LECTURE NOTES ON
THE K-THEORY OF OPERATOR ALGEBRAS

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Contents

1 Preliminaries on C^*-Algebras **5**
1.1 Basic definitions 5
1.1.1 Definitions C^*-algebra, $*$-algebra 5
1.1.2 Sub-C^* and sub-$*$-algebras 5
1.1.3 Ideals and quotients 5
1.1.4 The main examples 6
1.1.5 Short exact sequences 7
1.1.6 Adjoining a unit 8
1.2 Spectral theory 10
1.2.1 Spectrum 10
1.2.2 Continuous functional calculus 12
1.3 Matrix algebras and tensor products 13
1.4 Examples and Exercises 14

2 Projections and Unitaries **17**
2.1 Homotopy for unitaries 17
2.2 Equivalence of projections 21
2.3 Semigroups of projections 24
2.4 Examples and Exercises 25

3 The K_0-Group for Unital C^*-Algebras **27**
3.1 The Grothendieck Construction 27
3.2 Definition of the K_0-group of a unital C^*-algebra 29
3.2.1 Portrait of K_0 — the unital case 29
3.2.2 The universal property of K_0 29
3.2.3 Functoriality 30
3.2.4 Homotopy invariance 31
3.3 Examples and Exercises 31

4 K_0-Group — the General Case **37**
4.1 Definition of the K_0-Functor 37
4.1.1 Functoriality of K_0 37
4.1.2 Homotopy invariance of K_0 38
4.2 Further Properties 38
4.2.1 Portrait of K_0 38
4.2.2 (Half)exactness of K_0 39
4.3 Inductive Limits. Continuity and Stability of K_0 41
4.3.1 Increasing limits of C^*-algebras 41

4.3.2 Direct limits of $*$-algebras . . . 41
4.3.3 C^*-algebraic inductive limits . . . 42
4.3.4 Continuity of K_0 . . . 42
4.3.5 Stability of K_0 . . . 43
4.4 Examples and Exercises . . . 44

5 K_1-Functor and the Index Map 51
5.1 The K_1 Functor . . . 51
5.1.1 Definition of the K_1-group . . . 51
5.1.2 Properties of the K_1-functor . . . 52
5.2 The Index Map . . . 55
5.2.1 Fredholm index . . . 55
5.2.2 Definition of the index map . . . 58
5.2.3 The exact sequence . . . 58
5.3 Examples and Exercises . . . 60

6 Bott Periodicity and the Exact Sequence of K-Theory 67
6.1 Higher K-Groups . . . 67
6.1.1 The suspension functor . . . 67
6.1.2 Isomorphism of $K_1(A)$ and $K_0(SA)$ . . . 67
6.1.3 The long exact sequence of K-theory . . . 68
6.2 Bott Periodicity . . . 69
6.2.1 Definition of the Bott map . . . 69
6.2.2 The periodicity theorem . . . 70
6.3 The 6-Term Exact Sequence . . . 74
6.3.1 The 6-term exact sequence of K-theory . . . 74
6.3.2 The exponential map . . . 75
6.3.3 An explicit form of the exponential map . . . 75
6.4 Examples and Exercises . . . 76

7 Tools for the computation of K-groups 83
7.1 Crossed products, the Thom-Connes isomorphism and the Pimsner-Voiculescu sequence . . . 83
7.1.1 Crossed products . . . 83
7.1.2 Crossed products by $\mathbb{R}$ and by $\mathbb{Z}$ . . . 84
7.1.3 Irrational rotation algebras . . . 85
7.2 The Mayer-Vietoris sequence . . . 86
7.3 The Künneth formula . . . 89

Chapter 1

Preliminaries on C^*-Algebras

1.1 Basic definitions

1.1.1 Definitions C^*-algebra, $*$-algebra

Definition 1.1.1. *A C^*-algebra A is an algebra over $\mathbb{C}$ with involution $a \mapsto a^*$ (*-algebra), equipped with a norm $a \mapsto \|a\|$, such that A is a Banach space, and the norm satisfies $\|ab\| \leqq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ (C^*-property).*

Immediate consequence: $\|a^*\| = \|a\|$. a^* is called adjoint of a.

A C^*-algebra A is called *unital* if it has a multiplicative unit $1_A = 1$. Immediate consequence: $1^* = 1$, $\|1\| = 1$ ($\|1\| = \|1^2\| = \|1\|^2$). If A and B are C^*-algebras, a $*$-*homomorphism* $\varphi : A \to B$ is a linear multiplicative map commuting with the involution. If A and B are unital, φ is called unital if $\varphi(1_A) = 1_B$. A surjective φ is always unital.

A C^*-algebra A is called separable, if it contains a countable dense subset.

1.1.2 Sub-C^* and sub-$*$-algebras

A subset B of a C^*-algebra A is called sub-$*$-algebra, if it closed under all algebraic operations (including the involution). It is called sub-C^*-algebra, if it is also norm-closed. The norm closure of a sub-$*$-algebra is a sub-C^*-algebra (from continuity of the algebraic operations).

If F is a subset of a C^*-algebra A, the sub-C^*-algebra generated by F, denoted by $C^*(F)$, is the smallest sub-C^*-algebra containing F. It coincides with the norm closure of the linear span of all monomials in elements of F and their adjoints. A subset F is called self-adjoint, if $F^* := \{a^* \mid a \in F\} = F$.

1.1.3 Ideals and quotients

An ideal in a C^*-algebra is a norm-closed two-sided ideal. Such an ideal is always self-adjoint, hence a sub-C^*-algebra. ([D-J77, 1.8.2], [M-GJ90, 3.1.3]) If I is an ideal in a C^*-algebra A, the quotient $A/I = \{a + I \mid a \in A\}$ is a $*$-algebra, and also a C^*-algebra with respect to the norm $\|a + I\| := \inf\{\|a + x\| \mid x \in I\}$. I is obviously the kernel of the quotient map $\pi : A \to A/I$. ([M-GJ90, 3.1.4], [D-J77, 1.8.2])

A $*$-homomorphism $\varphi : A \to B$ of C^*-algebras is always norm-decreasing, $\|\varphi(a)\| \leqq \|a\|$. It is injective if and only if it is isometric. ([M-GJ90, 3.1.5]). $\operatorname{Ker}\varphi$ is an ideal in A,

$\mathrm{Im}\varphi$ a sub-C^*-algebra of B. ([M-GJ90, 3.1.6]). φ always factorizes as $\varphi = \varphi_0 \circ \pi$, with injective $\varphi_0 : A/\mathrm{Ker}\,\varphi \to B$.

A C^*-algebra is called *simple* if its only ideals are $\{0\}$ and A (trivial ideals).

1.1.4 The main examples

Example 1. Let X be a locally compact Hausdorff space, and let $C_0(X)$ be the vector space of complex-valued continuous functions that vanish at infinity, i.e., for all $\epsilon > 0$ exists a compact subset $K_\epsilon \subseteq X$ such that $|f(x)| < \epsilon$ for $x \notin K_\epsilon$. Equipped with the pointwise multiplication and the complex conjugation as involution, $C_0(X)$ is a $*$-algebra. With the norm $\|f\| := \sup_{x\in X}\{|f(x)|\}$, $C_0(X)$ is a (in general non-unital) commutative C^*-algebra.

Theorem 1.1.2. *(Gelfand-Naimark) Every commutative C^*-algebra is isometrically isomorphic to an algebra $C_0(X)$ for some locally compact Hausdorff space X.*

Idea of proof: X is the set of multiplicative linear functionals (characters (every character is automatically *-preserving, [D-J77, 1.4.1(i)], equivalently, the set of maximal ideals), with the weak-$*$-topology (i.e., the weakest topology such that all the functionals $\chi \mapsto \chi(a)$, $a \in A$, are continuous.

Additions:

(i) $C_0(X)$ is unital iff X is compact.

(ii) $C_0(X)$ is separable iff X is separable.

(iii) X and Y are homeomorphic iff $C_0(X)$ and $C_0(Y)$ are isomorphic.

(iv) Each proper continuous map $\eta : Y \to X$ induces a $*$-homomorphism $\eta^* : C_0(X) \to C_0(Y)$ ($\eta^*(f) = f \circ \eta$). Conversely, each $*$-homomorphism $\varphi : C_0(X) \to C_0(Y)$ induces a proper continuous map $\eta : Y \to X$ (map a character χ of $C_0(Y)$ to the character $\chi \circ \varphi$ of $C_0(X)$).

(v) There is a bijective correspondence between open subsets of X and ideals in $C_0(X)$ (the ideal to an open subset is the set of functions vanishing on the complement of the subset, to an ideal always corresponds the set of characters vanishing on the ideal, its complement in the set of all characters is the desired open set). If $U \subseteq X$ is open, then there is a short exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C_0(X) \longrightarrow C_0(X \setminus U) \longrightarrow 0, \tag{1.1.1}$$

where $C_0(U) \to C_0(X)$ is given by extending a function on U as 0 to all of X, and $C_0(X) \to C_0(X \setminus U)$ is the restriction, being surjective due to Stone-Weierstraß.

Example 2. Let $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ denote the set of all continuous linear operators on $\mathcal{H}$. Then $B(\mathcal{H})$ is an algebra with respect to addition, multiplication with scalars, and composition of operators, it is a $*$-algebra with the usual operator adjoint, and it is a C^*-algebra with respect to the operator norm.

Theorem 1.1.3. *(Gelfand-Naimark) Every C^*-algebra A is isometrically isomorphic to a closed C^*-subalgebra of some $B(\mathcal{H})$.*

Idea of proof: Consider the set of positive linear functionals ($\varphi(a^*a) \geq 0$) on A. Every such functional allows to turn the algebra into a Hilbert space on which the algebra is represented by its left action. Take as Hilbert space the direct sum of all these Hilbert spaces. Then the direct sum of these representations gives the desired injection.

1.1.5 Short exact sequences

A sequence of C^*-algebras and $*$-homomorphisms

$$\ldots \longrightarrow A_k \xrightarrow{\varphi_k} A_{k+1} \xrightarrow{\varphi_{k+1}} A_{k+2} \longrightarrow \ldots \tag{1.1.2}$$

is said to be *exact*, if $\mathrm{Im}\varphi_k = \mathrm{Ker}\,\varphi_{k+1}$ for all k. An exact sequence of the form

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{1.1.3}$$

is called *short exact*. Example: If $I \subseteq A$ is an ideal, then

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0 \tag{1.1.4}$$

is short exact (ι the natural embedding $I \to A$). If a short exact sequence (1.1.3) is given, then $\varphi(I)$ is an ideal in A, there is an isomorphism $\psi_/ : B \to A/\varphi(I)$, and the diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\varphi"] & A \arrow[r, "\psi"] \arrow[d, "\mathrm{id}"] & B \arrow[r] \arrow[d, "\psi_/"] & 0 \\
0 \arrow[r] & \varphi(I) \arrow[r, "\iota"] & A \arrow[r, "\pi"] & A/\varphi(I) \arrow[r] & 0
\end{tikzcd} \tag{1.1.5}$$

is commutative. If for a short exact sequence (1.1.3) exists $\lambda : B \to A$ with $\psi \circ \lambda = \mathrm{id}_B$, then the sequence is called *split exact*, and λ is called *lift* of ψ. Diagrammatic:

$$0 \longrightarrow I \xrightarrow{\varphi} A \underset{\lambda}{\overset{\psi}{\rightleftarrows}} B \longrightarrow 0. \tag{1.1.6}$$

Not all short exact sequences are split exact.
Example:

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\iota} C([0,1]) \xrightarrow{\psi} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0 \tag{1.1.7}$$

with $\psi(f) = (f(0), f(1))$ is an exact sequence. It does not split: Every linear map $\lambda : \mathbb{C} \oplus \mathbb{C} \to C([0,1])$ is determined by its values on the basis elements, $\lambda((1,0)) = f_1,\ \lambda((0,1)) = f_2$. The split condition means $f_1(0) = 1,\ f_1(1) = 0$ and $f_2(0) = 0,\ f_2(1) = 1$. If λ is to be a homomorphism, because of $(1,0)^2 = (1,0)$, we should have $f_1^2 = \lambda((1,0))^2 = \lambda((1,0)^2) = \lambda((1,0)) = f_1$, and analogously $f_2^2 = f_2$. However, a continuous function on a connected space is equal to its square if and only if it is either the constant function 1 or the constant function 0. Both is not the case for f_1 and f_2.
Geometric interpretation: ψ corresponds to the embedding of two points as end points of

the interval $[0,1]$. However, it is not possible to map this interval continuously onto the set $\{0,1\}$.

The direct sum $A \oplus B$ of two C^*-algebras is the direct sum of the underlying vector spaces, with component-wise defined multiplication and involution, and with the norm $\|(a,b)\| = \max(\|a\|,\|b\|)$. It is again a C^*-algebra. There are natural homomorphisms $\iota_A : A \to A \oplus B,\ a \mapsto (a,0)$, $\pi_A : A \oplus B \to A,\ (a,b) \mapsto a$, analogously $\iota_B,\ \pi_B$. Then

$$0 \longrightarrow A \xrightarrow{\iota_A} A \oplus B \underset{\iota_B}{\overset{\pi_B}{\rightleftarrows}} B \longrightarrow 0. \tag{1.1.8}$$

is a split exact sequence with lift ι_B. Not all split exact sequences come in this manner from direct sums.
Example (not presented in lecture).
Let

$$0 \longrightarrow A \xrightarrow{\varphi} E \xrightarrow{\psi} B \longrightarrow 0 \tag{1.1.9}$$

be an exact sequence. Then there exists an isomorphism $\theta : E \to A \oplus B$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"] & E \arrow[r, "\psi"] \arrow[d, "\theta"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\iota_A"] & A \oplus B \arrow[r, "\pi_B"] & B \arrow[r] & 0 \end{tikzcd} \tag{1.1.10}$$

commutative if and only there exists a homomorphism $\nu : E \to A$ such that $\nu \circ \varphi = \mathrm{id}_A$.
Proof: If $\theta : E \to A \oplus B$ makes the diagram commutative, then $\theta_{|\mathrm{Im}\varphi}$ is an injective map whose image is $\iota_A(A)$. $\nu := \pi_A \circ \theta_{|\mathrm{Im}\varphi} : E \to A$ fulfills $\nu \circ \varphi = \mathrm{id}_A$. If $\nu : E \to A$ with this property is given, put $\theta(e) = (\iota_A \circ \nu(e), \psi(e))$. θ is an isomorphism:
surjective: Let $a \in A$. Then $\varphi(a) \in \operatorname{Ker}\psi$, hence $\psi(\varphi(a)) = 0$. But, $\nu(\varphi(a)) = a$, i.e., $\theta(\varphi(a)) = (a,0)$. On the other hand, as $\pi_B \circ \theta = \psi$ and ψ is surjective, for any $b \in B$ exists $a' \in A$ such that $(a',b) \in \mathrm{Im}\theta$. Since $(a',0) \in \mathrm{Im}\theta$, also $(0,b) \in \mathrm{Im}\theta$ for any $b \in B$, thus finally all $(a,b) \in \mathrm{Im}\theta$.
injective: If $\psi(e) = 0$ with $e \neq 0$ then $e = \varphi(a)$ with $a \neq 0$, and $\nu(e) = \nu \circ \varphi(a) = a \neq 0$, thus $\iota_A \circ \nu(e) \neq 0$ by injectivity of ι_A. Otherwise, $\psi(e) \neq 0$ already means $\theta(e) \neq 0$. □

If this condition is satisfied, the upper sequence is isomorphic to the lower one, and thus also split. Counterexample (where the condition is not fulfilled)?

1.1.6 Adjoining a unit

Definition 1.1.4. *Let A be a $*$-algebra. Put $\tilde{A} = A \oplus \mathbb{C}$ (direct sum of vector spaces) and*

$$(a,\alpha)(b,\beta) := (ab + \beta a + \alpha b, \alpha\beta), \quad (a,\alpha)^* := (a^*, \bar{\alpha}). \tag{1.1.11}$$

Define $\iota : A \to \tilde{A}$ and $\pi : \tilde{A} \to \mathbb{C}$ by $\iota(a) = (a,0)$, $\pi(a,\alpha) = \alpha$ (i.e., $\iota = \iota_A$, $\pi = \pi_{\mathbb{C}}$ in the direct sum terminology used above).

Proposition 1.1.5. *With the operations just introduced, $\tilde{A}$ is a unital $*$-algebra with unit $1_{\tilde{A}} = (0,1)$. ι is an injective, π a surjective $*$-homomorphism.*

Proof. Straightforward. □

Sometimes ι is suppressed, and we write also $\tilde{A} = \{a + \alpha 1 \mid a \in A, \alpha \in \mathbb{C}\}$.

Let now A be a C^*-algebra, and let $\|.\|_A$ be the norm on A.

Note that the direct sum norm $\|(a, \alpha)\| = \max(\|a\|, |\alpha|)$ does in general not have the C^*-property (because $A \oplus \mathbb{C}$ does not have the direct sum product). Example: A unital, put $\alpha = 1$, $a = 1_A$, then $\|(a, \alpha)(a^*, \bar{\alpha})\| = \max(\|aa^* + \bar{\alpha}a^* + \alpha a^*\|, |\alpha|^2) = 3$, $\|(a, \alpha)\|^2 = \max(\|a\|^2, |\alpha|^2) = 1$.
Recall that the algebra $B(E)$ of linear operators on a Banach space E is a Banach space (algebra) with norm $\|b\| = \sup_{\|x\| \leqq 1} \|b(x)\|$ (see [RS72, Theorem III.2], [D-J73, 5.7]). Note that $(a, \alpha) \mapsto L_a + \alpha \mathrm{id}_A$, where $L_a(b) = ab$ for $a, b \in A$, defines a homomorphism φ of $\tilde{A}$ onto the subspace of all continuous linear operators of the form $L_a + \alpha \mathrm{id}_A$ in $B(A)$. This homomorphism is injective iff A is not unital. (exercise) Indeed, let A be not unital, and assume $L_a(b) + \alpha b = 0$ for all $b \in A$. If α would be $\neq 0$ then $-\frac{a}{\alpha}$ would be a left unit for A, thus also a right unit, hence a unit, contradicting the assumed non-unitality. Thus we have $\alpha = 0$, i.e. $ab = 0$ for all $b \in A$. In particular, $aa^* = 0$, hence $\|a^*\|^2 = \|aa^*\| = 0$, i.e., $\|a\| = \|a^*\| = 0$, i.e., $a = 0$. On the other hand, if A is unital, $(1_A, -1)$ is in the kernel of φ.

We have $\|a\| = \|L_a\|$ for $a \in A$: $\|L_a\| \leqq \|a\|$ is clear by the definition of the operator norm ($\|L_a\| = \sup_{\|b\| \leqq 1} \|ab\| \leqq \sup_{\|b\| \leqq 1} \|a\|\|b\| = \|a\|$), and $\|a\|^2 = \|aa^*\| = \|L_a(a^*)\| \leqq \|L_a\|\|a^*\|$, hence also $\|a\| \leqq \|L_a\|$. Thus it makes sense to define for non-unital A a norm on $\tilde{A}$ by transporting the norm of $B(A)$, i.e., we put $\|(a, \alpha)\|_{\tilde{A}} := \|L_a + \alpha \mathrm{id}_A\|$. For unital A, we note that $\tilde{A}$ is as a $*$-algebra isomorphic to $A \oplus \mathbb{C}$ (direct sum of C^*-algebras). The isomorphism is given by $(a, \alpha) \mapsto (a + \alpha 1_A, \alpha)$ (easy exercise). As before, we define the norm on $\tilde{A}$ by transport with the isomorphism. Note that $(-1_A, 1)$ is a projector in $A \oplus \mathbb{C}$.

Proposition 1.1.6. *$\tilde{A}$ is a unital C^*-algebra with norm $\|.\|_{\tilde{A}}$. $\iota(A)$ is a closed ideal in $\tilde{A}$.*

Proof. The additive and multiplicative triangle inequality come from these properties for the norm in $B(A)$ and $A \oplus \mathbb{C}$. Since $\{L_a | a \in A\}$ is closed and thus complete in $B(A)$, and $\{L_a | a \in A\}$ has codimension 1 in $\varphi(\tilde{A})$, the latter is also complete in the nonunital case, and it is obviously complete in the unital case. Also, it is obvious in the unital case that the norm has the C^*-property. To prove the latter for the nonunital case, we define the involution on $\varphi(\tilde{A})$ by transport with φ, i.e.,

$$(L_a + \alpha \mathrm{id}_A)^* := L_{a^*} + \bar{\alpha} \mathrm{id}_A. \tag{1.1.12}$$

Hence, by this definition $\varphi(\tilde{A})$ is a complete normed $*$-algebra. It remains to show that the C^*-property is satisfied. Let $\epsilon > 0$ and let $x = L_a + \alpha \mathrm{id}_A \in \varphi(\tilde{A})$. By the definition of the operator norm, there exists $b \in A$ with $\|b\| \leqq 1$ such that

$$\|x\|^2 = \|L_a + \alpha \mathrm{id}_A\|^2 \leqq \|(L_a + \alpha \mathrm{id}_A)(b)\|^2 + \epsilon. \tag{1.1.13}$$

The right hand side can be continued as follows:

$$\begin{aligned}
&= \|ab + \alpha b\|^2 + \epsilon \\
&= \|(ab + \alpha b)^*(ab + \alpha b)\| + \epsilon \\
&= \|(b^* a^* + \bar{\alpha} b^*)(ab + \alpha b)\| + \epsilon \\
&= \|b^*(L_{a^*} + \bar{\alpha}\mathrm{id}_A)(L_a + \alpha \mathrm{id}_A)(b)\| + \epsilon \\
&\leqq \|b^*\| \|(L_{a^*} + \bar{\alpha}\mathrm{id}_A)(L_a + \alpha \mathrm{id}_A)(b)\| + \epsilon \\
&\leqq \|b^*\| \|((L_{a^*} + \bar{\alpha}\mathrm{id}_A)(L_a + \alpha \mathrm{id}_A)\| \|b\| + \epsilon \\
&\leqq \|x^* x\| + \epsilon.
\end{aligned}$$

Thus we have $\|x\|^2 \leqq \|x^*x\| + \epsilon$ for any ϵ, hence $\|x\|^2 \leqq \|x^*x\|$. However, also $\|x^*x\| \leqq \|x^*\|\|x\|$ ($B(A)$ is a normed algebra). Exchanging the roles of x and x^*, we also obtain $\|x^*\|^2 \leqq \|x\|\|x^*\|$, together $\|x\| = \|x^*\|$. Going back to the inequalities, this also gives the C^*-property. □

For both the unital and nonunital case, we have $\tilde{A}/\iota(A) \cong \mathbb{C}$, and the sequence

$$0 \longrightarrow A \longrightarrow \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0, \tag{1.1.14}$$

with $\pi : \tilde{A} \to \mathbb{C}$ the quotient map and $\lambda : \mathbb{C} \to \tilde{A}$ given by $\alpha \mapsto (0, \alpha)$, is split exact. Note also that adjoining a unit is functorial: If $\varphi : A \to B$ is a homomorphism of C^*-algebras, there is a unique homorphism $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota_A"] \arrow[d, "\varphi"] & \tilde{A} \arrow[r, "\pi_A"] \arrow[d, "\tilde{\varphi}"] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & B \arrow[r, "\iota_B"] & \tilde{B} \arrow[r, "\pi_B"] & \mathbb{C} \arrow[r] & 0
\end{tikzcd} \tag{1.1.15}$$

commutative. It is given by $\tilde{\varphi}(a, \alpha) = (\varphi(a), \alpha)$. $\tilde{\varphi}$ is unit-preserving, $\tilde{\varphi}(0, 1) = (0, 1)$. If A is a sub-C^*-algebra of a unital C^*-algebra B whose unit 1_B is not in A, then $\tilde{A}$ is isomorphic to the sub-C^*-algebra $A + \mathbb{C}1_B$ of B (exercise).

1.2 Spectral theory

1.2.1 Spectrum

Let A be a unital C^*-algebra. Then the spectrum (with respect to A) of $a \in A$ is defined as

$$\mathrm{sp}(a) (= \mathrm{sp}_A(a)) := \{\lambda \in \mathbb{C} \mid a - \lambda 1_A \text{ is not invertible in } A\}. \tag{1.2.16}$$

Elementary statements about the spectrum, true already for a unital algebra, are:

(i) If $A = \{0\}$ then $\mathrm{sp}(0) = \emptyset$.

(ii) $\mathrm{sp}(\lambda 1_A) = \{\lambda\}$ for $\lambda \in \mathbb{C}$.

(iii) $a \in A$ is invertible iff $0 \notin \mathrm{sp}(a)$.

(iv) If $P \in \mathbb{C}[X]$ (polynomial in one variable with complex coefficients), then $\mathrm{sp}(P(a)) = P(\mathrm{sp}(a))$.

(v) If $a \in A$ is nilpotent, then $\mathrm{sp}(a) = \{0\}$ (if $A \neq \{0\}$).

(vi) If $\varphi : A \to B$ is a morphism of unital algebras over $\mathbb{C}$, then $\mathrm{sp}_B(\varphi(a)) \subseteq \mathrm{sp}_A(a)$.

(vii) If $(a, b) \in A \oplus B$ (direct sum of algebras), then $\mathrm{sp}_{A\oplus B}((a, b)) = \mathrm{sp}_A(a) \cup \mathrm{sp}_B(b)$. (Can be generalized to direct products.)

If A is the algebra of continuous complex-valued functions on a topological space, then the spectrum of any element is the set of values of the function. If A is the algebra of endomorphisms of a finite dimensional vector space over $\mathbb{C}$ then the spectrum of an element is the set of eigenvalues.

For a Banach algebra, the spectrum of an element is always a compact subset of $\mathbb{C}$ contained in the ball of radius $\|a\|$,

$$r(a) = \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(a)\} \leqq \|a\|. \tag{1.2.17}$$

Idea of proof: If $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$, hence $1 - \lambda^{-1}a$ is invertible (This uses: if $\|a\| < 1$ then $1 - a$ is invertible, with $(1 - a)^{-1} = 1 + a + a^2 + \ldots$ – Neumann series.) Thus $\lambda \notin \mathrm{sp}(a)$. The spectrum is closed because the set of invertible elements is open (use again the fact stated in parentheses).

The number $r(a)$ is called spectral radius of a. Using complex analysis, one can show that the spectrum is non-empty. The sequence $(\|a^n\|^{1/n})$ is convergent, and $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. If A is not unital, the spectrum of an element $a \in A$ is defined as the spectrum of $\iota(a) \in \tilde{A}$. In this case always $0 \in \mathrm{sp}(a)$ ($(a, 0)(b, \beta) = (ab + \beta a, 0) \neq (0, 1) = 1_{\tilde{A}}$).

Definition 1.2.1. *An element a of a C^*-algebra A is called*

- *normal if $aa^* = a^*a$,*
- *self-adjoint if $a = a^*$,*
- *positive if it is normal and* $\mathrm{sp}(a) \subseteq \mathbb{R}_+ (= [0, \infty[)$,
- *unitary if A is unital and $aa^* = a^*a = 1_A$.*
- *a projector if $a = a^* = a^2$.*

The set of positive elements is denoted by A^+.

The spectrum of a self-adjoint element is contained in $\mathbb{R}$, that of a unitary element is contained in $\mathbb{T}^1 = S^1$ (the unit circle, considered as a subset of $\mathbb{C}$), that of a projector is contained in $\{0, 1\}$ (exercises). An element a of a C^*-algebra A is positive if and only if it is of the form $a = x^*x$, for some $x \in A$. For normal elements, the above formula for the spectral radius reduces to $r(a) = \|a\|$. This allows to conclude

Proposition 1.2.2. *The C^*-norm of a C^*-algebra is unique.*

Proof. $\|a\|'^2 = \|a^*a\|' = r(a^*a) = \|a^*a\| = \|a\|^2$. □

Let us also note that every element is a linear combination of two self-adjoint elements, $a = \frac{1}{2}(a + a^*) + i\frac{1}{2i}(a - a^*)$ (this is the unique decomposition $a = h_1 + ih_2$, with h_1 and h_2 self-adjoint), and also a linear combination of four unitary elements.

The spectrum a priori depends on the ambient C^*-algebra. However, if B is a unital C^*-subalgebra of a unital C^*-algebra A, whose unit coincides with the unit of A, then the spectrum of an element of B with respect to B coincides with its spectrum with respect to A (exercise, use that the inverse of an element belongs to the smallest C^*-algebra containing that element, i.e., the C^*-subalgebra generated by that element). If A is not unital, or if the unit of A does not belong to B, then $\mathrm{sp}_A(b)\cup\{0\} = \mathrm{sp}_B(b)\cup\{0\}$ (exercise).

1.2.2 Continuous functional calculus

Let A be a unital C^*-algebra, and let $a \in A$ be normal. Then there is a unique C^*-isomorphism $j : C(\mathrm{sp}(a)) \to C^*(a, 1)$ mapping the identity map of $\mathrm{sp}(a)$ into a. Moreover, this isomorphism maps a polynomial P into $P(a)$ and the complex conjugation $z \mapsto \bar{z}$ into a^*. Therefore one writes $j(f) = f(a)$. One knows that $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$ (spectral mapping theorem).

If $\varphi : A \to B$ is $*$-homomorphism of unital C^*-algebras, then $\mathrm{sp}(\varphi(a)) \subseteq \mathrm{sp}(a)$ and $\varphi(f(a)) = f(\varphi(a))$ for $f \in \mathbb{C}(\mathrm{sp}(a))$.

If a C^*-algebra is realized as a subalgebra of $B(\mathcal{H})$, the functional calculus is realized for self-adjoint elements in terms of their spectral decompositions: If $a = \int \lambda dE_\lambda$ then $f(a) = \int f(\lambda)dE_\lambda$, where E_λ is the family of spectral measures belonging to a.

If a is a normal element of a non-unital C^*-algebra A, then $f(a)$ is a priori in $\tilde{A}$. We have $f(a) \in \iota(A) \simeq A$ iff $f(0) = 0$: When $\pi : \tilde{A} \to \mathbb{C}$ is the quotient mapping, we have $\pi(f(a)) = f(\pi(a)) = f(0)$.

Lemma 1.2.3. *Let $K \subseteq \mathbb{R}$ be compact and non-empty, and let $f \in C(K)$. Let A be a unital C^*-algebra, and let Ω_K be the set of self-adjoint elments of A with spectrum contained in K. Then the induced function*

$$f : \Omega_K \longrightarrow A, \quad a \mapsto f(a), \tag{1.2.18}$$

is continuous.

Proof. The map $a \mapsto a^n$, $A \to A$ is continuous (continuity of multiplication). Thus every complex polynomial f induces a continuous map $A \to A$, $a \mapsto f(a)$.

Now, let $f \in C(K)$, let $a \in \Omega_K$, and let $\epsilon > 0$. Then there is a complex polynomial g such that $|f(z) - g(z)| < \frac{\epsilon}{3}$ for every $z \in K$. By continuity discussed above, for every ϵ we find $\delta < 0$ such that $\|g(a) - g(b)\| \leqq \frac{\epsilon}{3}$ for $b \in A$ with $\|a - b\| \leqq \delta$. Since, moreover,

$$\|f(c) - g(c)\| = \|(f - g)(c)\| = \sup\{|(f - g)(z)| \mid z \in \mathrm{sp}(c)\} \leqq \frac{\epsilon}{3} \tag{1.2.19}$$

for $c \in \Omega_K$, we conclude $\|f(a) - f(b)\| = \|f(a) - g(a) + g(a) - g(b) + g(b) - f(b)\| \leqq \|f(a - g(a)\| + \|g(a) - g(b)\| + \|g(b) - f(b)\| \leqq \epsilon$ for $b \in \Omega_K$ with $\|a - b\| \leqq \delta$. □

1.3 Matrix algebras and tensor products

Let A_1, A_2 be C^*-algebras. The algebraic tensor product $A_1 \otimes A_2$ is a $*$-algebra with multiplication and adjoint given by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2, \tag{1.3.20}$$

$$(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*. \tag{1.3.21}$$

Problem: There may exist different norms with the C^*-property on this $*$-algebra, leading to different C^*-algebras under completion (though one can show that all norms with the C^*-property are cross norms, $\|a_1 \otimes a_2\| = \|a_1\|\|a_2\|$). We will restrict to the case where this problem is not there by definition: A C^*-algebra is called *nuclear* if for any C^*-algebra B there is only one C^*-norm on the algebraic tensor product $A \otimes B$. Examples: finite dimensional, commutative, type I (every non-zero irreducible representation in a Hilbert space contains the compact operators). If one of the tensor factors is nuclear, the unique C^*-norm on the algebraic tensor product coincides with the norm in $B(\mathcal{H})$ under a faithful representation of the completed tensor product.

We will mainly need the following very special situation. Let A be a C^*-algebra, and let $M_n(\mathbb{C})$ ($n \in \mathbb{N}$) be the algebra of complex $n \times n$-matrices. Then $A \otimes M_n(\mathbb{C})$ can be identified with $M_n(A)$, the $*$-algebra of $n \times n$-matrices with entries from A, with product and adjoint given according to the matrix structure. The unique C^*-norm on $A \otimes M_n(\mathbb{C}) = M_n(A)$ is defined using any injective $*$-homomorphism $\varphi : A \to B(\mathcal{H})$, and the canonical injective $*$-homomorphism $M_n(\mathbb{C}) \to B(\mathbb{C}^n)$, i.e., $\|a \otimes m\| = \|\varphi(a) \otimes m\|$, where on the right stands the norm in $B(\mathcal{H}) \otimes B(\mathbb{C}^n) = B(\mathcal{H} \otimes \mathbb{C}^n)$. One has the inequality (exercise):

$$\max \|a_{ij}\| \leqq \left\| \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \right\| \leqq \sum \|a_{ij}\|. \tag{1.3.22}$$

The following lemma will be needed later. It involves the C^*-algebra $C_0(X, A)$, see Exercice 6.

Lemma 1.3.1. *Let X be a locally compact Hausdorff space and let A be a C^*-algebra. Define for $f \in C_0(X)$, $a \in A$ an element $fa \in C_0(X, A)$ by*

$$(fa)(x) = f(x)a. \tag{1.3.23}$$

Then $\operatorname{span}\{fa \mid f \in C_0(X), a \in A\}$ *is dense in $C_0(X, A)$.*

Proof. Let $X^+ = X \cup \{\infty\}$ be the one-point compactification of X. Then

$$C_0(X, A) = \{f \in C(X^+, A) \mid f(\infty) = 0\}. \tag{1.3.24}$$

Let $f \in C_0(X, A)$, $\epsilon > 0$. There is an open covering $U_1, \dots, U_n$ of X^+ such that $\|f(x) - f(y)\| < \epsilon$ if $x, y \in U_k$. (Compactness of X^+, continuity of f.) Choose $x_k \in U_k$, with $x_k = \infty$ if $\infty \in U_k$. Let $(h_k)_{k=1}^n$ be a partition of unity subordinate to the covering (U_k), i.e., $h_k \in C(X^+)$, $\operatorname{supp} h_k \subseteq U_k$, $\sum_{k=1}^n h_k = 1$, $0 \leqq h_k \leqq 1$. (Note that every compact Hausdorff space is paracompact.) Then $\|f(x)h_k(x) - f_k(x_k)h_k(x)\| \leqq \epsilon h_k(x)$ for $x \in X$, $k = 1, \dots, n$. It follows that $\|f(x) - \sum_{k=1}^n f(x_k)h_k(x)\| \leqq \epsilon$, for $x \in X$. Put $a_k = f(x_k) \in A$. Then $\sum_{k=1}^n h_k a_k \in \operatorname{span}\{fa \mid f \in C_0(X), a \in A\}$, because $a_k = f(x_k) = 0$ if $\infty \in U_k$, and $\|f - \sum_{k=1}^n h_k a_k\| \leqq \epsilon$. $\square$

1.4 Examples and Exercises

EXERCISE 1.4.1. If A is a sub-C^*-algebra of a unital C^*-algebra B whose unit 1_B is not in A, then $\tilde{A}$ is isomorphic to the sub-C^*-algebra $A + \mathbb{C}1_B$ of B.

The map $(a, \alpha) \mapsto a + \alpha 1_B$ is the desired isomorphism: It is obviously surjective, and injectivity follows as injectivity of φ in the proof of Proposition 1.1.6: Let $a + \alpha 1_B = 0$. If $\alpha \neq 0$, then $1_B = -\frac{a}{\alpha} \in A$, contradicting the assumption. Thus $\alpha = 0 = a$. That the mapping is a $*$-homomorphism is straightforward.

EXERCISE 1.4.2. Let A be a unital C^*-algebra. Show the following.

(i) Let u be unitary. Then $\mathrm{sp}(u) \subseteq \mathbb{T}$.

(ii) Let u be normal, and $\mathrm{sp}(u) \subseteq \mathbb{T}$. Then u is unitary.

(iii) Let a be self-adjoint. Then $\mathrm{sp}(a) \subseteq \mathbb{R}$.

(iv) Let p be a projection. Then $\mathrm{sp}(p) \subseteq \{0, 1\}$.

(v) Let p be normal with $\mathrm{sp}(p) \subseteq \{0, 1\}$. Then p is a projector.

(i): $\|u\| = 1$, due to $\|u\|^2 = \|u^*u\| = \|1_A\| = 1$. Hence $|\lambda| \leqq 1$ for $\lambda \in \mathrm{sp}(u)$. By the spectral mapping theorem, $\lambda^{-1} \in \mathrm{sp}(u^{-1}) = \mathrm{sp}(u^*)$. But also $\|u^*\| = 1$, and thus $|\lambda^{-1}| \leqq 1$, so $|\lambda| = 1$.

(ii): Due to normality, there is a C^*-isomorphism $C(\mathrm{sp}(u)) \to C^*(u, 1)$, mapping $\mathrm{id}_{\mathrm{sp}(u)} \mapsto u$ and $\bar{\mathrm{id}}_{sp(u)} \mapsto u^*$. Hence $1_{\mathrm{sp}(u)} \mapsto u^*u = uu^* = 1 (= 1_A)$.

(iii): $a \in A$ is invertible iff a^* is invertible, thus $a - \lambda 1_A$ is invertible iff $a^* - \bar{\lambda}$ is invertible. Thus $\lambda \in \mathrm{sp}(a)$ iff $\bar{\lambda} \in \mathrm{sp}(a^*)$, and for $a = a^*$ the spectrum is invariant under complex conjugation. The series $\exp(ia) := \sum_{n=0}^{\infty} \frac{(ia)^n}{n!}$ is absolutely convergent, its adjoint is (due to continuity of the star operation) $\exp(-ia) = \sum_{n=0}^{\infty} \frac{(-ia)^n}{n!}$ and fulfills $\exp(ia)\exp(-ia) = 1_A = \exp(-ia)\exp(ia)$, so it is a unitary element in $C^*(a, 1)$, which means that $\exp(i\lambda) \in \mathbb{T}$ for $\lambda \in \mathrm{sp}(a)$, i.e., $\lambda \in \mathbb{R}$.

(iv): Let $p = p^* = p^2$. By (iii), $\mathrm{sp}(p)$ is real, and by the spectral mapping theorem we have $\mathrm{sp}(p) = \mathrm{sp}(p)^2$. This means that $\mathrm{sp}(p) \subseteq [0, 1]$. Using the isomorphism $C(\mathrm{sp}(p)) \to C^*(p, 1)$, we have $\mathrm{id}_{\mathrm{sp}(p)} = \mathrm{id}^2_{\mathrm{sp}(p)}$, thus $\mathrm{sp}(p) \subseteq \{0, 1\}$.

(v): Let p be normal, $\mathrm{sp}(p) \subseteq \{0, 1\}$. Then $\mathrm{id}_{\mathrm{sp}(p)} = \bar{\mathrm{id}}_{\mathrm{sp}(p)} = \mathrm{id}^2_{\mathrm{sp}(p)}$, and the same is true for p (using the isomorphism $C(\mathrm{sp}(p)) \to C^*(p, 1)$).

EXERCISE 1.4.3. Let A be a unital C^*-algebra, $a \in A$.

(i) a is invertible iff aa^* and a^*a are invertible. In that case, $a^{-1} = (a^*a)^{-1}a^* = a^*(aa^*)^{-1}$.

(ii) Let a be normal and invertible in A. Then there exists $f \in C(\mathrm{sp}(a))$ such that $a^{-1} = f(a)$, i.e., a^{-1} belongs to $C^*(a, 1)$.

(iii) Let $a \in A$ be invertible. Then a^{-1} belongs to $C^*(a, 1)$, the smallest unital C^*-subalgebra containing a.

(i): If a^{-1} exists, then also $a^{*-1} = a^{-1*}$ and $(aa^*)^{-1} = a^{*-1}a^{-1}$, $(a^*a)^{-1} = a^{-1}a^{*-1}$. If $(aa^*)^{-1}$ and $(a^*a)^{-1}$ exist, put $b := a^*(aa^*)^{-1}$ and $c := (a^*a)^{-1}a^*$. Then $ab = 1 = ca$ and, multiplying the left of these equalities by c from the left, the right one by b from the right, $cab = c$, $b = cab$. This means $b = c = a^{-1}$.

(ii): a invertible means that $0 \notin \mathrm{sp}(a)$. Thus, the function $\mathrm{id}_{\mathrm{sp}(a)}$ corresponding to a under the isomorphism $C(\mathrm{sp}(a)) \to C^*(a,1)$ is invertible, and the corresponding inverse is in $C^*(a,1)$.

(iii): aa^* and a^*a are normal (selfadjoint) and by (i) invertible in A. By (ii) their inverses are in the C^*-subalgebras generated by $\{aa^*, 1\}$ and $\{a^*a, 1\}$, thus also in $C^*(a,1)$. Again using (i) (considering $C^*(a,1)$ instead of A), we obtain $a^{-1} \in C^*(a,1)$.

Exercise 1.4.4. Show the uniqueness of the decomposition $a = h_1 + ih_2$, $h_{1,2}$ self-adjoint.

We have $a^* = h_1 - ih_2$, hence $h_1 = \frac{1}{2}(a + a^*)$ and $h_2 = \frac{1}{2i}(a - a^*)$.

Exercise 1.4.5. Let $\varphi : A \to B$ be a morphism of unital C^*-algebras.

(i) Show that $\mathrm{sp}(\varphi(a)) \subseteq \mathrm{sp}(a)$ for all $a \in A$, and that there is equality if φ is injective.

(ii) Show that $\|\varphi(a)\| \leqq \|a\|$, equality if φ is injective.

Let φ be not necessarily injective. If $a - \lambda 1_A$ is invertible, then $\varphi(a - \lambda 1_A) = \varphi(a) - \lambda 1_B$ is invertible (with inverse $\varphi((a - \lambda 1_A)^{-1})$). This shows $\mathbb{C} \setminus \mathrm{sp}(a) \subseteq \mathbb{C} \setminus \mathrm{sp}(\varphi(a))$. Thus we also have $r(\varphi(a^*a)) \leqq r(a^*a)$, which gives $\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \leqq r(a^*a) = \|a^*a\| = \|a\|^2$.

Let φ be injective, and let $a \in A$. With the isomorphisms $C(\mathrm{sp}(a^*a)) \to C^*(a^*a, 1)$ and $C(\mathrm{sp}(\varphi(a^*a))) \to C^*(\varphi(a^*a), 1)$, φ gives rise under to an injective C^*-homomorphism $\varphi_a : C(\mathrm{sp}(a^*a)) \to C(\mathrm{sp}(\varphi(a^*a)))$. One shows as in [D-J77, Proof of 1.8.1] that φ_a corresponds to a surjective continuous map $\psi_a : \mathrm{sp}(\varphi(a^*a)) \to \mathrm{sp}(a^*a)$. Now, the pull-back of any surjective continuous map is isometric: If $\psi : Y \to X$ is a surjective map of sets, and if $f : X \to \mathbb{C}$ is a function such that $\sup_{x \in X} |f(x)|$ exists, then $\sup_{x \in X} |f(x)| = \sup_{y \in Y} |f(\psi(y))|$. In our situation, each φ_a is isometric, which, using the above isomorphisms, just amounts to saying that φ is isometric. This proves both desired equalities.

Exercise 1.4.6. If A is a C^*-algebra, and X is a locally compact Hausdorff space, then let $C_0(X, A)$ denote the set of all continuous maps $f : X \to A$ such that $\|f\| := \sup_{x \in X} \|f(x)\|$ exists and f vanishes at infinity, i.e., $\forall \epsilon > 0\ \exists$ compact $K \subseteq X : \|f(x)\| < \epsilon$ for $x \in X \setminus K$. On $C(X, A)$, introduce operations of a $*$-algebra pointwise. Show that $C_0(X, A)$ is a C^*-algebra.

The algebraic properties, the triangle inequalities and the C^* property are easy to verify. The proof of completeness (convergence of Cauchy sequences) is standard (e.g., [D-J73, 7.1.3] or [RS72, Theorem I.23]). The idea is to show that the limit given by pointwise Cauchy sequences is indeed an element of $C_0(X, A)$. The only thing not proven in the above references is vanishing at infinity of the limit. This can be concluded from the following statement: Let $f \in C(X, A)$, $g \in C_0(X, A)$, $\|f - g\| < \epsilon/2$. Then there is a compact $K \subseteq X$ such that $\|f(x)\| < \epsilon$ for $x \in X \setminus K$. Indeed, since $g \in C_0(X, A)$, there is a compact $K \subseteq X$ such that $\|g(x)\| < \epsilon/2$ for $x \in X \setminus K$. Then $\|f(x)\| \leq \|f(x) - g(x)\| + \|g(x)\| < \epsilon/2 + \epsilon/2 = \epsilon$ for $x \in X \setminus K$.

Exercise 1.4.7. Let A be a unital C^*-algebra, $x \in M_2(A)$. Show that x commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ iff $x = \mathrm{diag}(a, b)$ for some $a, b \in A$. Then a, b are unitary iff x is unitary.

Exercise 1.4.8. Prove the inequalities (1.3.22).

Let $a^{(ij)}$ be the element of $M_n(A)$ which has a_{ij} at the intersection of the i-th row with the j-th column and zero at all other places. Let us first show $\|a^{(ij)}\| = \|a_{ij}\|$. In the identification $M_n(A) = A \otimes M_n(\mathbb{C})$ we have $a^{(ij)} = a_{ij} \otimes e_{ij}$, where $e_{ij} \in M_n(\mathbb{C})$ is the ij-th matrix unit. Thus, for an injective $*$-homomorphism $\varphi : A \to B(\mathcal{H})$, we have $\|a^{(ij)}\| = \|\varphi \otimes \mathrm{id}(a^{(ij)}\| = \|\varphi(a_{ij}) \otimes e_{ij}\| = \|\varphi(a_{ij}\|\|e_{ij}\| = \|a_{ij}\|$. Here, we have made use of the following facts: Every injective $*$-homomorphism of C^*-algebras is isometric (Exercice 5 (ii)), the norm of a tensor product of operators is the product of the norms of the factors (see e.g. [M-GJ90, p. 187]), and $\|e_{ij}\| = 1$ (easy to verify). This is enough to prove the right inequality: $\|(a_{ij})\| = \|\sum_{i,j} a^{(ij)}\| \leqq \sum_{i,j} \|a^{(ij)}\| = \sum_{i,j} \|a_{ij}\|$.

For the left inequality, we have

$$\begin{aligned} \|(a_{ij})\|^2 &= \sup_{\psi \in \mathcal{H}\otimes\mathbb{C}^n, \|\psi\|=1} \|\sum_{i,j} \varphi(a_{ij}) \otimes e_{ij}(\psi)\|^2 \\ &\geq \sup_{\psi=\psi_1\otimes\psi_2, \|\psi_1\|=\|\psi_2\|=1} \|\sum_{i,j} \varphi(a_{ij})(\psi_1) \otimes e_{ij}(\psi_2)\|^2. \end{aligned} \tag{1.4.25}$$

Now, choose $\psi_2 = e_k$, e_k an element of the canonical basis of $\mathbb{C}^n$. Then $e_{ij}(e_k) = \delta_{jk}e_i$, and the above inequality can be continued:

$$\geq \sup_{\|\psi\|=1} \|\sum_i \varphi(a_{ik})(\psi) \otimes e_i\|^2 = \sup_{\|\psi\|=1} \sum_i \|\varphi(a_{ik})(\psi)\|^2 \geq \max_i \|\varphi(a_{ik})\|^2 = \max_i \|a_{ik}\|^2. \tag{1.4.26}$$

(Note that $\|\sum_i \psi_i \otimes e_i\|^2 = \sum_i \|\psi_i\|^2$.) Since this is true for all k, we have the desired inequality.

Exercise 1.4.9. Let A be a unital C^*-algebra, and let $a \in M_n(A)$ be upper triangular, i.e.,

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}. \tag{1.4.27}$$

Show that a has an inverse in the subalgebra of upper triangular elements of $M_n(A)$ iff all diagonal elements a_{kk} are invertible in A.

Let all a_{kk} be invertible in A. Then $a_0 := \mathrm{diag}(a_{11}, \dots, a_{nn})$ is invertible in $M_n(A)$ (with inverse $a_0^{-1} = \mathrm{diag}(a_{11}^{-1}, \dots, a_{nn}^{-1})$), and $a = a_0 + N$ with nilpotent $N \in M_n(A)$. We can write $a = a_0 + N = a_0(1 + a_0^{-1}N)$ where in our concrete case $a_0^{-1}N$ is again nilpotent. Since for nilpotent m we have $(1 + m)^{-1} = 1 - m + m^2 - m^3 + \ldots \pm m^k$ for a certain $k \in \mathbb{N}$, a is invertible.

Conversely, assume that there exists an inverse b of a that is upper triangular. Then $ab = 1$ and $ba = 1$ give immediately that $b_{kk} = a_{kk}^{-1}$ for $k = 1, \dots, n$.

Note that there are invertible upper triangular matrices, whose diagonal elements are not invertible, and whose inverse is not upper triangular. Example: Let s be the unilateral shift, satisfying $s^*s = 1$. Neither s nor s^* is invertible. Nevertheless, the matrix $\begin{pmatrix} s & 1 \\ 0 & s^* \end{pmatrix}$ has the inverse $\begin{pmatrix} s^* & -1 \\ 1 - ss^* & s \end{pmatrix}$.

Exercise 1.4.10. Let A be a C^*-algebra, $a, b \in A$. Show that $\left\| \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| = \max\{\|a\|, \|b\|\}$.

Chapter 2

Projections and Unitaries

2.1 Homotopy for unitaries

Definition 2.1.1. *Let X be a topological space. Then $x, y \in X$ are homotopic in X, $x \sim_h y$ in X, if there exists a continuous map $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$.*

The relation $\sim_h$ is an equivalence relation on X (exercise). $f : t \mapsto f(t) = f_t$ as above is called continuous path from x to y. In a vector space, any two elements are homotopic: Take the path $t \mapsto (1-t)x + ty$.

Definition 2.1.2. *Let A be a unital C^*-algebra, and let $\mathcal{U}(A)$ denote the group of unitary elements of A. Then $\mathcal{U}_0(A) := \{u \in \mathcal{U}(A) \mid u \sim_h 1_A \text{ in } \mathcal{U}(A)\}$ (connected component of 1_A in $\mathcal{U}(A)$).*

Remark 2.1.3. If $u_1, u_2, v_1, v_2 \in \mathcal{U}(A)$ with $u_i \sim_h v_j$, $j = 1, 2$, then $u_1 u_2 \sim_h v_1 v_2$. Indeed, if $t \mapsto w_j(t)$ are continuous paths connecting u_j with v_j, then $t \mapsto w_1(t) w_2(t)$ is a continuous path connecting $u_1 u_2$ with $v_1 v_2$ (everything in $\mathcal{U}(A)$). ◇

Lemma 2.1.4. *Let A be a unital C^*-algebra.*

(i) If $h \in A$ is self-adjoint, then $\exp(ih) \in \mathcal{U}_0(A)$.

(ii) If $u \in \mathcal{U}(A)$ and $\mathrm{sp}(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(A)$.

(iii) If $u, v \in \mathcal{U}(A)$ and $\|u - v\| < 2$, then $u \sim_h v$.

Proof. (i) By the contiuous functional calculus, if $h = h^*$ and f is a continuous function on $\mathbb{R}$ with values in $\mathbb{T}$, then $f(h)^* = \bar{f}(h) = f^{-1}(h)$, i.e., $f(h)$ is unitary. In particular, $\exp(ih)$ is unitary. Now for $t \in [0,1]$ define $f_t : \mathrm{sp}(h) \to \mathbb{T}$ by $f_t(x) := \exp(itx)$. Then, by continuity of $t \mapsto f_t$, the path $t \mapsto f_t(h)$ in $\mathcal{U}(A)$ is continuous, thus $\exp(ih) = f_1(h) \sim_h f_0(h) = 1$.
(ii) If $\mathrm{sp}(u) \neq \mathbb{T}$, there exists $\theta \in \mathbb{R}$ such that $\exp(i\theta) \notin \mathrm{sp}(u)$. Note that $\varphi(\exp(it)) = t$ defines a continuous function φ on $\mathrm{sp}(u)$ with values in the open interval $]\theta, \theta + 2\pi[\subseteq \mathbb{R}$. We have $z = \exp(i\varphi(z))$ for $z \in \mathrm{sp}(u)$. Then $h = \varphi(u)$ is a self-adjoint element of A with $u = \exp(ih)$, and by (i) $u \in \mathcal{U}_0(A)$.
(iii) From $\|u - v\| < 2$ it follows that $\|v^* u - 1\| = \|v^*(u - v)\| < 2$ (since $\|v^*\| = 1$). Thus $-2 \notin \mathrm{sp}(v^* u - 1)$, i.e., $-1 \notin \mathrm{sp}(v^* u)$. Then, by (ii), $v^* u \sim_h 1$, hence $u \sim_h v$ (remark before the lemma). □

Corollary 2.1.5. *$\mathcal{U}(M_n(\mathbb{C}) = \mathcal{U}(M_n(\mathbb{C}))$, i.e., the unitary group in $M_n(\mathbb{C})$ is connected.*

Proof. Each unitary in $M_n(\mathbb{C})$ has finite spectrum, therefore the assumption of (ii) of Lemma 2.1.4 is satisfied. □

Lemma 2.1.6. *(Whitehead) Let A be a unital C^*-algebra, and $u, v \in \mathcal{U}(A)$. Then*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \text{ in } \mathcal{U}(M_2(A). \tag{2.1.1}$$

In particular,

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ in } \mathcal{U}(M_2(A)). \tag{2.1.2}$$

Proof. First note that the spectrum of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $\{1, -1\}$ (direct elementary computation). Thus by Lemma 2.1.4 (ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now write

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1.3}$$

Then, by Remark 2.1.3,

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1.4}$$

analogously

$$\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1.5}$$

thus $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}$. In particular, $\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$, thus

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.1.6}$$

□

Proposition 2.1.7. *Let A be a unital C^*-algebra.*

(i) $\mathcal{U}_0(A)$ is a normal subgroup of $\mathcal{U}(A)$.

(ii) $\mathcal{U}_0(A)$ is open and closed relative to $\mathcal{U}(A)$.

(iii) $u \in \mathcal{U}_0(A)$ iff there are finitely many self-adjoint $h_1, \ldots, h_n \in A$ such that

$$u = \exp(ih_1) \cdots \exp(ih_n). \tag{2.1.7}$$

Proof. (i): First note that $\mathcal{U}_0(A)$ is closed under multiplication by Remark 2.1.3. In order to show that with $u \in \mathcal{U}_0(A)$ also $u^{-1} \in \mathcal{U}_0(A)$ and $vuv^* \in \mathcal{U}_0(A)$ (for any $v \in \mathcal{U}(A)$), let $t \mapsto w_t$ be a continuous path from 1 to u in $\mathcal{U}(A)$. Then $t \mapsto w_t^{-1}$ and $t \mapsto vw_tv^*$ are continuous paths from 1 to u^{-1} and vuv^* in $\mathcal{U}(A)$.
(ii) and (iii): Let $G := \{\exp(ih_1)\cdots\exp(ih_n) \mid n \in \mathbb{N},\ h_k = h_k^* \in A\}$. By (i) and Lemma 2.1.4, (i), $G \subseteq \mathcal{U}_0(A)$. Since $\exp(ih)^{-1} = \exp(-ih)$, for $h = h^*$, G is a subgroup of $\mathcal{U}_0(A)$.

G is open relative to $\mathcal{U}(A)$: If $v \in G$ and $u \in \mathcal{U}(A)$ with $\|u - v\| < 2$, then $\|1 - uv^*\| = \|(u - v)\| < 2$, and by Lemma 2.1.4 (iii) and its proof, $\mathrm{sp}(uv^*) \neq \mathbb{T}$, and, by the proof of Lemma 2.1.4 (ii), there exists $h = h^* \in A$ such that $uv^* = \exp(ih)$. Thus $u = \exp(ih)v \in G$.

G is closed relative to $\mathcal{U}(A)$: $\mathcal{U}(A) \setminus G$ is a disjoint union of cosets Gu, with $u \in \mathcal{U}(A)$. Each Gu is homeomorphic to G, therefore Gu is open relative to $\mathcal{U}(A)$. Thus G is closed in $\mathcal{U}(A)$.

By the above, G is a nonempty subset of $\mathcal{U}_0(A)$, it is open and closed in $\mathcal{U}(A)$, consequently also in $\mathcal{U}_0(A)$. The latter is connected, hence $G = \mathcal{U}_0(A)$. This proves (ii) and (iii). ☐

Lemma 2.1.8. *Let A and B be unital C^*-algebras, and let $\varphi : A \to B$ be a surjective (thus unital) $*$-homorphism.*

(i) $\varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$.

(ii) $\forall u \in \mathcal{U}(B) \exists v \in \mathcal{U}_0(M_2(A))$:

$$\varphi_2(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \tag{2.1.8}$$

with $\varphi_2 : M_2(A) \to M_2(B)$ the extension of φ.

(iii) If $u \in \mathcal{U}(B)$ and there is $v \in \mathcal{U}(A)$ with $u \sim_h \varphi(v)$, then $u \in \varphi(\mathcal{U}(A))$.

Proof. Any unital $*$-homomorphism is continuous and maps unitaries into unitaries, hence $\varphi(\mathcal{U}_0(A)) \subseteq \mathcal{U}_0(B)$. Conversely, if $u \in \mathcal{U}_0(B)$, then by Proposition 2.1.7 (iii) there are self-adjoint $h_j \in B$ such that

$$u = \exp(ih_1)\cdots\exp(ih_n). \tag{2.1.9}$$

By surjectivity of φ, there are $a_j \in A$ with $\varphi(a_j) = h_j$. Then $k_j := \frac{a_j + a_j^*}{2}$ are self-adjoint and satisfy $\varphi(k_j) = h_j$. Put

$$v = \exp(ik_1)\cdots\exp(ik_n). \tag{2.1.10}$$

Then $\varphi(v) = u$ and $v \in \mathcal{U}_0(A)$ by Proposition 2.1.7 (iii). This proves (i).

(ii): By Lemma 2.1.4 we have $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \in \mathcal{U}_0(M_2(A))$. On the other hand, $\varphi_2 : M_2(A) \to M_2(B)$ is a surjective $*$-homomorphism, so (i) proves the desired claim.

(iii): If $u \sim_h \varphi(v)$, then $u\varphi(v^*) \in \mathcal{U}_0(B)$, and, by (i), $u\varphi(v^*) = \varphi(w)$ with $w \in \mathcal{U}_0(A)$. Hence $u = \varphi(wv)$, with $wv \in \mathcal{U}(A)$. ☐

Definition 2.1.9. *Let A be a unital C^*-algebra. The group of invertible elements in A is denoted by $GL(A)$. $GL_0(A) := \{a \in GL(A) \mid a \sim_h 1 \text{ in } GL(A)\}$.*

$\mathcal{U}(A)$ is a subgroup of $GL(A)$.

If $a \in A$, then there is a well-defined element $|a| = (a^*a)^{\frac{1}{2}}$, by the continuous functional calculus. $|a|$ is called absolute value of a.

Proposition 2.1.10. *Let A be a unital C^*-algebra.*

(i) If $a \in GL(A)$, then also $|a| \in GL(A)$, and $a|a|^{-1} \in \mathcal{U}(A)$.

(ii) Let $\omega : GL(A) \to \mathcal{U}(A)$ be defined by $\omega(a) = a|a|^{-1}$. Then ω is continuous, $\omega(u) = u$ for $u \in \mathcal{U}(A)$, and $\omega(a) \sim_h a$ in $GL(A)$ for every $a \in GL(A)$.

(iii) If $u, v \in \mathcal{U}(A)$ and if $u \sim_h v$ in $GL(A)$, then $u \sim_h v$ in $\mathcal{U}(A)$.

Proof. (i): If $a \in GL(A)$ then also $a^*, a^*a \in GL(A)$. Hence also $|a| = (a^*a)^{\frac{1}{2}} \in GL(A)$, with $|a|^{-1} = ((a^*a)^{-1})^{\frac{1}{2}}$. Then $a|a|^{-1}$ is invertible and unitary: $|a|^{-1}$ is self-adjoint and $|a|^{-1}a^*a|a|^{-1} = |a|^{-1}|a|^2|a|^{-1} = 1$.

(ii): Multiplication in a C^*-algebra is continuous, as well as the map $a \mapsto a^{-1}$ in $GL(A)$. (see [M-GJ90, Theorem 1.2.3]) Therefore to show continuity of ω, it is sufficient to show that $a \mapsto |a|$ is continuous. The latter is the composition of $a \mapsto a^*a$ and $h \mapsto h^{\frac{1}{2}}$ (for $h \in A^+$). The first of these maps is continuous by continuity of $*$ and the multiplication. Now it is sufficient to show the continuity of the square root on any bounded $\Omega \subseteq A^+$.This follows from Lemma 1.2.3, because each such Ω is contained in Ω_K, with $K = [0, R]$, $R = \sup_{h\in\Omega} \|h\|$.

For $u \in \mathcal{U}(A)$ we have $|u| = 1$, hence $\omega(u) = u$.

For $a \in GL(A)$, put $a_t := \omega(a)(t|a| + (1-t)1_A)$, $t \in [0, 1]$. This is a continuous path from $\omega(a) = a_0$ to $a = a_1$. It remains to show that $a_t \in GL(A)$, $t \in [0, 1]$. Since $|a|$ is positive and invertible, there is $\lambda \in]0, 1]$ with $|a| \geq \lambda 1_A$. Then, for each $t \in [0, 1]$, $t|a| + (1-t)1_A \geq \lambda 1_A$. (Properties of positive operators, use the isomorphism $C(\mathrm{sp}(a^*a)) \to C^*(a^*a, 1)$.) Hence $t|a| + (1-t)1_A$ and consequently a_t are invertible.

(iii) If $t \mapsto a_t$ is a continuous path in $GL(A)$ from u to v (unitaries), then $t \mapsto \omega(a_t)$ is such a path in $\mathcal{U}(A)$. □

Remark 2.1.11. (ii) of the above proposition says that $\mathcal{U}(A)$ is a retract of $GL(A)$. $\omega : GL(A) \to \mathcal{U}(A)$ is the corresponding retraction. (A subspace X of a topological space Y is called retract of Y if there is a continuous $r : Y \to X$ with $x \sim_h r(x)$ in Y $\forall x \in Y$ and $r(x) = x$ $\forall x \in X$.) ◇

Remark 2.1.12. (ii) also says that $a = \omega(a)|a|$, with unitary $\omega(a)$, for invertible a. This is called the (unitary) polar decomposition of a. For any $a \in A$, there is a polar decomposition $a = v|a|$, with a unique partial isometry v. ◇

Proposition 2.1.13. *Let A be a unital C^*-algebra. Let $a \in GL(A)$, and let $b \in A$ with $\|a - b\| < \|a^{-1}\|^{-1}$. Then $b \in GL(A)$,*

$$\|b^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a - b\|, \tag{2.1.11}$$

and $a \sim_h b$ in $GL(A)$.

Proof. We have

$$\|1 - a^{-1}b\| = \|a^{-1}(a-b)\| \leqq \|a^{-1}\|\|a-b\| < 1, \tag{2.1.12}$$

thus $(a^{-1}b)^{-1} = \sum_{k=0}^{\infty}(1-a^{-1}b)^k$ is absolutely convergent with norm $\|(a^{-1}b)^{-1}\| \leqq \sum_{k=0}^{\infty}\|1-a^{-1}b\|^k = (1-\|1-a^{-1}b\|)^{-1}$. Thus $b \in GL(A)$ with inverse $b^{-1} = (a^{-1}b)^{-1}a^{-1}$, and $\|b^{-1}\|^{-1} \geq \|(a^{-1}b)^{-1}\|^{-1}\|a^{-1}\|^{-1} \geq (1-\|1-a^{-1}b\|)\|a^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a-b\|$. For the last claim, put $c_t = (1-t)a + tb$ for $t \in [0,1]$. Then $\|a - c_t\| = t\|a-b\| < \|a^{-1}\|^{-1}$, therefeore $c_t \in GL(A)$ by the first part of the proof. □

2.2 Equivalence of projections

Definition 2.2.1. *The set of projections in a C^*-algebra A is denoted by $\mathcal{P}(A)$. A* partial isometry *is a $v \in A$ such that $v^*v \in \mathcal{P}(A)$. If v is a partial isometry, then vv^* is also a projection (exercise). v^*v is called the* support projection*, vv^* the* range projection *of v.*

If v is a partial isometry, put $p = v^*v$ and $q = vv^*$. then

$$v = qv = vp = qvp. \tag{2.2.13}$$

(exercise).

Lemma 2.2.2. *The following are equivalence relations on $\mathcal{P}(A)$:*

- *$p \sim q$ iff there exists $v \in A$ with $p = v^*v$ and $q = vv^*$ (*Murray-von Neumann equivalence*),*
- *$p \sim_u q$ iff there exists $u \in \mathcal{U}(A)$ with $q = upu^*$ (*unitary equivalence*).*

Proof. Transitivity of Murray-von Neumann: Let $p \sim q$ and $q \sim r$, and let v, w be partial isometries such that $p = v^*v$, $q = vv^* = w^*w$, $r = ww^*$. Put $z = wv$. Then $z^*z = v^*w^*wv = v^*qv = v^*v = p$, $zz^* = wvv^*w^* = wqw^* = ww^* = r$, i.e., $p \sim r$. The other claims are checked easily. □

Proposition 2.2.3. *Let $p, q\mathcal{P}(A)$, A unital. The following are equivalent:*

(i) $\exists u \in \mathcal{U}(\tilde{A}) : q = upu^*$,

(ii) $\exists u \in \mathcal{U}(A) : q = upu^*$,

(iii) $p \sim q$ *and* $1_A - p \sim 1_A - q$.

Proof. Let $f = 1_{\tilde{A}} - 1_A = (-1_A, 1)$. Then $\tilde{A} = A + \mathbb{C}f$ and $fa = af = 0\ \forall a \in A$.

(i) $\Longrightarrow$ (ii): Let $q = zpz^*$ for some $z \in \mathcal{U}(\tilde{A})$. Then $z = u + \alpha f$ for some $u \in A$ and $\alpha \in \mathbb{C}$. It is straightforward to show $u \in \mathcal{U}(A)$ and $q = upu^*$.

(ii) $\Longrightarrow$ (iii): Let $q = upu^*$ for $u \in \mathcal{U}(A)$. Put $v = up$ and $w = u(1_A - p)$. Then

$$v^*v = p, \quad vv^* = q, \quad w^*w = 1_A - p, \quad ww^* = 1_A - q. \tag{2.2.14}$$

(iii) $\Longrightarrow$ (i): Assume that there are partial isometries v, w satisfying (2.2.14). Then (2.2.13) gives by direct calculation $z := v+w+f \in \mathcal{U}(\tilde{A})$, and that $zpz^* = vpv^* = vv^* = q$. □

Note that one could prove (iii) $\Longrightarrow$ (ii) using the unitary $u = v + w \in \mathcal{U}(A)$.

Lemma 2.2.4. *Let A be a C^*-algebra, $p \in \mathcal{P}(A)$, and $a = a^* \in A$. Put $\delta = \|p - a\|$. Then*

$$\operatorname{sp}(a) \subseteq [-\delta, \delta] \cup [1 - \delta, 1 + \delta]. \tag{2.2.15}$$

Proof. We know $\operatorname{sp}(a) \in \mathbb{R}$ and $\operatorname{sp}(p) \in \{0, 1\}$. It suffices to show that for $t \in \mathbb{R}$ the assumption $\operatorname{dist}(t, \{0, 1\}) > \delta$ implies $t \notin \operatorname{sp}(a)$. Such a t is not in $\operatorname{sp}(p)$, i.e., $p - t1$ is invertible in $\tilde{A}$, and

$$\|(p - t1)^{-1}\| = \max(|-t|^{-1}, |1 - t|^{-1}) = d^{-1}. \tag{2.2.16}$$

(consider $p - t1$ as an element of $C(\operatorname{sp}(p)) \subseteq \mathbb{C}^2$.) Consequently,

$$\|(p - t1)^{-1}(a - t1) - 1\| = \|(p - t1)^{-1}(a - p)\| \leqq d^{-1}\delta < 1. \tag{2.2.17}$$

Thus $(p - t1)^{-1}(a - t1)$ is invertible, hence also $a - t1$ is invertible, i.e., $t \notin \operatorname{sp}(a)$. □

Proposition 2.2.5. *If $p, q \in \mathcal{P}(A)$, $\|p - q\| < 1$, then $p \sim_h q$.*

Proof. Put $a_t = (1 - t)p + tq$, $t \in [0, 1]$. Then $a_t = a_t^*$, $t \mapsto a_t$ is continuous, and

$$\min(\|a_t - p\|, \|a_t - q\|) \leqq \|p - q\|/2 < 1/2. \tag{2.2.18}$$

Thus by Lemma 2.2.4 $\operatorname{sp}(a_t) \subseteq K := [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$, with $\delta = \|p - q\|/2 < 1/2$, i.e., $a_t \in \Omega_K$ in the notation of Lemma 1.2.3. Then $f : K \to \mathbb{C}$, defined to be zero on $[-\delta, \delta]$ and one on $[1 - \delta, 1 + \delta]$, is continuous, and $f(a_t)$ is a projection for each $t \in [0, 1]$ because $f = f^2 = \bar{f}$. By Lemma 1.2.3, $t \mapsto f(a_t)$ is continuous, and $p = f(p) = f(a_0) \sim_h f(a_1) = f(q) = q$. □

Proposition 2.2.6. *Let A be a unital C^*-algebra, $a, b \in A$ selfadjoint. Suppose $b = zaz^{-1}$ for some invertible $z \in A$. Then $b = uau^*$, where $u \in \mathcal{U}(A)$ is the unitary in the polar decomposition $z = u|z|$ of z (see Remark 2.1.12).*

Proof. $b = zaz^{-1}$ is the same as $bz = za$, and also $z^*b = az^*$. Hence

$$|z|^2 a = z^* z a = z^* b z = a z^* z = a|z|^2, \tag{2.2.19}$$

a commutes with $|z|^2$. Thus a commutes with all elements of $C^*(1, |z|^2)$, in particular with $|z|^{-1}$. Therefore,

$$uau^* = z|z|^{-1}au^* = za|z|^{-1}u^* = bz|z|^{-1}u^* = buu^* = b. \tag{2.2.20}$$

□

Proposition 2.2.7. *Let A be a C^*-algebra, $p, q \in \mathcal{P}(A)$. Then $p \sim_h q$ in $\mathcal{P}(A)$ iff $\exists u \in \mathcal{U}_0(\tilde{A}) : q = upu^*$.*

Proof. Assume $q = upu^*$ for some $u \in \mathcal{U}_0(\tilde{A})$, and let $t \mapsto u_t$ be a continuous path in $\mathcal{U}_0(\tilde{A})$ connecting $1 (= 1_{\tilde{A}})$ and u. Then $t \mapsto u_t p u_t^*$ is a continuous path of projections in A (A is an ideal in $\tilde{A}$).

Conversely, if $p \sim_h q$, then there are $p = p_0, p_1, \ldots, p_n \in \mathcal{P}(A)$ such that $\|p_j - p_{j+1}\| < 1/2$ (the set $\{p_t \mid t \in [0, 1]\}$ is compact in the metric space $\mathcal{P}(A)$ and thus totally bounded, cf. [D-J73, 3.16]). Thus it is sufficient to consider only the case $\|p-q\| < 1/2$. The element $z := pq + (1-p)(1-q) \in \tilde{A}$ satisfies

$$pz = pq = zq \tag{2.2.21}$$

and $\|z-1\| = \|p(q-p)+(1-p)((1-q)-(1-p))\| \leqq 2\|p-q\| < 1$ (consider $2p-1$ under the isomorphism $C(\mathrm{sp}(p)) \to C^*(p,1)$), thus z is invertible and $z \sim_h 1$ by Proposition 2.1.13. If $z = u|z|$ is the unitary polar decomposition of z (Remark 2.1.12), then from formula (2.2.21) and Proposition 2.2.6 $p = uqu^*$. Eventually, it follows from Proposition 2.1.10 (ii) that $u \sim_h z \sim_h 1$ in $GL(\tilde{A})$, and from Proposition 2.1.10 (iii) that $u \in \mathcal{U}_0(\tilde{A})$. □

Proposition 2.2.8. *Let A be a C^*-algebra, $p, q \in \mathcal{P}(A)$.*

(i) $p \sim_h q \implies p \sim_u q$.

(ii) $p \sim_u q \implies p \sim q$.

Proof. (i): Immediate from Proposition 2.2.7.
(ii): If $upu^* = q$ for $u \in \mathcal{U}(\tilde{A})$, then $v = up \in A$, $v^*v = p$, and $vv^* = q$. □

Proposition 2.2.9. *Let A be a C^*-algebra, $p, q \in \mathcal{P}(A)$.*

(i) $p \sim q \implies \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ *in $M_2(A)$.*

(ii) $p \sim_u q \implies \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ *in $M_2(A)$.*

Proof. Let $v \in A$ such that $p = v^*v$, $q = vv^*$. Then (2.2.13) can be used to show that

$$u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}, \quad w = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \in \mathcal{U}(M_2(\tilde{A})). \tag{2.2.22}$$

Since

$$wu \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^*w^* = w \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} w^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.2.23}$$

on the other hand

$$wu = \begin{pmatrix} v + (1-q)(1-p) & (1-q)v^* \\ q(1-p) & 1-q+qv^* \end{pmatrix} \in \widetilde{M_2(A)}, \tag{2.2.24}$$

(i) is proved. Note that $\widetilde{M_2(A)}$ is considered as a unital subalgebra of $M_2(\tilde{A})$ via the map $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \alpha\right) \mapsto \begin{pmatrix} (a,\alpha) & (b,0) \\ (c,0) & (d,\alpha) \end{pmatrix}$, and that one has to check that wu, being a priori in $M_2(\tilde{A})$, is indeed in $\widetilde{M_2(A)}$.

(ii): The assumption means $q = upu^*$ for some $u \in \mathcal{U}(\tilde{A})$. By Lemma 2.1.6 there is a homotopy $t \mapsto w_t$ in $\mathcal{U}(M_2(\tilde{A})$ connecting $w_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with $w_0 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. Put $e_t = w_t \operatorname{diag}(p,0) w_t^*$. Then $e_t \in \mathcal{P}(M_2(A))$ (A is an ideal in $\tilde{A}$), $t \mapsto e_t$ is continuous, $e_0 = \operatorname{diag}(p,0)$, and $e_1 = \operatorname{diag}(q,0)$. □

Remark 2.2.10. Propositions 2.2.8 and 2.2.9 say that the three equivalence relations $\sim$, $\sim_u$ and $\sim_h$ are equivalent if one passes to matrix algebras. Otherwise, the implications $p \sim q \implies p \sim_u q$ and $p \sim_u q \implies p \sim_h q$ do not hold:

Let A be a unital C^*-algebra containing a non-unitary isometry s, i.e., $s^*s = 1 \neq ss^*$. Example: one-sided shift. Then s^*s and ss^* are projections, and by definition $s^*s \sim ss^*$. On the other hand, $1 - s^*s = 0 \nsim 1 - ss^* \neq 0$, because from $v^*v = 0$ follows $v = 0$ (C^*-property and thus also $vv^* = 0$. By Proposition 2.2.3 (iii), s^*s and ss^* cannot be unitarily equivalent.

Example of a unital C^*-algebra containing projections p, q with $p \sim_u q$ and $p \nsim_h q$: There exists a unital C^*-algebra B such that $M_2(B)$ contains $u \in \mathcal{U}(M_2(B)$ not being homotopic in $\mathcal{U}(M_2(B))$ to any $\mathrm{diag}(v, 1)$, $v \in \mathcal{U}(B)$. Then $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim_u u \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u^*$ in $M_2(B)$, but $p \nsim_h q$. Indeed, if one assumes $p \sim_h q$, then by Proposition 2.2.7 there is $w \in \mathcal{U}(M_2(B))$ such that $wqw^* = p$. Hence $(wu)p = p(wu)$, and (see exercise) $wu = \mathrm{diag}(a, b)$, with $a, b \in \mathcal{U}(B)$. From Lemma 2.1.6 and $w \in \mathcal{U}(M_2(B))$ we obtain $u \sim_h wu = \mathrm{diag}(a, b) \sim_h \mathrm{diag}(ab, 1)$, contradicting the original assumption about u. $\diamond$

2.3 Semigroups of projections

Definition 2.3.1. *Let A be a C^*-algebra, $n \in \mathbb{N}$. Put $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$ and $\mathcal{P}_\infty(A) = \cup_{n=1}^\infty \mathcal{P}_n(A)$ (disjoint union).*

Let $M_{m,n}(A)$ be the set of rectangular $m \times n$-matrices with entries from A. The adjoint of such a matrix is defined combining the matrix adjoint with the adjoint in A.

Definition 2.3.2. *Let $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$. Then $p \sim_0 q$ iff $\exists v \in M_{m,n}(A) : p = v^*v, q = vv^*$.*

$\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(A)$ and reduces for $m = n$ to the Murray-von Neumann equivalence on $\mathcal{P}(M_n(A))$.

Definition 2.3.3. *Define a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ by*

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}. \tag{2.3.25}$$

If $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$, then $p \oplus q \in \mathcal{P}_{n+m}(A)$.

Proposition 2.3.4. *Let A be a C^*-algebra, $p, q, r, p', q' \in \mathcal{P}_\infty(A)$.*

(i) $\forall n \in \mathbb{N} : p \sim_0 p \oplus 0_n$ (0_n the zero of $M_n(A)$),

(ii) if $p \sim_0 p'$ and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$,

(iii) $p \oplus q \sim_0 q \oplus p$,

(iv) if $p, q \in \mathcal{P}_n(A)$, $pq = 0$, then $p + q \in \mathcal{P}_n(A)$ and $p + q \sim_0 p \oplus q$,

(v) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.

Proof. (i): Let $m, n \in \mathbb{N}$, $p \in \mathcal{P}_m(A)$. Put $u_1 = \begin{pmatrix} p \\ 0 \end{pmatrix} \in M_{m+n,m}(A)$. Then $p = u_1^* u_1 \sim_0 u_1 u_1^* = p \oplus 0_n$.
(ii): If $p \sim_0 p'$ and $q \sim_0 q'$, then $\exists v, w : p = v^*v,\ p' = vv^*,\ q = w^*w,\ q' = ww^*$. Put $u_2 = \mathrm{diag}(v, w)$. Then $p \oplus q = u_2^* u_2 \sim_0 u_2 u_2^* = p' \oplus q'$.
(iii): Let $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$, and put $u_3 := \begin{pmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{pmatrix}$, with $0_{k,l}$ the zero of $M_{k,l}(A)$. Then $u_3 \in M_{n+m}(A)$, and $p \oplus q = u_3^* u_3 \sim_0 u_3 u_3^* = q \oplus p$.
(iv): If $pq = 0$ then $p + q$ is a projection (exercise). Put $u_4 = \begin{pmatrix} p \\ q \end{pmatrix} \in M_{2n,n}(A)$. Then $p + q = u_4^* u_4 \sim_0 u_4 u_4^* = p \oplus q$.
(v): trivial. □

Definition 2.3.5.

$$\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim_0 . \tag{2.3.26}$$

$[p]_{\mathcal{D}} \in \mathcal{D}(A)$ denotes the equivalence class of $p \in \mathcal{P}_\infty(A)$.

Lemma 2.3.6. *The formula*

$$[p]_{\mathcal{D}} + [q]_{\mathcal{D}} = [p \oplus q]_{\mathcal{D}} \tag{2.3.27}$$

defines a binary operation on $\mathcal{D}(A)$ making it an abelian semigroup.

Proof. This is immediate from Proposition 2.3.4.

2.4 Examples and Exercises

Exercise 2.4.1. Let $\varphi : A \to B$ be a surjective $*$-homomorphism of C^*-algebras. If $\varphi(a) = b$, then a is called lift of b.

(i) Any $b \in B$ has a lift $a \in A$ with $\|b\| = \|a\|$.

(ii) Any selfadjoint b has a selfadjoint lift a with $\|b\| = \|a\|$.

(iii) Any positive b has a positive lift a with $\|b\| = \|a\|$.

(iv) A normal element does not in general have a normal lift.

(v) A projection does not in general lift to a projection.

(vi) A unitary does not in general lift to a unitary.

(ii) For a lift x of b, also $a_0 := \frac{x+x^*}{2} = a_0^*$ is a lift of b. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} -\|b\| & t \leqq \|b\|, \\ t & -\|b\| \leqq t \leqq \|b\|, \\ \|b\| & t \geq \|b\|. \end{cases} \tag{2.4.28}$$

Put $a = f(a_0)$. Then $a = a^*$, $\mathrm{sp}(a) = \{f(t) \mid t \in \mathrm{sp}(a_0)\} \subseteq [-\|b\|, \|b\|]$ (by definition of f), and $\|a\| = r(a) \leqq r(b) = \|b\|$. Also, a is a lift of b, $\varphi(a) = \varphi(f(a_0)) = f(\varphi(a_0)) = f(b) = b$, because $f(t) = t$ for $t \in \mathrm{sp}(b)$. Finally, φ is norm-decreasing (as any $*$-homomorphism), thus also $\|b\| = \|\varphi(a)\| \leqq \|a\|$, hence $\|b\| = \|a\|$.

(i) For $b \in B$, $y := \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}$ is a self-adjoint element of $M_2(B)$ with $\|y\| = \|b\|$ ($\|y\|^2 = \|y^*y\| = \|\begin{pmatrix} bb^* & 0 \\ 0 & b^*b \end{pmatrix}\| = \max\{\|bb^*\|, \|b^*b\|\} = \|b\|^2$, using Exercice 10 of Chapter 1). By (ii), there is a self-adjoint lift $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ of y with $\|x\| = \|y\|$. $a = x_{12}$ is a lift of b, and by (1.3.22) $\|a\| \leqq \|x\| = \|y\| = \|b\|$. But also $\|b\| \leqq \|a\|$, thus $\|a\| = \|b\|$.

(iii) For a lift x of b, also $a_0 := (x^*x)^{1/2} \geq 0$ is a lift: $\varphi(a_0) = (\varphi(x^*)\varphi(x))^{1/2} = (b^*b)^{1/2} = b$. Put $a = f(a_0)$, with f from (2.4.28). Then a is normal, $\varphi(a) = b$ ($\varphi(a) = \varphi(f(a_0)) = f(\varphi(a_0)) = f(b) = b$), $\mathrm{sp}(a) \subseteq [0, \|b\|]$. Thus, $a \geq 0$, $\|a\| = \|b\|$.

(iv) Let s be the unilateral shift. Then $s^*s = 1$, $s^*s - ss^* = \mathrm{pr}_{e_0}$ is compact. Let $\pi : B(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}$ (Calkin-Algebra). Then $\pi(s)$ is normal ($\pi(\mathrm{pr}_{e_0}) = 0$), however, $\pi(s)$ has no lift to a normal operator: There is no normal operator N such that $s - N$ is compact.

(v) Let $A = C([0,1])$, $B = \mathbb{C} \oplus \mathbb{C}$, $\varphi(f) = (f(0), f(1))$. Then $q = (0,1) \in \mathcal{P}(\mathbb{C} \oplus \mathbb{C})$. However, there are no nontrivial projections in $C([0,1])$ ($\varphi(p) = q$ would mean $p(0) = 1$, $p(1) = 0$).

Exercise 2.4.2. Let A be a unital C^*-algebra,

$$a = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \in M_n(A).$$

Show: $a \in GL_n(A)$, $a \sim_h 1$ in $GL_n(A)$.

The first claim is immediate from Exercice 9 of Chapter 1. For the second claim, write $a = 1 + a_0$. Then $a_t = 1 + ta_0$ is a curve connecting a and 1 in $GL_n(A)$ (again by Exercice 9 of Chapter 1).

Exercise 2.4.3. Let A be a C^*-algebra, $p, q \in \mathcal{P}(A)$. Write $p \perp q$ if $pq = 0$. The following are equivalent:

(i) $p \perp q$,

(ii) $p + q \in \mathcal{P}(A)$,

(iii) $p + q \leqq 1$.

(i) $\implies$ (ii): $p + q$ is self-adjoint, and $(p+q)^2 = p^2 + pq + qp + q^2 = p + q$.

(ii) $\implies$ (iii): $1 - (p+q) = 1 - (p+q) - (p+q) + (p+q)^2 = (1 - (p+q))^2$.

(iii) $\implies$ (i): Use the general implication $a \leqq b \implies (c^*ac \leqq c^*bc,\ \forall c \in A)$ to conclude $p + q \leqq 1 \implies p(p+q)p \leqq p^2 = p \implies p + pqp \leqq p \implies pqp \leqq 0$. On the other hand, $pqp = pqqp \geq 0$, thus $pqqp = pqp = 0$, which is equivalent to $pq = qp = 0$.

More generally, for $p_1, \dots, p_n \in \mathcal{P}(A)$, the following are equivalent:

(i) $p_i \perp p_j$, for all $i \neq j$,

(ii) $p_1 + \dots + p_n \in \mathcal{P}(A)$,

(iii) $p_1 + \dots + p_n \leqq 1$.

Chapter 3

The K_0-Group for Unital C^*-Algebras

3.1 The Grothendieck Construction

Lemma 3.1.1. *Let $(S, +)$ be an abelian semigroup. Then the binary relation $\sim$ on $S \times S$ defined by*

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow \exists z \in S : x_1 + y_2 + z = x_2 + y_1 + z \tag{3.1.1}$$

is an equivalence relation.

Proof. The relation $\sim$ is clearly symmetric and reflexive. Transitivity: Let $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, i.e., $x_1 + y_2 + z = x_2 + y_1 + z,\ x_2 + y_3 + w = x_3 + y_2 + w$ for some $z, w \in S$. Then $x_1 + y_3 + (y_2 + z + w) = x_2 + y_1 + z + y_3 + w = x_3 + y_1 + (y_2 + z + w)$, i.e., $(x_1, y_1) \sim (x_3, y_3)$. □

Let $G(S) := (S \times S)/\sim$, and $< x, y >$ denote the class of (x, y).

Lemma 3.1.2. *The operation*

$$< x_1, y_1 > + < x_2, y_2 >=< x_1 + x_2, y_1 + y_2 > \tag{3.1.2}$$

is well-defined and yields an abelian group $(G(S), +)$. Inverse and zero are given by

$$- < x, y >=< y, x >, \quad 0 =< x, x > . \tag{3.1.3}$$

Proof. Straightforward. □

The group $(G(S), +)$ is called the *Grothendieck group* of S.

For $y \in S$, there is a map $\gamma : S \to G(S),\ x \mapsto < x + y, y >$ (Grothendieck map). It is independent of y and a homomorphism of abelian semigroups (additive).

Definition 3.1.3. *An abelian semigroup $(S, +)$ is said to have the* cancellation property *if from $x + z = y + z$ follows $x = y$ $(x, y, z \in S)$.*

Proposition 3.1.4. *Let $(S, +)$ be an abelian semigroup.*

(i) If H is an abelian group, $\varphi : S \to H$ additive, then there is a unique group homomorphism $\psi : G(S) \to H$ such that $\varphi = \psi \circ \gamma$ (universal property).

(ii) *If $\varphi : S \to T$ is a homomorphism (additive map) of abelian semigroups, then there is a unique group homomorphism $G(\varphi) : G(S) \to G(T)$ such that $\gamma_T \circ \varphi = G(\varphi) \circ \gamma_S$ (functoriality).*

(iii) $G(S) = \{\gamma_S(x) - \gamma_S(y) \mid x, y \in S\}$.

(iv) *For $x, y \in S$, $\gamma_S(x) = \gamma_S(y)$ iff $\exists z \in S$ such that $x + z = y + z$.*

(v) *Let $(H, +)$ be an abelian group, $\emptyset \neq S \subseteq H$. If S is closed under addition, then $(S, +)$ is an abelian semigroup with the cancellation property, and $G(S)$ is isomorphic to the subgroup H_0 generated by S, with $H_0 = \{x - y \mid x, y \in S\}$.*

(vi) *The map $\gamma_S : S \to G(S)$ is injective iff S has the cancellation property.*

Proof. (iii): For $< x, y > \in G(S)$ we have $< x, y > = < x, y > + < x+y, x+y > = < x+x+y, y+x+y > = < x+y, y > + < x, x+y > = < x+y, y > - < x+y, x > = \gamma_S(x) - \gamma_S(y)$.
(iv): If $x + z = y + z$, then by additivity of γ_S $\gamma_S(x) + \gamma_S(z) = \gamma_S(y) + \gamma_S(z)$, hence, since $G(S)$ is a group, $\gamma_S(x) = \gamma_S(y)$. Conversely, let $\gamma_S(x) = \gamma_S(y)$, in particular $< x + y, y > = < y + x, x >$, i.e., $\exists w \in S : (x + y) + x + w = (y + x) + y + w$. Thus $x + z = y + z$, with $z = x + y + w$.
(v): immediate from (iv).
(i): If ψ exists, it has to satisfy $\psi(< x, y >) = \varphi(x) - \varphi(y)$, in order to have $\psi \circ \gamma_S = \varphi$. Then additivity of ψ follows from additivity of φ, and uniqueness follows from (iii). To see that ψ exists, assume $< x_1, y_1 > = < x_2, y_2 >$, i.e., $\exists z \in S : x_1 + y_2 + z = x_2 + y_1 + z$. Then $\varphi(x_1) + \varphi(y_2) + \varphi(z) = \varphi(x_2) + \varphi(y_1) + \varphi(z)$ in H, by addivity of φ. Since H is a group, we have $\varphi(x_1) - \varphi(y_1) = \varphi(x_2) - \varphi(y_2)$, which shows that ψ is well-defined by $\psi(< x, y >) = \varphi(x) - \varphi(y)$.
(ii): $\gamma_T \circ \varphi : S \to G(T)$ is an additive map into the group $G(T)$, thus by (i) there is a unique group homomorphism $G(\varphi) : G(S) \to G(T)$ such that $\gamma_T \circ \varphi = G(\varphi) \circ \gamma_S$.
(vi): Any non-empty subset of an abelian group that is closed under addition is an abelian semigroup with cancellation property. The inclusion $\iota : S \to H$ is additive and gives by (i) rise to a group homomorphism $\psi : G(S) \to H$ such that $\psi \circ \gamma_S = \iota$, i.e., $\psi(\gamma_S(x)) = x$ for $x \in S$. By (iii), $\psi(G(S)) = \{x - y \mid x, y \in S\} = H_0$. If $\psi(\gamma_S(x) - \gamma_S(y)) = 0$, then $x = y$ and so $\gamma_S(x) - \gamma_S(y) = 0$, i.e., ψ is injective. □

Examples:

- $(\mathbb{N}, +)$ is an abelian semigroup with cancellation property, whose Grothendieck group is isomorphic to $(\mathbb{Z}, +)$.
- Let $(\mathbb{N} \cup \{\infty\}, +)$ be the abelian semigroup whose addition is defined by the usual addition in $\mathbb{N}$ and by $x + \infty = \infty = \infty + \infty$. Then $(\mathbb{N} \cup \{\infty\}, +)$ does not have the cancellation property, and the corresponding Grothendieck group is $\{0\}$. Indeed, from $x + \infty = \infty + \infty$ it does not follow that $x = \infty$, and $< x, y > = < x, x >$ for any $x, y \in \mathbb{N} \cup \{\infty\}$, because $x + x + \infty = y + x + \infty = \infty$ $\forall x, y \in \mathbb{N} \cup \{\infty\}$.

3.2 Definition of the K_0-group of a unital C^*-algebra

Definition 3.2.1. *Let A be a unital C^*-algebra. The $K_0(A)$ group is defined as the Grothendieck group of the semigroup $\mathcal{D}(A)$:*

$$K_0(A) \stackrel{def}{=} G(\mathcal{D}(A)).$$

We also define a map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$ by $[p]_0 = \gamma_{\mathcal{D}(A)}([p]_{\mathcal{D}})$ for $p \in \mathcal{P}_\infty(A)$.

Remark 3.2.2. Formally, this definition could be made for non-unital C^*-algebras as well, but it would not be appropriate, since the resulting K_0-functor would not be half-exact.

3.2.1 Portrait of K_0 — the unital case

We define a binary relation $\sim_s$ on $\mathcal{P}_\infty(A)$ as follows: $p \sim_s q$ iff there exists an $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. The relation $\sim_s$ is called *stable equivalence* and it is easy to verify that it is indeed an equivalence relation. Furthermore, the relation can be defined equivalently as $p \sim_s q$ iff $p \oplus 1_n \sim_0 q \oplus 1_n$ for some positive integer n. Indeed, if $p \oplus r \sim_0 q \oplus r$ for $r \in \mathcal{P}_n(A)$, then $p \oplus 1_n \sim_0 p \oplus r \oplus (1_n - r) \sim_0 q \oplus r \oplus (1_n - r) \sim_0 q \oplus 1_n$.

Proposition 3.2.3. *Let A be a unital C^*-algebra. Then*

(i) $K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A),\ n \in \mathbb{N}\}$,

(ii) $[p]_0 + [q]_0 = [p \oplus q]_0$ *for* $p, q \in \mathcal{P}_\infty(A)$, *and if p and q are orthogonal then* $[p]_0 + [q]_0 = [p + q]_0$,

(iii) $[0_A]_0 = 0$,

(iv) if $p, q \in \mathcal{P}_n(A)$ and $p \sim_h q$ in $\mathcal{P}_n(A)$ then $[p]_0 = [q]_0$,

(v) $[p]_0 = [q]_0$ *iff $p \sim_s q$ for $p, q \in \mathcal{P}_\infty(A)$.*

Proof. Straightforward. As an example, we only verify (v). If $[p]_0 = [q]_0$ then by part (iv) of Proposition [3.1.4] there is an $r \in \mathcal{P}_\infty(A)$ such that $[p]_{\mathcal{D}} + [r]_{\mathcal{D}} = [q]_{\mathcal{D}} + [r]_{\mathcal{D}}$. Hence $[p \oplus r]_{\mathcal{D}} = [q \oplus r]_{\mathcal{D}}$. Thus $p \oplus r \sim_0 q \oplus r$ and consequently $p \sim_s r$. Conversely, if $p \sim_s q$ then there is $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. Then $[p]_0 + [r]_0 = [q]_0 + [r]_0$ by part (ii) above and hence $[p]_0 = [q]_0$ since $K_0(A)$ is a group. ☐

3.2.2 The universal property of K_0

Proposition 3.2.4. *Let A be a unital C^*-algebra, let G be an abelian group, and let $nu : \mathcal{P}_\infty(A) \to G$ be a map satisfying the following conditions:*

(i) $\nu(p \oplus q) = \nu(p) + \nu(q)$,

(ii) $\nu(0_A) = 0$,

(iii) if $p \sim_h q$ in $\mathcal{P}_n(A)$ then $\nu(p) = \nu(q)$.

Then there exists a unique homomorphism $K_0(A) \to G$ such that the diagram

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[d, "{[\cdot]_0}"'] \arrow[dr] & \\ K_0(A) \arrow[r] & G \end{tikzcd} \tag{3.2.4}$$

is commutative.

Proof. At first we observe that if $p, q \in \mathcal{P}_\infty(A)$ and $p \sim_0 q$ then $\nu(p) = \nu(q)$. Indeed, let $p \in \mathcal{P}_k(A)$, $q \in \mathcal{P}_l(A)$. Take $n \geq \max\{k, l\}$ and put $p' = p \oplus 0_{n-k}$ and $q' = q \oplus 0_{n-l}$. We have $p' \sim_0 p \sim_0 q \sim_0 q'$ and hence $p' \sim q'$. Thus $p' \oplus 0_{3n} \sim_h q' \oplus 0_{3n}$ in $\mathcal{P}_{4n}(A)$ by Proposition 2.2.9. Hence

$$\nu(p) = \nu(p) + (4n-k)\nu(0) = \nu(p' \oplus 0_{3n}) = \nu(q' \oplus 0_{3n}) = \nu(q),$$

as required. Consequently, the map $\mathcal{D}(A) \to G$, $[p]_\mathcal{D} \mapsto \nu(p)$ is well-defined. Clearly, this map is additive. The rest follows from the univesal property of the Grothendieck construction (part (i) of Proposition 3.1.4). □

3.2.3 Functoriality

Now we observe that K_0 is a covariant functor from the category of unital C^*-algebras with (not necessarily unital) $*$-homomorphisms to the category of abelian groups.

Let $\varphi : A \to B$ be a (not necessarily unital) $*$-homomorphism between unital C^*-algebras. For each n it extends to a $*$-homomorphism $\varphi_n : M_n(A) \to M_n(B)$, and this yields a map $\varphi : \mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)$. Define $\nu : \mathcal{P}_\infty(A) \to K_0(B)$ by $\nu(p) = [\varphi(p)]_0$. Then ν satisfies the conditions of Proposition 3.2.4. Thus, there is a homomorphsm $K_0(\varphi) : K_0(A) \to K_0(B)$ such that $K_0(\varphi)([p]_0) = [\varphi(p)]_0$. That is, we have a commutative diagram

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[r, "\varphi"] \arrow[d, "{[\cdot]_0}"] & \mathcal{P}_\infty(B) \arrow[d, "{[\cdot]_0}"] \\ K_0(A) \arrow[r, "K_0(\varphi)"] & K_0(B) \end{tikzcd} \tag{3.2.5}$$

Proposition 3.2.5. *Let $\varphi : A \to B$, $\psi : B \to C$ be $*$-homomorphisms between unital C^*-algebras. Then*

(i) $K_0(id_A) = id_{K_0(A)}$,

(ii) $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$.

Proof. By definition, (i) and (ii) hold when applied to $[p]_0$, $p \in \mathcal{P}_\infty(A)$. Then use part (i) of Proposition 3.2.3. □

3.2.4 Homotopy invariance

Let A, B be C^*-algebras. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are *homotopic* $\varphi \sim_h \psi$ if there exist $*$-homomorphisms $\varphi_t : A \to B$ for $t \in [0,1]$ such that $\varphi_0 = \varphi$, $\varphi_1 = \psi$, and the map $[0,1] \ni t \mapsto \varphi_t(a) \in B$ is norm contnuous for each $a \in A$.

C^*-algebras A and B are *homotopy equivalent* if there exist $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \varphi \sim_h \mathrm{id}_A$ and $\varphi \circ \psi \sim_h \mathrm{id}_B$. In such a case we write $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$.

Proposition 3.2.6. *Let A, B be unital C^*-algebras.*

(i) If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms then $K_0(\varphi) = K_0(\psi)$.

(ii) If A is homotopy equivalent via $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ then $K_0(\varphi) : K_0(A) \to K_0(\psi)$ is an isomorphism with $K_0(\varphi)^{-1} = K_0(\psi)$.

Proof. Part (i) follows from Proposition 3.2.3. Part (ii) follows from part (i) and functoriality of K_0 (Proposition 3.2.5). $\square$

3.3 Examples and Exercises

EXAMPLE 3.3.1. $K_0(\mathbb{C}) \cong \mathbb{Z}$. Indeed, $\mathcal{D}(\mathbb{C}) \cong (\mathbb{Z}_+, +)$ and the Grothendieck group of $\mathbb{Z}_+$ is $\mathbb{Z}$.

EXAMPLE 3.3.2. If $\mathcal{H}$ is an infinite dimensional Hilbert space then $K_0(\mathcal{B}(\mathcal{H})) = 0$. Indeed, if $\mathcal{H}$ is separable then $\mathcal{D}(\mathcal{B}(\mathcal{H})) \cong \mathbb{Z}_+ \cup \{\infty\}$ with the addition in $\mathbb{Z}_+$ extended by $m + \infty = \infty + m = \infty + \infty = \infty$. The Grothendieck group of this semigroup is 0. The non-separable case is handled similarly.

EXERCISE 3.3.3. If X is a *contractible* compact, Hausdorff space then $K_0(C(X)) \cong \mathbb{Z}$. Hint: recall that X is contractible if there exists a point $x_0 \in X$ and a continuous map $\alpha : [0,1] \times X \to X$ such that $\alpha(0,x) = x$ and $\alpha(1,x) = x_0$ for all $x \in X$, and use Proposition 3.2.6 and Example 3.3.1.

EXAMPLE 3.3.4 (TRACES). Let A be a unital C^*-algebra. A bounded linear functional $\tau : A \to \mathbb{C}$ is a *trace* if $\tau(ab) = \tau(ba)$ for all $a, b \in A$. Hence $\tau(p) = \tau(q)$ if p, q are Murray-von Neumann equivalent projections. A trace τ is *positive* if $\tau(a) \geq 0$ for all $a \geq 0$. It is a *tracial state* if it is positive of norm 1.

A trace τ extends to a trace τ_n on $M_n(\mathbb{C})$ by $\tau_n([a_{i,j}]) = \sum_{i=1}^n \tau(a_i)$. Thus τ gives rise to a function $\tau : \mathcal{P}_\infty(A) \to \mathbb{C}$. By the universal property of K_0 this yields a group homomorphism $K_0(\tau) : K_0(A) \to \mathbb{C}$ such that $K_0(\tau)([p]_0) = \tau(p)$. If τ is positive then $K_0(\tau) : K_0(A) \to \mathbb{R}$ and $K_0([p]_0) \in \mathbb{R}_+$ for $p \in \mathcal{P}_\infty(A)$.

EXERCISE 3.3.5. If $n \in \mathbb{Z}_+$ then $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$, and the class of a minimal projection is a generator. In fact, let Tr be the standard matrix trace. Then $K_0(\mathrm{Tr}) : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ is an isomorphism.

EXERCISE 3.3.6. Let X be a connected, compact Hausdorff space. Show that there exists a surjective homomorphism

$$\dim : K_0(C(X)) \to \mathbb{Z}$$

such that $\dim([p]_0) = \mathrm{Tr}(p(x))$.

To this end, identify $M_n(C(X))$ with $C(X, M_n(\mathbb{C}))$. For each $x \in X$ the evaluation at x is a positive trace and hence, by Example 3.3.4 gives rise to a homomorphism from $K_0(C(X))$ to $\mathbb{R}$. If $p \in \mathcal{P}_\infty(C(X))$ then the function $x \mapsto \mathrm{Tr}(p(x)) \in \mathbb{Z}$ is continuous and locally constant, hence constant since X is connected. Finally, the homomorhism is surjective since $\dim([1]_0) = 1$.

EXERCISE 3.3.7. Let X be a compact Hausdorff space.

(1) By generalizing Exercise 3.3.6, show that there exists a surjective group homomorphism

$$\dim : K_0(C(X)) \to C(X, \mathbb{Z})$$

such that $\dim([p]_0)(x) = \mathrm{Tr}(p(x))$.

(2) Given $p \in \mathcal{P}_n(C(X))$ and $q \in \mathcal{P}_m(C(X))$ show that $\dim([p]_0) = \dim([q]_0)$ iff for each $x \in X$ there exists $v_x \in M_{m,n}(C(X))$ such that $v_x^* v_x = p(x)$ and $v_x v_x^* = q(x)$. Note that in general one cannot choose v_x so that the map $x \mapsto v_x$ be continuous.

(3) Show that if X is *totally disconnected* then the map dim is an isomorphism.

Recall that a space is totally disconnected if it has a basis for topology consisting of sets which are simultaneously open and closed. To prove the claim it sufficies (in view of part (1)) to show that dim is injective. To this end use part (ii) and total disconnectedness of X to find a partition of X into open and closed subsets $X = X_1 \cup X_2 \cup \ldots \cup X_k$ and rectangular matrices $v_1, v_2, \ldots, v_k$ over $C(X)$ such that $||v_i^* v_i - p(x)|| < 1$ and $||v_i v_i^* - q(x)|| < 1$ for all $x \in X_i$. From this deduce that $p \sim_0 q$.

EXERCISE 3.3.8. Let A be a unital C^*-algebra and let $\tau : A \to \mathbb{C}$ be a bounded linear functional. Show that the following conditions are equivalent:

(i) $\tau(ab) = \tau(ba)$ for all $a, b \in A$,

(ii) $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$,

(iii) $\tau(uyu^*) = \tau(y)$ for all $y \in A$ and all unitary $u \in A$.

(ii)$\Rightarrow$(iii) Suppose (ii) holds. At first consider $a \geq 0$ and set $x = u|a|^{1/2}$. Then $\tau(uau^*) = \tau(xx^*) = \tau(x^*x) = \tau(a)$. Then use the fact that every element of a C^*-algebra can be written as a linear combination of four positive elements.

(iii)$\Rightarrow$(i) Suppose (iii) holds. If $b \in A$ and u is a unitary in A then $\tau(ub) = \tau(u(bu)u^*) = \tau(bu)$. Then use the fact that every element of a unital C^*-algebra may be written as a linear combination of four untaries.

EXAMPLE 3.3.9. Let Γ be a countable discrete group with infinite conjugacy classes (an ICC group). Let $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be its left regular representation. Let $W^*(\Gamma)$ be the closure of the linear span of $\lambda(\Gamma)$ in the strong operator topology (that is, in the topology of pointwise convergence). It can be shown that there exists a unique tracial state τ on $W^*(\Gamma)$, and that this trace has the following properties:

(i) Two projections p, q are Murray-von Neumann equivalent in $W^*(\Gamma)$ iff $\tau(p) = \tau(q)$.

(ii) $\{\tau(p) : p \in \mathcal{P}(W^*(\Gamma))\} = [0, 1]$.

Deduce that $K_0(W^*(\Gamma) \cong (\mathbb{R}, +)$. The conslusion of this example remains valid if $W^*(\Gamma)$ is replaced by any factor von Neumann algebra of type II_1.

Example 3.3.10 (Matrix stability of K_0). Let A be a unital C^*-algebra and let n be a positive integer. Then

$$K_0(A) \cong K_0(M_n(A)).$$

More specifically, the $*$-homomorphism $\varphi : A \to M_n(A)$, $a \mapsto \mathrm{diag}(a, 0_{n-1})$ induces an isomorphism $K_0(\varphi) : K_0(A) \to K_0(M_n(A))$.

Indeed, we construct inverse to $K_0(\varphi)$, as follows. For each k let $\gamma_k : M_k(M_n(A)) \to M_{kn}(A)$ be the isomorphism which "erases parentheses". Define $\gamma : \mathcal{P}_\infty(M_n(A)) \to K_0(A)$ by $\gamma(p) = [\gamma_k(p)]_0$ for $p \in \mathcal{P}_k(M_n(A))$. The universal property of K_0 applied to γ yields a homomorphism $\alpha : K_0(M_n(A)) \to K_0(A)$ such that $\alpha([p]_0) = [\gamma(p)]_0$.We claim that $\alpha = K_0(\varphi)^{-1}$. To this end it sufficies to show that

(i) $\varphi_{kn}(\gamma_k(p)) \sim_0 p$ in $\mathcal{P}_\infty(M_n(A))$ for $p \in \mathcal{P}_k(M_n(A))$, and

(i) $\gamma_k(\varphi_k(p)) \sim_0 p$ in $\mathcal{P}(A)$ for $p \in \mathcal{P}_k(A)$.

Proof of (i). exercise.

Proof of (ii). Let $e_1, \ldots, e_{kn}$ be the standard basis in $\mathbb{C}^{kn}$ and let u be a permutation unitary such that $ue_i = e_{n(i-1)+1}$ for $i = 1, 2, \ldots, k$. Then $p \sim_0 p \oplus 0_{(n-1)k} = u^*\gamma_k(\varphi_k(p))u$ for all projections p in $\mathcal{P}_k(A)$.

Exercise 3.3.11. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are *orthogonal* if $\varphi(A)\psi(B) = \{0\}$. Show that if φ and ψ are orthogonal then $\varphi+\psi$ is a $*$-homomorphism and $K_0(\varphi+\psi) = K_0(\varphi) + K_0(\psi)$.

Exercise 3.3.12 (Cuntz algebras). Let $\mathcal{H}$ be a Hilbert space, let n be a positive integer bigger than 1, and let $S_1, \ldots, S_n$ be isometries on $\mathcal{H}$ whose range projections add up to the identity. Let $C^*(S_1, \ldots, S_n)$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\{S_1, \ldots, S_n\}$. It was proved by Cuntz in [C-J77] that this C^*-algebra is independent of the choice of such isometries. That is, if $T_1, \ldots, T_n$ is another family of isometries whose range projections add up to the identity then there is a $*$-isomorphism $\varphi : C^*(S_1, \ldots S_n) \to C^*(T_1, \ldots, T_n)$ such that $\varphi(S_j) = T_j$ for $j = 1, \ldots, n$. Thus defined C^*-algebra is denoted $\mathcal{O}_n$ and called *Cuntz algebra*. It is a simple, unital, separable C^*-algebra. Alternatively, $\mathcal{O}_n$ may be defined as the universal C^*-algebra generated by elements $S_1, \ldots, S_n$ subject to the relations:

(i) $S_i^*S_i = I$ for $i = 1, \ldots, n$,

(ii) $\sum_{i=1}^n S_iS_i^* = I$.

(1) Let u be a unitary in $\mathcal{O}_n$. There exists a unique unit preserving injective $*$-homomorphism $\lambda_u : \mathcal{O}_n \to \mathcal{O}_n$ (i.e. an *endomorphism* of $\mathcal{O}_n$) such that $\lambda_u(S_j) = uS_j$ for $j = 1, \ldots, n$. Moreover, if φ is an endomorphism of $\mathcal{O}_n$ then $\varphi = \lambda_u$ with $u = \sum_{i=1}^n \varphi(S_i)S_i^*$.

(2) Let σ be an endomorphism of $\mathcal{O}_n$ such that $\sigma(x) = \sum_{i=1}^n s_ixs_i^*$ (the *shift* endomorphism). Then $K_0(\sigma) : K_0(\mathcal{O}_n) \to K_0(\mathcal{O}_n)$ is the multiplication by n, that is $K_0(\sigma)(g) = ng$ for all $g \in K_0(\mathcal{O}_n)$. Hint: Use Exercise 3.3.11 and the following fact. If v is an isometry in a unital C^*-algebra A then the map $\mu : A \to A$, $\mu(x) = vxv^*$ is a $*$-homomorphism and

$K_0(\mu) = \mathrm{id}$. For the latter observe that $\mu_k : M_k(A) \to M_k(A)$ is given by $\mu_k(y) = v_k y v_k^*$, where $v_k = \mathrm{diag}(v, \dots, v)$.

(3) Let w be a unitary in $\mathcal{O}_n$ such that $\sigma = \lambda_w$. Then $w \sim_h 1$ in $\mathcal{U}(\mathcal{O}_n)$ and hence $\sigma \sim_h \mathrm{id}$. Consequently, $K_0(\sigma) = \mathrm{id}_{K_0(\mathcal{O}_n)}$. Hint: Note that w belongs to $M = \mathrm{span}\{S_i S_j S_k^* S_m^*\}$, and that M is a C^*-subalgebra of $\mathcal{O}_n$ isomorphic to $M_{n^2}(\mathbb{C})$.

(4) Combining (2) and (3) we get $(n-1)K_0(\mathcal{O}_n) = 0$. Thus, in particular, $K_0(\mathcal{O}_2) = 0$. In fact, Cuntz showed in [C-J81] that $K_0(\mathcal{O}_n) \cong \mathbb{Z}_{n-1}$ for all $n = 2, 3, \dots$

Exercise 3.3.13 (Properly infinite algebras). Let A be a unital C^*-algebra. A is called *properly infinite* if there exist two projections e, f in A such that $ef = 0$ and $1 \sim e \sim f$. For example, Cuntz algebras are properly infinite. For the reminder of this exercise assume A is properly infinite.

(1) A contains isometries S_1, S_2 whose range projections are orthogonal.

(2) A contains an infinite sequence $\{t_j\}$ of isometries with mutually orthogonal ranges. Hint: take $S_2^k S_1$ for $k = 0, 1, 2, \dots$

(3) For each natural number n let v_n be an element of $M_{1,n}(A)$ with entries $t_1, \dots, t_n$. Then $v_n^* v_n = 1_n$ and for $p \in \mathcal{P}_n(A)$ we have $p \sim_0 v_n p v_n^*$, with $v_n p v_n^*$ a projection in A.

(4) Let p, q be projections in A. Set

$$r = t_1 p t_1^* + t_2(1-q)t_2^* + t_3(1 - t_1 t_1^* - t_2 t_2^*)t_3^*.$$

Then r is a projection in A and $[r]_0 = [p]_0 - [q]_0$.

Conclude that

$$K_0(A) = \{[p]_0 : p \in \mathcal{P}(A)\}.$$

Exercise 3.3.14. If A is a separable, unital C^*-algebra then $K_0(A)$ is countable.

Exercise 3.3.15. Show that condition (iii) of Proposition 3.2.4 may be replaced by any of the following three conditions:

(i) $\forall n \forall p, q \in \mathcal{P}_n(A)$ if $p \sim_u q$ then $\nu(p) = \nu(q)$,

(ii) $\forall p, q \in \mathcal{P}_\infty(A)$ if $p \sim_0 q$ then $\nu(p) = \nu(q)$,

(iii) $\forall p, q \in \mathcal{P}_\infty(A)$ if $p \sim_s q$ then $\nu(p) = \nu(q)$.

Exercise 3.3.16. Let A be a unital C^*-algebra and let $a \in A$ be such that $a \geq 0$ and $||a|| \leqq 1$.

(1) Show that

$$p = \begin{pmatrix} a & \sqrt{a - a^2} \\ \sqrt{a - a^2} & 1 - a \end{pmatrix}$$

is a projection in $M_2(A)$.

(2) Show that $p \sim \mathrm{diag}(1, 0)$ in $M_2(A)$. Hint: consider

$$v = \begin{pmatrix} \sqrt{a} & \sqrt{1-a} \\ 0 & 0 \end{pmatrix}$$

(3) Let B be another unital C^*-algebra and let $\varphi : A \to B$ be a unit preserving, surjective $*$-homomorphism. Let q be a projection in B. Show that there is $a \geq 0$ in A such that $||a|| \leqq 1$ and $\varphi(a) = q$. Then use this a to define p as in (1) above and show that

$$\varphi(p) = \begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix}$$

Exercise 3.3.17 (Partial isometries). Show that for an element S of a C^*-algebra the following conditions are equivalent:

(i) S^*S is a projection,

(ii) SS^* is a projection,

(iii) $SS^*S = S$.

(i)$\Rightarrow$(iii) Show $(SS^*S - S)^*(SS^*S - S) = 0$.

An element S satisfying these conditions is called *partial isometry.*

Exercise 3.3.18. Let A be a unital C^*-algebra, a, b two elements of A, and p, q two projections in A. Show the following.

(i) $abb^*a^* \leqq ||b||^2 aa^*$.

(ii) $p \leqq q$ iff $pq = p$.

(i) Since $||b||^2 - bb^* \geq 0$ there is $x \in A$ such that $||b||^2 - bb^* = xx^*$. Thus

$$||b||^2 aa^* - abb^*a^* = a(||b||^2 - bb^*)a^* = axx^*a^* = (ax)(ax)^* \geq 0.$$

(ii) If $p \leqq q$ then $pqp - p = p(q-p)p \geq 0$, and hence $pqp \geq p$. But $pqp \leqq ||q||p = p$ (by part (i)). Thus $pqp = p$. Hence

$$(pq - p)(pq - p)^* = (pq - p)(qp - p) = pqp - pqp - pqp + p = 0$$

and consequently $pq - p = 0$.

Exercise 3.3.19. Let A be a unital C^*-algebra. Then the exact sequence

$$0 \longrightarrow A \xrightarrow{\imath} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

is split exact, with a splitting map $\lambda : \mathbb{C} \longrightarrow \tilde{A}$, and induces a split exact sequence

$$0 \longrightarrow K_0(A) \xrightarrow{K_0(\imath)} K_0(\tilde{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \longrightarrow 0,$$

with a splitting map $K_0(\lambda) : K_0(\mathbb{C}) \longrightarrow K_0(\tilde{A})$.

Hint: Let $f = 1_{\tilde{A}} - 1_A$, a projection such that $\tilde{A} = A \oplus \mathbb{C}f$ (direct sum of C^*-algebras). Let μ be the natural surjection from $\tilde{A}$ onto A and let $\lambda' : \mathbb{C} \to \tilde{A}$ be defined by $\lambda'(t) = tf$. Then we have the following identities: $\mathrm{id}_A = \mu \circ \imath$, $\pi \circ \imath = 0$, $\pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$, $\mathrm{id}_{\tilde{A}} = \imath \circ \mu + \lambda' \circ \pi$, and the maps $\imath \circ \mu$ and $\lambda' \circ \pi$ are orthogonal to one another (see Exercise 3.3.11). The claim follows from these identities and functoriality of K_0.

Example 3.3.20 (Algebraic definition of K_0). Let R be a unital ring. Recall that $e \in R$ is an *idempotent* if $e^2 = e$. We define $\mathcal{I}(R) = \{e \in R : e^2 = 2\}$, $\mathcal{I}_n(R) = \mathcal{I}(M_n(R))$, $\mathcal{I}_\infty(R) = \bigcup_{n=1}^{\infty} \mathcal{I}_n(R)$. We define a relation $\approx_0$ in $\mathcal{I}_\infty(R)$ as follows. If $e \in \mathcal{I}_n(R)$ and $f \in \mathcal{I}_m(R)$ then $e \approx_0 f$ iff there exist $a \in M_{n,m}(R)$ and $b \in M_{m,n}(R)$ such that $e = ab$ and $f = ba$. If this is the case then taking $a' = aba$ and $b = bab$ we may assume that a, b satisfy $aba = a$ and $bab = b$ (this we always assume in what follows). Claim: $\approx_0$ is an equivalence relation. For transitivity, let $e \approx_0 f$, $f \approx_0 g$ be idempotents and let c, d, x, y be matrices such that $e = cd$, $f = dc$, $f = xy$, $g = yx$. Then $(cx)(yd) = e$ and $(yd)(cx) = g$, and hence $e \approx_0 g$. Set $V(R) = \mathcal{I}_\infty(R)/_{\approx_0}$, anddenote the class of e by $[e]_V$.

Define a binary operation $\oplus$ on $\mathcal{I}_\infty(R)$ by $e \oplus f = \operatorname{diag}(e, f)$. This operation is well-defined on equivalence classes of $\approx_0$ and turns $V(R)$ into an abelian semigroup. Define$K_0(R)$ as the Grothendieck group of $(V(R), \oplus)$.

Now suppose A is a unital C^*-algebra. We show hat the two definitions of $K_0(A)$ coincide. In fact, the two semigoups $\mathcal{D}(A)$ and $V(A)$ are isomorphic. The proof proceeds in three steps.

(1) If $e \in \mathcal{I}_\infty(A)$ ten there exists a $p \in \mathcal{P}_\infty(A)$ such that $e \approx_0 p$. Indeed, let $e \in M_n(A)$, and set $h = 1_n + (e - e^*)(e - e^*)^*$. Then h is invertible and satisfies $eh = ee^*e = he$. Then $p = ee^*h^{-1}$ is a projection. Since $ep = p$ and $pe = e$, $e \approx_0 p$.

(2) If $p, q \in \mathcal{P}_\infty(A)$ then $p \sim_0 q$ iff $p \approx_0 q$. Indeed, suppose (after diagonalling adding zeros, if necessary) $p, q \in M_n(A)$ and $a, b \in M_n(A)$ are such that $p = ab$, $q = ba$, $a = aba$ (hence $a = paq$), $b = bab$ (hence $b = qbp$). Then $b^*b = (bab)^*b = (ab)^*b^*b = pb^*b$. It follows that b^*b belongs to the corner C^*-algebra $pM_n(A)p$. Since $p = (ab)^*ab = b^*(a^*a)b \leqq ||a||^2 b^*b$, the element b^*b is invertible in $pM_n(A)p$. Set $v = bp(b^*b)^{-1/2}$. We have $p = v^*v$ (straightforward calculation). In particular, v is a partial isometry. In fact, $b = v|b|$ is the polar decomposition of b in $M_n(A)$. Hence $bb^* = vb^*bv^*$.

It now suffices to show that $q = vv^*$. First note that $v = qv$ (directly follows from the definition of v). Thus

$$vv^* = qvv^*q \leqq ||v||^2 q = q = baa^*b^* \leqq ||a||^2 bb^* = ||a||^2 vb^*bv^* \leqq ||a||^2 ||b||^2 vv^*.$$

That is, vv^* and q are projections satisfying $vv^* \leqq q \leqq ||a||^2||b||^2 vv^*$. It follows that $vv^* = q$.

(3) The map $\mathcal{D}(A) \to V(A)$ given by $[p]_{\mathcal{D}} \mapsto [p]_V$ is a semigroup isomorphism. (exercise)

Chapter 4

K_0-Group — the General Case

4.1 Definition of the K_0-Functor

Definition 4.1.1. Let A be a non-unital C^*-algebra and let $\tilde{A}$ be its minimal unitization. We have a split-exact sequence

$$0 \longrightarrow A \longrightarrow \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$

Define $K_0(A) = \operatorname{Ker}(K_0(\pi))$, where $K_0(\pi) : K_0(\tilde{A}) \to K_0(\mathbb{C})$ is the map induced by π.

Thus, by definition, $K_0(A)$ is a subgroup of $K_0(\tilde{A})$ and hence an abelian group. If $p \in \mathcal{P}_\infty(A)$ then $[p]_0 \in K_0(\tilde{A})$. But $[p]_0 \in \operatorname{Ker} K_0(\pi)$ and hence $[p]_0 \in K_0(A)$. Thus, just as in the unital case, we have a map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$.

If A is unital then we can still form direct sum (of C^*-algebras) $\tilde{A} = A \oplus \mathbb{C}$. Let π be the natural surjection from $\tilde{A}$ onto $\mathbb{C}$. As shown in Exercise 3.3.19, we have $K_0(A) = \operatorname{Ker}(K_0(\pi))$. Thus, Definition 4.1.1 works equally well in the case of a unital C^*-algebra.

4.1.1 Functoriality of K_0

Let $\varphi : A \to B$ be a $*$-homomorphism. Then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"] & \tilde{A} \arrow[r] \arrow[d, "\tilde{\varphi}"] & \mathbb{C} \arrow[d, "="] \\ B \arrow[r] & \tilde{B} \arrow[r] & \mathbb{C} \end{tikzcd} \tag{4.1.1}$$

commutes. Functoriality of K_0 for unital C^*-algebras yields a commutative diagram

$$\begin{tikzcd} K_0(A) \arrow[r] & K_0(\tilde{A}) \arrow[r] \arrow[d, "K_0(\tilde{\varphi})"] & K_0(\mathbb{C}) \arrow[d, "="] \\ K_0(B) \arrow[r] & K_0(\tilde{B}) \arrow[r] & K_0(\mathbb{C}) \end{tikzcd} \tag{4.1.2}$$

and there exists exactly one map $K_0(\varphi) : K_0(A) \to K_0(B)$ which completes the diagram. Note that we have $K_0([p]_0) = [\varphi(p)]_0$ for $p \in \mathcal{P}_\infty(A)$.

Proposition 4.1.2. *Let $\varphi : A \to B$, $\psi : B \to C$ be $*$-homomorphsms between C^*-algebras. Then*

(i) $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$,

(ii) $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$.

Proof. Exercise — use functoriality of K_0 for unital C^*-algebras. □

Moreover, it is immediate from the definitions that K_0 of the zero algebra is 0 and K_0 of the zero homomorphism is the zero map.

4.1.2 Homotopy invariance of K_0

Proposition 4.1.3. *Let A, B be C^*-algebras.*

(i) If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms then $K_0(\varphi) = K_0(\psi)$.

(ii) If $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ is a homotopy then $K_0(\varphi)$ and $K_0(\psi)$ are isomorphisms and inverses of one another.

Proof. (i) Since φ and ψ are homotopic so are they unital extensions $\tilde{\varphi}$ and $\tilde{\psi}$ to $\tilde{A}$, whence $K_0(\tilde{\varphi}) = K_0(\tilde{\psi})$ by Proposition 3.2.6. Then $K_0(\varphi) = K_0(\psi)$ being the restrictions of these maps to $K_0(A)$. Part (ii) follows from part (i) and functoriality of K_0. □

4.2 Further Properties

4.2.1 Portrait of K_0

Let A be a $C*$-algebra and consider the split-exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0,$$

with the splitting map $\lambda : \mathbb{C} \to \tilde{A}$. Define the *scalar map* $s = \lambda \circ \pi : \tilde{A} \to \tilde{A}$, so that $s(a + t1) = t1$. Let $s_n : M_n(\tilde{A}) \to M_n(\tilde{A})$ be the natural extensions of s. The image of s_n is isomorphic to $M_n(\mathbb{C})$, and its elements are called *scalar matrices*. The scalar map is natural in the sense that for any $*$-homomorphism $\varphi : A \to B$ the diagram

$$\begin{tikzcd} \tilde{A} \arrow[r, "s"] \arrow[d, "\tilde{\varphi}"] & \tilde{A} \arrow[d, "\tilde{\varphi}"] \\ \tilde{B} \arrow[r, "s"] & \tilde{B}; \end{tikzcd} \tag{4.2.3}$$

commutes.

Proposition 4.2.1. Let A be a C^*-algebra.

(i) $K_0(A) = \{[p]_0 - [s(p)]_0 : p \in \mathcal{P}_\infty(\tilde{A})\}$.

(ii) If $p, q \in \mathcal{P}_\infty(\tilde{A})$ then the following are equivalent:

(a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$,

(b) there are k, l such that $p \oplus 1_k \sim_0 q \oplus 1_l$ in $\mathcal{P}_\infty(\tilde{A})$,

(c) there are scalar projections r_1, r_2 such that $p \oplus r_1 \sim_0 q \sim_0 r_2$ such that $p \oplus r_1 \sim_0 q \oplus r_2$.

(iii) If $p \in \mathcal{P}_\infty(\tilde{A})$ and $[p]_0 - [s(p)]_0 = 0$ then there is m such that $p \oplus 1_m \sim s(p) \oplus 1_m$.

(iv) If $\varphi : A \to B$ is a $*$-homomorphism then $K_0(\varphi)([p]_0 - [s(p)]_0) = [\tilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0$ for each $p \in \mathcal{P}_\infty(\tilde{A})$.

Proof. (i) It is clear that $[p]_0 - [s(p)]_0 \in \mathrm{Ker}\,(K_0(\pi)) = K_0(A)$. Conversely, let $g \in K_0(A)$, and let e, f be projections in $M_n(\tilde{A})$ such that $g = [e]_0 - [f]_0$. Put

$$p = \begin{pmatrix} e & 0 \\ 0 & 1_n - f \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}.$$

We have $[p]_0 - [q]_0 = [e]_0 + [1_n - f]_0 - [1_n]_0 = [e]_0 - [f]_0 = g$. As $q = s(q)$ and $K_0(\pi)(g) = 0$, we also have $[s(p)]_0 - [q]_0 = [s(p)]_0 - [s(q)]_0 = K_0(s)(g) = (K_0(\lambda) \circ K_0(\pi))(g) = 0$. Hence $g = [p]_0 - [s(p)]_0$.

(ii) (a)$\Rightarrow$(c) If $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ then $[p \oplus s(q)]_0 = [q \oplus s(p)]_0$ and hence $p \oplus s(q) \sim_s q \oplus s(p)$ in $\mathcal{P}_\infty(\tilde{A})$. Thus there is n such that $p \oplus s(q) \oplus 1_n \sim_0 q \oplus s(p) \oplus 1_n$, and it suffices to take $r_1 = s(q) \oplus 1_n$ and $r_2 = s(p) \oplus 1_n$.

(c)$\Rightarrow$(b) If r_1, r_2 are scalar projections in $\mathcal{P}_\infty(\tilde{A})$ of rank k and l, respectively, then $r_1 \sim_0 1_k$ and $r_2 \sim_0 1_l$. Thus $p \oplus 1_k \sim_0 q \oplus 1_l$.

(b)$\Rightarrow$(a) We have $[p \oplus 1_k]_0 - [s(p \oplus 1_k)]_0 = [p]_0 + [1_k]_0 - [s(p)]_0 - [1_k]_0 = [p]_0 - [s(p)]_0$ and likewise $[q \oplus 1_l]_0 - [s(q \oplus 1_l)]_0 = [q]_0 - [s(q)]_0$. Thus it suffices to show that $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ whenever $p \sim_0 q$. So let $p = v^*v$ and $q = vv^*$. Then $s(v)$ is a scalar rectangular matrix and $s(p) = s(v)^*s(v)$, $s(q) = s(v)s(v)^*$. Thus $s(p) \sim_0 s(q)$. Consequently $[p]_0 = [q]_0$ and $[s(p)]_0 = [s(q)]_0$.

(iii) If $[p]_0 - [s(p)]_0 = 0$ then $p \sim_s s(p)$ and hence there is m such that $p \oplus 1_m \sim s(p) \oplus 1_m$.

(iv) $K_0(\varphi)([p]_0 - [s(p)]_0) = K_0(\tilde{\varphi})([p]_0 - [s(p)]_0) = [\tilde{\varphi}(p)]_0 - [\tilde{\varphi}(s(p))]_0 = [\tilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0$. □

4.2.2 (Half)exactness of K_0

In this section we proof that the K_0 functor is half exact — a property of crucial importence. To this end, we first proof the following technical lemma. Another lemma we need is given in Exercise 4.4.5.

Lemma 4.2.2. Let $\psi : A \to B$ be a $*$-homomorphism between two C^*-algebras, and let $g \in \mathrm{Ker}\,(K_0(\psi))$.

(i) There is n, a projection $p \in \mathcal{P}_n(\tilde{A})$, and a unitary $u \in M_n(\tilde{B})$ such that

$$g = [p]_0 - [s(p)]_0 \quad \text{and} \quad u\tilde{\psi}(p)u^* = s(\tilde{\psi}(p)).$$

(ii) If ψ is surjective then there is a projection $p \in \mathcal{P}_\infty(\tilde{A})$ such that

$$g = [p]_0 - [s(p)]_0 \quad \text{and} \quad \tilde{\psi}(p) = s(\tilde{\psi}(p)).$$

Proof. (i) By virtue of Proposition 4.2.1, there is a projection $p_1 \in \mathcal{P}_k(\tilde{A})$ such that $g = [p_1]_0 - [s(p_1)]_0$, and we have $[\tilde{\psi}(p_1)]_0 - [s(\tilde{\psi}(p_1))]_0 = 0$. Thus $\tilde{\psi}(p_1) \oplus 1_m \sim s(\tilde{\psi}(p_1)) \oplus 1_m$ for some m, again by Proposition 4.2.1. Put $p_2 = p_1 \oplus 1_m$. Then $g = [p_2]_0 - [s(p_2)]_0$ and $\tilde{\psi}(p_2) = \tilde{\psi}(p_1) \oplus 1_m \sim s(\tilde{\psi}(p_1)) \oplus 1_m = s(\tilde{\psi}(p_2))$. Put $n = 2(k+m)$ and $p = p_2 \oplus 0_{k+m} \in \mathcal{P}_n(\tilde{A})$. Clearly, $[p]_0 - [s(p)]_0 = g$. By Proposition 2.2.9, there is a unitary u in $M_n(\tilde{A})$ such that $u\tilde{\psi}(p)u^* = s(\tilde{\psi}(p))$.

(ii) By virtue of part (i), there is n, a projection $p_1 \in \mathcal{P}_n(\tilde{A})$, and a unitary $u \in M_n(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\psi}(p_1)u^* = s(\tilde{\psi}(p_1))$. By Lemma 2.1.8, there exists a unitary $v \in M_{2n}(\tilde{A})$ such that $\tilde{\psi}(v) = \operatorname{diag}(u, u^*)$. Put $p = v \operatorname{diag}(p_1, 0_n)v^*$. Then

$$\tilde{\psi}(p) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \tilde{\psi}(p_1) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} s(\tilde{\psi}(p_1)) & 0 \\ 0 & 0 \end{pmatrix}$$

is a scalar matrix. Thus $s(\tilde{\psi}(p)) = \tilde{\psi}(p)$. Finally, $g = [p]_0 - [s(p)]_0$ since $p \sim_0 p_1$. □

Theorem 4.2.3. A short exact sequence of C^*-algebras

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{4.2.4}$$

induces an exact sequence

$$K_0(J) \xrightarrow{K_0(\varphi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B).$$

If the sequence (4.2.4) splits with a splitting map $\lambda : B \to A$, then there is a split-exact sequence

$$0 \longrightarrow K_0(J) \xrightarrow{K_0(\varphi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B) \longrightarrow 0 \tag{4.2.5}$$

with a splitting map $K_0(\lambda) : K_0(B) \to K_0(A)$.

Proof. Since the sequence (4.2.4) is exact, functoriality of K_0 yields $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi) = K_0(0) = 0$. Thus the image of $K_0(\varphi)$ is contained in the kernel of $K_0(\psi)$. Conversely, let $g \in \operatorname{Ker}(K_0(\psi))$. Then there is a projection p in $\mathcal{P}_\infty(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $\tilde{\psi}(p) = s(\tilde{\psi}(p))$ by part (ii) of Lemma 4.2.2. By part (ii) of Exercise 4.4.5, there is an element e in $M_\infty(\tilde{J})$ such that $\tilde{\varphi}(e) = p$. Since $\tilde{\varphi}$ is injective (by part (i) of Exercise 4.4.5), e must be a projection. Hence $g = [\tilde{\varphi}(e)]_0 - [s(\tilde{\varphi}(e))]_0 = K_0(\varphi)([e]_0 - [s(e)]_0)$ belongs to the image of $K_0(\varphi)$.

Now suppose the sequence (4.2.4) is split-exact. The sequence (4.2.5) is exact at $K_0(A)$ by part (i) above. Functoriality of K_0 yields $\mathrm{id}_{K_0(B)} = K_0(\mathrm{id}_B) = K_0(\psi) \circ K_0(\lambda)$ and hence the sequence is exact at $K_0(B)$. It remains to show that $K_0(\varphi)$ is injective. Let $g \in \operatorname{Ker}(K_0(\varphi))$. By part (i) of Lemma 4.2.2 there is n, a projection $p \in \mathcal{P}_n(\tilde{J})$, and a unitary $u \in M_n(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\varphi}(p)u^* = s(\tilde{\varphi}(p))$. Put $v = (\tilde{\lambda} \circ \tilde{\psi})(u^*)u$, a unitary in $M_n(\tilde{A})$ such that $\tilde{\psi}(v) = 1_n$. By Exercise 4.4.5, there is an element $w \in M_n(\tilde{J})$ such that $\tilde{\varphi}(w) = v$. Since $\tilde{\varphi}$ is injective w must be unitary. We have

$$\tilde{\varphi}(wpw^*) = v\tilde{\varphi}(p)v^* = (\tilde{\lambda} \circ \tilde{\psi})(u^*)s(\tilde{\varphi}(p))(\tilde{\lambda} \circ \tilde{\psi})(u) = (\tilde{\lambda} \circ \tilde{\psi})(u^* s(\tilde{\varphi}(p))u)$$

$$= (\tilde{\lambda} \circ \tilde{\psi})(\tilde{\varphi}(p)) = s(\tilde{\varphi}(p)) = \tilde{\varphi}(s(p)).$$

Since $\tilde{\varphi}$ is injective, we conclude that $wpw^* = s(p)$. Thus $p \sim_u s(p)$ in $M_n(\tilde{J})$ and hence $g = 0$. □

4.3 Inductive Limits. Continuity and Stability of K_0

4.3.1 Increasing limits of C^*-algebras

Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of C^*-algebras such that $A_i \subseteq A_{i+1}$. Then $A_\infty = \bigcup_{i=1}^{\infty} A_n$ is a normed $*$-algebra satisfying all the axioms of a C^*-algebra except perhaps of completeness. Let A be the completion of A_∞. Then A is a C^*-algebra, called the *increasing limit* of A_n.

4.3.2 Direct limits of $*$-algebras

Let A_i be an infinite sequence of $*$-algebras. Suppose that for each pair $j \leqq i$ there is given a $*$-homomorphism $\Phi_{ij} : A_j \to A_i$, and that the following coherence condition holds: $\Phi_{ij} = \Phi_{ik} \circ \Phi_{kj}$ whenever $j \leqq k \leqq i$, and $\Phi_{ii} = \mathrm{id}$. Let $\prod_i A_i$ be the product $*$-algebra, with coordinate-wise operations inherited from A_i's. Let $\sum_i A_i$ be the $*$-ideal of $\prod_i A_i$ consisting of sequences whose all but finitely many terms are 0, and let $\pi : \prod_i A_i \to \prod_i A_i / \sum_i A_i$ be the canonical surjection. Set

$$A_\infty = \pi\left(\{(a_i) \in \prod_i A_i : \exists i_0 \forall i : i \geq i_0 \Rightarrow a_i = \Phi_{i i_0}(a_{i_0})\}\right). \tag{4.3.6}$$

A_∞ is called *direct limit* of the *directed system* $\{A_i, \Phi_{ij}\}$ and denoted $\varinjlim\{A_i, \Phi_{ij}\}$. By definition, A_∞ is a $*$-algebra, and there exist canonical morphisms $\Phi_i : A_i \to A_\infty$ such that $A_\infty = \bigcup_i \Phi_i(A_i)$ and for all $j \leqq i$ the following diagram commutes:

$$\begin{tikzcd}
A_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"'] & A_\infty \\
A_i \arrow[ur, "\Phi_i"'] &
\end{tikzcd} \tag{4.3.7}$$

Indeed, for $x \in A_j$ define $\Phi_j(x) = \pi((a_i))$, where $a_i = 0$ if $i < j$ and $a_i = \Phi_{ij}(x)$ if $i \geq j$.

The direct limit $A_\infty = \varinjlim\{A_i, \Phi_{ij}\}$ has the following *universal property.* If B is a $*$-algebra and for each i there is a $*$-homomorphism $\Psi_i : A_i \to B$ such that $\Psi_i \circ \Phi_{ij} = \Psi_j$ for every $j \leqq i$, then there exists a unique $*$-homomorphism $\Lambda : A_\infty \to B$ such that the diagram

$$\begin{tikzcd}
A_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"'] \arrow[dr] & A_\infty \arrow[d, "\Lambda"] \\
A_i \arrow[ur] \arrow[r, "\Psi_i"'] & B
\end{tikzcd} \tag{4.3.8}$$

commutes.

Everything from this section may be generalized to the case of directed systems of $*$-algebras over *directed sets* rather than merely sequences. Furthermore, the same construction works for abelian groups (or even monoids) and their homomorphisms rather than $*$-algebras and $*$-homomorphisms.

4.3.3 C^*-algebraic inductive limits

Now suppose that each A_i is a C^*-algebra rater than just a $*$algebra. By definition, the product $\prod_i A_i$ consists of sequences (a_i) for which $||(a_i)|| = \sup\{||a_i||\}$ is finite. With this norm $\prod_i A_i$ is a C^*-alebra. Let $\sum_i A_i$ be the closure of the ideal of sequences whose all but finitely many terms are 0, and let $\pi : \prod_i A_i \to \prod_i A_i / \sum_i A_i$ be the canonical surjection. We define

$$\varinjlim\{A_i, \Phi_{ij}\} = \text{ the closure of } \pi\left(\{(a_i) \in \prod_i A_i : \exists i_0 \forall i : i \geq i_0 \Rightarrow a_i = \Phi_{ii_0}(a_{i_0})\}\right). \tag{4.3.9}$$

This definition is correct since $*$-homomorphisms between C^*-algebras are norm-decreasing. As before, there exist $*$-homomorphisms $\Phi_i : A_i \to A_\infty$ satisfing (4.3.7), and the universal property (4.3.8) holds.

4.3.4 Continuity of K_0

Theorem 4.3.1. *Let $\{A_i, \Phi_{ij}\}$ be an inductive sequence of C^*-algebras and let $A = \varinjlim\{A_i, \Phi_{ij}\}$. Then $\{K_0(A_i), K_0(\Phi_{ij})\}$ is a direct sequence of abelian groups and*

$$K_0(A) = K_0(\varinjlim\{A_i, \Phi_{ij}\}) \cong \varinjlim\{K_0(A_i), K_0(\Phi_{ij})\}.$$

Proof. W denote by $\Phi_i : A_i \to A = \lim A_i$ the canonical maps. Functoriality of K_0 implies that $\{K_0(A_i), K_0(\Phi_{ij})\}$ is a direct sequence of abelian groups. Let $\varphi_i : K_0(A_i) \to \lim K_0(A_i)$ be the canonical maps. Since for $j \leqq i$ we have $K_0(\Phi_j) = K_0(\Phi_i) \circ K_0(\Phi_{ij})$ by functoriality of K_0, the universal property of $\lim K_0(A_i)$ yields a unique homomorphism $\varphi : \lim K_0(A_i) \to K_0(A)$ such that $\varphi_i = \varphi \circ \varphi_j$ for all $j \leqq i$.

$$\begin{tikzcd} K_0(A_j) \arrow[r, "\varphi_j"] \arrow[d, "K_0(\Phi_{ij})"'] \arrow[dr] & \lim K_0(A_i) \arrow[d, "\varphi"] \\ K_0(A_i) \arrow[r, "K_0(\Phi_i)"'] \arrow[ur] & K_0(\lim A_i) \end{tikzcd} \tag{4.3.10}$$

We must show that φ is injective and surjective.

Injectivity of φ. Since $\lim K_0(A_i) = \bigcup_i \varphi_i(K_0(A_i))$, it suffices to show that $\varphi|_{\varphi_j(K_0(A_j))}$ is injective for all j. That is we must show that if $g \in K_0(A_j)$ and $K_0(\Phi_j)(g) = (\varphi \circ \varphi_j)(g) = 0$ in $K_0(A)$ then $\varphi_j(g) = 0$ in $\lim K_0(A_i)$. So let $g = [p]_0 - [s(p)]_0$ for some $p \in \mathcal{P}_n(\tilde{A}_j)$. Then $0 = K_0(\Phi_j)(g) = [\tilde{\Phi}_j(p)]_0 - [s(\tilde{\Phi}_j(p))]_0$ in $K_0(A)$. Hence there is m and a partial isometry $w \in M_{n+m}(\tilde{A})$ such that

$$ww^* = \tilde{\Phi}_j(p) \oplus 1_m \quad \text{and} \quad w^*w = s(\tilde{\Phi}_j(p)) \oplus 1_m.$$

By Exercise 4.4.12, there is $i \geq j$ and $x_i \in M_{n+m}(\tilde{A}_i)$ with $\tilde{\Phi}_i(x_i)$ close enough to w to ensure that

$$||\tilde{\Phi}_i(x_i)\tilde{\Phi}_i(x_i)^* - \tilde{\Phi}_j(p) \oplus 1_m|| < 1/2 \quad \text{and} \quad ||\tilde{\Phi}_i(x_i)^*\tilde{\Phi}_i(x_i) - s(\tilde{\Phi}_j(p)) \oplus 1_m|| < 1/2.$$

Since $\Phi_j = \Phi_i \circ \Phi_{ij}$, Exercise 4.4.17 implies that there is $k \geq i$ such that

$$||x_k x_k^* - \tilde{\Phi}_{kj}(p) \oplus 1_m|| < 1/2 \quad \text{and} \quad ||x_k^* x_k - s(\tilde{\Phi}_{kj}(p)) \oplus 1_m|| < 1/2,$$

where $x_k = \tilde{\Phi}_{ki}(x_i)$. By part (ii) of Exercise 4.4.18, $\tilde{\Phi}_{kj}(p) \oplus 1_m$ is equivalent to $s(\tilde{\Phi}_j(p)) \oplus 1_m$ in $M_{n+m}(\tilde{A}_m)$. Thus

$$K_0(\Phi_{kj})(g) = [\tilde{\Phi}_{kj}(p) \oplus 1_m]_0 - [s(\tilde{\Phi}_j(p)) \oplus 1_m]_0 = 0$$

in $K_0(A_k)$. Consequently, $\varphi_j(g) = (\varphi_k \circ K_0(\Phi_{kj}))(g) = 0$, as required.

Surjectivity of φ. Consider an element $[p]_0 - [s(p)]_0$ of $K_0(A)$, for some $p \in \mathcal{P}_k(\tilde{A})$. Take a small $\epsilon > 0$. By Exercise 4.4.12, there is n and $b_n \in M_k(\tilde{A}_n)$ such that $||\tilde{\Phi}_n(b_n) - p|| < \epsilon$. Put $a_n = (b_n + b_n^*)/2$ and $a_m = \tilde{\Phi}_{mn}(a_n)$ for $m \geq n$. Each a_m is self-adjoint and $||\tilde{\Phi}_m(a_m) - p|| < \epsilon$. We have $||\tilde{\Phi}_n(a_n - a_n^2)|| < \epsilon(3+\epsilon) < 1/4$ for sufficiently small ϵ. Thus, by Exercise 4.4.17, $||a_m - a_m^2|| < 1/4$ for sufficiently large m. By Exercise 4.4.18, there is a projection q in $M_k(\tilde{A}_m)$ such that $||a_m - q|| < 1/2$. We have $||\tilde{\Phi}_m(q) - p|| < 1/2 + \epsilon < 1$ and hence $\tilde{\Phi}_m(q)$ and p are equivalent. Thus

$$[p]_0 - [s(p)]_0 = [\tilde{\Phi}_m(q)]_0 - [s(\tilde{\Phi}_m(q))]_0 = K_0(\Phi_m)([q]_0 - [s(q)]_0).$$

Since $K_0(\Phi_m) = \varphi \circ \varphi_m$, surjectivity of φ follows. ☐

4.3.5 Stability of K_0

Proposition 4.3.2. *Let A be a C^*-algebra, and let p be minimal projection in $\mathcal{K}$. The map $\varphi : A \to A \otimes \mathcal{K}$ such that $\varphi(a) = a \otimes p$ induces an isomorphism $K_0(\varphi) : K_0(A) \to K_0(A \otimes \mathcal{K})$.*

Proof. For $n \geq m$ let $\Phi_{nm} : M_m(A) \to M_n(A)$ be the imbedding $\Phi_{nm}(a) = \mathrm{diag}(a, 0_{n-m})$. By Exercise 4.4.16, $A \otimes \mathcal{K}$ is isomorphic with the limit of the inductive sequence $\{M_n(A), \Phi_{nm}\}$. We have $\Phi_{n1} = \Phi_{nm} \circ \Phi_{m1}$ and hence $K_0(\Phi_{n1}) = K_0(\Phi_{nm}) \circ K_0(\Phi_{m1})$. Moreover, all the mapsare isomorphism on K_0, by Exercise 4.4.7. Let $\psi_n = K_0(\Phi_{n1})^{-1}$. Then $\psi_m = \psi_n \circ K_0(\Phi_{nm})$ for all $n \geq m$. Thus the universal property of direct limits yields a unique homomorphism $\Lambda : \varinjlim\{M_n(A), \Phi_{nm}\} \cong K_0(A \otimes \mathcal{K}) \to K_0(A)$ which fits into the commutative diagram

$$\begin{tikzcd}
K_0(M_m(A)) \arrow[r, "K_0(\Phi_m)"] \arrow[d, "K_0(\Phi_{nm})"'] \arrow[dr] & K_0(A \otimes \mathcal{K}) \arrow[d, "\Lambda"] \\
K_0(M_n(A)) \arrow[r, "\psi_n"'] \arrow[ur] & K_0(A)
\end{tikzcd} \tag{4.3.11}$$

where $\Phi_n : M_n(A) \to A \otimes \mathcal{K}$ are the canonical maps. It follows that Λ is an isomorphism. Furthermore, $\Lambda^{-1} = K_0(\varphi)$, as required. ☐

4.4 Examples and Exercises

EXAMPLE 4.4.1. Consider the exact sequence

$$0 \longrightarrow C_0((0,1)) \longrightarrow C([0,1]) \xrightarrow{\psi} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

We have $K_0(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z}^2$ and $K_0(C([0,1])) \cong \mathbb{Z}$ (since $[0,1]$ is contractible). Thus $K_0(\psi)$ cannot be *surjective.*

EXAMPLE 4.4.2. Let $\mathcal{H}$ be a separable Hilbert space,and let $\mathcal{K}$ be the ideal of compact operators on $\mathcal{H}$. There is an exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{\imath} \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})/\mathcal{K} \longrightarrow 0.$$

We already know that $K_0(\mathcal{B}(\mathcal{H})) = 0$ and we will see later that $K_0(\mathcal{K}) \cong \mathbb{Z}$. Thus $K_0(\imath)$ cannot be *injective.*

EXERCISE 4.4.3. Let $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}$ be the Calkin algebra (corresponding to a separable Hilbert space $\mathcal{H}$), and let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}$ be the natural surjection. Show the following.

(i) If $p \neq 0$ is a projection in $\mathcal{Q}$ then there is a projection $\tilde{p}$ in $\mathcal{B}(\mathcal{H})$ with infinite dimensional range such that $\pi(\tilde{p}) = p$.

(ii) Any two non-zero projections in $\mathcal{Q}$ are Murray-von Neumann equivalent.

(iii) For each positive integer n we have $\mathcal{B}(\mathcal{H}) \cong M_n(\mathcal{B}(\mathcal{H}))$, $\mathcal{K} \cong M_n(\mathcal{K})$, and $\mathcal{Q} \cong M_n(\mathcal{Q})$.

(iv) The semigroup $\mathcal{D}(\mathcal{Q})$ is isomorphic to $\{0, \infty\}$, with $\infty + \infty = \infty$.

(v) $K_0(\mathcal{Q}) = 0$.

(i) Hint: If p is a projection in $\mathcal{Q}$ then there exists $x = x^*$ in $\mathcal{B}(\mathcal{H})$ such that $\pi(x) = p$. Thus $x^2 - x$ is compact. Let $x^2 - x = \sum_n \lambda_n e_n$ be the spectral decomposition ($0 \neq \lambda_n \in \mathbb{R}$, $\lambda_n \to 0$, $\{e_n\}$ mutually orthogonal projections of finite rank, commuting with x). Correct each xe_n.

EXAMPLE 4.4.4. In this example we argue why Definition 3.2.1 would not be appropriate for non-unital C^*-algebras. Nameley, let A be a C^*-alebra (unital or not) and define $K_{00}(A)$ as the Grothendieck group of $\mathcal{D}(A)$. Thus, if A is unital the $K_{00}(A) = K_0(A)$, but in the non-unital case these two groups may be different. It can be shown that such defined K_{00} is a covariant functor. However, this functor has a serious defect of not being half-exact. Indeed, consider an exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \longrightarrow C(S^2) \longrightarrow \mathbb{C} \longrightarrow 0.$$

We have $K_0(\mathbb{C}) \cong \mathbb{Z}$, and it can be shown that $K_0(S^2) \cong \mathbb{Z}^2$ and $K_{00}(\mathbb{R}^2) = 0$ (for the latter see Exercise 4.3.4 below). Thus K_{00} cannot be half-exact.

EXERCISE 4.4.5. If $0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$ is an exact sequence of C^*-algebras then:

(i) $\tilde{\varphi}_n : M_n(\tilde{J}) \to M_n(\tilde{A})$ is injective,

(ii) $a \in M_n(\tilde{A})$ is in the image of $\tilde{\varphi}_n$ iff $\tilde{\psi}_n(a) = s_n(\tilde{\psi}_n(a))$.

EXERCISE 4.4.6. Let X be a connected, locally compact but not compact Hausdorff space. Then $K_{00}(C_0(X)) = 0$. To this end show that $\mathcal{P}_\infty(C_0(X)) = \{0\}$, as follows. Identify $M_n(C_0(X))$ with $C_0(X, M_n(\mathbb{C}))$, and let p be a projection in $\mathcal{P}_n(C_0(X))$. As usual, let Tr be the standard trace on $M_n(\mathbb{C})$. The function $x \mapsto \mathrm{Tr}(p(x))$ belongs to $C_0(X, \mathbb{Z})$ and hence it is the zero function, since X is connected and non-compact.

EXERCISE 4.4.7 (MATRIX STABILITY OF K_0). Let A be a C^*-algebra and let n be a positive integer. Then $K_0(A) \cong K_0(M_n(A))$. More specifically, the map $\varphi_A : A \to M_n(A)$, $a \mapsto \mathrm{diag}(a, 0_{n-1})$ induces an isomorphism $K_0(\varphi_A) \to K_0(M_n(A))$. Indeed, the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi_A"] & \tilde{A} \arrow[r] \arrow[d, "\varphi_{\tilde{A}}"] & \mathbb{C} \arrow[r] \arrow[d, "\varphi_{\mathbb{C}}"] & 0 \\ 0 \arrow[r] & M_n(A) \arrow[r] & M_n(\tilde{A}) \arrow[r] & M_n(\mathbb{C}) \arrow[r] & 0 \end{tikzcd} \tag{4.4.12}$$

commutes and has split-exact rows. Thus the diagram

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r] \arrow[d, "K_0(\varphi_A)"] & K_0(\tilde{A}) \arrow[r] \arrow[d, "K_0(\varphi_{\tilde{A}})"] & K_0(\mathbb{C}) \arrow[r] \arrow[d, "K_0(\varphi_{\mathbb{C}})"] & 0 \\ 0 \arrow[r] & K_0(M_n(A)) \arrow[r] & K_0(M_n(\tilde{A})) \arrow[r] & K_0(M_n(\mathbb{C})) \arrow[r] & 0 \end{tikzcd} \tag{4.4.13}$$

commutes and has split-exact rows. Hence the Five Lemma (or an easy diagram chasing) implies that $K_0(\varphi_A)$ is an isomorphism if both $K_0(\varphi_{\tilde{A}})$ and $K_0(\varphi_{\mathbb{C}})$ are. This reduces the proof to the unital case (see Exercise 3.3.10).

EXERCISE 4.4.8. Let A be a C^*-algebra, and denote by $\mathrm{Aut}(A)$ the group of $*$-automorphisms of A. If $\alpha \in \mathrm{Aut}(A)$ then $K_0(\alpha)$ is an automorphism of $K_0(A)$.

(i) If u is a unitary in $\tilde{A}$ then $\mathrm{Ad}(u) : A \to A$, $a \mapsto uau^*$, is an automorphism of A. Moreover, the map $\mathcal{U}(\tilde{A}) \to \mathrm{Aut}(A)$, $u \mapsto \mathrm{Ad}(u)$ is a group homomorphism, and $\mathrm{Inn}(A) = \{\mathrm{Ad}(u) : u \in \mathcal{U}(\tilde{A})\}$ is a normal subgroup of $\mathrm{Aut}(A)$.

(ii) If $\alpha \in \mathrm{Inn}(A)$ then $K_0(\alpha) = \mathrm{id}$.

(iii) An $\alpha \in \mathrm{Aut}(A)$ is *approximately inner* iff for any finite subset F of A and any $\epsilon > 0$ there is $\beta \in \mathrm{Inn}(A)$ such that $||\alpha(x) - \beta(x)|| < \epsilon$ for all $x \in F$. The collection of all approximately inner automorphisms of A is denoted $\overline{\mathrm{Inn}}(A)$.

Show that if A is separable then α is approximately inner iff there is a sequence $\beta_n \in \mathrm{Inn}(A)$ such that $\beta_n(a) \to \alpha(a)$ for each $a \in A$.

(iv) $\overline{\mathrm{Inn}}(A)$ is a normal subgroup of $\mathrm{Aut}(A)$, and $K_0(\alpha) = \mathrm{id}$ for each $\alpha \in \overline{\mathrm{Inn}}(A)$.

(v) Give examples of automorphisms of C^*-algebras which induce non-trivial automorphisms on K_0.

Example 4.4.9. Let A be a C^*-algebra. We define the *cone* CA and the *suspension* SA as follows:

$$\begin{aligned} CA &= \{f : [0,1] \to A : f \text{ continuous and } f(0) = 0\}, \\ SA &= \{f : [0,1] \to A : f \text{ continuous and } f(0) = f(1) = 0\}. \end{aligned}$$

There is a short exact sequence

$$0 \longrightarrow SA \longrightarrow CA \xrightarrow{\pi} A \longrightarrow 0,$$

with $\pi(f) = f(1)$. Furthermore, CA is homotopy equivalent to $\{0\}$. Indeed, with $t \in [0,1]$ set $\varphi_t : CA \to CA$ as $\varphi_t(f)(s) = f(st)$. Then for each $f \in CA$ the map $t \mapsto \varphi_t(f)$ is continuous, and $\varphi_0 = 0$, $\varphi_1 = \mathrm{id}$. We conclude that

$$K_0(CA) = 0.$$

Example 4.4.10 (Direct sums). For any two C^*-algebras A, B we have

$$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B).$$

More specifically, if i_A and i_B are the inclusions of A and B, respectively, into $A \oplus B$, then $K_0(i_A) \oplus K_0(B) : K_0(A) \oplus K_0(B) \to K_0(A \oplus B)$ is an isomorphism. Indeed, let π_A and π_B be the surjections from $A \oplus B$ onto A and B, respectively. The following diagram has exact rows (the bottom one by split-exactness of K_0) and commutes (since $\pi_B \circ i_A = 0$ and $\pi_B \circ i_B = \mathrm{id}_B$):

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r, "\alpha"] \arrow[d, "="] & K_0(A) \oplus K_0(B) \arrow[r, "\beta"] \arrow[d, "K_0(i_A) \oplus K_0(i_B)"] & K_0(B) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & K_0(A) \arrow[r, "K_0(i_A)"'] & K_0(A \oplus B) \arrow[r, "K_0(\pi_B)"'] & K_0(B) \arrow[r] & 0 \end{tikzcd} \tag{4.4.14}$$

An easy diagram chasing (or the Five Lemma) implies that $K_0(i_A) \oplus K_0(i_B)$ is an isomorphism.

Exercise 4.4.11. Let $\{A_i\}$ be a sequence of C^*-algebras, and let $a = (a_i) \in \prod_i A_i$. Then

$$||\pi(a)|| = \overline{\lim}||a_i||.$$

In particular, a belongs to $\sum_i A_i$ iff $\lim\limits_{i \to \infty} ||a_i|| = 0$.

Exercise 4.4.12. Let $A = \varinjlim\{A_i, \Phi_{ij}\}$. To each $x \in A$ and $\epsilon > 0$ there is an arbitrarily large index i and $x_i \in A_i$ such that

$$||x - \Phi_i(x_i)|| < \epsilon.$$

Example 4.4.13 (*UHF* algebras). Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of integers $p_n \geq 2$. For $1 \leqq j$ define $\Phi_{j+1,j} : \bigotimes_{n=1}^{j} M_{p_n}(\mathbb{C}) \to \bigotimes_{n=1}^{j+1} M_{p_n}(\mathbb{C})$ by $\Phi_{j+1,j}(x) = x \otimes I$. These are unital, injectve $*$-homomorphisms. Then for $1 \leqq j \leqq i$ define $\Phi_{ij} : \bigotimes_{n=1}^{j} M_{p_n}(\mathbb{C}) \to \bigotimes_{n=1}^{i} M_{p_n}(\mathbb{C})$ by $\Phi_{ij} = \Phi_{i,i-1} \circ \ldots \circ \Phi_{j+1,j}$. The inductive limit $\varinjlim\{A_i, \Phi_{ij}\}$ is called the *UHF* algebra corresponding to the supernatural number $(p_1 p_2 \cdots)$. These are simple, unital C^*-algebras, equipped with a unique tracial state. To learn much more about *UHF* algebras see [G-J60].

EXAMPLE 4.4.14 (AF-ALGEBRAS). For $n = 1, 2, \ldots$ let A_n be a finite dimensional C^*-algebra. Thus, A_n is isomorphic to a direct sum of matrix algebras

$$A_n \cong M_{k_1^n}(\mathbb{C}) \oplus \ldots \oplus M_{k_n^{r(n)}}(\mathbb{C}).$$

For $1 \leqq j$ let $\Phi_{j+1,j} : A_j \to A_{j+1}$ be a $*$-homomorphism, and define $\Phi_{ij} = \Phi_{i,i-1} \circ \ldots \circ \Phi_{j+1,j}$. The corresponding inductive limit $\varinjlim \{A_i, \Phi_{ij}\}$ is called an AF-algebra. To learn much more about AF-algebras and about *Bratteli diagrams* which describe them see [B-O72].

EXERCISE 4.4.15 (THE COMPACTS). Let $\mathcal{H}$ be a separable (infinite dimensional) Hilbert space. Denote by $\mathcal{F}$ the collection of all finite rank operators in $\mathcal{B}(\mathcal{H})$, and let $\mathcal{K}$ be the norm closure of $\mathcal{F}$ (the C^*-algebra of *compact* operators). Show the following.

(i) $\mathcal{F}$ is a two-sided $*$-ideal of $\mathcal{B}(\mathcal{H})$, but $\mathcal{F}$ is not norm closed in $\mathcal{B}(\mathcal{H})$.

(ii) $\mathcal{K}$ is a norm closed, two-sided $*$-ideal of $\mathcal{B}(\mathcal{H})$, and $\mathcal{K} \neq \mathcal{B}(\mathcal{H})$.

(iii) Let $\{\xi_n\}_{n=0}^\infty$ be an orthonoral basis of $\mathcal{H}$. For all i, j let E_{ij} be an operator defined by $E_{ij}(v) = \langle v, \xi_j \rangle \xi_i$. Then each E_{ij} is a rank one opeator with domain $\mathbb{C}\xi_j$ and range $\mathbb{C}\xi_i$. In particular, $\{E_{ii}\}$ are mutually orthogonal projections of rank one whose sum of the ranges densely spans the entire space $\mathcal{H}$. Furthermore, for each i we have $E_{ii}\mathcal{K}E_{ii} = \mathbb{C}E_{ii}$ (a projection in a C^*-algebra with this property is called *minimal*). The following identities hold:

$$E_{ij}E_{kn} = \delta_{jk}E_{in}, \qquad E_{ij}^* = E_{ji}. \tag{4.4.15}$$

(A collection of elements of a C^*-algebra satisfying (4.4.15) is called a *system of matrix units.*) Prove that the closed span of $\{E_{ij} : i, j = 0, 1, \ldots\}$ coincides with $\mathcal{K}$.

(iv) Let $\mathcal{H}'$ be another Hilbert space and let $\pi : \mathcal{K} \to \mathcal{H}'$ be a nondegenerate representation (i.e. a $*$-homomorphism such that $\pi(\mathcal{K})\mathcal{H}'$ is dense in $\mathcal{H}'$). Show that there exists a Hilbert space $\mathcal{H}_0$ and a unitary operator $U : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{H}_0$ such that for all $x \in \mathcal{K}$ we have

$$U\pi(x)U^* = x \otimes I_{\mathcal{H}_0}.$$

The dimension of $\mathcal{H}_0$ is called the *multiplicity* of π. Show that π is irreducible iff the multiplicity of π is one. Thus, in particular, the compacts admit precisely one (up to unitary equivalence) irreducible representation.

(v) $\mathcal{K}$ is the universal C^*-algebra for the relations (4.4.15).

(vi) $\mathcal{K}$ is a *simple* C^*-algebra, in the sense that the only closed, two-sided $*$-ideals of $\mathcal{K}$ are $\{0\}$ and $\mathcal{K}$. (In fact, it can be shown that every norm closed two-sided ideal of a C^*-algebra is automatically closed under $*$).

(vii) For each $j = 1, 2, \ldots$ let $\Phi_{j+1,j} : M_j(\mathbb{C}) \to M_{j+1}(\mathbb{C})$ be an imbedding into the upper-left corner, i.e. $\Phi_{j+1,j}(x) = \mathrm{diag}(x, 0)$. As usual, let $\Phi_{ij} = \Phi_{i,i-1} \circ \ldots \circ \Phi_{j+1,j}$ for $j \leqq i$. Show that

$$\mathcal{K} \cong \varinjlim \{M_n(\mathbb{C}), \Phi_{ij}\}.$$

EXERCISE 4.4.16. Let A be a C^*-algebra. For $n \geq m$ let $\Phi_{nm} : M_m(A) \to M_n(A)$ be the diagonal imbedding $\Phi_{nm}(a) = \mathrm{diag}(a, 0_{n-m})$. Show that the inductive limit of the drected sequence $\{M_n(A), \Phi_{nm}\}$ is isomorphic with $A \otimes \mathcal{K}$.

EXERCISE 4.4.17. Let $\{A_i, \Phi_{ij}\}$ be an inductive sequence of C^*-algebras and let $\Phi_i : A_i \to \varinjlim A_i$ be the canonical maps. Then for all n and $a \in A_n$ we have

$$||\Phi_n(a)|| = \lim_{m\to\infty} ||\Phi_{mn}(a)||.$$

EXERCISE 4.4.18. Let A be a C^*-algebra.

(i) If $a = a^*$ in A and $||a - a^2|| < 1/4$ then there is a projection $p \in A$ with $||a - p|| < 1/2$.

(ii) Let p be a projection in A, and let a be a self-adjoint eleent in A. Put $\delta = ||a - p||$. Then

$$\mathrm{sp}(a) \subseteq [-\delta, \delta] \cup [1 - \delta, 1 + \delta].$$

(iii) If p, q are projections in A such that there exists an element $x \in A$ with $||x^*x - p|| < 1/2$ and $||xx^* - q|| < 1/2$ then $p \sim q$.

(i) Use Gelfand Theorem.

(ii) Recall that the spectrum of a self-adjoint element consists of real numbers, and that the spectrum of a non-trivial projection is $\{0, 1\}$. Let t be a real number whose distance d to the set $\{0, 1\}$ is greater than δ. It suffices to show that $t - a$ is invertible in $\tilde{A}$. Indeed, for such a t the element $t - p$ is invertible in $\tilde{A}$ and

$$||(t - p)^{-1}|| = \max\{\frac{1}{|-t|}, \frac{1}{|1 - t|}\} = \frac{1}{d}.$$

Consequently,

$$||(t - p)^{-1}(t - a) - 1|| = ||(t - p)^{-1}(p - a)|| \leqq \frac{1}{d}\delta < 1.$$

Thus $(t - p)^{-1}(t - a)$ is invertible, and so is $t - a$.

(iii) Let $\Omega = \mathrm{sp}(x^*x) \cup \mathrm{sp}(xx^*)$. In view of part (ii) of this exercise, Ω is a compact subset of $[0, 1/2) \cup (1/2, 3/2)$. Let $f : \Omega \to \mathbb{R}$ be a continuous function which is 0 on $\Omega \cap [0, 1/2)$ and 1 on $\Omega \cap (1/2, 3/2)$. Then both $f(x^*x)$ and $f(xx^*)$ are projections. We have $||f(x^*x) - p|| \leqq ||f(x^*x) - x^*x|| + ||x^*x - p|| < 1/2 + 1/2 = 1$ and, likewise, $||f(xx^*) - q|| < 1$. Thus $f(x^*x) \sim p$ and $f(xx^*) \sim q$ by Proposition 2.2.5. So it suffices to show that $f(x^*x) \sim f(xx^*)$. To this end, first note that $xh(x^*x) = h(xx^*)x$ for every $h \in C(\Omega)$. Indeed, this is obviously true for polynomials, and the general case follows from the Stone-Weierstrass Theorem. Let $g \in C(\Omega)$, $g \geq 0$ be such that $tg(t)^2 = f(t)$ for all $t \in \Omega$. Set $w = xg(x^*x)$. Then

$$\begin{aligned} w^*w &= g(x^*x)x^*xg(x^*x) = f(x^*x), \\ ww^* &= xg(x^*x)^2x^* = g(xx^*)^2xx^* = f(xx^*), \end{aligned}$$

and the claim follows.

EXAMPLE 4.4.19. Let A be a unital Banach algebra, and let $a, b \in A$. Then

$$\mathrm{sp}(ab) \cup \{0\} = \mathrm{sp}(ba) \cup \{0\}.$$

Indeed, let $0 \neq \lambda \notin \mathrm{sp}(ab)$ and let $u = (\lambda - ab)^{-1}$. Then $1 - \lambda u + uab = 0$. Put $w = (1/\lambda)(1 + bua)$. We have

$$w(\lambda - ba) = \frac{1}{\lambda}(1+bua)(\lambda - ba) = 1 - \frac{1}{\lambda}ba + bua - \frac{1}{\lambda}buaba = 1 - \frac{1}{\lambda}b(1 - \lambda u + uab)a = 1.$$

Similarly $(\lambda - ba)w = 1$ and hence $w = (\lambda - ba)^{-1}$. Thus $\lambda \notin \mathrm{sp}(ba)$.

EXERCISE 4.4.20. In the category of abelian groups, consider a direct sequence $A_i = \mathbb{Z}$ with connecting maps $\Phi_{j+1,j}(1) = j$. Show that the corresponding limit is isomorphic to the additive group of $\mathbb{Q}$.

EXERCISE 4.4.21 (IRRATIONAL ROTATION ALGEBRAS). For an irrational number $\theta \in [0, 1)$ define A_θ as the universal C^*-algebra generated by two elements u, v, subject to the relations

$$vu = e^{2\pi i\theta} uv, \qquad uu^* = u^*u = 1 = vv^* = v^*v. \tag{4.4.16}$$

A_θ is called the *irrational rotation algebra* corresponding to the angle θ. Show the following.

(i) Let $\mathcal{L}^2(\mathbb{T})$ be the Hilbert space of square integrable functions on the circle group (with respect to the probability Haar measure dz). Set $\mathcal{H} = \mathcal{L}^2(\mathbb{T}) \otimes \mathcal{L}^2(\mathbb{T})$, and define two operators $U, V \in \mathcal{B}(\mathcal{H})$ by

$$(U\xi)(z_1, z_2) = z_1 \xi(z_1, z_2), \quad (V\xi)(z_1, z_2) = z_2 \xi(e^{2\pi i\theta} z_1, z_2).$$

Then U, V satisfy (4.4.16). Thus, there exists a representation $\pi : A_\theta \to \mathcal{B}(\mathcal{H})$ such that $\pi(u) = U$ and $\pi(v) = V$.

(ii) Let $\mathcal{A}_\theta$ be the dense $*$-subalgebra of A_θ generated by u, v. Then each element of $\mathcal{A}_\theta$ has the form

$$\sum_{n,m \in \mathbb{Z}} \lambda_{n,m} u^n v^m, \qquad \lambda_{n,m} \in \mathbb{C}.$$

(iii) Let ξ_0 be the unit vector in $\mathcal{H}$ such that $\xi_0(z_1, z_2) = 1$, and define $\tau(a) = \langle \pi(a)\xi_0, \xi_0 \rangle$, $a \in A_\theta$. Then $\tau(\sum_{n,m\in\mathbb{Z}} \lambda_{n,m} u^n v^m) = \lambda_{0,0}$ and hence $\tau(aa^*) = \tau(a^*a)$ for all $a \in \mathcal{A}_\theta$. Conclude that τ is a tracial state on A_θ.

(iv) For $f, g : \mathbb{T} \to \mathbb{R}$ let

$$p = f(u)v^* + g(u) + vf(u)$$

be a self-adjoint element of A_θ. Use an approximation of f and g by Laurent polynomials to show that

$$\tau(p) = \int_{\mathbb{T}} g(z)dz.$$

(v) Let $\varphi : \mathbb{T} \to \mathbb{T}$ be given by $\varphi(z) = e^{2\pi i\theta} z$. Then $vh(u) = (h \circ \varphi)(u)v$ for all $h \in C(\mathbb{T})$. Show that $p = p^2$ if and only if

$$(f \circ \varphi)f = 0, \quad (g + g \circ \varphi^{-1})f = f, \quad g = g^2 + f^2 + (f \circ \varphi)^2. \tag{4.4.17}$$

(vi) Let ϵ be such that $0 < \epsilon \leqq \theta < \theta + \epsilon \leqq 1$, and set

$$g(e^{2\pi it}) = \begin{cases} \epsilon^{-1}t, & 0 \leqq t \leqq \epsilon, \\ 1, & \epsilon < t \leqq \theta, \\ \epsilon^{-1}(\theta + \epsilon - t), & \theta < t \leqq \theta + \epsilon, \\ 0, & \theta + \epsilon < t \leqq 1, \end{cases}$$

for $t \in [0, 1]$. For such g one an find f such that (4.4.17) holds, and hence p is a projection. Then $\tau(p) = \theta$. Thus, the homomorphism $K_0(\tau) : K_0(A_\theta) \to \mathbb{R}$ contains $\mathbb{Z} \cup \theta\mathbb{Z}$ in its image.

In fact, it can be shown that $K_0(\tau)$ is an isomorphism of $K_0(A_\theta)$ onto $\mathbb{Z} \cup \theta\mathbb{Z} \cong \mathbb{Z}^2$.

The definition of A_θ makes sense for rational θ as well. However, the structure of the *rational rotation algebras* is completely different from the irrational ones. Namely, it can be shown that for an irrational θ the C^*-algebra A_θ is simple, while for a rational θ the C^*-algebra A_θ contains many non-trivial ideals. In the case $\theta = 0$ we have $A_0 \cong C(\mathbb{T}^2)$. Thus the rotation agebras A_θ are considered noncommutative analogues of the torus.

Chapter 5

K_1-Functor and the Index Map

5.1 The K_1 Functor

5.1.1 Definition of the K_1-group

Let A be a unital C^*-algebra. We denote

$$\begin{aligned} \mathcal{U}(A) &= \text{ the group of unitary elements of } A, \\ \mathcal{U}_n(A) &= \mathcal{U}(M_n(A)), \\ \mathcal{U}_\infty(A) &= \bigcup_{n=1}^{\infty} \mathcal{U}_n(A). \end{aligned}$$

We define a relation $\sim_1$ in $\mathcal{U}_\infty(A)$ as follows. For $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$ we have $u \sim_1 v$ iff there exists $k \geq \max\{n, m\}$ such that $\operatorname{diag}(u, 1_{k-n}) \sim_h \operatorname{diag}(v, 1_{k-m})$. Then $\sim_1$ is an equivalence relation in $\mathcal{U}_\infty(A)$ (exercise). We denote by $[u]_1$ the equivalence class of the unitary $u \in \mathcal{U}_\infty(A)$.

Lemma 5.1.1. *Let A b a unital C^*-algebra. Then*

(i) $[u]_1[v]_1 = [\operatorname{diag}(u, v)]_1$ is a well-defined associative binary operation on $\mathcal{U}_\infty(A)/_{\sim_1}$,

(ii) $[u]_1[v]_1 = [v]_1[u]_1$ for all $u, v \in \mathcal{U}_\infty(A)$,

(iii) $[u]_1[1_n]_1 = [1_n]_1[u]_1 = [u]_1$ for all n and all $u \in \mathcal{U}_\infty(A)$,

(iv) if $u, v \in \mathcal{U}_m(A)$ then $[u]_1[v]_1 = [uv]_1$.

Proof. Exercise — use Lemma 2.1.6. □

By the above lemma, $\mathcal{U}_\infty(A)/_{\sim_1}$ equipped with the multiplication $[u]_1[v]_1 = [\operatorname{diag}(u, v)]_1$ is an abelian group, with $[u]_1^{-1} = [u^*]_1$.

Definition 5.1.2. *If A is a C^*-algebra then we define*

$$K_1(A) = \mathcal{U}_\infty(\tilde{A})/_{\sim_1},$$

an abelian group with multiplication $[u]_1[v]_1 = [\operatorname{diag}(u, v)]_1$.

When A is *unital* then $K_1(A)$ may be defined simply as $\mathcal{U}_\infty(A)/_{\sim_1}$ (see Exercise 5.3.1. Also, instead of using equivalence classes of *unitaries* one could define K_1 with help of equivalence classes of *invertibles* (see Exercise 5.3.6). In particular, the polar decomposition $w = u|w|$ yields a well-defined map

$$[\cdot]_1 : \mathrm{GL}_\infty(A) \to K_1(A)$$

by $[w]_1 = [u]_1 = [w|w|^{-1}]_1$.

Proposition 5.1.3 (Universal property of K_1). Let A be a C^*-algebra, G an abelian group, and $\nu : \mathcal{U}_\infty(\tilde{A}) \to G$ a map such that:

(i) $\nu(\mathrm{diag}(u, v)) = \nu(u) + \nu(v)$,

(ii) $\nu(1) = 0$,

(iii) if $u, v \in \mathcal{U}_n(\tilde{A})$ and $u \sim_h v$ then $\nu(u) = \nu(v)$.

Then there exists a unique homomorphism $K_1(A) \to G$ making the diagram

$$\begin{tikzcd} \mathcal{U}_\infty(\tilde{A}) \arrow[d, "{[\cdot]_1}"'] \arrow[dr, "\nu"] & \\ K_1(A) \arrow[r] & G \end{tikzcd} \tag{5.1.1}$$

commutative.

Proof. Exercise. □

5.1.2 Properties of the K_1-functor

Let A, B be C^*-algebras and let $\varphi : A \to B$ be a $*$-homomorphism. Then φ extends to unital $*$-homomorphisms $\tilde{\varphi}_n : M_n(\tilde{A}) \to M_n(\tilde{B})$ and thus yields a map $\tilde{\varphi} : \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(\tilde{B})$. We define $\nu : \mathcal{U}_\infty(\tilde{A}) \to K_1(B)$ by $\nu(u) = [\tilde{\varphi}(u)]_1$ and use the univesal property of K_1 to conclude that there exists a unique homomorphism $K_1(\varphi) : K_1(A) \to K_1(B)$ such that $K_1([u]_1) = [\tilde{\varphi}(u)]_1$ for $u \in \mathcal{U}_\infty(\tilde{A})$.

Proposition 5.1.4 (Functoriality of K_1). *Let A, B, C be C^*-algebras and let $\varphi : A \to B$ and $\psi : B \to C$ be $*$-homomorphisms. Then*

(i) $K_1(\mathrm{id}_A) = \mathrm{id}_{K_1(A)}$,

(ii) $K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi)$.

Thus K_1 is a covariant functor.

Proof. Exercise. □

It is also clear from the definitions that K_1 of the zero algebra and the zero map are zero.

Proposition 5.1.5 (Homotopy invariance of K_1). *Let A, B be C^*-algebras.*

(i) If $\varphi, \psi : A \to B$ are homotopic $*$-homomorphisms then $K_1(\varphi) = K_1(\psi)$.

(ii) If A and B are homotopy equivalent then $K_1(A) \cong K_1(B)$. More specifically, if $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ is a homotopy then $K_1(\varphi)$ and $K_1(\psi)$ are isomorphisms and inverses of one another.

Proof. Exercise. □

Theorem 5.1.6 ((Half)exactness of K_1). *If*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\pi} B \longrightarrow 0 \tag{5.1.2}$$

is an exact sequence of C^*-algebras then the sequence

$$K_1(J) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\pi)} K_1(B) \tag{5.1.3}$$

is exact. If the sequence (5.1.2) is split-exact with a splitting map $\lambda : B \longrightarrow A$, then the sequence

$$0 \longrightarrow K_1(J) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\pi)} K_1(B) \longrightarrow 0 \tag{5.1.4}$$

is split-exact with a splitting map $K_1(\lambda) : K_1(B) \longrightarrow K_1(A)$.

Proof. $K_1(\pi) \circ K_1(\varphi) = K_1(\pi \circ \varphi) = K_1(0) = 0$ by functoriality of K_1, and hence $\mathrm{Im}(K_1(\varphi)) \subseteq \mathrm{Ker}\,(K_1(\pi))$. For the reverse inclusion, let $u \in \mathcal{U}_n(\tilde{A})$ and $K_1(\pi)([u]_1) = [1]_1$. Then there is m such that $\mathrm{diag}(\tilde{\pi}(u), 1_n) \sim_h 1_{n+m}$. By Lemma 2.1.8, there is $w \in \mathcal{U}_{n+m}(\tilde{A})$ such that $w \sim_h 1_{n+m}$ and $\tilde{\pi}(w) = \mathrm{diag}(\tilde{\pi}(u), 1_m)$. Thus $[u]_1 = [\mathrm{diag}(u, 1_n)w^*]_1$ and $\tilde{\pi}(\mathrm{diag}(u, 1_n)w^*) = 1_{n+m}$. By Exercise 4.4.5, there is $v \in \mathcal{U}_{n+m}(\tilde{J})$ such that $\tilde{\varphi}(v) = \mathrm{diag}(u, 1_n)w^*$. Thus $[u]_1 \in \mathrm{Im}(K_1(\varphi))$ and consequently $\mathrm{Ker}\,(K_1(\pi)) \subseteq \mathrm{Im}(K_1(\varphi))$. This shows that the sequence (5.1.3) is exact.

Now suppose that the sequence (5.1.2) is split-exact. Then the sequence (5.1.4) is exact at $K_1(A)$ by the preceding argument. By functoriality of K_1 we have $K_1(\pi) \circ K_1(\lambda) = \mathrm{id}_{K_1(B)}$, and hence (5.1.4) is exact at $K_1(B)$ (and $K_1(\lambda)$ is a splitting map). It remains to show that $K_1(\varphi)$ is injective. So let $u \in \mathcal{U}_n(\tilde{J})$ be such that $K_1(\varphi)([u]_1) = [1]_1$. Then there is m such that $\mathrm{diag}(\tilde{\varphi}(u), 1_m) \sim_h 1_{n+m}$. Let $t \mapsto w_t$ be a continuous path in $\mathcal{U}_{n+m}(\tilde{A})$ connecting $\mathrm{diag}(\tilde{\varphi}(u), 1_m)$ and 1_{n+m}. We would like to apply $\tilde{\varphi}^{-1}$ to w_t to conclude that $\mathrm{diag}(u, 1_m)$ is homotopic to the identity. In general, this is impossible since some of w_t may lie outside the range of $\tilde{\varphi}$. However, in the presence of a splitting map λ we can correct the path w_t by setting $v_t = w_t(\tilde{\lambda} \circ \tilde{\pi})(w_t^*)$. Then v_t is a continuous path in $\mathcal{U}_{n+m}(\tilde{A})$ connecting $\mathrm{diag}(\tilde{\varphi}(u), 1_m)(\tilde{\lambda} \circ \tilde{\pi})(\mathrm{diag}(\tilde{\varphi}(u^*), 1_m))$ and 1_{n+m}. Since $\tilde{\pi}(v_t) = 1_{n+m}$ for all t, Exercise 4.4.5 implies that each v_t is in the image of $\tilde{\varphi}$. Thus $\tilde{\varphi}^{-1}(v_t)$ is a continuous path in $\mathcal{U}_{n+m}(\tilde{J})$ connecting $\mathrm{diag}(u, 1_m)\tilde{\varphi}^{-1}((\tilde{\lambda} \circ \tilde{\pi})(\mathrm{diag}(\tilde{\varphi}(u^*), 1_m)))$ and 1_{n+m}. Since $\tilde{\varphi}^{-1}((\tilde{\lambda} \circ \tilde{\pi})(\mathrm{diag}(\tilde{\varphi}(u^*), 1_m)))$ is a scalar matrix, it is homotopic to the identity. Thus

$$[u]_1 = [\mathrm{diag}(u, 1_m)]_1 = [\mathrm{diag}(u, 1_m)\tilde{\varphi}^{-1}((\tilde{\lambda} \circ \tilde{\pi})(\mathrm{diag}(\tilde{\varphi}(u^*), 1_m)))]_1 = [1]_1,$$

and the map $K_1(\varphi)$ is injective. □

Proposition 5.1.7 (Continuity of K_1). *Let $A = \varinjlim\{A_i, \Phi_{ij}\}$ be the inductive limit of a sequence of C^*-algebras, and let $\Phi_i : A_i \to A$ be the canonical maps. Let $G = \varinjlim\{K_1(A_i), K_1(\Phi_{ij})\}$ be the inductive limit of the corresponding sequence of abelian groups,* *and let $\varphi_i : K_1(A_i) \to G$ be the canonical maps. Then there exists an isomorphism $\Lambda : G \to K_1(A)$ such that for all $i \geq j$ the diagram*

$$\begin{tikzcd} K_1(A_j) \arrow[r, "\varphi_j"] \arrow[d, "K_1(\Phi_{ij})"'] \arrow[dr] & G \arrow[d, "\Lambda"] \\ K_1(A_i) \arrow[r, "K_1(\Phi_i)"'] \arrow[ur] & K_1(A) \end{tikzcd} \tag{5.1.5}$$

is commutative.

Proof. The universal property of the direct limit G of the sequence $\{K_1(A_i), K_1(\Phi_{ij})\}$ yields a unique homomorphism $\Lambda : G \to K_1(A)$ making the diagram (5.1.5) commutative. We must show that Λ is surjective and injective.

Surjectivity. Let $u \in \mathcal{U}_n(\tilde{A})$. By part (ii) of Exercise (5.3.10), there is i and $w \in \mathcal{U}_n(\tilde{A}_i)$ such that $||u - \tilde{\Phi}_i(w)|| < 2$. Thus u and $\tilde{\Phi}_i(w)$ are homotopic in $\mathcal{U}_n(\tilde{A})$ by Lemma 2.1.4. Hence

$$[u]_1 = [\tilde{\Phi}_i(w)]_1 = K_1(\Phi_i)([w]_1) = (\Lambda \circ \varphi_i)([w]_1),$$

and Λ is surjective.

Injectivity. It suffices to show that for each j the restriction of Λ to the image of φ_j is injective. So let $u \in \mathcal{U}_n(\tilde{A}_j)$ be such that $(\Lambda \circ \varphi_j)([u]_1) = K_1(\Phi_j)([u]_1) = [\tilde{\Phi}_j(u)]_1 = [1]_1$ in $K_1(A)$. We must show that $\varphi_j([u]_1) = 0$ in G. Indeed, there is m such that $\mathrm{diag}(\tilde{\Phi}_j(u), 1_m) \sim_h 1_{n+m}$ in $\mathcal{U}_{n+m}(\tilde{A})$. By part (iii) of Exercise (5.3.10), there is $i \geq j$ such that $\mathrm{diag}(\tilde{\Phi}_{ij}(u), 1_m)$ is homotopic to 1_{n+m}. Thus $[\tilde{\Phi}_{ij}(u)]_1 = [\mathrm{diag}(\tilde{\Phi}_{ij}(u), 1_m)]_1 = [1]_1$. Consequently, $\varphi_j([u]_1) = (\varphi_i \circ K_1(\tilde{\Phi}_{ij}))([u]_1) = 0$, and Λ is injective. □

Proposition 5.1.8 (Stability of K_1). *Let A be a C^*-algebra.*

(i) For each $n \in \mathbb{N}$ we have

$$K_1(A) \cong K_1(M_n(A)).$$

More specifically, let $\psi : A \to M_n(A)$ be such that $\psi(a) = \mathrm{diag}(a, 0_{n-1})$. Then $K_1(\psi) : K_1(A) \to K_1(M_n(A))$ is an isomorphism.

(ii) Let $\mathcal{K}$ be the C^*-algebra of compact operators. Then

$$K_1(A) \cong K_1(A \otimes \mathcal{K}).$$

More specfically, let p be a minimal projection in $\mathcal{K}$ and let $\varphi : A \to A \otimes \mathcal{K}$ be the map such that $\varphi(a) = a \otimes p$. Then $K_1(\varphi) : K_1(A) \to K_1(A \otimes \mathcal{K})$ is an isomorphism.

Proof. (i) Exercise.

(ii) Since

$$A \otimes \mathcal{K} \cong A \otimes (\lim M_n(\mathbb{C})) \cong \lim M_n(A),$$

the claim follows from part (i) and continuity of K_1. □

5.2 The Index Map

5.2.1 Fredholm index

Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space. We denot by $\mathcal{F}$ the algebra of finite rank operators on $\mathcal{H}$ (a two-sided $*$-ideal in $\mathcal{B}(\mathcal{H})$), by $\mathcal{K}$ the C^*-algebra of compact operators on $\mathcal{H}$ (the norm closure of $\mathcal{F}$ and the only non-trivial, norm closed, two-sided ideal of $\mathcal{B}(\mathcal{H})$), by $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}$ the Calkin algebra, and by $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}$ the natural surjection.

Theorem 5.2.1 (Atkinson). *If $F \in \mathcal{B}(\mathcal{H})$ then the following conditions are equivalent.*

(i) Both $\mathrm{Ker}\,(F)$ *and* $\mathrm{Coker}\,(F)$ *are finite dimensional.*

(ii) There exists an operator $G \in \mathcal{B}(\mathcal{H})$ such that both $FG-1$ and $GF-1$ are compact.

(iii) The image $\pi(F)$ of F in the Calkin algebra $\mathcal{Q}$ is invertible.

Furthermore, if F satisfies the above conditions then the range of F is closed in $\mathcal{H}$.

Proof. Obviously, conditions (ii) and (iii) are equivalent.

(i)$\Rightarrow$(ii) We first observe that the image of F is a closed subspace of $\mathcal{H}$. Indeed, let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$ of smallest possible dimension such that $\mathrm{Im}(F) + \mathcal{H}_0 = \mathcal{H}$. Then $n = \dim(\mathcal{H}_0)$ is finite, since the cokernel of F is finite dimensional. Then the restriction of F to the orthogonal complement of its kernel is a bijection from $\mathrm{Ker}\,(F)^\perp$ onto $\mathrm{Im}(F)$, and it extends to a linear bijection $\tilde{F} : \mathrm{Ker}\,(F)^\perp \oplus \mathbb{C}^n \longrightarrow \mathrm{Im}(F) + \mathcal{H}_0 = \mathcal{H}$. By the Inverse Mapping Theorem, the inverse of $\tilde{F}$ is continuous. It follows that $\mathrm{Im}(F) = \tilde{F}(\mathrm{Ker}\,(F)^\perp)$ is closed in $\mathcal{H}$.

By the preceding argument, F yields a continuous linear bijection from $\mathrm{Ker}\,(F)^\perp$ onto $\mathrm{Im}(F)$ — a closed subspace of $\mathcal{H}$. Thus, by the Inverse Mapping Theorem, it has a continuous inverse $G : \mathrm{Im}(F) \to \mathrm{Ker}\,(F)^\perp$. Extend G to a bounded linear operator on $\mathcal{H}$ (still denoted G) by setting $G\xi = 0$ for $\xi \in \mathrm{Im}(F)^\perp$. Then both $FG-1$ and $GF-1$ are finite dimensional and (ii) holds.

(ii)$\Rightarrow$(i) Let K be a compact operator such that $GF = 1+K$. Then $\mathrm{Ker}\,(F) \subseteq \mathrm{Ker}\,(GF) = \mathrm{Ker}\,(1+K)$, and $\mathrm{Ker}\,(1+K)$ is the eigenspace of K corresponding to eigenvalue -1. Since K is compact this eigenspace is finite dimensional and so is the kernel of F. We also have $\mathrm{Im}(F) \supseteq \mathrm{Im}(FG) = \mathrm{Im}(1+K)$. Since $1+K$ can be written as an invertible plus a finite rank operator, its range has finite codimenson. Thus $\mathrm{Coker}\,(F)$ is finite dimensional. □

A bounded operator satisfying the conditions of Theorem 5.2.1 is called *Fredholm*. In particular, any invertible operator in $\mathcal{B}(\mathcal{H})$ is Fredholm. It follows immediately from Theorem 5.2.1 that if F, T are Fredholm and K is compact then the operators F^*, FT and $F+K$ are Fredholm.

If F, G are Fredholm operators satisfying condition (ii) of Theorem 5.2.1, then G is called *parametrix* of F.

Definition 5.2.2 (Fredholm index). *Let F be a Fredholm operaor. Then its* Fredholm index *is an integer defined as*

$$\mathrm{Index}(F) = \dim(\mathrm{Ker}\,(F)) - \dim(\mathrm{Coker}\,(F)).$$

Since $\dim(\mathrm{Coker}\,(F)) = \dim(\mathrm{Ker}\,(F^*))$, we have $\mathrm{Index}(F) = -\mathrm{Index}(F^*)$. If F is Fredholm and V is invertible then clearly $\mathrm{Index}(FV) = \mathrm{Index}(VF) = \mathrm{Index}(F)$ and $\mathrm{Index}(V) = 0$.

Let $\{\xi_n : n = 0, 1, \ldots\}$ be an orthonormal basis of $\mathcal{H}$. The operator $S \in \mathcal{B}(\mathcal{H})$ such that $S(\xi_n) = \xi_{n+1}$ is called *unilateral shift*. It is a Fredholm operator with index -1. Thus for any positive integer k we have $\mathrm{Index}(S^k) = -k$ and $\mathrm{Index}((S^*)^k) = k$.

In a finite dimensional Hilbert space all operators are compact and hence all operators are Fredholm. The rank-nullity theorem of elementary linear algebra may then be interpreted as saying that every Fredholm operator on a finite dimensional Hilbert space has index 0.

Theorem 5.2.3 (Riesz). *If F is Fredholm and K is compact then*

$$\mathrm{Index}(F + K) = \mathrm{Index}(F).$$

Proof. We first observe that if R is of finite rank then $\mathrm{Index}(1 + R) = 0$. Indeed, let $\mathcal{H}_0 = \mathrm{Im}(R) + \mathrm{Ker}\,(R)^\perp$. Then $\mathcal{H}_0$ is finite dimensional, and the restriction of both R and R^* to $\mathcal{H}_0^\perp$ is zero. Thus the index of $1 + R$ coincides with the index of its restriction to $\mathcal{H}_0$ and hence is 0.

Now let K be compact. Find R of finite rank such that $||K - R|| < 1$. Then $V = 1 + (K - R)$ is invertible. Hence

$$\mathrm{Index}(1 + K) = \mathrm{Index}(V + R) = \mathrm{Index}(V(1 + V^{-1}R)) = 0.$$

Let F be a Fredholm operator of index 0. Then there is a finite rank operator R such that R maps bijectively $\mathrm{Ker}\,(F)$ onto $\mathrm{Im}(F)^\perp = \mathrm{Ker}\,(F^*)$. Let $V = F + R$. Then V is a continuous linear bijection of $\mathcal{H}$ onto itself and hence it is an invertible operator. Thus if K is compact then

$$\mathrm{Index}(F + K) = \mathrm{Index}(V + (K - R)) = \mathrm{Index}(V(1 + V^{-1}(K - R))) = 0.$$

Finally, let F be an arbitrary Fredholm operator and let K be compact. Then $\mathrm{Index}(F \oplus F^*) = 0$ and hence $\mathrm{Index}((F + K) \oplus F^*) = 0$. Consequenty, $\mathrm{Index}(F + K) = -\mathrm{Index}(F^*) = \mathrm{Index}(F)$. □

We showed in the course of the proof of Theorem 5.2.3 that if F is a Fredholm operator with index 0 then there exists a finite rank operator R such that $F + R$ is invertible.

Corollary 5.2.4. *If F, T are Fredholm operators then*

$$\mathrm{Index}(FT) = \mathrm{Index}(F) + \mathrm{Index}(G).$$

Proof. Suppose first that $\mathrm{Index}(F) = 0$, and let R be an operator of finite rank such that $F + R$ is invertible. Then

$$\mathrm{Index}(FT) = \mathrm{Index}(FT + RT) = \mathrm{Index}((F + R)T) = \mathrm{Index}(T).$$

Now suppose that $\mathrm{Index}(F) = k > 0$, and let S be a unilateral shift on $\mathcal{H}$. Then $\mathrm{Index}(F \oplus S^k) = 0$ and hence

$$\mathrm{Index}(FT \oplus S^k) = \mathrm{Index}((F \oplus S^k)(T \oplus 1)) = \mathrm{Index}(T \oplus 1) = \mathrm{Index}(T).$$

Consequently, we have

$$\mathrm{Index}(FT) = -\mathrm{Index}(S^k) + \mathrm{Index}(T) = \mathrm{Index}(F) + \mathrm{Index}(T),$$

as required. □

In particular, if G is a parametrix of F then $\mathrm{Index}(G) = -\mathrm{Index}(F)$.

Proposition 5.2.5. *The index map is locally constant and continuous in norm.*

Proof. Let F be a Fredholm operator and let G be its parametrix. Let K be compact such that $FG = 1 + K$. It suffices to show that if T is a Fredholm operator with $||T - F|| < 1/||G||$ then $\mathrm{Index}(F) = \mathrm{Index}(T)$. Indeed, the operator $(T - F)G + 1$ is invertible, since its distance from the identity is less than 1. Thus

$$\mathrm{Index}(T)+\mathrm{Index}(G) = \mathrm{Index}(TG) = \mathrm{Index}((T-F+F)G) = \mathrm{Index}(((T-F)G+1)+K) = 0.$$

Thus $\mathrm{Index}(T) = -\mathrm{Index}(G) = \mathrm{Index}(F)$. □

If F, T are two Fredholm operators then we say that they are *homotopic* if there exists a norm continuos path from F to T consisting of Fredholm operators.

Proposition 5.2.6. *Two Fredholm operators are homotopic iff they have the same index.*

Proof. Let F and T be Fredholm operators.

Suppose that F and T are homotopic, and let $t \mapsto V_t$ be a continuous path of Fredholm operators from F to T. Then the map $t \mapsto \mathrm{Index}(V_t)$ is continuous and hence constant.

To show the converse we first observe that every Fredholm operator V with $\mathrm{Index}(V) = 0$ is homotopic to 1. Indeed, there is a finite rank operator such that $V + R$ is invertible. Then $t \mapsto V + tR$ is a path connecting V to an invertible element, and in $\mathcal{B}(\mathcal{H})$ the group of invertibles is path-connected.

Now suppose that $\mathrm{Index}(F) = \mathrm{Index}(T)$. Then both FT^* and T^*T have index 0 and thus are homotopic to 1. Consequently, the operators F, $F(T^*T) = (FT^*)T$ and T are homotopic. □

Let u be a unitary in $M_n(\mathcal{Q})$ and let $U \in M_n(B(\mathcal{H}))$ be such that $\tilde{\pi}(U) = u$. Then U is a Fredholm operator on $\oplus^n\mathcal{H}$. Define a map $\mu : \mathcal{U}_\infty(\mathcal{Q}) \to \mathbb{Z}$ by $\mu(u) = \mathrm{Index}(U)$. It follows from the properties of Fredholm operators that μ satisfies conditions (i)–(iii) of the universal property of K_1. Thus, there exists a homomorphism $\mathrm{Index} : K_1(\mathcal{Q}) \to \mathbb{Z}$ such that $\mathrm{Index}([u]_1) = \mu(U) = \mathrm{Index}(U)$. It is not difficult to see that Index is an somorphism. Thus $K_1(\mathcal{Q}) \cong \mathbb{Z}$.

Since $\mathcal{Q}$ is properly infinite, there is no need to go to matrices over $\mathcal{Q}$ and we have $K_1(\mathcal{Q}) = \{[u]_1 : u \in \mathcal{U}(\mathcal{Q})\}$ (see Exercise 5.3.8). Furthermore, every unitary u in $\mathcal{Q}$ lifts to a *partial isometry* U in $\mathcal{B}(\mathcal{H})$ (Exercise 5.3.11). Thus $\dim(\mathrm{Ker}\,(U))$ equals the rank of $1 - U^*U$ and can be identified with the element $[1 - U^*U]_0$ in $K_0(\mathcal{K})$. Likewise, $\dim(\mathrm{Coker}\,(U))$ equals the rank of $1 - UU^*$ and can be identified with the element $[1 - UU^*]_0$ in $K_0(\mathcal{K})$. Consequently, we can view the index map as an isomorphism

$$\mathrm{Index} : K_1(\mathcal{Q}) \to K_0(\mathcal{K}),$$

such that if U is a partial isometry lift of u then

$$\mathrm{Index}([u]_1) = [1 - UU^*]_0 - [1 - UU^*]_0.$$

5.2.2 Definition of the index map

Let
$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{5.2.6}$$
be an exact sequence of C^*-algebras. Let $u \in \mathcal{U}_n(\tilde{B})$. Then there exists a unitary V in $\mathcal{U}_{2n}(\tilde{A})$ such that
$$\tilde{\psi}(V) = \mathrm{diag}(u, u^*). \tag{5.2.7}$$
Then $\tilde{\psi}(V\,\mathrm{diag}(1_n, 0)V^*) = \mathrm{diag}(1_n, 0)$. Thus there exists a projection P in $\mathcal{P}_{2n}(\tilde{J})$ such that
$$\tilde{\varphi}(P) = V\,\mathrm{diag}(1_n, 0)V^*. \tag{5.2.8}$$
Since $(\tilde{\psi} \circ \tilde{\varphi})(P) = \mathrm{diag}(1_n, 0)$, it follows that $s(P) = \mathrm{diag}(1_n, 0)$, where s is the scalar map. Then there is a well-defined map
$$\mu : \mathcal{U}_\infty(\tilde{B}) \to K_0(J) \text{ such that } \mu(u) = [P]_0 - [s(P)]_0.$$
Indeed, suppose that $W \in \mathcal{U}_{2n}(\tilde{A})$ and $Q \in \mathcal{P}_{2n}(\tilde{J})$ are such that $\tilde{\psi}(W) = \mathrm{diag}(u, u^*)$ and $\tilde{\varphi}(Q) = W\,\mathrm{diag}(1_n, 0)W^*$. We must show that $[P]_0 - [s(P)]_0 = [Q]_0 - [s(Q)]_0$ in $K_0(J)$. Indeed since $\tilde{\psi}(VW^*) = 1_{2n}$ there is $Y \in \mathcal{U}_{2n}(\tilde{J})$ such that $\tilde{\varphi}(Y) = VW^*$. Since
$$\tilde{\varphi}(P) = VW^*\tilde{\varphi}(Q)(VW^*)^* = \tilde{\varphi}(YQY^*),$$
we have $P = YQY^*$ and the claim follows. That is, $\mu : \mathcal{U}_\infty(\tilde{B}) \to K_0(J)$ is well-defined.

This map μ satisfies conditions (i)–(iii) of Proposition 5.1.3. We only verify (iii), leaving (i) and (ii) as exercise. So let $u \sim_h w \in \mathcal{U}_n(\tilde{B})$, $U, W \in \mathcal{U}_{2n}(\tilde{A})$, $P, Q \in \mathcal{P}_{2n}(\tilde{J})$ be such that $\tilde{\psi}(U) = \mathrm{diag}(u, u^*)$, $\tilde{\psi}(W) = \mathrm{diag}(w, w^*)$, $\tilde{\varphi}(P) = U\,\mathrm{diag}(1_n, 0)U^*$ and $\tilde{\varphi}(Q) = W\,\mathrm{diag}(1_n, 0)W^*$ (that is, $\{u, U, P\}$ and $\{w, W, Q\}$ satisfy conditions (5.2.7) and (5.2.8), respectively). Then $u^*w \sim_h 1_n \sim_h uw^*$ and thus there exist $X, Y \in \mathcal{U}_n(\tilde{A})$ such that $\tilde{\psi}(X) = u^*w$ and $\tilde{\psi}(Y) = uw^*$. Put $Z = U\,\mathrm{diag}(X, Y)$, a unitary in $\mathcal{U}_{2n}(\tilde{A})$. We have $\tilde{\psi}(Z) = \mathrm{diag}(w, w^*)$ and $\tilde{\varphi}(P) = Z\,\mathrm{diag}(1_n, 0)Z^*$. Thus, by the definition of μ, we have $\mu(w) = [P]_0 - [s(P)]_0 = \mu(u)$. The universal property of K_1 now implies that there exists a homomorphism
$$\partial_1 : K_1(B) \longrightarrow K_0(J),$$
called the *index map*, such that
$$\partial_1([u]_1) = [P]_0 - [s(P)]_0.$$

5.2.3 The exact sequence

Theorem 5.2.7. *Let*
$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$
be an exact sequence of C^*-algebras. Then the sequence
$$\begin{tikzcd} K_0(J) \arrow[r, "K_0(\varphi)"] & K_0(A) \arrow[r, "K_0(\psi)"] & K_0(B) \\ K_1(B) \arrow[u, "\partial_1"] & K_1(A) \arrow[l, "K_1(\psi)"] & K_1(J) \arrow[l, "K_1(\varphi)"] \end{tikzcd} \tag{5.2.9}$$
is exact everywhere.

Proof. By virtue of half-exactness of K_0 and K_1, it suffices to prove that $\mathrm{Im}(K_1(\psi)) = \mathrm{Ker}\,(\partial_1)$ and $\mathrm{Im}(\partial_1) = \mathrm{Ker}\,(K_0(\varphi))$.

1. We show $\mathrm{Im}(K_1(\psi)) \subseteq \mathrm{Ker}\,(\partial_1)$. Indeed, if $U \in \mathcal{U}_n(\tilde{A})$ then $\mathrm{diag}(\tilde{\psi}(U), \tilde{\psi}(U)^*)$ lifts to a diagonal unitary $V = \mathrm{diag}(U, U^*)$ and $\tilde{\varphi}(1_n) = V\,\mathrm{diag}(1_n, 0)V^* = \mathrm{diag}(1_n, 0)$. Thus $\partial_1(K_1(\psi)([U]_1)) = \partial_1([\tilde{\psi}(U)]_1) = [1_n]_0 - [s(1_n)]_0 = 0$.

2. We show $\mathrm{Im}(K_1(\psi)) \supseteq \mathrm{Ker}\,(\partial_1)$. To simplify notation, we identify J with its image in A and thus put $\varphi = \mathrm{id}$. Let $u \in \mathcal{U}_n(\tilde{B})$ be such that $[u]_1 \in \mathrm{Ker}\,(\partial_1)$. By Exercise 5.3.15, there is a partial isometry $U \in M_{2n}(\tilde{A})$ such that

$$\tilde{\psi}(U) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$0 = \partial_1([u]_1) = [1_{2n} - U^*U]_0 - [1_{2n} - UU^*]_0 \text{ in } K_0(J).$$

Thus there is k and $w \in M_{2n+k}(\tilde{J})$ such that

$$w^*w = (1_{2n} - U^*U) \oplus 1_k \quad \text{and} \quad ww^* = (1_{2n} - UU^*) \oplus 1_k.$$

Hence

$$\tilde{\psi}(w^*w) = \tilde{\psi}(ww^*) = \begin{pmatrix} 0 & 0 \\ 0 & 1_{n+k} \end{pmatrix}$$

and $\tilde{\psi}(w)$ is a scalar matrix, since $w \in M_{2n+k}(\tilde{J})$. Consequently,

$$\tilde{\psi}(w) = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix},$$

with z a scalar unitary matrix in $M_{n+k}(\tilde{B})$. In particular, z is homotopic to 1_{n+k} in $\mathcal{U}_{n+k}(\tilde{B})$. Set

$$V = w + \begin{pmatrix} U & 0 \\ 0 & 0_k \end{pmatrix},$$

an element of $M_{2n+k}(\tilde{A})$. By Exercise 5.3.18, V is unitary. We have

$$\tilde{\psi}(V) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & 1_{n+k} \end{pmatrix}.$$

Thus $[u]_1 - [\tilde{\psi}(V)]_1 = K_1(\psi)([V]_1)$.

3. We show $\mathrm{Im}(\partial_1) \subseteq \mathrm{Ker}\,(K_0(\varphi))$. Indeed, let $u \in \mathcal{U}_n(\tilde{B})$ and let $V \in \mathcal{U}_{2n}(\tilde{A})$, $P \in \mathcal{P}_{2n}(\tilde{J})$ be such that (5.2.7) and (5.2.8) hold. Then we have

$$K_0(\varphi)([P]_0 - [s(P)]_0) = [\tilde{\varphi}(P)]_0 - [1_n]_0 = [V\,\mathrm{diag}(1_n, 0)V^*]_0 - [1_n]_0 = 0.$$

4. We show $\mathrm{Im}(\partial_1) \supseteq \mathrm{Ker}\,(K_0(\varphi))$. Let $g \in \mathrm{Ker}\,(K_0(\varphi))$. By Lemma 4.2.2, there is n, a projection $p \in \mathcal{P}_n(\tilde{J})$ and a unitary $w \in \mathcal{U}_n(\tilde{A})$ such that

$$g = [p]_0 - [s(p)]_0 \quad \text{and} \quad w\tilde{\varphi}(p)w^* = s(p).$$

Set $u_0 = \tilde{\psi}(w(1_n - \tilde{\varphi}(p)))$, a partial isometry in $M_n(\tilde{B})$. We have

$$\begin{aligned} u_0^* u_0 &= 1_n - \tilde{\psi}(\tilde{\varphi}(p)), \\ u_0 u_0^* &= 1_n - \tilde{\psi}(s(p)) = u_0^* u_0. \end{aligned}$$

Thus $u = u_0 + (1_n - u_0 u_0^*)$ is unitary in $M_n(\tilde{B})$. We want to show that $g = \partial_1([u]_1)$. To this end, we frst find a lift of $\mathrm{diag}(u, 0_n)$ to a suitable partial isometry in $M_{2n}(\tilde{A})$. Take

$$V_0 = \begin{pmatrix} w(1_n - \tilde{\varphi}(p)) & 0 \\ 0 & s(p) \end{pmatrix},$$

a partial isometry in $M_{2n}(\tilde{A})$ such that

$$\tilde{\psi}(V_0) = \begin{pmatrix} u_0 & 0 \\ 0 & s(p) \end{pmatrix}.$$

Set

$$Z = \begin{pmatrix} 1_n - s(p) & s(p) \\ s(p) & 1_n - s(p) \end{pmatrix},$$

a self-adjoint, unitary scalar matrix, and put $V = Z V_0 Z^*$. Then we have

$$\tilde{\psi}(V) = Z\tilde{\psi}(V_0)Z^* = Z \begin{pmatrix} u_0 & 0 \\ 0 & s(p) \end{pmatrix} Z^* = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, by Exercise 5.3.15,

$$\partial_1([u]_1) = [\tilde{\varphi}^{-1}(1_{2n} - V^*V)]_0 - [\tilde{\varphi}^{-1}(1_{2n} - VV^*)]_0 = [\tilde{\varphi}^{-1}(1_{2n} - V_0^*V_0)]_0 - [\tilde{\varphi}^{-1}(1_{2n} - V_0V_0^*)]_0 =$$

$$\left|\begin{pmatrix} p & 0 \\ 0 & 1_n - s(p) \end{pmatrix}\right]_0 - \left|\begin{pmatrix} s(p) & 0 \\ 0 & 1_n - s(p) \end{pmatrix}\right]_0 = [p]_0 - [s(p)]_0 = g.$$

That is, $g = \partial_1([u]_1)$, as required.

□

5.3 Examples and Exercises

EXERCISE 5.3.1. Let A be a *unital* C^*-algebra. We have $\tilde{A} = A \oplus \mathbb{C}f$, where $f = 1_{\tilde{A}} - 1_A$. Define a unital $*$-homomorphism $\mu : \tilde{A} \to A$ by $\mu(a + \lambda f) = a$. As usual, for each n extend μ to a unital $*$-homomorphism $M_n(\tilde{A}) \to M_n(A)$ (still denoted μ). This yields a map $\mu : \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(A)$. Show that there exists an isomorphism $K_1(A) \to \mathcal{U}_\infty(A)/_{\sim_1}$ making the diagram

$$\begin{tikzcd} \mathcal{U}_\infty(\tilde{A}) \arrow[r, "\mu"] \arrow[d, "{[\cdot]_1}"'] & \mathcal{U}_\infty(A) \arrow[d] \\ K_1(A) \arrow[r] & \mathcal{U}_\infty(A)/_{\sim_1} \end{tikzcd} \tag{5.3.10}$$

commutative. To this end, show the following:

(i) $\mu(\mathrm{diag}(u, v)) = \mathrm{diag}(\mu(u), \mu(v))$,

(ii) if $u, v \in \mathcal{U}_n(\tilde{A})$ then $\mu(u) \sim_h \mu(v)$ iff $u \sim_h v$,

(iii) if $u, v \in \mathcal{U}_\infty(\tilde{A})$ then $\mu(u) \sim_1 \mu(v)$ iff $u \sim_1 v$.

(ii) Let $\mu(u) \sim_h \mu(v)$. By the definition of μ, there exist unitary $u_0, v_0 \in \mathcal{U}_n(\mathbb{C}f)$ such that $u = \mu(u)+u_0$ and $v = \mu(v)+v_0$. Since the unitary group of $M_n(\mathbb{C})$ is path-connected we have $u_0 \sim_h v_0$ in $\mathcal{U}_n(\mathbb{C}f)$. It follows that $u \sim_h v$ in $\mathcal{U}_n(\tilde{A})$.

EXERCISE 5.3.2. Show the following.

(i) $K_1(\mathbb{C}) = 0$.

(ii) For any two C^*-algebras A, B we have $K_1(A \oplus B) \cong K_1(A) \oplus K_1(B)$.

(iii) If A is an AF-algebra (see Example 4.4.14) then $K_1(A) = 0$.

EXAMPLE 5.3.3. If $\mathcal{H}$ is an infinite dimensional Hilbert space then $K_1(\mathcal{B}(\mathcal{H})) = 0$. Indeed, since $\mathcal{U}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{U}(\mathcal{B}(\oplus^n \mathcal{H}))$, it suffices to show that every unitary in $\mathcal{B}(\mathcal{H})$ is homotopic to the identity. But this follows from the fact that for every unitary u in $\mathcal{B}(\mathcal{H})$ there is a self-adjoint $a \in \mathcal{B}(\mathcal{H})$ such that $u = \exp(ia)$. Indeed, one may take $a = \varphi(u)$, where $\varphi : \mathbb{T} \to [0, 2\pi)$ is a bounded Borel function such that $\varphi(e^{i\theta}) = \theta$.

EXERCISE 5.3.4. Let X be a compact Hausdorff space.

(i) For each n identify $M_n(C(X))$ with $C(X, M_n(\mathbb{C}))$ and define the *determinant* function $\det : M_n(C(X)) \to C(X)$. Show that det maps $\mathcal{U}_\infty(C(X))$ into $\mathcal{U}(C(X))$.

(ii) Let $\langle v \rangle$ denote the class of $v \in \mathcal{U}(C(X))$ in $\mathcal{U}(C(X))/\mathcal{U}_0(C(X))$. Apply the universal property of K_1 to the map $\mathcal{U}_\infty(C(X)) \ni u \mapsto \langle \det(u) \rangle$ to show that there exists a homomorphism $D : K_1(A) \to \mathcal{U}(C(X))/\mathcal{U}_0(C(X))$ such that $D([u]_1) = \langle \det(u) \rangle$.

(iii) Show that the sequence

$$0 \longrightarrow \mathrm{Ker}\,(D) \longrightarrow K_1(C(X)) \xrightarrow{D} \mathcal{U}(C(X))/\mathcal{U}_0(C(X)) \longrightarrow 0$$

is split-exact, with a splitting map $\omega : \mathcal{U}(C(X))/\mathcal{U}_0(C(X)) \to K_1(C(X))$ given by $\omega(\langle u \rangle) = [u]_1$.

(iv) Let $X = \mathbb{T}$. Recall that $\varphi : \mathbb{R} \to \mathbb{T}$, $\varphi(x) = e^{2\pi i x}$, is a covering map. Thus, if $u \in \mathcal{U}(C(\mathbb{T}))$ then there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $u(e^{2\pi i t}) = e^{2\pi i f(t)}$. If f, g are two such functions then $f - g$ is a constant integer. Thus, there is a well-defined map $\mu : \mathcal{U}(C(\mathbb{T})) \to \mathbb{Z}$ given by $\mu(u) = f(1)-f(0)$ (the *winding number* of u). Show that μ induces an isomorphism of $\mathcal{U}(C(\mathbb{T}))/\mathcal{U}_0(C(\mathbb{T}))$ and $\mathbb{Z}$ such that $\langle u \rangle \mapsto \mu(u)$.

(v) Conclude that there exists a surjective homomorphism from $K_1(C(\mathbb{T}))$ onto $\mathbb{Z}$. In fact, we will see later that $K_1(C(\mathbb{T})) \cong \mathbb{Z}$.

EXERCISE 5.3.5. If A is a separable C^*-algebra then $K_1(A)$ is countable.

EXERCISE 5.3.6. Let A be a unital C^*-algebra. Replacing unitaries $\mathcal{U}_\infty(A)$ with invertibles $\mathrm{GL}_\infty(A)$ one can repeat the constructions from Section 5.1.1 and define an abelian group $\mathrm{GL}_\infty(A)/_{\sim_1}$. Show that this group is isomorphic to $K_1(A) = \mathcal{U}_\infty(A)/_{\sim_1}$ (see Exercise 5.3.1). Hint: For $w \in \mathrm{GL}_\infty(A)$ let $w = u|w|$ be the *polar decomposition*. Define a map $[\cdot]_1 : \mathrm{GL}_\infty(A) \to K_1(A)$ by $[w]_1 = [u]_1 = [w|w|^{-1}]_1$ and use Proposition 2.1.10.

EXERCISE 5.3.7. Let A be a non-unital C^*-algebra, and let $s : \tilde{A} \to \tilde{A}$ be the scalar map $s(a + t1) = t1$. Define

$$\begin{aligned} \mathcal{U}^+(A) &= \{u \in \mathcal{U}(\tilde{A}) : s(u) = 1\}, \\ \mathcal{U}^+_n(A) &= \{u \in \mathcal{U}(M_n(\tilde{A})) : s_n(u) = 1_n\}, \\ \mathcal{U}^+_\infty(A) &= \bigcup_{n=1}^{\infty} \mathcal{U}^+_n(A). \end{aligned}$$

Proceeding as in Section 5.1.1, one can define an abelian group $\mathcal{U}^+_\infty(A)/_{\sim_1}$. Show that this group is isomorphic to $K_1(A)$.

EXERCISE 5.3.8. Let A be a unital C^*-algebra.

(i) Let u be unitary and let s be an isometry in A. Then $sus^* + (1 - ss^*)$ is unitary and we have

$$\begin{pmatrix} s & 1-ss^* \\ 0 & s^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & 1-ss^* \\ 0 & s^* \end{pmatrix}^* = \begin{pmatrix} sus^* + (1-ss^*) & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $[sus^* + (1 - ss^*)]_1 = [u]_1$.

(ii) Let $u_1, \ldots, u_n$ be unitary elements of A and let $s_1, \ldots, s_n$ be isometries in A with mutually orthogonal range projections. Then

$$u = s_1 u_1 s_1^* + \ldots + s_n u_n s_n^* + (1 - s_1 s_1^* - \ldots - s_n s_n^*)$$

is unitary. Use (i) to show that $[u]_1 = [u_1]_1 [u_2]_1 \ldots [u_n]_1$.

(iii) et $s_1, \ldots, s_n$ be isometres as in (ii). Put

$$V = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

hen V is an isometry in $M_n(A)$. Show that for any unitary $u \in \mathcal{U}_n(A)$ there is a unitary w in A such that

$$VuV^* + (1_n - VV^*) = \operatorname{diag}(w, 1_{n-1})$$

(iv) Let A be *properly infinite* (see Exercise 3.3.13). Show that

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}(A)\}.$$

EXAMPLE 5.3.9. The K_1-functor is not exact. Indeed, for a separable Hilbert space $\mathcal{H}$ the sequence

$$0 \longrightarrow \mathcal{B}(\mathcal{H}) \xrightarrow{\pi} \mathcal{B}(\mathcal{H})/\mathcal{K} \longrightarrow 0$$

of C^*-algebras is exact. But $K_1(\mathcal{B}(\mathcal{H})) = 0$ and we will see later that $K_1(\mathcal{B}(\mathcal{H})/\mathcal{K}) \cong \mathbb{Z}$. Thus $K_1(\pi)$ cannot be surjective. Likewise, there is an exact sequence

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\varphi} C([0,1]).$$

But $K_1(C([0,1])) = 0$ and we will see later that $K_1(C_0((0,1))) \cong \mathbb{Z}$. Thus $K_1(\varphi)$ cannot be injective.

EXERCISE 5.3.10. Let $A = \varinjlim\{A_i, \Phi_{ij}\}$ be the inductive limit of a sequence of C^*-algebras, and let $\Phi_i : A_i \to A$ be the canonical maps.

(i) For any invertible $y \in \tilde{A}$ and any $\epsilon > 0$ there is arbitrarily large i and invertible $z \in \tilde{A}_i$ such that $||y - \tilde{\Phi}_i(z)|| < \epsilon$.

(ii) For any unitary $u \in \tilde{A}$ and any $\epsilon > 0$ there is arbitrarily large i and unitary $w \in \tilde{A}_i$ such that $||u - \tilde{\Phi}_i(w)|| < \epsilon$.

(iii) If u is unitary in $\tilde{A}_j$ such that $\tilde{\Phi}_j(u) \sim_h 1$ in $\tilde{A}$, then there is arbitrarily large i such that $\tilde{\Phi}_{ij}(u) \sim_h 1$ in $\tilde{A}_i$.

(iv) Parts (i)–(iii) remain valid with $\tilde{A}$ and $\tilde{A}_i$ replaced by $M_n(\tilde{A})$ and $M_n(\tilde{A})$, respectively.

(i) First find k and $x, x' \in \tilde{A}_k$ so that both $||\tilde{\Phi}_k(x) - y||$ and $||\tilde{\Phi}_k(x') - y^{-1}||$ are small. Thus both $||\tilde{\Phi}_k(xx' - 1)||$ and $||\tilde{\Phi}_k(x'x - 1)||$ are small. Then, using Exercise 4.4.17, take i large enough so that both $||\tilde{\Phi}_{ik}(xx' - 1)||$ and $||\tilde{\Phi}_{ik}(x'x - 1)||$ are small. Then $z = \tilde{\Phi}_{ik}(x)$ is both left and right invertible, hence invertible, and $\tilde{\Phi}_i(z)$ approximates y.

(ii) This follows from part (i) and continuity of the polar decomposition (see Proposition 2.1.10).

(iii) Let w_t, $t \in [0, 1]$, be a continuous path of unitaries in $\tilde{A}$ connecting $w_0 = \tilde{\Phi}_j(u)$ and $w_1 = 1$. By compactness, there are $0 = t_0 < t_1 < \ldots < t_{k+1} = 1$ such that $||w_{t_{r+1}} - w_{t_r}|| < 2$ for all r. Applying repeatedly part (ii), find $m \geq j$ and unitary elements $v_1, \ldots, v_k$ in $\tilde{A}_i$ so close to $w_{t_1}, \ldots, w_{t_k}$, respectively, that all the norms: $||\tilde{\Phi}_j(u) - \tilde{\Phi}_m(v_1)||$, $||\tilde{\Phi}_m(v_k) - 1||$, and $\tilde{\Phi}_m(v_{r+1}) - \tilde{\Phi}_m(v_r)||$ for $r = 1, \ldots, k-1$ are less than 2. Then by Exercise 4.4.17, there is arbitrarily large $i \geq m$ such that all the norms $||\tilde{\Phi}_{ij}(u) - \tilde{\Phi}_{im}(v_1)||$, $||\tilde{\Phi}_{im}(v_k) - 1||$, and $\tilde{\Phi}_{im}(v_{r+1}) - \tilde{\Phi}_{im}(v_r)||$ for $r = 1, \ldots, k-1$ are less than 2. Now the claim follows from Lemma 2.1.4.

(iv) Exercise.

EXERCISE 5.3.11. Show that every unitary in the Calkin algebra $\mathcal{Q}$ lifts to a partial isometry in $\mathcal{B}(\mathcal{H})$. In fact, it can be lifted to an isometry or a coisometry.

EXERCISE 5.3.12. Let $\psi : A \to B$ be a surjective $*$-homomorphism of C^*-alebras. Show the following.

(i) For each $b = b^* \in B$ there is $a = a^* \in A$ such that $||a|| = ||b||$ and $\psi(a) = b$.

(i) For each $b \in B$ there is $a \in A$ such that $||a|| = ||b||$ and $\psi(a) = b$.

(i) Take any $t \in A$ with $\psi(t) = b$ and set $x = 1/2(t + t^*)$. Then $x = x^*$ and $\psi(x) = b$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(r) = r$ if $|r| \leqq ||b||$ and $|f(r)| = ||b||$ if $|r| \geq ||b||$. Put $a = f(x)$. Then $\psi(a) = \psi(f(x)) = f(\psi(x)) = f(b) = b$, and $||a|| \leqq ||b||$. But $||b|| = ||\psi(a)|| \leqq ||a||$ since $||\psi|| = 1$. Thus $||a|| = ||b||$.

(ii) Consider $\psi_2 : M_2(A) \to M_2(B)$, and put

$$y = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}.$$

Since $y = y^*$, there is $x = x^* \in M_2(A)$ such that $\psi_2(x) = y$ and $||x|| = ||y|| = ||b||$, by part (i). Let

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

and set $a = x_{12}$. Then $\psi(a) = b$ and $||a|| \leqq ||x|| = ||b||$. But $||b|| = ||\psi(a)|| \leqq ||a||$ since $||\psi|| = 1$. Thus $||a|| = ||b||$.

EXERCISE 5.3.13. Consider an exact sequence of C^*-algebras

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\psi} B \longrightarrow 0,$$

in which we identify J with its image in A. Let u be a unitary in $\mathcal{U}_n(\tilde{B})$. By part (ii) of Exercse 5.3.12, there is $a \in \mathcal{U}_n(\tilde{A})$ such that $\tilde{\psi}(a) = u$ and $||a|| = ||u|| = 1$. Then for any continuous function $f : \mathbb{R} \to \mathbb{C}$ we have $af(a^*a) = f(aa^*)a$. Use this to show that

$$V = \begin{pmatrix} a & (1_n - aa^*)^{1/2} \\ -(1_n - a^*a)^{1/2} & a^* \end{pmatrix}$$

is a unitary in $\mathcal{U}_{2n}(\tilde{A})$. Then show that

$$\tilde{\psi}(V) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

and

$$V \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} V^* = \begin{pmatrix} aa^* & -a(1_n - a^*a)^{1/2} \\ -(1_n - a^*a)^{1/2}a^* & 1_n - a^*a \end{pmatrix}.$$

Then write explicitly $\partial_1([u]_1)$.

EXERCISE 5.3.14. Consider an exact sequence of C^*-algebras

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\psi} B \longrightarrow 0,$$

in which we identify J with its image in A. Suppose that $u \in \mathcal{U}_n(\tilde{B})$ is such that there exists a *partial isometry* $U \in M_n(\tilde{A})$ with $\tilde{\psi}(U) = u$. Show the following.

(i) The element

$$V = \begin{pmatrix} U & 1_n - UU^* \\ 1_n - U^*U & U^* \end{pmatrix}$$

is unitary in $\mathcal{U}_{2n}(\tilde{A})$ and $\tilde{\psi}(V) = \mathrm{diag}(u, u^*)$.

(ii) Both $1_n - U^*U$ and $1_n - UU^*$ are projections in $M_n(J)$, and

$$\partial_1([u]_1) = [1_n - U^*U]_0 - [1_n - UU^*]_0.$$

EXERCISE 5.3.15. Consider an exact sequence of C^*-algebras

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\psi} B \longrightarrow 0,$$

in which we identify J with its image in A. Let $u \in \mathcal{U}_n(\tilde{B})$, and let $a \in M_n(\tilde{A})$ be such that $\tilde{\psi}(a) = u$ and $||a|| = ||u|| = 1$. Put

$$U = \begin{pmatrix} a & 0 \\ (1_n - a^*a)^{1/2} & 0 \end{pmatrix}.$$

Show that $U^*U = \mathrm{diag}(1_n, 0)$, which entails that U is a partial isometry. Then show that

$$\tilde{\psi}(U) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, show that

$$\partial_1([u]_1) = [1_{2n} - U^*U]_0 - [1_{2n} - UU^*]_0.$$

Exercise 5.3.16. Consider an exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \longrightarrow C(\mathbb{D}) \longrightarrow C(S^1) \longrightarrow 0,$$

with the map $C(\mathbb{D}) \to C(S^1)$ given by the restriction. In the corresponding exact sequence

$$\begin{tikzcd} K_0(C_0(\mathbb{R}^2)) \arrow[r] & K_0(C(\mathbb{D})) \arrow[r] & K_0(C(S^1)) \\ K_1(C(S^1)) \arrow[u, "\partial_1"] & K_1(C(\mathbb{D})) \arrow[l] & K_1(C_0(\mathbb{R}^2)) \arrow[l] \end{tikzcd} \tag{5.3.11}$$

we have $K_1(C(\mathbb{D})) = 0$, since $C(\mathbb{D})$ and $\mathbb{C}$ are homotopy equivalent. Show that the map $K_0(C(\mathbb{D})) \to K_0(C(S^1))$ is injective, and conclude that $\partial_1 : K_1(C(S^1)) \longrightarrow K_0(C_0(\mathbb{R}^2))$ is an isomorphism. Then calculate $\partial_1([z]_1)$ and thus find a generator of $K_0(C_0(\mathbb{R}^2))$.

Exercise 5.3.17 (Naturality of the index map). Show that every commutative diagram of C^*-algebras, with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram of abelian groups, with exact rows:

$$\begin{tikzcd} K_1(J_1) \arrow[r] \arrow[d] & K_1(A_1) \arrow[r] \arrow[d] & K_1(B_1) \arrow[r, "\partial_1"] \arrow[d] & K_0(J_1) \arrow[r] \arrow[d] & K_0(A_1) \arrow[r] \arrow[d] & K_0(B_1) \arrow[d] \\ K_1(J_2) \arrow[r] & K_1(A_2) \arrow[r] & K_1(B_2) \arrow[r, "\partial_1"'] & K_0(J_2) \arrow[r] & K_0(A_2) \arrow[r] & K_0(B_2) \end{tikzcd}$$

Exercise 5.3.18. Show that if partial isometries $v_1, \ldots, v_n$ in a unital C^*-algebra satisfy

$$v_1v_1^* + \ldots + v_nv_n^* = 1 = v_1^*v_1 + \ldots + v_n^*v_n$$

then $u = v_1 + \ldots + v_n$ is unitary. Hint: use Exercise 2.4.3.

Chapter 6

Bott Periodicity and the Exact Sequence of K-Theory

6.1 Higher K-Groups

6.1.1 The suspension functor

Recall that the *suspension* SA of a C^*-algebra A is defined as

$$SA = \{f \in C([0,1], A) : f(0) = f(1) = 0\},$$

and is isomorphic to $C_0((0,1), A) \cong C_0(\mathbb{R}) \otimes A$ (cf. Lemma 1.3.1). If $\varphi : A \to B$ is a $*$-homomorphism between two C^*-algebras, then $S\varphi : SA \to SB$, given by $(S\varphi(f))(t) = \varphi(f(t))$ is a $*$-homomorphism between their suspensions. It is not difficult to verif that this yields a covariant functor from the category of C^*-algebras into itself.

Proposition 6.1.1. *The suspension functor S is exact. That is, if*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

is an exact sequence of C^*-algebras then the sequence

$$0 \longrightarrow SJ \xrightarrow{S\varphi} SA \xrightarrow{S\psi} SB \longrightarrow 0$$

is exact.

Proof. Exercise. $\square$

6.1.2 Isomorphism of $K_1(A)$ and $K_0(SA)$

Let A be a C^*-algebra. We define a map

$$\theta_A : K_1(A) \longrightarrow K_0(SA),$$

as follows. By Exercise 6.4.1, each element of $K_1(A)$ is represented by a unitary $u \in \mathcal{U}_n(\tilde{A})$ (for some n) such that $s(u) = 1_n$. For such a u we can find a continuous function

$v : [0,1] \to \mathcal{U}_{2n}(\tilde{A})$ such that $v(0) = 1_{2n}$, $v(1) = \mathrm{diag}(u, u^*)$ and $s(v(t)) = 1_{2n}$ for all $t \in [0,1]$. We put

$$p = v \begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix} v^*,$$

a projection in $\mathcal{P}_{2n}(\widetilde{SA})$ with $s(p) = \mathrm{diag}(1_n, 0_n)$. We set

$$\theta_A([u]_1) = [p]_0 - [s(p)]_0.$$

Theorem 6.1.2. *For any C^*-algebra A, the map*

$$\theta_A : K_1(A) \longrightarrow K_0(SA)$$

is an isomorphism. Furthermore, if B is a C^*-algebra and $\varphi : A \to B$ is a $*$-homomorphism then the diagram

$$\begin{array}{ccc} K_1(A) & \xrightarrow{K_1(\varphi)} & K_1(B) \\ {\scriptstyle\theta_A}\downarrow & & \downarrow{\scriptstyle\theta_B} \\ K_0(SA) & \xrightarrow[K_0(S\varphi)]{} & K_0(SB) \end{array} \tag{6.1.1}$$

is commutative.

Proof. Recall from Example 4.4.9 the exact sequence

$$0 \longrightarrow SA \longrightarrow CA \xrightarrow{\pi} A \longrightarrow 0, \tag{6.1.2}$$

where CA is the cone over A. Since CA is homotopy equivalent to $\{0\}$ we have $K_0(CA) = K_1(CA) = 0$. Let $\partial_1 : K_1(A) \to K_0(SA)$ be the index map associated with the extension (6.1.2). It follows from Theorem 5.2.7 that ∂_1 is an isomorphism. Thus, it suffices to identify ∂_1 with θ_A (exercise). □

6.1.3 The long exact sequence of K-theory

For each natural number $n \geq 2$ we define inductively a covariant functor from the category of C^*-algebras t the category of abelian groups as follows. $K_n(A) = K_{n-1}(SA)$, and if $\varphi : A \to B$ is a $*$-homomorphism then $K_n(\varphi) = K_{n-1}(S\varphi)$. It is clear that such defined functor is half-exact.

Now suppose that

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{6.1.3}$$

is an exact sequence of C^*-algebras. Then $\partial_1 : K_1(B) \to K_0(J)$ is the index map. We define higher index maps

$$\partial_n : K_n(B) \to K_{n-1}(J),$$

as follows. Applying $n-1$ times the suspension functor to sequence (6.1.3), we get an exact sequence

$$0 \longrightarrow S^{n-1}J \xrightarrow{S^{n-1}\varphi} S^{n-1}A \xrightarrow{S^{n-1}\psi} S^{n-1}B \longrightarrow 0. \tag{6.1.4}$$

Let $\overline{\partial}_1 : K_1(S^{n-1}B) \to K_0(S^{n-1}J)$ be the index map associated with (6.1.4). By the definition of higher K-functors, we have $K_n(B) = K_1(S^{n-1}B)$ and $K_{n-1}(J) = K_1(S^{n-2}J)$. By Theorem 6.1.2, there is an isomorphism $\theta_{S^{n-2}J} : K_1(S^{n-2}J) \to K_0(S^{n-1}J)$. We define

$$\partial_n = \theta^{-1}_{S^{n-2}J} \circ \overline{\partial}_1.$$

Such defined higher index maps have naturality analogous to the one enjoyed by the usual index (cf. Exercise 5.3.17).

Proposition 6.1.3. *Every short exact sequence*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

of C^*-algebras induces a long exact sequence on K-theory:

$$\ldots \xrightarrow{\partial_{n+1}} K_n(J) \xrightarrow{K_n(\varphi)} K_n(A) \xrightarrow{K_n(\psi)} K_n(B) \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} K_0(J) \xrightarrow{K_0(\varphi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B).$$

Proof. Exercise. □

This Proposition serves only as an intermediate step towards the fundamental 6-term exact sequence of K-theory. The point is that $K_{n+2} \cong K_n$ (as we will see in the next section), and the apparently infinite sequence from Proposition 6.1.3 shrinks to a much more useful finite one, which contains only K_0 and K_1.

6.2 Bott Periodicity

In this section, we prove the fundamental result of Bott that $K_0(A) \cong K_1(SA)$ for any C^*-algebra A. Combined with Theorem 6.1.2, it says $K_{n+2}(A) \cong K_n(A)$ — the Bott periodicity.

6.2.1 Definition of the Bott map

We begin by defining a Bott map

$$\beta_A : K_0(A) \longrightarrow K_1(SA)$$

for unital C^*-algebras A, and then reduce the general case to the unital one. So let A be a unital C^*-algebra. We use the obvious identification

$$SA = \{f : \mathbb{T} \to A : f \text{ continuous}, \ f(1) = 0\}.$$

Thus, elements of $M_n(SA)$ may be identified with continuous *loops* $f : \mathbb{T} \to M_n(A)$ such that $f(1) = 0$. It follows that $M_n(\widetilde{SA})$ may be identified with continuous functions $f : \mathbb{T} \to M_n(A)$ such that $f(1) \in M_n(\mathbb{C}1_A)$.

For any natural n and any projection $p \in \mathcal{P}_n(A)$ we define a *projection loop* $f_p : \mathbb{T} \to \mathcal{U}_n(A)$ by

$$f_p(z) = zp + (1_n - p), \quad z \in \mathbb{T}.$$

Clearly, we have $f_p \in \mathcal{U}_n(\widetilde{SA})$. By the universal property of K_0 we get a homomorphism $\beta_A : K_0(A) \longrightarrow K_1(SA)$ such that

$$\beta_A([p]_0) = [f_p]_1,$$

called the *Bott map.*

Now if $\varphi : A \to B$ is a unital $*$-homomorphism, then $\widetilde{S\varphi}(f_p)(z) = \varphi(f_p(z)) = f_{\varphi(p)}(z)$ for all $z \in \mathbb{T}$. Hence the diagram

$$\begin{tikzcd} K_0(A) \arrow[r, "K_0(\varphi)"] \arrow[d, "\beta_A"'] & K_0(B) \arrow[d, "\beta_B"] \\ K_1(SA) \arrow[r, "K_1(S\varphi)"'] & K_1(SB) \end{tikzcd} \tag{6.2.5}$$

is commutative. This is the *naturality* of the Bott map.

Finally, suppose that A does not have a unit. Then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r] & K_0(\tilde{A}) \arrow[r] \arrow[d, "\beta_{\tilde{A}}"'] & K_0(\mathbb{C}) \arrow[r] \arrow[d, "\beta_{\mathbb{C}}"] & 0 \\ 0 \arrow[r] & K_1(SA) \arrow[r] & K_1(S\tilde{A}) \arrow[r] & K_1(\mathbb{C}) \arrow[r] & 0 \end{tikzcd} \tag{6.2.6}$$

with split-exact rows. It follows that there is exactly one map $\beta_A : K_0(A) \to K_1(SA)$ which completes the diagram. By Exercise 6.4.2, we have

$$\beta_A([p]_0 - [s(p)]_0) = [f_p f^*_{s(p)}]_1.$$

6.2.2 The periodicity theorem

The following teorem is considered a central result of K-theory.

Theorem 6.2.1. *For any C^*-algebra A, the Bott map*

$$\beta_A : K_0(A) \longrightarrow K_1(SA)$$

is an isomorphism.

Proof. It suffices to prove the theorem for unital C^*-algebras. Indeed, the general case follows from the unital one and (6.2.6) through a diagram chase. Thus assume A is unital. It will be convenient for us to use the description of $K_1(SA)$ as the collection of suitable equivalence classes in $GL_\infty(\widetilde{SA})$ (see Exercise 5.3.6). We must show that the Bott map $\beta_A : K_0(A) \longrightarrow K_1(SA)$ is both surjective and injective.

Surjectivity. We consider the following subsets of $GL_\infty(\widetilde{SA})$:

$$\begin{aligned} GL^n &= \{f : \mathbb{T} \to GL_n(A) : f \text{ continuous and } f(1) \in M_n(\mathbb{C}1_A)\}, \\ LL^n_m &= \{f \in GL^n : f \text{ a Laurent polynomial in } z \text{ with coefficients in} \\ &\qquad M_n(A) \text{ and } \deg(f) \leqq m\}, \\ PL^n_m &= \{f \in LL^n_m : f \text{ a polynomial}\}, \\ PRL^n &= \{f_p : p \in \mathcal{P}_n(A)\}. \end{aligned}$$

Elements of GL^n, LL^n_m, PL^n_m and PL^n_1 are called *invertible loops*, *Laurent loops*, *polynomial loops* and *linear loops*, respectively. We have

$$PRL^n \subseteq PL^n_1 \subseteq \bigcup_m PL^n_m \subseteq \bigcup_m LL^n_m \subseteq GL^n$$

and $K_1(SA) = \{[f]_1 : f \in GL^n,\ n \in \mathbb{N}\}$.

Step 1. $\bigcup_m LL^n_m$ is dense in GL^n. Indeed, $\operatorname{span}\{z^k : k \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$ by the Stone-Weierstrass theorem. Hence $\operatorname{span}\{z^k : k \in \mathbb{Z}\} \otimes A$ is dense in $C(\mathbb{T}) \otimes A$, and this easily implies that Laurent loops are a dense subset of invertible loops.

Step 2. By virtue of Step 1, it suffices to show that the range of β_A contains the equivalence classes of all Laurent loops. But each Laurent loop is a quotient of two polynomal loops. Thus, it suffices to show that the range of β_A contains the classes of all polynomial loops. To this end, we show that for each $n, m \in \mathbb{N}$ there is a continuous map

$$\mu^n_m : PL^n_m \longrightarrow PL^{mn+n}_1$$

such that $\mu^n_m(f) \sim_h \operatorname{diag}(f, 1_{mn})$ within PL^{mn+n}_k for all $f \in PL^n_k$, $k \leqq m$. Indeed, let $f(z) = \sum_{j=0}^{m} a_j z^j$, with $a_j \in M_n(A)$ for all $j = 0, \dots, m$. For each z, we define

$$\tilde{\mu}^n_m(f)(z) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{m-1} & a_m \\ -z & 1 & 0 & \dots & 0 & 0 \\ 0 & -z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -z & 1 \end{pmatrix},$$

an element of $M_{m+1}(M_n(A))$. (In the above matrix we wrote 1 for 1_n and z for $z1_n$.) Cleary, $\tilde{\mu}^n_m(f)(z) = T_0 + T_1 z$ for some $T_0, T_1 \in M_{mn+n}(A)$, and the map $f \mapsto \tilde{\mu}^n_m(f)$ is continuous. We claim the following:

(i) $\tilde{\mu}^n_m(f)(z)$ is invertible for all z,

(ii) $\tilde{\mu}^n_m(f)(1) \sim_h 1_{mn+n}$,

(iii) $\tilde{\mu}^n_m(f) \sim_h \operatorname{diag}(f, 1_{mn})$.

Once the properties (i)–(iii) are established, we obtain the desired map μ^n_m by setting $\mu^n_m(f) = (\tilde{\mu}^n_m(f)(1))^{-1}\tilde{\mu}^n_m(f)$.

In order to prove properties (i)–(iii), we consider matrices

$$A_m = \begin{pmatrix} 1 & 0 & 0 & \dots & -a_m \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad A_{m-1} = \begin{pmatrix} 1 & 0 & \dots & -(a_{m-1} + a_m z) & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \dots,$$

$$A_1 = \begin{pmatrix} 1 & -(a_1 + a_2 z + \dots + a_m z^{m-1}) & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

and matrices B_k having 1's on the main diagonal, z in the entry in column k and row $k+1$, and 0's elsewhere. Then we have

$$A_1 A_2 \cdots A_m \tilde{\mu}_m^n(f)(z) = \begin{pmatrix} f(z) & 0 & \dots & 0 & 0 \\ -z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -z & 1 \end{pmatrix}$$

and

$$A_1 A_2 \cdots A_m \tilde{\mu}_m^n(f)(z) B_m B_{m-1} \cdots B_1 = \operatorname{diag}(f(z), 1_{mn}). \tag{6.2.7}$$

Since $f(z)$ and all of the matrices $A_1, \dots, A_m, B_1, \dots, B_m$ are invertible for all z, (6.2.7) implies (i). Furthermore, each of the A_j and B_j matrices may be continuously deformed to the identity within the set polynomial loops by multiplying the sole off-diagonal entry with a parameter $t \in [0,1]$. Thus (6.2.7) implies (ii) and (iii).

Step 3. By virtue of Step 2, it suffices to show that the range of β_A contains the equivalence classes of all linear loops. This will follow if we show that there exists a continuous retraction

$$\nu : PL_1^n \longrightarrow PRL^n$$

such that $\nu(f) \sim_h f$ within PL_1^n for all $f \in PL_1^n$. Indeed, let $f(z) = a_0 + a_1 z$. Then $f(1) = a_0 + a_1$ is an invertible element of $M_n(\mathbb{C}1_A)$, and we can put $g = f(1)^{-1} f$. Then

$$g(z) = 1_n + b(z-1),$$

with $b = (a_0 + a_1)^{-1} a_1$. When $z \neq 1$ we can write

$$g(z) = (1-z)\left(\frac{1}{1-z} 1_n - b\right),$$

and since $g(z)$ is invertible for all $z \in \mathbb{T}$ we see that $1/(1-z) \notin \operatorname{sp}(b)$ if $z \in \mathbb{T} \setminus \{1\}$. Since the function $z \mapsto 1/(1-z)$ maps $\mathbb{T} \setminus \{1\}$ onto the line $\{\lambda \in \mathbb{C} : \Re(\lambda) = 1/2\}$, we see that

$$\operatorname{sp}(b) \subseteq \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \Re(\lambda) = 1/2\}.$$

For $t \in [0,1]$ consider a function

$$g_t(z) = \begin{cases} tz & \text{if } \Re(z) < 1/2, \\ tz + (1-t) & \text{if } \Re(z) > 1/2. \end{cases}$$

Each function g_t is holomorphic on an open neighbourhood of $\operatorname{sp}(b)$ and thus the holomorphic function calculus (see Exercise 6.4.3) yields elements $g_t(b) \in M_n(A)$, which depend continuously on the parameter t. Since the image of $g_t(z)$ does not intersect the line $\{\lambda \in \mathbb{C} : \Re(\lambda) = 1/2\}$, the elements

$$h_t(z) = 1_n + g_t(b)(z-1) = (1-z)\left(\frac{1}{1-z} 1_n - g_t(b)\right)$$

are invertible. We have $g_1(z) = z$ and thus $g_1(b) = b$. On the other hand, $g_0(z)^2 = g_0(z)$ and thus $e = g_0(b)$ is an idempotent. Consequently, $t \mapsto h_t$ is a homotopy within PL_1^n between g and the idempotent loop $1_n + e(z-1)$. Now we can deform the idempotent e to a projection, as follows (cf. Exercise 3.3.20).

Lemma 6.2.2. *Let B be a unital C^*-algebra. Recall that $\mathcal{I}(B)$ denotes the set of idempotents in B and $\mathcal{P}(B)$ denotes the set of projections (i.e. self-adjoint idempotents) in B. Then we have the following.*

(i) For every idempotent $e \in B$ the element

$$\rho(e) = ee^*(1 + (e - e^*)(e^* - e))^{-1}$$

is a projection.

(ii) The map $\rho : \mathcal{I}(B) \to \mathcal{P}(B)$, defined in (i), is a continuous retraction. In particular, $\rho(e) \sim_h e$ in $\mathcal{I}(B)$ for every idempotent e.

(iii) If $p, q \in \mathcal{P}(B)$ and $p \sim_h q$ in $\mathcal{I}(B)$, then $p \sim_h q$ in $\mathcal{P}(B)$.

Proof. (i) Put $w = 1+(e-e^*)(e^*-e)$. Then w is positive and invertible, thus $\rho(e) = ee^*w^{-1}$ is well-defined. A straightforward calculation yields $ew = ee^*e = we$ and $e^*w = e^*ee^* = we^*$. Thus $ee^*w = (ee^*)^2 = wee^*$ and $ee^*w^{-1} = w^{-1}ee^*$. This implies that ee^*w^{-1} is self-adjoint and that

$$\rho(e)^2 = ee^*w^{-1}ee^*w^{-1} = (ee^*)^2w^{-2} = ee^*w^{-1} = \rho(e).$$

Whence $\rho(e)$ is a projection.

(ii) Clearly, ρ is a continuous map and $\rho(p) = p$ if p is a projection.

To see that $\rho(e) \sim_h e$ in $\mathcal{I}(B)$, set $u_t = 1 - t(e - \rho(e))$ for $t \in [0, 1]$. Since $\rho(e)e = e$ an $e\rho(e) = \rho(e)$, we have $(e - \rho(e))^2 = 0$. Therefore u_t is invertible with the inverse $u_t^{-1} = 1 + t(e - \rho(e))$. Thus $u_t^{-1}eu_t$ is an idempotent for all $t \in [0, 1]$, and we have

$$e = u_0^{-1}eu_0 \sim_h u_1^{-1}eu_1 = (1 + (e - \rho(e)))e(1 - (e - \rho(e))) = \rho(e).$$

(iii) If $t \mapsto e_t$ is a continuous path in $\mathcal{I}(B)$ from $e_0 = p$ to $e_1 = q$, then $t \mapsto \rho(e_t)$ is a continuous path in $\mathcal{P}(B)$ from $\rho(e_0) = p$ to $\rho(e_1) = q$. □

Let $\rho : \mathcal{I}_n(A) \to \mathcal{P}_n(A)$ be the map defined in Lemma 6.2.2 (with $B = M_n(A)$). Then $\nu(f) = 1_n + \rho(e)(z - 1)$ yields the desired map $\nu : PL_1^n \longrightarrow PRL^n$.

Injectivity. Let $p, q \in \mathcal{P}_n(A)$ and assume that $\beta_A([p]_0 - [q]_0) = [f_pf_q^*]_1 = [1]_1$ in $K_1(SA)$. Then, after increasing n if necessary, we have $f_p \sim_h f_q$ in GL^n. It suffices to show that there exists $m \in \mathbb{N}$ such that $\mathrm{diag}(p, 1_m) \sim_h \mathrm{diag}(q, 1_m)$ in $\mathcal{P}_{n+m}(A)$.

As a first step, we observe that there exists a polygonal (i.e. piece-wise linear) homotopy $t \mapsto h_t$ from f_p to f_q such that all h_t are Laurent loops with a uniform bound on both positive and negative degrees. (This follows from the density of Laurent loops in invertible loops via a routine compactness argument — exercise.) Thus there are $m, k \in \mathbb{N}$ such that $z^mh_t \in PL_k^n$ for all k. Since $z^mf_p \sim_h f_{\mathrm{diag}(p,1_m)}$ in PL_m^{m+n} (exercise), we see that $f_{\mathrm{diag}(p,1_m)}$ and $f_{\mathrm{diag}(q,1_m)}$ are homotopic in PL_{m+k}^{m+n}. Let $t \mapsto e_t$ be such a homotopy. Then applying the maps μ_{m+k}^{m+n} and ν, constructed in steps 2 and 3, respectively, of the proof of surjectivity, we get a homotopy $t \mapsto \nu(\mu_{m+k}^{m+n}(e_t)) = f_{p_t}$ from $f_{\mathrm{diag}(p,1_m)}$ to $f_{\mathrm{diag}(q,1_m)}$ in projection loops. Since the map $f_{p_t} \mapsto p_t$ is continuous (exercise), we finally see that $\mathrm{diag}(p, 1_m)$ and $\mathrm{diag}(g, 1_m)$ are homotopic via a path of projections. Consequently, $[p]_0 = [q]_0$ in $K_0(A)$, as required. □

Combining Theorems 6.1.2 and 6.2.1 we get

$$K_j(SA) \cong K_{1-j}(A)$$

for any C^*-algebra A and $j = 0, 1$. Thus, for any natural n we have

$$K_{n+2}(A) \cong K_n(A).$$

Furthermore, naturality of the maps θ_* and β_* easily implies that the functors K_{n+2} and K_n are isomorphic.

6.3 The 6-Term Exact Sequence

6.3.1 The 6-term exact sequence of K-theory

With the Bott periodicity theorem at hand, we are now ready to present the 6-term exact sequence of K-theory — a tool of paramount importance in applications. Let

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be an exact sequence of C^*-algebras. Applying the suspension functor, we obtain the exact sequence

$$0 \longrightarrow SJ \xrightarrow{S\varphi} SA \xrightarrow{S\psi} SB \longrightarrow 0.$$

Denote by $\partial : K_1(SB) \to K_0(SJ)$ the corresponding index map. Let $\theta_J : K_1(J) \to K_0(SJ)$ and $\beta_B : K_0(B) \to K_1(SB)$ be the isomorphisms from Therems 6.1.2 and 6.2.1, respectively. Then the *exponential map*

$$\partial_0 : K_0(B) \longrightarrow K_1(J)$$

is defined as the unique homomorphism making the diagram

$$\begin{tikzcd} K_0(B) \arrow[r, "\partial_0"] \arrow[d, "\beta_B"'] & K_1(J) \arrow[d, "\theta_J"] \\ K_1(SB) \arrow[r, "\partial"'] & K_0(SJ) \end{tikzcd} \tag{6.3.8}$$

commutative.

Theorem 6.3.1. *Let*

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be an exact sequence of C^*-algebras. Then the sequence

$$\begin{tikzcd} K_0(J) \arrow[r, "K_0(\varphi)"] & K_0(A) \arrow[r, "K_0(\psi)"] & K_0(B) \arrow[d, "\partial_0"] \\ K_1(B) \arrow[u, "\partial_1"] & K_1(A) \arrow[l, "K_1(\psi)"] & K_1(J) \arrow[l, "K_1(\varphi)"] \end{tikzcd} \tag{6.3.9}$$

is exact everywhere.

Proof. By virtue of Theorem 5.2.7, it suffices to show exactness at $K_0(B)$ and $K_1(J)$. It turns out that at this stage this requires nothing more but a diagram chase.

To prove exactness of (6.3.9) at $K_0(B)$, consider the commutative (due to naturality of the Bott map) diagram

$$\begin{tikzcd} K_0(A) \arrow[r, "K_0(\psi)"] \arrow[d, "\beta_A"'] & K_0(B) \arrow[r, "\partial_0"] \arrow[d, "\beta_B"'] & K_1(J) \arrow[d, "\theta_J"] \\ K_1(SA) \arrow[r, "K_0(S\psi)"'] & K_1(SB) \arrow[r, "\partial"'] & K_0(SJ) \end{tikzcd} \tag{6.3.10}$$

All the vertical arrows are isomorphisms, and the bottom row is exact by Theorem 5.2.7. Thus the top row is exact.

To prove exactness of (6.3.9) at $K_1(J)$, consider the commutative (due to naturality of the θ_* map) diagram

$$\begin{tikzcd} K_0(B) \arrow[r, "\partial_0"] \arrow[d, "\beta_B"'] & K_1(J) \arrow[r, "K_1(\varphi)"] \arrow[d, "\theta_J"'] & K_1(A) \arrow[d, "\theta_A"] \\ K_1(SB) \arrow[r, "\partial"'] & K_0(SJ) \arrow[r, "K_0(S\varphi)"'] & K_0(SA) \end{tikzcd} \tag{6.3.11}$$

All the vertical arrows are isomorphisms, and the bottom row is exact by Theorem 5.2.7. Thus the top row is exact. □

6.3.2 The exponential map

6.3.3 An explicit form of the exponential map

Proposition 6.3.2. *Let $0 \to J \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$ be exact and let $\partial_0 : K_0(B) \to K_1(J)$ be the associated exponential map. Then*

(i) If $p \in \mathcal{P}_n(\tilde{B})$ and $x = x^* \in M_n(\tilde{A})$ such that $\tilde{\psi}(x) = p$ then $\exists! u \in \mathcal{U}_n(\tilde{J})$ such that $\tilde{\varphi}(u) = \exp(2\pi i x)$, and we have

$$\partial_0([p]_0 - [s(p)]_0) = -[u]_1. \tag{6.3.12}$$

(ii) Suppose that A is unital. If $p \in \mathcal{P}_n(B)$ and $x = x^* \in M_n(A)$ such that $\psi(x) = p$, then $\exists! u \in \mathcal{U}_n(\tilde{J})$ such that $\tilde{\varphi}(u) = \exp(2\pi i x)$, and we have

$$\partial_0([p]_0) = -[u]_1. \tag{6.3.13}$$

Proof. Part (i) follows from (ii) by a diagram chase. So we prove (ii). For simplicity, assume $J \subseteq A$ and $\varphi = \mathrm{id}$. Suppose A unital then and let $p \in \mathcal{P}_n(B)$. There is $x = x^* \in M_n(A)$ such that $\psi(x) = p$. Then $\psi(\exp(2\pi i x)) = \exp(2\pi i \psi(x)) = \exp(2\pi i p) = 1_n$, hence $\exp(2\pi i x) \in \mathcal{U}_n(\tilde{J})$. We must show that

$$\theta_J([\exp(-2\pi i x)]_1) = (\partial_1 \circ \beta_B)([p]_0), \tag{6.3.14}$$

where $\partial_1 : K_1(SB) \to K_0(SJ)$ is the index map corresponding to $0 \to SJ \to SA \to SB \to 0$. We identify SB with $\{f \in C([0,1], B) | f(0) = f(1) = 0\}$. Thus $M_k(\widetilde{SB}) = \{f \in C([0,1], M_k(B)) | f(0) = f(1) \in M_k(\mathbb{C}1_B)\}$, and $f_p \in \mathcal{U}(\widetilde{SB})$ is $f_p(t) = e^{2\pi it}p + 1_n - p, \quad t \in [0,1]$. Let $v \in \mathcal{U}_{2n}(\widetilde{SA})$ be such that $\widetilde{S\psi}(v) = \begin{pmatrix} f_p & 0 \\ 0 & f_p^* \end{pmatrix}$. Then $v : [0,1] \to \mathcal{U}_{2n}(A)$ is a continuous map such that $v(0) = v(1) \in M_2n(\mathbb{C}1_A)$, and $\psi(v(t)) = \begin{pmatrix} f_p(t) & 0 \\ 0 & f_p^*(t) \end{pmatrix}$. As $f_p(0) = f_p(1) = 1$, we have $v(0) = v(1) = 1_{2n}$. With $x = x^* \in M_n(A)$ a lift of p, put $z(t) = \exp(2\pi itx)$ for $t \in [0,1]$. $t \mapsto z(t) \in \mathcal{U}_n(A)$ is continuous and $\psi(z(t)) = f_0(t)$. Hence

$$\psi\left(v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix}\right) = 1_{2n}, \quad s\left(v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix}\right) = 1_{2n}. \tag{6.3.15}$$

Thus $w(t) = v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix}$ is a unitary element in $\mathcal{U}_{2n}(\tilde{J})$. We have $w(0) = 1_{2n}$ and $w(1) = \begin{pmatrix} \exp(-2\pi ix) & 0 \\ 0 & \exp(2\pi ix) \end{pmatrix}$. Thus, by the definition of θ_J, we have

$$\theta_J([\exp(-2\pi ix)]_1) = \left[w\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}w^*\right]_0 - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right]_0. \tag{6.3.16}$$

We also have

$$w(t)\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}w(t)^* = v(t)\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}v(t)^*, \tag{6.3.17}$$

and the unitary v was chosen so that

$$\widetilde{S\psi}(v) = \begin{pmatrix} f_p & 0 \\ 0 & f_p^* \end{pmatrix}. \tag{6.3.18}$$

So, by the definition of the index map, we get

$$\partial_1([f_p]_1) = \theta_J([\exp(-2\pi ix]_1). \tag{6.3.19}$$

□

6.4 Examples and Exercises

Exercise 6.4.1. Let A be a C^*-algebra. Show that every class in $K_1(A)$ contains a unitary $u \in \mathcal{U}_n(\tilde{A})$ normalized so that $s(u) = 1_n$, where s is the scalar map.

Exercise 6.4.2. Show that if A is a non-unital C^*-algebra then for any $p \in \mathcal{P}_n(\tilde{A})$ we have

$$\beta_A([p]_0 - [s(p)]_0) = [f_p f_{s(p)}^*]_1.$$

Exercise 6.4.3 (Holomorphic function calculus). Let $\gamma_1, \ldots, \gamma_n$ be a finite collection of continuous and piece-wise continuously differentialble paths $\gamma_k : [a_k, b_k] \to \mathbb{C}$.

We assume that each γ_k is closed, i.e. $\gamma_k(a_k) = \gamma_k(b_k)$. A *contour* is a finite collection $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. If f is a piece-wise continuous, complex function defined on $\mathrm{Im}(\Gamma) = \bigcup_k \gamma_k([a_k, b_k])$, then there is a well-defined integral

$$\int_\Gamma f(z)dz = \int_{\gamma_1} f(z)dz + \ldots + \int_{\gamma_n} f(z)dz.$$

If $z_0 \notin \mathrm{Im}(\Gamma)$ then the *index* of z_0 with respect to Γ is the integer defined as

$$\mathrm{Ind}_\Gamma(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z - z_0}.$$

Let K be a compact subset of an open set $\Omega \subseteq \mathbb{C}$. Then we say that Γ *surrounds* K in Ω if $\mathrm{Im}(\Gamma) \subseteq \Omega \setminus K$ and

$$\mathrm{Ind}_\Gamma(z) = \begin{cases} 1, & z \in K, \\ 0, & z \in \mathbb{C} \setminus \Omega. \end{cases}$$

Let A be a unital C^*-algebra. If $a \in A$ and Γ is a contour surrounding $\mathrm{sp}(a)$ in an open set Ω, then for every holomorphic function $f : \Omega \to \mathbb{C}$ there is a well-defined Riemann integral

$$f(a) = \frac{1}{2\pi i}\int_\Gamma f(z)(z1_A - a)^{-1}dz.$$

This integral yields a unique element $f(a)$ of A such that for every continuous functional $\varphi : A \to \mathbb{C}$ we have

$$\varphi(f(a)) = \frac{1}{2\pi i}\int_\Gamma f(z)\varphi((z1_A - a)^{-1})dz.$$

The mapping $f \mapsto f(a)$ is called the *holomorphic function calculus* for a. It has the following properties (see [T-M79, M-GJ90, P-GK79]).

(i) The map $f \mapsto f(a)$ is a unital algebra homomorphism.

(ii) If g is a holomorphic function on $f(\Omega)$ then $(g \circ f)(a) = g(f(a))$.

(iii) $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$.

(iv) If f_n is a sequence of holomorphic functions on Ω converging almost uniformly to a function g, then g is holomorphic on Ω and

$$||f_n(a) - g(a)|| \longrightarrow 0.$$

Note that the holomorphic function calculus applies to an arbitrary element a of a C^*-algebra, not just a self-adjoint one. If a is self-adjoint then the holomorphic function calculus is compatible with the continuous function calculus via the Gelfand transform.

EXERCISE 6.4.4. By virtue of Theorems 6.1.2 and 6.2.1, we have

$$K_0(C_0(\mathbb{R}^{2n})) \cong K_1(C_0(\mathbb{R}^{2n+1})) \cong K_0(\mathbb{C}) \cong \mathbb{Z},$$

$$K_1(C_0(\mathbb{R}^{2n})) \cong K_0(C_0(\mathbb{R}^{2n+1})) \cong K_1(\mathbb{C}) = 0,$$

for all $n \in \mathbb{N}$.

EXERCISE 6.4.5. For each natural number $n \geq 1$, find a split-exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^n) \longrightarrow C(S^n) \longrightarrow \mathbb{C} \longrightarrow 0.$$

Then use Exercise 6.4.4 and split-exactness of K_* to determine the K-groups of $C(S^n)$ for all spheres S^n.

EXERCISE 6.4.6. By Exercise 6.4.5, we have $K_0(C(\mathbb{T})) \cong K_1(C(\mathbb{T})) \cong \mathbb{Z}$. Use the isomorphism $C(\mathbb{T}^{n+1}) \cong C(\mathbb{T}) \otimes C(\mathbb{T}^n)$ to find a split-exact sequence

$$0 \longrightarrow SC(\mathbb{T}^n) \longrightarrow C(\mathbb{T}^{n+1}) \longrightarrow C(\mathbb{T}^n) \longrightarrow 0.$$

Then use split-exactness of K_* to determine the K-groups of $C(\mathbb{T}^n)$ for all tori $\mathbb{T}^n$.

EXERCISE 6.4.7. Let

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be an exact sequence of C^*-algebras. Show that if every projection in $\mathcal{P}_\infty(\tilde{B})$ lifts to a projection in $\mathcal{P}_\infty(\tilde{A})$ then $\partial_0 : K_0(B) \to K_1(J)$ is the zero map.

EXERCISE 6.4.8 (TOEPLITZ ALGEBRA). Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{\xi_n : n = 0, 1, 2, \ldots\}$. Let $S \in \mathcal{B}(\mathcal{H})$, $S(\xi_n) = \xi_{n+1}$ be the unilateral shift. We define the *Toeplitz algebra* $\mathcal{T}$ as the C^*-algebra generated by S. It can be shown [C-L67] that $\mathcal{T}$ is the universal C^*-algebra for the relation $S^*S = 1$, and that if T is a *proper isometry* on a Hilbert space then there exists a $*$-isomorphism $\mathcal{T} = C^*(S) \to C^*(T)$ such that $T \mapsto S$.

(i) Show that the closed two-sided ideal of $\mathcal{T}$ generated by $1 - SS^*$ coincides with the algebra $\mathcal{K}(\mathcal{H})$.

(ii) Let $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{K}$ be the natural surjection. Show that $\mathcal{T}/\mathcal{K}$ is isomorphic to $C(S^1)$ and $\pi(S)$ may be identified with the generator z. There is an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

(iii) By Exercise 6.4.5, $K_0(C(S^1)) \cong \mathbb{Z}$ (with a generator $[1]_0$) and $K_1(C(S^1)) \cong \mathbb{Z}$ (with a generator $[z]_1$, the class of the identity map $z \mapsto z$). Calculate $\partial_1([z]_1)$ and show that the index map

$$\partial_1 : \mathbb{Z} \cong K_1(C(S^1)) \longrightarrow K_0(\mathcal{K}) \cong \mathbb{Z}$$

is an isomorphism.

(iv) Use (iii) and the exact sequence from Theorem 6.3.1 to show that

$$K_0(\mathcal{T}) \cong \mathbb{Z}, \quad K_1(\mathcal{T}) = 0.$$

Find the generator of $K_0(\mathcal{T})$.

EXERCISE 6.4.9. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the compact operators on $\mathcal{H}$, and let $S^n \in \mathcal{B}(\mathcal{H})$ be an isometry with cokernel of dimension n, for some natural number n. Let $C^*(S^n, \mathcal{K})$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by S^n and $\mathcal{K}$. Show that there exists an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(S^n, \mathcal{K}) \longrightarrow C(S^1) \longrightarrow 0$$

and determine the K-theory of $C^*(S^n, \mathcal{K})$.

EXERCISE 6.4.10. Let $\mathbb{R}P^2$ be the real projective plane. Find an exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \longrightarrow C(\mathbb{R}P^2) \longrightarrow C(S^1) \longrightarrow 0$$

and determine the K-theory of $C(\mathbb{R}P^2)$.

EXERCISE 6.4.11. Find an exact sequence

$$0 \longrightarrow C(S^1) \otimes C_0(\mathbb{R}^2) \longrightarrow C(S^3) \longrightarrow C(S^1) \longrightarrow 0.$$

Then apply to it the 6-term exact sequence of K-theory and thus calculate in an alternative way the K-groups of the 3-sphere (cf. Exercise 6.4.5).

EXERCISE 6.4.12. Let $\mathcal{H}$ be a separable Hilbert space. Consider two operators $T, U \in \mathcal{B}(\mathcal{H})$ such that T is a proper isometry (i.e. $T^*T = 1 \neq TT^*$) and U is a partial unitary on $1 - TT^*$ with full spectrum (i.e. $U^*U = UU^* = 1 - TT^*$ and $\mathrm{sp}(U) = S^1 \cup \{0\}$). Let A be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T and S.

(i) Let J be the closed two-sided ideal of A generated by U. Show that J is isomorphic to $C(S^1) \otimes \mathcal{K}$, with $\mathcal{K}$ the C^*-algebra of compact operators.

(ii) Let $\pi : A \to A/J$ be the natural surjection. Show that A/J is generated (as a C^*-algebra) by the unitary element $\pi(T)$. Show that $\mathrm{sp}(\pi(T))$ contains the entire unit circle, and thus A/J is isomorphic to $C(S^1)$.

(iii) By (i) and (ii) above, there is an exact sequence

$$0 \longrightarrow C(S^1) \otimes \mathcal{K} \longrightarrow A \longrightarrow C(S^1) \longrightarrow 0.$$

Apply the 6-term exact sequence and calculate the K-theory of A.

EXAMPLE 6.4.13. (Mirror-disc-type quantum two-spheres)
Consider, for $p \in]0, 1[$, the $*$-algebra

$$\mathcal{O}(D_p) := \mathbb{C}\langle x, x^* \rangle / J, \tag{6.4.20}$$

where J is the $*$-ideal generated by $x^*x - pxx^* - (1 - p)$. This is called $*$-algebra of the quantum disc (see [KL93], where a two-parameter family of such quantum discs is considered). It is not hard to see that $\|\rho(x)\| = 1$ in any bounded representation ρ, so that one can form the C^*-closure $C(D_p)$ of $\mathcal{O}(D_p)$. Moreover $\mathcal{O}(D_p)$ is faithfully imbedded in $C(D_p)$. (There is exactly one faithful irreducible representation, up to unitary equivalence.) It is known that $C(D_p)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$, so all the C^*-algebras $C(D_p)$ are isomorphic. There is a $*$-homomorphism $\varphi : C(D_p) \to C(S^1)$, sending the generator x to the unitary generator u of $C(S^1)$. Consider for any $q \in]0, 1[$ a second copy $\mathcal{O}(D_q)$, with generator y.

Definition 6.4.14. *Let $\alpha : \mathcal{O}(S^1) \to \mathcal{O}(S^1)$ denote the $*$-automorphism defined by $u \mapsto u^*$. Define*

$$\mathcal{O}(S^2_{pq}) := \{(f, g) \in \mathcal{O}(D_p) \oplus \mathcal{O}(D_q) \mid \varphi(f) = \alpha \circ \varphi(g)\}. \tag{6.4.21}$$

This is called the $*$-algebra of the mirror-disc-type quantum two-sphere.

Proposition 6.4.15.

$$\mathcal{O}(S^2_{pq}) \cong \mathbb{C}\langle C, C^*, D, D^*, E, E^*\rangle/J, \tag{6.4.22}$$

where the $*$-ideal J is generated by the relations

$$\begin{aligned} C^*C &= 1 - pD - E, \\ CC^* &= 1 - D - qE, \\ DC &= pCD, \\ EC &= q^{-1}CE, \\ DE &= 0, \\ D &= D^*, \\ E &= E^*. \end{aligned}$$

The isomorphism is given by $(x, y^*) \mapsto C$, $(1 - xx^*, 0) \mapsto D$, $(0, 1 - yy^*) \mapsto E$.

Proposition 6.4.16. *The following is a complete list (up to unitary equivalence) of irreducible $*$-representations of $\mathcal{O}(S^2_{pq})$ in some Hilbert space:*

(i) ρ_+, acting on a separable Hilbert space $\mathcal{H}$ with orthonormal basis $e_0, e_1, \ldots$ according to

$$\begin{aligned} \rho_+(C)e_k &= \sqrt{1 - p^{k+1}}e_{k+1}, \\ \rho_+(D)e_k &= p^k e_k, \\ \rho_+(E) &= 0. \end{aligned}$$

(ii) ρ_-, acting on $\mathcal{H}$ by

$$\begin{aligned} \rho_-(C)e_k &= \sqrt{1 - q^k}e_{k-1}, \\ \rho_-(D)e_k &= 0, \\ \rho_-(E)e_k &= q^k e_k. \end{aligned}$$

(iii) An S^1-family ρ_μ, acting on $\mathbb{C}$ by

$$\begin{aligned} \rho_\mu(C) &= \mu, \\ \rho_\mu(D) &= 0, \\ \rho_\mu(E) &= 0. \end{aligned}$$

One can again show that there is a uniform bound on the norm of the generators for all bounded $*$-representations, so that one can form a C^*-closure $C(S^2_{pq})$ of $\mathcal{O}(S^2_{pq})$ using bounded $*$-representations. $\rho_+ \oplus \rho_-$ is a faithful representation of $\mathcal{O}(S^2_{pq})$ as well as of $C(S^2_{pq})$, so that $\mathcal{O}(S^2_{pq})$ is faithfully imbedded in $C(S^2_{pq})$. Moreover, the closed ideals J_D, J_E generated by D, E are isomorphic to $\mathcal{K}$ ($\rho_+(J_D) = \mathcal{K} = \rho_-(J_E)$), they have zero intersection, and $(\rho_+ \oplus \rho_-)(J_D + J_E) = \mathcal{K} \oplus \mathcal{K}$. Finally, there is an exact sequence

$$0 \to \mathcal{K} \oplus \mathcal{K} \to C(S^2_{pq}) \xrightarrow{\psi} C(S^1) \to 0, \tag{6.4.23}$$

where ψ is defined by $C \mapsto u$, $D \mapsto 0$, $E \mapsto 0$. This exact sequence can be used to compute the K-theory of $C(S^2_{pq})$:

Proposition 6.4.17. *$K_0(C(S^2_{pq}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $K_1(C(S^2_{pq})) = 0$.*

Proof. With $K_0(C(S^1)) \cong \mathbb{Z} \cong K_1(C(S^1))$, $K_0(\mathcal{K}) \cong \mathbb{Z}$, $K_1(\mathcal{K}) = 0$, we obtain from the standard six-term exact sequence corresponding to (6.4.23)

$$0 \to K_1(C(S^2_{pq}) \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \to K_0(C(S^2_{pq})) \xrightarrow{K_0(\psi)} \mathbb{Z} \to 0. \tag{6.4.24}$$

Let us compute the index map ∂. It is determined by its value on the generator $[u]_1 \in K_1(C(S^1))$,

$$\partial([u]_1) = [1 - b^*b]_0 - [1 - bb^*]_0, \tag{6.4.25}$$

where $b \in C(S^2_{pq})$ is any partial isometry with $\psi(b) = u$. Identify $C(S^2_{pq}) \cong (\rho_+ \oplus \rho_-)(C(S^2_{pq})$. Then $b = (s, s^*)$, s the one-sided shift, is a continuous function of $(\rho_+ \oplus \rho_-)(C)$ such that $b - (\rho_+ \oplus \rho_-)(C) \in \mathcal{K} \oplus \mathcal{K}$: $b = (\rho_+(C)|\rho_+(C)|^{-1}, \rho_-(C)|\rho_-(C)|^-)$ with $|\rho_-(C)|^- e_k := \begin{cases} 0 & k = 0 \\ \frac{1}{\sqrt{1-q^k}} e_k & k > 0. \end{cases}$ Then $b^*b = (s^*s, ss^*) = 1 - p_2$, $bb^* = (ss^*, s^*s) = 1 - p_1$, where $p_2 = (p_{e_0}, 0)$, $p_1 = (0, p_{e_0})$ can be considered as the generators $(0, 1)$, $(1, 0)$ of $\mathbb{Z} \oplus \mathbb{Z} \cong K_0(\mathcal{K} \oplus \mathcal{K})$. Then $\partial([u]_1) = [p_2]_0 - [p_1]_0 = (0, 1) - (1, 0)$, so that ∂ is injective, and we can conclude that $K_1(C(S^2_{pq})) = 0$. We are left with the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{K_0(j)} K_0(C(S^2_{pq}) \xrightarrow{K_0(\psi)} \mathbb{Z} \to 0. \tag{6.4.26}$$

As $\mathbb{Z}$ is a free module over itself, this sequence splits, and $K_0(C(S^2_{pq})) \cong \mathrm{Im} K_0(j) \oplus \mathbb{Z}$. There remains the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{K_0(j)} \mathrm{Im} K_0(j) \to 0. \tag{6.4.27}$$

Here, $K_0(j)$ is determined by its values on $(1, 0)$ and $(0, 1)$, however, $(1, 0) - (0, 1) \in \mathrm{Ker}\, K_0(j) = \mathrm{Im}\partial$, i.e., $K_0(j)(1, 0) = K_0(j)(0, 1)$, consequently $\mathrm{Im} K_0(j) = \{nK_0(j)(1, 0) | n \in \mathbb{Z}\} \cong \mathbb{Z}$. It follows that $K_0(C(S^2_{pq})) \cong \mathbb{Z} \oplus \mathbb{Z}$. ☐

Chapter 7

Tools for the computation of K-groups

7.1 Crossed products, the Thom-Connes isomorphism and the Pimsner-Voiculescu sequence

7.1.1 Crossed products

Let G be a locally compact abelian group. Then $C_c(G) = \{f \in C(G) | \mathrm{supp}(f) \text{ compact}\}$ is a $*$-algebra with respect to

$$\begin{aligned} (f * g)(s) &= \int_G f(t)g(t-s)dt, & (7.1.1)\\ f^*(s) &= \overline{f(s^{-1})}, & (7.1.2) \end{aligned}$$

where the integration is with respect to the Haar measure. The universal norm on $C_c(G)$,

$$\|f\| = \sup\{\|\pi(f)\| | \pi : C_c(G) \to B(\mathcal{H}) \text{ a } *\text{-representation}\}, \tag{7.1.3}$$

is well-defined since (one can show that)

$$\|f\| \leq \|f\|_1 = \int_G |f(t)|dt. \tag{7.1.4}$$

The completion of $C_c(G)$ with respect to $\|.\|$ is the group C^*-algebra $C^*(G)$ of G. By Gelfand's theorem, since $C^*(G)$ is abelian, there is a locally compact Hausdorff space Ω such that $C^*(G) \cong C_0(\Omega)$. Ω may be identified with $\hat{G} = \{\chi : G \to \mathbb{T} | \chi \text{ continuous }, \chi(s+t) = \chi(s)\chi(t)\}$, the dual group of G. $\hat{G}$ is equipped with the topology of almost uniform convergence. Every $\chi \in \hat{G}$ yields a multiplicative functional of $C^*(G)$ by

$$\omega_\chi(f) = \int_G \chi(t)f(t)dt. \tag{7.1.5}$$

Thus we have $C^*(G) \cong C_0(\hat{G})$ via the Gelfand transform. Now suppose that A is a C^*-algebra and $\alpha : G \to Aut(A)$ is a homomorphism such that $G \ni t \mapsto \alpha_t(x) \in A$ is continuous $\forall x \in A$. Then (A, G, α) is called a C^*-dynmaical system. The vector space $\{f \in C(G, A) | \mathrm{supp}(f) \text{ compact}\}$ becomes a $*$-algebra with

$$\begin{aligned} (f * g)(s) &= \int_G f(t)\alpha_t(g(s-t))dt, && (7.1.6) \\ f^*(s) &= \alpha_s(f(s^{-1})^*). && (7.1.7) \end{aligned}$$

Note that even if both G and A are abelian, this algebra may be noncommutative if the action α is nontrivial. The universal norm $\|.\|$ on this $*$-algebra is defined as the supremum over the norms in all $*$-representations. $A \rtimes_\alpha G$ is by definition the C^*-algebraic closure of $A \otimes C_c(G)$ with respect to $\|.\|$. If $\alpha : G \to Aut(A)$ is trivial, i.e., $\alpha_t(x) = x$, $\forall x$, then we have $A \rtimes_\alpha G \cong A \otimes C^*(G) \cong A \otimes C_0(\hat{G})$ (C^*-algebra isomorphisms). For a given action $\alpha : G \to Aut(A)$ there exists a canonical *dual action* $\hat{\alpha} : \hat{G} \to Aut(A \rtimes_\alpha G)$ such that

$$\hat{\alpha}_\chi(f)(t) = \langle \chi, t \rangle f(t) \tag{7.1.8}$$

for $f \in C(G, A)$ with compact support.

Theorem 7.1.1. *(Takesaki-Takai duality)*

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes \mathcal{K}, \tag{7.1.9}$$

if G is infinite.

The dual acion is functorial in the follwoing sense: If $\alpha : G \to Aut(A)$ and $\beta : G \to Aut(B)$ are actions and $\rho : A \to B$ is a G-equivariant $*$-homomorphism, then there exists a $*$-homomorphism $\hat{\rho} : A \rtimes_\alpha G \to B \rtimes_\beta G$ such that

$$(\hat{\rho} f)(s) = \rho(f(s)) \tag{7.1.10}$$

for $f : G \to A$, and ρ is equivariant with respect to $\hat{\alpha}$ and $\hat{\beta}$.

7.1.2 Crossed products by $\mathbb{R}$ and by $\mathbb{Z}$

Theorem 7.1.2. *(Connes) For any action $\alpha : \mathbb{R} \to Aut(A)$, we have*

$$K_j(A) \cong K_{1-j}(A \rtimes_\alpha \mathbb{R}). \tag{7.1.11}$$

In the special case of a trivial action, we have $K_j(A) \cong K_{1-j}(A \otimes C_0(\mathbb{R}))$ (Bott periodicity). Intuitively, the Connes-Thom isomorphism can be explained as follows: "Any action of $\mathbb{R}$ may be continuously deformed to a trivial one. Then the result follows from the Bott periodicity since K-theory is insensitive to continuous deformations". This can be made precise with the help of KK-equivalence.

Theorem 7.1.3. *(Pimsner-Voiculescu) If $\alpha \in Aut(A)$, then there is an exact sequence*

$$\begin{tikzcd} K_0(A) \arrow[r, "\mathrm{id} - K_0(\alpha^{-1})"] & K_0(A) \arrow[r, "i_0"] & K_0(A \rtimes_\alpha) \arrow[d] \\ K_1(A \rtimes_\alpha) \arrow[u] & K_1(A) \arrow[l, "i_1"'] & K_1(A) \arrow[l, "\mathrm{id} - K_1(\alpha^{-1})"'] \end{tikzcd} \tag{7.1.12}$$

where i_0, i_1 are the natural imbeddings.

Proof. (idea, Connes) Define $M_\alpha = \{f \in C(\mathbb{R}, M) | f(1) = \alpha(f(0)\}$ (mapping torus of α). $\mathbb{R}$ acts on M_α by $(\beta_t f)(s) = f(s-t)$. By a result of Green,

$$A \rtimes_\alpha \mathbb{Z} \simeq_{\text{Morita}} M_\alpha \rtimes_\beta \mathbb{R}. \tag{7.1.13}$$

Hence, by the Connes-Thom isomorphism,

$$K_j(A \rtimes_\alpha \mathbb{Z}) \cong K_j(M_\alpha \rtimes_\beta \mathbb{R}) \cong K_{1-j}(M_\alpha). \tag{7.1.14}$$

Now, there is an exact sequence

$$0 \to SA \to M_\alpha \to A \to 0, \tag{7.1.15}$$

and the 6-term exact sequence yields

$$\begin{tikzcd} K_1(A) \cong K_0(SA) \arrow[r] & K_1(A \rtimes_\alpha \mathbb{Z}) \cong K_0(M_\alpha) \arrow[r] & K_0(A) \arrow[d, "\partial_1"] \\ K_1(A) \arrow[u, "\partial_0"] & K_0(A \rtimes_\alpha \mathbb{Z}) \cong K_1(M_\alpha) \arrow[l] & K_1(sA) \cong K_0(A). \arrow[l] \end{tikzcd} \tag{7.1.16}$$

One can calculate the connecting maps as

$$\partial_* = \mathrm{id} - K_*(\alpha^{-1}). \tag{7.1.17}$$

7.1.3 Irrational rotation algebras

Let us recall that, for $\theta \in \mathbb{R}$, the rotation algebra A_θ is defined to be the universal C^*-algebra $C^*(u, v)$ generated by two unitaries u, v such that

$$vu = e^{2\pi i\theta} uv. \tag{7.1.18}$$

We have seen that there is a trace $\tau : A_\theta \to \mathbb{R}$, and that the image of $K_0(\tau)$ contains $\mathbb{Z} \cup \theta\mathbb{Z}$. Notice that $C^*(v) \cong C(S^1)$ and that $\alpha_\theta := Ad_v$ is an automorphism of $C(S^1)$ such that

$$\alpha_\theta v = e^{2\pi i\theta} v. \tag{7.1.19}$$

It can be shown that

$$A_\theta \cong C(S^1) \rtimes_{\alpha_\theta} \mathbb{Z}. \tag{7.1.20}$$

The Pimsner-Voiculescu sequence is

$$\begin{tikzcd} K_0(C(S^1)) \arrow[r, "\mathrm{id} - K_0(\alpha_\theta^{-1})"] & K_0(C(S^1)) \arrow[r] & K_0(A_\theta) \arrow[d] \\ K_1(A_\theta) \arrow[u] & K_1(C(S^1)) \arrow[l] & K_1(C(S^1)). \arrow[l, "\mathrm{id} - K_1(\alpha_\theta^{-}1)"'] \end{tikzcd} \tag{7.1.21}$$

$K_0(C(S^1))$ is generated by $[1]_0$, hence $\mathrm{id} - K_0(\alpha_\theta^{-1})$ is the zero map. Likewise, $e^{2\pi i\theta} u \sim_h u$, hence $\mathrm{id} - K_1(\alpha_\theta^{-1})$ is also the zero map. Consequently, since $K_j(C(S^1)) \cong \mathbb{Z}$, we get

$$K_0(A_\theta) \cong \mathbb{Z}^2 \cong K_1(A_\theta). \tag{7.1.22}$$

Furthermore, $[u]_1, [v]_1$ are generators of $K_1(A_\theta)$ and $K_0(\tau) : K_0(A_\theta) \cong \mathbb{Z}^2 \to \mathbb{Z} \cup \theta\mathbb{Z}$ is an isomorphism.

7.2 The Mayer-Vietoris sequence

The Mayer-Vietoris sequence in the classical case of topological spaces concerns relating the (co)homologies of a space that is glued from two (or more) subspaces to the (co)homologies of the subspaces and the way they are glued together. In the context of differential forms and De Rham cohomologies, it is natural (due to differentiability) to consider open subspaces. In the purely topological setting and in the realm of Gelfand theory for compact spaces, it seems to be more natural (also easier) to consider closed subsets. Thus we are trying to generalize the following situation to a noncommutative setting: There is a compact Hausdorff space X that is the union of two compact subspaces, which have a certain intersection. Diagrammatically:

$$\begin{tikzcd} X & X_1 \arrow[l] \\ X_2 \arrow[u] & X_1 \cap X_2, \arrow[l] \arrow[u] \end{tikzcd} \tag{7.2.23}$$

where the maps are injections of sets.

Dually, by Gelfand theory there is the following diagram:

$$\begin{tikzcd} C(X) \arrow[r] \arrow[d] & C(X_1) \arrow[d] \\ C(X_2) \arrow[r] & C(X_1 \cap X_2), \end{tikzcd} \tag{7.2.24}$$

where the maps are the natural restriction maps. In fact, it is almost obvious that $C(X) \cong \{(f_1, f_2) \in C(X_1) \oplus C(X_2) \mid f_1|X_1 \cap X_2 = f_2|X_1 \cap X_2\}$. Thus we are led to consider the following commutative diagram of unital C^*-algebras:

$$\begin{tikzcd} A \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & B_1 \arrow[d, "\pi_1"] \\ B_2 \arrow[r, "\pi_2"'] & D, \end{tikzcd} \tag{7.2.25}$$

where $A = \{(b_1, b_2) \in B_1 \oplus B_2 \mid \pi_1(b_1) = \pi_2(b_2)\}$, with π_1, π_2 surjective $*$-homomorphisms, pr_1 and pr_2 the restrictions of the natural projections $B_1 \oplus B_2 \to B_1$ and $B_1 \oplus B_2 \to B_2$ to the subspace $A \subseteq B_1 \oplus B_2$. A is called the pullback of B_1 and B_2 (over D), or the fiber product of B_1 and B_2 (over D), and the diagram (7.2.25) is called a pull-back diagram. We have

Theorem 7.2.1. *Corresponding to (7.2.25), there is a six-term exact sequence*

$$\begin{tikzcd} K_0(A) \arrow[r, "{(K_0(\mathrm{pr}_1), K_0(\mathrm{pr}_2))}"] & K_0(B_1)\oplus K_0(B_2) \arrow[r, "K_0(\pi_2)-K_0(\pi_1)"] & K_0(D) \arrow[d] \\ K_1(D) \arrow[u] & K_1(B_1)\oplus K_1(B_2) \arrow[l, "K_1(\pi_2)-K_1(\pi_1)"] & K_1(A)). \arrow[l, "{(K_1(\mathrm{pr}_1), K_1(\mathrm{pr}_2))}"] \end{tikzcd} \tag{7.2.26}$$

Proof. (partial, based on [BHMS], which is in turn based on ideas of Atiyah and Hirzebruch, see [?]) Define $\hat{A} \subseteq B_1 \oplus B_2 \oplus C([0,1], D)$ by

$$\hat{A} = \{(b_1, b_2, \omega \mid b_1 \in B_1, b_2 \in B_2, \omega(0) = \pi_1(b_1), \omega(1) = \pi_2(b_2)\} \tag{7.2.27}$$

Put

$$C_0(]0,1[, D) = \{\omega \in C((0,1), D \mid \omega(0) = \omega(1) = 0\}. \tag{7.2.28}$$

Then the sequence

$$0 \to C_0(]0,1[, D) \to \hat{A} \to B_1 \oplus B_2 \to 0 \tag{7.2.29}$$

is exact, where the map $C_0(]0,1[, D) \to \hat{A}$ is $\omega \mapsto (0,0,\omega)$, and the map $\hat{A} \to B_1 \oplus B_2$ is $(b_1, b_2, \omega) \mapsto (b_1, b_2)$. Exactness of this sequence at $C_0(]0,1[, D)$ and $\hat{A}$ is obvious, at $B_1 \oplus B_2$ it is due to the fact that $\omega \in C([0,1], D)$ can have any independent values $\omega(0), \omega(1) \in D$ (any two elements in a vector space are homotopic). As $C_0(]0,1[, D)$ is just the suspension of D, we have

$$K_j(C_0(]0,1[, D) \cong K_{1-j}(D), \quad j = 0,1. \tag{7.2.30}$$

We will show that

$$K_j(\hat{A}) \cong K_j(A), \quad j = 0,1. \tag{7.2.31}$$

Then (7.2.30) and (7.2.31) together allow to conclude that the 6-term exact sequence corresponding to the exact sequence (7.2.29) has the form

$$\begin{tikzcd} K_1(D) \arrow[r] & K_0(A) \arrow[r] & K_0(B_1\oplus B_2)) \arrow[d] \\ K_1(B_1\oplus B_2) \arrow[u] & K_1(A) \arrow[l] & K_0(D). \arrow[l] \end{tikzcd} \tag{7.2.32}$$

which after a counter-clockwise rotation about one position gives just the claim of the theorem. It remains to prove (7.2.31). Our goal is to show that the map $i : A \to \hat{A}$, $(b_1, b_2) \mapsto (b_1, b_2, \underline{\pi_1(b_1)})$, where $\underline{\pi_1(b_1)}$ is the constant path at $\pi_1(b_1)$, is a K-isomorphism. Consider the ideal $I_1 := \mathrm{Ker}\,(\mathrm{pr}_1 : A \to B_1) = \{(0, b_2) \in A\} = \{(0, b_2) \in B_1 \oplus B_2 \mid \pi_2(b_2) = 0\} \subseteq A$, being also isomorphic to $\mathrm{Ker}\,\pi_2$ ($I_1 \ni (0, b_2) \mapsto b_2 \in \mathrm{Ker}\,\pi_2$ being the isomorphism). The image of I_1 under i in A is $\hat{I}_1 = \{(0, b_2, \underline{0}) \mid \pi_2(b_2) = 0\}$. $\hat{I}_1$ is isomorphic to I_1, and is also an ideal in $\hat{A}$. Thus we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I_1 \arrow[r] \arrow[d, "j_1"'] & A \arrow[r] \arrow[d, "j"'] & A/I_1 \arrow[r] \arrow[d, "k"] & 0 \\ 0 \arrow[r] & \hat{I}_1 \arrow[r] & \hat{A} \arrow[r] & \hat{A}/\hat{I}_1 \arrow[r] & 0. \end{tikzcd} \tag{7.2.33}$$

Here, j_1 is an isomorphism, and both j and k are injective. Let us show that k is a homotopy equivalence: First let us note that

$$\begin{aligned}
\hat{A}/\hat{I}_1 &= \{(b_1, b_2, \omega) \mid \omega = \pi_1(b_1), \omega(1) = \pi_2(b_2)\}/\{(0, b_2, \underline{0}) \mid \pi_2(b_2) = 0\} \\
&\cong \{(b_1, \omega) \mid b_1 \in B_1, \omega(0) = \pi_1(b_1)\} =: \hat{B}_1.
\end{aligned}$$

The isomorphism is given by factorizing the map $(b_1, b_2, \omega) \mapsto (b_1, \omega)$, $\hat{A} \to \hat{B}_1$, whose kernel is $\{(0, b_2, \underline{0}) \mid \pi_2(b_2) = 0\}$, and which is obviously surjective. Define

$$\varphi : \hat{B}_1 \to B_1 \cong A/I_1, \quad \psi : b_1 \to \hat{B}_1, \tag{7.2.34}$$

by

$$\varphi(b_1, \omega) = b_1, \quad \varphi(b_1) = (b_1, \underline{\pi_1(b_1)}). \tag{7.2.35}$$

Then $\varphi \circ \psi = \mathrm{id}_{B_1}$, $\psi \circ \varphi(b_1, \omega) = (b_1, \underline{\pi_1(b_1)})$, and the homomorphisms $\varphi_t : \hat{B}_1 = \hat{A}/\hat{I}_1 \to \hat{B}_1$ defined by $\varphi_t(b_1, \omega) = (b_1, (1-t)\omega + t\underline{\pi_1(b_1)})$ satisfies $\varphi_0 = \mathrm{id}$, $\varphi_1 = \psi \circ \varphi$. This proves that A/I_1 and $\hat{A}/\hat{I}_1$ are homotopy equivalent and that $K_j(k)$ are isomorphisms. Thus from the above commutative diagram (7.2.33) we obtain another commutative diagram by combining two 6-term exact sequences:

$$\begin{tikzcd}
K_0(I_1) \arrow[rr] \arrow[dr, "K_0(j_1)"] & & K_0(A) \arrow[rr] \arrow[d, "K_0(j)"'] & & K_0(A/I_1) \arrow[dl, "K_0(k)"'] \arrow[ddd] \\
& K_0(\hat{I}_1) \arrow[r] & K_0(\hat{A}) \arrow[r] & K_0(\hat{A}/\hat{I}_1) \arrow[d] & \\
& K_1(\hat{A}/\hat{I}_1) \arrow[u] & K_1(\hat{A}) \arrow[l] & K_1(\hat{I}_1) \arrow[l] & \\
K_1(A/I_1) \arrow[uuu] \arrow[ur, "K_1(k)"] & & K_1(A) \arrow[ll] \arrow[u, "K_1(j)"] & & K_1(I_1). \arrow[ll] \arrow[ul, "K_1(j_1)"']
\end{tikzcd} \tag{7.2.36}$$

The diagram has two exact circles, and since $K_i(j_1)$ and $K_i(k)$ are isomorphisms, we obtain from the Five Lemma that also $K_i(j)$ are isomorphisms. Thus we have proved the desired isomorphy $K_i(A) \cong K_i(\hat{A})$. $\square$

Let us describe the connecting morphisms. For the morphism $K_0(D) \to K_1(A)$, let $P \in M_n(D)$ be an idempotent. Choose $P_1 \in M_n(B_1)$ and $P_2 \in M_n(B_2)$ such that $\pi_1(P_1) = P = \pi_2(P_2)$. (Here, π_1 and π_2 are the obvious extensions to matrices, which are also surjective.) Then $(e^{2\pi i P_1}, e^{2\pi i P_2}) \in B_1 \oplus B_2$ is in fact in $B_1 \oplus_D B_2$, because $e^{2\pi i P_1} \mapsto e^{2\pi i P} = I_n + (1 - e^{2\pi i})P = I_n$, $e^{-2\pi i P_2} \mapsto e^{-2\pi i P} = I_n + (1 - e^{-2\pi i})P = I_n$. Thus we have constructed the invertible element $(e^{2\pi i P_1}, e^{-2\pi i P_2}) \in M_n(A)$. The so-constructed map $P \mapsto (e^{2\pi i P_1}, e^{-2\pi i P_2})$ defines the desired morphism $K_0(D) \to K_1(A)$. If P is assumed to be selfadjoint, then P_1 and P_2 can be chosen to be selfadjoint (by Exercice 2.4.1 (ii)). Then the construction gives a unitary in $M_n(A)$. Note that without the minus sign on one side the resulting element $(e^{2\pi i P_1}, e^{2\pi i P_2}) = e^{2\pi i (P_1, P_2)} \in M_n(A)$ is homotopic to the identity (by the homotopy $[0, 1] \ni t \mapsto e^{2\pi i (tP_1, tP_2)}$) and leads to a trivial map $K_0(D) \to K_1(A)$.

In order to construct the connecting morphism $K_1(D) \to K_0(A)$, let θ be an invertible in $M_n(D)$. Think of θ as acting on the right on $D \oplus D \ldots \oplus D$. Consider the set $M_\theta := \{((v_1,\ldots,v_n),(w_1,\ldots,w_n)) \in B_1 \oplus \cdots \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_2 \mid (\pi_1(v_1),\ldots,\pi_1(v_1))\theta = (\pi_2(w_1),\ldots,\pi_2(w_n))\}$. M_θ is a finitely generated projective module over A. The connecting morphism we are looking for is now $[\theta] \mapsto [M_\theta] - [n] : K_1(D) \to K_0(A)$, where $[n]$ denotes the class of the free module of rank n over A. (We make use of the correspondence between idempotents and finitely generated projective modules.)

Example 7.2.2. Consider the circle S^1 as a union of two closed intervals, $S^1 = I \cup I$. Then we have a pull-back diagram

$$\begin{tikzcd} C(S^1) \arrow[r] \arrow[d] & C(I) \arrow[d] \\ C(I) \arrow[r] & \mathbb{C} \oplus \mathbb{C}, \end{tikzcd} \tag{7.2.37}$$

and a corresponding Mayer-Vietoris six-term exact sequence

$$\begin{tikzcd} K_0(C(S^1)) \arrow[r] & K_0(C(I)) \oplus K_0(C(I)) \arrow[r] & K_0(\mathbb{C} \oplus \mathbb{C})) \arrow[d] \\ K_1(\mathbb{C} \oplus \mathbb{C}) \arrow[u] & K_1(C(I) \oplus C(I)) \arrow[l] & K_1(C(S^1)). \arrow[l] \end{tikzcd} \tag{7.2.38}$$

Let us take for granted that $K_0(\mathbb{C}) = \mathbb{Z} = K_0(C(I))$ and $K_1(\mathbb{C}) = 0 = K_1(C(I))$. Then the above diagram is reduced to

$$0 \to K_0(C(S^1)) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to K_1(C(S^1)) \to 0. \tag{7.2.39}$$

We have to determine $K_0(\pi_2) - K_0(\pi_1) : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. $\pi_1 = \pi_2 : C(I) \to \mathbb{C} \oplus \mathbb{C}$ is the map $f \mapsto (f(0), f(1))$. The generator of $K_0(C(I))$ is $[1]_0$, so $K_0(\pi_1)$ is determined by $K_0(\pi_1)([1]_0) = ([1]_0, [1]_0)$ (where the 1 on the right is $1 \in \mathbb{C}$). It follows that $K_0(\pi_2) - K_0(\pi_1)$ has on the generators $([1]_0, 0)$ and $(0, [1]_0)$ of $K_0(C(I) \oplus C(I))$ the values $-([1]_0, [1]_0)$ and $([1]_0, [1]_0)$. Thus $\mathrm{Im}(K_0(\pi_2) - K_0(\pi_1))$ is the diagonal $\Delta \subseteq \mathbb{Z} \oplus \mathbb{Z}$, and $K_1(C(S^1)) \cong \mathbb{Z} \oplus \mathbb{Z}/\Delta \cong \mathbb{Z}$. On the other hand, also $\mathrm{Ker}\,(K_0(\pi_2) - K_0(\pi_1)) = \Delta$, because $([1]_0, [1]_0) \mapsto -([1]_0, [1]_0) + ([1]_0, [1]_0) = 0$ and $(n[1]_0, m[1]_0) \mapsto (m-n)([1]_0, [1]_0) \neq 0$ for $m \neq n$. So Δ is the image of the injective map $K_0(C(S^1)) \to \mathbb{Z} \oplus \mathbb{Z}$, i.e., $K_0(C(S^1)) \cong \Delta \cong \mathbb{Z}$.

7.3 The Künneth formula

In classical cohomology theory, say for differential forms, the Künneth formula states that the cohomology of a product of two manifolds is the (graded) tensor product of the cohomologies of the two factors,

$$H^*(M \times N) \cong H^*(M) \hat{\otimes} H^*(N). \tag{7.3.40}$$

For C^*-algebras, the product of noncommutative spaces corresponds to the tensor product of the algebras, and the following theorem generalizes the classical Künneth formula:

Theorem 7.3.1. *Let A, B be C^*-algebras, and assume that $K_*(B)$ is torsion-free and that A is separable and type I. Then*

$$K_*(A \otimes B) \cong K_*(A) \hat{\otimes} K_*(B). \tag{7.3.41}$$

Note that the formula (7.3.41) explicitly means

$$\begin{aligned} K_0(A \otimes B) &\cong (K_0(A) \otimes K_0(B)) \oplus (K_1(A) \otimes K_1(B)), \\ K_1(A \otimes B) &\cong (K_0(A) \otimes K_1(B)) \oplus (K_1(A) \otimes K_0(B)). \end{aligned}$$

Note also that there is no question about the kind of tensor product $A \otimes B$, because every separable type I C^*-algebra is nuclear. Also, there are more general statements without assumptions about torsion, but still assuming nuclearity of at least one of the factors (see [B-B98] and [S-C6]).

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