K-theory of operator algebras - written examination

5 1. Compute $K_*(C(S^1))$ and $K_*(S^2)$.

5 2. Using the homotopy invariance of K-groups and remembering $K_*(S^1)$ and $K_*(S^2)$, prove that $S^2 \times S^1$ is not homotopic to S^3.

3 3. Using the functoriality of K-theory, prove that there is no retraction of the two-dimensional disc D to S^1.

4 4. For any action of $\mathbb{Z}$ on the two-disc D, compute $K_*(C(D) \rtimes \mathbb{Z})$.

4 5. Let $\mathcal{T}$ denote the Toeplitz algebra, and let s denote its generator (one-sided shift). Then there is a $*$-homorphism $\sigma : \mathcal{T} \to C(S^1)$, given by $\sigma(s) = u$, u the unitary generator of $C(S^1)$. Let $\mathcal{T} \oplus_\sigma \mathcal{T} := \{(f, g) \in \mathcal{T} \oplus \mathcal{T} \mid \sigma(f) = \sigma(g)\}$ (pull-back). Taking for granted that $K_0(\mathcal{T}) \simeq \mathbb{Z}$ is generated by $[1]_0$, but assuming no knowledge of $K_1(\mathcal{T})$, show that $K_0(\mathcal{T} \oplus_\sigma \mathcal{T})$ contains $\mathbb{Z}$ as a direct summand.

4 6. Give an example of a C^*-algebra A for which $K_0(A) = 0$, $K_1(A) \neq 0$. Show that if $K_0(A) = 0$, $K_1(A) \neq 0$, then A cannot be the C^*-algebra of a directed graph with finitely many vertices.