Committee Scoring Rules: Axiomatic Characterization and Hierarchy

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Committee scoring voting rules are multiwinner analogues of positional scoring rules, which constitute an important subclass of single-winner voting rules. We identify several natural subclasses of committee scoring rules, namely, weakly separable, representation-focused, top-k-counting, OWA-based, and decomposable rules. We characterize SNTV, Bloc, and k-Approval Chamberlin–Courant as the only nontrivial rules in pairwise intersections of these classes. We provide some axiomatic characterizations for these classes, where monotonicity properties appear to be especially useful. The class of decomposable rules is new to the literature. We show that it strictly contains the class of OWA-based rules and describe some of the applications of decomposable rules.

CCS Concepts: • **Computing methodologies** → **Multi-agent systems**; **Cooperation and coordination**; • **Applied computing** → **Economics**; • **Theory of computation** → *Algorithmic game theory and mechanism design*;

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1 INTRODUCTION

Axiomatic studies of multiwinner voting rules go back to Felsenthal and Maoz [34] and Debord [22], but a systematic work on the topic has begun only recently and on several different fronts. New results appear within social choice theory, computer science, artificial intelligence, and a

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number of other fields (see the work of Faliszewski et al. [31] for more details on the history as well as recent progress). The reason for this explosion of interest from a number of research communities is the wide range of applications of multiwinner voting rules, on the one hand, and the corresponding richness and diversity of the spectrum of those rules on the other. Typically, social-choice theorists study normative properties of various multiwinner rules, computer scientists investigate feasibility of computing the election results, and researchers working within artificial intelligence use multiwinner elections as a versatile tool (e.g., useful in genetic algorithms [29], for ranking search results [78], or for providing personalized recommendations [51]). Yet, there is a growing interplay between these areas and an increased need for a new level of comprehension of results obtained in all of them. In this article, we partially address this need by linking syntactic features of certain families of committee scoring rules with their normative properties. The syntactic features of the rules are useful, e.g., for establishing their computational properties [32, 75], or for viewing those rules as achieving certain optimization goals (which allows one to consider these rules as tools for certain tasks from artificial intelligence and operation research). The normative properties, however, are useful for understanding the "behavior" of these rules and the settings for which they may be appropriate.

The model of multiwinner elections studied in this article is as follows. We are given a set of candidates, a collection of voters—each with a preference order in which the candidates are ranked from the best to the worst—and an integer k. A multiwinner rule maps this input to a subset of k candidates (i.e., a committee; we discuss tie-breaking later) that, in some sense, best reflects the voters' preferences. For example, the Single Non-Transferable Vote rule (the SNTV rule) chooses k candidates that are top-ranked most frequently, whereas the Bloc rule selects k candidates that are ranked most frequently among top k positions (equivalently, under Bloc each voter names members of his or her favorite committee, and those that are mentioned most often are selected). Naturally, there are many other multiwinner rules to choose from, defined in various ways.

In this article, we focus on the class of committee scoring rules, introduced by Elkind et al. [25] as multiwinner generalizations of classic positional scoring rules. The main idea of committee scoring rules is essentially the same as in the single-winner case: Each voter gives each committee a score based on the positions of the members of this committee in the voter's ranking, scores from individual voters are aggregated into the societal scores of the committees, and the committee(s) with the highest score wins. Committee scoring rules appear to form a remarkably rich class that includes both very simple rules, such as SNTV and Bloc, and rather sophisticated ones, such as the rule of Chamberlin and Courant [17] or variants of the Proportional Approval Voting rule [46]. As these rules tend to be very different in nature, they are suitable for different purposes, such as selecting a diverse committee, selecting a committee that proportionally represents the electorate, or selecting a committee consisting of k individually best candidates. This richness is the main strength of the class of committee scoring rules, but to choose rules for given settings wisely, it is important to understand the internal structure of the class. Understanding this structure is the main goal of the current article.[1]

So far, researchers have identified the following subclasses of committee scoring rules (we provide their formal definitions in Sections 2 and 3; here we give intuitions only). (Weakly) separable rules, introduced by Elkind et al. [25], are those rules where we compute a separate score for each candidate (using a single-winner scoring rule) and then pick k candidates with the top scores (for example, using Plurality scores leads to SNTV).[2] Representation-focused rules, also introduced by

[1]Understanding committee scoring rules in the "external" context of other multiwinner rules was the goal of Skowron et al. [77], who characterized this class axiomatically.

[2]If the underlying single-winner scoring rule does not depend on the size of the committee (as in the case of SNTV), then the rule is referred to as separable. If there is such dependence (as in the case of Bloc), then the rule is weakly separable.

Elkind et al. [25], are similar in spirit to the Chamberlin–Courant rule, whose aim is to ensure that in the elected committee each voter's most preferred committee member (his or her representative) is ranked as high as possible. However, top-k-counting rules, introduced by Faliszewski et al. [32], capture rules where each voter evaluates the quality of a committee by the number of members of that committee that he or she ranks among the top k ones; Bloc is a prime example of a top-k-counting rule. Finally, the class of OWA-based rules—introduced by Skowron et al. [75], also studied in the approval-based election model [2, 5, 49]—contains all the previously mentioned classes. Under these rules a voter calculates the score of a committee as the *ordered weighted average* (OWA) of the scores of the candidates in that committee.[3] In this article, we also introduce the class of *decomposable* committee scoring rules that strictly contains all the OWA-based ones, has interesting applications, and appears to be easier to work with axiomatically.

All the above classes have been defined purely in terms of the syntactic features of the functions used to calculate the scores of the committees.[4] These syntactic features are important if, for example, one wants to assess some computational properties of the rules (e.g., it is known that weakly separable rules are polynomial-time computable [25], that representation-focused rules tend to be NP-hard to compute [51, 65, 75], and that the structure of the functions used within OWA-based rules affects the ability to compute their results approximately [75]). Such syntactic features are also essential when we view committee scoring rules as specifying optimization goals for particular applications (for example, since under the Chamberlin–Courant rule each voter's score depends solely on his or her representative in the elected committee, this rule is particularly suitable in the context of deliberative democracy [17], for targeted advertising [51, 52], or for certain facility location problems [88]). Nonetheless, these syntactic features do not tell us much about the behavior of the rules.

Our first result reinforces the syntactic hierarchy of committee scoring rules. We show that the class of committee scoring rules strictly contains the class of decomposable rules, which, in turn, strictly contains the class of OWA-based ones, and that the class of OWA-based rules strictly contains the classes of (weakly) separable rules, representation-focused rules, and top-k-counting rules. For each pair of the latter three classes, we show that their intersection contains exactly one, previously known, non-trivial voting rule. See Figure 1 for a visualization of the syntactic hierarchy of committee scoring rules.

Our second, and the main, result establishes a link between several levels of the syntactic hierarchy and respective normative properties; we establish axiomatic characterizations of some of the studied subclasses of committee scoring rules. Until now, the only result of this form was a characterization of fixed-majority consistent committee scoring rules as those top-k-counting rules whose scoring functions satisfy (a relaxed variant of) the convexity property [32]. Here, our main result is that many of the syntactic properties of our rules nicely correspond to certain types of monotonicity. Specifically, we focus on the *committee enlargement monotonicity*[5] property, which requires that if we increase the size of the committee sought in the election, then the new winning committee should be a superset of the old winning committee, and on variants of the *non-crossing monotonicity* property, which requires that if we shift forward some members of a winning committee within any vote in a way that does not affect the positions of the remaining members of

[3]See the original work of Yager [83] for a general discussion of OWAs, and, e.g., the works of Kacprzyk et al. [45] or Goldsmith et al. [41] for their other applications in voting.

[4]A notable exception is the class of top-k-counting rules, which were discovered while characterizing those committee scoring rules that satisfy the fixed-majority property [32].

[5]This notion is also known as *committee monotonicity* [25] and *enlargement consistency* [7]. We chose a name that is more informative than the former but that is not tied to the realm of resolute rules, as the latter. In the literature on apportionment rules, a related property is often called *house monotonicity* [6, 66].

this committee, then this committee should still win. We show that committee enlargement monotonicity characterizes exactly the class of separable rules among committee scoring rules, and that non-crossing monotonicity characterizes the class of weakly separable ones. Then, we introduce top-member monotonicity (a variant of non-crossing monotonicity restricted within each vote to shifting only the highest-ranked member of the winning committee) and show that together with narrow-top consistency (which requires that if there are k candidates that are ever ranked in the top position within each vote, then these candidates should form a winning committee) it characterizes the class of representation-focused rules. Finally, we show that if a committee scoring rule is prefix-monotone (i.e., satisfies a yet another restricted variant of non-crossing monotonicity) then it must be decomposable.

The article is organized as follows. In Section 2, we describe the model of multiwinner elections, define the class of committee scoring rules, provide their basic properties, and show several examples of committee scoring rules. Section 3 is devoted to structural properties of the classes of committee scoring rules. Here, we build the hierarchy of the classes and show results regarding containments and intersections among them. In Section 4, we switch to axiomatic properties and provide several axiomatic characterizations of the rules in our classes. Finally, we discuss related work in Section 5 and conclude in Section 6.

2 MULTIWINNER ELECTIONS AND COMMITTEE SCORING RULES

In this section, we set the stage for the discussions provided throughout the rest of the article by providing preliminary definitions as well as introducing the class of committee scoring rules. For each positive integer t, we write $[t]$ to denote $\{1, \dots, t\}$. By $\mathbb{R}_+$, we mean the set of nonnegative real numbers.

2.1 Preliminaries

An election is a pair $E = (C, V)$, where $C = \{c_1, \dots, c_m\}$ is a set of candidates and $V = (v_1, \dots, v_n)$ is a collection of voters. Each voter v_i has a *preference order* $\succ_i$ (also referred to as a *vote*), expressing his or her ranking of the candidates, from the most to the least desirable one. Given a voter v and a candidate c, by $\mathrm{pos}_v(c)$, we mean the position of c in v's preference order (the top-ranked candidate has position 1, the next one has position 2, and so on).

A *multiwinner voting rule* is a function $\mathcal{R}$ that given an election $E = (C, V)$ and a committee size k, $1 \leq k \leq |C|$, returns a family $\mathcal{R}(E, k)$ of size-k subsets of C, i.e., the set of committees that tie as winners of the election (we use the nonunique-winner model or, in other words, we assume that multiwinner rules are irresolute). We provide a few concrete examples of multiwinner rules in Section 2.2.

Most of the multiwinner rules that we study are based on single-winner scoring functions. A *single-winner scoring function* for m candidates is a nonincreasing function $\gamma : [m] \to \mathbb{R}_+$ that assigns a score value to each position in a preference order. Given a preference order $\succ_i$ and a candidate c, by the γ-score of c (given by voter v_i), we mean the value $\gamma(\mathrm{pos}_{v_i}(c))$. The two most commonly used scoring functions are the Borda scoring function,[6]

$$\beta_m(i) = m - i,$$

and the t-Approval scoring function,

$$\alpha_t(i) = \begin{cases} 1 & \text{if } i \leq t \\ 0 & \text{otherwise.} \end{cases}$$

In particular, α_1 is known as the Plurality scoring function.

[6]Sometimes in the literature the Borda scoring function is alternatively defined as $\beta_m(i) = m - i + 1$.

Committee scoring functions generalize single-winner scoring functions to the multiwinner setting in a natural way, by assigning scores to the positions of the whole committees. Formally, given a vote v and a committee S of size k, the *committee position* of S in v, denoted $\mathrm{pos}_v(S)$, is a sequence $(i_1, \ldots, i_k)$ that results from sorting the set $\{\mathrm{pos}_v(c) \mid c \in S\}$ in the increasing order. We write $[m]_k$ to denote the set of all such length-k increasing sequences of numbers from $[m]$ (i.e., we write $[m]_k$ to denote the set of all possible committee positions for the case of m candidates and committees of size k). Given two committee positions from $[m]_k$, $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$, we say that I *weakly dominates* J, $I \succeq J$, if for each $t \in [k]$ it holds that $i_t \leq j_t$ (we say that I *dominates* J, denoted $I \succ J$, if at least one of these inequalities is strict[7]). Below, we define committee scoring functions formally.

Definition 2.1 (Elkind et al. [25]). A committee scoring function for m candidates and a committee size k is a function $f_{m,k} : [m]_k \to \mathbb{R}_+$ such that for each two sequences $I, J \in [m]_k$, if I weakly dominates J then $f_{m,k}(I) \geq f_{m,k}(J)$.

Let $f = (f_{m,k})_{k \leq m}$ be a family of committee scoring functions, where each $f_{m,k}$ is a function for m candidates and committees of size k. Given an election $E = (C, V)$ with m candidates and a committee S of size k, we define the $f_{m,k}$-score of S to be

$$f_{m,k}\text{-score}_E(S) = \sum_{v_i \in V} f_{m,k}(\mathrm{pos}_{v_i}(S)).$$

When f is clear from the context, we often speak of the score of a committee instead of its f-score. Given the above notation, we are ready to define committee scoring rules formally.

Definition 2.2. Let $f = (f_{m,k})_{k \leq m}$ be a family of committee scoring functions (with one function for each m and k, $k \leq m$). Committee scoring rule $\mathcal{R}_f$ is a multiwinner voting rule that given an election $E = (C, V)$ and committee size k, outputs all size-k committees with the highest $f_{|C|,k}$-score.

We say that a committee scoring rule $\mathcal{R}_f$ is *degenerate* if there is a number of candidates m and a committee size k such that $f_{m,k}$ is a constant function. As a consequence, a degenerate rule returns all size-k committees for every election with m candidates. The *trivial committee scoring* rule is a degenerate rule that returns the set of all size-k committees for all elections and all sizes k (naturally, it is defined by a family of constant functions).

2.2 Examples of Committee Scoring Rules

Many well-known multiwinner rules are, in fact, committee scoring rules; below, we provide several such examples. For each of the rules, we provide the family of committee scoring functions used in its definition, discuss these functions intuitively, and mention some applications.

SNTV, Bloc, and k-Borda. These three rules use the following committee scoring functions:

$$f^{\mathrm{SNTV}}_{m,k}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \alpha_1(i_t) = \alpha_1(i_1),$$

$$f^{\mathrm{Bloc}}_{m,k}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \alpha_k(i_t), \text{ and}$$

[7]In previous papers on committee scoring rules, the notions of weak dominance and dominance were conflated. We believe that giving them clear, separate meanings will help in providing more crisp arguments and discussions.

$$f_{m,k}^{k\text{-Borda}}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \beta_m(i_t).$$

That is, under the SNTV rule, we choose k candidates with the highest Plurality scores; under Bloc, we choose k candidates with the highest k-Approval scores; and under k-Borda, we choose k candidates with the highest Borda scores. On the intuitive level, under SNTV each voter names his or her favorite committee member, under Bloc each voter names all the k members of his or her favorite committee, and under k-Borda each voter ranks all the candidates and assigns them scores in a way that corresponds linearly to their position in the ranking. SNTV and Bloc are sometimes used in political elections (with the former used, e.g., in the parliamentary elections in Puerto Rico, and with the latter often used for various local elections in many countries). k-Borda and other rules based on similar scoring schemes are often used to determine finalists of competitions (e.g., the finalists of the Eurovision Song Contest are selected using a system very close to k-Borda).

The Chamberlin–Courant rule. Under the Chamberlin–Courant rule (the β-CC rule), the score that a voter v assigns to a committee S depends only on how v ranks his or her favorite member of S (referred to as v's *representative* in S). The Chamberlin–Courant rule seeks committees in which each voter ranks his or her representative as high as possible. Formally, the rule uses functions

$$f_{m,k}^{\beta\text{-CC}}(i_1, \ldots, i_k) = \beta_m(i_1).$$

This is the variant of the rule originally proposed by Chamberlin and Courant [17], but, subsequently, other authors (e.g., Procaccia et al. [65], Betzler at al. [9], and Faliszewski et al. [33]) considered other ones, based on other single-winner scoring functions. In particular, we will be interested in the k-Approval Chamberlin–Courant rule (α_k-CC), which is defined through functions

$$f_{m,k}^{\alpha_k\text{-CC}}(i_1, \ldots, i_k) = \alpha_k(i_1).$$

Intuitively, both variants of the Chamberlin–Courant rule seek committees of diverse candidates that "cover" as broad a spectrum of voters' views as possible. Lu and Boutilier [51] considered (generalized variants of) the Borda-based Chamberlin–Courant rule in the context of recommendation systems.

The PAV rule. The Proportional Approval Voting rule (the PAV rule) was originally defined by Thiele [81] in the approval setting (where instead of ranking the candidates, the voters indicate which ones they accept as committee members; for recent discussions of the rule see the overview of Kilgour [46] and the works of Aziz et al. [2] and Lackner and Skowron [49]). We model it as a committee scoring rule α_t-PAV, where t is a parameter, defined using scoring functions of the form

$$f_{m,k}^{\alpha_t\text{-PAV}}(i_1, \ldots, i_k) = \alpha_t(i_1) + \frac{1}{2}\alpha_t(i_2) + \frac{1}{3}\alpha_t(i_3) + \cdots + \frac{1}{k}\alpha_t(i_k).$$

PAV is particularly well-suited for electing parliaments. Indeed, Brill et al. [14] have shown that it generalizes the D'Hondt apportionment method, which is used for this purpose in many countries (e.g., in France and Poland). A number of recent works [2, 4, 14, 49] explain why the harmonic sequence used within the PAV scoring function ensures that the elected committee represents the voters proportionally.

Naturally, there are many other committee scoring rules, and we will discuss some of them throughout the article. Nonetheless, the above few suffice to illustrate our main points. There is also a number of other multiwinner rules that are not committee scoring rules, such as STV (see, e.g., the work of Tideman and Richardson [82]), Monroe [56], Minimax Approval Voting [10], or rules that are stable in the sense of Gehrlein [40]. We do not discuss them in this article, but we provide some literature pointers in Section 5.

2.3 Basic Features of Committee Scoring Rules

The class of committee scoring rules is very rich and there are only a few basic properties shared by all the rules in this class. Below we discuss several such properties that will be useful throughout this article.

From our point of view, the most important common feature of committee scoring rules is that they are uniquely defined by their scoring functions (up to linear transformations). Formally, we have the following lemma (we provide the proof in the Appendix).

Lemma 2.1. *Let $\mathcal{R}_f$ and $\mathcal{R}_g$ be two committee scoring rules defined by committee scoring functions $f = (f_{m,k})_{k \le m}$ and $g = (g_{m,k})_{k \le m}$, respectively. If $\mathcal{R}_f = \mathcal{R}_g$, then for each m and k, $k \le m$, there are two values, $a_{m,k} \in \mathbb{R}_+$ and $b_{m,k} \in \mathbb{R}$, such that for each $I \in [m]_k$, we have that $f_{m,k}(I) = a_{m,k} \cdot g_{m,k}(I) + b_{m,k}$.*

Due to Lemma 2.1, to show that two committee scoring rules are distinct it suffices to show that their scoring functions are not linearly related. In particular, this will be very useful when we will be showing that certain rules cannot be represented using scoring functions of a given form.

The second common feature of committee scoring rules is nonimposition, which requires that for every committee there is some election where it wins uniquely. Formally, we have the following definition.

Definition 2.3. Let $\mathcal{R}$ be a multiwinner rule. We say that $\mathcal{R}$ has the nonimposition property if for each candidate set C and each subset W of C, there is an election $E = (C, V)$ such that $\mathcal{R}(E, |W|) = \{W\}$.

Nonimposition is such a basic property that it is hardly surprising that all non-degenerate committee scoring rules (i.e., all rules defined through non-constant scoring functions) satisfy it. We prove the next lemma in the Appendix.

Lemma 2.2. *Let $\mathcal{R}_f$ be a committee scoring rule. It satisfies the nonimposition property if and only if it is non-degenerate.*

While at first sight nonimposition and Lemma 2.2 seem hardly exciting, in fact they are sufficient to illustrate intriguing differences between single-winner voting rules and their multiwinner counterparts. For example, one can verify that all nontrivial single-winner scoring rules satisfy the following extended variant of the nonimposition property: For every candidate set C and its subset S, there is an election $E = (C, V)$ where exactly the candidates from S tie as winners. An analogous result does not hold for committee scoring rules, even for the case of two committees (in which case it could be dubbed as 2-nonimposition; the example below is due to Lackner and Skowron [49]).

Example 2.1. Suppose that we want to select a committee of size k using the Bloc rule and that there are two disjoint, winning committees, W_1 and W_2 (e.g., consider an election with two voters, the first ranking the members of W_1 on top and the second ranking the member of W_2 on top). Then, each candidate in W_1 has the same k-Approval score as each candidate in W_2 (otherwise, one of the committees would not be winning) and, in consequence, each committee $W \subset W_1 \cup W_2$ of size k also is winning. Thus, W_1 and W_2 are not the two unique winning committees.

The fact that, in general, 2-nonimposition does not hold for committee scoring rules is quite disappointing, because many results would be far easier to prove if we could assume that it is always possible to construct an election where two arbitrary given committees are the only winning ones. However, it is possible to construct elections where two size-k committees W_1 and W_2 are the only winning ones, provided that they share $k-1$ candidates (and, indeed, this fact is used in the proof of Lemma 2.1).

There are a few more common properties of committee scoring rules. For example, they all satisfy the *candidate monotonicity* property [25], which requires that if we shift forward a member of a winning committee then, afterward, this candidate still belongs to some winning committee (but possibly quite a different one; see the work of Bredereck et al. [12]). Also, all committee scoring rules are *consistent* in the sense that if two elections E_1 and E_2 (over the same candidate set) have some common winning committees, then these are exactly the winning committees in an election obtained by merging the voter collections of E_1 and E_2.[8] The former property is related to our discussions in Section 4 and the latter one is often useful as a tool when proving various results (and, indeed, it is crucial in characterizing the class of committee scoring rules axiomatically [77]).

2.4 The T-Shirt Store Example

In Section 2.2, we have provided a number of examples of committee scoring rules, and we have discussed some of their applications, focusing mostly on political elections. However, committee scoring rules have far more varied applications (see, e.g., the overview of Faliszewski et al. [31]), most of which have nothing to do with politics. Below, we describe a simplified business-inspired scenario where committee scoring rules may be useful. We use this example to guide our way through the different types of committee scoring rules discussed in this article.

Example 2.2. Consider a T-shirt store that needs to decide which shirts to put on offer and let C be the set of T-shirts that the store can order from its suppliers ($|C| = m$). Since the store has limited space, it can only put k different T-shirts on display, and it wants to pick them in a way that would maximize its revenue (i.e., the number of T-shirts sold). We assume that every customer knows all the designs (say, from a website) and ranks all T-shirts from the best one to the worst one. Let us say that a customer considers a T-shirt to be "very good" if it is among the top k T-shirts (of course, this is an arbitrary choice, made for the sake of simplifying the example).

How should the store decide which T-shirts to put on display? This depends on how the customers behave. Consider a customer that ranks the available T-shirts on positions $i_1 < i_2 < \cdots < i_k$. If this is a very picky customer that only buys a T-shirt if it is the very best among all possible ones (according to his or her opinion), then the number of T-shirts this customer buys is given by $f^{\mathrm{SNTV}}_{m,k}(i_1, \ldots, i_k) = \alpha_1(i_1)$. However, if this customer were to buy one copy of each T-shirt he or she considered as "very good," he or she would buy $f^{\mathrm{Bloc}}_{m,k}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \alpha_k(i_t)$ T-shirts. It is also possible that a customer would buy only one T-shirt, provided he or she considered it as "very good." The number of T-shirts bought by such a customer would be $f^{\alpha_k\text{-CC}}_{m,k}(i_1, \ldots, i_k) = \alpha_k(i_1)$. Depending on which type of customers the store expects to have, it should choose its selection of T-shirts either using SNTV, Bloc, or k-Approval Chamberlin–Courant. (Surely, other types of customers are possible as well, and we will discuss some of them later. It is also likely that the store would face a mixture of different types of customers, but this is beyond our study.)

[8] The axiom of consistency is sometimes also called *reinforcement*.

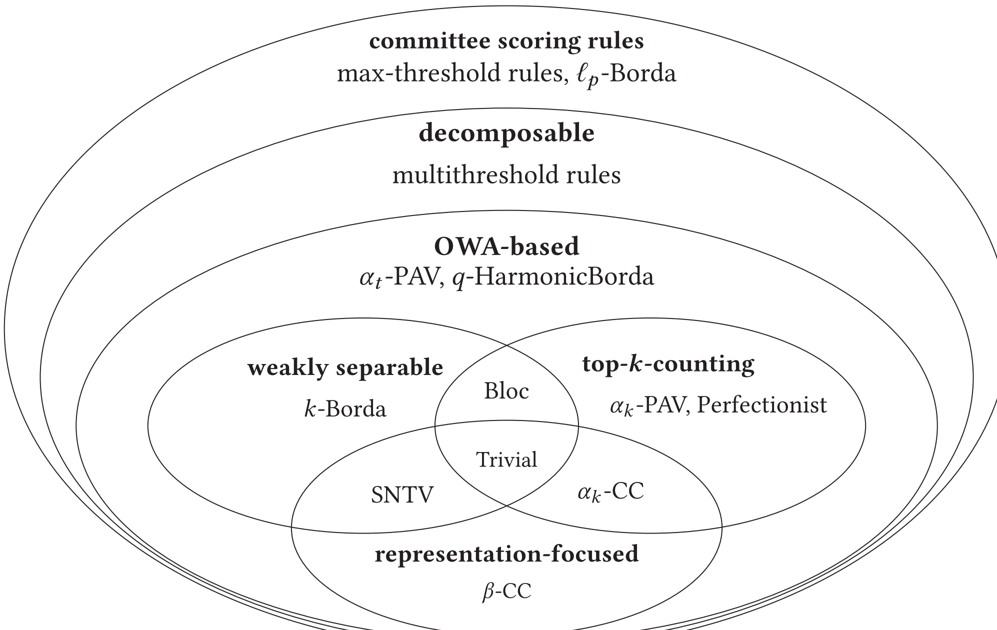

Fig. 1. The hierarchy of committee scoring rules.

3 HIERARCHY OF COMMITTEE SCORING RULES

In this section, we describe the classes of committee scoring rules that were studied to date, introduce a new class—the class of decomposable rules—and investigate how all these classes relate to each other, forming a hierarchy. In Figure 1, we present the relations between the classes discussed in this section, with examples of notable rules. The classes are defined by setting restrictions on the scoring functions so, in other words, in this section, we are interested in the syntactic hierarchy of committee scoring rules. Later, in Section 4, we will consider semantic properties.

3.1 Separable and Weakly Separable Rules

We say that a family of committee scoring functions $f = (f_{m,k})_{k \le m}$ is *weakly separable* if there exists a family of (single-winner) scoring functions $(\gamma_{m,k})_{k \le m}$ with $\gamma_{m,k} : [m] \to \mathbb{R}_+$ such that for every $m \in \mathbb{N}$ and every committee position $I = (i_1, \dots, i_k) \in [m]_k$, we have

$$f_{m,k}(i_1, \dots, i_k) = \sum_{t=1}^{k} \gamma_{m,k}(i_t).$$

A committee scoring rule $\mathcal{R}_f$ is *weakly separable* if it is defined through a family of weakly separable scoring functions f. In other words, if a rule is weakly separable, then we can compute the score of each candidate independently, using the single-winner scoring function $\gamma_{m,k}$, and pick the k candidates with the highest scores. In consequence, it is possible to compute winning committees for all weakly separable rules in polynomial time, provided that their underlying single-winner scoring functions are polynomial-time computable [25].[9]

[9]There is a subtlety here as there may be exponentially many winning committees. However, by listing the scores of all the candidates, we provide enough information to, e.g., enumerate all the winning committees in time proportional to the

If for all m we have $\gamma_{m,1} = \cdots = \gamma_{m,m}$, then we say that the family f and the corresponding committee scoring rule $\mathcal{R}_f$ are *separable*, without the "weakly" qualification. Thus, separable rules use the same scoring function for each value of the size of a committee to be elected. Interestingly, separable rules have some axiomatic properties that other weakly separable rules lack [25]—we will discuss this further in Section 4.

The notion of (weakly) separable rules was introduced by Elkind et al. [25]; they pointed out that SNTV and k-Borda are separable, whereas Bloc is only weakly separable.

3.2 Representation-Focused Rules

A family of committee scoring functions $f = (f_{m,k})_{k \le m}$ is *representation-focused* if there exists a family of (single-winner) scoring functions $(\gamma_{m,k})_{k \le m}$ such that for every $m \in \mathbb{N}$ and every committee position $I = (i_1, \ldots, i_k) \in [m]_k$, we have

$$f_{m,k}(i_1, \ldots, i_k) = \gamma_{m,k}(i_1).$$

This means that the score that a committee receives from a voter depends only on the position of the most preferred member of this committee in the voter's preference ranking—such a member can be viewed as a representative of the voter in the committee. A committee scoring rule $\mathcal{R}_f$ is *representation-focused* if it is defined through a family of representation-focused scoring functions f. The notion of representation-focused rules was introduced by Elkind et al. [25]; β-CC is the archetypal example of a representation-focused committee scoring rule and all the representation-focused rules can be seen as variants of the Chamberlin–Courant rule.

SNTV is both separable and representation-focused, and it is the only non-degenerate committee scoring rule with this property.

Proposition 3.1. *SNTV is the only non-degenerate committee scoring rule that is (weakly) separable and representation-focused.*

Proof. It is easy to verify that SNTV is separable and representation-focused. For the other direction, let $\mathcal{R}$ be a rule that is separable and representation focused. It follows that $\mathcal{R} \equiv \mathcal{R}_f \equiv \mathcal{R}_{f'}$ for some families of committee scoring functions f and f', such that $f_{m,k}(i_1, \ldots, i_k) = \phi_{m,k}(i_1) + \cdots + \phi_{m,k}(i_k)$ and $f'_{m,k}(i_1, \ldots, i_k) = \gamma_{m,k}(i_1)$. Every linear transformation of f' has the same form (i.e., it only depends on i_1), so by Lemma 2.1 (linearly transforming g, if necessary) we can assume that $f = f'$.

Without loss of generality, we can assume that $m > k$. For each committee position $(i_1, \ldots, i_k)$ with $i_1 = 1$, we have that

$$\phi_{m,k}(i_1) + \cdots + \phi_{m,k}(i_k) = \gamma_{m,k}(i_1),$$

and, so, we can conclude that $\phi_{m,k}(2) = \cdots = \phi_{m,k}(m)$. Since $\mathcal{R}$ is non-degenerate, we have that $\phi_{m,k}(1) > \phi_{m,k}(m)$, and so that $\phi_{m,k}(1) > \phi_{m,k}(2)$. This is sufficient to conclude that $\mathcal{R}$ is equivalent to SNTV. □

Generally, representation-focused rules are NP-hard to compute (SNTV is one obvious exception). This fact was first shown by Procaccia et al. [65] in the approval-based setting, and then by Lu and Boutilier [51] for β-CC. Since then, various means of computing the results under the Chamberlin–Courant rule and its variants were studied in quite some detail [9, 20, 28, 33, 48, 59, 76, 79].

number of these committees, or to perform many other tasks related to winner determination (such as computing the score of a winning committee).

3.3 Top-k-Counting Rules

A committee scoring rule $\mathcal{R}_f$, defined by a family $f = (f_{m,k})_{k \leq m}$, is *top-k-counting* if there exists a sequence of nondecreasing functions $(g_{m,k})_{k \leq m}$, with $g_{m,k} : \{0, \ldots, k\} \to \mathbb{R}_+$, such that

$$f_{m,k}(i_1, \ldots, i_k) = g_{m,k}(|\{i_t \mid i_t \leq k\}|).$$

That is, the value $f_{m,k}(i_1, \ldots, i_k)$ depends only on the number of committee members that the given voter ranks among his or her top k positions. We refer to the functions $g_{m,k}$ as the *counting functions*. Top-k-counting rules were introduced by Faliszewski et al. [32].

Remark 1. It would be quite natural to require that all counting functions for a given committee size were the same, that is, that for each $k \in \mathbb{N}$ it held that $g_{k,k} = g_{k+1,k} = g_{k+2,k} = \cdots$. Following Faliszewski et al. [32], we formally do not make this requirement, but we expect it to hold for all natural top-k-counting rules.

Top-k-counting rules include, for example, the Bloc rule, α_k-PAV, and α_k-CC, where Bloc uses the linear counting functions $g^{\text{Bloc}}_{m,k}(i) = i$, α_k-PAV uses counting functions $g^{\alpha_k\text{-PAV}}_{m,k}(i) = \sum_{t=1}^{i} \frac{1}{t}$, and α_k-CC uses counting functions

$$g^{\text{CC}}_{m,k}(i) = \begin{cases} 1 & \text{if } i \geq 1 \\ 0 & \text{if } i = 0. \end{cases}$$

As an extreme example of a top-k-counting rule, Faliszewski et al. [32] introduced the Perfectionist rule, which uses counting functions

$$g^{\text{Perf}}_{m,k}(i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Perfectionist is extreme in the sense that a voter assigns a point to a committee exactly if he or she ranks all the members of this committee as k best ones.

Example 3.1. Let us recall our T-shirt store example (Example 2.2). Consider a particularly snobbish customer, who is willing to buy a shirt from a store only if he or she views all the available shirts as very good (recall that we defined "very good" to mean being ranked among top k positions). Then if $i_1, \ldots, i_k$ are the positions of the available shirts in the customer's ranking, the number of shirts that the store should expect to sell to such a customer is

$$f_{m,k}(i_1, \ldots, i_k) = g^{\text{Perf}}_{m,k}(|\{i_t \mid i_t \leq k\}|) = \alpha_k(i_k).$$

Thus, if the store expects such customers, then it should use the Perfectionist rule to choose its merchandise (and, possibly, it should also increase its prices!).

Bloc is the only nontrivial rule that is both weakly separable and top-k-counting, and α_k-CC is the only nontrivial rule that is both representation-focused and top-k-counting.

Proposition 3.2. *Bloc is the only nontrivial rule that is weakly separable and top-k-counting.*

Proof. By combining Lemma 2.1 and the results of Faliszewski et al. [32], we obtain that the top-k-counting rule defined by a family of linear counting functions is the only weakly separable top-k-counting rule, and this rule is exactly Bloc. □

Proposition 3.3. *α_k-CC is the only nontrivial rule that is representation-focused and top-k-counting.*

Proof. It is easy to verify that α_k-CC is top-k-counting and representation-focused. For the other direction, let $\mathcal{R}$ be a rule that is both top-k-counting and representation-focused. It follows that $\mathcal{R} \equiv \mathcal{R}_f \equiv \mathcal{R}_{f'}$ for two committee scoring functions, f and f', with the syntactic structures witnessing that the rule is top-k-counting and representation-focused, respectively. Thus, there exist families of functions $(\gamma_{m,k})_{k\le m}$ and $(g_{m,k})_{k\le m}$ such that for each $k \le m$, we have that $f_{m,k}(i_1,\dots,i_k) = \gamma_{m,k}(i_1)$ and $f'_{m,k}(i_1,\dots,i_k) = g_{m,k}(|\{i_t \mid i_t \le k\}|)$ (the existence of the families $(\gamma_{m,k})_{k\le m}$ and $(g_{m,k})_{k\le m}$ follows from the definitions of the the classes of representation-focused and top-k-counting rules, respectively). Since each linear transformation of f is a representation-focused scoring function, by Lemma 2.1, we can assume that $f = f'$.

For each $i \in [m-k+1]$ let $L(i)$ denote the sequence $(i, m-k+2, m-k+3, \dots, m)$. For each $i, j > k$, we have that

$$\gamma_{m,k}(i) = f_{m,k}(L(i)) = f'_{m,k}(L(i)) = g_{m,k}(0) = f'_{m,k}(L(j)) = f_{m,k}(L(j)) = \gamma_{m,k}(j).$$

By the same reasoning, we can prove that for each $i, j \le k$, we have $\gamma_{m,k}(i) = \gamma_{m,k}(j)$. Since the rule is nontrivial, we know that for some i, j it holds that $\gamma_{m,k}(i) \ne \gamma_{m,k}(j)$. This is sufficient to claim that $\mathcal{R}$ is equivalent to α_k-CC. □

Faliszewski et al. [32] show that top-k-counting rules tend to be NP-hard to compute, but point out several polynomial-time computable exceptions, including Bloc and Perfectionist. They also observe that for rules with concave counting functions there are polynomial-time constant-factor approximation algorithms, whereas for rules with convex counting functions such algorithms may be missing (under standard complexity-theoretic assumptions).

3.4 OWA-Based Rules

Skowron et al. [75] introduced a class of multiwinner rules based on ordered weighted average (OWA) operators. Similar rules for approval-based ballots were first considered in the 19th Century by Thiele [81] and more recently were studied by Aziz et al. [2, 5] and Lackner and Skowron [49] (see also the discussion by Kilgour [46]). Elkind and Ismaili [26] use OWA operators to define a different class of multiwinner rules, which we do not consider in this article.

We provide intuition for the OWA-based rules by using our T-shirts store example.

Example 3.2. Let us say that a customer views a T-shirt as “good enough” if it is among the top 10% of the shirts available on the market. Suppose that a customer identifies the best T-shirt available in the store and buys it with probability 1, provided it is “good enough.” Then he or she also finds the second best T-shirt and buys it with probability $1/2$ (again, provided that it is “good enough”), the third best shirt with probability $1/3$, and so on, all the way to the kth best T-shirt, which he or she buys with probability $1/k$ (if it is “good enough”). If $i_1,\dots,i_k$ are the positions (in the customer’s preference order) of the T-shirts that the store puts on display, then the expected number of T-shirts he or she buys is given by the function

$$f_{m,k}(i_1,\dots,i_k) = 1 \cdot \alpha_{0.10m}(i_1) + 1/2 \cdot \alpha_{0.10m}(i_2) + \cdots + 1/k \cdot \alpha_{0.10m}(i_k).$$

Thus, to maximize its revenue, the store should find a winning committee for the election where the T-shirts are the candidates, where the voters are the customers, and where we use committee scoring rule $\mathcal{R}_f$ based on $f = (f_{m,k})_{k\le m}$. This multiwinner voting rule is $\alpha_{0.10m}$-PAV, a variant of the Proportional Approval Voting rule.

Now let us define OWA-based rules formally. An OWA operator Λ of dimension k is a sequence $\Lambda = (\lambda_1,\dots,\lambda_k)$ of nonnegative real numbers.

Definition 3.1. Let $\Lambda = (\Lambda^{m,k})_{k\le m}$ be a sequence of OWA operators such that $\Lambda^{m,k} = (\lambda_1^{m,k},\dots,\lambda_k^{m,k})$ has dimension k. Let $\gamma = (\gamma_{m,k})_{k\le m}$ be a family of single-winner scoring functions. Then, γ and Λ define a family $f = (f_{m,k})_{k\le m}$ of committee scoring functions such that for each $(i_1, \dots, i_k) \in [m]_k$, we have

$$f_{m,k}(i_1, \dots, i_k) = \sum_{t=1}^{k} \lambda_t^{m,k} \gamma_{m,k}(i_t).$$

We refer to committee scoring rules $\mathcal{R}_f$ defined through f in this way as OWA-based.

It is known that weakly separable, representation-focused, and top-k-counting rules are OWA-based. The first class is defined using OWA operators $(1, \dots, 1)$, the second one uses OWA operators $(1, 0, \dots, 0)$, and the last one contains rules that use k-Approval single-winner scoring functions and any OWA operator (the argument that shows this is due to Faliszewski et al. [32, Proposition 3] and requires a bit more effort than for the previous two classes). As a corollary to the preceding propositions, we get the following.

Corollary 3.4. *Each of the classes of separable, top-k-counting, and representation-focused rules is strictly contained in the class of OWA-based rules.*

Proof. Containment follows from the paragraph above. Strictness follows, as we have Bloc as the unique rule in the intersection of the classes of top-k-counting and weakly separable rules; SNTV as the unique rule in the intersection of the classes of weakly separable and representation-focused rules; and α_k-CC as the unique rule in the intersection of the classes of top-k-counting and representation-focused rules. It follows that Bloc is not representation-focused; SNTV is not top-k-counting; and α_k-CC is not weakly separable. We get the claim by noticing that Bloc, SNTV, and α_k-CC, are all OWA-based. □

Naturally, there are also OWA-based rules that do not belong to any of the above-mentioned classes. For example, this is the case for α_t-PAV rules (provided that the parameter t is not equal to the committee size k, e.g., if it is fixed as a constant) or for the related q-HarmonicBorda rules (the q-HB rules), defined by the following scoring functions ($q \in \mathbb{R}_+$ is a parameter):

$$f_{m,k}^{q\text{-HB}}(i_1, \dots, i_k) = \beta_m(i_1) + \frac{1}{2^q}\beta_m(i_2) + \frac{1}{3^q}\beta_m(i_3) + \cdots + \frac{1}{k^q}\beta_m(i_k).$$

The q-HarmonicBorda rules were introduced by Faliszewski et al. [30], who were looking for various means of achieving a compromise between the k-Borda rule and the Chamberlin–Courant rule (0-HB is k-Borda, and as q becomes larger and larger, q-HB becomes more and more similar to β-CC).

Proposition 3.5. *Neither α_t-PAV nor q-HB is weakly separable, nor representation-focused, nor top-k-counting, for any choice of constants $t \in \mathbb{N}$ and $q \in \mathbb{R}_+$.*

To prove Proposition 3.5 it suffices to show that the committee scoring functions of these rules cannot be expressed as linear transformations of weakly separable, representation-focused, and top-k-counting scoring functions, and invoke Lemma 2.1. We omit the details of this simple but somewhat tedious task.

Skowron et al. [75] have shown that OWA-based rules are typically NP-hard to compute (with the clear exception of, e.g., weakly separable rules and the Perfectionist rule). They have also linked the properties of the OWA operators with the ability to approximate the rules (generally speaking, if the OWA operators for a given rule are non-increasing, then there are polynomial-time constant-factor approximation algorithms for this rule, and, otherwise, they are typically missing[10]).

[10]However, there are exceptions. For example, viewed as an OWA-based rule, Perfectionist uses OWA operators $(0, \dots, 0, 1)$ but still is polynomial-time computable. This is because, as a top-k-counting rule, Perfectionist uses a very restrictive single-winner scoring function, and is not captured by the results of Skowron et al. [75].

3.5 Decomposable Rules

We introduce the following class that naturally generalizes the class of OWA-based rules and resort to our T-shirt store example to help the reader rationalize it.

Definition 3.2. Let $\gamma = (\gamma^{(t)}_{m,k})_{t \leq k \leq m}$ be a family of single-winner scoring functions. These functions define a family of committee scoring functions $f = (f_{m,k})_{k \leq m}$ such that for each committee position $(i_1, \ldots, i_k) \in [m]_k$, we have

$$f_{m,k}(i_1, \ldots, i_k) = \textstyle\sum_{t=1}^{k} \gamma^{(t)}_{m,k}(i_t).$$

We refer to committee scoring rules $\mathcal{R}_f$ defined through f in this way as decomposable.

At first glance, decomposable rules seem very similar to the weakly separable ones. The difference is that for fixed m and k and two different values t and t', for decomposable rules the functions $\gamma^{(t)}_{m,k}$ and $\gamma^{(t')}_{m,k}$ can be completely different. It is apparent that OWA-based rules are decomposable. We will see that this containment is strict.

Example 3.3. Let us recall from Example 3.2 that a customer considers a T-shirt to be "good enough" if it is among the best 10% of all shirts and let us say that a shirt is "great" if it is among the top 1% of all shirts. A customer buys two "great" T-shirts, or one "at least good enough" T-shirt (if there are no two "great" T-shirts on display). Naturally, the customer picks the best T-shirt(s) he can find (respecting the above constraints). If $i_1, \ldots, i_k$ are the positions (in the customer's preference order) of the T-shirts that the store puts on display, then the number of T-shirts he or she buys is given by function:

$$f_{m,k}(i_1, \ldots, i_k) = \alpha_{0.10m}(i_1) + \alpha_{0.01m}(i_2).$$

Thus, to maximize its revenue, the store should find a winning committee for the election where the T-shirts are the candidates, the voters are the customers, and where we use decomposable committee scoring rule $\mathcal{R}_f$ based on $f = (f_{m,k})_{k \leq m}$.

We refer to decomposable rules defined through committee scoring functions of the form

$$f_{m,k}(i_1, \ldots, i_k) = \lambda^k_1 \alpha_{t_{m,k,1}}(i_1) + \cdots + \lambda^k_k \alpha_{t_{m,k,k}}(i_k),$$

where $\Lambda_k = (\lambda^k_1, \ldots, \lambda^k_k)$ are OWA operators and $t_{m,k,1}, \ldots, t_{m,k,k}$ are sequences of integers from $[m]$, as *multithreshold* rules (we put no constraints on $t_{m,k,1}, \ldots, t_{m,k,k}$; both increasing and decreasing sequences are natural).

Proposition 3.6. *The committee scoring rule defined through the multithreshold functions $f_{m,k}(i_1, \ldots, i_k) = \alpha_{p_1}(i_1) + \alpha_{p_2}(i_2)$, for $p_1, p_2 \in \{2, \ldots, m-k-2\}$, $p_1 > p_2 + 1 \geq 3$, is not OWA-based.*

Proof. Let us fix p_1, p_2, m, and k that satisfy the requirements from the statement of the theorem. For the sake of contradiction, assume that our multithreshold function is OWA-based. By Lemma 2.1, we infer that there exists a committee scoring function $g_{m,k}$ of the form

$$g_{m,k}(i_1, \ldots, i_k) = \lambda_1 \gamma(i_1) + \lambda_2 \gamma(i_2),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are two numbers and γ is a single-winner scoring function, such that for each committee position $I = (i_1, \ldots, i_k)$ it holds that $f_{m,k}(I) = g_{m,k}(I)$; this follows because, by Lemma 2.1, the OWA-based committee scoring functions for our rule have to depend on i_1 and i_2 only, and by applying appropriate linear transformations, we can assume that these OWA-based committee scoring functions equal $f_{m,k}$.

Let us now consider two committee positions $I' = (p_2, p_1 + 1, \ldots)$ and $I'' = (p_2, p_1, \ldots)$. We see that:

$$f_{m,k}(I') - f_{m,k}(I'') = (\alpha_{p_1}(p_2) + \alpha_{p_2}(p_1 + 1)) - (\alpha_{p_1}(p_2) + \alpha_{p_2}(p_1)) = \alpha_{p_2}(p_1 + 1) - \alpha_{p_2}(p_1) = 0,$$

and, thus, it must also be the case that

$$g_{m,k}(I') - g_{m,k}(I'') = (\lambda_1\gamma(p_2) + \lambda_2\gamma(p_1 + 1)) - (\lambda_1\gamma(p_2) + \lambda_2\gamma(p_1)) = \lambda_2(\gamma(p_1 + 1) - \gamma(p_1)) = 0.$$

However, for committee positions $J' = (p_1 + 1, p_1 + 2, \ldots)$ and $J'' = (p_1, p_1 + 2, \ldots)$, we have

$$f_{m,k}(J') - f_{m,k}(J'') = (\alpha_{p_1}(p_1 + 1) + \alpha_{p_2}(p_1 + 2)) - (\alpha_{p_1}(p_1) + \alpha_{p_2}(p_1 + 2)) < 0,$$

and, consequently,

$$\begin{aligned} g_{m,k}(J') - g_{m,k}(J'') &= (\lambda_1\gamma(p_1 + 1) + \lambda_2\gamma(p_1 + 2)) - (\lambda_1\gamma(p_1) + \lambda_2\gamma(p_1 + 2)) \\ &= \lambda_1(\gamma(p_1 + 1) - \gamma(p_1)) < 0. \end{aligned}$$

Since we have both $\lambda_2(\gamma(p_1 + 1) - \gamma(p_1)) = 0$ and $\lambda_1(\gamma(p_1 + 1) - \gamma(p_1)) < 0$, we conclude that $\lambda_2 = 0$. However, for committee positions $L' = (p_2 - 1, p_2 + 1, \ldots)$ and $L'' = (p_2 - 1, p_2, \ldots)$, we have

$$f_{m,k}(L') - f_{m,k}(L'') = (\alpha_{p_1}(p_2 - 1) + \alpha_{p_2}(p_2 + 1)) - (\alpha_{p_1}(p_2 - 1) + \alpha_{p_2}(p_2)) < 0$$

and

$$g_{m,k}(L') - g_{m,k}(L'') = (\lambda_1\gamma(p_2 - 1) + 0 \cdot \gamma(p_2 + 1)) - (\lambda_1(p_2 - 1) + 0 \cdot (p_2)) = 0,$$

which is a contradiction and completes the proof. □

We generally expect decomposable rules to be NP-hard, but even among these rules there are polynomial-time computable ones (that are not OWA-based). For example, in their discussion of top-k-counting rules, Faliszewski et al. [32] mention a multithreshold rule that uses scoring functions that mix SNTV and Perfectionist:

$$f_{m,k}^{\mathrm{SNTV+Perf}}(i_1, \ldots, i_k) = f_{m,k}^{\mathrm{SNTV}}(i_1, \ldots, i_k) + f_{m,k}^{\mathrm{Perf}}(i_1, \ldots, i_k) = \alpha_1(i_1) + \alpha_k(i_k).$$

Briefly put, each winning committee under this rule is either an SNTV winning committee or is ranked on top k positions by some voter, and it suffices to check all such possibilities (thus, e.g., it is possible to compute some winning committee in polynomial time). One can show that this rule is not OWA-based using the same approach as in Proposition 3.6.

3.6 Beyond Decomposable Rules

Naturally, there are also committee scoring rules that go beyond the class of decomposable rules. Below, we provide two examples, starting with one inspired by our T-shirt store.

Example 3.4. In this example, the store does not want to maximize its direct revenue (i.e., the number of T-shirts sold), but the number of happy customers (in hope of increased future revenue). Let us say that a customer is happy if he or she finds at least two "good enough" T-shirts or at least one "great" T-shirt (recall that "at least good enough" shirts are among top 10% of all available ones, and "great" shirts are among the top 1%). Then the store should use the committee scoring function

$$f_{m,k}(i_1, \ldots, i_k) = \max(\alpha_{0.01m}(i_1), \alpha_{0.10m}(i_2)).$$

We refer to multithreshold rules with summation replaced by the max operator as *max-threshold* rules. Using an approach similar to that from Proposition 3.6, one can show that there are max-threshold rules that are not decomposable (we omit details).

In their search for rules between k-Borda and β-CC, Faliszewski et al. [30] introduced the class of ℓ_p-Borda rules, based on the following scoring functions ($p \geq 1$ is a parameter):

$$f_{m,k}^{\ell_p\text{-Borda}}(i_1, \ldots, i_k) = \sqrt[p]{\beta_m^p(i_1) + \cdots + \beta_m^p(i_k)}.$$

While the motivation for these rules is the same as for the q-HarmonicBorda rules, they behave quite differently (see the work of Faliszewski et al. [30] for a detailed discussion).

Corollary 3.7. *There are committee scoring rules that are not decomposable.*

Throughout the rest of the article, we will not venture outside the class of decomposable rules. However, the above two examples show that there are interesting rules there that also deserve to be studied carefully.

4 AXIOMATIC PROPERTIES OF COMMITTEE SCORING RULES

After exploring the universe of committee scoring rules from a syntactic (structural) perspective, we now consider axiomatic properties of the observed classes. Specifically, we will use two types of monotonicity notions—non-crossing monotonicity (together with its relaxations) and committee enlargement monotonicity—to characterize several of the classes and to gain insights regarding some others. Indeed, various monotonicity concepts have long been used in social choice (with Maskin monotonicity [53] being perhaps the most important example), and we follow this tradition.

4.1 Non-Crossing Monotonicity and Its Relaxations

Elkind et al. [25] introduced two monotonicity notions for multiwinner rules, namely candidate monotonicity (recall Section 2.3) and non-crossing monotonicity. In the former, we require that if we shift forward a candidate from a winning committee in some vote, then this candidate still belongs to some winning committee after the shift, but possibly to a different one. In the latter, we require that the whole committee remains winning, but we forbid shifts were members of the winning committee pass each other (i.e., after a shift none of the committee members gets worse and some get better). More formally, we have the following definition.

Definition 4.1 (Elkind et al. [25]). A multiwinner rule $\mathcal{R}$ is non-crossing monotone if for each election $E = (C, V)$ and each $k \in [|C|]$ the following holds: if $c \in W$ for some $W \in \mathcal{R}(E, k)$, then for each E' obtained from E by shifting c forward by one position in some vote without passing another member of W, we still have $W \in \mathcal{R}(E', k)$.

Elkind et al. [25] have shown that weakly separable rules are non-crossing monotone, and we will now show that the converse is also true. However, before we proceed to the proof, we introduce the following notation (that will also be useful in further analysis):

> Consider an arbitrary number of candidates m and a size of committee $k \in [m]$. For each $t \in [k]$ and $p \in [m]$, let $P_{m,k}(t, p)$ be the set of committee positions from $[m]_k$ that have their t-th element equal to p and such that they do not include position $p - 1$. We set $P_{m,k}(p) = \bigcup_{t \leq k} P_{m,k}(t, p)$.

For example, if $m = 5$ and $k = 3$, then $P_{5,3}(1, 4) = \emptyset$, $P_{5,3}(2, 4) = \{(1, 4, 5), (2, 4, 5)\}$, $P_{5,3}(3, 4) = \{(1, 2, 4)\}$, and $P_{5,3}(4) = P_{5,3}(1, 4) \cup P_{5,3}(2, 4) \cup P_{5,3}(3, 4) = \{(1, 4, 5), (2, 4, 5), (1, 2, 4)\}$.

Intuitively, $P_{m,k}(t, p)$ is a collection of committee positions in which the tth committee member stands on position p and where shifting him or her without passing another committee member is possible. Similarly, $P_{m,k}(p)$ is a collection of committee positions in which there is *some* committee member on position p and it is possible to shift him to position $p-1$ without passing another committee member.

Theorem 4.1. *Let $\mathcal{R}_f$ be a committee scoring rule. $\mathcal{R}_f$ is non-crossing monotone if and only if it is weakly separable.*

Proof. Let $\mathcal{R}_f$ be a committee scoring rule defined through a family $f = (f_{m,k})_{k \leq m}$ of scoring functions $f_{m,k} : [m]_k \to \mathbb{R}$. Due to the results of Elkind et al. [25], it suffices to show that if $\mathcal{R}_f$ is non-crossing monotone then it is weakly separable. So let us assume that $\mathcal{R}_f$ is non-crossing monotone.

Let us fix the number of candidates m and the committee size $k \in [m]$. Let $E = (C, V)$ be an election with candidate set $C = \{c_1, \ldots, c_m\}$ and collection of voters $V = (v_1, \ldots, v_{m!})$, with one voter for each possible preference order. By symmetry, every size-k subset W of C is a winning committee under $\mathcal{R}_f$.

Consider an arbitrary integer $p \in \{2, \ldots, m\}$, two arbitrary committee positions $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ from $P_{m,k}(p)$, and an arbitrary vote v from the election. (If $P_{m,k}(p)$ is empty, then $m = k$ follows, and so the rule is weakly separable. If $P_{m,k}(p)$ has a single element, then our reasoning still can be applied, since I and J do not have to be distinct.) Let $C(I)$ be the set of candidates that v ranks at positions $i_1, \ldots, i_k$, and let $C(J)$ be defined analogously for the case of J. Let E' be the election obtained by shifting in v the candidate currently in position p one position up. Finally, let I' and J' be committee positions obtained from I and J by replacing the number p with $p-1$ (it is possible to do so as I and J are both from $P_{m,k}(p)$).

Since, by assumption, $\mathcal{R}_f$ is non-crossing monotone, it must be the case that $C(I)$ and $C(J)$ are winning committees under $\mathcal{R}_f$ also in election E'. The difference of the scores of committee $C(I)$ in elections E' and E is $f_{m,k}(I') - f_{m,k}(I)$, and the difference of the scores of committee $C(J)$ in E' and E is $f_{m,k}(J') - f_{m,k}(J)$. It must be the case that

$$f_{m,k}(I') - f_{m,k}(I) = f_{m,k}(J') - f_{m,k}(J) \geq 0.$$

However, since the choice of p and the choices of I and J within $P_{m,k}(p)$ were completely arbitrary, it must be the case that there is a function $h_{m,k}$ such that for each $p \in \{2, \ldots, m\}$, each sequence $U \in P_{m,k}(p)$, and each committee position U' obtained from U by replacing position p with $p-1$, we have

$$h_{m,k}(p-1) = f_{m,k}(U') - f_{m,k}(U),$$

and the values of $h_{m,k}$ are non-negative.

Our goal now is to construct a single-winner scoring function $\gamma_{m,k}$ such that for each committee position $(\ell_1, \ldots, \ell_k) \in [m]_k$ it holds that

$$f_{m,k}(\ell_1, \ldots, \ell_k) = \gamma_{m,k}(\ell_1) + \gamma_{m,k}(\ell_2) + \cdots + \gamma_{m,k}(\ell_k).$$

We define $\gamma_{m,k}$ by requiring that (a) for each $p \in \{2, \ldots, m\}$, we have $\gamma_{m,k}(p-1) - \gamma_{m,k}(p) = h_{m,k}(p-1)$ (so $\gamma_{m,k}$ is a non-increasing function), and (b) value $\gamma_{m,k}(m)$ is such that $\gamma_{m,k}(m) + \gamma_{m,k}(m-1) + \cdots + \gamma_{m,k}(m-(k-1)) = f_{m,k}(m-(k-1), \ldots, m-1, m)$ (so that $\gamma_{m,k}$ indeed correctly describes the $f_{m,k}$-score of the committee ranked at the k bottom positions as a sum of the scores of the candidates).

We fix some committee position $(\ell_1, \ldots, \ell_k)$ from $[m]_k$. We know that, due to the choice of $\gamma_{m,k}(m)$, for $R = (r_1, \ldots, r_k) = (m-k+1, \ldots, m)$ it does hold that $f_{m,k}(r_1, \ldots, r_k) = \gamma_{m,k}(r_1) + \cdots + \gamma_{m,k}(r_k)$. Now, we can see that this property also holds for $R' = (r_1 - 1, r_2, \ldots, r_k)$. The reason is that

$$\gamma_{m,k}(m-k) - \gamma_{m,k}(m-k+1) = h_{m,k}(m-k) = f_{m,k}(R') - f_{m,k}(R).$$

Thus, for R', we have $f_{m,k}(R') = \gamma_{m,k}(r_1 - 1) + \gamma_{m,k}(r_2) + \cdots + \gamma_{m,k}(r_k)$. We can proceed in this way, shifting the top member of the committee up by sufficiently many positions, to obtain $R'' = (\ell_1, r_2, \ldots, r_k)$, where (by the same argument as above) we have

$$f_{m,k}(R'') = \gamma_{m,k}(\ell_1) + \gamma_{m,k}(r_2) + \cdots + \gamma_{m,k}(r_k).$$

Then, we can do the same to position r_2, and keep decreasing it until we get ℓ_2. Then the same for the third position, and so on, until the k-th position. Finally, we get

$$f_{m,k}(\ell_1, \ldots, \ell_k) = \gamma_{m,k}(\ell_1) + \cdots + \gamma_{m,k}(\ell_k).$$

This proves our claim and completes the proof. □

Non-crossing monotonicity is particularly natural when we seek committees of individually excellent candidates (for example, when we seek finalists of a competition or when we are interested in some shortlisting tasks [25, 31]). Indeed, if we have a committee W where we view each member as good enough to be selected and one of the members of W improves its performance without hurting the performance of any of the others, then it is perfectly natural to expect that all members of W are still good enough to be selected. Theorem 4.1 justifies axiomatically that if we are looking for a committee scoring rule for selecting individually excellent candidates, then we should look within the class of weakly separable rules. In fact, Elkind et al. [25] pointed out that we should focus on separable rules only, and we will provide axiomatic justification for this view in Section 4.2.

4.1.1 Prefix Monotonicity and Decomposable Rules. Based on the idea of non-crossing monotonicity, we can define other similar notions. In this section, we introduce and discuss one of them, which we call *prefix monotonicity*. Intuitively, if a rule satisfies the prefix monotonicity condition, then shifting forward a group of highest-ranked members of a winning committee within a given vote never prevents this committee from winning.

Definition 4.2. A multiwinner rule $\mathcal{R}$ satisfies *t-prefix monotonicity*, $t \geq 0$, if for each election $E = (C, V)$ and each committee size k, $t \leq k \leq |C|$, the following holds: For every $W \in \mathcal{R}(E, k)$, and every E' obtained from E by shifting by one position forward (to a more preferred position) in some vote each of the top-ranked t members of W (according to this vote), we have that $W \in \mathcal{R}(E', k)$. We say that $\mathcal{R}$ satisfies *prefix monotonicity* if it satisfies t-prefix monotonicity for every $t \in \mathbb{N}$.[11]

Example 4.1. Consider an election E that contains a vote $v : a \succ b \succ c \succ d \succ e$, and let $k = 3$. Further, assume that committee $W = \{b, c, d\}$ is winning in E. Let E' be an election obtained from E by replacing vote v with $v' : b \succ c \succ a \succ d \succ e$. Then, 2-prefix monotonicity requires that W is also winning in E'.

Prefix monotonicity is a relaxation of non-crossing monotonicity and, in consequence, all weakly separable rules satisfy it. In the remaining part of this section, we will show that only decomposable rules can be prefix-monotone (and mostly, though not only, those based on convex functions). Before we prove this statement, let us first prove one more technical lemma, which will allow us to reuse some of the reasoning later on. The lemma uses the same high-level idea as the first part of the proof of Theorem 4.1, yet it is more involved and differs in a number of details. (Recall that $P_{m,k}(t, p)$ used in the statement of the lemma was defined right before Theorem 4.1.)

Lemma 4.2. *Let $\mathcal{R}_f$ be a committee scoring rule and let t be an integer such that for each $x \in [t]$ this rule is x-prefix monotone. Then, for every number of candidates m and size of the committee k,*

[11]Note that 0-prefix monotonicity is an empty concept; as such, every rule satisfies it.

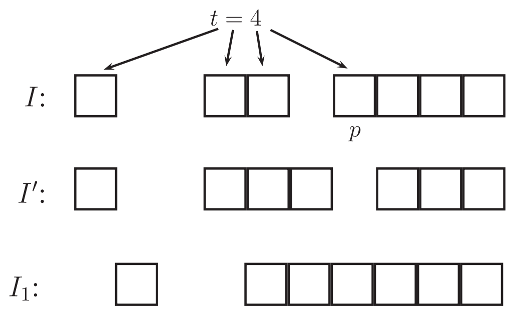

Fig. 2. An example showing how the sequences of positions I, I', and I_1 from the proof of Lemma 4.2 are related.

there exists a function h_t such that for each $p \in [m]$, each $U \in P_{m,k}(t,p)$, with $p \geq t$, and committee position U' obtained from U by replacing position p with $p-1$, we have

$$h_t(p-1) = f_{m,k}(U') - f_{m,k}(U) \geq 0.$$

Proof. Consider an arbitrary integer $p \in [m]$ and two arbitrary (possibly equal) committee positions $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ from $P_{m,k}(t,p)$, such that I and J have the first t elements equal. Let I' and J' be the committee positions obtained from I and J, respectively, by replacing the element p with $p-1$ (by the choice of I and J, it is possible to do so). Let I_1 and J_1 be the committee positions obtained from I and J, respectively, by increasing every element with value lower than p by one (in particular, when $t = 1$ we have $I_1 = I$ and $J_1 = J$). The way the sequences I' and I_1 are constructed from I is depicted in Figure 2 (in this example $t = 4$) and is presented below, also for J' and J_1 ($i_t = p$, $i_{t-1} \leq p-2$, $j_1 = i_1, \ldots, j_t = i_t$):

$$\begin{aligned}
I &= (i_1, i_2 \ldots, i_{t-1}, p, i_{i+1}, \ldots, i_k), & J &= (i_1, i_2 \ldots, i_{t-1}, p, j_{i+1}, \ldots, j_k),\\
I' &= (i_1, i_2 \ldots, i_{t-1}, p-1, i_{i+1}, \ldots, i_k), & J' &= (i_1, i_2 \ldots, i_{t-1}, p-1, j_{i+1}, \ldots, j_k),\\
I_1 &= (i_1+1, i_2+1 \ldots, i_{t-1}+1, p, i_{i+1}, \ldots, i_k), & J_1 &= (i_1+1, i_2+1 \ldots, i_{t-1}+1, p, j_{i+1}, \ldots, j_k).
\end{aligned}$$

As in the proof of Theorem 4.1, we construct an election $E = (C, V)$ with candidate set $C = \{c_1, \ldots, c_m\}$ and $m!$ voters $v_1, \ldots, v_{m!}$, one for each possible preference order. By symmetry, every size-k subset W of C is a winning committee of E under $\mathcal{R}_f$. Further, consider an arbitrary vote v from the election; let $C(I_1)$ and $C(J_1)$ be the committees that v ranks on positions I_1 and J_1, respectively. As all other committees, $C(I_1)$ and $C(J_1)$ are winning in E. Let us shift in v by one position forward each candidate from $C(I_1)$ that stands on a position with value lower than p. After such an operation, committee $C(I_1)$ will have position I and committee $C(I_1)$ will have position J. Since, by assumption, $\mathcal{R}_f$ is $(t-1)$-prefix-monotone, and exactly $t-1$ candidates have changed positions, it must be the case that $C(I_1)$ and $C(J_1)$ are still winning under $\mathcal{R}_f$. It must be the case that

$$f_{m,k}(I) - f_{m,k}(I_1) = f_{m,k}(J) - f_{m,k}(J_1) \geq 0.$$

By a similar reasoning, using the fact that $\mathcal{R}_f$ is t-prefix-monotone, we also conclude that

$$f_{m,k}(I') - f_{m,k}(I_1) = f_{m,k}(J') - f_{m,k}(J_1) \geq 0.$$

From the two above equalities, we get the following (the final inequality follows, because J' dominates J):

$$f_{m,k}(I') - f_{m,k}(I) = f_{m,k}(J') - f_{m,k}(J) \geq 0. \tag{1}$$

Recall that in the above equality I and J have the first $t-1$ elements equal. We would like to obtain the same relation even if the prefixes of I and J differ. Thus, now we will show how to change one element in the prefix of I and I' to an element differing by one, so that the equality still holds. By repeating this operation sufficiently many times, we can conclude that the equality does not depend on the prefix of I. For the sake of concreteness, we will show how to change i_{t-1} to $i_{t-1}+1$ in the prefix of I (this assumes that $i_{t-1}+1 < p-1$). A change of any other element in the prefix can be performed analogously. We proceed as follows. Let us define

$$I_{\mathrm{new}} = (i_1, i_2 \dots, i_{t-1}+1, p, i_{i+1}, \dots, i_k), \qquad L_{\mathrm{new}} = (i_1, i_2 \dots, i_{t-1}+1, p-1, i_{i+1}, \dots, i_k),$$
$$I'_{\mathrm{new}} = (i_1, i_2 \dots, i_{t-1}, p, i_{i+1}, \dots, i_k), \qquad L'_{\mathrm{new}} = (i_1, i_2 \dots, i_{t-1}, p-1, i_{i+1}, \dots, i_k).$$

In particular, observe that $I'_{\mathrm{new}} = I$ and that $L'_{\mathrm{new}} = I'$. Similarly as before, by using $(t-1)$-prefix-monotonicity and $(t-2)$-prefix-monotonicity, we obtain that

$$f_{m,k}(I'_{\mathrm{new}}) - f_{m,k}(I_{\mathrm{new}}) = f_{m,k}(L'_{\mathrm{new}}) - f_{m,k}(L_{\mathrm{new}}) \geq 0. \tag{2}$$

Adding inequalities Equations (1) and (2), we get

$$f_{m,k}(I') - f_{m,k}(I) + f_{m,k}(I'_{\mathrm{new}}) - f_{m,k}(I_{\mathrm{new}}) = f_{m,k}(J') - f_{m,k}(J) + f_{m,k}(L'_{\mathrm{new}}) - f_{m,k}(L_{\mathrm{new}}),$$

which is equivalent to

$$f_{m,k}(L_{\mathrm{new}}) - f_{m,k}(I_{\mathrm{new}}) = f_{m,k}(J') - f_{m,k}(J).$$

However, we can see that L_{new} and I_{new} are simply I' and I where one element of the prefix, i_{t-1}, is replaced with $i_{t-1}+1$. By our previous discussion, it follows that we can prove that $f_{m,k}(I') - f_{m,k}(I) = f_{m,k}(J') - f_{m,k}(J)$ even if I and J have different prefixes.

Since the choice of p, I, and J (within $P_{m,k}(t,p)$) is completely arbitrary, it must be the case that for each t there exists a function h_t such that for each $p \in \{t+1, \dots, m\}$, each sequence $U \in P_{m,k}(t,p)$, and each sequence U' obtained from U by replacing position p with $p-1$, we have

$$h_t(p-1) = f_{m,k}(U') - f_{m,k}(U) \geq 0.$$

The final inequality follows from Equation (1). □

We are ready to show that only decomposable rules can satisfy prefix-monotonicity.

Theorem 4.3. *Let $\mathcal{R}_f$ be a committee scoring rule. If $\mathcal{R}_f$ is prefix-monotone, then it must be decomposable.*

Proof. Let $f = (f_{m,k})_{k \leq m}$ be a family of committee scoring functions such that $\mathcal{R}_f$ is prefix-monotone. Let us fix the number of candidates m and the committee size k. For each $t \in [k]$, let h_t be the function constructed in Lemma 4.2.

Our goal is to provide single-winner scoring functions $\gamma^{(1)}_{m,k}, \dots, \gamma^{(k)}_{m,k}$ such that for each committee position $(\ell_1, \dots, \ell_k)$, we have

$$f_{m,k}(\ell_1, \dots, \ell_k) = \gamma^{(1)}_{m,k}(\ell_1) + \gamma^{(2)}_{m,k}(\ell_2) + \cdots + \gamma^{(k)}_{m,k}(\ell_k). \tag{3}$$

To this end, for each $t \in [k]$, we define $\gamma^{(t)}_{m,k} : \{t, \dots, m-k+t\} \to \mathbb{R}$ so that:[12]

(1) The values $\gamma^{(k)}_{m,k}(m), \gamma^{(k-1)}_{m,k}(m-1), \dots, \gamma^{(1)}_{m,k}(m-(k-1))$ are such that

$$f_{m,k}(m-(k-1), \dots, m-1, m) = \gamma^{(k)}_{m,k}(m) + \gamma^{(k-1)}_{m,k}(m-1) + \ldots + \gamma^{(1)}_{m,k}(m-(k-1))$$

[12]Formally, γ_t must be defined on $[m]$ but it actually never has a chance to calculate values $\gamma_t(s)$, where $s < t$ or $s > m-k+t$, so these values of γ_t can be chosen arbitrarily.

(so Equation (3) holds for the committee position where the candidates are ranked at the k bottom positions).

(2) For each $p \in \{t+1, \dots, m-k+t\}$, we have $\gamma_{m,k}^{(t)}(p-1) - \gamma_{m,k}^{(t)}(p) = h_t(p-1)$. (By Lemma 4.2, we have $h_t(p-1) \geq 0$, so $\gamma_{m,k}^{(t)}$ is nonincreasing.)

There may be many different ways to define functions $\gamma_{m,k}^{(1)}, \dots, \gamma_{m,k}^{(k)}$ satisfying the above conditions, and we choose one of them arbitrarily.

To show that Equation (3) holds, we use the same approach as in the second part of the proof of Theorem 4.1. Specifically, we note that if Equation (3) holds for some committee position $R = (r_1, \dots, r_k)$ and $R' = (r_1, \dots, r_t - 1, \dots, r_k)$ also is a valid committee position for some $t \in [k]$, then (by definition of h_t), we have

$$\begin{aligned} f_{m,k}(R') &= f_{m,k}(R) + h_t(r_t - 1) \\ &= \gamma_{m,k}^{(1)}(r_1) + \cdots + \gamma_{m,k}^{(t-1)}(r_{t-1}) + (\gamma_{m,k}^{(t)}(r_t) + h_t(r_t - 1)) + \gamma_{m,k}^{(t+1)}(r_{t+1}) + \cdots + \gamma_{m,k}^{(k)}(r_k) \\ &= \gamma_{m,k}^{(1)}(r_1) + \cdots + \gamma_{m,k}^{(t-1)}(r_{t-1}) + \gamma_{m,k}^{(t)}(r_t - 1) + \gamma_{m,k}^{(t+1)}(r_{t+1}) + \cdots + \gamma_{m,k}^{(k)}(r_k). \end{aligned}$$

Since Equation (3) holds for committee position $(m-(k-1), \dots, m)$, applying the above argument inductively proves that Equation (3) holds for all committee positions. □

Theorem 4.3 states that decomposability is a necessary condition for a committee scoring rule to be prefix-monotone. However, as the following example shows, it is not sufficient.

Example 4.2. The k-Approval Chamberlin–Courant rule (α_k-CC), defined by committee scoring functions $f_{m,k}^{\alpha_k\text{-CC}}(i_1, \dots, i_k) = \alpha_k(i_1)$, is a decomposable rule that is not prefix-monotone. Indeed, consider $k = 2$ and an election with four candidates $\{a, b, c, d\}$ that includes one vote for each possible ranking of these four candidates. This election contains $4! = 24$ votes and, in particular, vote $v : a \succ b \succ c \succ d$. By the symmetry of the rule, we see that for such election each committee is winning, including $W = \{b, c\}$ and $W' = \{c, d\}$. If α_k-CC were prefix-monotone, then shifting b and c by one position forward in v (to obtain $b \succ c \succ a \succ d$) should keep W winning. Doing so, however, does not change the score of W and increases the score of W', so W no longer wins. This shows that α_k-CC is not prefix-monotone.

However, if we assume that the single-winner scoring functions underlying a decomposable rule are, in a certain sense, convex, then we obtain a sufficient condition for this rule to be prefix-monotone.

Proposition 4.4. *Let $\mathcal{R}_f$ be a decomposable committee scoring rule defined through a family of scoring functions $f_{m,k}(i_1, \dots, i_k) = \gamma_{m,k}^{(1)}(i_1) + \gamma_{m,k}^{(2)}(i_2) + \cdots + \gamma_{m,k}^{(k)}(i_k)$, where $\gamma = (\gamma_{m,k}^{(t)})_{t \leq k \leq m}$ is a family of single-winner scoring functions. A sufficient condition for $\mathcal{R}_f$ to be prefix-monotone is that for each m and each $k \in [m]$, we have that*

(i) for each $i \in [k]$ and each $p, p' \in [m-1]$, $p < p'$, it holds that

$$\gamma_{m,k}^{(i)}(p) - \gamma_{m,k}^{(i)}(p+1) \geq \gamma_{m,k}^{(i)}(p') - \gamma_{m,k}^{(i)}(p'+1), \text{ and} \tag{4}$$

(ii) for each $i, j \in [k]$, $j > i$, and each $p \in [m]$, $j \leq p < m - (k - i)$, it holds that

$$\gamma_{m,k}^{(i)}(p) - \gamma_{m,k}^{(i)}(p+1) \geq \gamma_{m,k}^{(j)}(p) - \gamma_{m,k}^{(j)}(p+1). \tag{5}$$

Intuitively, condition (i) says that the functions in the family γ are convex, and condition (ii) says that, for each m and k, if $i < j$ then $\gamma_{m,k}^{(i)}$ decreases not slower than $\gamma_{m,k}^{(j)}$.

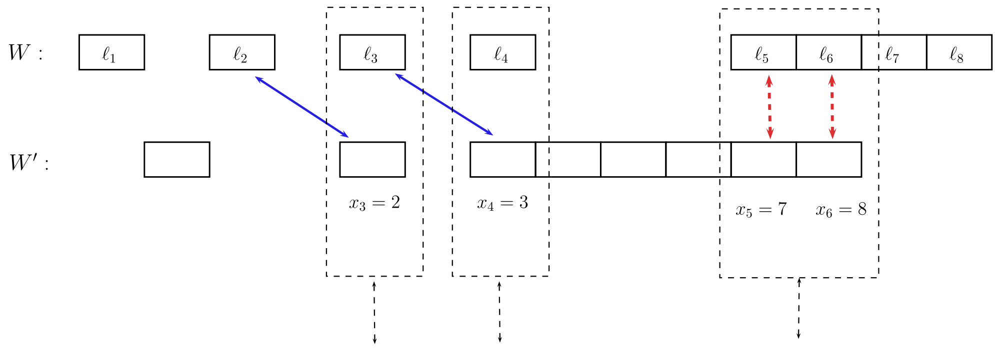

Fig. 3. Illustration of the notation used in Proposition 4.4.

Proof. Let $\mathcal{R}_f$ be defined as in the statement of the proposition and fix the number of candidates m and the committee size k. Consider an election E where a committee W is a winner. Let j be a number from $[k]$ and let E' be an election obtained from E by shifting forward by one position each of the first j members of W in some vote v. We will show that W is a winning committee in E'. Let $(\ell_1, \ldots, \ell_k)$ be the committee position of W in v (in election E). In comparison with E, in E' the score of W is increased by

$$\sum_{t=1}^{j} \left(\gamma_{m,k}^{(t)}(\ell_t - 1) - \gamma_{m,k}^{(t)}(\ell_t)\right).$$

Let us now assess by how much the score of some other committee, W', can increase. Let us fix $t \in [j]$, and let c_t be the candidate standing at position ℓ_t in v (in particular, $c_t \in W$). If $c_t \notin W'$, then shifting c_t one position up has no positive effect on the score of W'. Consider the case when $c_t \in W'$. Let x_t denote the position of c_t within W' according to v (for instance, if c_t is the most preferred among members of W' in v, then $x_t = 1$). This notation is illustrated in Figure 3. Now, we consider two cases:

Case 1 ($x_t \geq t$). Condition (5) implies that

$$\gamma_{m,k}^{(x_t)}(\ell_t - 1) - \gamma_{m,k}^{(x_t)}(\ell_t) \leq \gamma_{m,k}^{(t)}(\ell_t - 1) - \gamma_{m,k}^{(t)}(\ell_t).$$

Thus, the increase of the score of W' due to shifting c_t one position up is not greater than the increase of the score of W due to shifting c_t one position up. We assign c_t in W to c_t in W'; this assignment is shown with a bold dashed arrow in Figure 3) and, intuitively, it means that the increase of the score of W due to shifting c_t "compensates for" the increase of the score of W' due to shifting the assigned candidate.

Case 2 ($x_t < t$). Now, we observe that due to (4), we have

$$\gamma_{m,k}^{(x_t)}(\ell_t - 1) - \gamma_{m,k}^{(x_t)}(\ell_t) \leq \gamma_{m,k}^{(x_t)}(\ell_{x_t} - 1) - \gamma_{m,k}^{(x_t)}(\ell_{x_t}).$$

Thus, the increase of the score of W' due to shifting c_t one position up is not greater than the increase of the score of W due to shifting the candidate at position $\ell_{x_t} < \ell_t$, call such a candidate c, one position up. We assign c_t in W' to c in W; this assignment is depicted with a solid arrow in Figure 3.

From the above reasoning, we see that for each $t \in [j]$ the increase of the score of W' due to shifting c_t one position up is no greater than the increase of the score of W due to shifting some other candidate c_r ($r \leq t$) one position up; in such case, we say that c_r is assigned to c_t and that c_r compensates for c_t. Further, we note that each candidate $c_t \in W'$ is assigned to a different "compensating" candidate (see Figure 3 and consider how the assignment is defined, starting from the highest values of t and decreasing t one by one). We conclude that the score of W' increases in E' by a value that is not greater than the increase of the score of W. Since W' was chosen arbitrarily, we get that W is a winner in E', which completes the proof. □

For example, the Chamberlin–Courant rule based on the Borda scoring function is decomposable, satisfies the two conditions imposed by Proposition 4.4, and, thus, is prefix-monotone. Using a similar reasoning one can verify that a convex combination of the Chamberlin–Courant rule and the k-Borda rule defined by $f^{\mathrm{comb}}_{m,k}(i_1, \dots, i_k) = \lambda \sum_{j=1}^{k} \beta_m(i_j) + (1 - \lambda)\beta_m(i_1)$ also satisfies prefix-monotonicity for each $\lambda \in [0, 1]$. However, as explained in Example 4.2, the k-Approval Chamberlin–Courant rule is decomposable, but not prefix-monotone. Indeed, since the function a_k is not convex, it violates condition (i) from Proposition 4.4.

As far as applications of multiwinner voting goes, prefix monotonicity does not seem to have as clear-cut interpretation as non-crossing monotonicity. Nonetheless, in the next section, we will see how its relaxed variant is useful in characterizing representation-focused rules (and how this characterization can be interpreted in the context of diversity-oriented committee elections).

4.1.2 Top-Member Monotonicity and Representation-Focused Rules. Our goal in this section is to provide an axiomatic characterization of representation-focused rules. The first tool that we employ for this task is 1-prefix monotonicity (recall Definition 4.2), which we rename as *top-member monotonicity*. Intuitively, top-member monotonicity requires that if in some vote v we shift forward the highest-ranked member of a given winning committee, then this committee remains as a winning one. Since top-member monotonicity is a relaxed variant of non-crossing monotonicity (and of prefix monotonicity), it is satisfied by all weakly separable rules and alone is insufficient to characterize representation-focused rules. Thus, we will also use the notion of narrow-top consistency, defined below (which, in fact, is a relaxed form of the solid coalitions property of Elkind et al. [25], itself motivated by a much stronger notion of Dummet [23]).

Definition 4.3. A multiwinner rule $\mathcal{R}$ satisfies *narrow-top consistency* if for each election $E = (C, V)$ and each $k \in [|C|]$ the following holds: If there exists a set of k candidates S, such that each voter in V ranks some candidate from S first and each member of S is ranked first by some voter, then $S \in \mathcal{R}(E, k)$.

Together, top-member monotonicity and narrow-top consistency exactly characterize the class of representation-focused rules (within the class of committee scoring rules). We prove this result formally below, and then we explain the roles of both our axioms intuitively.

Theorem 4.5. *Let $\mathcal{R}_f$ be a committee scoring rule. $\mathcal{R}_f$ is representation-focused if and only if it satisfies top-member monotonicity and narrow-top consistency.*

Proof. It is apparent that each representation-focused rule satisfies both top-member monotonicity and narrow-top consistency. Suppose that $\mathcal{R}_f$ is a committee scoring rule, defined through a family $f = (f_{m,k})_{k \le m}$ of scoring functions $f_{m,k} : [m]_k \to \mathbb{R}_+$, that satisfies these two properties. We will show that $\mathcal{R}_f$ is representation-focused.

Let us fix the number of candidates m and the committee size k. Since $\mathcal{R}_f$ satisfies 1-prefix monotonicity (top-member monotonicity), by Lemma 4.2, we have that there exists a function h such that for each $p \in [m]$, each $U \in P_{m,k}(1, p)$ and the committee position U', obtained from U by replacing position p with $p - 1$, we have $h(p - 1) = f_{m,k}(U') - f_{m,k}(U) \ge 0$.

Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ be such that $i_1 = j_1$. We will show that $f_{m,k}(I) = f_{m,k}(J)$, which is sufficient to prove that $\mathcal{R}_f$ is representation focused. For the sake of contradiction, let us assume that this is not the case, and, without loss of generality, let us assume that $f_{m,k}(I) > f_{m,k}(J)$. There exists a positive integer η such that $\eta f_{m,k}(I) > \eta f_{m,k}(J) + k f_{m,k}(1, \ldots, k)$.

Let us fix a vote v and let W and W' denote the committees that stand in v on positions I and J, respectively. Note that they have a common member d who stands on position $i_1 = j_1$ and is highest ranked by v in both committees. Consider an election E with η copies of vote v and with k votes such that for each candidate $c \in W'$ there is one vote who ranks c first and the remaining candidates in some fixed, arbitrary way. In this election the score of W is at least equal to $\eta f_{m,k}(I)$ and the score of W' is at most equal to $\eta f_{m,k}(J) + k f_{m,k}(1, \ldots, k)$. Thus, the score of W is higher than the score of W'. If $i_1 = j_1 = 1$, then we get a contradiction immediately, since by the narrow-top consistency W' must be winning.

If $i_1 = j_1 \neq 1$, then we construct election E' by shifting, in each copy of v, the highest-ranked member of $\in W \cap W'$ to the top position. In comparison to E, the scores of committees W and W' in E' increase by the same value $\eta\big(h(1) - h(i_1)\big)$. As a result, W has a higher score than W' also in E'. This, however, contradicts narrow-top consistency, since all top positions in this election are occupied by candidates from W'. This proves that $f_{m,k}(I) = f_{m,k}(J)$, and completes the reasoning. □

The axioms used in Theorem 4.5 are logically independent. For example, k-Borda satisfies top-member monotonicity but is not narrow-top consistent. However, the max-threshold rule from Example 3.4 is narrow-top consistent, but violates top-member monotonicity.[13]

Let us now explain intuitively the interplay between top-member monotonicity and narrow-top consistency in the characterization of representation-focused rules. If we applied similar reasoning as we used in the proofs of Theorems 4.1 and 4.3 to top-member monotonicity, then we could show that if a committee scoring rule $\mathcal{R}_f$ is top-member monotone then its scoring functions are of the form:

$$f_{m,k}(i_1, \ldots, i_k) = \gamma_{m,k}(i_1) + g_{m,k-1}(i_2, \ldots, i_k), \tag{6}$$

where $\gamma = (\gamma_{m,k})_{k \le m}$ is a family of single-winner scoring functions and $g = (g_{m,k-1})_{k-1 \le m}$ is a family of committee scoring functions. Requiring that $\mathcal{R}_f$ is also narrow-top consistent ensures that the functions $g_{m,k}$ are, in fact, constant, and in consequence gives that $\mathcal{R}_f$ is representation-focused. Since all decomposable committee scoring rules are already of the form presented in Equation (6) (and, in fact, they are of a far more restricted form), we have the following corollary (the proof follows directly from the preceding reasoning, but straightforward calculations also show it directly; we omit these details).

Corollary 4.6. *If a decomposable committee scoring rule is narrow-top consistent, then it is representation focused.*

Representation-focused rules generally, and the Chamberlin Courant rule specifically, are often considered in the context of selecting diverse committees [25, 31]. While there is no clear definition of what a "diverse committee" is, researchers often use this term intuitively, to mean that as many voters as possible can find a committee member that they rank highly (if a voter v ranks some committee member c highly, then we could say that c "covers" the views of v, so some authors speak of "diversity/coverage"; see the works of Ratliff and Saari [68], Bredereck et al. [11], Celis et al. [16] and Izsak et al. [43] for a different view regarding diverse committees). Theorem 4.5

[13] To see the former, note that the scoring function for this rule uses scores 0 and 1 only, and that it gives score 1 to committee positions of the form $(1, i_2, \ldots, i_k)$. To see the latter, it suffices to consider an election with all $m!$ possible votes and a situation where shifting a top member of a committee does not change this committee's score, but improves the score of some other committee; such situations are easy to create.

justifies the use of representation-focused rules to seek committees that are diverse in this sense. Indeed, if there is a committee such that every voter ranks one of its members on top, then certainly this committee "covers" the "diverse" views of all the voters; narrow-top consistency ensures that this committee is selected. However, if there is a committee W, and we agree that it "covers" the views of sufficiently many voters, then if some voter ranks his or her highest-ranked committee member even higher (i.e., this voter realizes that the candidate represents his or her views even better), then certainly we should still view W as "covering" the views of sufficiently many voters; this is ensured by top-member monotonicity.

4.2 Committee Enlargement Monotonicity and Separable Rules

In this section, we consider the committee enlargement monotonicity axiom. While it is markedly different from the notions that we used in the previous sections, it still has a clear monotonicity flavor: Informally speaking, it requires that if W is a size-k winning committee for some election, then there also is a size-$(k + 1)$ winning committee for this election that includes all the members of W (the actual definition is more complicated due to possible ties; its exact form is due to Elkind et al. [25], but it was already studied by Barberà and Coelho [7] for resolute multiwinner rules, and in the literature on apportionment rules it is well-known as *house monotonicity* [6, 66]).

Definition 4.4 (Elkind et al. [25]). A multiwinner election rule $\mathcal{R}$ satisfies *committee enlargement monotonicity* if for each m and k, $1 \leq k < m$, and for each election E the following two conditions hold:

(1) for each $W \in \mathcal{R}(E, k)$ there exists $W' \in \mathcal{R}(E, k + 1)$ such that $W \subseteq W'$;
(2) for each $W \in \mathcal{R}(E, k + 1)$ there exists $W' \in \mathcal{R}(E, k)$ such that $W' \subseteq W$.

Remark 2. Barberà and Coelho [7] introduced a variant of this notion for resolute rules and called it *enlargement monotonicity*, whereas Elkind et al. [25] introduced the variant for non-resolute rules (in particular, adding Item (2) to the definition) under the name *committee monotonicity*. We decided to use the name *committee enlargement monotonicity* as it is more informative than the latter (indeed, *committee monotonicity* could well refer to k-prefix monotonicity), but is not tied to the world of resolute rules like the former.

This section is almost completely dedicated to showing that in the class of committee scoring rules, committee enlargement monotonicity characterizes exactly the class of separable rules.

Theorem 4.7. *Let $\mathcal{R}_f$ be a committee scoring rule. $\mathcal{R}_f$ is committee enlargement monotone if and only if $\mathcal{R}_f$ is separable.*

Before we provide the proof of Theorem 4.7, we first introduce useful notation and tools. Given two elections $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$, by $E_1 + E_2$, we mean election $(C, V_1 + V_2)$, whose voter collection is obtained by concatenating the voter collections of E_1 and E_2. For an election $E = (C, V)$ and a positive integer λ, by λE, we mean election $(C, \lambda V)$, whose voter collection consists of λ concatenated copies of V. We will heavily rely on the properties of the following elections (let C be some set of m candidates; this set will always be clear from the context when we use the notation introduced below):

(1) For each candidate $c \in C$, by $\zeta(c)$, we denote the election with $(m - 1)!$ voters who all rank c as their most preferred candidate, followed by each possible permutation of the remaining $m - 1$ candidates.
(2) For each subset $S \subseteq C$, we define election $\zeta(S)$ to be $\sum_{c \in S} \zeta(c)$ (i.e., it is a concatenation of the elections $\zeta(c)$ for each $c \in S$).

The next two lemmas describe which committees win in elections $\zeta(c)$ and $\zeta(S)$.

Lemma 4.8. *Fix m and k, and consider a non-degenerate committee scoring rule $\mathcal{R}$ defined through a scoring function $f_{m,k}$. The set of winners for $\zeta(c)$ consists of all committees that contain c.*

Proof. Since $\mathcal{R}$ is non-degenerate, there exists i such that $f_{m,k}(i+1,\ldots,i+k) > f_{m,k}(i+2,\ldots,i+k+1)$. By the fact that election $\zeta(c)$ is symmetric with respect to all the candidates except c, we see that all committees that contain c have the same $f_{m,k}$-score. Similarly, all committees that do not contain c also have the same score. Consider a committee W such that $c \notin W$. Let c' be an arbitrary member of W and let $W' = (W \setminus \{c'\}) \cup \{c\}$. Naturally, in each vote the position of committee W' dominates that of W. Further, there exists a vote where W has committee position $(i+2,\ldots,i+k+1)$, and W' has position $(1,i+2,\ldots,i+k)$. From this vote W gets score $f_{m,k}(i+2,\ldots,i+k+1)$ and W' gets score $f_{m,k}(1,i+2,\ldots,i+k) \geq f_{m,k}(i+1,\ldots,i+k)$. Thus, the score of W' in $\zeta(c)$ is higher than that of W. This completes the proof. □

Lemma 4.9. *Fix m, k, and $S \subseteq C$, and consider a non-degenerate committee scoring rule $\mathcal{R}$ defined through a scoring function $f_{m,k}$. If $|S| \geq k$, then the set of winning committees of $\zeta(S)$ consists of all the committees W such that $W \subseteq S$. Otherwise, it consists of all the committees W such that $S \subseteq W$.*

Proof. Consider election $\zeta(c)$ and let x and y denote the scores of committees, respectively, containing c and not containing c. From Lemma 4.8 it follows that $x > y$. Consider the case when $|S| \geq k$ (the proof for the other case follows by analogous reasoning). The score of a committee W such that $W \subseteq S$ is equal to $kx + (|S|-k)y$. For each committee W' with $W' \nsubseteq S$, its score is at most equal to $(k-1)x + (|S|-k+1)y < kx + (|S|-k)y$. □

In the following observation, we analyze the scores of candidates and committees in the elections we will be using in the proof of Theorem 4.7.

Observation 1. *Consider two committees, W_1 and W_2, with $W_1 \setminus W_2 = \{c_1\}$ and $W_2 \setminus W_1 = \{c_2\}$. By symmetry of our construction, for each single-winner scoring function $f_{m,1}$, the $f_{m,1}$-scores of the candidates c_1 and c_2 are the same in election $\zeta(W_1 \cup W_2)$, are the same in election $\zeta(W_1 \cap W_2)$, and are the same in election $\zeta(\{c_1,c_2\})$. Further, in each of these three elections, the $f_{m,1}$-scores of any two candidates $c, c' \in W_1 \cap W_2$ are equal. If $f_{m,1}$ is nontrivial, then in $\zeta(W_1 \cup W_2)$, $\zeta(W_1 \cap W_2)$ and $\zeta(\{c_1,c_2\})$ the $f_{m,1}$-scores of candidates c_1 and c_2 are, respectively, the same, lower, and higher than the $f_{m,1}$-score of any other candidate $c \in W_1 \cap W_2$. Also, for each committee scoring function $f_{m,k}$, the $f_{m,k}$-scores of committees W_1 and W_2 are the same in $\zeta(W_1 \cup W_2)$, are the same in $\zeta(W_1 \cap W_2)$, and are the same in $\zeta(\{c_1,c_2\})$. In $\zeta(W_1 \cap W_2 \setminus \{c\})$, where $c \neq c_1, c_2$, the $f_{m,k}$-scores of W_1 and W_2 are equal, and the $f_{m,1}$-score of c is lower than the $f_{m,1}$-score of any other candidate from $W_1 \cap W_2$.*

In the next lemma, we handle the possibility that the rule $\mathcal{R}_f$ in Theorem 4.7 may be trivial. (Note that it is not the case that if a committee enlargement monotone multiwinner rule always outputs all size-1 committees then it also always outputs all size-k committees for larger values of k. The result below excludes this behavior for the subclass of committee scoring rules.)

Lemma 4.10. *Suppose that $\mathcal{R}_f$ is a committee scoring rule defined by a family $f = (f_{m,k})_{k \leq m}$ of scoring functions, such that $\mathcal{R}_f$ is committee enlargement monotone and $f_{m,1}$ is constant. Then $f_{m,k}$ is constant for every $k \leq m$.*

Proof. $\mathcal{R}_f$ is trivial for $k = 1$, and we will show that, in fact, it is trivial for all k. The proof follows by induction. Let us assume that $\mathcal{R}_f$ is trivial for some $k = p-1$, i.e., that $f_{m,p-1}$ is constant. For the sake of contradiction let us assume that $f_{m,p}$ is not trivial, hence $f_{m,p}(1,\ldots,p) > f_{m,p}(m-p+1,\ldots m)$. Let i be the smallest positive integer such that $f_{m,p}(i+1,\ldots,i+p) > f_{m,p}(i+2,\ldots i+p+1)$. Consider an election where a certain candidate c is always in position

$i + p + 1$, the positions $i + p + 2, \dots, m$ are also always occupied by the same candidates, and on positions $1, \dots, i + p$ there are always the same candidates, call the set of these candidates S, but in all possible permutations. We can see that the $f_{m,p}$-scores of committees that consist only of the candidates from S are higher than the $f_{m,p}$-scores of committees that contain c (the reasoning is very similar to the one given in the proof of Lemma 4.8). This, however, contradicts committee enlargement monotonicity, since by our inductive assumption, for $k = p - 1$ all committees were winning, and so for $k = p$ there should be at least one winning committee containing c. □

We are nearly ready to present the proof of Theorem 4.7. The final piece of notation that we will need is as follows. Given two committee positions $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$, we will sometimes treat them as sets rather than sequences. For example, by $|I \cap J|$ we will mean the number of single-candidate positions that occur within both I and J, and we will say that $i \in I$ if there is some t such that $i = i_t$.

Proof of Theorem 4.7. Each separable committee scoring rule is committee enlargement monotone, and we focus on proving the converse.

Let $\mathcal{R}_f$ be the committee enlargement monotone committee scoring rule defined through a family $f = (f_{m,k})_{k \leq m}$ of scoring functions. Let us fix the number of candidates in our elections to be m. $\mathcal{R}_f$ assigns a score to each committee of each size and, in particular, for $k = 1$, given an election $E = (C, V)$ it assigns $f_{m,1}$-score to each candidate (i.e., to each singleton committee):

$$f_{m,1}\text{-score}_E(c) = \sum_{v_i \in V} f_{m,1}(\mathrm{pos}_{v_i}(c)).$$

We will show by induction on k that for each election E, a size-k committee W is winning under $\mathcal{R}_f$ if and only if it consists of candidates with the k highest $f_{m,1}$-scores.

The base for the induction, for $k = 1$, follows immediately from the definition of $\mathcal{R}_f$. Now, to prove the inductive step, let us assume that for each $k < p$ and for each election E it holds that $W \in \mathcal{R}_f(E, k)$ if and only if it consists of k candidates with the highest $f_{m,1}$-scores. We will show that this is also the case for $k = p$. By Lemma 4.10, we may assume that $\mathcal{R}_f$ for $k = 1$ is nontrivial.

Our first task is to show that whenever

$$f_{m,p}(1, \dots, p) = f_{m,p}(i + 1, \dots, i + p) \tag{7}$$

for some $i \in [m]$, then $f_{m,1}(1) = \cdots = f_{m,1}(i + p)$. For the sake of contradiction let us assume that $f_{m,p}(1, \dots, p) = f_{m,p}(i + 1, \dots, i + p)$ and $f_{m,1}(1) > f_{m,1}(i + p)$. We first show that it must hold that $f_{m,1}(1) = \cdots = f_{m,1}(i + p - 1)$. To see why this is the case, consider an election with a single vote $c_1 \succ c_2 \succ \cdots \succ c_m$. In such an election, committee $\{c_1, \dots, c_p\}$ always wins and, since $f_{m,p}(1, \dots, p) = f_{m,p}(i + 1, \dots, i + p)$, we have that committee $W_i = \{c_{i+1}, \dots, c_{i+p}\}$ also wins. By committee enlargement monotonicity, we know that some size-$(p - 1)$ subcommittee of W_i wins for committee size $p - 1$ and, in particular, by weak dominance, we get that certainly $W_i' = \{c_{i+1}, \dots, c_{i+p-1}\}$ wins. Thus, by the inductive hypothesis it must be the case that

$$f_{m,1}(1) + \cdots + f_{m,1}(p - 1) = f_{m,1}(i + 1) + \cdots + f_{m,1}(i + p - 1),$$

which implies that $f_{m,1}(1) = f_{m,1}(i + p - 1)$ and, thus, that $f_{m,1}(1) = f_{m,1}(2) = \cdots = f_{m,1}(i + p - 1)$. Since we assumed that $f_{m,1}(1) > f_{m,1}(i + p)$, it must be the case that $f_{m,1}(i + p - 1) > f_{m,1}(i + p)$.

Now, we show that the assumption that $f_{m,1}(i + p - 1) > f_{m,1}(i + p)$ also leads to a contradiction. Consider an election E with two votes:

$$v_1 : c_1 \succ c_2 \succ \cdots \succ c_{i+p-1} \succ c_{i+p} \succ \cdots \succ c_m,$$
$$v_2 : c_1 \succ c_2 \succ \cdots \succ c_{i+p} \succ c_{i+p-1} \succ \cdots \succ c_m,$$

which differ only in the order of c_{i+p-1} and c_{i+p}. For each $j < i+p-1$, we have $f_{m,1}$-score$_E(c_j) = 2f_{m,1}(j)$ and this value is higher than the $f_{m,1}$-scores of c_{i+p-1} and c_{i+p}. By the inductive hypothesis, this means that for $k = p-1$ there is no winning committee that contains either c_{i+p-1} or c_{i+p}. Thus, by committee enlargement monotonicity, we infer that no winning committee for $k = p$ contains both c_{i+p-1} and c_{i+p}. However, for $k = p$ due to (7) the $f_{m,p}$-score of committee $\{c_{i+1}, \ldots, c_{i+p-1}, c_{i+p}\}$ is the highest among committees of size p, which gives a contradiction.

Next, let $i_{\min}$ be the smallest value of i such that $f_{m,p}(i+1, \ldots, i+p) > f_{m,p}(i+2, \ldots, i+p+1)$. We will show that $f_{m,1}(i_{\min}+p) > f_{m,1}(i_{\min}+p+1)$. Again, for the sake of contradiction, let us assume that this is not the case and $f_{m,1}(i_{\min}+p) = f_{m,1}(i_{\min}+p+1)$. By our previous reasoning and by the definition of $i_{\min}$, we have that $f_{m,1}(1) = \cdots = f_{m,1}(i_{\min}+p)$. Consider an election where a fixed candidate c stands on position $i_{\min}+p+1$ and some set of $i_{\min}+p$ candidates stands on the first $i_{\min}+p$ positions in all possible permutations. In such an election there is no winning committee of size p that contains c. Indeed, for each committee that contains c there is at least one vote where this committee stands on a position with score at most $f_{m,p}(i_{\min}+2, \ldots, i_{\min}+p+1)$ and, so, it receives a strictly lower score than a committee that consists of the candidates that in each vote stand among the top $i_{\min}+p$ positions. However, by the inductive hypothesis, a winning committee of size $p-1$ containing c does exist. This contradicts committee enlargement monotonicity.

By the above reasoning, we can find two committee positions I^* and J^*, for committees of size p, such that $|I^* \cap J^*| = p-1$, $f_{m,p}(I^*) > f_{m,p}(J^*)$, and $\sum_{i \in I^*} f_{m,1}(i) > \sum_{i \in J^*} f_{m,1}(i)$ (for example, $I^* = (i_{\min}+1, \ldots, i_{\min}+p)$ and $J^* = (i_{\min}+2, \ldots, i_{\min}+p+1)$, where $i_{\min}$ is as in the previous paragraph). Let us arrange all committee positions from $[m]_k$ in a sequence $\mathcal{S}$ so that for each two consecutive elements I and J in $\mathcal{S}$ it holds that $|I \cap J| = p-1$. Such an arrangement always exists—see the construction based on Johnson graphs in Lemma 8 of the work of Skowron et al. [77]. We claim that for each two consecutive elements of sequence $\mathcal{S}$, call them I and J, it holds that

$$\frac{f_{m,p}(I) - f_{m,p}(J)}{f_{m,p}(I^*) - f_{m,p}(J^*)} = \frac{\sum_{i \in I} f_{m,1}(i) - \sum_{i \in J} f_{m,1}(i)}{\sum_{i \in I^*} f_{m,1}(i) - \sum_{i \in J^*} f_{m,1}(i)}. \tag{8}$$

(Note that the above expression is well defined. There is no division by zero, because we selected I^* and J^* so that $f_{m,p}(I^*) \neq f_{m,p}(J^*)$ and $\sum_{i \in I^*} f_{m,1}(i) \neq \sum_{i \in J^*} f_{m,1}(i)$.) For the sake of contradiction, let us assume that equality (8) does not hold for some I and J, and let us assume that there exist $x, y \in \mathbb{N}$ such that

$$\frac{f_{m,p}(I) - f_{m,p}(J)}{f_{m,p}(I^*) - f_{m,p}(J^*)} > \frac{x}{y} > \frac{\sum_{i \in I} f_{m,1}(i) - \sum_{i \in J} f_{m,1}(i)}{\sum_{i \in I^*} f_{m,1}(i) - \sum_{i \in J^*} f_{m,1}(i)}. \tag{9}$$

(Later on, we will explain how to handle the case when $<$ is replaced with $>$ in the above inequality.) Let W_1 and W_2 be two fixed committees with $|W_1 \cap W_2| = p-1$. Let $W_1 \setminus W_2 = \{c_1\}$ and $W_2 \setminus W_1 = \{c_2\}$. We construct election Q in which there are x votes where W_1 stands on position I^* and W_2 on position J^*, and y votes where W_1 stands on position J and W_2 on position I. In Q, the score of W_2 is equal to $xf_{m,p}(J^*) + yf_{m,p}(I)$, and the score of W_1 is equal to $xf_{m,p}(I^*) + yf_{m,p}(J)$. By inequality Equation (9), we see that the $f_{m,p}$-score of W_2 in Q is greater than the $f_{m,p}$-score of W_1, yet the sum of the $f_{m,1}$-scores of the members of W_2 is lower than that of the members of W_1, which means that the $f_{m,1}$-score of c_1 in Q is greater than the $f_{m,1}$-score of c_2 (these are the only candidates in which the two committees differ). If inequality Equation (9) were reversed (i.e., if we replaced both occurrences of "$>$" with "$<$"), then the same construction would still work, but we would have to reverse the roles of W_1 and W_2 and of c_1 and c_2.

We construct election Q_s by taking each possible permutation σ of the candidates from $W_1 \cap W_2$ and by concatenating all elections of the form $\sigma(Q)$ (where $\sigma(Q)$ is an election that results from

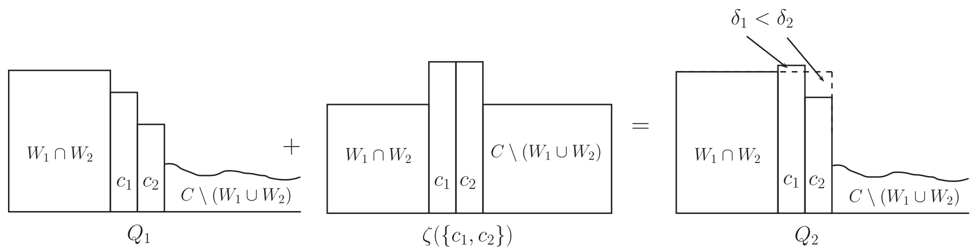

Fig. 4. The construction of election Q_2 from Q_1 in case (i). The "x-axis" corresponds to candidates and the "y-axis" corresponds to their $f_{m,1}$-scores. Here, δ_1 and δ_2 denote, respectively, the differences between the scores of c and c_1 and the difference between the scores of c and c_2. The shape of the election $\zeta(\{c_1, c_2\})$ is justified in Observation 1.

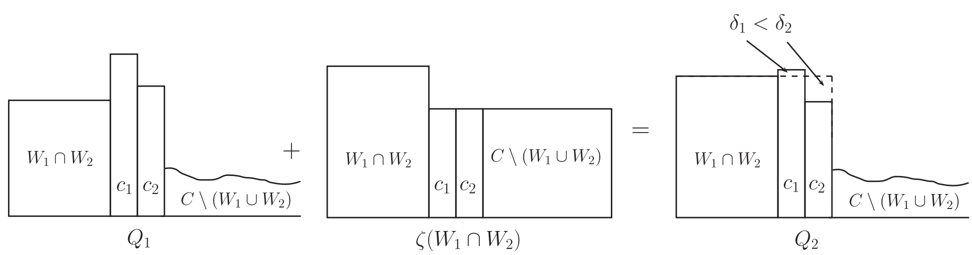

Fig. 5. The construction of election Q_2 from Q_1 in case (ii); the interpretation of the figure is the same as for Figure 4. The shape of the election $\zeta(W_1 \cap W_2)$ is justified in Observation 4.

applying σ to the candidates in all the preference orders within Q). Thus, intuitively, Q_s can be viewed as a symmetric version of Q, where symmetry is with respect to the candidates in $W_1 \cap W_2$. In particular in Q_s it holds that:

(a) the $f_{m,p}$-score of W_2 is higher than the $f_{m,p}$-score of W_1,
(b) the $f_{m,1}$-score of c_1 is higher than that of c_2, and
(c) the $f_{m,1}$-scores of all the candidates from $W_1 \cap W_2$ are equal.

There exists $\lambda \in \mathbb{N}$ such that in election $Q_1 = \lambda\zeta(W_1 \cup W_2) + Q_s$ each candidate from $W_1 \cup W_2$ has higher $f_{m,1}$-score than each candidate outside of $W_1 \cup W_2$. By Observation 1, it is clear that the $f_{m,1}$-score of candidate c_1 in Q_1 is higher than that of candidate c_2. Intuitively, this transformation allows us to focus only on the candidates from $W_1 \cup W_2$.

Now, let c be a fixed arbitrary candidate from $W_1 \cap W_2$. We construct election Q_2 using Q_1 in the following way.

(i) If in Q_1 the $f_{m,1}$-score of c is higher than the $f_{m,1}$-score of c_1, then we define Q_2 as a linear combination $Q_2 = \lambda_1 Q_1 + \lambda_2\zeta(\{c_1, c_2\})$ (this is depicted in Figure 4).
(ii) Otherwise, i.e., if in Q_1 the $f_{m,1}$-score of c_1 is at least as high as the $f_{m,1}$-score of c, then we define Q_2 as a linear combination $Q_2 = \lambda_1 Q_1 + \lambda_2\zeta(W_1 \cap W_2)$ (this is depicted in Figure 5).

In each of these two cases, we choose the coefficients λ_1 and λ_2 so that in Q_2 it holds that the $f_{m,1}$-score of c_1 is higher than that of c, which is higher than the $f_{m,1}$-score of c_2. Further, we choose λ_1 and λ_2 so that the difference between the $f_{m,1}$-scores of c and c_1 is smaller than the difference between the $f_{m,1}$-scores of c and c_2. Formally,

$$f_{m,1}\text{-score}_{Q_2}(c_1) - f_{m,1}\text{-score}_{Q_2}(c) < f_{m,1}\text{-score}_{Q_2}(c) - f_{m,1}\text{-score}_{Q_2}(c_2). \tag{10}$$

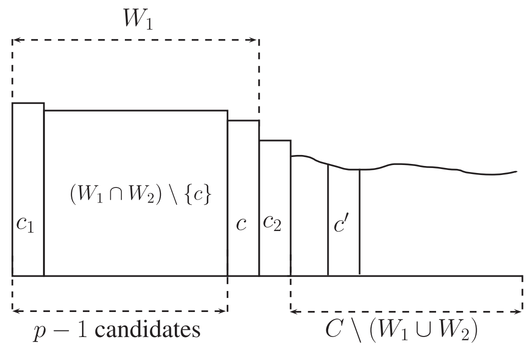

Fig. 6. Illustration of election Q_3. The interpretation of the figure is the same as for Figure 4.

Why is it possible to choose such λ_1 and λ_2? We will give a formal argument for Case (i) and it will be clear that this reasoning can be repeated for Case (ii). Let

$$\Delta_1 = f_{m,1}\text{-score}_{Q_1}(c) - f_{m,1}\text{-score}_{Q_1}(c_1) \quad \text{and} \quad \Delta_2 = f_{m,1}\text{-score}_{Q_1}(c_1) - f_{m,1}\text{-score}_{Q_1}(c_2).$$

Further, let Δ_3 denote the difference between the $f_{m,1}$-scores of the candidates from $\{c_1, c_2\}$ and the $f_{m,1}$-scores of the candidates outside of $\{c_1, c_2\}$ in $\zeta(\{c_1, c_2\})$. Naturally, there exist natural numbers $p, q \in \mathbb{N}$ such that

$$\Delta_1 < \frac{p}{q}\Delta_3 < \Delta_1 + \frac{1}{2}\Delta_2.$$

We set $\lambda_1 = q$ and $\lambda_2 = p$, and from the above inequality, we get that

$$\lambda_1\Delta_1 < \lambda_2\Delta_3 < \lambda_1\left(\Delta_1 + \frac{1}{2}\Delta_2\right). \tag{11}$$

Observe that

$$\begin{aligned} f_{m,1}\text{-score}_{Q_2}(c_1) - f_{m,1}\text{-score}_{Q_2}(c) &= -\lambda_1\Delta_1 + \lambda_2\Delta_3 > 0, \\ f_{m,1}\text{-score}_{Q_2}(c) - f_{m,1}\text{-score}_{Q_2}(c_2) &= \lambda_1(\Delta_1 + \Delta_2) - \lambda_2\Delta_3 > -\lambda_1\Delta_1 + \lambda_2\Delta_3 \\ &= f_{m,1}\text{-score}_{Q_2}(c_1) - f_{m,1}\text{-score}_{Q_2}(c). \end{aligned}$$

(The second inequality above is equivalent to $2\lambda_1\Delta_1 + \lambda_1\Delta_2 - 2\lambda_2\Delta_3 > 0$, and thus follows from inequality Equation (11).)

Next, we construct Q_3 as $Q_3 = \lambda_4 Q_2 + \zeta(W_1 \cap W_2 \setminus \{c\})$, where λ_4 is a very large number so that in Q_3, we still have that $f_{m,1}\text{-score}_{Q_3}(c_1) > f_{m,1}\text{-score}_{Q_3}(c) > f_{m,1}\text{-score}_{Q_3}(c_2)$ and that in Q_3 inequality Equation (10) still holds, yet the $f_{m,1}$-score of c is slightly lower than the $f_{m,1}$-scores of the other candidates from $W_1 \cap W_2$. Election Q_3 is depicted in Figure 6.

Given the $f_{m,1}$-scores of the candidates in Q_3, by our inductive assumption the unique size-$(p-1)$ winning committee consists of c_1 and all the candidates from $W_1 \cap W_2 \setminus \{c\}$. By committee enlargement monotonicity, we conclude that all size-p winning committees for Q_3 are of the form $\{c_1\} \cup (W_1 \cap W_2 \setminus \{c\}) \cup \{c'\}$, where c' is some other candidate. Let W' be one such winning committee. We know that it cannot be the case that $c' = c$ (i.e., W_1 cannot be winning in Q_3). This is so, because in Q_3 the $f_{m,p}$-score of W_2 is higher than the $f_{m,p}$-score of W_1 (since it was higher already in Q_s, and we added only elections that are symmetric with respect to W_1 and W_2—this symmetry follows from Observation 1). Thus, by the properties of the $f_{m,1}$-scores of the candidates (see Figure 6), c' must be some candidate such that

$$f_{m,1}\text{-score}_{Q_3}(c') \leq f_{m,1}\text{-score}_{Q_3}(c_2),$$

and, in particular, c' may simply be c_2 (but it also may be some other candidate). From the above inequality and from inequality Equation (10) (which holds for Q_3 as well), we get that

$$f_{m,1}\text{-score}_{Q_3}(c_1) - f_{m,1}\text{-score}_{Q_3}(c) < f_{m,1}\text{-score}_{Q_3}(c) - f_{m,1}\text{-score}_{Q_3}(c').$$

We note that in Q_3 committee W' has higher score than any committee containing c (otherwise, since W' is winning in Q_3 and by the above analysis, it would mean that W_1 is winning in Q_3, which is not the case).

Next, we construct election Q_3' by swapping candidates c_1 and c' in each vote in Q_3. Committee W' is also winning in Q_3' and thus it has higher score in Q_3' than any committee containing c. Similarly, by symmetry, we infer that

$$f_{m,1}\text{-score}_{Q_3'}(c') - f_{m,1}\text{-score}_{Q_3'}(c) < f_{m,1}\text{-score}_{Q_3'}(c) - f_{m,1}\text{-score}_{Q_3'}(c_1).$$

Finally, we construct election Q_4 by taking one copy of Q_3 and one copy of Q_3'. Observe that

$$\begin{aligned} f_{m,1}\text{-score}_{Q_4}(c) &= f_{m,1}\text{-score}_{Q_3}(c) + f_{m,1}\text{-score}_{Q_3'}(c) \\ &> f_{m,1}\text{-score}_{Q_3}(c_1) - f_{m,1}\text{-score}_{Q_3}(c) + f_{m,1}\text{-score}_{Q_3}(c') \\ &\quad + f_{m,1}\text{-score}_{Q_3'}(c') - f_{m,1}\text{-score}_{Q_3'}(c) + f_{m,1}\text{-score}_{Q_3'}(c_1) \\ &= f_{m,1}\text{-score}_{Q_4}(c_1) + f_{m,1}\text{-score}_{Q_4}(c') - f_{m,1}\text{-score}_{Q_4}(c). \end{aligned}$$

We can rewrite the above inequality as

$$f_{m,1}\text{-score}_{Q_4}(c) > \frac{1}{2}(f_{m,1}\text{-score}_{Q_4}(c_1) + f_{m,1}\text{-score}_{Q_4}(c')).$$

Since $f_{m,1}\text{-score}_{Q_4}(c_1) = f_{m,1}\text{-score}_{Q_4}(c')$ (election Q_4 is symmetric with respect to c_1 and c'), we get that $f_{m,1}\text{-score}_{Q_4}(c) > f_{m,1}\text{-score}_{Q_4}(c')$ and $f_{m,1}\text{-score}_{Q_4}(c) > f_{m,1}\text{-score}_{Q_4}(c_1)$. Thus, the $f_{m,1}$-score of c in Q_4 is among the $f_{m,1}$-scores of the $(p-1)$ top-scoring candidates. From our inductive assumption and from committee enlargement monotonicity, we infer that each winning committee in Q_4 must contain c. This contradicts the fact that W' is winning in Q_4, and proves Equation (8). Thus, setting

$$\alpha = \frac{f_{m,p}(I^*) - f_{m,p}(J^*)}{\sum_{i \in I^*} f_{m,1}(i) - \sum_{i \in J^*} f_{m,1}(i)},$$

we get that for any two consecutive elements, I and J, on path $\mathcal{S}$ it holds that

$$f_{m,p}(I) - f_{m,p}(J) = \alpha \left(\sum_{i \in I} f_{m,1}(i) - \sum_{i \in J} f_{m,1}(i) \right). \tag{12}$$

By a simple induction over the path $\mathcal{S}$, we can show that the above equality holds for any I and J (I and J do not have to be consecutive elements in $\mathcal{S}$). Consequently, we get that $f_{m,p}$ is a linear transformation of the function $g_{m,p}(I) = \sum_{i \in I} f_{m,1}(i)$; thus, they yield the same committee scoring rule. This proves our inductive step, and completes the proof. □

Elkind et al. [25] and Barberà and Coelho [7] point out that committee enlargement monotonicity is an extremely natural requirement for multiwinner rules whose role is to select committees of individually excellent candidates. For example, if some k candidates are good enough to be shortlisted for receiving some award, then increasing k should not lead to any of them losing their nominations. Thus, intuitively, Theorem 4.7 says that if one is interested in a committee scoring rule for choosing individually excellent candidates, then one should look within the class of separable rules. This refines and reinforces the recommendation provided by Theorem 4.1, which suggested looking among weakly separable rules.

Yet, one could challenge this recommendation. For example, SNTV is separable, but it is also representation-focused and there is some evidence that its behavior is closer to that of the Chamberlin–Courant rule (which is seen as selecting committees representing a diverse spectrum of opinions; recall the discussion after Theorem 4.5), than to that of, say, k-Borda (which is seen as selecting individually excellent candidates). Such evidence is provided, for example, by Elkind et al. [24], who evaluated a number of committee scoring rules experimentally, by computing their results on elections obtained from several two-dimensional Euclidean models and presenting them graphically. Nonetheless, in real-life settings even SNTV is sometimes used for choosing individually excellent candidates. As a piece of anecdotal evidence, let us mention that while preparing this article, we have run into a news article that listed the 10 best ski-jumpers of all time. The criterion for inclusion on that list was the number of times a given sportsman had won an individual competition of the ski-jumping World Cup. In other words, all the individual World Cup competitions that ever took place were seen as "voters," ranking all the sportsmen from the winner to the loser, and then SNTV was used to select the "committee" of 10 best ski-jumpers of all time.[14]

Theorems 4.1 and 4.7 have yet another interesting consequence. They imply that committee enlargement monotonicity of a committee scoring rule implies its non-crossing monotonicity. This is somewhat surprising, since the two variants of monotonicity seem almost unrelated as one describes how the result of an election changes if we increase the size of the committee and the other one—what happens when we shift a member of a winning committee in a preference relation of a voter.

Corollary 4.11. *If a committee scoring rule is committee enlargement monotone, then it is also non-crossing monotone.*

Finally, we note that outside of the class of committee scoring rules there exist various natural and important rules satisfying committee enlargement monotonicity. For example, we can speak of the "best-k" multiwinner variant of the Kemeny rule, by defining it to output committees of the k best candidates according to the Kemeny ranking. Indeed, every social welfare function generates such a committee enlargment monotone rule and the other way round [25, 31] (there are some subtleties related to tie-breaking here, which we omit). Greedy algorithms for committee scoring rules[15] are a particularly interesting class of committee enlargement monotone rules. They are very useful, for example, when we need to provide a ranking of the candidates so that every prefix of this ranking forms a good committee according to some criterion, such as proportional representation of the voters [78].

5 RELATED WORK

Over the past few years, multiwinner voting has attracted significant interest within the computational social choice literature, but it has also been studied for much longer within social choice theory and within economics. Below, we briefly review this literature (for a more detailed review, we point the readers to the overview of Faliszewski et al. [31]; we have also mentioned many related papers in the context of respective results).

[14]Naturally, the sets of ski-jumpers participating in the contests were often different. Formally, we would say that the participating sportsmen were ranked according to their result in the competition and all the non-participating ones were ranked below, in some arbitrary order. Since we are using SNTV, the order in which the non-participants are ranked is irrelevant.

[15]Greedy (also referred to as *sequential*) variants of committee scoring rules build winning committees in k steps. They start with an empty committee and in each step they add to the current committee a candidate that increases its score (according to the particular committee scoring function) most.

Axiomatic studies of voting rules were initiated by Arrow [1], and in a somewhat more narrow framework, by May [54]. Single-winner scoring rules are perhaps the best understood among single-winner election systems. Axiomatic characterizations of this class were provided, e.g., by Gärdenfors [39], Smith [80], and Young [86], and in a more general setting, by Myerson [57] and Pivato [64]. More specific axiomatic characterizations of single-winner scoring rules include those of the Borda rule [38, 42, 80, 84], of the Plurality rule [19, 69], of the Antiplurality rule [8], and of the Approval Voting [35, 72] (see also the overviews of Chebotarev and Shamis [18] and of Merlin [55]). Classic works on axiomatic properties of multiwinner rules include those of Dummett [23], Gehrlein [40], Felsenthal and Maoz [34], Debord [22], Ratliff [67], and Barberà and Coelho [7].

Our work mostly builds on that of Elkind et al. [25], where the authors introduced the class of committee scoring rules and many of the notions on which we rely, such as candidate monotonicity, non-crossing monotonicity, and committee enlargement-monotonicity (regarding the latter one, see also the work of Barberà and Coelho [7]). In particular, Elkind et al. [25] identified the classes of (weakly) separable and representation-focused rules and provided some of their basic features. OWA-based rules were introduced by Skowron et al. [75], who analyzed their computational properties (and who, in fact, studied a somewhat more general model). Faliszewski et al. [30, 32] introduced the class of top-k-counting rules and the ℓ_p-Borda and q-HarmonicBorda rules.

Recently Skowron et al. [77] characterized the class of committee scoring rules using the axioms of consistency, symmetry, continuity, and weak efficiency. Our article can be seen as complementary to theirs: They study committee scoring rules as opposed to all the other multiwinner rules, whereas we focus on the internal structure of the class.

Aziz et al. [2, 5] studied a class of approval-based rules that is very similar to the class of committee scoring rules (the class was first introduced by Thiele [81] in the 19th Century but was forgotten for some time; some of Thiele's rules were recalled by Kilgour [46] and then by Aziz et al.). Lackner and Skowron [49] studied axiomatic properties of these rules and highlighted their axiomatic similarity to committee scoring rules. Recently, monotonicity notions similar to those studied in this article were also considered in the context of approval-based multiwinner rules [49, 70]. For more general discussions of the properties of approval-based rules, we point the reader to the work of Kilgour and Marshall [47].

The study of computational properties of committee scoring rules, in general, and of specific rules, such as Chamberlin–Courant and Proportional Approval Voting, has attracted significant attention. This line of work has started with the paper of Procaccia et al. [65], who have shown that an approval-based variant of Chamberlin–Courant is NP-hard to compute. The same result for the classic, Borda-based variant was shown by Lu and Boutilier [51]. Betzler et al. [9] considered parameterized complexity of the rule, whereas the study of approximation algorithms was initiated by Lu and Boutilier [51], who have given a polynomial-time $(1 - \frac{1}{e})$-approximation algorithm (this algorithm, based on the greedy procedure of Nemhauser et al. [58] for submodular functions, has since then been adapted to other committee scoring rules as well). Skowron et al. [76] improved this result by providing a polynomial-time approximation scheme for the Borda-based variant; Skowron and Faliszewski [74] gave an FPT approximation scheme for the approval-based variant (and argued why the $(1 - \frac{1}{e})$-approximation algorithm is the best we can hope for among polynomial-time algorithms). The complexity of Chamberlin–Courant was also studied in much depth for various restricted domains, including the single-peaked domain [9, 20, 59], the single-crossing domain [79], and a number of others [48, 60, 87]. Faliszewski et al. [28, 33] considered a number of heuristic algorithms.

Proportional Approval Voting (PAV) received a bit less attention than the Chamberlin–Courant rule, but due to the work of Aziz et al. [2] on justified representation, it is now being studied with increasing interest (briefly put, Aziz et al. have shown that PAV is remarkably good at providing committees that represent the voters proportionally, as also confirmed by Brill et al. [14]; see the work of Brill et al. [13] for another rule with similar properties). The rule was shown to be NP-hard to compute [5, 75], but the standard greedy $(1 - \frac{1}{e})$-approximation algorithm works for it. Very recently, Byrka et al. [15] have shown a different, apparently much more powerful algorithm. The rule was also considered in the context of restricted domains [59]. FPT approximation schemes for PAV and other OWA-based rules were provided by Skowron [73].

More general computational results regarding committee scoring rules were provided by Skowron et al. [75], who studied the complexity and approximability of OWA-based rules, by Faliszewski et al. [32], for top-k-counting rules, by Peters [59], for OWA-based rules in the single-peaked domain, and by Faliszewski et al. [28], who introduced several general-purpose heuristic algorithms.

Naturally, there exist many interesting multiwinner rules beyond the class of committee scoring rules. These include, for example, Single Transferable Vote (see, e.g., the work of Tideman and Richardson [82]), a number of rules based on the Condorcet principle [3, 7, 21, 27, 36, 37, 40, 67, 71], Monroe's rule [56], and different variants of the rule invented by Phragmén [13, 44, 61, 62, 63]. For an overview of electoral systems used to select committees of representatives in practice, we refer the reader to the book of Lijphart and Grofman [50].

6 CONCLUSION

We have studied the class of committee scoring rules and explored the interesting hierarchy formed by its subclasses studied to date (including the class of decomposable rules introduced in this article). We have highlighted several fundamental properties of committee scoring rules, ranging from the nonimposition property (i.e., that for every committee and every nontrivial committee scoring rule, there is an election where this committee wins uniquely under this rule) to quite a varied landscape of monotonicity notions. This allowed us to partially match syntactic properties of such rules to their normative properties.

There is a number of follow-up directions for this research. For example, whole-committee monotonicity (where all the members of the committee are shifted forward) is an interesting property. The axiomatic characterization of OWA-based rules remains an open problem. Further, it is interesting to see whether there exist other properties, e.g., those that relate to proportionality of representation, that can be used to characterize different (subclasses of) committee scoring rules, or other multiwinner election systems (the works of Aziz et al. [2] and Lackner and Skowron [49] made some headway in this direction). A formal axiomatic study that would allow to compare committee scoring rules to other multiwinner election systems is an important, yet challenging question.

Our work provides axiomatic characterizations of several prominent rules within the class of committee scoring rules. An interesting open question is whether these characterizations can be extended to full characterizations within the class of all multiwinner rules. Indeed, Skowron et al. [77] provided a characterization of the class of committee scoring rules, but their results cannot be easily combined with ours. The main obstacle is that Skowron et al. [77] considered the model where election rules map voters' preferences to weak orders over committees, whereas in our model election rules output sets of winning committees (adapting their proof to our setting seems to be a highly nontrivial task; indeed, even in the world of single-winner rules where characterizations of scoring rules exist for both models, the proofs used to obtain them are quite different [85, 86]). Lackner and Skowron [49] showed a technique that allows one to translate characterizations from one model to the other, but their technique is not always applicable. In particular, it uses the 2-nonimposition property, which some committee scoring rules fail (see Section 2.3).

A APPENDIX

In this section, we prove two basic properties of committee scoring rules. The first one is non-imposition, which requires that for every committee there is an election where this committee wins uniquely (recall Definition 2.3 in Section 2.3). We show that all non-degenerate committee scoring rules have the nonimposition property. In the proof, we use elections $\zeta(S)$ introduced in Section 4.2.

LEMMA 2.2. *Let $\mathcal{R}_f$ be a committee scoring rule. It satisfies the nonimposition property if and only if it is non-degenerate.*

PROOF. A degenerate rule does not satisfy the nonimposition property. If $\mathcal{R}_f$ is non-degenerate, then for each committee W, by Lemma 4.9, election $\zeta(W)$ witnesses that $\mathcal{R}_f$ satisfies nonimposition. □

The next observation will be useful a bit later.

OBSERVATION 2. *Consider two committees, W_1 and W_2, such that $|W_1 \cap W_2| = k - 1$. In election $\zeta(W_1 \cup W_2) + \zeta(W_1 \cap W_2)$, committees W_1 and W_2 are the only winning ones. Indeed, by Lemma 4.9, we know that all W with $W \subseteq W_1 \cup W_2$ are winning in $\zeta(W_1 \cup W_2)$ and that all W with $W_1 \cap W_2 \subseteq W$ are winning in $\zeta(W_1 \cap W_2)$. The only two committees winning in both elections are W_1 and W_2. Since committee scoring rules satisfy consistency [77], we conclude that W_1 and W_2 are the only winners in $\zeta(W_1 \cup W_2) + \zeta(W_1 \cap W_2)$.*

Next, we give a proof of Lemma 2.1, by showing that two committee scoring functions (for a given number of candidates m and size k of the committees) define the same rule (for these m and k) if and only if they are linearly related.

LEMMA 2.1. *Let $\mathcal{R}_f$ and $\mathcal{R}_g$ be two committee scoring rules defined by committee scoring functions $f = (f_{m,k})_{k \le m}$ and $g = (g_{m,k})_{k \le m}$, respectively. If $\mathcal{R}_f = \mathcal{R}_g$, then for each m and k, $k \le m$, there are two values, $a_{m,k} \in \mathbb{R}_+$ and $b_{m,k} \in \mathbb{R}$, such that for each $I \in [m]_k$, we have that $f_{m,k}(I) = a_{m,k} \cdot g_{m,k}(I) + b_{m,k}$.*

PROOF. Let us fix m and k. Let $I_{\max} = (1, 2, \ldots, k)$ and $I_{\min} = (m - k + 1, m - k + 2, \ldots, m)$ be two committee positions, the former consisting of top k positions and the latter consisting of bottom k ones. The statement of the lemma clearly holds when $f_{m,k}(I_{\max}) = f_{m,k}(I_{\min})$ as then $\mathcal{R}_f$ is trivial, and so g must be constant. Thus, from now on, we assume that $f_{m,k}(I_{\max}) > f_{m,k}(I_{\min})$. Let $h_{m,k}$ be a linear transformation of $g_{m,k}$ such that $f_{m,k}(I_{\max}) = h_{m,k}(I_{\max})$ and $f_{m,k}(I_{\min}) = h_{m,k}(I_{\min})$. It is apparent that h and g implement the same multiwinner rule. We will show that $f_{m,k} = h_{m,k}$, which is sufficient to complete the proof. For the sake of contradiction let us assume that this is not the case.

Since $f_{m,k} \neq h_{m,k}$, there must exist I^* such that $f_{m,k}(I^*) \neq h_{m,k}(I^*)$; let us assume that $f_{m,k}(I^*) > h_{m,k}(I^*)$. There exists a sequence $\mathcal{S}$ of committee positions from $[m]_k$, starting with $I_{\max}$, containing I^*, and ending in $I_{\min}$, such that for each two consecutive elements, I and J, in the sequence (i.e., when J appears right after I in the sequence) it holds that

(i) $|I \cap J| = k - 1$, and
(ii) I dominates J.

For instance, for $m = 5$, $k = 2$ and $I^* = (2, 4)$, the sequence $\mathcal{S}$ could be $((1, 2), (1, 3), (1, 4), (2, 4), (2, 5), (3, 5), (4, 5))$ (note that this sequence does not need to contain all possible committee positions and, thus, it is easy to form it).

Consider function $\psi = f_{m,k} - h_{m,k}$. Since $f_{m,k}(I_{\max}) = h_{m,k}(I_{\max})$ and $f_{m,k}(I_{\min}) = h_{m,k}(I_{\min})$, we have that $\psi(I_{\min}) = 0$ and $\psi(I_{\max}) = 0$. Additionally, we know that $\psi(I^*) > 0$. Thus, there exist committee positions $I, J, I', J' \in [m]_k$ such that J is right after I and J' is right after I' in the sequence $\mathcal{S}$, and such that $\psi(I) \le 0$, $\psi(J) > 0$, $\psi(I') > 0$, and $\psi(J') \le 0$ (it might be the case that $J = I'$). That is

$$f_{m,k}(I) \le h_{m,k}(I), \quad f_{m,k}(J) > h_{m,k}(J), \quad f_{m,k}(I') > h_{m,k}(I'), \quad \text{and} \quad f_{m,k}(J') \le h_{m,k}(J').$$

Combining these inequalities, and taking into account that I dominates J, and that I' dominates J', we get that

$$0 \le f_{m,k}(I) - f_{m,k}(J) < h_{m,k}(I) - h_{m,k}(J) \quad \text{and} \quad f_{m,k}(I') - f_{m,k}(J') > h_{m,k}(I') - h_{m,k}(J') \ge 0.$$

This means that there exist two positive integers, $x, y \in \mathbb{N}$, such that

$$\frac{f_{m,k}(I) - f_{m,k}(J)}{f_{m,k}(I') - f_{m,k}(J')} < \frac{y}{x} < \frac{h_{m,k}(I) - h_{m,k}(J)}{h_{m,k}(I') - h_{m,k}(J')}$$

and, in consequence,

$$x(f_{m,k}(I) - f_{m,k}(J)) < y(f_{m,k}(I') - f_{m,k}(J')) \quad \text{and} \quad x(h(I) - h_{m,k}(J)) > y(h_{m,k}(I') - h_{m,k}(J')).$$

Let us fix two committees, W_1 and W_2, with $|W_1 \cap W_2| = k - 1$, and consider an election E with $x + y$ voters, where in x votes W_1 stands on position I and W_2 on position J, and in y votes W_1 stands on position J' and W_2 stands on position I'. We can add to E a sufficient number of copies of election $\zeta(W_1 \cup W_2) + \zeta(W_1 \cap W_2)$ (recall Section 4.2 for the definition of ζ). By Observation 2, we know that in election $\zeta(W_1 \cup W_2) + \zeta(W_1 \cap W_2)$ only committees W_1 and W_2 are winning. Consequently, if we add a sufficient number of copies of this election to E, we can ensure that in E only W_1, W_2, or both W_1 and W_2 can be winners. Since the elections that we added to E are symmetric with respect to W_1 and W_2, the outcome of the election (i.e., whether W_1 or W_2 is winning) depends only on election E. However, according to f committee W_1 has lower score than W_2, so the latter should be winning. Yet, by looking at h, we come to the opposite conclusion. This gives a contradiction and proves that $f = h$. This completes the proof. □

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