ABSTRACT. The results on existence and uniqueness for a problem obtained by finite element discretization of the micropolar equations are presented. Next, Newton's method for the discrete system is analyzed. In each iteration step, symmetric block diagonal preconditioner is used for nonsymmetric indefinite algebraic problem. We prove that the condition number of transformed problem is independent of the mesh size.

MIXED FINITE ELEMENT METHOD FOR MICROPOLAR FLUIDS

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1. INTRODUCTION.

In this paper, we consider a finite element discretization of the following system of partial differential equations with Dirichlet boundary conditions:

$$
\left\{
\begin{array}{ll}
-(\nu+\nu_r)\Delta u + (u\cdot\nabla)u + \nabla p = 2\nu_r \operatorname{curl}\omega + f & \text{in } \Omega\\
\operatorname{div} u = 0 & \text{in } \Omega\\
-(c_a+c_d)\Delta\omega + (u\cdot\nabla)\omega - (c_0+c_d-c_a)\nabla \operatorname{div}\omega + 4\nu_r\omega = 2\nu_r \operatorname{curl} u + g & \text{in } \Omega\\
u = u_0 & \text{on } \partial\Omega\\
\omega = \omega_0 & \text{on } \partial\Omega
\end{array}
\right.
\tag{1.1}
$$

This system is a mathematical model of stationary flow of viscous micropolar fluids, which are supposed to model the motion of solid particle suspension in a liquid. Such model is a basis for more complicated ones, widely used in applied sciences, for example in the theory of lubrication [Bayada-Lukaszewicz], [Khonsari] or in the theory of blood flow [Popel].

The unknowns are the velocity vector u, the pressure p and the internal microrotation vector ω. By f we denote the external force, while the angular momentum force is denoted by g. The constant $\nu > 0$ denotes usual kinematic viscosity. Other constants: ν_r, c_a, c_d, c_0 are introduced by the model as additional viscosities connected with the existence of the field of internal rotation ω. For short we define $c_1 = c_a + c_d$ and $c_2 = c_0 + c_d - c_a$. It follows from physical considerations [Petrosyan] that ν_r, c_1 and c_2 are positive.

Recently, substantial progress has been made in investigating theoretical properties of stationary micropolar fluid equations and related systems, cf. [Łukaszewicz],[Bayada-Lukaszewicz], [Krzyżanowski]. It has been proved in [Łukaszewicz], that in the case of homogenous boundary conditions and bounded domain Ω with smooth boundary, there exists a weak solution of (1.1) for any $f, g \in L^2(\Omega)$ and any positive ν, ν_r, c_1, c_2. This solution is also unique provided the data f, g are relatively small with respect to ν, ν_r, c_1, c_2.

Here, we are going to provide counterparts of the existence and uniqueness results of [Lukaszewicz] for the case of mixed finite element discretization of system (1.1). We also give an approximation result for the discrete solution, using now standard theory of Brezzi, Rappaz and Raviart [Girault-Raviart]. It appears that if the solution is far from a bifurcation point, then there exists a locally unique solution of the discrete problem, and the error is of optimal order with respect to the mesh size h. To the author's knowledge, these results are new in micropolar fluids context.

We also investigate the Newton's method for the resulting system of discrete algebraic nonlinear equations. We conclude that the method is quadratically convergent, provided the initial guess is sufficiently close to the solution, and the convergence rate is independent of h.

Each iteration step requires solution of a linear system with a nonsymmetric matrix, which is badly conditioned with respect to the mesh size h. We propose a preconditioning method for the linear system, based on a

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preconditioner for the Laplacian. The theory and methods for preconditioning the discrete Laplacian are well developed, so our preconditioner can be easily constructed and implemented, using, for example, a domain decomposition preconditioner or a multigrid method, see the recent Domain Decomposition Conference or Copper Mountain Conference on Multigrid Methods Proceedings for the most up-to-date references.

Block diagonal preconditioners for Stokes-like problems have been considered by numerous authors before, see, for example, [D'yakonov], [Bramble-Pasciak1], [Bramble-Pasciak2] [Silvester-Wathen], [Rusten-Winther], [Elman-Silvester] or [Klawonn]. However, our analysis of the preconditioned system does not rely on the symmetry nor on the positive definiteness of the matrix. On the other hand, the preconditioned system *is* symmetric and positive definite with respect to some auxiliary scalar product, so standard iterative methods, like conjugate gradient method, can be used for this system. We also remark that this approach can be easily extended onto stabilized methods for solving general saddle point problems.

The paper is organized as follows. In Section 2 we introduce the notation and formulate the problem in variational setting. Then we define the discrete problem, using mixed Galerkin method. For simplicity of presentation we pose the discrete problem in particular finite element spaces, using the well known Taylor-Hood element. Then in Section 3 we give a general existence result for the discrete problem. Local uniqueness and approximation results are proved, using the nonsingular solutions branches theory, in Section 4. Section 5 is devoted to the analysis of Newton's algorithm for solving the discrete system, while in Section 6 we propose a preconditioner for the linear problems arising on each step of the Newton's method, and prove a theorem on the condition number of the preconditioned problem.

2. Notation. Variational formulation.

Throughout the paper we shall assume that Ω is an open, bounded polygon or polyhedron with Lipschitz continuous boundary in R^N, where $N = 3$ is the dimension of the physical domain. The differential operators $\Delta, \nabla, \mathrm{curl}, \mathrm{div}$, which already appeared in (1.1), are defined in a standard way, see [Girault-Raviart].

We shall use several function spaces, which properties are described, for example, in [Adams]. By $H^k(\Omega)$ we shall denote the usual Sobolev spaces, identifying $H^0(\Omega)$ with the $L^2(\Omega)$ space of square integrable functions. The standard norm in $H^k(\Omega)$ shall be denoted by $||\cdot||_k$, while the seminorm – by $|\cdot|_k$.

By $H^1_0(\Omega)$ we shall understand the subspace of $H^1(\Omega)$ of functions with their trace on $\partial\Omega$ equal to zero, while by $L^2_0(\Omega)$ we denote the subspace of $L^2(\Omega)$, defined as

$$L^2_0(\Omega) = \{w \in L^2(\Omega) : \int_\Omega w = 0\}.$$

From Poincare – Friedrichs inequality, the H^1 norm is equivalent in $H^1_0(\Omega)$ or in $H^1(\Omega)\cap L^2_0(\Omega)$ to the seminorm

$$|u|_1 = |u|_{1,2} = ||\nabla u||_0.$$

For a positive integer k, we denote the inner product in the Cartesian product of k instances of $L^2(\Omega)$ spaces, $L^2(\Omega) \times L^2(\Omega) \ldots \times L^2(\Omega) = [L^2(\Omega)]^k$ by brackets:

$$(u, v) := \sum_{i=1}^{k} \int_\Omega u_i\, v_i\, dx$$

for any $u, v \in [L^2(\Omega)]^k$, while for the inner product in $[H^1_0(\Omega)]^k$ we use double brackets:

$$((u, v)) := \sum_{j=1}^{k} (\nabla u_j, \nabla v_j) = \sum_{j=1}^{k} \sum_{i=1}^{k} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}$$

for any $u, v \in [H^1_0(\Omega)]^k$. Let us define two auxiliary trilinear forms related to the convective term in (1.1),

$$d_1(u,\ v,\ w) := ((u\cdot\nabla)v, w) = \sum_{i,j=1}^{N} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j, \tag{2.1}$$

and, following [Temam],

$$d_2(u,\, v,\, w) := \frac{1}{2}\Big(((u\cdot\nabla)v, w) - ((u\cdot\nabla)w, v)\Big) \tag{2.2}$$

for any $u, v, w \in [H^1(\Omega)]^N$. Note that these forms are continuous on $[H^1(\Omega)]^N \times [H^1(\Omega)]^N \times [H^1(\Omega)]^N$ and if $\operatorname{div} u = 0$ then $d_1(u,\,\cdot,\,\cdot) \equiv d_2(u,\,\cdot,\,\cdot)$. The form $d_2(\cdot,\,\cdot,\,\cdot)$ is by the definition skew-symmetric with respect to the last two arguments (which reflects the skew-symmetry of $((u\cdot\nabla)v, w)$ on the solution u of (1.1)). In particular, we have

$$d_2(u,\, v,\, v) = 0. \tag{2.3}$$

By "C" we denote a generic positive constant, which, where necessary, we shall distinguish by subscripts.

Where there is no risk of confusion, we shall write H^k, H^1_0, L^2_0 instead of $H^k(\Omega)$, $H^1_0(\Omega)$, $L^2_0(\Omega)$ and use the same symbols for N-fold products of such spaces.

2.1. The discrete problem. We pose our original problem (1.1) in the variational form, using the following function spaces:

$V := [H^1_0(\Omega)]^N$, where $N = 3$ is the dimension of $\Omega \subset R^N$,
$W := L^2_0(\Omega)$.

In the rest of the paper we shall assume that the data for Problem 2.1 satisfy $f, g \in [L^2(\Omega)]^N$ and the boundary conditions on u, ω are homogenous. Then the variational formulation of (1.1) takes the form:

Problem 2.1. Find $(u, p, \omega) \in V \times W \times V$, such that for all (v, q, ξ) in $V \times W \times V$ it holds

$$\begin{cases} (\nu+\nu_r)(\nabla u, \nabla v) + d(u,\, u,\, v) - (p, \operatorname{div} v) = 2\nu_r(\operatorname{curl}\omega, v) + (f, v), \\ (\operatorname{div} u, q) = 0, \\ c_1(\nabla\omega, \nabla\xi) + d(u,\, \omega,\, \xi) + c_2(\operatorname{div}\omega, \operatorname{div}\xi) + 4\nu_r(\omega, \xi) = 2\nu_r(\operatorname{curl} u, \xi) + (g, \xi), \end{cases} \tag{2.4}$$

where $d(\cdot,\,\cdot,\,\cdot)$ denotes either the form $d_1(\cdot,\,\cdot,\,\cdot)$ or $d_2(\cdot,\,\cdot,\,\cdot)$.

We approximate Problem 2.1 in finite dimensional subspaces $V_h \subset V$, $W_h \subset W$, using a mixed Galerkin method:

Problem 2.2. Find $(u_h, p_h, \omega_h) \in V_h \times W_h \times V_h$, such that

$$\begin{cases} (\nu+\nu_r)(\nabla u_h, \nabla v) + d(u_h,\, u_h,\, v) - (p_h, \operatorname{div} v) = 2\nu_r(\operatorname{curl}\omega_h, v) + (f, v), \\ (\operatorname{div} u_h, q) = 0, \\ c_1(\nabla\omega_h, \nabla\xi) + d(u_h,\, \omega_h,\, \xi) + c_2(\operatorname{div}\omega_h, \operatorname{div}\xi) + 4\nu_r(\omega_h, \xi) = 2\nu_r(\operatorname{curl} u_h, \xi) + (g, \xi), \end{cases} \tag{2.5}$$

for all (v, q, ξ) in $V_h \times W_h \times V_h$.

2.2. Finite element assumptions. In our analysis we shall assume that the finite dimensional spaces V_h, W_h are particular finite element spaces.

We cover $\bar\Omega$ with a quasi-uniform triangulation [Ciarlet] $\mathcal{T}_h$, dividing $\bar\Omega$ into tetrahedra K:

$$\bigcup_{K\in\mathcal{T}_h} K = \bar\Omega.$$

The mesh parameter h is defined as

$$h = \max_{K\in\mathcal{T}_h} \operatorname{diam} K.$$

Let $P_j(K)$ denote the space of polynomials of degree not greater than j on a single element $K \in \mathcal{T}_h$. We define the finite element spaces V_h, W_h as follows.

For approximation of the velocity and pressure we shall use the Taylor – Hood finite elements (see, e.g. [Brezzi-Fortin]), so that

$$V_h = \{v \in V \cap C(\bar{\Omega}) : v\big|_K \in P_2(K) \quad \forall K \in \mathcal{T}_h\},$$

and

$$W_h = \{w \in W \cap C(\bar{\Omega}) : w\big|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\}.$$

The microrotation field is approximated in the same space V_h as the velocity. Obviously, V_h, W_h satisfy the *inf-sup* condition [Bercovier-Pironneau].

Remark 2.1. We consider these particular finite element spaces just for simplicity of the presentation. Results stated below easily extend (with slight modifications) to other conforming finite elements satisfying the inf-sup condition.

3. Existence result.

In the case when the form $d(\cdot,\,\cdot,\,\cdot)$ is skew-symmetric with respect to the last two arguments, i.e. it is chosen as $d_2(\cdot,\,\cdot,\,\cdot)$, then the following general existence result holds:

Theorem 3.1. *For any $f, g \in L^2$ and any positive ν, ν_r, c_1, c_2 there exists at least one triple $(u_h, p_h, \omega_h) \in V_h \times W_h \times V_h$ which solves the discrete nonlinear system (2.5). Moreover, the solution is unique, provided the data f, g are sufficiently small with respect to ν, ν_r, c_1, c_2.*

Remark 3.1. The "small data" assumption in Theorem 3.1 reflects similar requirements of the uniqueness statement for the continuous case, see [Lukaszewicz]. However, the restrictions on the data we get from our analysis are (quantitatively) stronger than those in [Lukaszewicz].

Proof. The uniqueness part of the theorem is proved using standard arguments, in a similar way as in the continuous case, see [Lukaszewicz]. We omit the details.

The proof of existence is based on a variant of Brouwer's fixed point theorem, see [Lions]. Let us introduce an auxiliary space of discrete divergence free functions,

$$V_h^0 = \{v \in V_h : (\operatorname{div} v, w) = 0 \quad \forall w \in W_h\}.$$

This is a finite dimensional Hilbert space with inner product $((\cdot,\cdot))$ derived from V_h, i.e. $((u,v)) = (\nabla u, \nabla v)$ for any $u, v \in V_h$. We define a nonlinear operator $P : V_h^0 \to V_h^0$ by the following identity:

$$((P(u), v)) = (\nu + \nu_r)(\nabla u, \nabla v) + d_1(u,\, u,\, v) - 2\nu_r(\operatorname{curl} \omega, v) - (f, v) \quad \text{for all } v \in V_h^0, \tag{3.1}$$

where $\omega \in V_h$ satisfies

$$c_1(\nabla\omega, \nabla\xi) + d_1(u,\, \omega,\, \xi) + c_2(\operatorname{div}\omega, \operatorname{div}\xi) + 4\nu_r(\omega, \xi) = 2\nu_r(\operatorname{curl} u, \xi) + (g, \xi) \tag{3.2}$$

for all $\xi \in V_h$.

From (2.3) it follows that the left hand side of (3.2) defines a bilinear form, which is continuous and V_h-elliptic. As the right hand side of equation (3.2) represents a bounded linear functional of $\xi \in V_h$, it follows from Lax-Millgram theorem that for given $u \in V_h^0$ there exists a unique $\omega \in V_h$, for which (3.2) holds. Then we conclude that the right hand side of (3.1) is a well defined bounded linear functional on V_h^0, hence Riesz theorem guarantees $P(u)$ is a uniquely determined element of V_h^0 for all $u \in V_h^0$.

Lemma 3.2. *There exists $r > 0$ such that for all $u \in V_h^0$ with $|u|_1 = r$,*

$$((P(u), u)) \geq 0.$$

Proof. First, we estimate the L^2 norm of ω from the definition of $P(u)$. Setting $\xi = \omega$ in (3.2) we obtain, using (2.3),

$$c_1|\omega|_1^2 + c_2||\operatorname{div}\omega||_0^2 + 4\nu_r||\omega||_0^2 \leq 2\nu_r|(\operatorname{curl} u, \omega)| + ||g||_0||\omega||_0, \tag{3.3}$$

which, since $||\operatorname{curl} u||_0 \leq |u|_1$ for u, yields in H_0^1,

$$\begin{aligned} 4\nu_r||\omega||_0 &\leq 2\nu_r||\operatorname{curl} u||_0 + ||g||_0 \\ &\leq 2\nu_r|u|_1 + ||g||_0. \end{aligned} \tag{3.4}$$

Then from (3.1) we have, using (2.3) and the fact that $(\operatorname{curl} u, \omega) = (\operatorname{curl}\omega, u)$ for u, ω in H_0^1,

$$\begin{aligned} ((P(u), u)) &= (\nu + \nu_r)|u|_1^2 - 2\nu_r(\operatorname{curl}\omega, u) - (f, u) \\ &\geq (\nu + \nu_r)|u|_1^2 - 2\nu_r||\omega||_0|u|_1 - C||f||_0|u|_1 \\ &\geq |u|_1\big((\nu + \nu_r)|u|_1 - 2\nu_r(\frac{1}{2}|u|_1 + \frac{1}{\nu_r}||g||_0) - C||f||_0\big), \end{aligned}$$

whence

$$((P(u), u)) \geq |u|_1\big(\nu|u|_1 - C_1(||f||_0 + ||g||_0)\big).$$

Now, ν is positive, so for $r > \frac{C_1}{\nu}(||f||_0 + ||g||_0) > 0$ we obtain $((P(u), u)) \geq 0$, which ends the proof of the lemma. $\square$

Using the above lemma, it follows (see [Lions], for example) that there is at least one $u \in V_h^0$, for which $P(u) = 0$.

Observe that if for some $u \in V_h^0$, $P(u) = 0$ in V_h^0, then, u and corresponding ω satisfy the first and the third equation of (2.5), with v, ξ varying over V_h^0 and V_h, respectively. Obviously, the second equation of (2.5) is implicitly satisfied, since $u \in V_h^0$.

In other words, there exists at least one pair $(u, \omega) \in V_h \times V_h$ that satisfies

$$\begin{aligned} &(\nu + \nu_r)(\nabla u, \nabla v) + d_1(u, u, v) = 2\nu_r(\operatorname{curl}\omega, v) + (f, v), \\ &(\operatorname{div} u, q) = 0, \\ &c_1(\nabla\omega, \nabla\xi) + d_1(u, \omega, \xi) + c_2(\operatorname{div}\omega, \operatorname{div}\xi) + 4\nu_r(\omega, \xi) = 2\nu_r(\operatorname{curl} u, \xi) + (g, \xi), \end{aligned} \tag{3.5}$$

for all $(v, q, \xi) \in V_h^0 \times W_h \times V_h$. Then from the inf-sup condition (Lemma I.4.1 in [Girault-Raviart]) we conclude that the original problem (2.5) admits at least one solution $(u, p, \omega) \in V_h \times W_h \times V_h$. $\square$

4. Approximation of nonsingular solutions of Problem 2.1

In this section we derive local uniqueness and approximation results for Problem 2.2, using the theory of branches of nonsingular solutions of nonlinear equations

$$F(\lambda, \mathcal{U}) \equiv \mathcal{U} + T \cdot G(\lambda, \mathcal{U}) = 0, \tag{4.1}$$

as described in the book [Girault-Raviart].

Let $d(\cdot,\cdot,\cdot)$ be either $d_1(\cdot,\cdot,\cdot)$ or $d_2(\cdot,\cdot,\cdot)$ and let us denote $\lambda = (\nu + \nu_r)^{-1}$. Then Problem 2.1 is equivalent to finding the solution $\mathcal{U} = (u, \lambda p, \omega)$ of

$$F(\lambda, \mathcal{U}) = 0, \tag{4.2}$$

where F is defined as in (4.1), with G and T chosen as follows. We define $G : R_+ \times (V \times W \times V) \to H^{-1} \times H^{-1}$ as

$$G(\lambda, \mathcal{U}) = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

where

$$\begin{aligned} <\theta_1, v> &= \lambda \cdot \big(d(u, u, v) - 2\nu_r(\operatorname{curl}\omega, v) - (f, v)\big) \qquad \forall v \in V, \\ <\theta_2, v> &= d(u, \omega, v) - 2\nu_r(\operatorname{curl} u, v) - (g, v) \qquad \forall v \in V. \end{aligned}$$

Here, $< \cdot, \cdot >$ denotes the dual pairing between H_0^1 and H^{-1}. The linear operator $T : H^{-1} \times H^{-1} \to V \times W \times V$ is then defined by

$$T \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = (u^*, p^*, \omega^*),$$

where the triple (u^*, p^*, ω^*) satisfies an identity

$$\left\{ \begin{array}{ll} (\nabla u^*, \nabla v) - (p^*, \operatorname{div} v) =< \theta_1, v > & \forall v \in V, \\ (\operatorname{div} u^*, w) = 0 & \forall w \in W, \\ c_1(\nabla \omega^*, \nabla \xi) + c_2(\operatorname{div} \omega^*, \operatorname{div} \xi) + 4\nu_r(\omega^*, \xi) =< \theta_2, \xi > & \forall \xi \in V. \end{array} \right. \tag{4.3}$$

Lemma 4.1. *The mapping T is well defined and is a bounded linear operator from $H^{-1} \times H^{-1}$ onto $V \times W \times V$.*

Proof. The first two equations of (4.3) are Stokes equations in mixed variational setting. The third equation is a V-elliptic one, independent of the previous two. The existence of a unique pair $(u_h^*, p_h^*) \in V \times W$ satisfying the first two equations follows from the theory of saddle point problems [Girault-Raviart], while Riesz theorem guarantees the existence and uniqueness of $\omega_h^* \in V$, for which the third equation of (4.3) holds. □

Next, we define the approximate operator $T_h : H^{-1} \times H^{-1} \to V_h \times W_h \times V_h$ as

$$T_h \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = (u_h^*, p_h^*, \omega_h^*),$$

where $(u_h^*, p_h^*, \omega_h^*) \in V_h \times W_h \times V_h$ is a solution of a variational problem in finite dimensional space

$$\left\{ \begin{array}{ll} (\nabla u_h^*, \nabla v) - (p_h^*, \operatorname{div} v) =< \theta_1, v > & \forall v \in V_h, \\ (\operatorname{div} u_h^*, w) = 0 & \forall w \in W_h, \\ c_1(\nabla \omega_h^*, \nabla \xi) + c_2(\operatorname{div} \omega_h^*, \operatorname{div} \xi) + 4\nu_r(\omega_h^*, \xi) =< \theta_2, \xi > & \forall \xi \in V_h. \end{array} \right. \tag{4.4}$$

Lemma 4.2. *The mapping T_h is well defined and is a bounded linear operator from $H^{-1} \times H^{-1}$ onto $V_h \times W_h \times V_h$.*

Proof. The proof is analogous to the proof of the previous lemma. We use the fact that V_h, W_h satisfy the inf-sup condition. □

Having T_h and G defined, it can be easily verified that $\mathcal{U}_h = (u_h, \lambda p_h, \omega_h)$ satisfies

$$F_h(\lambda, \mathcal{U}_h) \equiv \mathcal{U}_h + T_h \cdot G(\lambda, \mathcal{U}_h) = 0 \tag{4.5}$$

if and only if (u_h, p_h, ω_h) is the solution of Problem 2.2.

Now, we shall examine some properties of mappings introduced above.

Lemma 4.3. *Suppose $\mathcal{W} = (v, q, z)$ is an arbitrary element of $V \times W \times V$. Then for any $\lambda \in R_+$ and any $\mathcal{U} = (u, p, \omega) \in V \times W \times V$,*

$$D_{\mathcal{U}} G(\lambda, \mathcal{U}) \cdot \mathcal{W} \in L^{3/2} \times L^{3/2},$$

where $D_{\mathcal{U}} G$ denotes Fréchet derivative of G with respect to the second argument.

Proof. We apply an argument similar to [Girault-Raviart], Theorem IV.4.1. The first derivative of G with respect to $\mathcal{U}$ is given by

$$D_{\mathcal{U}} G(\lambda, \mathcal{U}) \cdot \mathcal{W} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{4.6}$$

where

$$< \eta_1, \xi >= \lambda\big(d(u,\, v,\, \xi) + d(v,\, u,\, \xi) - 2\nu_r(\operatorname{curl} z, \xi)\big) \qquad \forall \xi \in V, \tag{4.7}$$

and

$$< \eta_2, \xi >= d(u,\, z,\, \xi) + d(v,\, w,\, \xi) - 2\nu_r(\operatorname{curl} v, \xi) \qquad \forall \xi \in V. \tag{4.8}$$

Now, let us consider the case when the form $d(\cdot,\,\cdot,\,\cdot)$ is defined as $d_1(\cdot,\,\cdot,\,\cdot)$. Then η_1, η_2 may be identified with $\eta_1 = \lambda\big((u\cdot\nabla)v + (v\cdot\nabla)u - 2\nu_r \operatorname{curl} z\big)$, $\eta_2 = (u\cdot\nabla)z + (v\cdot\nabla)w - 2\nu_r \operatorname{curl} v$. Since $H_0^1 \hookrightarrow L^6$, it can be easily shown using Hölder inequality that η_1 and η_2 belong to $L^{3/2}$.

If $d(\cdot,\,\cdot,\,\cdot)$ is chosen equal to $d_2(\cdot,\,\cdot,\,\cdot)$, then the situation is quite similar. Integrating by parts the convective term, we obtain

$$((u\cdot\nabla)w, v) = -\Big((\operatorname{div} u\cdot w, v) + ((u\cdot\nabla)v, w)\Big),$$

so, substituting into the definition of $d_2(\cdot,\,\cdot,\,\cdot)$ in (2.2), we obtain

$$d_2(u,\,v,\,w) = ((u\cdot\nabla)w, v) + \frac{1}{2}(\operatorname{div} u\cdot w, v),$$

whence $\eta_1 = \lambda\big((u\cdot\nabla)v + (v\cdot\nabla)u + \frac{1}{2}\operatorname{div} u\cdot v + \frac{1}{2}\operatorname{div} v\cdot u - 2\nu_r \operatorname{curl} z\big)$, and $\eta_2 = (u\cdot\nabla)z + (v\cdot\nabla)w + \frac{1}{2}\operatorname{div} u\cdot z + \frac{1}{2}\operatorname{div} v\cdot w - 2\nu_r \operatorname{curl} v$. Again, using Hölder inequality and the imbedding $H_0^1 \hookrightarrow L^6$ we prove $\eta_1,\,\eta_2 \in L^{3/2}$. □

Lemma 4.4. *The mapping G is twice differentiable, and its second derivative with respect to $\mathcal{U}$ is independent of $\mathcal{U}$, namely, for $\mathcal{W} = (w, r, \tau)$ and $\mathcal{V} = (v, q, z)$ in $V\times W\times V$,*

$$D^2_{\mathcal{U}\mathcal{U}} G(\lambda, \mathcal{U})\cdot(\mathcal{W}, \mathcal{V}) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{4.9}$$

where $\eta_1, \eta_2 \in H^{-1}$ are such that

$$< \eta_1, \xi >= \lambda\big(d(w,\,v,\,\xi) + d(v,\,w,\,\xi)\big) \qquad \forall \xi \in V, \tag{4.10}$$

$$< \eta_2, \xi >= d(w,\,z,\,\xi) + d(v,\,\tau,\,\xi) \qquad \forall \xi \in V. \tag{4.11}$$

Proof. The lemma follows directly from (4.6) – (4.8). □

Lemma 4.5. *Let $\theta_1, \theta_2 \in H^{-1}$. Denote by $\mathcal{U} = (u, p, \omega) \in V\times W\times V$ the solution of (4.3), i.e.*

$$\mathcal{U} = T\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

and by $\mathcal{U}_h = (u_h, p_h, \omega_h)$ – the solution of approximate problem (with T replaced by T_h). If $(u, p, \omega) \in H^{l+1}\times H^l \times H^{l+1}$ for $l \in \{1, 2\}$, then

$$\begin{aligned} ||u - u_h||_1 + ||p - p_h||_0 &\le Ch^l(||u||_{l+1} + ||p||_l), \\ ||\omega - \omega_h||_1 &\le Ch^l||\omega||_{l+1}. \end{aligned} \tag{4.12}$$

Proof. The first inequality follows from the approximation properties of Taylor-Hood finite element and the general theory of mixed Galerkin method for Stokes equations, see [Girault-Raviart], Theorem II.1.8. The second inequality follows from the general theory of finite element approximation of V-elliptic second order partial differential equations [Ciarlet]. □

We are now in position to prove the main result of this section, concerning approximation of branches of nonsingular solutions of (4.2) (cf. [Girault-Raviart], Section IV.3.1). This theorem also provides local uniqueness result.

Theorem 4.6. *Let Λ be a compact interval in R_+ and assume that*

$$\{(\lambda, (u(\lambda), \lambda p(\lambda), \omega(\lambda)) : \lambda \in \Lambda\}$$

is a branch of nonsingular solutions of (4.2) such that $\mathcal{U}(\lambda) = (u(\lambda), \lambda p(\lambda), \omega(\lambda))$ satisfies

$$u(\lambda) \in H^{l+1}, \quad p(\lambda) \in H^l, \quad \omega(\lambda) \in H^{l+1}$$

for $l = 1$ or $l = 2$ and for all $\lambda \in \Lambda$. Then there exists a neighbourhood $\mathcal{O}$ of the origin in $V\times W\times V$ and h_0 (small enough), such that for $h \le h_0$ there exists a unique smooth function $\lambda \in \Lambda \to (u_h(\lambda), \lambda p_h(\lambda), \omega_h(\lambda)) \in V_h \times W_h \times V_h$ such that:

(i) $\{(\lambda,(u_h(\lambda),\lambda p_h(\lambda),\omega_h(\lambda)):\lambda\in\Lambda\}$, *is a branch of nonsingular solutions of (4.5),*
(ii) $(u(\lambda),\lambda p(\lambda),\omega(\lambda))-(u_h(\lambda),\lambda p_h(\lambda),\omega_h(\lambda))\in\mathcal{O}$ *for all* $\lambda\in\Lambda$,
(iii) *there exists a constant* $C>0$, *independent of* h *and* λ, *such that*

$$|u_h(\lambda)-u(\lambda)|_1+|\lambda|||p_h(\lambda)-p(\lambda)||_0+|\omega_h(\lambda)-\omega(\lambda)|_1 \\ \leq Ch^l(||u(\lambda)||_{l+1}+|\lambda|||p(\lambda)||_l+||\omega(\lambda)||_{l+1})$$

for all $\lambda\in\Lambda$.

Proof. The theorem follows from Theorem IV.3.3. in [Girault-Raviart]. The mapping G is C^∞-regular and satisfies assumptions of this theorem, due to Lemma 4.3 together with compact imbedding $L^{3/2}\hookrightarrow\hookrightarrow H^{-1}$ and by (4.9), (4.10) and (4.11). Lemma 4.5 guarantees that T_h posseses required approximating properties. $\square$

5. Newton's method for solving the discrete problem.

Problem 2.2, obtained from finite element discretization of Problem 2.1, corresponds to equation (4.5) in a finite dimensional space. In this section we present analysis of the Newton's method for solving this equation for fixed $\lambda\in\Lambda$, where Λ is a compact interval in R_+.

We shall consider an iterative method for solving (2.5), derived from the Newton's iteration for (4.5). Starting from an initial guess $\mathcal{U}_h^0=(u_h^0,\lambda p_h^0,\omega_h^0)$ we define the iteration step $\mathcal{U}_h^n\mapsto\mathcal{U}_h^{n+1}$ as the solution of the following (linear) equation,

$$\mathcal{U}_h^{n+1}=\mathcal{U}_h^n-[D_\mathcal{U}F_h(\lambda,\mathcal{U}_h^n)]^{-1}\cdot F_h(\lambda,\mathcal{U}_h^n),\qquad n=0,1,\ldots$$

which is, in our case, equivalent to solving some variational problem. This yields the iterative method for solution of Problem 2.2.

Newton's algorithm. Given $(u_h^n,p_h^n,\omega_h^n)\in V_h\times W_h\times V_h$, find $(u_h^{n+1},p_h^{n+1},\omega_h^{n+1})\in V_h\times W_h\times V_h$, which satisfies

$$\tag{5.1}\begin{cases}(\nu+\nu_r)(\nabla u_h^{n+1},\nabla v)+d(u_h^n,\,u_h^{n+1},\,v)+d(u_h^{n+1},\,u_h^n,\,v)-(p_h,\operatorname{div}v)\\ \qquad=2\nu_r(\operatorname{curl}\omega_h^{n+1},v)+d(u_h^n,\,u_h^n,\,v)+(f,v),\\ (\operatorname{div}u_h^{n+1},q)=0,\\ c_1(\nabla\omega_h^{n+1},\nabla\xi)+d(u_h^n,\,\omega_h^{n+1},\,\xi)+d(u_h^{n+1},\,\omega_h^n,\,\xi)+c_2(\operatorname{div}\omega_h^{n+1},\operatorname{div}\xi)\\ \qquad+4\nu_r(\omega_h^{n+1},\xi)=2\nu_r(\operatorname{curl}u_h^{n+1},\xi)+d(u_h^n,\,\omega_h^n,\,\xi)+(g,\xi)\end{cases}$$

for all $(v,q,\xi)\in V_h\times W_h\times V_h$. The following theorem guarantees local quadratic convergence of algorithm (5.1) to the solution of the discrete system. The speed of convergence is affected by parameter λ, but is not dependent on the mesh parameter h.

Theorem 5.1. *Assume that* $\mathcal{U}_h(\lambda)=(u_h(\lambda),\lambda p_h(\lambda),\omega_h(\lambda))$, *for* $\lambda\in\Lambda$, *is as in Theorem 4.6 with* h_0 *sufficiently small. For fixed* λ *in* Λ *there exists* $r>0$ *such that for all* $h\leq h_0$ *the Newton's method defines a unique sequence* $\{\mathcal{U}_h^n\}_{n\in N}$, *which is convergent to* $\mathcal{U}_h$ *and all contained in a ball* $B(\mathcal{U}_h,r)$, *provided the initial guess* $\mathcal{U}_h^0$ *lies in* $B(\mathcal{U}_h,r)$. *Moreover, the convergence speed is quadratic,*

$$||\mathcal{U}_h^{n+1}-\mathcal{U}_h||_{V\times W\times V}\leq C||\mathcal{U}_h^{n+1}-\mathcal{U}_h||^2_{V\times W\times V},$$

with constant C *independent of* h.

Proof. It suffices to check the assumptions of classical theorem on the convergence of Newton's method in finite dimensional Banach space for the case when the existence of solution is *a priori* guaranteed. Since $\mathcal{U}_h(\lambda)$ lies on a branch of nonsingular solutions, then $D_\mathcal{U}F_h(\lambda,\mathcal{U}_h(\lambda))$ is an isomorphism on $V_h\times W_h\times V_h$, moreover,

$$\tag{5.2}||[D_\mathcal{U}F_h(\lambda,\mathcal{U}_h(\lambda))]^{-1}||_{L(V\times W\times V)}\leq C_\lambda$$

and C_λ is independent of h (see (3.32) in Chapter IV, Paragraph 3 in [Girault-Raviart]). Then (5.2) yields

$$||[D_\mathcal{U}F_h(\lambda,\mathcal{U})]^{-1}||_{L(V\times W\times V)}\leq 2C_\lambda,$$

provided $U \in V_h \times W_h \times V_h$ is such that

$$||D_{\mathcal{U}} F_h(\lambda, \mathcal{U}) - D_{\mathcal{U}} F_h(\lambda, \mathcal{U}_h(\lambda))||_{L(V\times W\times V)} < \frac{1}{2C_\lambda}.$$

This holds for $||\mathcal{U} - \mathcal{U}_h(\lambda)||_{V\times W\times V} \leq r'$ with $r' > 0$ sufficiently small. Then the theorem follows from the fact that the second derivative $D^2_{\mathcal{U}\mathcal{U}} F_h(\lambda, \mathcal{U})$ is independent of $\mathcal{U}$ and bounded independently of h. $\square$

6. A preconditioning method for Newton's iteration step.

In this section we propose and analyse a preconditioning method for one step of Newton's method, described in the previous section. Performing one step of Newton's algorithm,

$$(u_h^n, p_h^n, \omega_h^n) \mapsto (u_h^{n+1}, p_h^{n+1}, \omega_h^{n+1}),$$

requires solution of variational problem (5.1). In fact, we are actually dealing with a family of such problems, indexed by the mesh parameter h. In this section we are concerned with preconditioning these problems with respect to h, so that the resulting (preconditioned) problem is given by a linear symmetric positive definite operator whose condition number is independent of h.

Problem (5.1) may be stated in another form (for short, we denote $(u_h^{n+1}, p_h^{n+1}, \omega_h^{n+1})$ by (u, p, ω), keeping in mind that (u, p, ω) depends on h and n)

$$\begin{cases} a(u,\, v) + b(v,\, p) + t_a(\omega,\, v) = \langle\langle \phi, v\rangle\rangle, \\ b(u,\, w) = \langle\langle \rho, w\rangle\rangle, \\ c(\omega,\, \xi) + t_c(u,\, \xi) = \langle\langle \psi, \xi\rangle\rangle, \end{cases} \tag{6.1}$$

with (v, w, ξ) varying in $V_h \times W_h \times V_h$. Here $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the dual pairing between V_h and V_h', or W_h and W_h', respectively. We use the same symbol for these two different pairings, since it will be always clear form the context, which of them we have in mind.

In our case, $a(\cdot,\,\cdot) : V \times V \to R$ is a continuous, bilinear form, defined as

$$a(u,\, v) \equiv (\nu + \nu_r)(\nabla u, \nabla v) + d(u_h^n,\, u,\, v) + d(u,\, u_h^n,\, v),$$

$b(\cdot,\,\cdot) : V \times W \to R$ is a continuous, bilinear form, defined as

$$b(u,\, w) \equiv -(\operatorname{div} u, w),$$

$c(\cdot,\,\cdot) : V \times V \to R$ is a continuous, bilinear form, defined as

$$c(\omega,\, \xi) \equiv c_1(\nabla\omega, \nabla\xi) + d(u_h^n,\, \omega,\, \xi) + c_2(\operatorname{div}\omega, \operatorname{div}\xi) + 4\nu_r(\omega, \xi),$$

$t_a(\cdot,\,\cdot) : V \times V \to R$ is a continuous, bilinear form, defined as

$$t_a(\omega,\, v) \equiv -2\nu_r(\operatorname{curl}\omega, v),$$

$t_c(\cdot,\,\cdot) : V \times V \to R$ is a continuous, bilinear form, defined as

$$t_c(u,\, \xi) \equiv -2\nu_r(\operatorname{curl} u, \xi) + d(u,\, \omega_h^n,\, \xi).$$

Lemma 6.1. *Suppose that the assumptions of Theorem 5.1 hold with $h_0 > 0$ and $r > 0$ sufficiently small. Then for any $(\phi, \rho, \psi) \in V_h' \times W_h' \times V_h'$ there exists a unique solution (u_h, p_h, ω_h) of (6.1) and there holds*

$$||u_h||_1 + ||p_h||_0 + ||\omega_h||_1 \leq C(||\phi||_{V_h'} + ||\rho||_{W_h'} + ||\psi||_{V_h'})$$

with C independent of ϕ, ρ, ψ and h.

Proof. The lemma follows from the fact that under assumptions of Theorem 5.1, the derivative of F_h is an isomorphism, whose inverse is bounded independently of h. $\square$

Let us remark, that if (6.1) is used to make the Newton's iteration step of (5.1), then the right hand side of the problem is

$$\begin{aligned}
\langle\langle \phi, v\rangle\rangle &\equiv d(u_h^n,\, u_h^n,\, v) + (f, v),\\
\langle\langle \rho, w\rangle\rangle &\equiv 0,\\
\langle\langle \psi, \xi\rangle\rangle &\equiv d(u_h^n,\, \omega_h^n,\, \xi) + (g, \xi),
\end{aligned}$$

so that $\phi, \psi \in H^{-1} \subset V_h'$.

Let us define linear operators corresponding to forms introduced above,

$$\begin{array}{lll}
A : V_h \to V_h', & \langle\langle Au, v\rangle\rangle = a(u,\, v) & \forall u, v \in V,\\
B : V_h \to W_h, & \langle\langle Bu, w\rangle\rangle = b(u,\, w) & \forall w \in W, u \in V,\\
C : V_h \to V_h', & \langle\langle Cu, v\rangle\rangle = c(u,\, v) & \forall u, v \in V,\\
T_A : V_h \to V_h', & \langle\langle T_A u, v\rangle\rangle = t_a(u,\, v) & \forall u, v \in V,\\
T_C : V_h \to V_h', & \langle\langle T_C u, v\rangle\rangle = t_c(u,\, v) & \forall u, v \in V,
\end{array}$$

and their duals

$$\begin{array}{ll}
A' : V_h \to V_h', & \langle\langle A'u, v\rangle\rangle = a(v,\, u),\\
B' : W \to V_h', & \langle\langle B'p, v\rangle\rangle = b(v,\, p),\\
C' : V_h \to V_h', & \langle\langle C'u, v\rangle\rangle = c(v,\, u),\\
T_A' : V_h \to V_h', & \langle\langle T_A' u, v\rangle\rangle = t_a(v,\, u),\\
T_C' : V_h \to V_h', & \langle\langle T_C' u, v\rangle\rangle = t_c(v,\, u).
\end{array}$$

Then we may give (6.1) another formulation:

Find (u, p, ω) in $V_h \times W_h \times V_h$, such that

$$\left\{\begin{aligned}
Au + B'p + T_A\omega &= \phi && \text{in } V_h',\\
Bu &= \rho && \text{in } W_h',\\
C\omega + T_C u &= \psi && \text{in } V_h'.
\end{aligned}\right. \tag{6.2}$$

6.1. Preconditioned system. Before we proceed, let us introduce some more notation. The inner product in W is obviously the L^2-product $(\cdot,\cdot)$. We remind that we define the inner product in V as

$$((u, v)) := (\nabla u, \nabla v)$$

for $u, v \in V$. The inner products in W_h, V_h are then inherited from W, V. With the inner product $((\cdot,\cdot))$ we associate the discrete Laplace operator, $\Delta_h : V_h \to V_h'$, defined by

$$\langle\langle \Delta_h u, v\rangle\rangle \equiv ((u, v)).$$

We also define canonical mapping $J : W_h \to W_h'$ (the "mass matrix" operator),

$$\langle\langle Jp, q\rangle\rangle \equiv (p, q).$$

Then its inverse, $J^{-1} : W_h' \to W_h$ obviously satisfies $(J^{-1}g, p) = \langle\langle g, p\rangle\rangle$ for all $p \in W_h$.

Let $A_0 : V_h \to V_h'$ be a good preconditioner for the discrete Laplace operator Δ_h, cf. [Bramble-Pasciak1], so that

(i) A_0 is easy to invert,
(ii) $\langle\langle A_0 u, v\rangle\rangle = \langle\langle A_0 v, u\rangle\rangle$ for all $u, v \in V_h$,

(iii) there exist constants $\alpha_1, \alpha_2 > 0$, independent of h, such that

$$\alpha_1 \langle\langle \Delta_h u, u\rangle\rangle \le \langle\langle A_0 u, u\rangle\rangle \le \alpha_2 \langle\langle \Delta_h u, u\rangle\rangle \tag{6.3}$$

for all $u \in V_h$.

Similarly, we introduce (cf. [Klawonn]) a preconditioner for the "mass matrix" operator, $J_0 : W_h \to W_h'$, with properties:

(iv) J_0 is easy to invert,
(v) $\langle\langle J_0 p, q\rangle\rangle = \langle\langle J_0 q, p\rangle\rangle$ for all $p, q \in W_h$,
(vi) there exist constants $\beta_1, \beta_2 > 0$, independent of h, such that

$$\beta_1 \langle\langle J p, p\rangle\rangle \le \langle\langle J_0 p, p\rangle\rangle \le \beta_2 \langle\langle J p, p\rangle\rangle \tag{6.4}$$

for all $p \in W_h$.

Introducing the notation

$$\mathcal{M}_0^{-1} = \begin{pmatrix} A_0^{-1} & 0 & 0 \\ 0 & J_0^{-1} & 0 \\ 0 & 0 & A_0^{-1} \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} A & B' & T_A \\ B & 0 & 0 \\ T_C & 0 & C \end{pmatrix}, \tag{6.5}$$

we define a preconditioned version of (6.2) as follows, obtaining yet another form of problem (5.1).

Problem 6.1. Find $(u, p, \omega) \in V_h \times W_h \times V_h$ such that

$$\mathcal{M}_0^{-1}\mathcal{M}'\mathcal{M}_0^{-1}\mathcal{M} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} = \mathcal{F} \qquad \text{in } V_h \times W_h \times V_h,$$

where

$$\mathcal{F} = \mathcal{M}_0^{-1}\mathcal{M}'\mathcal{M}_0^{-1} \begin{pmatrix} \phi \\ \rho \\ \psi \end{pmatrix}.$$

Let us introduce the inner product on $V_h \times W_h \times V_h$, denoted by $[\cdot,\cdot]$,

$$\left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} v \\ q \\ z \end{pmatrix} \right] \equiv \langle\langle A_0 u, v\rangle\rangle + \langle\langle J_0 p, q\rangle\rangle + \langle\langle A_0 \omega, z\rangle\rangle. \tag{6.6}$$

Lemma 6.2. *The operator $\mathcal{P} = \mathcal{M}_0^{-1}\mathcal{M}'\mathcal{M}_0^{-1}\mathcal{M}$ with $\mathcal{M}_0$ and $\mathcal{M}$ defined by (6.5) is a self-adjoint operator with respect to the scalar product $[\cdot,\cdot]$ defined in (6.6).*

Proof. Indeed, $\mathcal{P}$ may be written as

$$\begin{aligned} \mathcal{P} = & \begin{pmatrix} A_0^{-1} & 0 & 0 \\ 0 & J_0^{-1} & 0 \\ 0 & 0 & A_0^{-1} \end{pmatrix} \times \\ & \times \begin{pmatrix} A'A_0^{-1}A + B'J_0^{-1}B + T_C'A_0^{-1}T_C & A'A_0^{-1}B' & A'A_0^{-1}T_A + T_C'A_0^{-1}C \\ BA_0^{-1}A & BA_0^{-1}B' & BA_0^{-1}T_A \\ T_A'A_0^{-1}A + C'A_0^{-1}T_C & T_A'A_0^{-1}B' & T_A'A_0^{-1}T_A + C'A_0^{-1}C \end{pmatrix}. \end{aligned} \tag{6.7}$$

□

The main result of this section is an estimate of the condition number of operator $\mathcal{P}$ in the norm induced by $[\cdot,\cdot]$.

Theorem 6.3. *Let $\mathcal{P}$ be defined as Lemma 6.2. Suppose Lemma 6.1 holds and A_0, J_0 satisfy assumptions (i) – (vi) of this section. Then there exist positive constants m_1, m_2, independent of h, but dependent on $\alpha_1, \alpha_2, \beta_1, \beta_2$, such that*

$$m_1 \left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] \le \left[\mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] \le m_2 \left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] \tag{6.8}$$

for any $(u,p,\omega) \in V_h \times W_h \times V_h$.

Proof. By (6.6) and (6.7), we have

$$\begin{aligned} \left[\mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] &= \langle\langle Au, A_0^{-1} Au \rangle\rangle + \langle\langle Bu, J_0^{-1} Bu \rangle\rangle + \langle\langle T_C u, A_0^{-1} T_C u \rangle\rangle \\ &+ 2 \langle\langle Au, A_0^{-1} B'p \rangle\rangle + \langle\langle B'p, A_0^{-1} B'p \rangle\rangle \\ &+ 2 \langle\langle Au, A_0^{-1} T_A \omega \rangle\rangle + 2 \langle\langle T_C u, A_0^{-1} C\omega \rangle\rangle \\ &+ 2 \langle\langle B'p, A_0^{-1} T_A \omega \rangle\rangle + \langle\langle T_A \omega, A_0^{-1} T_A \omega \rangle\rangle \\ &+ \langle\langle C\omega, A_0^{-1} C\omega \rangle\rangle \end{aligned}$$

whence

$$\begin{aligned} \left[\mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] &= \langle\langle Au + B'p + T_A\omega, A_0^{-1}(Au + B'p + T_A\omega) \rangle\rangle + \langle\langle Bu, J_0^{-1} Bu \rangle\rangle \\ &+ \langle\langle T_C u + C\omega, A_0^{-1}(T_C u + C\omega) \rangle\rangle. \end{aligned} \tag{6.9}$$

Since there exist constants $\gamma_1, \gamma_2 > 0$, and $\delta_1, \delta_2 > 0$, independent of h, such that

$$\begin{aligned} \gamma_1 ||\phi||^2_{V_h'} \le \langle\langle \phi, A_0^{-1} \phi \rangle\rangle \le \gamma_2 ||\phi||^2_{V_h'} \qquad \forall \phi \in V_h', \\ \delta_1 ||q||_0^2 \le \langle\langle q, J_0^{-1} q \rangle\rangle \le \delta_2 ||q||_0^2 \qquad \forall q \in W_h', \end{aligned} \tag{6.10}$$

then we obtain a lower bound,

$$\left[\mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] \ge \gamma_1 ||Au + B'p + T_A\omega||^2_{V_h'} + \delta_1 ||Bu||_0^2 + \gamma_1 ||T_C u + C\omega||^2_{V_h'}.$$

Now, Lemma 6.1 implies that if (u,p,ω) solves

$$\begin{cases} Au + B'p + T_A\omega = \tilde{\phi} & \text{in } V_h', \\ Bu = \tilde{\rho} & \text{in } W_h', \\ C\omega + T_C u = \tilde{\psi} & \text{in } V_h', \end{cases}$$

then

$$||u||_1 + ||p||_0 + ||\omega||_1 \le C(||\tilde{\phi}||_{V_h'} + ||\tilde{\rho}||_{W_h'} + ||\tilde{\psi}||_{V_h'}) \tag{6.11}$$

with C independent of h. Therefore, taking (6.11), (6.3) and (6.4) into account, we obtain

$$\left[\mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] \ge C \left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right].$$

Similarly we can prove the upper bound. Indeed, (6.10) together with (6.9) yield

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \leq \gamma_2||Au + B'p + T_A\omega||^2_{V_h'} + \delta_2||Bu||^2_0 + \gamma_2||T_C u + C\omega||^2_{V_h'}$$
$$\leq C(||Au||^2_{V_h'} + ||B'p||^2_{V_h'} + ||T_A\omega||^2_{V_h'} + ||Bu||^2_0 + ||T_C u||^2_{V_h'} + ||C\omega||^2_{V_h'}).$$

Estimating each term in the sum we arrive at

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \leq C\left[\left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right]\right].$$

This completes the proof. □

Remark 6.1. Since the resulting system is symmetric and positive definite, we can apply conjugate gradient algorithm for solving it iteratively. By Theorem 6.3, the number of iterations required for reducing the residual by a given factor is independent of h. As it has been pointed out by [Bramble-Pasciak1], computing the inner product during the CG step requires only one solution of a system with the operator

$$\begin{bmatrix} A_0 & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & A_0 \end{bmatrix}.$$

Remark 6.2. Many authors contributed to the area of numerical solution of saddle point problems, just to mention a few [D'yakonov], [Bramble-Pasciak1], [Bramble-Pasciak2] [Rusten-Winther], [Silvester-Wathen], [Elman-Silvester], [Klawonn], addressing mostly (if not exclusively) the symmetric operator case. The idea of symmetrizing the saddle point system with the aid of a preconditioner for the Laplacian has been considered previously in, for example, [D'yakonov] and [Bramble-Pasciak1]. However, our analysis remains valid for nonsymmetric operators.

Since in case of $v_r = c_1 = c_2 = 0$ our system reduces to Newton's linearization of the Navier-Stokes equations, our preconditioner applies also to this particular case. Moreover, this preconditioning method generalizes to the case of abstract saddle point equations with nonsymmetric, indefinite diagonal part. Different preconditioning methods for these problems will be analysed in a forthcoming report.

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