# NONCOMMUTATIVE GEOMETRY AND PHYSICS

ALAIN CONNES

## 1. Introduction

The two existing theories which successfully encode our knowledge of space-time are :

- General Relativity
- The Standard Model

General relativity describes space-time as far as large scales are concerned (*cf.* Figure 1 and [2] for many more suggestive thoughts and pictures) and is based on the geometric paradigm discovered by Riemann. It replaces the flat (pseudo) metric of Poincaré, Einstein, and Minkowski,

$$ds^2 = -\,dt^2 + dx^2 + dy^2 + dz^2$$

by a curved space-time metric whose components form the gravitational potential $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu$$

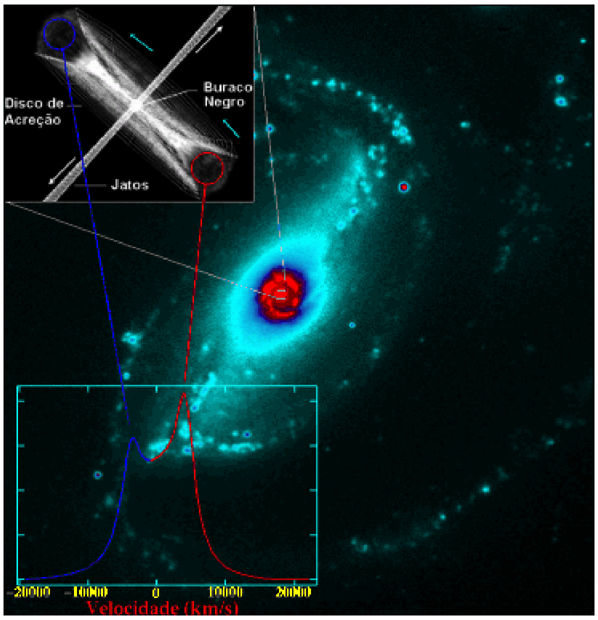


Figure 1. Suspected black hole in center of galaxy

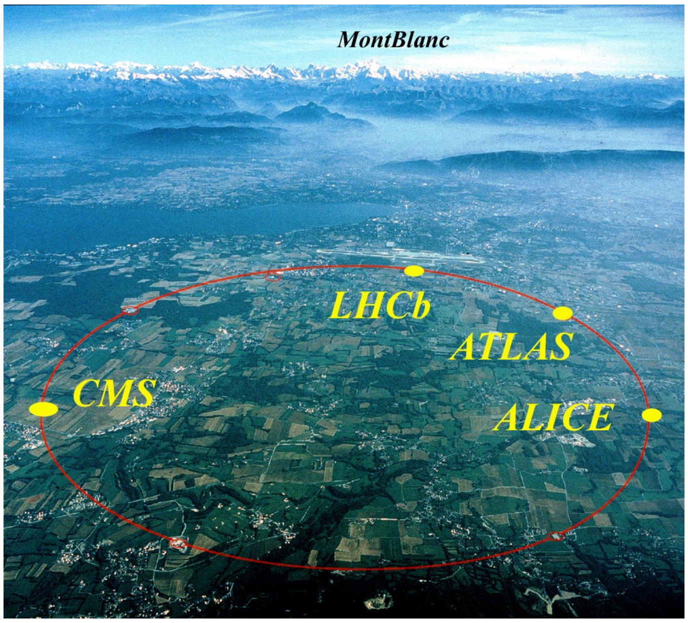


FIGURE 2. CERN collision ring

The basic Einstein-Hilbert action principle given by the action

$$S_E[\, g_{\mu\nu}] = \frac{1}{G} \int_M r \sqrt{g}\; d^4x$$

which holds in empty space with the possible addition of a cosmological term, is replaced in the presence of matter by the combination

$$S = S_E + S_{SM} \tag{1}$$

where $S_{SM}$ is the standard model action which encapsulates our knowledge of space-time at small scales as uncovered by the high energy experiments such as those performed at CERN (Figure 2).

The transition

$$\text{Classical} \to \text{Quantum}$$

is very simple to formulate in terms of the Feynman integral which affects each classical field configuration with the probability amplitude

$$e^{i\,\frac{S}{\hbar}}$$

While this prescription works remarkably well for the quantization of the classical fields involved in the standard model provided one uses the technique of renormalization,

this latter perturbative technique fails dramatically when one tries to deal with the gravitational field $g_{\mu\nu}$.
In many ways this result is not surprising. Indeed many of the basic notions of the traditional formalism of Quantum Field Theory (QFT), such as particles, scattering matrices, etc... heavily rely on the flat geometry of Minkowski space and the related Poincaré symmetry group. Treating the quantization of the $g_{\mu\nu}$ in the same way would -if successful- produce a quantum field theory of the $g_{\mu\nu}$ on Minkowski space: a strange result indeed when viewed from the geometric standpoint! The technical reason for the notorious difficulty of quantizing the $g_{\mu\nu}$ in the traditional perturbative way is the clash with either renormalizability or unitarity.
In some sense this clash contains a serious warning, namely that one should not try to rush but rather meditate the lessons of both general relativity and QFT before even starting to compute something. In this very short essay we shall describe a "spectral" point of view on geometry which allows to start taking into account the lessons from both sides. We shall first do that for renormalization and explain in rough outline the content of our recent collaborations with Dirk Kreimer and Matilde Marcolli. As far as general relativity is concerned, since the functional integral cannot be treated in the traditional perturbative manner, it relies heavily as a "sum over geometries" on the chosen paradigm of geometric space. This will give us the occasion to discuss, in the light of noncommutative geometry, the issue of "observables" in gravity and our joint work with Ali Chamseddine on the spectral action, with a first attempt to write down a functional integral on the space of noncommutative geometries.

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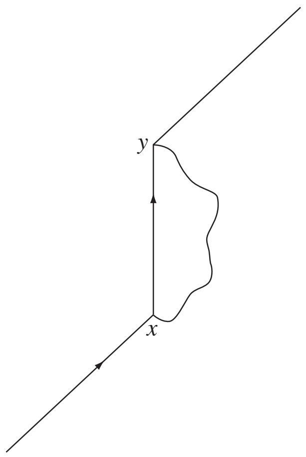


FIGURE 3. Feynman Graph

## 2. Lessons from renormalization

In QFT the recipe of Dirac and Feynman gives the *probability amplitude* of a classical field configuration $A$ as

$$e^{i\,\frac{S(A)}{\hbar}}$$

where the classical action is the integral of the Lagrangian density

$$S\,(A) = \int \mathcal{L}\,(A)\,d^4x$$

One implements this recipe using perturbation theory. The *perturbative expansion* generates integrals $U(\Gamma)$ labeled by Feynman graphs $\Gamma$. It was recognized very early on (already by Oppenheimer around 1930 in trying to compute higher order effects in the Dirac theory of spontaneous and induced atomic transitions) that, as a rule, these integrals are divergent.

Around 1947 and in a close interplay between experimental results (such as the Lamb shift) and theory (as developed by Schwinger, Feynman, Dyson) the technique of renormalization was successfully applied to overcome the difficulty created by the divergencies. We refer to Schwinger's book on quantum electrodynamics and its introduction for a description of the legacy of difficulties that came from the point-like nature of the electron.

Already in the nineteen'th century, around 1830, Green had shown that one needs to modify Newton's law $F = m\,a$ when dealing with an object moving in a fluid. Thus for instance for a spherical object moving in an incompressible fluid one needs to replace

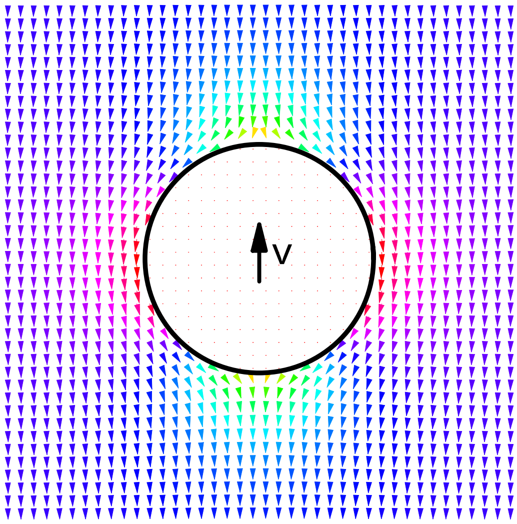


FIGURE 4. Hydrodynamics

its inertial mass $m$ by the *renormalized* mass $m \to m + \frac{1}{2}M$ where $M$ is the mass of the fluid corresponding to the volume of the ball (as in Archimedes law). While in this macroscopic case the correction $\frac{1}{2}M$ is finite, the point-like nature of the electron entails that the correction $\delta m$ to its inertial mass due to the self-energy of the perturbation it generates in the electromagnetic field is infinite. What saves the day then is that since there is no way to extract the electron from the electromagnetic field, one only cares about the sum $m + \delta m$ so that the value of $m$ (even if infinite) is irrelevant.
The explicit formulas which allow to concretely perform the renormalization procedure were gradually obtained by Bogoliubov, Parasiuk, Hepp and Zimmermann. They provide an inductive procedure which is based on three steps. Given a graph $\Gamma$, one first "prepares" $\Gamma$, by replacing the unrenormalized value $U(\Gamma)$ by a sum involving suitably defined (not necessarily connected) subgraphs $\gamma \subset \Gamma$ and the contracted graphs $\Gamma/\gamma$ or cograph, obtained by collapsing each connected component of $\gamma$ to a single vertex,

$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)$$

where the counterterms $C(\gamma)$ are defined inductively by

$$C(\Gamma) = -T(\overline{R}(\Gamma)) = -T\left(U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)\right)$$

where $T$ is the pole part in dimensional regularization. Finally the renormalized value of the graph $\Gamma$ is given by

$$R(\Gamma) = \overline{R}(\Gamma) + C(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)$$

While this procedure is perfectly justified from the physics standpoint it took a long time to uncover its precise meaning from the mathematical standpoint. This was obtained in two key steps which are

(1) The Hopf algebra of Feynman graphs
(2) Renormalization as Birkhoff decomposition

The Hopf algebra structure hidden behind the combinatorics of Feynman graphs was discovered by D. Kreimer who first formulated the hierarchical structure of subgraphs in terms of rooted trees. In our joint work [16] we formulated the Hopf algebra directly in terms of Feynman graphs. As an algebra, the Hopf algebra $\mathcal{H}$ is the free commutative algebra generated by one particle irreducible (1PI) graphs. In order to define the coproduct

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

it is enough to specify it on 1PI graphs. One sets

$$\Delta\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$

where the subgraphs $\gamma \subset \Gamma$ and the cographs $\Gamma/\gamma$ are the same as those involved in the BPHZ preparation procedure. It is a quite remarkable fact that the obtained coproduct is coassociative *i.e.* that it fulfills

$$(\Delta \otimes \mathrm{id})\, \Delta = (\mathrm{id} \otimes \Delta)\, \Delta\,.$$

By construction, a commutative Hopf algebra is the algebra of coordinates on a group, and in the case of a graded connected Hopf algebra such as the Hopf algebra of graphs the corresponding group is a projective limit of unipotent Lie groups. While this shows that there are very interesting mathematical structures underlying the perturbative expansion it left open the problem of giving a conceptual understanding of the BPHZ formulas.

This problem was solved in our joint work [16]. This gives a precise unexpected relation between renormalization and a basic geometric procedure called the Birkhoff decomposition which originates in the problem of classifying holomorphic bundles on the sphere. A complex vector bundle $E$ of dimension $n$ on the Riemann sphere is obtained by clutching together two trivial bundles on the lower and upper hemispheres $C_{\pm}$, using a map $\gamma(z) \in \mathrm{GL}(n, \mathbb{C})$ defined on the common boundary $C$ (Figure 5). The Birkhoff decomposition of the map $\gamma$ is the factorization

$$\gamma(z) = \gamma_{-}(z)^{-1}\, \lambda(z)\, \gamma_{+}(z)\,, \qquad z \in C$$

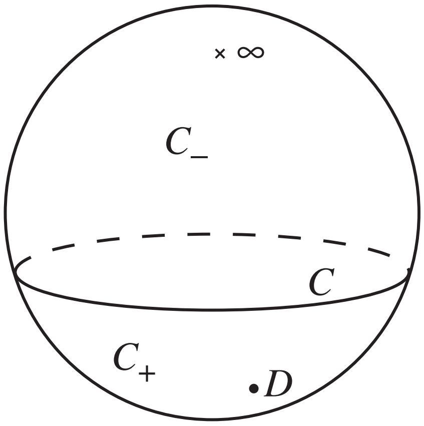


FIGURE 5. Birkhoff decomposition

where $\gamma_\pm$ are holomorphic maps from $C_\pm$ to $\mathrm{GL}(n,\mathbb{C})$ while $\lambda$ is a diagonal map of the form,

$$\lambda(z) = \begin{pmatrix} z^{k_1} & & & \\ & z^{k_2} & & \\ & & \ddots & \\ & & & z^{k_n} \end{pmatrix}$$

The original holomorphic vector bundle $E$ is then isomorphic to the sum of the line bundles of degree $k_j$ obtained by the simple clutching function $\lambda$. Thus the geometric meaning of the Birkhoff decomposition is the Birkhoff-Grothendieck theorem asserting that the holomorphic bundles on $P_1(\mathbb{C})$ are isomorphic to direct sums of holomorphic line bundles.

When one replaces the group $\mathrm{GL}(n,\mathbb{C})$ by a prounipotent simply connected complex Lie group $G$, the Birkhoff decomposition of a map $\gamma(z) \in G$ defined on the common boundary $C$, takes the simpler form

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z)\,, \qquad z \in C$$

Simplifying further one can let $C$ be a circle of infinitesimal radius around a point $z_0 = 0 \in P_1(\mathbb{C})$. One can then use the Hopf algebra $\mathcal{H}$ of coordinates on $G$ to encode the map $\gamma$ as a homomorphism $\phi : \mathcal{H} \to \mathbb{C}\{z\}[z^{-1}]$ to the field $K = \mathbb{C}\{z\}[z^{-1}]$ of convergent Laurent series. The maps $\gamma : P_1(\mathbb{C})\backslash\{z_0\} \to G$ which are holomorphic on $C_- = P_1(\mathbb{C})\backslash\{z_0\}$ are encoded by homomorphisms such that $\phi(\mathcal{H}) \subset \mathbb{C}([z^{-1}])$. The maps $\gamma$ which take a finite value at $z_0 = 0$ are encoded by homomorphisms such that $\phi(\mathcal{H}) \subset \mathbb{C}\{z\}$ where $\mathbb{C}\{z\} \subset K$ is the subring of Laurent series which are regular at $z_0 = 0$.

The conceptual meaning of the *BPHZ* combinatorial procedure, which is the main result of our joint work [16], is given below, using the following notation for the coproduct in a graded connected Hopf algebra $\mathcal{H}$

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$$

**Theorem 2.1.** (1) *Let $\mathcal{H}$ be a graded connected Hopf algebra and $\phi : \mathcal{H} \to K$ be an algebra homomorphism. The Birkhoff decomposition of the corresponding loop is obtained recursively from the equalities*

$$\phi_-(X) = -T\left(\phi(X) + \sum \phi_-(X')\phi(X'')\right)$$

*and*

$$\phi_+(X) = \phi(X) + \phi_-(X) + \sum \phi_-(X')\phi(X'').$$

(2) *When $\mathcal{H}$ is the Hopf algebra of graphs and $\phi = U$ the homomorphism associated to the unrenormalized value of graphs in dimensional regularization then $\phi_-$ gives the counterterms $C$ and $\phi_+$ gives the renormalized value $R$.*

Put in other words the combinatorial recipe given by the BPHZ procedure coincides with the Birkhoff decomposition of the loop giving the unrenormalized value of the theory. While this allows to understand in a really conceptual and simple way the procedure currently used by physicists in their computations it also ties up with a central idea in mathematics, namely the *Riemann-Hilbert* correspondence. Loosely speaking such a correspondence relates objects of a differential theoretic nature–such as differential systems–with finite dimensional representations of a "monodromy" group $G$. It plays a central role in the Riemann-Hilbert problem for which the Birkhoff decomposition was invented. In the simplest example of a system with regular singularities the group $G$ is indeed the monodromy group but the Riemann-Hilbert correspondence makes sense in a much wider setting and allows to give meaning to a group which plays the same role as the monodrmy in the irregular singular case ([24], [32]). The Riemann-Hilbert correspondence underlying renormalization was unveiled in our joint work with Matilde Marcolli ([20], [19]). Quite surprisingly the group $U$ whose representations classify the natural differential systems arising from renormalization turned out to be essentially the *Cosmic Galois Group* which had been conjectured by Cartier who wrote in [4]:

> *"La parenté de plus en plus manifeste entre le groupe de Grothendieck–Teichmüller d'une part, et le groupe de renormalisation de la Théorie Quantique des Champs n'est sans doute que la première manifestation d'un groupe de symétrie des constantes fondamentales de la physique, une espèce de groupe de Galois cosmique!"*

In fact the Lie algebra involved in the dream of Cartier is the free graded Lie algebra

$$\mathcal{H}_c = \mathcal{U}(\mathcal{F}(3,5,7,\cdots)_\bullet)^\vee$$

with generators of odd degrees while for the group $U$ which appears in our work the Lie algebra is the free graded Lie algebra

$$\mathcal{H} = \mathcal{U}(\mathcal{F}(1,2,3,\cdots)_\bullet)^\vee$$

with generators of all integer degree. The group $U$ appears in our work from the classification of equisingular flat connections and is the *Universal Symmetry Group* of renormalizable theories. It acts on the coupling constants of any such theory and it contains the usual renormalization group as a natural one parameter subgroup $\mathbb{R} \subset U$. In simple rather rough terms, what happens is that the role of the $\beta$ function attached to each coupling is now played by a single element of the Lie algebra of the group $G$ associated to Feynman graphs. This Lie algebra element $\beta$ admits homogeneous components $\beta_n$ in each power of the Planck constant $\hbar$, corresponding to the grading of the Hopf algebra of graphs by their loop number, and the generators $e_{-n}$ of the universal Lie algebra (of the group $U$) are mapped to $\beta_n$, which gives the first vertical arrow in the following diagram of group homomorphisms which allows for $U$ to act at the formal level on the coupling constants:

*Cosmic Galois Group U*
↓
*Group G associated to Feynman graphs*
↓
*Group of formal diffeomorphisms of coupling constants*

The Lie subalgebra generated under the graph Lie bracket by the components $\beta_n$ plays the role of the Galois group of a given theory [22]. It is still a mystery to find a precise relation of the group $U$ with the (abstractly isomorphic) motivic Galois group (in the sense of mixed Tate motives) of the scheme $S_4$ of 4-cyclotomic integers

$$G_{\mathcal{M}_T}(\mathcal{O}), \quad \mathcal{O} = \mathbb{Z}[i][\frac{1}{2}]$$

and to clarify the number theoretic flavor of the first vertical arrow in the above diagram.

The geometric space which is the support of the equisingular connections is a two dimensional complex space $B$. It is a principal bundle with structure group the complex multiplicative group $\mathbb{G}_m = \mathbb{C}^*$. The base is an infinitesimal disk $\Delta$ around $0 \in \mathbb{C}$. The equisingular connections are singular on the fiber over $0 \in \mathbb{C}$. They are connections on filtered vector bundles and we refer to [20], [19] for the precise definitions. At the physics level the parameter $z$ in the base $\Delta$ is the same as the $z$ in the dimensional regularization procedure (Dim-Reg) used to get away from the singularity by replacing the dimension $D$ of space-time in the evaluation of Feynman graphs by $D - z$. The generic element in the fiber over $z \in \Delta$ is of the form $v = \mu^z \hbar$ where $\mu$ is a mass scale and $\hbar$ is the Planck constant. The principal bundle action of the group $\mathbb{G}_m = \mathbb{C}^*$ is by rescaling of $\hbar$ *i.e.* by $\hbar \frac{\partial}{\partial \hbar}$. The singularity of the connections coming from QFT computations are governed by the following universal behavior of a flat section[1]:

$$\gamma_U(z, v) = \mathrm{T}\mathbf{e}^{-\frac{1}{z}\int_0^{\mathbf{v}} \mathbf{u}^{\mathbf{Y}}(\mathbf{e})\,\frac{\mathbf{du}}{\mathbf{u}}} \in U \tag{2}$$

which falls in the "irregular singular" case of the theory of differential equations. It provides a universal formula for the counterterms when using the combination of Dim-Reg with minimal substraction. There is a strong analogy between the role of the *exponential torus* in the theory developed by Deligne, Ramis and Martinet ([24], [32]) for formal solutions of differential systems and the role of the cosmic Galois group in the renormalization procedure. In both cases the groups appear from the Riemann-Hilbert correspondence developed by Grothendieck as a natural generalization of the ideas of Galois to the higher dimensional set-up. In both cases the basic intuition of Galois of an underlying *théorie de l'ambiguïté* is present. In the case of renormalization physicists have from the very start of the theory recognized that there is no way (except experimental tests) to break the ambiguity which is inherent to the specific values of the parameters in a renormalizable theory. Galois was well aware of a similar phenomenon in the theory of equations including in the higher dimensional case as witnessed by his last writing:

> *"Tu sais, mon cher Auguste, que ces sujets ne sont pas les seuls que j'aie explorés. Mes principales méditations depuis quelque temps étaient dirigées sur l'application à l'analyse transcendante de la théorie de l'ambiguïté. Il s'agissait de voir a priori dans une relation entre des quantités ou fonctions transcendantes quels échanges on pouvait faire, quelles quantités on pouvait substituer aux quantités données sans que la relation pût cesser d'avoir lieu. Cela fait reconnaitre tout de suite l'impossibilité de beaucoup d'expressions que l'on pourrait chercher. Mais je n'ai pas le temps et mes idées ne sont pas encore assez développées sur ce terrain qui est immense"*

A typical example (*cf.* [33]) of an "échange" which illustrates this type of ambiguity is the substitution

$$e^{-\frac{1}{z}} \rightarrow \lambda\, e^{-\frac{1}{z}}$$

which comes from the action of the exponential torus on the infinitely flat terms[2] involved in a formal solution of a differential system.

[1] where $Te$ denotes the time ordered exponential [1]

[2] *i.e.* all terms of the Taylor expansion vanish


FIGURE 6. Galois

The main lesson one learns from the above developments is that one should not consider the divergences of QFT as unwanted nuisances but rather as the signature of subtle symmetries of Galois type which prevent one from making simple predictions unless they are carefully taken into account. It also shows that it is worthwhile to give a precise geometric support to the dimensional regularization and to understand in a more geometric manner the universal behavior of counterterms as dictated by (2). As we shall briefly explain below this can be done within the new framework provided by noncommutative geometry.

## 3. Noncommutative Geometry

We shall first give a brief introduction to noncommutative geometry from the physics perspective. Our standpoint will be to view it as a very economical way of describing the geometry of space-time at the effective level *i.e.* up to the length scales of the inverse of a 100 GEV. We shall explain the spectral paradigm of NCG in close connection with the issue of measuring distances and start by drawing the parallel between the transition from the riemannian $g_{\mu\nu}$ paradigm to the NCG spectral paradigm and the evolution followed by the standard of length in the metric system.

3.1. **Why noncommutative spaces ?** The full action (1) of gravity coupled with matter

$$S = S_E + S_{SM}$$

admits a huge natural group of symmetries. The group of gauge invariance for the Einstein action $S_E$ is the group $\mathrm{Diff}(M)$ of diffeomorphisms of the manifold $M$ and the gauge invariance of the action is simply the manifestation of its geometric nature. The full group $\mathcal{U}$ of invariance of the action (1) is however richer than the group $\mathrm{Diff}(M)$ of diffeomorphisms of the manifold $M$ since one needs to include the group $\mathcal{G}$ of gauge transformations of the matter sector. By construction the group $\mathcal{G}$ is a group of maps from $M$ to the small gauge group $G$ which as far as we know, *i.e.* up to energies of the order of 100 GEV, is $G = U(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. The group $\mathrm{Diff}(M)$ acts on $\mathcal{G}$ by permutations and the full group $\mathcal{U}$ of symmetries of $S$ is the semi-direct product

$$\mathcal{U} = \mathcal{G} \rtimes \mathrm{Diff}(M)$$

and always contains the huge normal subgroup $\mathcal{G}$. Rather than postponing the addition of the matter action $S_{SM}$ it is natural to try and find a space $X$ whose group of diffeomorphisms is simply $\mathcal{U}$. This search is bound to fail if one looks for an ordinary manifold since by a mathematical result due to J. Mather and W. Thurston the connected component of the identity in $\mathrm{Diff}(M)$ is always a simple group, excluding a semi-direct product structure as that of $\mathcal{U}$. But noncommutative spaces of the simplest kind readily give the answer, modulo a few subtle points. To understand what happens note that for ordinary manifolds the algebraic object corresponding to a diffeomorphism is just an automorphism $\alpha \in \mathrm{Aut}(\mathcal{A})$ of the algebra of coordinates. When an algebra is not commutative it admits "trivial" automorphisms, called *inner* given by the formula

$$\alpha(x) = u\,x\,u^{-1}\,, \quad \forall x \in \mathcal{A}$$

where $u$ is an invertible element of $\mathcal{A}$. When $\mathcal{A}$ is an involutive algebra the element $u$ is taken to be unitary ($i.e.$ $u\,u^* = u^*\,u = 1$) so that $\alpha$ preserves the involution. Moreover the inner automorphisms form a subgroup

$$\mathrm{Int}(\mathcal{A}) \subset \mathrm{Aut}(\mathcal{A})$$

which is always a normal subgroup of $\mathrm{Aut}(\mathcal{A})$. Let us take the simplest example where we take for $\mathcal{A}$ the algebra

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C})) = C^\infty(M) \otimes M_n(\mathbb{C})$$

of smooth maps from a manifold $M$ to the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices. One then shows that the group $\mathrm{Int}(\mathcal{A})$ in that case is locally isomorphic to the group $\mathcal{G}$ of

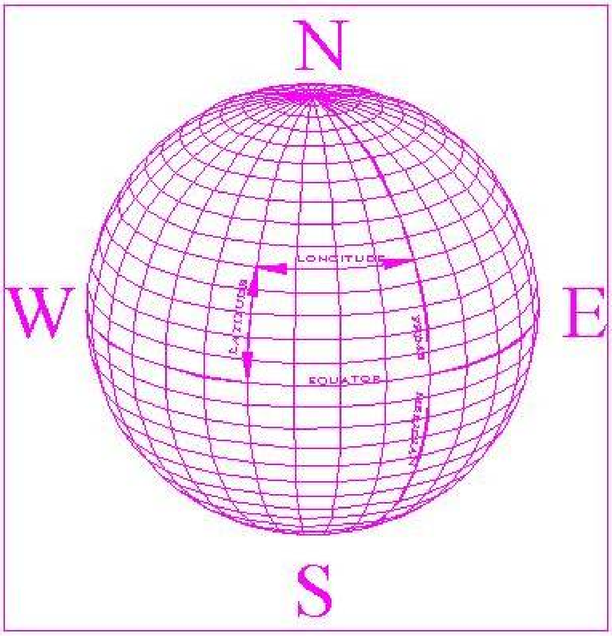


FIGURE 7. Meridian

smooth maps from $M$ to the small gauge group $G = PSU(n)$ (quotient of $SU(n)$ by its center) and that the general exact sequence

$$1 \to \mathrm{Int}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1$$

becomes identical to the exact sequence governing the structure of the group $\mathcal{U}$, namely

$$1 \to \mathcal{G} \to \mathcal{U} \to \mathrm{Diff}(M) \to 1$$

It is quite striking that the terminology coming from physics: internal symmetries agrees so well with the mathematical one of inner automorphisms. In the general case only automorphisms that are unitarily implemented in Hilbert space will be relevant but modulo this subtlety one can see at once from the above example the advantage of treating noncommutative spaces on the same footing as the ordinary ones. The next step is to properly define the notion of metric for such spaces and we shall first indulge in a short historical description of the evolution of the definition of the "unit of length" in physics. This will prepare the ground for the introduction to the spectral paradigm of NCG in the following section.

3.2. **A brief history of the metric system.** The notion of geometry is intimately tied up with the measurement of length and it was never obvious how to reach some agreement on a physical unit of length which would unify the numerous existing choices.

FIGURE 8. The meridian from Barcelone to Dunkerque

Around the end of the 18th century France and England decided to try and find a unit that would be invariable in time and acceptable by all nations. In 1790, Talleyrand proposed to define the unit using the length of a pendulum beating the second at sea level (as already proposed by Picard, in 1670) at 45° of latitude. But this solution was quickly abandoned. One year later several scientists in France including Monge, Lagrange and Laplace agreed on the definition of the unit of length in the metric system, the "mètre", as being $10^{-7}$ times the quarter of the meridian of the earth (Figure 7). An expedition was sent out to measure the arc of the meridian from Barcelone to Dunkerque (Figure 8) while the corresponding angle ($\sim 9.5°$) was determined using reference stars. A concrete realization of the "mètre", called the "mètre-étalon" was also realized in the form a platinum bar (Figure 9) which was deposited near Paris in a specific place (from 1889 this place was the pavillon de Breteuil) and the inaccuracy of the measurement of the meridian forced to redefine the "mètre" as the length of the "mètre-étalon". This definition held until 1960.

Already in 1927, at the seventh conference on the metric system, in order to take into account the inevitable natural variations of the concrete "mètre-étalon", the idea emerged to compare it with a reference wave length (the red line of Cadmium). Around 1960 the reference to the "mètre-étalon" was finally abandoned and a new definition of the "mètre" was adopted as $1650763, 73$ times the wave length of the radiation corresponding to the transition between the levels 2p10 and 5d5 of the Krypton 86Kr. In 1967 the second was defined as the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of Caesium-133. Finally in 1983 the "mètre" was defined as the distance traveled by light in 1/299792458

FIGURE 9. The “mètre-étalon”

second. In fact the speed of light is just a conversion factor[3] and to define the “mètre” one gives it the specific value of

$$c = 299792458\, m/s$$

In other words the “mètre” is defined as a certain fraction $\frac{9192631770}{299792458} \sim 30.6633...$ of the wave length of the radiation coming from the transition between the above hyperfine levels of the Caesium atom.
The advantages of the new standard of length are many. First by not being tied up with any specific location it is in fact available anywhere where Caesium is. The choice of Caesium as opposed to Helium or Hydrogen which are much more common in the universe is of course still debatable.

While it would be difficult to communicate our standard of length with other extra terrestrial civilizations if they had to make measurements of the earth (such as its size) the spectral definition can easily be encoded in a probe and sent out. In fact spectral patterns (Figure 10) provide a perfect “signature” of chemicals, and a universal information available anywhere where these chemicals can be found.

3.3. **Spectral Geometry.** It is thus natural to wonder wether one can adapt the basic paradigm of geometry to the new standard of length. The Riemannian paradigm is based on the Taylor expansion in local coordinates $x^\mu$ of the square of the line element, in the form

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{3}$$

[3]In particular, since it is a mere convention, it could have been taken to be $300\,000\,000\, m/s$. It is sad (*cf.* the excellent account `http://kolmogorov.unex.es/~navarro/res/notices.pdf`) that while Grothendieck was asking the right question: “what is the mètre” and rightly saying that the convention $c = 300\,000\,000\, m/s$ would have been simpler, the standard reaction to his query was to see there the clear symptom of a deranged mind.

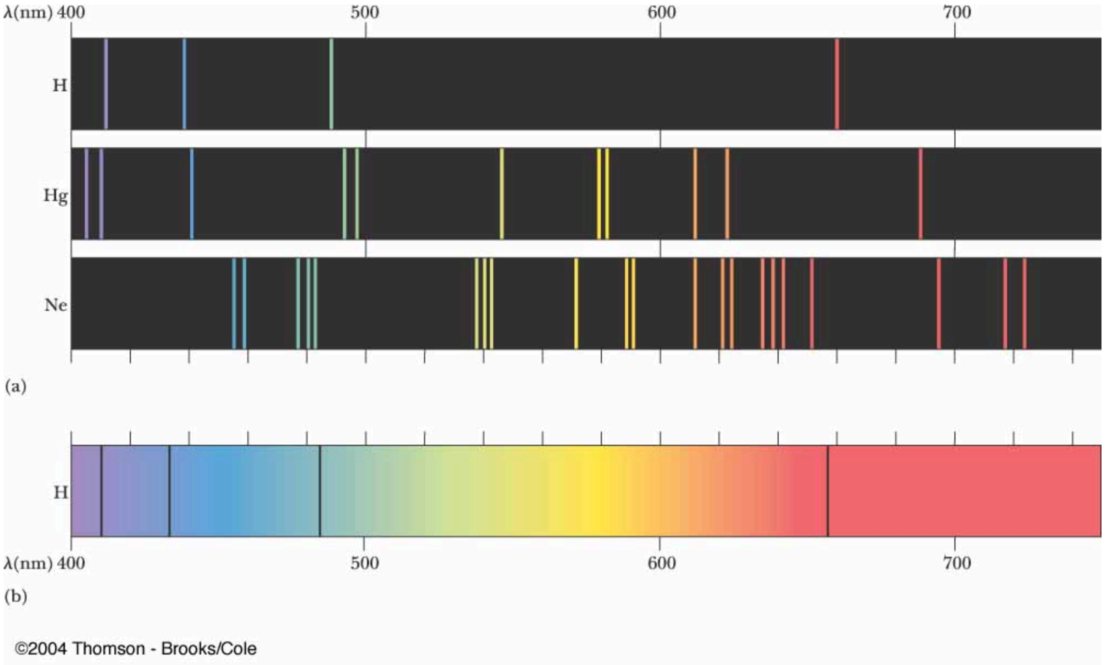


FIGURE 10. Wave length of spectral lines

and the measurement of the distance $d(x, y)$ between two points is given by the geodesic formula

$$d(A, B) = \operatorname{Inf} \int_\gamma ds \tag{4}$$

where the infimum is taken over all paths from $A$ to $B$.

In noncommutative geometry the first basic change of paradigm has to do with the classical notion of a "real variable" which one would normally describe as a real valued function $f$ on a set $X$ *i.e.* as a map $f : X \to \mathbb{R}$. In fact quantum mechanics provides a very convenient substitute. It is given by a self-adjoint operator $H$ in Hilbert space. Note that the choice of Hilbert space $\mathcal{H}$ is irrelevant here since all separable infinite dimensional Hilbert spaces are isomorphic. All the usual attributes of real variables such as their range, the number of times a real number is reached as a value of the variable etc... have a perfect analogue in the quantum mechanical setting. The range is the spectrum of the operator $H$, and the spectral multiplicity $n(\lambda)$ gives the number of times a real number $\lambda \in \mathbb{R}$ is reached.

As in the classical framework, a space $X$ is described by the corresponding algebra $\mathcal{A}$ of coordinates which is now concretely represented as operators in a fixed Hilbert space $\mathcal{H}$. What is surprising in the new set-up is that it gives a natural home for "infinitesimals". Indeed it is perfectly possible for an operator to be "smaller than $\epsilon$ for any $\epsilon$" without being zero. This happens when the norm of the restriction of the operator to subspaces of finite codimension tends to zero when these subspaces decrease (under the natural filtration by inclusion). The corresponding operators are called "compact" and they share with naive infinitesimals all the expected algebraic properties. Indeed they form a two-sided ideal $\mathcal{K}$ of the algebra of bounded operators in $\mathcal{H}$ and the only property of the naive infinitesimal calculus that needs to be dropped is the commutativity.

| **Space** $X$ | **Algebra** $\mathcal{A}$ |
|---|---|
| **Real variable** $x^\mu$ | **Self-adjoint operator** $H$ |
| **Infinitesimal** $dx$ | **Compact operator** $\epsilon$ |
| **Integral of infinitesimal** | $\int\!\!\!\!- \epsilon =$ **Coefficient of** $\log(\Lambda)$ **in** $\mathrm{Tr}_\Lambda(\epsilon)$ |
| **Line element** $\sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$ | **ds = Fermion propagator** |

It is important to explain what is gained in dropping such a useful rule as commutativity. We shall explain this point for a specific infinitesimal namely the "line element" $ds$ which defines the geometry through the measurement of distances. If an infinitesimal commutes with a variable with connected range it follows that the corresponding variable $x$ affects a specific value. In particular with $x^\mu$ the coordinates and assuming that they commute with each other and with the line element, the latter is forced to be "localized" somewhere which is very inconvenient. When the hypothesis of commutativity is dropped it is no longer the case that the line element $ds$ needs to be localized and in fact it is precisely the lack of commutation of $ds$ with the coordinates that allows to measure distances. Thus in noncommutative geometry the basic classical formula (4) is replaced by the following

$$d(A,B) = \operatorname{Sup}\{|f(A) - f(B)|\,;\, f \in \mathcal{A}\,,\ \|[D,f]\| \leq 1\} \tag{5}$$

where $D$ is the inverse of the line element $ds$. Note that one should not confuse the "line element" $ds$ with the unit of length. In the classical framework, the latter allows one to give a numerical value to the distance between nearby points in the form (3). Multiplying the unit of length by a scalar $\lambda$ one divides the line element $ds$ by $\lambda$ since $ds$ is measured by its ratio with the unit of length.

A noncommutative geometry is given by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *i.e.* by an involutive algebra $\mathcal{A}$ concretely represented as operators in Hilbert space $\mathcal{H}$ and the line element $ds = 1/D$.

| | |
|---|---|
| **Geodesic equation** | $\frac{d\psi(t)}{dt} = i\,\lvert D\rvert\,\psi(t)$ |
| **Geodesic Flow** | $e^{i\,t\,\lvert D\rvert}$ |
| **Geodesic distance** | $d(A,B) = \operatorname{Sup}\{\lvert f(A) - f(B)\rvert$ $f \in \mathcal{A}\,,\ \lVert[D,f]\rVert \leq 1\}$ |
| **Volume form** | $⨍ f\,\lvert ds\rvert^n$ |
| **Einstein action** | $⨍ f\,\lvert ds\rvert^{n-2}$ |

The traditional notions of geometry all have natural analogues in the spectral framework. Some of these analogues are summarized in the above table and we refer to [10] for more details. The dimension of a noncommutative geometry is not a number but a spectrum, the *dimension spectrum* (*cf.* [23]) which is the subset $\Pi$ of the complex plane $\mathbb{C}$ at which the spectral functions have singularities. Under the hypothesis that the dimension spectrum is *simple i.e.* that the spectral functions have at most simple poles, the residue at the pole defines a far reaching extension (*cf.* [23]) of the fundamental integral in noncommutative geometry given by the Dixmier trace (*cf.* [10]). This extends the Wodzicki residue from pseudodifferential operators on a manifold to the general framework of spectral triples, and gives meaning to $⨍ T$ in that context. It is simply given by

$$⨍ T = \operatorname{Res}_{s=0} \operatorname{Tr}\left(T\,|D|^{-s}\right). \tag{6}$$

3.4. **Inner fluctuations of the metric.** Exactly as the inner automorphisms of a noncommutative space correspond to the internal symmetries of physics (section 3.1), the metric of a noncommutative space admits natural inner fluctuations, which generate a natural foliation of the space of metrics, and correspond to the gauge bosons (other than the graviton). These inner fluctuations appear through the simple issue of Morita equivalence. Given an algebra $\mathcal{A}$, a Morita equivalent algebra $\mathcal{B}$ is the algebra of endomorphisms of a finite projective (right) module $\mathcal{E}$ over $\mathcal{A}$

$$\mathcal{B} = \operatorname{End}_{\mathcal{A}}(\mathcal{E})$$

If $\mathcal{A}$ acts in the Hilbert space $\mathcal{H}$ then $\mathcal{B}$ acts in a natural manner in the tensor product

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

which is a Hilbert space provided that $\mathcal{E}$ is hermitian. But to define the analogue $D'$ of the operator $D$ for $(\mathcal{B}, \mathcal{H}')$ requires the choice of a *hermitian connection* $\nabla$ on $\mathcal{E}$. The point is that the formula

$$D'(\xi \otimes \eta) = \xi \otimes D\eta$$

is not compatible with the tensor product over $\mathcal{A}$ since in general the operator $D$ does not commute with elements of $\mathcal{A}$. A connection is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ satisfying the Leibniz rule

$$\nabla(\xi a) = (\nabla\xi)a + \xi \otimes da\,, \quad \forall\, \xi \in \mathcal{E}\ ,\ a \in \mathcal{A},$$

where $da = [D, a]$ and where

$$\Omega^1_D = \{\sum a_j\,[D, b_j]\ ;\ a_j,\, b_j \in \mathcal{A}\}$$

which is by construction a bimodule over $\mathcal{A}$. One then lets

$$D'(\xi \otimes \eta) = \xi \otimes D\eta + (\nabla\xi)\,\eta$$

and this combination is well defined in $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$.

Any algebra $\mathcal{A}$ is Morita equivalent to itself (with $\mathcal{E} = \mathcal{A}$) and when one applies the above construction in that special case one gets the inner deformations of the spectral geometry. These replace the operator $D$ by

$$D \to D + A \tag{7}$$

where $A = A^*$ is an arbitrary selfadjoint element of $\Omega^1_D$.

The above discussion of the inner fluctuations of the metric adapts to the presence of the additional structure given by the charge conjugation operator (real structure) which is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, with the property that

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon''\gamma J \,(\text{even case}). \tag{8}$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of $n \mod 8$ (*cf.* [12]) given by $(-1, 1, 1)$ for $n = 4$. Moreover, the action of $\mathcal{A}$ satisfies the commutation rule

$$[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}, \tag{9}$$

where

$$b^0 = Jb^* J^{-1} \qquad \forall b \in \mathcal{A}, \tag{10}$$

defines a right $\mathcal{A}$-module structure on $\mathcal{H}$ by

$$\xi\, b = \, b^0\, \xi\,, \quad \forall\, \xi \in \mathcal{H}\,, \quad b \in \mathcal{A}$$

The operator $D$ satisfies the "order one" condition,

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}\,. \tag{11}$$

The unitary group of the algebra $\mathcal{A}$ then acts by the "adjoint representation" in $\mathcal{H}$ in the form

$$\xi \in \mathcal{H} \to u\, \xi\, u^*\,, \quad \forall\, \xi \in \mathcal{H}\,, \quad u \in \mathcal{A}\,, \quad u\, u^* = u^*\, u = 1\,,$$

and the perturbation (7) gets replaced by

$$D \to D + A + J\, A\, J^{-1} \tag{12}$$

3.5. **Dimensional regularization and spaces of dimension $z$.** When expanded in terms of the free generators $e(-n)$ of the Lie algebra of the cosmic Galois group $U$, the universal singular frame gives the following expression,

$$\gamma_U(-z, v) = \sum_{n \geq 0} \sum_{k_j > 0} \frac{e(-k_1)e(-k_2)\cdots e(-k_n)}{k_1\,(k_1 + k_2)\cdots(k_1 + k_2 + \cdots + k_n)}\, v^{\sum k_j}\, z^{-n} \tag{13}$$

whose coefficients are strikingly similar to the coefficients which appear in the local index formula in NCG ([23]). This index formula gives an analogue of the Pontrjagin classes in the general NCG framework, and is expressed in terms of a cyclic cocycle whose components are of the form

$$\varphi_n(a^0, \ldots, a^n) := \sum_k c_{n,k} ⨍ \gamma\, a^0 [D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)}\, |D|^{-n-2|k|}, \quad \forall a^j \in \mathcal{A}.$$

where $T^{(k_i)} = \nabla^{k_i}(T)$ with $\nabla(T) = D^2 T - T D^2$. The summation index $k$ is a multi-index with $k_j \geq 0$, $|k| = k_1 + \ldots + k_n$, and the coefficients are

$$c_{n,k} = \frac{(-1)^{|k|}}{2}\, (k_1! \ldots k_n!)^{-1}\, ((k_1 + 1) \ldots (k_1 + k_2 + \ldots + k_n + n))^{-1}\, \Gamma\left(|k| + n/2\right).$$

Motivated by the similarity between the coefficients $c_{n,k}$ and those of the universal singular frame, we have shown in our joint work with Matilde Marcolli [21], how to develop dimensional regularization (*Dim-Reg*) in a concrete explicit manner in the framework of NCG. We construct noncommutative spaces $X_z$ of dimension $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, such that the dimension spectrum of $X_z$ is reduced to the complex number $z$ and whose inverse line element $D_z$ fulfills

$$\mathrm{Trace}(e^{-\lambda D_z^2}) = \pi^{z/2} \lambda^{-z/2}, \quad \forall \lambda \in \mathbb{R}_+^* .$$

This allows to define Dim-Reg in NCG but we also show in [21] that it allows to treat chiral anomalies in QFT and corresponds exactly to the t'Hooft-Veltman and Breitenlohner-Maison prescription. This works inasmuch as one restricts attention to those Feynman graphs with only fermionic internal lines. It corresponds to taking the product of the standard geometry of (Euclidean) space-time by the space $X_z$. The product of two spectral triples is obtained in general as follows

$$\mathcal{H}'' = \mathcal{H} \otimes \mathcal{H}', \quad D'' = D \otimes 1 + \gamma_5 \otimes D' . \tag{14}$$

The evanescent gauge potentials [9] appear naturally from the inner fluctuations of this product metric. The relation between chiral anomalies and the local index formula in NCG will be discussed in full in [21]. From the physics standpoint the long range goal is to model the universal formula for the counterterms (13) of renormalizable theories as corresponding to a universal correction of the standard euclidean geometry obtained from the products with the spaces $X_z$ following the universal singular frame when $z \to 0$. Much more concrete work will be needed to implement this idea but as we shall see shortly the "effective geometry" of space-time coming from the nuance between QED and the standard model is also described as a product of the form (14) of the standard euclidean geometry by a zero dimensional noncommutative space. Thus one can then combine both corrections and get a suggestive form of the effective geometry of space-time using noncommutative geometry.

**Remark 3.1.** As far as describing a good model of the effective geometry of space-time it is worthwhile to note that the spectral framework of NCG allows to take into account the dressing of the (euclidean) fermion propagator[4] as quantum corrections of the geometry. In the resulting noncommutative geometry $(\mathcal{A}, \mathcal{H}, D(\mu))$ the inverse line element $D(\mu)$ depends explicitly on the energy scale $\mu$ at which the measurements are performed. The dimension (more precisely the dimension spectrum) is then also a function of $\mu$. It is tempting to investigate models in which the dependence in $\mu$ plays a role not only at high energies (short distances) but also in the infrared domain.

[4]the same point applies to the Dirac Hamiltonian which has to do with the geometry of "space" as opposed to space-time

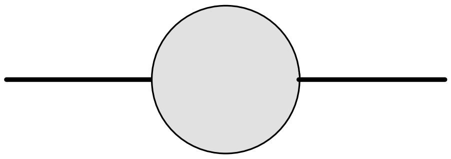

FIGURE 11. Dressed line element.

## 4. Observables in gravity and the spectral action

We shall give in this section a brief description of our joint work with Ali Chamseddine on the spectral action principle [5], [6], [7], [8].

4.1. **The spectral action principle.** The starting point is the discussion of observables in gravity. By the principle of gauge invariance the only quantities which have a chance to be observable in gravity are those which are invariant under the gauge group *i.e.* the group of diffeomorphisms of the space-time $M$. Assuming first that we deal with a classical manifold (and Wick rotate to euclidean signature for simplicity), one can form a number of such invariants (under suitable convergence conditions) as the integrals of the form

$$\int_M F(K)\sqrt{g}\,d^4x \tag{15}$$

where $F(K)$ is a scalar invariant function[5] of the Riemann curvature $K$. We refer to [27] for other more complicated examples of such invariants, where those of the form (15) appear as the *single integral* observables *i.e.* those which add up when evaluated on the direct sum of geometric spaces. Now while in theory a quantity like (15) is observable it is almost impossible to evaluate since it involves the knowledge of the entire space-time and is in that way highly non localized. On the other hand, spectral datas[6] (Figure 10) are available in localized form anywhere, and are (asymptotically) of the form (15) when they are of the additive form

$$\mathrm{Trace}\,(f(D/\Lambda)), \tag{16}$$

where $D$ is the Dirac operator and $f$ is a positive even function of the real variable while the parameter $\Lambda$ fixes the mass scale.
The spectral action principle asserts that the fundamental action functional $S$ that allows to compare different geometric spaces at the classical level and is used in the functional integration to go to the quantum level, is itself of the form (16). The detailed form of the even function $f$ is largely irrelevant since, assuming[7] that the dimension

[5]the scalar curvature is one example of such a function but there are many others

[6]the datas of Figure 10 are intimately related to the Dirac Hamiltonian, hence to the geometry of "space"

[7]this hypothesis fails for conical singularities

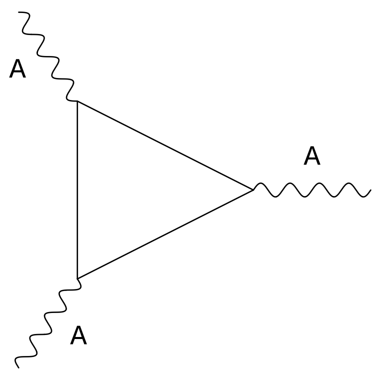


FIGURE 12. The triangle graph.

spectrum is *simple*, the spectral action (16) can be expanded in decreasing powers of the scale $\Lambda$ in the form

$$\text{Trace}\,(f(D/\Lambda)) \sim \sum_{k \in \Pi^+} f_k\, \Lambda^k \fint |D|^{-k} \, + \, f(0)\, \zeta_D(0) + \, o(1), \tag{17}$$

where the function $f$ only appears through the scalars

$$f_k = \int_0^\infty f(v)\, v^{k-1}\, dv. \tag{18}$$

The term independent of the parameter $\Lambda$ is the value at $s = 0$ (regularity at $s = 0$ is assumed) of the zeta function,

$$\zeta_D(s) = \, \text{Tr}\,(|D|^{-s})\,. \tag{19}$$

The terms involving negative powers of $\Lambda$ involve the full Taylor expansion of $f$ at 0.

As explained above the gauge potentials make good sense in the framework of NCG and come from the inner fluctuations of the metric. This gives meaning to the Feynman graphs all of whose internal lines are fermionic lines such as the triangle graph of Figure 12. In [8] we analyze how the spectral action behaves under the inner fluctuations. The main results are

- In dimension $\leq 4$ the variation of the spectral action under inner fluctuations gives the local counterterms for the fermionic graphs

$$\zeta_{D+A}(0) - \zeta_D(0) = -\fint AD^{-1} + \frac{1}{2}\fint (AD^{-1})^2 - \frac{1}{3}\fint (AD^{-1})^3 + \frac{1}{4}\fint (AD^{-1})^4,$$

- Assuming that the tadpole graph vanishes the above variation is the sum of a Yang-Mills action and a Chern-Simons action relative to a cyclic 3-cocycle on the algebra $\mathcal{A}$.

More precisely the variation under inner fluctuations of the scale independent terms of the spectral action is given (*cf.* [8] for precise notations) in dimension 4 by

$$\zeta_{D+A}(0) - \zeta_D(0) = \frac{1}{4}\int_{\tau_0} (dA + A^2)^2 - \frac{1}{2}\int_{\psi} (AdA + \frac{2}{3}A^3). \tag{20}$$

Here $\tau_0$ is a Hochschild 4-cocycle and $\psi$ a cyclic 3-cocycle both on the algebra $\mathcal{A}$. They are both computed as residues (*cf.* [8]) and under the above hypothesis, they both vanish unless the dimension is $\geq 3$ while the fluctuation (20) has a chance to be positive for all $A$ only in dimension 4.

The term in $\Lambda^2$ in the spectral action (17) is proportional to $⨍ ds^2$ which, in the usual 4-dimensional Riemannian case by [30], gives the Einstein-Hilbert action functional with the physical sign for the euclidean functional integral provided the moment $f_2 > 0$ (which is the case if $f$ is a positive function).

The term in $\Lambda^4$ in the spectral action (17) is proportional to $⨍ ds^4$ which, in the 4-dimensional case, gives a cosmological term. As we shall see later the natural constraint in the set-up of the functional integral will provide a homological meaning to this term *cf.* equation (29) below.

4.2. **Functional integral.** The formulation of the Feynman integral for gravity is highly dependent upon the chosen geometric set-up. Loosely speaking one should, in the Euclidean framework, perform the functional integral on the space of all geometries and the "no boundary proposal" ([29]) suggests to take all the 4-dimensional geometries with a fixed three dimensional boundary. The standard Riemannian geometric paradigm contains as basic ingredients a manifold $M$ and the Riemannian metric given in local coordinates by the $g_{\mu\nu}$. It thus seems at first sight that since the functional integral involves the sum over all geometries one would first need a good control of the classification of four manifolds before being able to even get started. In our spectral framework, it is the operator $D$ that contains all the relevant spectral information needed to evaluate the spectral action functional. But it is crucial to also include the matter fields in order to take into account the full action $S$ of gravity and matter (1)

$$S = S_E + S_{SM}$$

The key reason for dealing with noncommutative spaces is that a very simple modification of the geometry of space-time obtained by making the product (in the sense of (14)) of ordinary Euclidean space-time by a zero dimensional geometry gives ([6], [31])

- The group $\mathcal{U}$ of symmetries of gravity and matter (1) as the group of automorphisms (implementable in Hilbert space).
- The gauge bosons (including the Higgs) as the inner fluctuations (section (3.4)) of the metric.
- The action $S = S_E + S_{SM}$ as the spectral action.

In fact the spectral action contains additional terms such as a cosmological term, a term in $r\,H^2$ with $r$ the scalar curvature and $H$ the Higgs field, and a term in $C^2$ with $C$ the Weyl curvature. Many of the free parameters of the standard model such as the Yukawa masses and the Cabbibo-Kobayashi-Maskawa mixing matrix are simply encoded by the zero dimensional geometry $(\mathcal{A}_f, \mathcal{H}_f, D_f)$ where the letter $f$ stands for "finite". The obtained values of the remaining couplings are physical in both their signs and size but cannot have much significance until the renormalization group is brought into play. In order to achieve this one needs to test the spectral action under the renormalization group and the simplest way to proceed is to set up a functional integral using the spectral action in the exponent. Since the latter is spectral it admits a huge group of invariance, namely the group of unitary operators in Hilbert space. The difficulty is to specify precisely the integration variable, which roughly speaking should be the self-adjoint operator $D$ corresponding to the most general noncommutative geometry of the correct dimension, since there is no reason to restrict a priori the amount of noncommutativity in the space-time coordinates. Since the inner fluctuations (section 3.4) act on $D$ in a linear way it is natural to take $D[D]$ as the formal integration measure but one needs to write down the constraints fulfilled by $D$. We gave in [12] (*cf.* [25]) abstract conditions on an operator $D$ to be a Dirac operator on a given smooth manifold $M$. We explained in [12] how to formulate these conditions in the noncommutative case and expressed the hope that, in the commutative case, the stated conditions single out not only the Dirac operators (*cf.* [12] and [25] for the proof) but also the smooth manifolds among general compact spaces. This latter hope is still unproved and in some sense things would be more interesting if one obtained in that way a larger class of "pseudo-manifolds".

The first observation we use to get started in setting up a functional integral on "all four geometries" is that the following set of datas that come from a compact four Riemannian spin manifold $(M, g)$ are in fact independent (up to isomorphism) of the choice of $(M, g)$, they are[8]

- The Hilbert space $\mathcal{H}$ of $L^2$-spinors
- The $\gamma_5$ operator $\gamma$ in $\mathcal{H}$
- The charge conjugation operator $J$
- The decreasing filtration $\mathcal{H}^s \subset \mathcal{H}^{s'}$, $s > s'$ of $L^2$-spinors by Sobolev spaces

At the algebraic level (and as a consequence of working in dimension 4) the operators $\gamma$ and $J$ fulfill the simple rules

$$\gamma^2 = 1\,, \quad J^2 = -1\,, \quad J\gamma = \gamma\, J$$

with $\gamma$ self-adjoint, while $J$ is an antilinear isometry of $\mathcal{H}$. The role of the decreasing filtration by Sobolev spaces is to allow to define smoothness and not just the usual $C^0$ set-up of $C^*$-algebras. Its origin lies in the notion of Rigged Hilbert space or Gelfand triple ([26]).

[8]One can also fix the sign $\mathrm{Sign}(D) = F$ of the Dirac operator, *cf.* below.

We can thus define a universal algebra $\mathcal{C}$ of "would be coordinates" as follows, it consists of bounded operators $a$ in $\mathcal{H}$ which together with their adjoints fulfill the following conditions,

$$[\gamma, a] = 0\,, \quad a\,\mathcal{H}^s \subset \mathcal{H}^s\,, \quad \forall s \tag{21}$$

which are certainly fulfilled by coordinates of a compact four Riemannian spin manifold. So far the whole set-up is canonical and we now describe the two variables for the functional integral which roughly correspond to

1) The manifold
2) The metric

1) The variable playing the role of the manifold is more precisely a "volume form" and in specific terms is a Hochschild 4-cycle[9]

$$c \in Z_4(\mathcal{C}, \mathcal{C}) \tag{22}$$

which at the heuristic level gives a four dimensional projection of the noncommutative space described by the universal algebra $\mathcal{C}$.
2) The role of the "metric" is played by the operator $D$ with

$$D = D^*\,, \quad \gamma\, D = -\,D\,\gamma\,, \quad J\,D = \,D\,J\,, \tag{23}$$

which fulfills the linear condition

$$D\,\mathcal{H}^s \subset \mathcal{H}^{s-1}\,, \quad \forall s\,, \tag{24}$$

so that in particular its domain is $\mathcal{H}^1$.
The main relation is the following polynomial equation which relates $D$, $c$ and $\gamma$,

$$\langle c, D^{(4)}\rangle = \gamma \tag{25}$$

where the pairing $\langle c, D^{(4)}\rangle$ is defined as follows

$$\langle c, D^{(4)}\rangle = \sum a_0(k)[D,\, a_1(k)]\,\ldots[D,\, a_4(k)] \tag{26}$$

(In the case of an ordinary compact oriented spin manifold $M$, one obtains the cycle $c$ as follows, one lets $U_k$ be an open cover of $M$ by domains of local coordinates $x_j(k)$ (extended

[9]A general Hochschild $n$-cycle can be written in the form

$$c = \sum a_0(k) \otimes\, a_1(k)\, \otimes \ldots \otimes\, a_n(k)$$

where the $a_j(k) \in \mathcal{C}$ and the cycle condition is $b\,c = 0$ where the Hochschild boundary is

$$b(c) = \sum(\sum_0^{n-1} (-1)^j\, a_0(k)\otimes\ldots\otimes a_j(k)\, a_{j+1}(k)\, \otimes\ldots\otimes a_n(k) + (-1)^n\, a_n(k) a_0(k)\otimes\ldots\otimes\ldots\otimes a_{n-1}(k))$$

smoothly outside $U_k$) and $x(k)$ be a smooth partition of unity subordinated to the covering. Then up to normalization,

$$c = \sum_{k,\sigma} \epsilon(\sigma)\, \sqrt{g}\, x(k) \otimes\, x_{\sigma(1)}(k)\, \otimes \ldots \otimes\, x_{\sigma(4)}(k)$$

where the sum runs over all permutations $\sigma$ of $\{1,\ldots 4\}$ and $\epsilon(\sigma)$ is the signature of the permutation).

To obtain the algebra $\mathcal{A}$ of coordinates on the four dimensional projection associated to the cycle $c$, one can start from the subalgebra of $\mathcal{C}$ generated by the components $a_j(k)$ of the cycle $c$. These are easy to obtain linearly from $c$ using the contraction

$$\alpha^{(j)}(c) = \langle\, c\,,\, \mathrm{id}_j \otimes \alpha \rangle \in \mathcal{C}$$

where $\alpha \in \mathcal{C}^{*\otimes 4}$ is a linear form on the tensor power $\mathcal{C}^{\otimes 4}$ and the tensor $c$ is contracted except at the $j$-th factor. One then writes the basic commutation relations (9) and (11) as follows

$$[a,\, J\, b^*\, J^{-1}] = 0\,, \quad [[D, a],\, J\, b^*\, J^{-1}] = 0\,, \quad \forall\; a \,=\, \alpha^{(i)}(c)\,,\; b \,=\, \beta^{(j)}(c) \tag{27}$$

where both conditions are bilinear in $c$ and the second is linear in $D$.

One can then begin to investigate a functional integral of the form

$$\langle \mathcal{O}\, \rangle = \mathcal{N} \int \mathcal{O}\; e^{-\mathrm{Tr}(f(D)) - <\bar{\psi},\, D\,\psi> - \gamma(c,D)}\; D[\psi] D[\bar{\psi}] D[D] D[c]$$

where the term $\gamma(c, D)$ implements the constraints (25), (27), and $\mathcal{O}$ is a unitarily invariant function $\mathcal{O}(\psi, \bar{\psi}, c, D)$ of the variables of integration. A word of warning before one gets started to say that this is just a "first shot" showing that there is a way to formulate a functional integral involving the spectral action on the space of all 4-dimensional noncommutative geometries, but that many variants and refinements of the basic idea are possible. Rather than using the undetermined observable $\mathcal{O}$ it is preferable to add a source term in the exponent and experiment in trying to compute the resulting effective action. One can also truncate the integral and work for instance with a fixed value of the cycle $c$ associated to a standard four geometry. One obtains in that way a form of unimodular gravity on a fixed background manifold since $c$ specifies both the manifold and the volume form. In particular the cosmological term is canceled by the overall normalization factor $\mathcal{N}$ and the term $\mathrm{Tr}(f(D))$ is positive as long as the function $f$ is positive. It is natural to take $f(x) = g(x^2)$ with $g$ decreasing so that it makes $ds^2 = D^{-2}$ as small as possible. Note that by (24) the operator $D$ cannot be too big either.

The above functional admits a huge group of invariance, namely the subgroup $\mathcal{U}$ of the unitary group of $\mathcal{C}$ determined by the condition

$$\mathcal{U} = \{U \in \mathcal{C}\,;\; U\, U^* = U^*\, U = 1\,, \quad U\, J\, U^{-1} =\, J\}$$

It is of course necessary to treat the above functional integral as one treats gauge theories *i.e.* developing the appropriate BRS formalism.

In particular the unitary group $\mathcal{G}$ of the subalgebra $\mathcal{A} \subset \mathcal{C}$ maps to the large symmetry group $\mathcal{U}$ by the following "adjoint" representation,

$$u \in \mathcal{G} \to u\, JuJ^{-1} = u\, u^{*0}\,, \quad (u\, u^{*0}\xi = \, u\, \xi\, u^*\,, \quad \forall \xi \in \mathcal{H})$$

and this is the way the group of internal symmetries acts for the standard model [12].

Instead of fixing $c$ and performing the integration over $D$ one can go the other way and fix $D$ first, using the natural gauge fixing that makes this operator diagonal in a given orthonormal basis of $\mathcal{H}$ and the theory of random (symplectic) matrices to express the measure $D[D]$ in terms of the eigenvalues. At fixed $D$ one can bring in all the arsenal of noncommutative geometry [10] to analyze the existence of cycles $c$ fulfilling (25) and hopefully to start integrating over such $c$. One can restrict to the following "regular" subalgebra of $\mathcal{C}$,

$$\mathcal{C}(D) = \{a \in \mathcal{C}\,;\; t \to F_t(a)\,,\; t \to F_t([D,a]) \;\in\; C^\infty(\mathbb{R}, \mathcal{C})\} \tag{28}$$

where we use the notation

$$F_t(x) = e^{it|D|}x\, e^{-it|D|}$$

By [11] the map $t \in \mathbb{R} \to F_t(x)$ plays the role of the geodesic flow. In dimension 4 the typical behavior of the eigenvalues[10] $\lambda_n$ of $|D|$ is $\lambda_n \sim C\, n^{1/4}$ for some $C > 0$, as follows from the H. Weyl theorem. Assuming this behavior one then has (by [10] IV.2.$\gamma$, Theorem 8) the following homological interpretation of the cosmological term

$$f_4\, \Lambda^4 ⨍ D^{-4}$$

in the spectral action (17): assuming (25) one has

$$⨍ D^{-4} = \langle c, \phi\rangle \tag{29}$$

The right hand side is the pairing between cycles and cocycles in Hochschild cohomology, and $\phi$ is the cyclic four cocycle $\phi$ which is given by (*cf.* [10] IV.1.$\alpha$)

$$\phi(a_0, \ldots, a_4) = \mathrm{Tr}'(\gamma\, a_0[F, a_1] \ldots [F, a_4])\,, \quad \forall a_j \in \mathcal{C}(D)$$

where $\mathrm{Tr}'(T) = \frac{1}{2}\mathrm{Tr}(F(FT + TF))$ and the operator $F$ is the sign of the operator $D$. This operator $\mathrm{Sign}(D) = F$ fulfills

$$F^2 = 1\,, \quad F\,\gamma = -\gamma\, F\,, \quad J\, F = F\, J$$

and the whole data $(\mathcal{H}, \mathcal{H}^s, \gamma, J, F)$ is in fact unique up to isomorphism. This is intimately related to the higher dimensional form of the universal Grassmanian (*cf.* [34]). In particular one sees that the Hochschild class of $c$ is non-zero in $H_4(\mathcal{C}(D), \mathcal{C}(D))$ if $⨍ D^{-4} \neq 0$ which is the case for the above behavior of the eigenvalues.

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[10] in increasing size

**Remark 4.1.** We have neglected in the above discussion of the constraints the qualitative one corresponding to (28) but the $C^1$ condition should suffice, and means that

$$[D,a]\,\mathcal{H}^s \subset \mathcal{H}^s\,,\quad [D^2,[D,a]]\,\mathcal{H}^s \subset \mathcal{H}^{s-1}\,,\quad \forall s\,,\quad \forall\, a = \alpha^{(i)}(c) \tag{30}$$

which is still polynomial in both $c$ and $D$.

**Remark 4.2.** In the above set-up the cocycle $c$ encodes both the algebra $\mathcal{A}$ *i.e.* the manifold, as well as the volume form. To understand the simplest instance of the polynomial relation (25) one can check that its simple one dimensional analogue specifies the geometry of the circle $M = S^1$ of length $2\pi$ by the equations

$$u^{-1}\,[D,u] = 1\,,\quad u\,u^* = u^*\,u = 1$$

which (taking the real structure $J$ into account) admit only one irreducible representation in Hilbert space. Since (25) is linear in $c$, it is tempting to use the linear space structure of the Hochschild cocycles to help in setting up the functional integral as we did above. This allows in particular for a priori strange "superpositions" of actual geometries (but the constraints (27) will no longer hold). One should remember however that this sets the same linearity on the "volume forms" $dv = \sqrt{g}d^4x$ and one might need a Fadeev-Popov determinant to adjust to the correct choice in the transition from unimodular gravity to the standard one.

**Remark 4.3.** It remains in particular to understand a good reason why the algebra $\mathcal{A}$ should have a slight amount of noncommutativity, such as the one obtained for the geometric realization of the standard model. This question motivated the results of [18], [13], [14], [15] on the structure of "noncommutative spheres" characterized by the following quantization of the Hochschild cycle $c$: one requires that

$$\exists\; e = e^2 = e^* \in M_q(\mathcal{A})\,,\quad c = \mathrm{Ch}_2(e)$$

while the lower components $\mathrm{Ch}_j(e)$ $(j = 0,1)$ of the Chern character $\mathrm{Ch}(e)$ of the projection $e$ are all required to vanish. The algebra $\mathcal{A}$ is then generated by the matrix components of $e$ and it is quite remarkable that besides the usual sphere, this problem singles out a large class of noncommutative manifolds. Both the computation (*cf.* [21]) of chiral anomalies [3] in the general framework of NCG, as well as the potential role of quantum group extensions (at cubic root of 1) of the (euclidean form of the) Lorentz group[11] give directions for finer work to handle this question.

[11]while the Lorentz group is no longer a symmetry of space-time in the set-up of general relativity, it still plays a crucial role at the infinitesimal local level as the structure group of the natural frame bundle which encodes the principle of equivalence.

## References


[1] H. Araki, *Expansional in Banach algebras*, Ann. Sci. École Norm. Sup. (4) 6 (1973), 67–84.

[2] J. Baez, *Fundamental Physics: where we stand today*, Talk given in Luminy in February 2006 http://math.ucr.edu/home/baez/where_we_stand/where_we_stand.pdf

[3] J.S. Bell, R. Jackiw, *A PCAC puzzle*, Nuovo Cimento, Vol.60A (1969) 47–60.

[4] P. Cartier, *A mad day's work: from Grothendieck to Connes and Kontsevich. The evolution of concepts of space and symmetry*, Bull. Amer. Math. Soc. (N.S.) 38 (2001), no. 4, 389–408.

[5] A. Chamseddine, A. Connes, *Universal Formula for Noncommutative Geometry Actions: Unification of Gravity and the Standard Model*, Phys. Rev. Lett. 77, 486804871 (1996).

[6] A. Chamseddine, A. Connes, *The Spectral Action Principle*, Comm. Math. Phys. 186, 731-750 (1997).

[7] A. Chamseddine, A. Connes, *Scale Invariance in the Spectral Action*, hep-th/0512169 to appear in Jour. Math. Phys

[8] A. Chamseddine, A. Connes, *Inner fluctuations of the spectral action*, hep-th/0605011.

[9] J. Collins, *Renormalization*, Cambridge Monographs in Math. Physics, Cambridge University Press, 1984.

[10] A. Connes, *Noncommutative geometry*, Academic Press (1994). ftp://ftp.alainconnes.org/book94bigpdf.pdf

[11] A. Connes, *Geometry from the spectral point of view.* Lett. Math. Phys. 34 (1995), no. 3, 203–238.

[12] A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, Comm. Math. Phys. (1995)

[13] A. Connes, M. Dubois-Violette, *Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples.* Comm. Math. Phys. 230 (2002), no. 3, 539–579.

[14] A. Connes, M. Dubois-Violette, *Moduli space and structure of noncommutative 3-spheres.* Lett. Math. Phys. 66 (2003), no. 1-2, 91–121.

[15] A. Connes, M. Dubois-Violette, *Non commutative finite-dimensional manifolds II. Moduli space and structure of noncommutative 3-spheres.* ArXiv math.QA/0511337.

[16] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem.* Comm. Math. Phys. 210 (2000), no. 1, 249–273.

[17] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The $\beta$-function, diffeomorphisms and the renormalization group.* Comm. Math. Phys. 216 (2001), no. 1, 215–241.

[18] A. Connes and G. Landi, *Noncommutative manifolds, the instanton algebra and isospectral deformations.* Comm. Math. Phys. 221 (2001), no. 1, 141–159.

[19] A. Connes, M. Marcolli, *From Physics to Number theory via Noncommutative Geometry, II: Renormalization, the Riemann–Hilbert correspondence, and motivic Galois theory*, to appear in "Frontiers in Number Theory, Physics, and Geometry" Vol.II. Preprint hep-th/0411114.

[20] A. Connes, M. Marcolli, *Renormalization and motivic Galois theory*, International Math. Research Notices, (2004), no. 76, 4073–4091.

[21] A. Connes, M. Marcolli, *Dimensional regularization, anomalies, and noncommutative geometry*, in preparation.

[22] A. Connes, M. Marcolli, *book in preparation*

[23] A. Connes, H. Moscovici, *The local index formula in noncommutative geometry*, GAFA, Vol. 5 (1995), 174–243.

[24] P. Deligne, *Equations differentielles à points singuliers réguliers*, Lecture Notes in Mathematics 163, Springer 1970.

[25] H. Figueroa, J.M. Gracia-Bondía, J. Varilly, *Elements of Noncommutative Geometry*, Birkhäuser, 2000.

[26] I. M. Gelfand, N. J. Vilenkin, *Generalized Functions Vol IV*, Academic press, New-York, London, San-Diego(1964)

[27] S. Giddings, D. Marolf, J. Hartle, *Observables in effective gravity*, hep-th/0512200.

[28] G. Green, *Researches on the Vibrations of Pendulums in Fluid Media*, Royal Society of Edinburgh Transactions (1836) p 315-324.

[29] J. Hartle, S. Hawking, *Wave function of the universe*. Phys. Rev. D 28, 2960 (1983)

[30] D. Kastler, *The Dirac operator and gravitation*, Commun. Math. Phys. 166 (1995), 633-643.

[31] D. Kastler, *Noncommutative geometry and fundamental physical interactions: The Lagrangian level*, Journal. Math. Phys. 41 (2000), 3867-3891.

[32] J. Martinet, J.P. Ramis, *Elementary acceleration and multisummability, I*, Ann. Inst. Henri Poincaré, Vol.54 (1991) 331–401.

[33] J-P. Ramis, *Séries divergentes et théories asymptotiques*. Bull. Soc. Math. France, 121(Panoramas et Syntheses, suppl.) 74, 1993.

[34] G. Segal, G. Wilson, *Loop Groups and equations of KDV type* Publ.math. de l'IHES, 61 (1985) 5–65.

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