2j}\mathbb{C}^n, \quad \Lambda^{odd}\mathbb{C}^n = \bigoplus_j \Lambda^{2j+1}\mathbb{C}^n$$

$$\wedge + \iota \colon \mathbb{C}^n \times \Lambda^{ev}\mathbb{C}^n \to \mathbb{C}^n \times \Lambda^{odd}\mathbb{C}^n$$

$$(v, w) \mapsto (v, v \wedge w + \iota(v)w), \quad v \in \mathbb{C}^n, \ w \in \Lambda^{ev}\mathbb{C}^n.$$

Now in the special case of trivial bundle $E = M \times \mathbb{R}^n$ we have

$$(M, F^0, F^1, \sigma) \sim (M, F^0, F^1, \sigma) \times \mathbb{R}^{n\wedge}.$$

However this construction has enouch naturality, so that it can be done even when E is not trivial. Let $E \to M$ be smooth complex vector bundle. Then we have

$$\begin{array}{ccc} E & & T^*E \\ \downarrow & , & \downarrow \rho \\ M & & T^*M \end{array}$$

and T^*E is a $\mathbb{C}$-vector bundle on T^*M.

Set

$$\Lambda^{ev} := \bigoplus_j \Lambda^{2j}(T^*E), \quad \Lambda^{odd} := \bigoplus_j \Lambda^{2j+1}(T^*E)$$

and then form a symbol datum

$$(E, \rho^*[(F^0 \hat{\otimes} \Lambda^{ev}) \oplus (F^1 \hat{\otimes} \Lambda^{odd})], \rho^*[(F^1 \hat{\otimes} \Lambda^{ev}) \oplus (F^0 \hat{\otimes} \Lambda^{odd})], \sigma \#(\wedge + \iota))$$

$$\sim (M, F^0, F^1, \sigma).$$

In the formula above we use external tensor product of vector bundles and external tensor product of symbols, which we describe next. For a pair of vector bundles $E \to X$, $F \to Y$ their external tensor product is a bundle

$$\begin{array}{c} E\hat{\otimes}F \\ \big\downarrow \\ X \times Y \end{array}$$

with fiber

$$(E\hat{\otimes}F)_{(x,y)} = E_x \otimes_{\mathbb{C}} F_y.$$

Then the external product of symbol data is defined as follows

$$(M, F^0, F^1, \sigma) \times (W, E^0, E^1, \theta) :=$$

$$M \times W, (F^0\hat{\otimes}E^0) \oplus (F^1\hat{\otimes}E^1), (F^1\hat{\otimes}E^0) \oplus (F^0\hat{\otimes}E^1), \sigma\#\theta)$$

$$T^*(M \times W) = T^*M \times T^*W,$$

$$\sigma\#\theta = \begin{bmatrix} \sigma\hat{\otimes}I_{E^0} & -I_{F^1}\hat{\otimes}\theta^* \\ I_{F^0}\hat{\otimes}\theta & \sigma^*\hat{\otimes}I_{E^1}, \end{bmatrix}$$

where I is the identity map.

$$\mathrm{Support}(\sigma\#\theta) = \mathrm{Support}(\sigma) \cup \mathrm{Support}(\theta).$$

Now we can put

$$A := \{(M, F^0, F^1, \sigma)\}/\sim.$$

A is an abelian group with the addition defined as

$$(M, F^0, F^1, \sigma) + (W, E^0, E^1, \theta) = (M \cup W, F^0 \cup E^0, F^1 \cup E^1, \sigma \cup \theta),$$

the inverse

$$-(M, F^0, F^1, \sigma) = (M, F^1, F^0, \sigma^*),$$

and the identity being any datum (M, F^0, F^1, σ) with $\mathrm{Support}(\sigma) = \emptyset$, for example

$$(M, F, F, \mathrm{id}).$$

Now we can state and proof the

Theorem 1.23 (Atiyah-Singer). *Let $E^0, E^1 \to M$ be smooth $\mathbb{C}$-vector bundles on a smooth manifold M. For any elliptic pseudodifferential operator*

$$D\colon C^\infty(M, E^0) \to C^\infty(M, E^1),$$

with symbol datum

$$(T^*M, \pi^*E^0, \pi^*E^1, \sigma), \quad \pi\colon T^*M \to M$$

$$\mathrm{Index}(D) = (ch(\sigma) \cup \pi^* \mathrm{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}))[T^*M].$$

Remark 1.24. We do not assume that M is compact. If it is so, then one can use elliptic differential operator D.

Proof. (An outline) By the lemma (1.19) it is sufficient to show that the two maps

$$\varphi\colon A \to \mathbb{Z}, \ \ \tau\colon A \to \mathbb{Z}$$

given by the formulas

$$\varphi(M, F^0, F^1, \sigma) := (ch(\sigma) \cup \pi^* \operatorname{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}))[T^*M],$$

$$\tau(M, F^0, F^1, \sigma) = \operatorname{Index}(D),$$

satisfy assumptions of the lemma, and therefore $\varphi = \tau$. To do this we have to check that each of them is

1. well defined, that is compatible with the equivalence relation $\sim$,
2. integer valued,
3. additive, that is homomorbism of abelian groups.

Moreover for φ we have to check that it is "1-1" and "onto". Finally that there exists nonzero element of A on which both agree.

It is easy to check that φ is well defined, additive, and $\varphi(\mathbb{R}^{1\wedge}) = 1$, that is φ is onto. It follows from the naturality of the Chern character and the Todd class used in the formula. The more difficult part is to check that it is integer valued and one to one. To prove that it is so one needs a

Lemma 1.25. *Any symbol datum (M, F^0, F^1, σ) is equivalent to a symbol datum whose manifold is $\mathbb{R}^n$*

$$(M, F^0, F^1, \sigma) \sim (\mathbb{R}^n, G^0, G^1, \eta)$$

Proof. (An outline) Embed M into $\mathbb{R}^n$ for sufficiently large n in sauch way that M is a closed subset and C^∞ manifold of $\mathbb{R}^n$.

Next step is to use the normal bundle ν of M in $\mathbb{R}^n$ and do vector bundle modification by ν

$$(M, F^0, F^1, \sigma) \sim (\nu, -, -, -).$$

Now ν is an open subset of $\mathbb{R}^n$ and one can do excision "in reverse"

$$(\nu, -, -, -) \sim (\mathbb{R}^n, -, -, -).$$

$\square$

Any vector bundle on $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ is trivial so we can assume that

$$(M, F^0, F^1, \sigma) \sim (\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta).$$

Furthermore we can assume that $l \geqslant n$, for if $l < n$ let $r := n - l$, and then

$$(\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta) \sim$$
$$(\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta) \cup (\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \mathrm{id}) \sim$$
$$(\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^{l+r}, (T^*\mathbb{R}^n) \times \mathbb{C}^{l+r}, \eta \oplus \mathrm{id}).$$

Thus we can assume that

$$(M, F^0, F^1, \sigma) \sim (\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta), \ \ l \geqslant n.$$

$$T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

Mapping η can be considered as

$$\eta\colon \mathbb{R}^n \to M_l(\mathbb{C}) = \{\, l \times l \text{ matrices } [\lambda_{ij}] \bigm| \lambda_{ij} \in \mathbb{C}\}.$$

There extist a compact set $\Delta \in \mathbb{R}^n$ with

$$\eta(\xi) \in \mathrm{GL}(l, \mathbb{C}) \ \ \forall\, \xi \in \mathbb{R}^{2n} - \Delta.$$

Making an evident homotopy (if necessary) of η we may assume

$$\eta(\xi) \in \mathrm{GL}(l, \mathbb{C}) \ \ \forall\, ||\xi|| \geqslant 1.$$

Then

$$\eta|_{S^{2n-1}}\colon S^{2n-1} \to \mathrm{GL}(l, \mathbb{C}), \quad l \geqslant n.$$

$$\varphi((\mathbb{R}^n, (T^*\mathbb{R}^n) \times \mathbb{C}^l, (T^*\mathbb{R}^n) \times \mathbb{C}^l, \eta)) = \beta(\eta|_{S^{2n-1}})$$

for β defined in section (1.1), and we have

$$\beta(\eta|_{S^{2n-1}}) \in \mathbb{Z},$$

so φ is integer valued.

Suppose now that

$$\beta(\eta|_{S^{2n-1}}) = 0,$$

then

$$[\eta|_{S^{2n-1}}] = 0 \text{ in } \pi_{2n-1}(\mathrm{GL}(l, \mathbb{C})).$$

By making a homotopy of η w obtain

$$\tilde{\eta}\colon \mathbb{R}^{2n} \to M_l(\mathbb{C})$$

with

$$\tilde{\eta}(\mathbb{R}^{2n}) \in \mathrm{GL}(l, \mathbb{C}).$$

Such $\tilde{\eta}$ in an abelian group A is equal to 0, so this proves that φ is one to one.

Now for (M, F^0, F^1, σ) let D be any elliptic pseudodifferential operator whose symbol (up to homotopy of σ) is σ.

$$\tau(M, F^0, F^1, \sigma) = \mathrm{Index}(D) = \dim\ker D - \dim\operatorname{coker} D.$$

It is obvious that it is integer valued. Also it is easy to check that τ is a homomorphism of abelian groups. The difficult part is to check that it is well defined, that is index does not change, when we do any of five steps defining equivalence relation $\sim$ on symbol data. $\square$

1.4 Index theorem for families of operators

Let $\mathcal{H}$ be a Hilbert space, $T\colon \mathcal{H} \to \mathcal{H}$ a Fredholm operator (has finite dimensional kernel and cokernel). The space of Fredholm operators we denote $\mathcal{F}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$.

Theorem 1.26. *For compact space X*

$$[X, \mathcal{F}(\mathcal{H})] \simeq \mathrm{K}^0(X).$$

We describe only a map from the homotoy classes $[X, \mathcal{F}(\mathcal{H})]$ to $\mathrm{K}^0(X)$. Let $p \in X$, $f\colon X \to \mathcal{F}(\mathcal{H})$

$$f(p) \in \mathcal{F}(\mathcal{H})$$

Each $f(p)$ has finite dimensional kernel and cokernel. Let $\mathcal{N}(f(p))$ be the nullspace of $f(p)$ and $\mathcal{R}(f(p))$ its range. Then we have mappings

$$p \mapsto \mathcal{N}(f(p)),$$

$$p \mapsto \mathcal{R}(f(p))^{\perp},$$

If the dimensions of spaces $\mathcal{N}(f(p))$, $\mathcal{R}(f(p))^{\perp}$ are locally constant functions on X, then we have two vector bundles N, $R^{\perp}$ over X, both subbundles of infinite dimensional vector bundle $X \times \mathcal{H}$. The formal difference of isomorphism classes

$$[N] - [R^{\perp}]$$

is an element of K-theory of X. The contruction needs to be modified if the dimensions of $\mathcal{N}(f(p))$ or $\mathcal{R}(f(p))^{\perp}$ are not locally constant functions on X.

Let W, X be smooth manifolds without boundary, X compact. Suppose we are given submersion $\pi\colon W \to X$ and fibers of π are compact submanifolds. Suppose also that we have an elliptic differential operator on each fiber. Then we can form a kernel bundle and cokernel bundle which formal difference is an element of $\mathrm{K}^0(X)$. This element we call an index for a given family of operators.

We will give an idea of proof of the Bott periodicity theorem stated as

Theorem 1.27. *There is an isomorphism*

$$\beta\colon\ \mathrm{K}^0(X) \to \mathrm{K}^0(X \times \mathbb{R}^2)$$

$$\begin{tikzcd} E^0 \arrow[rr] \arrow[dr] & & E^1 \arrow[dl] \\ & M & \end{tikzcd} \quad \otimes \quad \begin{tikzcd} \mathbb{R}^2 \times \mathbb{C} \arrow[rr] \arrow[dr] & & \mathbb{R}^2 \times \mathbb{C} \arrow[dl] \\ & \mathbb{R}^2 & \end{tikzcd}$$

The map β is a multiplication by some element $b \in \mathrm{K}^0(\mathbb{R}^2) \simeq \mathrm{K}^0(S^2)$. There is an isomorphism

$$\mathrm{K}^0(X \times \mathbb{R}^2) \to \widetilde{\mathrm{K}}(X \times S^2)$$

(X is without distinguished point). Consider a complex vector bundle on $X \times S^2$

$$\begin{tikzcd} E \arrow[d] \\ X \times S^2 \end{tikzcd}$$

For each $p \in X$ we have a vector bundle on S^2

$$\begin{tikzcd} E_p = E|_{\{p\} \times S^2} \arrow[d] \\ S^2 \end{tikzcd}$$

The Dirac operator on S^2 can be represented as

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

We are interested only in

$$D_+ \colon C^\infty(S^2, L^+) \to C^\infty(S^2, L^-)$$

Where $L \to S^2$ is a line bundle. We can tensor the Dirac operator with the bundle E on S^2 and obtain for each $p \in X$

$$D_+ \otimes E_p \colon C^\infty(S^2, L^+ \otimes E_p) \to C^\infty(S^2, L^- \otimes E_p).$$

This gives a family of elliptic operators parametrized by X. It has an index in $\mathrm{K}^0(X)$, so we have defined a map

$$\alpha\colon\ \mathrm{K}^0(X \times \mathbb{R}^2) \to \mathrm{K}^0(X).$$

It can be proved that it is an inverse of β.

We list properties of maps β and α.

$$\beta\colon\ \mathrm{K}^0(X) \to \mathrm{K}^0(X \times \mathbb{R}^2)$$

1. it is functorial in X
2. it is $\mathrm{K}^0(X)$-module homomorphism
3. For $X = \mathrm{pt}$
$$\beta\colon\ \mathrm{K}^0(\mathrm{pt}) \to \mathrm{K}^0(\mathbb{R}^2) \simeq \mathbb{Z}$$
$$\beta(1) = b.$$

$$\alpha\colon\ \mathrm{K}^0(X \times \mathbb{R}^2) \to \mathrm{K}^0(X).$$

1. it is functorial in X
2. it is $\mathrm{K}^0(X)$-module homomorphism
3. For $X = \mathrm{pt}$
$$\alpha\colon\ \mathrm{K}^0(\mathbb{R}^2) \to \mathrm{K}^0(\mathrm{pt}) \simeq \mathbb{Z}$$
$$\alpha(b) = 1.$$

After proving above properties it is clear that β is an isomorphism and α is its inverse.