Graph C*-algebras

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Contents

1 Universal graph C*-algebras

Let G be a directed graph with

$$\begin{aligned} G^0 &\quad - \quad \text{vertices,} \\ G^1 &\quad - \quad \text{edges,} \\ r, s\colon G^1 \to G^0 &\quad - \quad \text{range and source of an edge.} \end{aligned}$$

Definition 1.1. *The universal C^*-algebra $C^*(G)$ is given by generators*

$$\{p_v \bigm| v \in G^0\}, \quad \{s_e \bigm| e \in G^1\},$$

with the following relations:

- *p_v are mutually orthogonal projections i.e. $p_v^2 = p_v^* = p_v$ and $p_v p_w = 0$ for $v \neq w$,*
- *$s_e^* s_e = p_{r(e)}$ and $s_e^* s_f = 0$ for $e \neq f$,*
- *if the set $\{e \bigm| s(e) = v\}$ is nonempty (v is not a sink) and finite then*

$$p_v = \sum_{\{e \bigm| s(e)=v\}} s_e s_e^*,$$

- *$s_e s_e^* \leqslant p_{s(e)}$.*

Example 1.2. Some known C^*-algebras arise in this way.

1. If G is only one vertex, then there is one generator $p = p^2 = p^*$. In this case $C^*(G) = \mathbb{C}$.

Figure 1: $\mathbb{C}$

2. G with one vertex and one edge (loop). Generators:

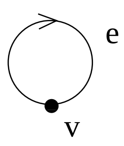


Figure 2: $C(S^1)$

$$p = p^2 = p^*, \quad s$$

relations:

$$s^* s = p = ss^*, \quad sp = ps, \quad s = ss^* s.$$

Then

$$C^*(G) = C^*(1, u) = C(S^1), \quad u - \text{unitary}.$$

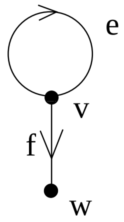


Figure 3: Toeplitz algebra $\mathcal{T}$

3. G with two vertices and two edges like on the picture (3).

$$p_v = p_v^2 = p_v^*, \;\; p_w = p_w^2 = p_w^*$$
$$s_e^* s_e = p_v, \;\; s_f^* s_f = p_w$$
$$p_v = s_e s_e^* + s_f s_f^*.$$

   $C^*(G)$ is isomorphic to the Toeplitz algebra - the universal C*-algebra for the relation $s^*s = 1$. The isomorphism is given by $s \mapsto s_e + s_f$.

4. G with three vertices and three edges like on a picture (4).

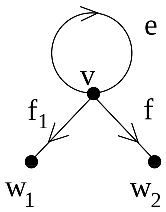


Figure 4: $C(S^2_{0\infty})$

$$p_v = p_v^2 = p_v^*, \;\; p_{w_i} = p_{w_i}^2 = p_{w_i}^*, \;\; i = 1, 2,$$
$$p_v p_{w_i} = 0, \;\; p_{w_1} p_{w_2} = 0,$$
$$s_e^* s_e = p_v = s_e s_e^* + s_{f_1} s_{f_1}^* + s_{f_2} s_{f_2}^*,$$
$$s_{f_1}^* s_{f_1} = p_{w_1}, \;\; s_{f_2}^* s_{f_2} = p_{w_2}.$$

   $C^*(G)$ is isomorphic to the quantum sphere

$$S^2_{0\infty}: \;\; B^*B = 1 - A^2, \; A = A^*, \; BB^* = 1, \; BA = 0$$

   and the isomorphism is given by

$$\begin{aligned} A &\mapsto p_{w_1} - p_{w_2}, \\ B &\mapsto s_e^* + s_{f_1}^* + s_{f_2}^*. \end{aligned}$$

   We denote this graph by $G_{S^2_{0\infty}}$.

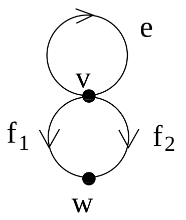


Figure 5: $C(\mathbb{R}P^2_q)$

5. In the example (4) we glue the vertices w_1, w_2 into one w obtaining graph G like on a picture (5).

$$p_v = p_v^2 = p_v^*, \;\; p_w = p_w^2 = p_w^*,$$
$$p_v p_w = 0,$$
$$s_e^* s_e = p_v = s_e^* s_e + s_{f_1} s_{f_1}^* + s_{f_2} s_{f_2}^*,$$
$$s_{f_1}^* s_{f_1} = p_w = s_{f_2}^* s_{f_2}.$$

Define $\mathbb{Z}_2$-action on the graph in the example (4).

$$s_e \mapsto -s_e, \;\; s_{f_1} \mapsto -s_{f_2}, \;\; s_{f_2} \mapsto -s_{f_1}$$

Then

$$p_v \mapsto p_v, \;\; p_{w_1} \mapsto p_{w_2}, \;\; p_{w_2} \mapsto p_{w_1}.$$

This action corresponds to

$$A \mapsto -A, \;\; B \mapsto -B$$

under the identification

$$C^*(G_{S^2_{0\infty}}) \simeq C(S^2_{0\infty}).$$

If we take the quotient $C(S^2_{0\infty})/\mathbb{Z}_2$ we obtain $C(\mathbb{R}P^2_q)$ - quantum projective space. On the other hand the quotient of the graph C*-algebra $C^*(G_{S^2_{0\infty}})$ by the defined action is the graph C*-algebra for our graph, which we now can denote $C^*(G_{\mathbb{R}P^2_q})$. The isomorphism is given by

$$p_v \mapsto p_v, \;\; p_w \mapsto p_{w_1} + p_{w_2},$$
$$s_e \mapsto s_e s_e, \;\; s_{f_1} \mapsto s_e(s_{f_1} + s_{f_2}), \;\; s_{f_2} \mapsto s_{f_1} - s_{f_2}.$$

Note that this $\mathbb{Z}_2$ action is not induced from a graph automorphism.

6. G with one vertex and n edges like on the picture (6).

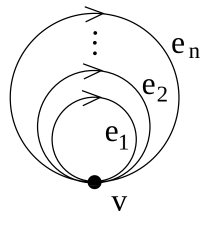


Figure 6: Cuntz algebra O_n

$$r(e_k) = s(e_k) = v, \quad k = 1, \dots, n,$$

$$p = s_{e_k}^* s_{e_k} = \sum_{k=1}^{n} s_{e_k} s_{e_k}^*,$$

$$s_{e_k}^* s_{e_{k'}} = 0 \text{ for } k \neq k'.$$

When $p = 1$ then $C^*(G)$ is the Cuntz algebra O_n - the universal C*-algebra for the relations

$$s_k^* s_k = 1, \quad k = 1, \dots n, \quad \sum_{k=1}^{n} s_k s_k^* = 1.$$

7. G with n vertices and $(n-1)$ edges in the straight segment as in the picture (7).

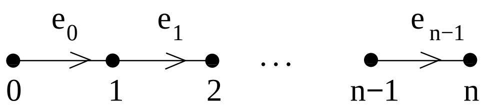


Figure 7: $M_n(\mathbb{C})$

$$s(e_k) = k, \quad r(e_k) = k + 1 \text{ for } k = 1, \dots, n-1,$$

$$p_k = s_{e_k} s_{e_k}^*, \quad p_{k+1} = s_{e_k}^* s_{e_k} \text{ for } k = 1, \dots n-1,$$

$$s_{e_k}^* s_{e_{k'}} = 0 \text{ for } k \neq k'.$$

$C^*(G)$ is the algebra of complex matrices $n \times n$, that is $M_n(\mathbb{C})$.

8. Similarly to the previous example we take straight segment, but infinite in both directions. Vertices are indexed by integers as in the picture (8).

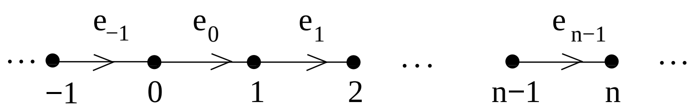


Figure 8: Compact operators $\mathcal{K}$

$$s(e_k) = k, \quad r(e_k) = k + 1, \quad k \in \mathbb{Z},$$

$$p_k = s_{e_k} s_{e_k}^*, \quad p_{k+1} = s_{e_k}^* s_{e_k},$$

$$s_{e_k}^* s_{e_{k'}} = 0 \text{ for } k \neq k'.$$

We obtain algebra of compact operators $\mathcal{K}$, the limit of the algebras in the preceeding example.

9. G with n vertices and n edges forming a cycle as in the picture (9).

$$s(e_k) = k, \quad r(e_k) = k + 1 \text{ for } k = 1, \dots, n-1, \quad r(e_n) = 1,$$

$$p_k = s_{e_k} s_{e_k}^*, \quad p_{k+1} = s_{e_k}^* s_{e_k},$$

$$s_{e_k}^* s_{e_{k'}} = 0 \text{ for } k \neq k'.$$

We obtain algebra of matrices over the algebra of functions on the circle, $C^*(G) = M_n(C(S^1))$.

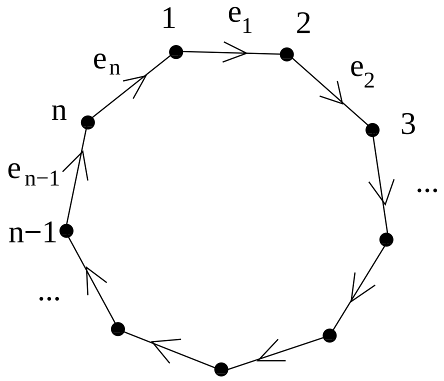


Figure 9: $M_n(C(S^1))$

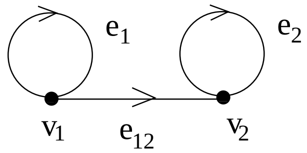


Figure 10: $C(\mathrm{SU}_q(2))$

10. G with two vertices with loops and connected by one edge.

$$p_{v_i} = p_{v_i}^* = p_{v_i}^2, \quad i = 1, 2, \quad p_{v_1} p_{v_2} = 0,$$

$$p_{v_1} = s_{e_{11}}^* s_{e_{11}} = s_{e_{11}}^* s_{e_{11}} + s_{e_{12}} s_{e_{11}}^*,$$

$$p_{v_2} = s_{e_{22}}^* s_{e_{22}} = s_{e_{12}}^* s_{e_{12}} = s_{e_{22}} s_{e_{22}}^*,$$

$$s_{e_{11}}^* s_{e_{12}} = 0, \quad s_{e_{11}}^* s_{e_{22}} = 0, \quad s_{e_{12}}^* s_{e_{22}} = 0.$$

We obtain C*-algebra for quantum SU(2), that is $C(\mathrm{SU}_q(2)) \simeq C(\mathrm{SU}_0(2))$, which is generated by two elements a, b satisfying the relations

$$a^* a + b^* b = 1, \quad a a^* + q^2 b^* b = 1,$$

$$ab = qba, \quad ab^* = qb^* a, \quad b^* b = bb^*.$$

The isomorphism is given by

$$\begin{array}{rcl} a & \mapsto & s_{e_{11}}^* + s_{e_{12}}^*, \\ b & \mapsto & s_{e_{22}}. \end{array}$$

11. The example (10) can be treated as the C*-algebra of the quantum sphere S_q^3. Now we present graph C*-algebra for the quantum sphere S_q^7, which is next generalized to arbitrary odd dimension. We take a graph G with four vertices with loops and each vertex is connected with all vertices with the greater index as in the picture (11). The C*-algebra for the quantum sphere S_q^7 is generated by the four elements z_1, z_2, z_3, z_4

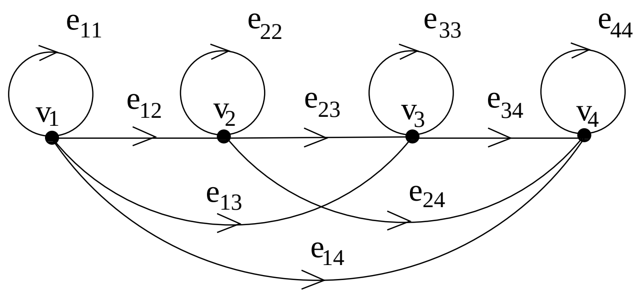


Figure 11: $C(S_q^7)$

satisfying the relations

$$\begin{aligned}
z_j z_i &= q z_i z_j \text{ for } i < j, \\
z_j^* z_i &= q z_i z_j^* \text{ for } i \neq j, \\
z_1^* z_1 &= z_1 z_1^* + (1-q^2)(z_2 z_2^* + z_3 z_3^* + z_4 z_4^*), \\
z_2^* z_2 &= z_2 z_2^* + (1-q^2)(z_3 z_3^* + z_4 z_4^*), \\
z_3^* z_3 &= z_3 z_3^* + (1-q^2) z_4 z_4^*, \\
z_4^* z_4 &= z_4 z_4^*, \\
z_1 z_1^* + z_2 z_2^* + z_3 z_3^* + z_4 z_4^* &= 1.
\end{aligned}$$

For $q = 0$ we have the isomorphism $C^*(G) \simeq C(S_0^7)$ given by

$$\begin{aligned}
z_1 &\mapsto s_{e_{11}} + s_{e_{12}} + s_{e_{13}} + s_{e_{14}}, \\
z_2 &\mapsto s_{e_{22}} + s_{e_{23}} + s_{e_{24}}, \\
z_3 &\mapsto s_{e_{33}} + s_{e_{34}}, \\
z_4 &\mapsto s_{e_{44}}.
\end{aligned}$$

12. As in the example (11) we take a graph with n vertices and edge between v_i and v_j if and only if $i \leqslant j$ as in the picture (12).

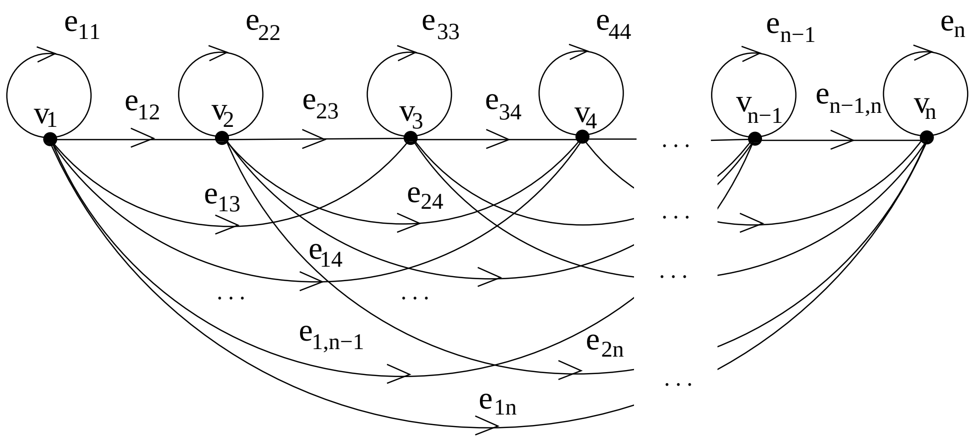


Figure 12: $C(S_q^{2n-1})$

$$v_1, \dots, v_n,$$

$$e_{ij}, \;\; j = i, \ldots, n, \;\; s(e_{ij}) = v_i, \;\; r(e_{ij}) = v_j.$$

The C*-algebra for the quantum sphere S_q^{2n-1} is generated by the n elements $z_1, \ldots, z_n$ satisfying the relations

$$\begin{aligned}
z_j z_i &= q z_i z_j \text{ for } i < j, \\
z_j^* z_i &= q z_i z_j^* \text{ for } i \neq j, \\
z_i^* z_i &= z_i z_i^* + (1 - q^2) \left(\sum_{j>i} z_j z_j^* \right) \text{ for } i = 1, \ldots, n, \\
\sum_{i=1}^{n} z_i z_i^* &= 1.
\end{aligned}$$

For $q = 0$ we have the isomorphism $C^*(G) \simeq C(S_0^{2n-1})$ given by

$$z_i \mapsto \sum_{j=i}^{n} s_{e_{ij}}, \;\; i = 1, \ldots, n.$$

13. We take a similar graph G to the one in the example (11), but with infinitely many paralell edges $v_i \to v_j$ for $i \leqslant j$.

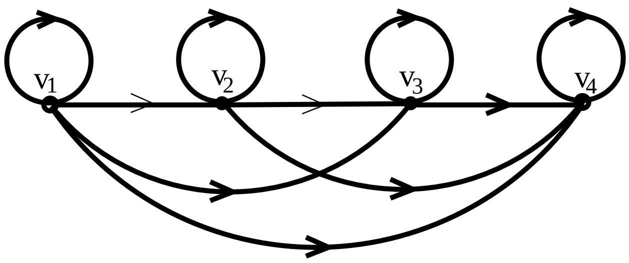


Figure 13: $C(\mathbb{C}P_q^3)$

14. We take a similar graph G to the one in the example (12), but with infinitely many paralell edges $v_i \to v_j$ for $i \leqslant j$.

15. If we modify the graph for the quantum sphere S_q^5 by adding two additional vertices w_1, w_2 and edges from each vertex v_1, v_2, v_3 to both of the added ones, then we obtain graph for the sphere S_q^6 as in the picture (15).

16. The example (15) can be generalized to arbitrary even dimension just by adding two vertices w_1, w_2 to the graph of the sphere S_q^{2n-1}. We have $n+2$ vertices $v_1, \ldots, v_n$ and w_1, w_2. Edges e_{ij} are from v_i to v_j whenever $i \leqslant j$ and g_{ik} are between v_i and w_k for $k = 1, 2$. More precisely for $i = 1, \ldots, n$ we have

$$s(e_{ij}) = v_i, \;\; r(e_{ij}) = v_j, \;\; j = i, \ldots, n,$$

$$s(g_{ik}) = v_i, \;\; r(g_{ik}) = w_k, \;\; k = 1, 2.$$

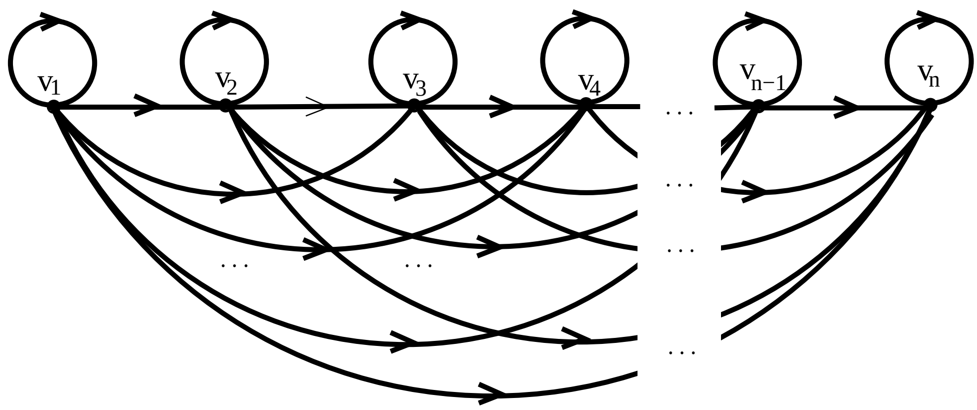


Figure 14: $C(\mathbb{C}P_q^{n-1})$

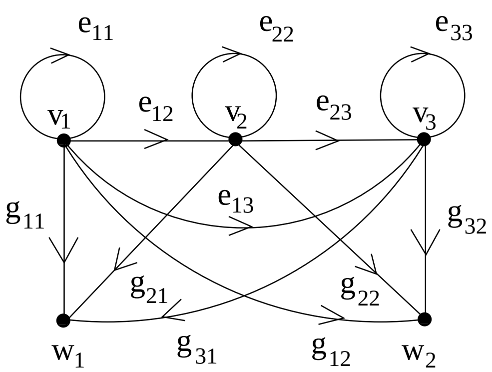


Figure 15: $C(S_q^6)$

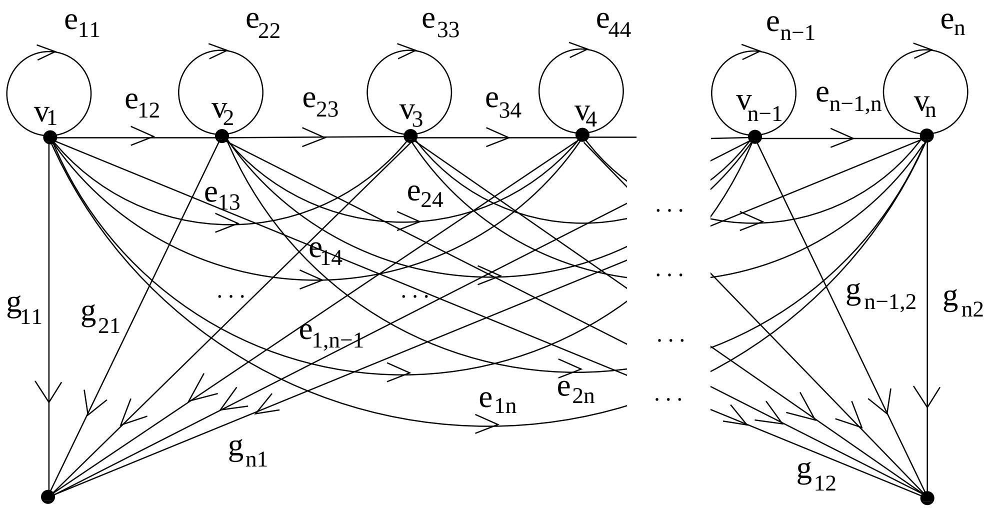


Figure 16: $C(S_q^{2n})$

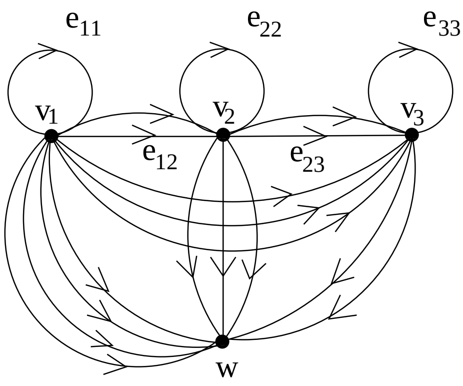


Figure 17: $C(\mathbb{R}P^6_q)$

17. In the example (15) we identify vertices w_1, w_2 and leave v_1, v_2, v_3 unchanged. The edges are as in the picture. General construction is described in the next example (18)

18. In the example (16) we identify vertices w_1, w_2 and leave $v_1, \ldots, v_n$ unchanged. The edges of the new graph are pairs (h_1, h_2) of edges from (16) such that $r(h_1) = s(h_2)$ and $r(h_2) \neq w_1, w_2$. Additionally we have edges f_{ik} from v_i to w for $i = 1, \ldots, n$ and $k = 1, \ldots, n+2-i$. The picture is analogous to the (17).

2 Computation of K-theory

The main tool for the computation of K-theory groups of the graph C*-algebras is the following

Theorem 2.1. *Let G be a directed graph and let $G^0_+ \subset G^0$ be the collection of vertices that emit at least one and at most finitely many edges. Let $\mathbb{Z}G^0_+$ and $\mathbb{Z}G^0$ be the free abelian groups on free generators G^0_+ and G^0. Let $A_G \colon \mathbb{Z}G^0_+ \to \mathbb{Z}G^0$ be the map defined by the formula*

$$A_G(v) := \left(\sum_{e \in G^1,\, s(e)=v} r(e) \right) - v.$$

Then

$$\begin{aligned} \mathrm{K}_0(C^*(G)) &\simeq \operatorname{coker} A_G \\ \mathrm{K}_1(C^*(G)) &\simeq \ker A_G \end{aligned}$$

The proof of this theorem will be postponed to the section (3), and now we compute the K-theory groups of the graph C*-algebras for the examples from section (1).

Example 2.2. 1. $\mathrm{K}_*(\mathbb{C})$

$$\begin{aligned} G^0 &= \{v\} \\ G^0_+ &= \emptyset \end{aligned}$$

$$A_G \colon \emptyset \to \mathbb{Z}$$

In this case A_G is from the empty set, but still we can write

$$\begin{aligned} \mathrm{K}_0(\mathbb{C}) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(\mathbb{C}) &= \ker A_G &= 0 \end{aligned}$$

2. $\mathrm{K}_*(C(S^1))$

$$\begin{aligned} G^0 &= \{v\} \\ G^0_+ &= \{v\} \end{aligned}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z}$$

$$v \quad \mapsto \quad v - v = 0$$

$$\begin{array}{rclcl} \mathrm{K}_0(C(S^1)) & = & \operatorname{coker} A_G & = & \mathbb{Z} \\ \mathrm{K}_1(C(S^1)) & = & \ker A_G & = & \mathbb{Z} \end{array}$$

3. $\mathrm{K}_*(\mathcal{T})$

$$\begin{aligned} G^0 &= \{v, w\} \\ G^0_+ &= \{v\} \end{aligned}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

$$v \quad \mapsto \quad v + w - v = w$$

$$\begin{array}{rclcl} \mathrm{K}_0(\mathcal{T}) & = & \operatorname{coker} A_G & = & \mathbb{Z} \\ \mathrm{K}_1(\mathcal{T}) & = & \ker A_G & = & 0 \end{array}$$

4. $\mathrm{K}_*(C(S^2_{0\infty}))$

$$\begin{array}{rcl} G^0 & = & \{v, w_1, w_2\} \\ G^0_+ & = & \{v\} \end{array}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$v \quad \mapsto \quad v + w_1 + w_2 - v = w_1 + w_2$$

$$\begin{array}{rclcl} \mathrm{K}_0(C(S^2_{0\infty})) & = & \operatorname{coker} A_G & = & \mathbb{Z} \oplus \mathbb{Z} \\ \mathrm{K}_1(C(S^2_{0\infty})) & = & \ker A_G & = & 0 \end{array}$$

5. $\mathrm{K}_*(C(\mathbb{R}P^2_q))$

$$\begin{array}{rcl} G^0 & = & \{v, w\} \\ G^0_+ & = & \{v\} \end{array}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

$$v \quad \mapsto \quad v + 2w - v = 2w$$

$$\begin{array}{rclcl} \mathrm{K}_0(C(\mathbb{R}P^2_q)) & = & \operatorname{coker} A_G & = & \mathbb{Z} \oplus \mathbb{Z}_2 \\ \mathrm{K}_1(C(\mathbb{R}P^2_q)) & = & \ker A_G & = & 0 \end{array}$$

6. $\mathrm{K}_*(O_n)$

$$\begin{aligned} G^0 &= \{v\} \\ G^0_+ &= \{v\} \end{aligned}$$

$$A_G\colon \mathbb{Z} \to \mathbb{Z}$$

$$v \mapsto nv - v = (n-1)v$$

$$\begin{aligned} \mathrm{K}_0(O_n) &= \operatorname{coker} A_G &= \mathbb{Z}_{n-1} \\ \mathrm{K}_1(O_n) &= \ker A_G &= 0 \end{aligned}$$

7. $\mathrm{K}_*(M_n(\mathbb{C}))$

$$\begin{aligned} G^0 &= \{v_1, v_2, \ldots, v_{n-1}\} \\ G^0_+ &= \{v_1, v_2, \ldots, v_n\} \end{aligned}$$

$$A_G\colon \mathbb{Z}^{n-1} \to \mathbb{Z}^n$$

$$v_i \mapsto v_{i+1} - v_i \text{ for } i = 1, \ldots, n-1$$

$$\begin{aligned} \mathrm{K}_0(M_n(\mathbb{C})) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(M_n(\mathbb{C})) &= \ker A_G &= 0 \end{aligned}$$

8. $\mathrm{K}_*(\mathcal{K})$

$$\begin{aligned} G^0 &= \{v_i \mid i \in \mathbb{Z}\} \\ G^0_+ &= \{v_i \mid i \in \mathbb{Z}\} \end{aligned}$$

$$A_G\colon \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \to \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$$

$$v_i \mapsto v_{i+1} - v_i \text{ for } i \in \mathbb{Z}$$

$$\begin{aligned} \mathrm{K}_0(\mathcal{K}) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(\mathcal{K}) &= \ker A_G &= 0 \end{aligned}$$

Ramark 2.3. If we take direct product instead of direct sum, then there will be nontrivial kernel.

9. $\mathrm{K}_*(M_n(S^1))$

$$\begin{aligned} G^0 &= \{v_1, v_2, \ldots, v_n\} \\ G^0_+ &= \{v_1, v_2, \ldots, v_n\} \end{aligned}$$

$$A_G\colon \mathbb{Z}^n \to \mathbb{Z}^n$$

$$\begin{aligned} v_i &\mapsto v_{i+1} - v_i \text{ for } i = 1, \ldots, n-1, \\ v_n &\mapsto v_1 - v_n \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(M_n(S^1)) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(M_n(S^1)) &= \ker A_G &= \mathbb{Z} \end{aligned}$$

10. $\mathrm{K}_*(C(\mathrm{SU}_q(2)))$

$$\begin{aligned} G^0 &= \{v_1, v_2\} \\ G^0_+ &= \{v_1, v_2\} \end{aligned}$$

$$A_G\colon \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$$

$$\begin{aligned} v_1 &\mapsto v_1 + v_2 - v_1 = v_2, \\ v_2 &\mapsto v_2 - v_2 = 0 \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(C(\mathrm{SU}_q(2))) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(C(\mathrm{SU}_q(2))) &= \ker A_G &= \mathbb{Z} \end{aligned}$$

11. $\mathrm{K}_*(C(S^7_q))$

$$\begin{aligned} G^0 &= \{v_1, v_2, v_3, v_4\} \\ G^0_+ &= \{v_1, v_2, v_3, v_4\} \end{aligned}$$

$$A_G\colon \mathbb{Z}^4 \to \mathbb{Z}^4$$

$$\begin{aligned} v_1 &\mapsto v_1 + v_2 + v_3 + v_4 - v_1 = v_2 + v_3 + v_4 \\ v_2 &\mapsto v_2 + v_3 + v_4 - v_2 = v_3 + v_4 \\ v_3 &\mapsto v_3 + v_4 - v_3 = v_4 \\ v_4 &\mapsto v_4 - v_4 = 0 \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(C(S^7_q)) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(C(S^7_q)) &= \ker A_G &= \mathbb{Z} \end{aligned}$$

12. $\mathrm{K}_*(C(S^{2n-1}_q))$

$$\begin{aligned} G^0 &= \{v_i \mid i = 1, \dots, n\} \\ G^0_+ &= \{v_i \mid i = 1, \dots, n\} \end{aligned}$$

$$A_G\colon \mathbb{Z}^n \to \mathbb{Z}^n$$

$$v_i \mapsto \sum_{j \geqslant i} v_j - v_i = \sum_{j > i} v_j$$

$$\begin{aligned} \mathrm{K}_0(C(S^{2n-1}_q)) &= \operatorname{coker} A_G &= \mathbb{Z} \\ \mathrm{K}_1(C(S^{2n-1}_q)) &= \ker A_G &= \mathbb{Z} \end{aligned}$$

13. $\mathrm{K}_*(C(\mathbb{C}P^3_q))$

$$\begin{aligned} G^0 &= \{v_1, v_2, v_3, v_4\} \\ G^0_+ &= \emptyset \end{aligned}$$

$$A_G\colon \emptyset \to \mathbb{Z}^4$$

$$\begin{aligned} \mathrm{K}_0(C(\mathbb{C}P^3_q)) &= \operatorname{coker} A_G &= \mathbb{Z}^4 \\ \mathrm{K}_1(C(\mathbb{C}P^3_q)) &= \ker A_G &= 0 \end{aligned}$$

14. $\mathrm{K}_*(C(\mathbb{C}P_q^{n-1}))$

$$\begin{aligned} G^0 &= \{v_i \mid i = 1, \ldots, n\} \\ G^0_+ &= \emptyset \end{aligned}$$

$$A_G \colon \emptyset \to \mathbb{Z}^n$$

$$\begin{aligned} \mathrm{K}_0(C(\mathbb{C}P_q^{n-1})) &= \operatorname{coker} A_G &= \mathbb{Z}^n \\ \mathrm{K}_1(C(\mathbb{C}P_q^{n-1})) &= \ker A_G &= 0 \end{aligned}$$

15. $\mathrm{K}_*(C(S_q^6))$

$$\begin{aligned} G^0 &= \{v_1, v_2, v_3, w_1, w_2\} \\ G^0_+ &= \{v_1, v_2, v_3\} \end{aligned}$$

$$A_G \colon \mathbb{Z}^3 \to \mathbb{Z}^5$$

$$\begin{aligned} v_1 &\mapsto v_1 + v_2 + v_3 + w_1 + w_2 - v_1 = v_2 + v_3 + w_1 + w_2 \\ v_2 &\mapsto v_2 + v_3 + w_1 + w_2 - v_2 = v_3 + w_1 + w_2 \\ v_3 &\mapsto v_3 + w_1 + w_2 - v_3 = w_1 + w_2 \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(C(S_q^6)) &= \operatorname{coker} A_G &= \mathbb{Z} \oplus \mathbb{Z} \\ \mathrm{K}_1(C(S_q^6)) &= \ker A_G &= 0 \end{aligned}$$

16. $\mathrm{K}_*(C(S_q^{2n}))$

$$\begin{aligned} G^0 &= \{v_1, \ldots, v_n, w_1, w_2\} \\ G^0_+ &= \{v_1, \ldots, v_n\} \end{aligned}$$

$$A_G \colon \mathbb{Z}^n \to \mathbb{Z}^{n+2}$$

$$v_i \mapsto \sum_{j \geqslant i} v_j + w_1 + w_2 - v_i = \sum_{j > i} v_j + w_1 + w_2$$

$$\begin{aligned} \mathrm{K}_0(C(S_q^{2n})) &= \operatorname{coker} A_G &= \mathbb{Z} \oplus \mathbb{Z} \\ \mathrm{K}_1(C(S_q^{2n})) &= \ker A_G &= 0 \end{aligned}$$

17. $\mathrm{K}_*(C(\mathbb{R}P_q^6))$

$$\begin{aligned} G^0 &= \{v_1, v_2, v_3, w\} \\ G^0_+ &= \{v_1, v_2, v_3\} \end{aligned}$$

$$A_G \colon \mathbb{Z}^3 \to \mathbb{Z}^4$$

$$\begin{aligned} v_1 &\mapsto \quad = \\ v_2 &\mapsto \quad = \\ v_3 &\mapsto \quad = \end{aligned}$$

$$\begin{aligned} \mathrm{K}_0(C(\mathbb{R}P_q^6)) &= \operatorname{coker} A_G &= \\ \mathrm{K}_1(C(\mathbb{R}P_q^6)) &= \ker A_G &= \end{aligned}$$

18. $\mathrm{K}_*(C(\mathbb{R}P_q^{2n}))$

$$\begin{aligned} G^0 &= \{v_1, \ldots, v_n, w\} \\ G^0_+ &= \{v_1, \ldots, v_n\} \end{aligned}$$

$$A_G \colon \mathbb{Z}^n \to \mathbb{Z}^{n+1}$$

$$v_i \quad \mapsto \quad =$$

$$\begin{aligned} \mathrm{K}_0(C(\mathbb{R}P_q^{2n})) &= \text{coker } A_G &= \\ \mathrm{K}_1(C(\mathbb{R}P_q^{2n})) &= \ker A_G &= \end{aligned}$$

3 Proof of the theorem (2.1)

Proof. There are seven steps in the proof, which we will sketch here.

1. Gauge action γ.

$$\gamma\colon \mathrm{U}(1) = S^1 \to \mathrm{Aut}(C^*(G))$$

$$\begin{aligned} \gamma_z(s_e) &= zs_e, \\ \gamma_z(p_v) &= p_v. \end{aligned}$$

2. $C^*(G) \rtimes_\gamma \mathrm{U}(1) \simeq C^*(G \times \mathbb{Z})$.

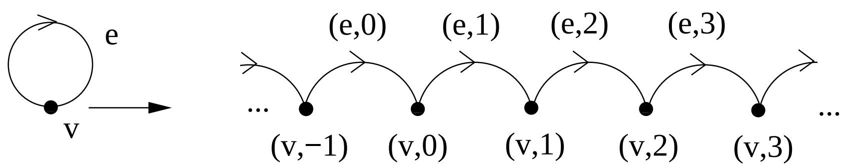
   We construct the new graph $G \times \mathbb{Z}$

$$\begin{aligned} (G \times \mathbb{Z})^0 &= G^0 \times \mathbb{Z}, \\ (G \times \mathbb{Z})^1 &= G^1 \times \mathbb{Z}. \end{aligned}$$

   It has no loops and

$$s(e, n) = (s(e), n - 1), \quad r(e, n) = (r(e), n).$$

   Each loop is resolved in the infinite segment



3. $C^*(G \times \mathbb{Z})$ is AF.

   It follows that $\mathrm{K}_1(C^*(G \times \mathbb{Z})) = 0$.

4. Dual action $\hat{\gamma}$.

$$\hat{\gamma}\colon \mathbb{Z} \to \mathrm{Aut}(C^*(G) \rtimes_\gamma \mathrm{U}(1))$$

$$\hat{\gamma}_\chi(f)(t) = \langle \chi, t \rangle f(t), \text{ where } f\colon \mathrm{U}(1) \to C^*(G).$$

5. Takesaki-Takai duality.

$$(C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z} \simeq C^*(G) \times \mathcal{K}.$$

From the stability of K_* it follows that

$$\mathrm{K}_*((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z}) \simeq \mathrm{K}_*(C^*(G)).$$

6. Pimsner-Voiculescu sequence.

The Pimsner-Voiculescu sequence is as follows

$$\begin{tikzcd}
\mathrm{K}_0((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \arrow[r] & \mathrm{K}_0((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \arrow[r] & \mathrm{K}_0((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z}) \arrow[d] \\
\mathrm{K}_1((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z}) \arrow[u] & \mathrm{K}_1((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \arrow[l] & \mathrm{K}_1((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \arrow[l]
\end{tikzcd}$$

where the maps are given by the formulas

$$\mathrm{K}_*(C^*(G) \rtimes_\gamma \mathrm{U}(1)) \xrightarrow{\mathrm{id} - \mathrm{K}_*(\hat{\gamma}^{-1})} \mathrm{K}_*(C^*(G) \rtimes_\gamma \mathrm{U}(1)),$$

$$\mathrm{K}_*(C^*(G) \rtimes_\gamma \mathrm{U}(1)) \xrightarrow{\mathrm{id} - \mathrm{K}_*(\beta^{-1})} \mathrm{K}_*((C^*(G) \rtimes_\gamma \mathrm{U}(1)) \rtimes_{\hat{\gamma}} \mathbb{Z}),$$

and the map $\beta\colon \mathbb{Z} \to \mathrm{Aut}(C^*(G \times \mathbb{Z}))$ is given by

$$\begin{aligned}
\beta_m(p_{(v,n)}) &= p_{(v,n+m)}, \\
\beta_m(s_{(e,n)}) &= s_{(e,n+m)}.
\end{aligned}$$

Using preceeding computations we can write the sequence as

$$\begin{tikzcd}
\mathrm{K}_0(C^*(G \times \mathbb{Z})) \arrow[r, "\mathrm{id} - \mathrm{K}_0(\hat{\gamma}^{-1})"] & \mathrm{K}_0(C^*(G \times \mathbb{Z})) \arrow[r, "1 - \mathrm{K}_0(\beta^{-1})"] & \mathrm{K}_0(C^*(G)) \arrow[d] \\
\mathrm{K}_1(C^*(G)) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

7. Computation of the kernel and cokernel of $1 - \mathrm{K}_0(\hat{\gamma}^{-1})$.

$\square$