# Mathematical and numerical analysis of problems in microcontinuum mechanics

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## Abstract

In this thesis, we discuss and solve certain open problems in microcontinuum mechanics, focusing ourselves on mathematical models of micropolar fluids. These models have many important applications, e.g. in lubrication theory, blood rheology, or in modelling of suspensions.

In the first part, we prove existence, regularity and uniqueness of solutions of a differential problem describing a stationary flow of a mixture of micropolar fluids with barodiffusion. Using Schauder fixed point argument, we prove the existence result in Sobolev spaces and derive the regularity and norm estimates of solutions. Next, applying those properties, we show the uniqueness of solutions under additional assumptions on the data and parameters of the problem.

The second part is devoted to the analysis of a mixed finite element method for a stationary micropolar fluid flow. Using the nonsingular branches theory, we prove that the discrete problem has a locally unique solution, with optimal approximation properties. Then, we analyse the Newton method for solving the discrete nonlinear algebraic problem and prove that under the nonsingularity assumption the method has optimal convergence rate.

In the last part of the thesis, we propose and analyse a class of parallel numerical algorithms for nonsymmetric saddle point problems encountered in computational fluid dynamics. We study block preconditioners, which may be based on the domain decomposition method for elliptic problems and we prove their optimality with respect to the mesh parameter. We apply this approach for solving the micropolar equations and present numerical experiments which confirm theoretical results.

Most of the results of this thesis have been published or accepted for publication, or appeared as technical reports. The first part of the thesis is based on the article [**57**]. The results concerning the mixed finite elements for micropolar equations and block diagonal preconditioners, previously presented in a report [**59**], have been accepted for publication in [**61**]. Abstract analysis of various block preconditioners for nonsymmetric problems has appeared in [**60**].

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# Contents

CHAPTER 1

# Overview

Nonlinear systems of partial differential equations play a vital role in mathematical modelling of numerous phenomena which have a lot of applications in the industry and the natural sciences. A very important class of such systems comes from mathematical models of a viscous fluid flow. Stimulated by numerous applications, there is a growing interest in the analysis and numerical solution of *complex* fluid flows. This covers:

- investigation of their mathematical properties, such as the existence, uniqueness, regularity and stability, etc.;
- discretization of the governing equations by finite difference, finite element or other techniques, and analysis of resulting finite dimensional problems;
- construction, analysis and implementation of effective algorithms for computation of the numerical solution.

In this thesis, we are going to address certain problems concerning these three areas and related to mathematical models of fluids with microstructure. The motion of this important class of fluids cannot be described by the classical Navier–Stokes equation. In the following sections of this chapter, we introduce the notion of a microcontinuum fluid and present an overview of the main results of this thesis, with their relations to other works in this field.

## 1.1. Fluids with microstructure

The motion of a viscous, incompressible fluid with constant density is described by the *Navier–Stokes* system of partial differential equations[1] :

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ \operatorname{div} u = 0. \end{cases} \tag{1.1.1}$$

The kinematic viscosity is denoted by $\nu$ and the unknown functions are the velocity of the fluid $u$ and the pressure $p$. Wide applicability of (1.1.1) in fluid flow modelling makes it an object of particular interest to the mathematicians. Although the analysis

[1]For simplicity, we assume that the density is equal to one.

of (1.1.1) is not straightforward, there has been a lot of mathematical work done on the Navier–Stokes system during past decades, both in the stationary and the evolutionary case. The most important results for the Navier–Stokes and Stokes equations are collected in monographs [**91**], [**46**], [**44**].

The Navier–Stokes system is insufficient for modelling complex fluids such as suspensions. Among many models which take some insight beyond the classical Navier–Stokes equations, there is a *microcontinuum fluid* model [**42**], [**3**]. In this model, developed in the 1960s, it is assumed that any volume element ("particle") of the continuous medium contains many microelements ("subscale structure") which can translate, rotate and even deformate independently of the macrovolume motion.

It has been verified that the microcontinuum approach leads to better understanding of some physical problems, see [**3**], [**4**], [**51**] and the references therein. Also, the experiments show good agreement with theoretical predictions. At present, successful applications of the microcontinuum fluid theory cover

- lubrication theory,
- blood rheology,
- analysis of behaviour of certain polymer solutions,
- liquid crystals properties.

The general microfluid as introduced by Eringen in [**42**] is characterized by 22 viscosity and material parameters and contains 19 unknowns. Many subclasses obtained by simplification of the microcontinuum model have been considered by numerous authors, see [**3**], [**4**]. An interesting and useful class is derived by neglecting the deformation of microelements. This leads to so called *micropolar fluid*, a stationary motion which is governed by the following system of partial differential equations:

$$\begin{cases} -(\nu+\nu_r)\Delta u + (u\cdot\nabla)u + \nabla p = 2\nu_r \operatorname{curl}\omega + f, \\ \\ \operatorname{div} u = 0, \\ \\ -(c_a+c_d)\Delta\omega + (u\cdot\nabla)\omega - (c_0+c_d-c_a)\nabla \operatorname{div}\omega + 4\nu_r\omega = \\ \\ \qquad = 2\nu_r \operatorname{curl} u + z. \end{cases} \tag{1.1.2}$$

The new unknown $\omega$ represents the microrotation in the fluid. The constant parameters $\nu_r, c_0, c_a, c_d$ denote additional viscosities derived by physical considerations. The importance of the micropolar fluid model has been recognized several years ago in the lubrication theory (see for example [**50**], [**51**], [**10**]). It has been observed that a bearing lubricated with a micropolar fluid shows higher load carrying capacity than a bearing with usual Newtonian fluid. This fact is being used in the design of modern lubricants: their performance is enhanced by specific polymer additives or minute balls suspended in the oil.

Recently, the governing system of partial differential equations for the micropolar fluid has also gained the interest of the mathematicians. The existence, uniqueness and regularity of stationary and evolutionary micropolar equations with Dirichlet boundary conditions have been studied in [**69**], [**68**]. Moreover, mathematical properties of a micropolar fluid in a porous medium have been considered in [**2**].

## 1.2. Micropolar fluids with barodiffusion

More complicated systems arise when, in addition to micropolar equations (1.1.2), diffusion effects are taken into account. In certain important cases, for example in the blood flow, the diffusion of red blood cells is driven not by the temperature but by the gradient of the pressure. Such flows are called *flows with barodiffusion*. A relatively simple model of a mixture of two micropolar fluids with barodiffusion is described by the following set of partial differential equations [**80**], [**81**], [**70**]:

$$
\begin{cases}
-(\nu+\nu_r)\Delta u + (u\cdot\nabla)u + \nabla p = 2\nu_r \operatorname{curl}\omega + f + cg, \\[2ex]
\operatorname{div} u = 0, \\[2ex]
-(c_a+c_d)\Delta\omega + (u\cdot\nabla)\omega - (c_0+c_d-c_a)\nabla \operatorname{div}\omega + 4\nu_r\omega = \\[2ex]
\qquad = 2\nu_r \operatorname{curl} u + z, \\[2ex]
-\operatorname{div}(D(c)\nabla c) + u\cdot\nabla c = \operatorname{div}(K(c)\nabla p).
\end{cases}
\tag{1.2.3}
$$

Here, $D$ and $K$ denote diffusion coefficients, dependent on the unknown concentration $c$.

Up to now, only a simplified model with barodiffusion has been mathematically analysed in [**70**], neglecting the effects of microrotation. In Chapter 2 we show that it is possible to obtain similar results for the full system (1.2.3).

We prove theorems on the existence, uniqueness and regularity of weak solutions of a Dirichlet boundary value problem for (1.2.3), in certain Sobolev spaces, see Theorem 2.2.1 and Theorem 2.2.2. We assume that $D$ is continuous, bounded and positive, and $K$ is Lipschitz continuous and satisfies additional condition given by physical considerations. In our proofs, we take advantage of relations between different coefficients in the model. The equations are coupled by parameter $\nu_r$, which stands for additional viscosity introduced by the microrotation of particles. An interesting observation is that the way this parameter enters the system is crucial for the analysis of (1.2.3).

The results of Chapter 2 have been published in [**57**]. They generalize earlier developments for Newtonian fluids with barodiffusion [**70**] and extend those for micropolar fluids [**69**] to this more complex case.

The proof of the existence result in Chapter 2 is based on the application of fixed point argument to a mapping $\Phi(u^*, c^*) = (u, c)$, obtained from a partial linearization of (1.2.3):

$$
\begin{cases}
-(\nu + \nu_r)\Delta u + (u^* \cdot \nabla)u + \nabla p = 2\nu_r \operatorname{curl} \omega + f + c^* g, \\
\\
\operatorname{div} u = 0, \\
\\
-(c_a + c_d)\Delta\omega + (u^* \cdot \nabla)\omega - (c_0 + c_d - c_a)\nabla \operatorname{div} \omega + 4\nu_r\omega = \\
\\
\qquad = 2\nu_r \operatorname{curl} u^* + z, \\
\\
-\operatorname{div}(D(c^*)\nabla c) + u^* \cdot \nabla c = \operatorname{div}(K(c)\nabla p).
\end{cases}
$$

In subsequent steps, we examine certain properties of $\Phi$, verifying the assumptions of Schauder fixed point theorem. This requires careful norm estimates on the components of the linearized solution and taking the advantage of precise relations between different coefficients in (1.2.3). The procedure also leads to a regularity result for the solution of the nonlinear problem. The proof of the uniqueness requires some more regularity assumptions and is based on estimating the difference between two solutions corresponding to the same data. Similarly to simpler cases of the stationary Navier–Stokes equations [**91**] or the micropolar system [**69**], we prove the uniqueness under additional assumption that the external forces $f$, $g$, $z$ are relatively small.

## 1.3. Numerical analysis and algorithms

Many applications of the microcontinuum model require numerical simulations in order to study the behaviour of solution of a particular problem. Furthermore, to justify the theory, comparisons should be made between the numerical solutions and the experimental data.

However, even the Navier–Stokes system is a challenging problem for the numerical computation. The problem is nonlinear, there are several unknown fields to calculate (ten for a 3-D flow) and the continuity constraint $\operatorname{div} u = 0$ lead to additional numerical difficulties. Sometimes the flow pattern can be very complex (turbulence, boundary layers), demanding fine meshing with a huge number of nodes and using special discretization techniques.

Numerical analysis of the Navier–Stokes system, including the existence, approximation and stability is well developed, see for example the classical monograph by Girault and Raviart [**46**]. The first papers analysing the finite difference method for the stationary and evolutionary Navier–Stokes system appeared in late 1960s, see [**64**], [**91**]. Starting from works by Ladyzhenskaya [**63**], Babuška [**5**] and Brezzi [**20**], it is well known that the primitive variable formulation of the Stokes system as a saddle point problem requires a balance between the spaces where the velocity and the pressure are approximated. Recently, some authors proposed alternative methods, see [**43**], [**75**], circumventing the Ladyzhenskaya–Babuška–Brezzi condition by using a Galerkin/Least–Squares technique, while still retaining good approximation and stability properties. Among other techniques there are the formulation of a least squares problem for a first order system derived from (1.1.1), see [**12**], [**26**], [**18**]; the reformulation of the problem in new unknowns; the use of a nonlinear conjugate gradient method or a projection method, and many more, see [**46**] or [**82**] and references therein.

**1.3.1. Mixed FEM for the micropolar equations.** The micropolar system (1.1.2) contains the Navier–Stokes equations, so all the numerical difficulties related to the latter remain. Some results of numerical simulations obtained for the micropolar equations can be found in [**10**] and [**51**], however, without rigorous mathematical justification of numerical schemes being used.

In Chapter 3, we provide an in-depth analysis of a mixed finite element approximation to the micropolar system (1.1.2) with Dirichlet boundary conditions, for possibly wide class of data, see Theorem 3.5.1 and Theorem 3.6.6. We restrict ourselves only to pure micropolar fluid, i.e. without barodiffusion effects. The main goal of Chapter 3 is to provide conditions under which the discrete problem is solvable and has good approximation properties. At first we prove a general existence result for the discrete equations. The key difficulty is the nonlinearity of the equations and therefore their nonuniqueness in general. To overcome this problem, we consider an approximation of a nonsingular branch of solutions of (1.1.2). We generalize the classical result for the Navier–Stokes equations by Brezzi, Rappaz and Raviart, see [**46**, Chapter IV, Section 3], to the micropolar fluids. In Theorem 3.6.6, we prove that far from the bifurcation point the discrete problem is locally uniquely solvable and it has optimal approximation properties.

Under the same nonsingularity assumptions, we study the Newton method for the discrete system, showing its local quadratic convergence in an appropriate norm, see Theorem 3.8.1.

The results from Chapter 3 have appeared in the form of a technical report [**59**] and have been submitted for publication.

**1.3.2. Efficient parallel block preconditioners.** From the application point of view it is very important to provide an efficient algorithm for solving the discrete nonlinear system. Since the advent of massively parallel computers, very complex partial differential equations can – potentially – be solved within a reasonable time. This aim can only be achieved if fast and scalable parallel algorithms are implemented to solve the partial differential equation, see [**78**], [**31**]. We keep this strategy in mind when developing in Chapters 4 and 5 numerical algorithms for solving algebraic problems arising from the discretization of differential problems considered in this thesis.

It is well realized that the use of iterative methods is vital for solving systems of linear algebraic equations encountered in the Navier–Stokes system solution. However, these problems are usually very ill conditioned with respect to the mesh size $h$ and the Reynolds number as well, so that a good preconditioner is required. Let us consider the Navier–Stokes system (1.1.1) discretized in the primitive variables $(u, p)$. Then the Newton method, and some other linearizations as well, lead to a sequence of saddle point problems of the following block structure:

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

Due to the presence of convective terms in the Newton linearization, the matrix $A$ is nonsymmetric. In the symmetric, positive definite case ($A = A^T > 0$), the theory for the block preconditioners of the form:

$$\begin{pmatrix} A_0 & 0 \\ 0 & J_0 \end{pmatrix}, \qquad \begin{pmatrix} A_0 & 0 \\ B & -J_0 \end{pmatrix},$$

has successfully been developed, starting from papers by D'yakonov [**35**] and Bramble and Pasciak [**16**], [**17**] with recent contributions in the 1990s by Elman, Rusten, Silvester, Wathen and Winther [**85**], [**38**], [**39**], [**83**] and by Klawonn [**52**], [**53**], [**54**]. It has been shown in [**52**] that it is possible to use fast block preconditioners $A_0, J_0$ based on the domain decomposition or the multigrid method, or on the incomplete factorizations [**85**]. Domain decomposition algorithms, such as the additive Schwarz method [**34**], [**33**] are very efficient parallel methods, so implementing them as preconditioners $A_0$ and $J_0$ results in an algorithm which has both a high level of parallelism and a good (small) condition number.

There are also numerical indications that block preconditioners are robust in the (nonsymmetric) case of the Oseen equation, which is kind of a linearized Navier–Stokes equation [**37**], [**56**]. However, except [**56**] and [**40**], where some special cases are discussed, there is no general theory explaining this behaviour. Therefore, in Chapter 4 we propose and analyse efficient algorithms for the numerical solution of *nonsymmetric saddle point problems*. The new general theory on the block preconditioners for nonsymmetric saddle point problems presented in Chapter 4 focuses on the diagonal or triangular preconditioners based on symmetric diagonal blocks. We show that classical preconditioners for the Laplacian and the mass matrix are sufficient for obtaining a system with the condition number bounded independently of the mesh parameter $h$, see Theorems 4.5.1, 4.5.2 and 4.6.1.

We believe that this approach may be more robust than designing a custom method for solving the nonsymmetric saddle point problem. Block preconditioners are easy to construct and to implement. Moreover, one can build the preconditioner basing on well known domain decomposition algorithms for *symmetric positive definite* problems, so this strategy leads the application programmer to efficient code reuse.

In Chapter 5 we apply block preconditioners to systems arising from the linearization of the micropolar equations and analyse them in Theorems 5.2.2 and 5.3.1. Theoretical developments in Chapters 4 and 5 are accompanied by results of computational experiments, performed using the author's software based on the PETSc package [**88**].

A paper containing partial results on block preconditioners for the micropolar equations has been accepted for publication in reviewed proceedings of The Ninth International Symposium on Domain Decomposition Methods in Science and Engineering [**61**]. An extended version has already appeared in [**60**].

## 1.4. Summary

In this thesis, the following open problems related to the microcontinuum mechanics have been solved.

1. Existence, uniqueness, regularity and norm estimates in Sobolev spaces have been proved for weak solutions of the barodiffusion problem in micropolar fluids. It has been shown that earlier developments for the Newtonian fluids can be generalized to the non-Newtonian case with no additional restrictions.
2. A mixed finite element method for the micropolar equations has been analysed. Our results on the existence, local uniqueness and optimal approximation generalize known theorems for the Navier–Stokes equations. Furthermore, the Newton method for solving the discrete nonlinear system has been analysed. For sufficiently large viscosity, the proposed numerical scheme can be successfully used to approximate the solutions of the micropolar equations.
3. A class of block preconditioners for nonsymmetric saddle point problems has been proposed and analysed. These preconditioners are robust and have been proved to be optimal with respect to the mesh size. Applications of this theory to the Oseen, Navier–Stokes and micropolar equations have been discussed, together with numerical experiments, confirming theoretical predictions.

## 1.5. Further perspectives

The present work is an important step towards the analysis of more sophisticated problems in the microcontinuum mechanics. A natural continuation would be to study efficient numerical schemes for the evolutionary micropolar problems or for the full system (1.2.3). Applications in blood flow research make it interesting to consider more realistic setting where the domain boundary is allowed to move: for example, a heart model [**79**]. Some theoretical work in this direction has already been done in the author's joint paper with G. Łukaszewicz, [**72**], which is beyond the scope of this thesis.

Applications of micropolar theory in the design of modern lubricated bearings encourage the development of efficient numerical methods for the flow in thin films. It is also challenging to consider nonstandard boundary conditions in (1.1.2), like those introduced by Bessonov [**10**], [**11**], both from the point of view of the mathematical theory and numerical analysis.

We also believe it can be possible to extend a part of the theory for the block preconditioners to systems obtained from the Lagrange multipliers formulation of the mortar method for the second order elliptic partial differential equations, see [**9**].

## 1.6. Plan of remaining chapters

In the following chapters we discuss in details problems related to mathematical models of microfluids, briefly introduced above. Each chapter begins with an introductory section, usually followed by a section with some background results.

In Chapter 2, we study the barodiffusion problem for micropolar fluids. At first we introduce the notion of a weak solution, make general assumptions and formulate main theorems. Next, we discuss properties of linearized problem, preparing for the proof of the existence theorem in Section 2.7. The last section is devoted to the proof of the uniqueness result.

The mixed finite element approximation of micropolar fluid equations is analysed in Chapter 3. In Section 3.5, we prove a general existence result for the discrete problem. Local uniqueness and approximation results are proved in Section 3.6, using the nonsingular solutions branches theory. We show that the nonsingularity assumption is non-void in Section 3.7. Section 3.8 is devoted to the analysis of Newton's method for solving the discrete system.

In the last two chapters, we propose and analyse block preconditioners for a class of saddle point problems. We develop general abstract theory in Chapter 4. Section 4.5 deals with preconditioners which transform the system into a symmetric positive definite one, thus suitable for conjugate gradient iteration, while in Section 4.6 we analyse a preconditioner for the GMRES method. The last section in Chapter 4 contains the results of numerical experiments and their discussion.

In Chapter 5, we apply the general theory from Chapter 4 to the problem obtained after the linearization of discrete micropolar equations discussed in Chapter 3. Again, we conclude this chapter with numerical experiments.

Throughout the thesis we use several function spaces, which are listed in Appendix A, together with their selected properties. The symbol "Const" denotes a generic positive constant, possibly different at different places. If necessary, we distinguish these constants by adding a subscript.

CHAPTER 2

# Mathematical analysis of a micropolar fluid with barodiffusion

In this chapter, we consider the boundary value problem for a stationary flow of micropolar fluids with barodiffusion. We prove the existence of weak solutions for possibly large class of data, determine conditions under which these solutions are unique, and study their regularity properties. These results were published in [**57**].

## 2.1. Introduction

The basic problem in the micropolar fluid theory, the homogeneous Dirichlet boundary value problem for an incompressible, isothermic micropolar fluid flow (1.1.2), has been studied by Łukaszewicz in [**69**] and [**68**]. It has been shown that for $f$, $z$ in $L^2(\Omega)$ the stationary micropolar problem admits at least one solution, which is unique, if only the Newtonian viscosity $\nu$ is large enough[1]. The results of this work were recently applied in lubrication theory, leading to a more general Reynolds equation describing the pressure in a thin film of a lubricant in a model bearing [**7**]. Another limit case, the micropolar flow in a porous medium, has been analysed in [**2**].

In literature some extensions of the micropolar model are considered. One of such extensions [**73**] takes into account the heat convection in the fluid, adding the energy equation to the micropolar system and assuming that the external fields may depend on the temperature.

In this chapter, we study another model of specific microfluid behaviour: the barodiffusion problem for micropolar fluids (1.2.3). We consider the isothermic flow of a mixture of two viscous micropolar fluids and assume that total density of the mixture is a positive constant. Fluid suspensions (blood, for example) belong to this type of mixtures [**81**]. Their densities and temperatures are constant and the diffusion effect is due to changes in the pressure and the concentration of components. Their microstructure is reflected by plugging the microrotation equation into the model.

[1] We give precise statement of these results in Section 3.7.

Below we introduce a system of partial differential equations which describe the stationary isothermic flow of a mixture of two viscous incompressible micropolar fluids with velocities $u^i$, internal rotations $\omega^i$ and densities $\rho^i$, $i = I, II$. The mixture occupies a bounded domain $\Omega \subset R^3$.

Under assumption that coefficients characterizing isotropic properties of the fluid are constant, some of them equal to zero, the motion of the mixture can be described, see [**80**], [**81**], [**70**], by the following boundary value problem[2]:

$$-(\nu + \nu_r)\Delta u + (u \cdot \nabla)u + \nabla p = 2\nu_r \operatorname{curl} \omega + f + cg \qquad \text{in } \Omega\,, \tag{2.1.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega\,, \tag{2.1.2}$$

$$\begin{aligned} -(c_a + c_d)\Delta\omega + (u \cdot \nabla)\omega - (c_0 + c_d - c_a)\nabla \operatorname{div} \omega + 4\nu_r\omega &= \\ = 2\nu_r \operatorname{curl} u + z & \qquad \text{in } \Omega\,, \end{aligned} \tag{2.1.3}$$

$$-\operatorname{div}(D(c)\nabla c) + u \cdot \nabla c = \operatorname{div}(K(c)\nabla p) \qquad \text{in } \Omega\,, \tag{2.1.4}$$

$$u = 0 \qquad \text{on } \partial\Omega\,, \tag{2.1.5}$$

$$\omega = 0 \qquad \text{on } \partial\Omega\,, \tag{2.1.6}$$

$$c = h \qquad \text{on } \partial\Omega\,. \tag{2.1.7}$$

The unknown functions are $u = (u_1, u_2, u_3)$, $p$, $\omega = (\omega_1, \omega_2, \omega_3)$ and $c$, which denote, respectively, the mean mass velocity vector $u = (\rho^I u^I + \rho^{II} u^{II})/\rho$, the pressure, the internal rotation vector $\omega = (\rho^I \omega^I + \rho^{II}\omega^{II})/\rho$ and the concentration of the first component $c = \rho^I/\rho$ in the mixture.

By $f = (f_1, f_2, f_3)$ we denote the external force per unit mass acting on the first component, by $g = (g_1, g_2, g_3)$ the difference between the forces acting on the second and the first components, and by $z = (z_1, z_2, z_3)$ we denote the external mean mass angular momentum vector. $D$ and $K$ are the diffusion and barodiffusion coefficients, assumed to be dependent on the concentration $c$. Constants $\nu$, $\nu_r$, $c_0$, $c_a$, $c_d$ correspond to different viscosities introduced by physical considerations. Usual viscosity is

[2]For simplicity, we assume that total density of the mixture $\rho = \rho^I + \rho^{II}$ and the micro inertia coefficient are equal to one.

denoted by $\nu > 0$, while $\nu_r$, $c_0$, $c_a$, $c_d$ denote additional viscosities connected with appearance of internal rotation field $\omega$ in the model considered. These constants satisfy $\nu_r > 0$, $c_a + c_d > 0$, $c_0 + c_d > c_a > 0$, see [**69**].

Let us remark that if $c = 0$ or $c = 1$ in $\Omega$, then the problem reduces to one–component problem. If $\nu_r = c_0 = c_a = c_d = 0$ and $z = 0$, then the problem reduces to the one considered in [**70**].

The governing system of partial differential equations for the stationary flow of a micropolar fluid with barodiffusion (2.1.1)–(2.1.7) has not been analysed with respect to its mathematical properties yet. Up to now, only simplified models of barodiffusion in mixtures, neglecting their micropolar behaviour, have been studied, see [**70**]. One of the aims of this chapter is to verify if it is possible to extend those results onto full differential system (2.1.1)–(2.1.7).

We prove the existence and regularity of weak solutions of a Dirichlet boundary value problem for the considered system of equations, as well as the uniqueness of solutions, under additional assumptions on the data. Actually, these theorems generalize the results from [**70**] and extend those for micropolar fluids [**69**] to this more complex case.

In what follows we use the notation introduced in Appendix A.

## 2.2. Main results

We constrain ourselves to an open, bounded, simply-connected domain $\Omega \subset R^3$, with smooth boundary. Furthermore, we assume the following:

(A1) $D : R \to R$ is a continuous function such that
$$0 < m \leq D(t) \leq M \qquad \forall t \in R,$$
for some constants $m$, $M$;

(A2) $K : R \to R$ is a Lipschitz continuous function with Lipschitz constant $L_K > 0$:
$$|K(t) - K(s)| \leq L_K |t - s| \qquad \forall t, s \in R,$$
and $K$ satisfies
$$K(t) = 0 \qquad \text{for } t \notin (0,1)\,;$$

(A3) $f$, $g \in L^q(\Omega)$ and $q$ is a fixed real number greater than 3;

(A4) $z \in L^2(\Omega)$;

(A5) $h \in C^{0,1}(\partial\Omega)$ and satisfies $0 \leq h \leq 1$.

Assumption (A1) is in agreement with the thermodynamical constraint $D > 0$, while (A2) reflects the fact that there is no diffusion flux in pure fluid. From the definition of the concentration $c$ it follows that physically meaningful values of $c$ are within the range $[0, 1]$, which explains the second part of assumption (A5).

We shall proceed with the following weak formulation of the problem:

PROBLEM 2.2.1. Find $(u, p, \omega, c) \in V^0(\Omega) \times W^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \times (W^{1,2}(\Omega) \cap C^0(\bar{\Omega}))$, such that $\int_\Omega p = 0$, $c = h$ on $\partial\Omega$ and

$$
\begin{cases}
(\nu + \nu_r) \displaystyle\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega (u \cdot \nabla) u \phi + \int_\Omega p \operatorname{div} \phi = \\
\qquad = 2\nu_r \displaystyle\int_\Omega \operatorname{curl} \omega \phi + \int_\Omega (f + cg)\phi \\
(c_a + c_d) \displaystyle\int_\Omega \nabla \omega \cdot \nabla \xi + \int_\Omega (u \cdot \nabla)\omega \xi + (c_0 + c_d - c_a) \int_\Omega \operatorname{div} \omega \operatorname{div} \xi + 4\nu_r \int_\Omega \omega \xi = \\
\qquad = 2\nu_r \displaystyle\int_\Omega \operatorname{curl} u \xi + \int_\Omega z\xi, \\
\displaystyle\int_\Omega D(c) \nabla c \nabla \varphi + \int_\Omega u \cdot \nabla c \varphi = -\int_\Omega K(c) \nabla p \nabla \varphi.
\end{cases}
$$

for all $\phi \in W_0^{1,2}(\Omega)$, $\xi \in W_0^{1,2}(\Omega)$ and $\varphi \in W_0^{1,2}(\Omega)$.

If $(u, p, \omega, c)$ are sufficiently smooth functions, then the weak problem is equivalent to the classical formulation (2.1.1) – (2.1.7).

Our aim in this chapter is to prove existence and uniqueness results for possibly wide class of data.

THEOREM 2.2.1 (Existence and regularity). *If assumptions (A1)–(A5) hold, then there exists a solution $(u, p, \omega, c)$ of Problem 2.2.1, which belongs to $W^{2,\tilde{q}}(\Omega) \times W^{1,\tilde{q}}(\Omega) \times W^{2,2}(\Omega) \times (W^{1,2}(\Omega) \cap C^0(\bar{\Omega}))$ where $\tilde{q} = \min\{q, 6\}$. Moreover, $0 \leq c \leq 1$, and there exists a polynomial $P : R \to R$ with property*

(P1) *$P$ is increasing on $[0, \infty)$ and $P(0) = 0$,* *such that*

$$\| u \|_{2,\tilde{q}} + \| p \|_{1,\tilde{q}} + \| \omega \|_{2,2} \leq P(|f|_q + |g|_q + |z|_2)\,.$$

THEOREM 2.2.2 (Uniqueness). *Let the assumptions of Theorem 2.2.1 hold. Suppose that $h \in W^{1-\frac{1}{r},r}(\partial\Omega)$, $3 < r \leq 6$, and that the diffusion coefficient is a Lipschitz continuous function,*

$$|D(t) - D(s)| \leq L_D |t - s| \qquad \forall t, s \in R,$$

*for some positive constant $L_D$. Then the solution of Problem 2.2.1 is unique, provided $|f|_q + |g|_q + |z|_2$ and $[h]_{1-\frac{1}{r},r}$ are sufficiently small.*

Below we outline the idea of the proof of these theorems. Complete proofs are given in the following sections.

The proof of the existence theorem is based on the application of Schauder fixed–point theorem (see Section 2.3.3) to a mapping $\Phi$ of some suitable subset of $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ into itself, $\Phi(u, b) = (u^*, c)$, defined as follows: for given (sufficiently regular) $u$, $b$, we obtain $\omega \in W_0^{1,2}(\Omega)$ as the unique solution of

$$(c_a + c_d) \int_\Omega \nabla\omega \cdot \nabla\xi + \int_\Omega (u \cdot \nabla)\omega\xi + (c_0 + c_d - c_a) \int_\Omega \operatorname{div}\omega \operatorname{div}\xi + 4\nu_r \int_\Omega \omega\xi =$$

$$= 2\nu_r \int_\Omega \operatorname{curl} u\xi + \int_\Omega z\xi \qquad \text{for all } \xi \in W_0^{1,2}(\Omega)\,.$$

With this $\omega$ we solve linearized Navier–Stokes equation, obtaining a unique pair $(u^*, p) \in V^0(\Omega) \times L^2(\Omega)$ which satisfies

$$-(\nu + \nu_r)\Delta u^* + (u \cdot \nabla)u^* + \nabla p = 2\nu_r \operatorname{curl}\omega + f + bg \qquad \text{in } \Omega$$

in the distribution sense, with $\int_\Omega p = 0$. We show that $u^* \in C^0(\bar{\Omega}) \cap W^{2,2}(\Omega)$, $p \in W^{1,2}(\Omega)$. With this $p$ we solve in $W^{1,2}(\Omega)$,

$$\int_\Omega D(b)\nabla c \nabla\phi + \int_\Omega u \cdot \nabla c\phi = -\int_\Omega K(c)\nabla p \nabla\phi \qquad \text{for all } \phi \in W_0^{1,2}(\Omega)\,,$$

with respect to $c$, with Dirichlet boundary condition $c = h$ on $\partial\Omega$; then we show $c \in C^0(\bar{\Omega})$, and thus obtain $(u^*, c) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ defining $\Phi(u, b)$.

From the above procedure it follows that if $\Phi$ has got a fixed point, $\Phi(u, c) = (u, c)$, then, together with corresponding $\omega$ and $p$, it is a solution of Problem 2.2.1.

In Sections 2.4, 2.5 and 2.6 we examine subsequent steps of this procedure, proving existence, uniqueness and regularity theorems for $\omega$, $(u^*, p)$ and $c$, respectively. In Section 2.7 we prove that the mapping $\Phi$ is well defined and satisfies assumptions of Schauder fixed point theorem. This requires careful norm estimates and taking advantage of precise relations between different coefficients in the equations. As a by-product of the procedure we obtain the regularity result.

The proof of the uniqueness, provided in Section 2.8, requires some more regularity and is based on estimating the difference between two solutions corresponding to the same data. As in the case of the Navier–Stokes equations [**91**], [**46**] or the micropolar problem [**69**], we have to assume that the external forces are relatively small.

We shall distinguish between constants only when it is necessary, indexing them with subscripts then. Before we proceed further with the proofs, let us first recall some basic tools from functional analysis.

## 2.3. Background results from functional analysis

In this section, we provide basic theorems from functional analysis, which we refer to later in the thesis. This section serves as a reference and is included for convenience of the reader.

Let us begin with existence theorems suitable for linear and nonlinear problems. These results may be found for example in [**45**], [**46**], [**21**], [**91**], [**65**].

**2.3.1. Lax–Milgram theorem.** Let $V$ denote a Hilbert space with a norm $||\cdot||$. Let $\langle\cdot,\,\cdot\rangle$ be the duality pairing between $V$ and its dual space, $V^*$. For a bilinear form $a(\cdot,\,\cdot) : V \times V \to R$ consider the following variational problem

PROBLEM 2.3.1. Given a linear continuous functional $f \in V^*$, find $u \in V$ which satisfies

$$a(u,\,v) = \langle f,\,v\rangle \qquad \forall v \in V.$$

THEOREM 2.3.1 (Lax–Milgram, see [**46**, Theorem I.1.7]). *If $a(\cdot,\,\cdot)$ is continuous and elliptic on $V$, that is, there exist positive constants $m, M$ such that*

$$a(u,\,v) \le M \cdot ||u|| \cdot ||v|| \qquad \forall u, v \in V,$$

*and*

$$a(u,\,u) \ge m \cdot ||u||^2 \qquad \forall u \in V,$$

*then there exists a unique solution of Problem 2.3.1. Moreover the solution $u$ satisfies*

$$||u|| \le \frac{1}{m} \cdot ||f||_{V^*},$$

*where $||\cdot||_{V^*}$ denotes the norm in the dual space $V^*$.*

**2.3.2. Babuška–Brezzi theorem.** Let $V$ and $W$ denote two Hilbert spaces, with norms $||\cdot||$ and $|\cdot|$, respectively. We assume that these norms are induced by inner products in these spaces and introduce two bilinear, continuous forms

$$a(\cdot,\,\cdot): V\times V \to R, \qquad b(\cdot,\,\cdot): V\times W \to R.$$

We consider a *saddle point problem*:

PROBLEM 2.3.2. Find $(u, p) \in V \times W$, such that

$$\begin{cases} a(u,\,v) + b(v,\,p) & = \langle f,\,v\rangle\,, \\ b(u,\,w) & = \langle g,\,w\rangle\,, \end{cases} \tag{2.3.8}$$

holds for all $(v, w) \in V \times W$. Here, $f$ and $g$ are given continuous linear functionals on $V$ and $W$, respectively.

In order to state the existence and uniqueness result for (2.3.8), we introduce a subspace of $V$,

$$V^0 = \{u \in V : b(u,\,w) = 0 \quad \forall w \in W\}.$$

THEOREM 2.3.2 (Babuška–Brezzi, see [**46**, Corollary I.4.1] ). *If the following conditions are satisfied:*

(i) *form $a(\cdot,\,\cdot)$ is $V^0$-elliptic,*

$$\exists \alpha > 0 \quad a(u,\,u) \geq \alpha ||u||^2 \qquad \forall u \in V^0,$$

(ii) *form $b(\cdot,\,\cdot)$ satisfies the Ladyzhenskaya–Babuška–Brezzi (LBB) condition,*

$$\exists \beta > 0 \quad \sup_{v\in V, v\neq 0} \frac{b(v,\,p)}{||v||} \geq \beta |p| \qquad \forall p \in W,$$

*then there exists a unique solution of (2.3.8).*

*Furthermore, the solution $(u, p)$ of Problem 2.3.2 satisfies*

$$||u|| \leq \frac{1}{\alpha}\cdot ||f||_{V^*} + \gamma \cdot |g|_{W^*},$$

$$|p| \leq \gamma \cdot ||f||_{V^*} + \frac{||a||}{\beta}\cdot \gamma \cdot |g|_{W^*},$$

*where* $\gamma = \dfrac{1}{\beta}\left(1 + \dfrac{||a||}{\beta}\right)$.

For a more general existence theorem for saddle point problems, see [**21**, Theorem II.1.1].

**2.3.3. Fixed point theorems.** Here, we recall theorems which guarantee that a nonlinear mapping $\Psi$ in a Banach space has a fixed point, i.e. there exists $x^*$ satisfying $\Psi(x^*) = x^*$. We use these facts in order to prove the existence of solutions to nonlinear variational problems considered in the thesis.

THEOREM 2.3.3 (Brouwer, see [**46**, Theorem IV.1.1]). *Let $K$ denote a nonempty, compact and convex subset of a finite dimensional space and let $\Psi$ be a continuous mapping of $K$ into itself. Then $\Psi$ has a fixed point.*

THEOREM 2.3.4 (Schauder, see [**45**, Corollary 10.2]). *Let $K$ be a nonempty, bounded, closed and convex subset in a Banach space and let $\Psi : K \to K$ be a completely continuous mapping. Then $\Psi$ has a fixed point.*

A very useful corollary of Theorem 2.3.3 gives necessary conditions for existence of a zero of a nonlinear mapping in a finite dimensional Hilbert space.

THEOREM 2.3.5 (see [**46**, Corollary IV.1.1]). *Let $H$ be a finite dimensional Hilbert space with scalar product denoted by $((\cdot,\cdot))$ and the corresponding norm by $||\cdot||$. If $\Phi : H \to H$ is a continuous mapping with property:*

$$\exists M > 0 \quad \forall u \in H, ||u|| = M \qquad ((\Phi(u), u)) \geq 0,$$

*then there exists at least one $u^* \in H$ such that*

$$\Phi(u^*) = 0, \qquad ||u^*|| \leq M.$$

**2.3.4. Stokes equations.** Let us consider the *Stokes problem*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\Omega$ be a bounded and simply connected open subset in $R^N$ with a Lipschitz continuous boundary. If $f \in H^{-1}(\Omega)$, this system of partial differential equations may be given a weak formulation which fits into the saddle point framework discussed in Section 2.3.2:

$$\begin{cases} (\nabla u,\, \nabla v) + (p,\, \operatorname{div} v) = \langle f,\, v\rangle & \forall v \in H_0^1(\Omega), \\ (\operatorname{div} u,\, w) = 0 & \forall w \in L_0^2(\Omega). \end{cases} \tag{2.3.9}$$

Here, $(\cdot,\, \cdot)$ denotes the inner product in $L_0^2(\Omega)$, see Appendix A.

From Theorem 2.3.2 we conclude

THEOREM 2.3.6 (see [**46**, Theorem I.5.1]). *There exists a unique solution* $(u,p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ *of the Stokes problem (2.3.9). Moreover,*

$$||u||_{1,2} + |p|_2 \leq \text{Const}\, ||f||_{-1,2}.$$

This theorem has a strong extension:

THEOREM 2.3.7 (Cattabriga, see [**91**, Chapter I, Proposition 2.3]). *Suppose $m$ is an integer and $m \geq -1$. Let $\Omega$ be an open bounded subset of $R^2$ or $R^3$, $\Omega \in C^r$, where $r = \max\{m+2, 2\}$. If $f \in W^{m,\alpha}(\Omega)$, $\alpha > 1$, then there exists a unique pair $(u,p) \in W_0^{1,2}(\Omega) \times L_0^2(\Omega)$, which is the generalized solution of the Stokes problem and*

$$(u,p) \in W^{m+2,\alpha}(\Omega) \times W^{m+1,\alpha}(\Omega),$$

*and*

$$||u||_{m+2,\alpha} + ||p||_{m+1,\alpha} \leq \text{Const}\, ||f||_{m,\alpha},$$

*with* $\text{Const} = \text{Const}(\alpha, m, \Omega)$.

**2.3.5. Navier–Stokes equations.** The stationary Navier–Stokes equations with homogeneous Dirichlet boundary condition on the velocity read as

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose $\Omega$ is an open, bounded, simply-connected subset of $R^N$, $N \leq 3$, with Lipschitz boundary. Constant $\nu > 0$ denotes the kinematic viscosity of the fluid. Let us define a trilinear continuous form on $H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$,

$$d_1(u,\, v,\, w) = \int_\Omega (u\cdot\nabla)v\, w = \sum_{i,j=1}^N \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j,$$

and let $||d_1||$ denote the norm of $d_1(\cdot,\,\cdot,\,\cdot)$. Observe that if $u \in V^0$, then (see [**91**, Lemma II.1.3])

$$d_1(u,\, v,\, w) = -d_1(u,\, w,\, v) \qquad \forall v, w \in W_0^{1,2}(\Omega) \tag{2.3.10}$$

and, in particular, $d_1(u,\, v,\, v) = 0$ for $v \in W_0^{1,2}(\Omega)$.

The Navier–Stokes problem may be written in a weak form,

PROBLEM 2.3.3. Given $f \in H^{-1}(\Omega)$, find a pair $(u,p) \in H_0^1(\Omega) \times L_0^2(\Omega)$, which satisfies

$$\begin{aligned} \nu(\nabla u,\, \nabla v) + d_1(u,\, u,\, v) - (p,\, \operatorname{div} v) = \langle f,\, v\rangle \qquad & \forall v \in H_0^1(\Omega), \\ (\operatorname{div} u,\, w) = 0 \qquad & \forall w \in L_0^2(\Omega). \end{aligned} \tag{2.3.11}$$

The following theorems ensure the existence and uniqueness of weak solutions of Navier–Stokes equations.

THEOREM 2.3.8 (see [**46**, Theorem IV.2.1]). *Problem 2.3.3 admits at least one solution.*

THEOREM 2.3.9 (see [**46**, Theorem IV.2.2]). *If* $||d_1|| \cdot ||f||_{-1} \leq \nu^2$, *then the solution* $(u, p)$ *of Problem 2.3.3 is unique.*

## 2.4. Microrotation equation

In this section we consider the linearized microrotation equation, see (2.2.1). Precisely, in what follows we shall analyse the following problem in $\omega$.

PROBLEM 2.4.1. For given $u \in V^0(\Omega)$, $z \in L^2(\Omega)$, find $\omega \in W_0^{1,2}(\Omega)$ such that

$$\begin{aligned}(c_a + c_d)\int_\Omega \nabla\omega \cdot \nabla\xi + \int_\Omega (u\cdot\nabla)\omega\xi + (c_0 + c_d - c_a)\int_\Omega \operatorname{div}\omega \operatorname{div}\xi + \\ + 4\nu_r\int_\Omega \omega\xi = 2\nu_r\int_\Omega \operatorname{curl} u\xi + \int_\Omega z\xi\end{aligned} \tag{2.4.12}$$

is satisfied for all $\xi \in W_0^{1,2}(\Omega)$.

LEMMA 2.4.1. *If* $u \in V^0(\Omega)$, *then there exists exactly one* $\omega \in W_0^{1,2}(\Omega)$ *satisfying (2.4.12).*

PROOF. The bilinear form $F : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to R$, defined as

$$\begin{aligned}F(\omega,\xi) = (c_a + c_d)\int_\Omega \nabla\omega \cdot \nabla\xi + \int_\Omega (u\cdot\nabla)\omega\xi + \\ + (c_0 + c_d - c_a)\int_\Omega \operatorname{div}\omega \operatorname{div}\xi + 4\nu_r\int_\Omega \omega\xi\end{aligned}$$

is continuous and coercive, the latter due to positivity of the coefficients and the fact that if $u \in V^0(\Omega)$ then

$$\int_\Omega (u\cdot\nabla)\omega\xi = -\int_\Omega (u\cdot\nabla)\xi\omega$$

for all $\omega$, $\xi \in W_0^{1,2}(\Omega)$, see (2.3.10). Thus, by Lax–Milgram lemma (Theorem 2.3.1), there exists unique $\omega \in W_0^{1,2}(\Omega)$ for which

$$F(\omega,\xi) = 2\nu_r\int_\Omega \operatorname{curl} u\xi + \int_\Omega z\xi\,,$$

as the right–hand side of this equation defines a continuous linear functional of the variable $\xi \in W_0^{1,2}(\Omega)$. $\square$

LEMMA 2.4.2. *Let $\omega_1$, $\omega_2$ be solutions of (2.4.12) corresponding to $u_1$, $u_2 \in V^0(\Omega) \cap C^0(\bar{\Omega})$. Then*

$$\| \omega_1 - \omega_2 \|_1 \leq (\mathrm{Const} + |\omega_2|_2)|u_1 - u_2|_\infty \,.$$

PROOF. Subtracting equation (2.4.12) in $\omega_1$ from the equation in $\omega_2$, we obtain

$$(c_a + c_d) \int_\Omega \nabla(\omega_1 - \omega_2) \cdot \nabla\xi + \int_\Omega (u_1 \cdot \nabla)\omega_1\xi - \int_\Omega (u_2 \cdot \nabla)\omega_2\xi +$$

$$+ (c_0 + c_d - c_a) \int_\Omega \operatorname{div}(\omega_1 - \omega_2) \operatorname{div}\xi + 4\nu_r \int_\Omega (\omega_1 - \omega_2)\xi =$$

$$= 2\nu_r \int_\Omega \operatorname{curl}(u_1 - u_2)\xi = 2\nu_r \int_\Omega (u_1 - u_2) \operatorname{curl}\xi$$

for all $\xi \in W_0^{1,2}(\Omega)$. In particular, setting $\xi = \omega_1 - \omega_2$ we have

$$(c_a + c_d) \| \omega_1 - \omega_2 \|_1^2 \leq 2\nu_r |u_1 - u_2|_2 |\operatorname{curl}(\omega_1 - \omega_2)|_2 +$$

$$+ \left| \int_\Omega (u_1 \cdot \nabla)\omega_1(\omega_1 - \omega_2) - \int_\Omega (u_2 \cdot \nabla)\omega_2(\omega_1 - \omega_2) \right|$$

$$\leq 2\nu_r |u_1 - u_2|_2 |\nabla(\omega_1 - \omega_2)|_2 + \left| \int_\Omega ((u_1 - u_2) \cdot \nabla)\omega_2(\omega_1 - \omega_2) \right| +$$

$$+ \left| \int_\Omega (u_1 \cdot \nabla)(\omega_1 - \omega_2)(\omega_1 - \omega_2) \right| ,$$

so, by (2.3.10),

$$\| \omega_1 - \omega_2 \|_1^2 \leq 2\nu_r \,\mathrm{Const}\, |u_1 - u_2|_\infty \| \omega_1 - \omega_2 \|_1 +$$

$$+ |u_1 - u_2|_\infty \| \omega_1 - \omega_2 \|_1 \, |\omega_2|_2 \,.$$

This yields

$$\| \omega_1 - \omega_2 \|_1 \leq (\mathrm{Const} + |\omega_2|_2) \cdot |u_1 - u_2|_\infty \,.$$

$\square$

LEMMA 2.4.3. *If $\omega$ is a solution of (2.4.12), then*

$$|\omega|_2 \leq \frac{1}{2} \parallel u \parallel_1 + \frac{1}{4\nu_r}|z|_2, \quad \parallel \omega \parallel_1 \leq \frac{2\nu_r}{c_a + c_d} \cdot \left( \parallel u \parallel_1 + \frac{\text{Const}}{2\nu_r}|z|_2 \right).$$

PROOF. Setting $\xi = \omega$ in (2.4.12) and proceeding in a standard way, we obtain desired inequalities. □

LEMMA 2.4.4. *If $u$ in problem (2.4.12) belongs to $V^0(\Omega) \cap C^0(\bar{\Omega})$, then corresponding to it $\omega$ belongs to $W^{2,2}(\Omega)$.*

PROOF. The proof follows from the estimates of solutions of strongly elliptic systems. Considering

$$-(c_a + c_d)\Delta\omega - (c_0 + c_d - c_a)\nabla \operatorname{div} \omega + 4\nu_r\omega = l\,,$$

where $l = 2\nu_r \operatorname{curl} u + z + (u \cdot \nabla)\omega$, we find that $l \in L^2(\Omega)$. Thus, due to Nečas' estimates (see also [**69**, Section 4]), $\omega \in W^{2,2}(\Omega)$ and

$$\parallel \omega \parallel_{2,2} \leq \text{Const}(2\nu_r |\operatorname{curl} u|_2 + |u|_\infty \parallel \omega \parallel_1 + |z|_2)\,.$$

□

## 2.5. Linearized Navier–Stokes equation

In this section we consider the following problem:

PROBLEM 2.5.1. For given $f, g \in L^q(\Omega)$, $q > 3$, and $\omega \in W_0^{1,2}(\Omega)$, $b \in C^0(\bar{\Omega})$, $u \in V^0(\Omega)$, $\omega$ being the solution of (2.4.12) with the same $u$, find $(u^*, p) \in V^0(\Omega) \times L_0^2(\Omega)$ which satisfy

$$-(\nu + \nu_r)\Delta u^* + (u \cdot \nabla)u^* + \nabla p = 2\nu_r \operatorname{curl} \omega + f + bg \tag{2.5.13}$$

in the distribution sense on $\Omega$.

LEMMA 2.5.1. *There exists a unique solution $(u^*, p) \in V^0(\Omega) \times L_0^2(\Omega)$ of Problem 2.5.1.*

PROOF. Let us consider the following weak form[3] of (2.5.13):

$$(\nu + \nu_r)\int_\Omega \nabla u^* \cdot \nabla v + \int_\Omega (u \cdot \nabla)u^* v = 2\nu_r \int_\Omega \operatorname{curl} \omega v + \int_\Omega (f + bg)v \tag{2.5.14}$$

for all $v \in V^0(\Omega)$.

[3]The pressure term disappears, since $\operatorname{div} v = 0$.

For fixed $u$, $f$, $b$, $g$ the left–hand side of (2.5.14) defines a bilinear form $F_1$ on $V^0(\Omega)\times V^0(\Omega)$,

$$F_1(u^*,v)=(\nu+\nu_r)\int_\Omega \nabla u^*\cdot\nabla v+\int_\Omega (u\cdot\nabla)u^*v\,,$$

which is coercive and continuous, and the right–hand side of (2.5.14) defines a continuous linear functional of $v\in V^0(\Omega)$. Indeed,

$$F_1(u^*,v)\le(\nu+\nu_r)|\nabla u^*|_2|\nabla v|_2+\text{Const}\,|u|_4|\nabla u^*|_2|v|_4\le \text{Const}\,||u^*||_1||v||_1$$

and $F_1(u^*,u^*)=(\nu+\nu_r)|\nabla u^*|_2^2+\int_\Omega(u\cdot u^*)u^*=(\nu+\nu_r)||u^*||_1^2$, by (2.3.10). Therefore, by Lax–Milgram lemma it follows that there exists a unique $u^*\in V^0(\Omega)$, which satisfies (2.5.14). Now, as $\tilde f=-(\nu+\nu_r)\Delta u^*+(u\cdot\nabla)u^*-2\nu_r\operatorname{curl}\omega-f-bg$ belongs to $H^{-1}(\Omega)$, and $\langle\tilde f,v\rangle=0$ for all $v\in V^0(\Omega)$, we can conclude, basing upon Propositions I.1.1. and I.1.2. in **[91]** that there exists unique $p\in L_0^2(\Omega)$ such that $\nabla p=\tilde f$ in the distribution sense. $\square$

LEMMA 2.5.2. *There exist positive constants $R_1$, $R_2$, $R_3$ such that if $u\in A$,*

$$A=\{v\in V^0(\Omega)\cap C^0(\bar\Omega):\parallel v\parallel_1\le R_1\,,\quad |v|_q\le R_2\,,\quad |v|_\infty\le R_3\}$$

*then Problem 2.5.1 has (unique) solution $(u^*,p)\in(W^{2,\tilde q}(\Omega)\cap A)\times W^{1,\tilde q}(\Omega)$, where $\tilde q=\min\{q,6\}$ and $\int_\Omega p=0$. Moreover*

$$\parallel u^*\parallel_{2,\tilde q}+\parallel p\parallel_{1,\tilde q}\le \text{Const}(R_3(R_2^2+R_1R_3+R_1)+R_1)\,. \tag{2.5.15}$$

PROOF. Substituting $v=u^*$ in (2.5.14) we obtain, since $|\operatorname{curl}u|_2\le|\nabla u|_2$ for any $u\in W_0^{1,2}(\Omega)$,

$$(\nu+\nu_r)\parallel u^*\parallel_1^2\le 2\nu_r|\omega|_2\parallel u^*\parallel_1+(|f|_2+|b|_\infty|g|_2)|u^*|_2$$

and, since $|u^*|_2\le\text{Const}\parallel u^*\parallel_1$ for any $u^*\in W_0^{1,2}(\Omega)$, see Theorem A.0.1, we have

$$(\nu+\nu_r)\parallel u^*\parallel_1\le 2\nu_r|\omega|_2+\text{Const}(|f|_2+|b|_\infty|g|_2)\,.$$

Now, by Lemma 2.4.3,

$$\begin{aligned}\parallel u^*\parallel_1&\le\frac{\nu_r}{\nu+\nu_r}\parallel u\parallel_1+\frac{\text{Const}}{\nu+\nu_r}(|f|_q+|b|_\infty|g|_q)+\frac{|z|_2}{2(\nu+\nu_r)}\\&\le\frac{\nu_r}{\nu+\nu_r}\parallel u\parallel_1+\frac{\text{Const}_1}{\nu+\nu_r}(|f|_q+|b|_\infty|g|_q+|z|_2)\,.\end{aligned} \tag{2.5.16}$$

Now, if $\parallel u\parallel_1\le R_1$, and $R_1$ is large enough, namely

$$R_1\ge\frac{\text{Const}_1}{\nu}(|f|_q+|b|_\infty|g|_q+|z|_2) \tag{2.5.17}$$

for $\mathrm{Const}_1$ the same as in (2.5.16), then also

$$\| u^* \|_1 \leq \frac{\nu_r}{\nu+\nu_r} R_1 + \frac{\nu}{\nu+\nu_r} R_1 = R_1 \,.$$

Using Hölder inequality and imbedding $W_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we conclude that $(u \cdot \nabla)u^* \in L^{3/2}(\Omega)$ and

$$|(u\cdot\nabla)u^*|_{3/2} \leq |u|_6 |\nabla u^*|_2 \leq \mathrm{Const} \, \| u \|_1 \| u^* \|_1 \leq \mathrm{Const}\, R_1^2 \,.$$

In purpose to state greater regularity of the solution of (2.5.13), we will apply Cattabriga's estimate (see Theorem 2.3.7) of the solution $(u^*, p)$, $\int_\Omega p = 0$, of the Stokes equation

$$\begin{cases} -(\nu+\nu_r)\Delta u^* + \nabla p = \tilde{f} & \text{in } \Omega\,, \\ \operatorname{div} u^* = 0 & \text{in } \Omega\,, \\ u^* = 0 & \text{on } \partial\Omega\,. \end{cases}$$

In our case, $\tilde{f} = f + bg + 2\nu_r \operatorname{curl}\omega - (u\cdot\nabla)u^*$. We have just proved that $\tilde{f} \in L^{3/2}(\Omega)$; therefore, Cattabriga's estimate yields $(u^*, p) \in W^{2,3/2}(\Omega) \times W^{1,3/2}(\Omega)$ and

$$\| u^* \|_{2,3/2} + \| p \|_{1,3/2} \leq \mathrm{Const} \, \| \tilde{f} \|_{0,3/2} \leq \mathrm{Const}(R_1^2 + |f|_q + |b|_\infty |g|_q + \| \omega \|_1)\,;$$

hence, due to Lemma 2.4.3 and (2.5.17)

$$\| u^* \|_{2,3/2} \leq \mathrm{Const}(R_1^2 + R_1)\,.$$

Then, since $W^{2,3/2}(\Omega) \hookrightarrow L^s(\Omega)$, $s \geq 1$, we get

$$|u^*|_q \leq \mathrm{Const}(R_1^2 + R_1) = R_2\,. \tag{2.5.18}$$

In this way we have proved that if $\| u \|_1 \leq R_1$, then $\| u^* \|_1 \leq R_1$ and $|u^*|_q \leq R_2$, with $R_2$ defined by (2.5.18). Now, to prove that also $|u^*|_\infty \leq R_3$ for some $R_3$, it suffices to note that $(u\cdot\nabla)u^* \in L^2(\Omega)$, and

$$|(u\cdot\nabla)u^*|_2 \leq |u|_6 |\nabla u^*|_3 \leq \mathrm{Const} \, \| u \|_1 \| u^* \|_{2,3/2} \leq \mathrm{Const}\, R_1 R_2 \,,$$

which follows from the imbeddings $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and $W^{2,3/2}(\Omega) \hookrightarrow W^{1,3}(\Omega)$. Then, once again from Cattabriga's estimate, $(u^*, p) \in W^{2,2}(\Omega) \times W^{1,2}(\Omega)$ and

$$\| u^* \|_{2,2} + \| p \|_{1,2} \leq \mathrm{Const}(R_1 R_2 + \| \omega \|_1 + |f|_q + |b|_\infty |g|_q) \leq$$

$$\leq \mathrm{Const}(R_1 R_2 + R_1)\,.$$

Then, since $W^{2,2}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, we have

$$|u^*|_\infty \leq \text{Const}(R_1 R_2 + R_1) = R_3\,.$$

Therefore, if $u \in A$, with constants $R_1$, $R_2$, $R_3$ defined as above, then also $u^* \in A$. It remains to prove that $(u^*, p) \in W^{2,\tilde{q}}(\Omega) \times W^{1,\tilde{q}}(\Omega)$, where $\tilde{q} = \min\{q, 6\}$.

First of all, let us notice that

$$|\operatorname{curl}\omega|_r \leq \text{Const}\ |\nabla\omega|_r \leq \text{Const}\ \|\ \omega\ \|_{1,r}$$

for all $\omega \in W^{1,r}(\Omega)$, $r \geq 1$. Indeed, we have

$$\begin{aligned}
|\operatorname{curl}\omega|_r &= \left( |\frac{\partial\omega_3}{\partial x_2} - \frac{\partial\omega_2}{\partial x_3}|_r^r + |\frac{\partial\omega_1}{\partial x_3} - \frac{\partial\omega_3}{\partial x_1}|_r^r + |\frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2}|_r^r \right)^{1/r} \\
&\leq \text{Const}\left( |\frac{\partial\omega_3}{\partial x_2} - \frac{\partial\omega_2}{\partial x_3}|_r + |\frac{\partial\omega_1}{\partial x_3} - \frac{\partial\omega_3}{\partial x_1}|_r + |\frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2}|_r \right) \\
&\leq \text{Const} \sum_{i,j=1}^{3} |\frac{\partial\omega_i}{\partial x_j}|_r \leq \text{Const}\left( \sum_{i,j=1}^{3} |\frac{\partial\omega_i}{\partial x_j}|_r^r \right)^{1/r} = \text{Const}\ |\nabla\omega|_r.
\end{aligned}$$

Let us take $r$ such that $\frac{1}{r} = \frac{1}{q} + \frac{1}{3}$. Since $q > 3$, then $r \in \left(\frac{3}{2}, 3\right)$. By Lemma 2.4.4 and imbedding $W^{2,2}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ we have, from Cattabriga's estimate

$$\begin{aligned}
\|\ u^*\ \|_{2,r} + \|\ p\ \|_{1,r} &\leq \text{Const}(|(u\cdot\nabla)u^*|_r + |\operatorname{curl}\omega|_r + |f|_q + |b|_\infty |g|_q) \\
&\leq \text{Const}(|u|_q |\nabla u^*|_3 + \|\ u\ \|_1 + |u|_\infty \|\ \omega\ \|_1 + |z|_2 + \\
&\quad + |f|_q + |b|_\infty |g|_q) \\
&\leq \text{Const}(R_2^2 + R_1 R_3 + R_1)
\end{aligned}$$

because we have previously assumed that $u \in A$. From imbedding $W^{2,r}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ we conclude that $(u\cdot\nabla)u^* \in L^q(\Omega)$, as

$$\begin{aligned}
|(u\cdot\nabla)u^*|_q \leq |u|_\infty |\nabla u^*|_q &\leq \text{Const}\ R_3 \ \|\ u^*\ \|_{2,r} \\
&\leq \text{Const}\ R_3(R_2^2 + R_1 R_3 + R_1)
\end{aligned}$$

and, applying Cattabriga's estimate we obtain, since $W^{2,2}(\Omega) \hookrightarrow W^{1,\tilde{q}}(\Omega)$,

$$\begin{aligned}\| u^* \|_{2,\tilde{q}} + \| p \|_{1,\tilde{q}} &\leq \mathrm{Const}(|(u\cdot\nabla)u^*|_{\tilde{q}} + |\operatorname{curl}\omega|_{\tilde{q}} + |f|_{\tilde{q}} + |b|_\infty |g|_{\tilde{q}}) \\ &\leq \mathrm{Const}(|(u\cdot\nabla)u^*|_q + \| \omega \|_{1,\tilde{q}} + |f|_q + |b|_\infty |g|_q) \\ &\leq \mathrm{Const}(R_3(R_2^2 + R_1R_3 + R_1) + R_1R_3 + R_1),\end{aligned}$$

which ends the proof. □

Lemma 2.5.3. *Let us consider a product $A \times B \subset C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$, where $A$ is defined as in Lemma 2.5.2, and*

$$B = \{ b \in C^0(\bar{\Omega}) : |b|_\infty \leq M_1 \}$$

*for some positive real $M_1$. Let us assume that sequences $(u_n)_{n=1}^\infty \subset A$ and $(b_n)_{n=1}^\infty \subset B$, If $(u_n, b_n) \to (u, b)$ in $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ topology, then $(u_n^*, p_n) \to (u^*, p)$ in $C^0(\bar{\Omega}) \times W^{1,2}(\Omega)$, where $(u_n^*, p_n)$ and $(u^*, p)$ are solutions of problem (2.5.13) corresponding to $(u_n, b_n)$ and $(u, b)$, respectively.*

Proof. First of all, we note that $A$ is closed in $C^0(\bar{\Omega})$. The only step in the proof of the closedness which requires mentioning, is to show that if $u_n \to u$ in $C^0(\bar{\Omega})$, $u_n \in A$ for all $n$, then $u \in V^0(\Omega)$.

As $\| u_n \|_1 \leq R_1$ for all $n \in N$, there exists a subsequence $(u_\tau) \subset (u_n)$, which converges weakly in $W_0^{1,2}(\Omega)$: for $i, k = 1, 2, 3$, there exist $\xi_{i,k} \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \frac{\partial (u_\tau)_i}{\partial x_k}\phi \to \int_\Omega \frac{\partial \xi_{i,k}}{\partial x_k}\phi \qquad \text{for all } \phi \in W_0^{1,2}(\Omega).$$

On the other hand,

$$\int_\Omega \frac{\partial (u_\tau)_i}{\partial x_k}\phi = -\int_\Omega (u_\tau)_i \frac{\partial \phi}{\partial x_k} \to -\int_\Omega u_i \frac{\partial \phi}{\partial x_k}$$

for all $\phi \in W_0^{1,2}(\Omega)$, since $u_\tau \to u$ in $C^0(\bar{\Omega})$. Combining these two facts we conclude that $\dfrac{\partial u_i}{\partial x_k} = \dfrac{\partial \xi_{i,k}}{\partial x_k} \in L^2(\Omega)$, so $u \in W^{1,2}(\Omega)$. Moreover, from above weak convergence it follows that $||\nabla u||_1 \leq R_1$. By similar argument we prove that $\operatorname{div} u = 0$. The trace of $u$ on $\Omega$ is equal to zero, because $u_n = 0$ on $\partial\Omega$ and $u_n \to u$ in $C^0(\bar{\Omega})$.

Now we are ready to begin the proof of convergence stated in our lemma.

The difference $(u^* - u_n^*, p - p_n)$ satisfies Stokes equation

$$\begin{cases} -(\nu+\nu_r)\Delta(u^*-u_n^*) + \nabla(p-p_n) = \tilde{f}_n \qquad \text{in } \Omega\,, \\ \operatorname{div}(u^*-u_n^*) = 0 \qquad \text{in } \Omega\,, \end{cases} \tag{2.5.19}$$

with $u^* - u_n^* = 0$ on $\partial\Omega$, $\int_\Omega (p-p_n) = 0$, and

$$\tilde{f} = (u_n \cdot \nabla)u_n^* - (u\cdot\nabla)u^* + 2\nu_r \operatorname{curl}(\omega-\omega_n) + g(b-b_n).$$

In particular, as the distributions in (2.5.19) are regular ($\tilde{f} \in L^2(\Omega)$), we can multiply (2.5.19) by $(u^* - u_n^*)$ and integrate the product over $\Omega$, obtaining

$$\begin{aligned}
&(\nu+\nu_r) \parallel u^* - u_n^* \parallel_1^2 \leq \\
&\qquad \leq 2\nu_r \int_\Omega |(\omega-\omega_n)\operatorname{curl}(u^*-u_n^*)| + + \int_\Omega |(b-b_n) g (u^*-u_n^*)| + \\
&\qquad\qquad + \left| \int_\Omega (u_n\cdot\nabla)u_n^*(u^*-u_n^*) - \int_\Omega (u\cdot\nabla)u^*(u^*-u_n^*) \right| \\
&\qquad \leq 2\nu_r |\omega-\omega_n|_2 \parallel u^*-u_n^* \parallel_1 + |b-b_n|_\infty |g|_2 |u^*-u_n^*|_2 \\
&\qquad\qquad + \left| \int_\Omega ((u-u_n)\cdot\nabla)u^*(u^*-u_n^*) \right| \\
&\qquad \leq \operatorname{Const}(|\omega-\omega_n|_2 \parallel u^*-u_n^* \parallel_1 + |b-b_n|_\infty |g|_q \parallel u^*-u_n^* \parallel_1 + \\
&\qquad\qquad + |u-u_n|_\infty |\nabla u^*|_2 \parallel u^*-u_n^* \parallel_1)\,.
\end{aligned}$$

Hence,

$$\parallel u^*-u_n^* \parallel_1 \leq \operatorname{Const}(|\omega-\omega_n|_2 + |b-b_n|_\infty |g|_q + |u-u_n|_\infty |\nabla u^*|_2)\,.$$

Since $u_n \in A$, then by Lemmas 2.4.2 and 2.4.3 we have $|\omega-\omega_n|_2 \to 0$ if $|u-u_n|_\infty \to 0$, so from the above inequality we conclude that

$$\parallel u^*-u_n^* \parallel_1 \to 0 \quad \text{as} \quad |u-u_n|_\infty \to 0 \quad \text{and} \quad |b-b_n|_\infty \to 0\,.$$

This implies $\tilde{f}_n \to 0$ in $L^2(\Omega)$, because

$$
\begin{aligned}
|\tilde{f}_n|_2 \leq \mathrm{Const}(&|((u-u_n)\cdot\nabla)u^*|_2 + |(u_n\cdot\nabla)(u^*-u_n^*)|_2 + 2\nu_r|\operatorname{curl}(\omega-\omega_n)|_2 + \\
&+ |g|_q|b-b_n|_\infty) \\
\leq \mathrm{Const}(&|u-u_n|_\infty \parallel u^* \parallel_1 + |u_n|_\infty \parallel u^*-u_n^* \parallel_1 + \parallel \omega-\omega_n \parallel_1 + \\
&+ |g|_q|b-b_n|_\infty)
\end{aligned}
$$

and the right–hand side of the inequality tends to zero with $n$ tending to infinity, as $(u_n, b_n) \to (u, b)$ in $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$, and $|u|_\infty \leq R_3$.

From Cattabriga's estimate, $u_n^* \to u^*$ in $W^{2,2}(\Omega)$, therefore in $C^0(\bar{\Omega})$, and $p_n \to p$ in $W^{1,2}(\Omega)$. □

## 2.6. Diffusion equation

In this section we provide an excerpt from a book [**71**], containing a lemma on existence, regularity and uniqueness of $c$ satisfying some linearized form of (2.1.4) with boundary conditions $c = h$ on $\partial\Omega$. This result is due to Łukaszewicz and can also be found in [**70**].

LEMMA 2.6.1. *Let $u \in V^0(\Omega)\cap C^0(\bar{\Omega})$, $b \in C^0(\bar{\Omega})$, $h \in C^{0,1}(\partial\Omega)$ and $p \in W^{1,\tilde{q}}(\Omega)$, where $\tilde{q}$ is defined as in Lemma 2.5.2. Let $D$, $K$ satisfy the assumptions (A1), (A2). Then there exists a unique solution $c \in W^{1,2}(\Omega)$ of the following problem:*

$$
\int_\Omega D(b)\nabla c\nabla\phi + \int_\Omega u\cdot\nabla c\phi = -\int_\Omega K(c)\nabla p\nabla\phi \qquad \forall\phi \in W_0^{1,2}(\Omega)\,, \tag{2.6.20}
$$

$$
c = h \qquad \text{on } \partial\Omega \tag{2.6.21}
$$

*and it satisfies inequality*

$$
\parallel c \parallel_1 \leq \mathrm{Const}([h]_{1/2,2} + |K|_\infty|\nabla p|_2)\,, \tag{2.6.22}
$$

*where* $\mathrm{Const} = \mathrm{Const}(|u|_\infty)$.

*Moreover, $c \in C^{0,\alpha}$ for some $\alpha \in (0,1]$, and $0 \leq c \leq 1$ in $\Omega$, provided $0 \leq h \leq 1$ on $\partial\Omega$, and*

$$
|c|_{0,\alpha,\Omega} \leq \mathrm{Const}\left(|h|_{0,1,\partial\Omega} + \frac{1}{m}([h]_{1/2,2} + |K|_\infty|\nabla p|_{\tilde{q}})\right)\,, \tag{2.6.23}
$$

*for* Const *depending on $|u|_\infty$ and $\tilde{q}$.*

## 2.7. Existence

In this section we prove Theorem 2.2.1.

PROOF. We consider a mapping

$$\Phi : C^0(\bar\Omega) \times C^0(\bar\Omega) \supset A \times B \to C^0(\bar\Omega) \times C^0(\bar\Omega)\,,$$

where $A$ is defined as in Lemma 2.5.2, and $B = \{b \in C^0(\bar\Omega) : |b|_\infty \leq 1\}$. Let us define $\Phi$ as

$$\Phi(u, b) = (u^*, c)\,,$$

where $u^*$, $c$ are the solutions of (2.5.13) and (2.6.20)–(2.6.21), respectively. In view of Lemmas 2.4.1, 2.5.1 and 2.6.1, the mapping is well defined. Lemmas 2.5.2 and 2.6.1 guarantee that $\Phi(A\times B) \subset A\times B$. Moreover, from (2.5.15) and (2.6.23), $\Phi(A\times B)$ is precompact in $C^0(\bar\Omega)\times C^0(\bar\Omega)$: by (2.5.15) we have $\| u^* \|_{2,\tilde q} \leq$ Const and the compact imbedding $W^{2,\tilde q}(\Omega) \hookrightarrow C^0(\bar\Omega)$ ($\tilde q > 3$) implies that the projection $\Pi_A(\Phi(A\times B))$ is precompact in $C^0(\bar\Omega)$. Similarly, (2.6.23) implies $\Pi_B(\Phi(A\times B))$ is precompact, by Arzela–Ascoli theorem.

Now, we shall show that the mapping $\Phi$ is continuous.

We have to prove that if $(u_n, b_n) \in A\times B$, $n = 1, 2, \ldots$ and $(u_n, b_n) \to (u, b) \in A\times B$ in $C^0(\bar\Omega)\times C^0(\bar\Omega)$, then $(u_n^*, c_n) \to (u^*, c)$ in $C^0(\bar\Omega)\times C^0(\bar\Omega)$, as $n \to \infty$.

By Lemma 2.5.3 $u_n^* \to u^*$ in $C^0(\bar\Omega)$ and $p_n \to p$ in $W^{1,2}(\Omega)$. Furthermore, from Lemmas 2.4.2, 2.4.3 we also have $\omega_n \to \omega$ in $W_0^{1,2}(\Omega)$. It remains to prove that $c_n \to c$ in $C^0(\bar\Omega)$.

For all $n$, concentrations $c_n$ satisfy identities

$$(2.7.24)\quad \int_\Omega D(b_n)\nabla c_n \nabla\phi + \int_\Omega u_n\cdot\nabla c_n\phi = -\int_\Omega K(c_n)\nabla p_n\nabla\phi \qquad \forall\phi\in W_0^{1,2}(\Omega)\,.$$

We shall show that the limit function $c$ satisfies a suitable identity.

As $(c_n)$ is bounded in $W^{1,2}(\Omega)$ and in $C^{0,\alpha}(\bar\Omega)$, there exists a subsequence $(c_\tau) \subset (c_n)$ and $\tilde c$ such that

$$(2.7.25)\qquad c_\tau \to \tilde c \text{ in } C^0(\bar\Omega) \qquad \text{and weakly in } W^{1,2}(\Omega)\,.$$

For $\phi \in W_0^{1,2}(\Omega)$ we have

$$\int_\Omega (D(b_\tau)\nabla c_\tau - D(b)\nabla\tilde c)\nabla\phi = \int_\Omega D(b)\nabla(c_\tau - \tilde c)\nabla\phi +$$

$$+\int_\Omega (D(b_\tau) - D(b))\nabla c_\tau\nabla\phi\,.$$

The first term in the right hand side tends to zero as $\tau \to \infty$ because $c_\tau \to \tilde{c}$ weakly in $W^{1,2}(\Omega)$. The second term also tends to zero, since $(c_\tau)$ is bounded in $\|\cdot\|_1$, $D$ is continuous and $b_\tau \to b$ in $C^0(\bar{\Omega})$. Hence,

$$\int_\Omega D(b_\tau)\nabla c_\tau \nabla\phi \to \int_\Omega D(b)\nabla\tilde{c}\nabla\phi, \qquad \tau \to \infty\,.$$

Similarly, we prove

$$\int_\Omega K(c_\tau)\nabla p_\tau \nabla\phi \to \int_\Omega K(\tilde{c})\nabla p\nabla\phi, \qquad \tau \to \infty\,.$$

We also have

$$\int_\Omega u_\tau \cdot \nabla c_\tau \phi \to \int_\Omega u \cdot \nabla\tilde{c}\phi, \qquad \tau \to \infty$$

because

$$\int_\Omega (u_\tau \cdot \nabla c_\tau - u \cdot \nabla\tilde{c})\phi = \int_\Omega u \cdot \nabla(c_\tau - \tilde{c})\phi + \int_\Omega (u_\tau - u)\cdot \nabla c_\tau \phi$$

and $c_\tau \to \tilde{c}$ weakly in $W^{1,2}(\Omega)$, $u_\tau \to u$ in $C^0(\bar{\Omega})$, $\nabla c_\tau$ is bounded in $|\cdot|_2$.

From the unique solvability of (2.6.20)–(2.6.21) we conclude that $\tilde{c} = c$, and the whole sequence $c_n \to c$ in $C^0(\bar{\Omega})$. (The latter is easy to prove by *reductio ad absurdum*: if not, then there would exist a subsequence $(c_\xi)$ such that $|c_\xi - c|_\infty > \varepsilon > 0$. Then we could choose another subsequence , $(c_\pi) \subset (c_\xi)$, which would be weakly convergent in $W^{1,2}(\Omega)$ and strongly in $C^0(\bar{\Omega})$, $c_\pi \to \tilde{\tilde{c}}$. However, once again passing to the limit in (2.7.24), we conclude that $\tilde{\tilde{c}} = c$ which is a contradiction to our assumption $|c_\pi - c|_\infty > \varepsilon > 0$.) Hence, $\Phi$ is continuous.

Therefore, the mapping $\Phi$ satisfies the assumptions of Schauder fixed–point theorem (see Theorem 2.3.4) and there exists $(u,c) \in A \times B$ such that

$$\Phi(u,c) = (u,c)$$

and $u$, $c$ together with $p$ and $\omega$ solve Problem 2.2.1 and they have the regularity stated in Theorem 2.2.1.

Norm estimates of $u$, $p$, $\omega$ and $c$ follow almost immediately from Lemmas 2.4.4 and 2.6.1, and inequality (2.5.15). $\square$

## 2.8. Uniqueness

In this section we prove Theorem 2.2.2. Before proceeding with the proof, we state (cf. [**70**]) higher regularity of $c$, provided the assumptions of Theorem 2.2.2 are satisfied.

LEMMA 2.8.1. *Let $h \in W^{1-1/r,r}(\partial\Omega)$, $3 < r \leq 6$. Then the weak solution $c$ of (2.1.4), (2.1.7) belongs to $W^{1,r}(\Omega)$ and*

$$\| c \|_{1,r} \leq \mathrm{Const}(|c|_\infty |u|_r + |K|_\infty |\nabla p|_r + |D|_\infty [h]_{1-1/r,r}) . \tag{2.8.26}$$

PROOF. As $h \in W^{1-1/r,r}(\partial\Omega)$, $3 < r \leq 6$, there exists an extension $\xi \in W^{1,r}(\Omega)$ such that $\xi = h$ on $\partial\Omega$. Let us set $\tilde{c} = c - \xi$. Then $\tilde{c} \in W_0^{1,2}(\Omega)$, and

$$\int_\Omega D(c) \nabla \tilde{c} \nabla \phi = \int_\Omega \tilde{g} \nabla \phi \qquad \text{for all } \phi \in W_0^{1,2}(\Omega) , \tag{2.8.27}$$

where $\tilde{g} = cu - K(c)\nabla p - D(c)\nabla\xi \in L^r(\Omega)$. Since $c \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha > 0$, we have $D(c) \in C^0(\bar{\Omega})$. Then Theorem 11.1 in [**86**] yields that problem (2.8.27) has a unique solution $\tilde{c} \in W_0^{1,r}(\Omega)$, with

$$\| \tilde{c} \|_{1,r} \leq \mathrm{Const}(|\tilde{g}|_r + |\tilde{c}|_r) .$$

Therefore, $c \in W^{1,r}(\Omega)$ and

$$\| c \|_{1,r} \leq \| \tilde{c} \|_{1,r} + \mathrm{Const}[h]_{1-1/r,r} \leq \mathrm{Const}(|\tilde{g}|_r + |\tilde{c}|_r + [h]_{1-1/r,r}) .$$

Since (2.8.27) is also uniquely solvable in $W_0^{1,2}(\Omega)$ with

$$|\nabla \tilde{c}|_2 \leq \frac{\mathrm{Const}}{m} |\tilde{g}|_r ,$$

then, as $W^{1,2}(\Omega) \hookrightarrow L^r(\Omega)$ for $1 \leq r \leq 6$, we have

$$|\tilde{c}|_r \leq \mathrm{Const} \, \| \tilde{c} \|_1 \leq \mathrm{Const} \, |\tilde{g}|_r ,$$

which ends the proof. □

PROOF. (of Theorem 2.2.2) Let us observe that any solution of Problem 2.2.1 satisfies estimates from Theorem 2.2.1, so in fact they are *a priori* estimates. This follows from Lemmas 2.4.3, 2.5.2 and 2.6.1.

Assuming that $(u_1, p_1, \omega_1, c_1)$ and $(u_2, p_2, \omega_2, c_2)$ are two solutions of Problem 2.2.1, we can see that the difference $(u_1 - u_2, p_1 - p_2, \omega_1 - \omega_2, c_1 - c_2)$ satisfies

$$\begin{aligned}(\nu + \nu_r) \int_\Omega \nabla(u_1 - u_2) \cdot \nabla v + \int_\Omega (u_1 \cdot \nabla)(u_1 - u_2) v = \\ = \int_\Omega ((u_2 - u_1) \cdot \nabla) u_2 v + 2\nu_r \int_\Omega \mathrm{curl}(\omega_1 - \omega_2) v + \\ + \int_\Omega (c_1 - c_2) g v \qquad \text{for all } v \in V^0(\Omega) ,\end{aligned} \tag{2.8.28}$$

$$(2.8.29) \qquad (c_a + c_d)\int_\Omega \nabla(\omega_1 - \omega_2)\cdot\nabla\xi + \int_\Omega (u_1\cdot\nabla)(\omega_1-\omega_2)\xi +$$

$$\int_\Omega ((u_1-u_2)\cdot\nabla)\omega_2\xi +$$

$$+ (c_0 + c_d - c_a)\int_\omega \operatorname{div}(\omega_1-\omega_2)\operatorname{div}\xi + 4\nu_r\int_\Omega(\omega_1-\omega_2)\xi =$$

$$= 2\nu_r \int_\Omega \operatorname{curl}(u_1-u_2)\xi \qquad \text{for all } \xi\in W_0^{1,2}(\Omega)\,,$$

$$(2.8.30) \qquad \int_\Omega D(c_1)\nabla(c_1-c_2)\nabla\phi = \int_\Omega G\cdot\nabla\phi \qquad \text{for all } \phi\in W_0^{1,2}(\Omega)\,,$$

where $G = c_1(u_1-u_2)+(c_1-c_2)u_2+(K(c_2)-K(c_1))\nabla p_2+K(c_1)\nabla(p_2-p_1)+(D(c_2)-D(c_1))\nabla c_2$. From estimates of solutions of elliptic equations [**45**] applied to (2.8.30) we deduce that

$$|c_1-c_2|_\infty \le \frac{\mathrm{Const}}{m}|G|_r\,,$$

where $3 < r \le \tilde{q}$. Our aim is to estimate $|G|_r$ in terms of $|c_1-c_2|_\infty$. By Theorem 2.2.1 and Lemma 2.5.2,

$$|c_1|_\infty \le 1 \quad \text{and} \quad |u_2|_r \le \mathrm{Const}(R+R^2) \quad \text{and} \quad |\nabla p_2|_r \le \mathrm{Const}\, P(R)\,,$$

where $R = |f|_q + |g|_q + |z|_2$ and $P$ is the same as in Theorem 2.2.1. Hence,

$$(2.8.31) \qquad |G|_r \le (|u_1-u_2|_r + \mathrm{Const}(R+R^2+L_K\cdot P(R))|c_1-c_2|_\infty +$$

$$+|K|_\infty|\nabla(p_1-p_2)|_r + L_D|\nabla c_2|_r|c_1-c_2|_\infty)\,.$$

Now, we shall estimate $|u_1-u_2|_r$.

Substituting $v = u_1 - u_2$ in (2.8.28) we have

(2.8.32)
$$
\begin{aligned}
(\nu + \nu_r) \parallel u_1 - u_2 \parallel_1^2 \leq \int_\Omega |((u_1 - u_2)\cdot \nabla) u_2 (u_1 - u_2)| &+ \\
+ 2\nu_r \parallel u_1 - u_2 \parallel_1 |\omega_1 - \omega_2|_2 &+ \\
+ \text{Const}\, |c_1 - c_2|_\infty |g|_q \parallel u_1 - u_2 \parallel_1 & \\
\leq \text{Const} \parallel u_1 - u_2 \parallel_1^2 \parallel u_2 \parallel_1 + 2\nu_r |\omega_1 - \omega_2|_2 \parallel u_1 - u_2 \parallel_1 &+ \\
+ \text{Const}\, |c_1 - c_2|_\infty |g|_q \parallel u_1 - u_2 \parallel_1 & .
\end{aligned}
$$

From (2.8.29) it follows that

$$
\begin{aligned}
|\omega_1 - \omega_2|_2 &\leq \text{Const} \parallel u_1 - u_2 \parallel_1 |\nabla \omega_2|_4 + \frac{1}{2} \parallel u_1 - u_2 \parallel_1 \\
&\leq \text{Const} \parallel u_1 - u_2 \parallel_1 \parallel \omega_2 \parallel_{2,2} + \frac{1}{2} \parallel u_1 - u_2 \parallel_1
\end{aligned}
$$

and from Lemmas 2.4.4 and 2.4.3 we have

$$
\parallel \omega_2 \parallel_{2,2} \leq \text{Const}(\parallel u_2 \parallel_1 + |u_2|_\infty \parallel \omega_2 \parallel_1 + |z|_2) \leq \text{Const}\, P_1(R)\,,
$$

where $P_1$ is a polynomial with property (P1), hence, substituting into (2.8.32) we obtain

$$
\begin{aligned}
(\nu + \nu_r) \parallel u_1 - u_2 \parallel_1^2 \leq \text{Const}_2\, R \parallel u_1 - u_2 \parallel_1^2 &+ \\
+ \text{Const}_3\, P_1(R) \parallel u_1 - u_2 \parallel_1^2 &+ \\
+ \nu_r \parallel u_1 - u_2 \parallel_1^2 + \text{Const}\, R |c_1 - c_2|_\infty \parallel u_1 - u_2 \parallel_1 & .
\end{aligned}
$$

If $R$ is small enough so that the inequality

$$
\nu - (\text{Const}_2\, R + \text{Const}_3\, P_1(R)) \geq \gamma
$$

holds for some positive constant $\gamma$, we then have

$$
\gamma \parallel u_1 - u_2 \parallel_1 \leq \text{Const}\, R |c_1 - c_2|_\infty \, .
$$

Therefore,

$$|u_1 - u_2|_r \leq \text{Const}\, R|c_1 - c_2|_\infty\,. \tag{2.8.33}$$

Further, we shall estimate the term $|\nabla(p_1 - p_2)|_r$. We have

$$\begin{cases} -(\nu + \nu_r)\Delta(u_1 - u_2) + \nabla(p_1 - p_2) = \tilde{f} & \text{in } \Omega\,, \\ \operatorname{div}(u_1 - u_2) = 0 & \text{in } \Omega\,, \\ u_1 - u_2 = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\tilde{f} = 2\nu_r \operatorname{curl}(\omega_1 - \omega_2) + (c_1 - c_2)g + ((u_2 - u_1)\cdot\nabla)u_2 + (u_1 \cdot \nabla)(u_2 - u_1)$. From Cattabriga's estimate it follows that

$$\| p_1 - p_2 \|_{1,r} \leq \text{Const}\, |\tilde{f}|_r \leq \text{Const}\, |\nabla(\omega_1 - \omega_2)|_r + |c_1 - c_2|_\infty |g|_r +$$

$$+ |u_2 - u_1|_r |\nabla u_2|_\infty + |u_1|_\infty |\nabla(u_2 - u_1)|_r\,.$$

By Theorem 2.2.1 and Lemma 2.5.2, $|u_1|_\infty$, $|\nabla u_2|_\infty \leq P_2(R)$ for some polynomial $P_2$ with property (P1), hence

$$|\nabla(p_1 - p_2)|_r \leq \text{Const}\, |\nabla(\omega_1 - \omega_2)|_r + \text{Const}\, R|c_1 - c_2|_\infty + \tag{2.8.34}$$

$$+ \text{Const}\, R \cdot P_2(R)|c_1 - c_2|_\infty + P_2(R)|\nabla(u_2 - u_1)|_r\,.$$

To estimate $|\nabla(\omega_1 - \omega_2)|_r$, we apply Nečas' estimate of solution of strongly elliptic system

$$-(c_a + c_d)\Delta(\omega_1 - \omega_2) - (c_0 + c_d - c_a)\nabla \operatorname{div}(\omega_1 - \omega_2) + 4\nu_r(\omega_1 - \omega_2) = \tilde{l}\,,$$

where

$$\tilde{l} = (u_1 \cdot \nabla)(\omega_2 - \omega_1) + ((u_2 - u_1)\cdot\nabla)\omega_2 + 2\nu_r \operatorname{curl}(u_1 - u_2)\,.$$

From Lemma 2.4.3 and estimates derived from (2.8.29) we have $\tilde{l} \in L^2$ and

$$
\begin{aligned}
|\tilde{l}|_2 &\leq |u_1|_\infty |\nabla(\omega_1-\omega_2)|_2 + |u_2-u_1|_4 |\nabla\omega_2|_4 + 2\nu_r \parallel u_1-u_2 \parallel_1 \\
&\leq P_2(R) \parallel \omega_1-\omega_2 \parallel_1 + \text{Const} \parallel u_1-u_2 \parallel_1 \parallel \omega_2 \parallel_{2,2} + 2\nu_r \parallel u_1-u_2 \parallel_1 \\
&\leq P_2(R)\,\text{Const} \parallel u_1-u_2 \parallel_1 (1+ \parallel \omega_2 \parallel_1) + \text{Const} \parallel u_1-u_2 \parallel_1 \parallel \omega_2 \parallel_{2,2} + \\
&\quad + 2\nu_r \parallel u_1-u_2 \parallel_1 \\
&\leq \text{Const}\, P_3(R) \cdot \parallel u_1-u_2 \parallel_1 ;
\end{aligned}
$$

hence

$$
\begin{aligned}
|\nabla(\omega_1-\omega_2)|_r &\leq \text{Const} \parallel \omega_1-\omega_2 \parallel_{2,2} \leq \text{Const}\, |\tilde{l}|_2 \\
&\leq \text{Const}\, P_3(R) ||u_1-u_2||_1 \leq \text{Const}\, P_3(R)\cdot R \cdot |c_1-c_2|_\infty \qquad (2.8.35) \\
&\leq \text{Const}\, P_4(R) |c_1-c_2|_\infty,
\end{aligned}
$$

and $P_4$ is a polynomial with property (P1).

To estimate $|\nabla(u_1-u_2)|_r$, we shall apply Cattabriga's estimate once again to (2.8).

Since $\tilde{f} \in L^r(\Omega)$, we have by definition

$$
\parallel \tilde{f} \parallel_{-1,r} = \sup \left\{ \left| \int_\Omega \tilde{f}\phi \right| : \parallel \phi \parallel_{1,s} \leq 1 \right\},
$$

where $\frac{1}{s}+\frac{1}{r}=1$, and, since $r \in (3,6]$, it follows that $s \in \left[\frac{6}{5},\frac{3}{2}\right)$. In consequence, $W^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$. Before we estimate $\parallel \tilde{f} \parallel_{-1,r}$, let us observe that

$$
\int_\Omega (u\cdot\nabla)v\phi = -\int_\Omega (u\cdot\nabla)\phi v
$$

for all $u \in V^0(\Omega)$, $v \in W_0^{1,r}(\Omega)$, $\phi \in W_0^{1,s}(\Omega)$ with $r$, $s$ as above.

Now, for $\phi \in W_0^{1,s}(\Omega)$, $\| \phi \|_{1,s} \leq 1$, we have by Hölder inequality,

$$\left| \int_\Omega \tilde{f} \phi \right| \leq 2\nu_r |\omega_1 - \omega_2|_r |\operatorname{curl} \phi|_s + \text{Const}\, |c_1 - c_2|_\infty |g|_q |\phi|_2 +$$

$$+ |u_1 - u_2|_r |\nabla \phi|_s$$

$$\leq \big(\text{Const}\, \| \omega_1 - \omega_2 \|_1 + \text{Const}\, R|c_1 - c_2|_\infty + 2P_2(R)|u_1 - u_2|_r\big) \cdot ||\phi||_{1,s}$$

$$\leq \text{Const}\, \| u_1 - u_2 \|_1 \,(1 + \| \omega_2 \|_1) + \text{Const}\, R|c_1 - c_2|_\infty +$$

$$+ \text{Const}\, P_2(R) \, \| u_1 - u_2 \|_1$$

$$\leq \text{Const}\, P_5(R)|c_1 - c_2|_\infty$$

and polynomial $P_5$ has property (P1). Therefore, $\| \tilde{f} \|_{-1,r} \leq P_5(R)|c_1 - c_2|_\infty$ and hence, by Cattabriga's inequality,

$$|\nabla(u_1 - u_2)|_r \leq \text{Const}\, P_5(R)|c_1 - c_2|_\infty \,, \tag{2.8.36}$$

which, together with (2.8.34) and (2.8.35) yields

$$\begin{aligned} |\nabla(p_1 - p_2)|_r &\leq \text{Const}(P_4(R) + R + R \cdot P_2(R) + P_2(R)P_5(R))|c_1 - c_2|_\infty = \\ &= P_6(R)|c_1 - c_2|_\infty \end{aligned} \tag{2.8.37}$$

and, of course, $P_6$ is a polynomial with property (P1).

Now, substituting (2.8.33), (2.8.37) and (2.8.26) into (2.8.31) we have

$$|G|_r \leq \text{Const}(P_2(R) + P(R) + P_6(R) + [h]_{1-1/r,r})|c_1 - c_2|_\infty \,.$$

Hence,

$$|c_1 - c_2|_\infty \leq \frac{\text{Const}}{m} |G|_r \leq (P_7(R) + \text{Const}_4 [h]_{1-1/r,r})|c_1 - c_2|_\infty$$

and again $P_7$ is a polynomial with property (P1).

Therefore, if only $R = |f|_q + |g|_q + |z|_2$ and $[h]_{1-1/r,r}$ are small enough so that

$$P_7(R) + \text{Const}_4 [h]_{1-1/r,r} < 1 \,,$$

then we arrive at

$$|c_1 - c_2|_\infty < |c_1 - c_2|_\infty \, ,$$

which means that $c_1 = c_2$, and, in consequence, Problem 2.2.1 is uniquely solvable. $\square$

CHAPTER 3

# Mixed finite element method for stationary flow of a micropolar fluid

In this chapter, we prove existence and uniqueness results for a finite element discretization of micropolar equations. Then we analyse the Newton method for solution of the discrete system. These results appeared in [**59**].

## 3.1. Introduction

We consider a mixed finite element discretization of the following system of partial differential equations with Dirichlet boundary conditions, which describes the stationary flow of a micropolar fluid:

(3.1.1)

$$\begin{cases} -(\nu+\nu_r)\Delta u + (u\cdot\nabla)u + \nabla p = 2\nu_r \operatorname{curl}\omega + f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ -(c_a+c_d)\Delta\omega + (u\cdot\nabla)\omega - (c_0+c_d-c_a)\nabla\operatorname{div}\omega + 4\nu_r\omega = 2\nu_r\operatorname{curl} u + g & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \\ \omega = \omega_0 & \text{on } \partial\Omega \end{cases}$$

For short, we denote $c_1 = c_a + c_d, c_2 = c_0 + c_d - c_a$. Following [**69**], we assume that $c_1, c_2 > 0$.

Some results of numerical simulations obtained for the micropolar equations can be found in [**10**] and [**51**], however, without rigorous mathematical justification of the numerical schemes being used. In [**13**] spectral approximation of evolutionary micropolar problem has been considered, with no straightforward relations to numerical approximation.

Here, we provide an in-depth analysis of a mixed finite element approximation of the micropolar system (1.1.2) with Dirichlet boundary conditions.

It has been proved in [**69**], that in the case of homogeneous boundary conditions and bounded domain $\Omega$ with sufficiently smooth boundary, there exists a weak solution of (3.1.1) for any $f, g \in L^2(\Omega)$ and any positive $\nu, \nu_r, c_1, c_2$. This solution is also unique provided the data $f, g$ are relatively small with respect to $\nu, \nu_r, c_1, c_2$. We formulate these theorems precisely in Section 3.7.

In this chapter, we establish counterparts of results mentioned above, for a mixed finite element discretization of (3.1.1), see Theorem 3.5.1. Using theory of nonsingular branches by Brezzi, Rappaz and Raviart [**46**, Chapter IV.3], we also prove an approximation result for the discrete solution. In Theorem 3.6.6, we show that if the solution is far from a bifurcation point, then there exists a locally unique solution of the discrete problem, and the error is of optimal order with respect to the mesh size $h$. To the author's knowledge, these results are new in the micropolar fluids context and they generalize analogous results for the Navier–Stokes equations.

We also investigate the Newton method for the solution of the resulting system of discrete algebraic nonlinear equations. Assuming that the initial guess is sufficiently close to the solution, we prove that the method is quadratically convergent and that its convergence rate is independent of $h$.

Each iteration step requires solution of a linear system with a nonsymmetric indefinite matrix, which is ill conditioned with respect to the mesh size $h$. Employing block preconditioners, in Chapter 5 we propose and analyse a preconditioning method for the linear system.

A natural question arises, if it is possible to generalize our results to the barodiffusion problem for a micropolar fluid (2.1.1)–(2.1.7). A source of severe difficulties is the barodiffusion term $\operatorname{div}(K(c)\nabla p)$. Our preliminary results on this subject [**58**] are rather disappointing, due to very strong (and, in our opinion, too restrictive) assumptions. For this complicated problem, where an implicit algebraic condition $0 \le c \le 1$ must be satisfied (otherwise the solution has no physical meaning), we believe that the approximation scheme has to be enhanced in some way, beyond the classical formulation.

The chapter is organized as follows. We begin with auxiliary theorems related to the finite element method for the Stokes and Navier–Stokes equations. In Section 3.3, we introduce the notation, while in Section 3.4 we formulate the problem in variational setting and define the discrete problem using mixed Galerkin method. For simplicity of presentation we discretize the problem using the well known Taylor-Hood element. In Section 3.5 we prove a general existence result for the discrete problem. Local uniqueness and approximation results are proved in Section 3.6, using the nonsingular solutions branches theory. In Section 3.7, we show that the nonsingularity assumption is non-void. Section 3.8 is devoted to the analysis of Newton's method for solving the discrete system.

## 3.2. Background results on mixed approximation of the Stokes and Navier–Stokes equations

**3.2.1. Approximation.** We discretize the three-dimensional Stokes problem, see Section 2.3.4, with the popular Taylor–Hood element, see Figure 3.2.1. To this end, let us assume that $\Omega$ is a polyhedron with Lipschitz boundary.

We cover $\bar{\Omega}$ with a quasi-uniform shape regular triangulation [**90**] $\mathcal{T}_h$, dividing $\bar{\Omega}$ into tetrahedra $K$:

$$\bigcup_{K\in\mathcal{T}_h} K = \bar{\Omega},$$

with mesh parameter $h$. In addition, we assume, cf. [**8**], [**90**], that each $K \in \mathcal{T}_h$ has at least one vertex inside $\Omega$. For approximation of the velocity $u$ and the pressure $p$ we shall use the Taylor – Hood finite element spaces $V_h, W_h$ (see, e.g. [**21**] and Figure 3.2.1),

$$V_h = \{v \in V \cap C^0(\bar{\Omega}) : v\big|_K \in P_2(K) \quad \forall K \in \mathcal{T}_h\},$$

and

$$W_h = \{w \in W \cap C^0(\bar{\Omega}) : w\big|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\}.$$

It is well known that then $V_h$ and $W_h$ satisfy the Ladyzhenskaya-Babuška-Brezzi condition, see [**90**], [**8**].

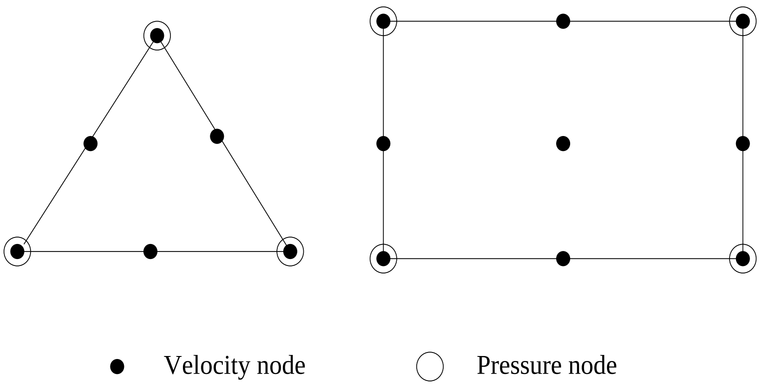


FIGURE 3.2.1. Taylor–Hood finite elements in 2-D: triangular elements $P_2$–$P_1$ (left) and rectangular elements $Q_2$–$Q_1$ (right).

The discrete Stokes problem is then defined as follows:

PROBLEM 3.2.1. Given $f \in H^{-1}(\Omega)$, find a pair $(u_h, p_h) \in V_h \times W_h$, which satisfies

$$\begin{cases} (\nabla u_h,\, \nabla v) + (p_h,\, \operatorname{div} v) = \langle f,\, v\rangle & \forall v \in V_h, \\ (\operatorname{div} u_h,\, w) = 0 & \forall w \in W_h. \end{cases} \tag{3.2.2}$$

Here, and in what follows, $(\cdot,\,\cdot)$ denotes the inner product in $L^2$; $\langle\cdot,\,\cdot\rangle$ is the dual pairing between $H_0^1$ and $H^{-1}$, see Appendix A.

THEOREM 3.2.1 (see e.g. [**46**, Theorem II.4.3] and [**90**]). *If the solution $(u, p)$ of Stokes system (2.3.9) belongs to $H^{k+1}(\Omega) \times H^k(\Omega)$, $k = 1$ or 2, then there exists a unique solution of Problem 3.2.1. Moreover, this solution satisfies the following error estimate:*

$$|u - u_h|_{1,2} + |p - p_h|_0 \leq \text{Const} \cdot h^k \left(||u||_{k+1,2} + ||p||_{k,2}\right).$$

Under slightly stronger assumptions one can also obtain higher approximation order on $u$ in $L^2$-norm.

For a general abstract result on approximation of a saddle point problem, see [**21**, Proposition II.2.4 and Theorem VI.5.1], or [**46**, Theorem II.1.1].

**3.2.2. Nonsingular branches theory.** Solutions of Navier-Stokes equations, guaranteed by Theorem 2.3.8, may be, in general, not unique. However, in practice those multiple solutions are usually well isolated, the fact which is mathematically formulated in [**92**]. Therefore, one may attempt to approximate the solution in a neighbourhood where it is unique. Since often the solutions depend continuously on the viscosity $\nu$, a *continuation method* may be used to trace the isolated branch of solutions.

In this paragraph, we briefly introduce the theory on the approximation of branches of nonsingular solutions of a class of nonlinear problems, developed by Brezzi, Rappaz and Raviart, see [**46**, Chapter IV.3]. In particular, Navier-Stokes equations fit into this class.

We consider a mapping $F : \Lambda \times X \to X$, where $X$ is a Banach space and $\Lambda$ is a compact interval of positive reals, $R_+$. We assume that $F$ is a $C^2$-mapping. We want to find pairs $(\lambda, \mathcal{U})$ which solve

$$F(\lambda, \mathcal{U}) = 0. \tag{3.2.3}$$

DEFINITION 3.2.1. We shall say that $\{(\lambda, \mathcal{U}(\lambda)) : \lambda \in \Lambda\}$ is a *nonsingular branch of solutions* of (3.2.3), if

1. $F(\lambda, \mathcal{U}(\lambda)) = 0$ for all $\lambda \in \Lambda$,
2. $\lambda \mapsto \mathcal{U}(\lambda)$ is a continuous mapping from $\Lambda$ into $X$,

3. $D_{\mathcal{U}}F(\lambda,\mathcal{U}(\lambda)) : X \to X$ is an isomorphism onto $X$ for all $\lambda \in \Lambda$.

Let us restrict ourselves to mappings of the form

$$F(\lambda,\mathcal{U}) \equiv \mathcal{U} + T \cdot G(\lambda,\mathcal{U}),$$

where $T : Y \to X$ is a bounded linear operator, $G : \Lambda \times X \to Y$ is a $C^2$-mapping, and $Y$ is a Banach space.

In order to approximate problem (3.2.3) we introduce a family of bounded linear operators $T_h : Y \to X$, $h > 0$, which approximate operator $T$ in the following sense:

$$\lim_{h\to 0^+} ||(T_h - T)f||_X = 0 \qquad \forall f \in X.$$

Furthermore, we assume that the range of $T_h$ is contained in a finite-dimensional subspace $X_h$ of $X$ (as it is in the case of finite element approximation).

We define the approximate problem (3.2.3) by

$$F_h(\lambda,\mathcal{U}_h) \equiv \mathcal{U}_h + T_h \cdot G(\lambda,\mathcal{U}_h) = 0. \tag{3.2.4}$$

Before we state the general existence and approximation theorem for (3.2.4), we additionally assume that there exists another Banach space $Z$, compactly and continuously imbedded in $Y$, such that $D_{\mathcal{U}}G(\lambda,\mathcal{U})$ is a bounded linear operator from $X$ into $Z$. Note that under these assumptions, $D_{\mathcal{U}}F(\lambda,\mathcal{U}) = I + T \cdot D_{\mathcal{U}}G(\lambda,\mathcal{U})$ is a compact perturbation of the identity operator.

THEOREM 3.2.2 (Brezzi, Rappaz and Raviart, see [**46**, Theorem IV.3.3]). *In the framework defined above, suppose that $\{(\lambda,\mathcal{U}(\lambda)) : \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (3.2.3). In addition, assume that $D^2_{\mathcal{U}\mathcal{U}}G$ is bounded on all bounded subsets of $\Lambda \times X$. Then there exists a neighbourhood $\mathcal{O}$ of the origin in $X$ and $h_0$ (small enough), such that for all $h \le h_0$ there exists a unique $C^2$-mapping $\lambda \mapsto \mathcal{U}_h(\lambda)$ with the following properties:*

(i) $\{(\lambda,\mathcal{U}(\lambda)) : \lambda \in \Lambda\}$ *is a branch of nonsingular solutions of (3.2.4),*

(ii) $\mathcal{U}_h(\lambda) - \mathcal{U}(\lambda) \in \mathcal{O}$ *for all* $\lambda \in \Lambda$,

(iii) *there exists* Const *independent of* $h, \lambda$, *for which*

$$||\mathcal{U}_h(\lambda) - \mathcal{U}(\lambda)||_X \le \text{Const}\, ||(T_h - T) \cdot G(\lambda,\mathcal{U}(\lambda))||_X$$

*for all* $\lambda \in \Lambda$.

## 3.3. General assumptions

Throughout this chapter we assume that $\Omega$ is an open, bounded polyhedron in $R^3$ with Lipschitz continuous boundary.

Let us define two auxiliary trilinear forms related to the convective term in (3.1.1),

$$d_1(u,\, v,\, w) = ((u\cdot\nabla)v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j, \tag{3.3.5}$$

already introduced in Section 2.3.4 and, following [**91**],

$$d_2(u,\, v,\, w) = \frac{1}{2}\Big( ((u\cdot\nabla)v, w) - ((u\cdot\nabla)w, v) \Big) \tag{3.3.6}$$

for any $u, v, w \in [H^1(\Omega)]^3$. Note that these forms are continuous on $[H^1(\Omega)]^3 \times [H_0^1(\Omega)]^3 \times [H_0^1(\Omega)]^3$ and if $\operatorname{div} u = 0$ then $d_1(u,\, \cdot,\, \cdot) \equiv d_2(u,\, \cdot,\, \cdot)$. By definition, the form $d_2(\cdot,\, \cdot,\, \cdot)$ is skew-symmetric with respect to the last two arguments (this reflects the analogous property of $((u\cdot\nabla)v, w)$ on the solution $u$ of (3.1.1)). In particular, we have

$$d_2(u,\, v,\, v) = 0 \qquad \forall u \in V^0, v \in H_0^1(\Omega). \tag{3.3.7}$$

As in previous chapters, "Const" denotes a generic positive constant, which, if necessary, will be distinguished using subscripts.

## 3.4. Discrete problem

We pose our original problem (3.1.1) in a variational form, using the following function spaces:

$$V := [H_0^1(\Omega)]^3, \qquad W := L_0^2(\Omega).$$

According to notation introduced in Appendix A, we denote the usual inner product in $L^2$ and $W$ by $(\cdot,\, \cdot)$, and in $V$ by $((\cdot, \cdot)) = (\nabla\cdot,\, \nabla\cdot)$. In the rest of this chapter, we assume that the data for problem (3.1.1) consist of $f, g \in [L^2(\Omega)]^3$ and the boundary conditions on $u, \omega$ are homogeneous. Then the variational formulation of (3.1.1) takes the form:

PROBLEM 3.4.1. Find $(u,p,\omega) \in V\times W\times V$ such that for all $(v,q,\xi)$ in $V\times W\times V$

$$
\tag{3.4.8}
\begin{cases}
(\nu+\nu_r)(\nabla u,\nabla v)+d(u,\,u,\,v)-(p,\operatorname{div} v)=2\nu_r(\operatorname{curl}\omega,v)+(f,v),\\[2ex]
(\operatorname{div} u,q)=0,\\[2ex]
c_1(\nabla\omega,\nabla\xi)+d(u,\,\omega,\,\xi)+c_2(\operatorname{div}\omega,\operatorname{div}\xi)+4\nu_r(\omega,\xi)=2\nu_r(\operatorname{curl} u,\xi)+(g,\xi),
\end{cases}
$$

where $d(\cdot,\,\cdot,\,\cdot)$ denotes either the form $d_1(\cdot,\,\cdot,\,\cdot)$ or $d_2(\cdot,\,\cdot,\,\cdot)$.

We triangulate $\Omega$ as described in Section 3.2. For approximation of the velocity $u$ and the pressure $p$ we use the Taylor – Hood finite element spaces $V_h, W_h$ (see Section 3.2). The microrotation field $\omega$ is approximated in the same space $V_h$ as the velocity $u$. The mixed finite element formulation of the approximate problem (3.1.1) is then as follows:

PROBLEM 3.4.2. Find $(u_h,p_h,\omega_h) \in V_h\times W_h\times V_h$, such that

$$
\tag{3.4.9}
\begin{cases}
(\nu+\nu_r)(\nabla u_h,\nabla v)+d(u_h,\,u_h,\,v)-(p_h,\operatorname{div} v)\\[1ex]
\qquad\qquad\qquad\qquad =2\nu_r(\operatorname{curl}\omega_h,v)+(f,v),\\[2ex]
(\operatorname{div} u_h,q)=0,\\[2ex]
c_1(\nabla\omega_h,\nabla\xi)+d(u_h,\,\omega_h,\,\xi)+c_2(\operatorname{div}\omega_h,\operatorname{div}\xi)+4\nu_r(\omega_h,\xi)\\[1ex]
\qquad\qquad\qquad\qquad =2\nu_r(\operatorname{curl} u_h,\xi)+(g,\xi),
\end{cases}
$$

for all $(v,q,\xi)$ in $V_h\times W_h\times V_h$.

## 3.5. Existence

In the case when the form $d(\cdot,\,\cdot,\,\cdot)$ is skew-symmetric with respect to the last two arguments, that is, $d(\cdot,\,\cdot,\,\cdot)=d_2(\cdot,\,\cdot,\,\cdot)$, the following general existence result holds:

THEOREM 3.5.1. *For any $f,g\in L^2$ and any positive $\nu,\nu_r,c_1,c_2$ there exists at least one triple $(u_h,p_h,\omega_h)\in V_h\times W_h\times V_h$ which solves the discrete nonlinear system (3.4.9) with $d(\cdot,\,\cdot,\,\cdot)=d_2(\cdot,\,\cdot,\,\cdot)$. Moreover, the solution is unique, provided that the $L^2$-norms of the data $f,g$ are sufficiently small with respect to $\nu,\nu_r,c_1,c_2$.*

PROOF. The "small data" assumption in Theorem 3.5.1 reflects similar requirements for the continuous case, see [**69**]. Actually, the proof of uniqueness is similar to that in [**69**], therefore we omit the details here.

The proof of existence is based on a variant of Brouwer's fixed point theorem, see Theorem 2.3.5. Let us introduce an auxiliary space of weakly divergence free[1] functions,

$$V_h^0 = \{v \in V_h : (\operatorname{div} v, w) = 0 \quad \forall w \in W_h\}.$$

This is a finite dimensional Hilbert space with the inner product $((\cdot,\cdot))$ derived from $V_h$, i.e. $((u, v)) = (\nabla u, \nabla v)$ for any $u, v \in V_h$. We define a nonlinear operator $P : V_h^0 \to V_h^0$ as follows:

$$((P(u), v)) = (\nu + \nu_r)(\nabla u, \nabla v) + d_2(u,\, u,\, v) - 2\nu_r(\operatorname{curl}\omega, v) - (f, v), \quad \forall v \in V_h^0, \tag{3.5.10}$$

where $\omega \in V_h$ satisfies

$$\begin{aligned} c_1(\nabla\omega, \nabla\xi) + d_2(u,\, \omega,\, \xi) + c_2(\operatorname{div}\omega, \operatorname{div}\xi) + 4\nu_r(\omega, \xi) \\ = 2\nu_r(\operatorname{curl} u, \xi) + (g, \xi), \quad \forall \xi \in V_h. \end{aligned} \tag{3.5.11}$$

It follows from (3.3.7) that the left hand side of (3.5.11) defines a bilinear form, which is continuous and $V_h$-elliptic. Since the right hand side of equation (3.5.11) represents a bounded linear functional of $\xi \in V_h$, it follows from Lax-Milgram theorem that for given $u \in V_h^0$ there exists a unique $\omega \in V_h$ for which (3.5.11) holds. Then we conclude that the right hand side of (3.5.10) is a well defined bounded linear functional on $V_h^0$, and hence Riesz theorem guarantees that $P(u)$ is a uniquely determined element of $V_h^0$ for all $u \in V_h^0$.

LEMMA 3.5.2. *There exists $r > 0$ such that for all $u \in V_h^0$ with $||u||_1 = r$,*

$$((P(u), u)) \geq 0.$$

PROOF. First, we estimate the $L^2$ norm of $\omega$ from the definition of $P(u)$. Setting $\xi = \omega$ in (3.5.11) and using (3.3.7), we get

$$c_1||\omega||_1^2 + c_2|\operatorname{div}\omega|_2^2 + 4\nu_r|\omega|_2^2 \leq 2\nu_r|(\operatorname{curl} u, \omega)| + |g|_2|\omega|_2, \tag{3.5.12}$$

which yields, along with $|\operatorname{curl} u|_2 \leq ||u||_1$ for $u$ in $H_0^1$,

$$4\nu_r|\omega|_2 \leq 2\nu_r|\operatorname{curl} u|_2 + |g|_2 \leq 2\nu_r||u||_1 + |g|_2. \tag{3.5.13}$$

[1]Note, however, that in general $V_h^0$ is *not* a subspace of $V^0$.

Then, using the fact that $(\operatorname{curl} u, \omega) = (\operatorname{curl} \omega, u)$ for $u, \omega$ in $H_0^1$, we obtain from (3.5.10)

$$
\begin{aligned}
((P(u), u)) &= (\nu + \nu_r)||u||_1^2 - 2\nu_r(\operatorname{curl}\omega, u) - (f, u) \\
&\geq (\nu + \nu_r)||u||_1^2 - 2\nu_r|\omega|_2||u||_1 - \text{Const}\,|f|_2||u||_1 \\
&\geq ||u||_1 \left( (\nu + \nu_r)||u||_1 - 2\nu_r\big(\frac{1}{2}||u||_1 + \frac{1}{4\nu_r}|g|_2\big) - \text{Const}\,|f|_2 \right),
\end{aligned}
$$

and hence

$$((P(u), u)) \geq ||u||_1\big(\nu||u||_1 - \text{Const}_1(|f|_2 + |g|_2)\big).$$

Now, $\nu$ is positive, so for $r > \dfrac{\text{Const}_1}{\nu}(|f|_2 + |g|_2) > 0$ we obtain $((P(u), u)) \geq 0$, which completes the proof of the lemma. $\square$

Using Lemma 3.5.2, it follows from Theorem 2.3.5 that there is at least one $u \in V_h^0$, for which $P(u) = 0$.

Observe that if $P(u) = 0$ for some $u \in V_h^0$, then $u$ and corresponding $\omega$ satisfy the first and the third equation of (3.4.9), with $v, \xi$ varying over $V_h^0$ and $V_h$, respectively. Obviously, the second equation of (3.4.9) is implicitly satisfied, since $u \in V_h^0$.

In other words, there exists at least one pair $(u, \omega) \in V_h \times V_h$ that satisfies

$$
(3.5.14) \qquad
\begin{cases}
(\nu + \nu_r)(\nabla u, \nabla v) + d_2(u,\, u,\, v) = 2\nu_r(\operatorname{curl}\omega, v) + (f, v), \\
(\operatorname{div} u, q) = 0, \\
c_1(\nabla\omega, \nabla\xi) + d_2(u,\, \omega,\, \xi) + c_2(\operatorname{div}\omega, \operatorname{div}\xi) + 4\nu_r(\omega, \xi) \\
\qquad\qquad = 2\nu_r(\operatorname{curl} u, \xi) + (g, \xi),
\end{cases}
$$

for all $(v, q, \xi) \in V_h^0 \times W_h \times V_h$. Hence from the LBB condition (see [**46**, Lemma I.4.1] and Section 2.3.4) we conclude that the original problem (3.4.9) admits at least one solution $(u, p, \omega) \in V_h \times W_h \times V_h$. $\square$

### 3.6. Approximation of nonsingular solutions of Problem 3.4.1

In this section we prove local uniqueness and approximation results for Problem 3.4.2 using the theory of branches of nonsingular solutions of nonlinear equations

$$F(\lambda, \mathcal{U}) \equiv \mathcal{U} + T \cdot G(\lambda, \mathcal{U}) = 0, \tag{3.6.15}$$

as described in Section 3.2.

Let $d(\cdot,\,\cdot,\,\cdot)$ be either $d_1(\cdot,\,\cdot,\,\cdot)$ or $d_2(\cdot,\,\cdot,\,\cdot)$ and let us set $\lambda = (\nu + \nu_r)^{-1}$. Then Problem 3.4.1 is equivalent to finding a solution $\mathcal{U}(\lambda) = (u(\lambda), \lambda p(\lambda), \omega(\lambda))$ of

$$F(\lambda, \mathcal{U}(\lambda)) = 0, \tag{3.6.16}$$

where $F$ is defined as in (3.6.15), with $G$ and $T$ chosen as follows. We define $G : R_+ \times (V \times W \times V) \to H^{-1} \times H^{-1}$ by

$$G(\lambda, \mathcal{U}) = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

where

$$< \theta_1, v >= \lambda \cdot \big(d(u,\, u,\, v) - 2\nu_r(\operatorname{curl} \omega, v) - (f, v)\big) \qquad \forall v \in V,$$
$$< \theta_2, v >= d(u,\, \omega,\, v) - 2\nu_r(\operatorname{curl} u, v) - (g, v) \qquad \forall v \in V.$$

Here, $< \cdot, \cdot >$ denotes the dual pairing between $H_0^1$ and $H^{-1}$. The linear operator $T : H^{-1} \times H^{-1} \to V \times W \times V$ is defined by

$$T \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = (u^*, p^*, \omega^*),$$

where the triple $(u^*, p^*, \omega^*)$ satisfies

$$\begin{cases} (\nabla u^*, \nabla v) - (p^*, \operatorname{div} v) =< \theta_1, v > & \forall v \in V, \\ \\ (\operatorname{div} u^*, w) = 0 & \forall w \in W, \\ \\ c_1(\nabla \omega^*, \nabla \xi) + c_2(\operatorname{div} \omega^*, \operatorname{div} \xi) + 4\nu_r(\omega^*, \xi) =< \theta_2, \xi > & \forall \xi \in V. \end{cases} \tag{3.6.17}$$

LEMMA 3.6.1. *The mapping $T$ is well defined and is a bounded linear operator from $H^{-1} \times H^{-1}$ onto $V \times W \times V$.*

PROOF. The first two equations of (3.6.17) are Stokes equations in mixed variational setting. The third equation which is independent of the first two is a $V$-elliptic one. The existence of a unique pair $(u^*, p^*) \in V \times W$ satisfying the first two equations follows from the theory of saddle point problems, see Theorem 2.3.6, while Riesz theorem guarantees the existence and uniqueness of $\omega^* \in V$ for which the third equation of (3.6.17) holds. $\square$

Next, we define the approximate operator $T_h : H^{-1} \times H^{-1} \to V_h \times W_h \times V_h$ as

$$T_h \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = (u_h^*, p_h^*, \omega_h^*),$$

where $(u_h^*, p_h^*, \omega_h^*) \in V_h \times W_h \times V_h$ is a solution of the following variational problem

$$\left\{ \begin{array}{ll} (\nabla u_h^*, \nabla v) - (p_h^*, \operatorname{div} v) = < \theta_1, v > & \forall v \in V_h, \\ (\operatorname{div} u_h^*, w) = 0 & \forall w \in W_h, \\ c_1(\nabla \omega_h^*, \nabla \xi) + c_2(\operatorname{div} \omega_h^*, \operatorname{div} \xi) + 4\nu_r(\omega_h^*, \xi) = < \theta_2, \xi > & \forall \xi \in V_h. \end{array} \right. \tag{3.6.18}$$

Lemma 3.6.2. *The mapping $T_h$ is well defined and is a bounded linear operator from $H^{-1} \times H^{-1}$ onto $V_h \times W_h \times V_h$.*

Proof. The proof is analogous to that of the previous lemma. We use the fact that $V_h$ and $W_h$ satisfy the LBB condition. □

Having $T_h$ and $G$ defined, it can be easily verified that $\mathcal{U}_h = (u_h, \lambda p_h, \omega_h)$ satisfies

$$F_h(\lambda, \mathcal{U}_h) \equiv \mathcal{U}_h + T_h \cdot G(\lambda, \mathcal{U}_h) = 0 \tag{3.6.19}$$

if and only if $(u_h, p_h, \omega_h)$ is the solution of Problem 3.4.2.

Now, we shall examine some properties of mappings introduced above.

Lemma 3.6.3. *Suppose $\mathcal{W} = (v, q, z)$ is an arbitrary element of $V \times W \times V$. Then for any $\lambda \in R_+$ and any $\mathcal{U} = (u, p, \omega) \in V \times W \times V$,*

$$D_{\mathcal{U}} G(\lambda, \mathcal{U}) \cdot \mathcal{W} \in L^{3/2} \times L^{3/2},$$

*where $D_{\mathcal{U}} G$ denotes Fréchet derivative of $G$ with respect to the second argument.*

Proof. We apply an argument similar to that in [**46**], Theorem IV.4.1. The first derivative of $G$ with respect to $\mathcal{U}$ is given by

$$D_{\mathcal{U}} G(\lambda, \mathcal{U}) \cdot \mathcal{W} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{3.6.20}$$

where

$$< \eta_1, \xi >= \lambda\big(d(u,\, v,\, \xi) + d(v,\, u,\, \xi) - 2\nu_r(\operatorname{curl} z, \xi)\big) \qquad \forall \xi \in V, \tag{3.6.21}$$

and

$$< \eta_2, \xi >= d(u,\, z,\, \xi) + d(v,\, w,\, \xi) - 2\nu_r(\operatorname{curl} v, \xi) \qquad \forall \xi \in V. \tag{3.6.22}$$

Now, let us consider the case when the form $d(\cdot,\,\cdot,\,\cdot) = d_1(\cdot,\,\cdot,\,\cdot)$. Then $\eta_1, \eta_2$ may be identified with $\eta_1 = \lambda\big((u\cdot\nabla)v + (v\cdot\nabla)u - 2\nu_r \operatorname{curl} z\big)$, $\eta_2 = (u\cdot\nabla)z + (v\cdot\nabla)w - 2\nu_r \operatorname{curl} v$. Since $H_0^1 \hookrightarrow L^6$, it can be easily shown using Hölder inequality that $\eta_1$ and $\eta_2$ belong to $L^{3/2}$.

If $d(\cdot,\,\cdot,\,\cdot) = d_2(\cdot,\,\cdot,\,\cdot)$, then the situation is quite similar. Integrating by parts the convective term, we get

$$((u\cdot\nabla)w, v) = -\Big((\operatorname{div} u \cdot w, v) + ((u\cdot\nabla)v, w)\Big),$$

so, substituting into the definition of $d_2(\cdot,\,\cdot,\,\cdot)$ in (3.3.6), we obtain

$$d_2(u,\, v,\, w) = ((u\cdot\nabla)w, v) + \frac{1}{2}(\operatorname{div} u\cdot w, v),$$

and hence $\eta_1 = \lambda\big((u\cdot\nabla)v + (v\cdot\nabla)u + \frac{1}{2}\operatorname{div} u \cdot v + \frac{1}{2}\operatorname{div} v\cdot u - 2\nu_r \operatorname{curl} z\big)$ and $\eta_2 = (u\cdot\nabla)z + (v\cdot\nabla)w + \frac{1}{2}\operatorname{div} u\cdot z + \frac{1}{2}\operatorname{div} v\cdot w - 2\nu_r \operatorname{curl} v$. Again, using Hölder inequality and the imbedding $H_0^1 \hookrightarrow L^6$ we prove $\eta_1,\, \eta_2 \in L^{3/2}$. □

LEMMA 3.6.4. *The mapping $G$ is twice differentiable, and its second derivative with respect to $\mathcal{U}$ is independent of $\mathcal{U}$, that is, for $\mathcal{W} = (w, r, \tau)$ and $\mathcal{V} = (v, q, z)$ in $V\times W\times V$,*

$$D^2_{\mathcal{U}\mathcal{U}} G(\lambda, \mathcal{U})\cdot(\mathcal{W}, \mathcal{V}) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{3.6.23}$$

*where $\eta_1, \eta_2 \in H^{-1}$ are such that*

$$< \eta_1, \xi >= \lambda\big(d(w,\, v,\, \xi) + d(v,\, w,\, \xi)\big) \qquad \forall \xi \in V, \tag{3.6.24}$$

$$< \eta_2, \xi >= d(w,\, z,\, \xi) + d(v,\, \tau,\, \xi) \qquad \forall \xi \in V. \tag{3.6.25}$$

PROOF. The lemma follows directly from (3.6.20) – (3.6.22). □

LEMMA 3.6.5. *Let $\theta_1, \theta_2 \in H^{-1}$. Denote by $\mathcal{U} = (u, p, \omega) \in V\times W\times V$ the solution of (3.6.17), i.e.*

$$\mathcal{U} = T\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

*and by $\mathcal{U}_h = (u_h, p_h, \omega_h)$ – the solution of the approximate problem (with $T$ replaced by $T_h$). If $(u, p, \omega) \in H^{l+1}\times H^l \times H^{l+1}$ for $l \in \{1, 2\}$, then*

$$\begin{aligned} ||u - u_h||_{1,2} + |p - p_h|_2 &\le \text{Const}\, h^l(||u||_{l+1,2} + ||p||_{l,2}), \\ ||\omega - \omega_h||_{1,2} &\le \text{Const}\, h^l ||\omega||_{l+1,2}. \end{aligned} \tag{3.6.26}$$

PROOF. The first inequality follows from the approximation properties of Taylor-Hood finite element and the general theory of mixed Galerkin method for Stokes equations, see Theorem 3.2.1. The second inequality follows from the general theory of finite element approximation of $V$-elliptic second order partial differential equations, see for example [**29**]. □

We are now in a position to prove the main result of this section, concerning approximation of branches of nonsingular solutions of (3.6.16) (cf. Section 3.2). This theorem also provides a local uniqueness result.

THEOREM 3.6.6. *Let $\Lambda$ be a compact interval in $R_+$ and assume that*

$$\{(\lambda, (u(\lambda), \lambda p(\lambda), \omega(\lambda)) : \lambda \in \Lambda\}$$

*is a branch of nonsingular solutions of (3.6.16) such that $\mathcal{U}(\lambda) = (u(\lambda), \lambda p(\lambda), \omega(\lambda))$ satisfies*

$$u(\lambda) \in H^{l+1}(\Omega), \quad p(\lambda) \in H^l(\Omega), \quad \omega(\lambda) \in H^{l+1}(\Omega),$$

*for $l = 1$ or $l = 2$ and for all $\lambda \in \Lambda$. Then there exists a neighbourhood $\mathcal{O}$ of the origin in $V \times W \times V$ and $h_0$ (small enough), such that for $h \leq h_0$ there exists a unique smooth function $\lambda \in \Lambda \mapsto (u_h(\lambda), \lambda p_h(\lambda), \omega_h(\lambda)) \in V_h \times W_h \times V_h$ such that:*

(i) *$\{(\lambda, (u_h(\lambda), \lambda p_h(\lambda), \omega_h(\lambda)) : \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (3.6.19),*

(ii) *$(u(\lambda), \lambda p(\lambda), \omega(\lambda)) - (u_h(\lambda), \lambda p_h(\lambda), \omega_h(\lambda)) \in \mathcal{O}$ for all $\lambda \in \Lambda$,*

(iii) *there exists* Const $> 0$, *independent of $h$ and $\lambda$, such that*

$$\begin{aligned} ||u_h(\lambda) - u(\lambda)||_{1,2} + |\lambda| \cdot |p_h(\lambda) - p(\lambda)|_2 + ||\omega_h(\lambda) - \omega(\lambda)||_{1,2} \\ \leq \text{Const}\, h^l \big(||u(\lambda)||_{l+1,2} + |\lambda| ||p(\lambda)||_{l,2} + ||\omega(\lambda)||_{l+1,2}\big) \end{aligned}$$

*for all $\lambda \in \Lambda$.*

PROOF. The theorem follows from Theorem 3.2.2. The mapping $G$ is $C^\infty$-regular and satisfies assumptions of this theorem, due to Lemma 3.6.3 together with compact imbedding $L^{3/2} \hookrightarrow H^{-1}$ and by (3.6.23), (3.6.24) and (3.6.25). Lemma 3.6.5 guarantees that $T_h$ has required approximating properties. □

## 3.7. Nonsingular solutions of micropolar equations

We prove the existence of nonsingular solutions of Problem 3.4.1 in any nonempty, open, bounded and simply-connected domain $\Omega$ with Lipschitz boundary, provided the data are sufficiently small. This extends and generalizes the result for the Navier–Stokes system [**46**, Lemma IV.3.2] and shows that the nonsingularity assumption in Theorem 3.6.6 is non-void.

Before we proceed, let us first recall the existence and uniqueness results for micropolar equations, due to Łukaszewicz.

THEOREM 3.7.1 (Łukaszewicz, [**69**, Theorems 0.1 and 0.2]). *Assume that $\Omega$ is a bounded open simply connected subset of $R^3$, with Lipschitz boundary. If $f$ and $z$ belong to $L^2(\Omega)$, then there exists a weak solution of Problem 3.4.1. The solution $(u, \omega)$ satisfies*

$$
\begin{aligned}
|u|_{1,2} &\le \frac{1}{\nu}\left(\mathrm{Const}\,|f|_2 + \frac{1}{2}|z|_2\right), \\
|\omega|_{1,2} &\le \frac{\mathrm{Const}}{c_1}\left(\mathrm{Const}\,\frac{\nu_r}{\nu}|f|_2 + \left(\frac{\nu_r}{\nu} + \mathrm{Const}\right)|z|_2\right),
\end{aligned}
\tag{3.7.27}
$$

*with* Const *dependent only on the diameter of $\Omega$. Moreover, if $\nu$ is sufficiently large, then the solution is unique.*

REMARK 3.7.1. Originally, the above theorem was proved for smooth domain $\Omega$. However, the proofs of both theorems carry over for only Lipschitz continuous boundary – all one needs for the proof are $H^1(\Omega)$ estimates and imbedding theorems, see Theorem A.0.2. Higher smoothness of the boundary was required only for the regularity theorem, see [**69**, Theorem 0.3].

Let us recall that we can reformulate Problem 3.4.1 as a zero-finding problem for a mapping $F(\lambda, \mathcal{U}) \equiv \mathcal{U} + T \cdot G(\lambda, \mathcal{U})$, defined in Section 3.6. In what follows we shall show that there exists a nonsingular branch of solutions for $F$, thus the assumption of Theorem 3.6.6 is non-void.

Let us consider a closed interval $[\nu_{\min}, \nu_{\max}]$, $0 < \nu_{\min} \le \nu_{\max}$, and suppose that $\nu \in [\nu_{\min}, \nu_{\max}]$. We assume that other parameters $\nu_r, c_1, c_2$ are fixed. Then $\lambda = (\nu + \nu_r)^{-1}$ varies within $\Lambda = [(\nu_{\max} + \nu_r)^{-1}, (\nu_{\min} + \nu_r)^{-1}]$, which is of course a compact interval in $R_+$. If $\nu_{\min}$ is big enough so that the uniqueness statement of Theorem 3.7.1 holds, then we can consider a well defined set $\mathcal{U}(\lambda) = (u(\lambda), \lambda p(\lambda), \omega(\lambda))$ of (unique) solutions of Problem 3.4.1 for $\lambda \in \Lambda$. In what follows we shall consider this branch of solutions $\{(\lambda, \mathcal{U}(\lambda)) : \lambda \in \Lambda\}$.

THEOREM 3.7.2. *If $\nu_{\min}$ is large enough and $|f|_2 + |z|_2$ is sufficiently small, then there exists a nonsingular branch of solutions of Problem 3.4.1.*

PROOF. According to the definition of branches of nonsingular solutions, see Definition 3.2.1, we have to check whether there exists a continuous branch of solutions

$\{(\lambda, \mathcal{U}(\lambda)) : \lambda \in \Lambda\}$, such that $D_{\mathcal{U}}F(\lambda, \mathcal{U}(\lambda)) : X \to X$ is an isomorphism onto $X$ for all $\lambda \in \Lambda$.

It is clear that $F$ is a $C^2$-mapping, since $G$ is quadratic. Moreover, Lemma 3.6.4 guarantees that $D^2_{\mathcal{U}\mathcal{U}}G$ is bounded on all bounded subsets of $\Lambda \times (V \times W \times V)$. Therefore it is sufficient to check if the mapping $\lambda \mapsto \mathcal{U}(\lambda)$ is continuous from $\Lambda$ into $V \times W \times V$ and to check the nonsingularity assumption.

Let us assume that $\nu_{\min}$ is large enough so that the solution of Problem 3.4.1 is unique and thus the branch $\{\lambda, \mathcal{U}(\lambda) : \lambda \in \Lambda\}$ is well defined.

Let us consider $(\lambda_n)_{n=1}^{\infty} \subset \Lambda$ and suppose that $\lambda_n \to \lambda_0 \in \Lambda$ for $n \to \infty$. Then, obviously, corresponding sequence $\nu_n = \dfrac{1}{\lambda_n} - \nu_r$ converges to $\nu_0 = \dfrac{1}{\lambda_0} - \nu_r$.

The difference of solutions $(u_n, p_n, \omega_n)$ and $(u_0, p_0, \omega_0)$, corresponding to $\lambda_n$ and $\lambda_0$, respectively, satisfies

$$
(3.7.28) \quad \left\{
\begin{aligned}
&(\nu_0 + \nu_r) \cdot ((u_0 - u_n,\, v)) + (\nu_0 - \nu_n) \cdot ((u_n,\, v)) + \\
&\qquad\qquad + d(u_0 - u_n,\, u_0,\, v) + d(u_n,\, u_0 - u_n,\, v) - (p_0 - p_n,\, \operatorname{div} v) \\
&\qquad = 2\nu_r(\operatorname{curl}(\omega_0 - \omega_n),\, v), \\
&(\operatorname{div}(u_0 - u_n),\, w) = 0, \\
&c_1((\omega_0 - \omega_n,\, \xi)) + d(u_0 - u_n,\, \omega_0,\, \xi) + d(u_n,\, \omega_0 - \omega_n,\, \xi) + \\
&\qquad\qquad + c_2(\operatorname{div}(\omega_0 - \omega_n,\, \operatorname{div} \xi) + 4\nu_r(\omega_0 - \omega_n,\, \xi) \\
&\qquad = 2\nu_r(\operatorname{curl}(u_0 - u_n),\, \xi).
\end{aligned}
\right.
$$

for all $(v, w, \xi) \in V \times W \times V$. Substituting $(v, w, \xi) = (u_0 - u_n, p_0 - p_n, \omega_0 - \omega_n)$, we obtain estimates

(3.7.29)
$$(\nu_0 + \nu_r)|u_0 - u_n|_{1,2} \le \mathrm{Const}_1\, |u_0|_{1,2}|u_0 - u_n|_{1,2} + 2\nu_r|\omega_0 - \omega_n|_2 + \mathrm{Const}\, |\nu_0 - \nu_n|\, |u_n|_{1,2},$$

and

(3.7.30)
$$c_1|\omega_0 - \omega_n|^2_{1,2} + 4\nu_r|\omega_0 - \omega_n|^2_2 \le 2\nu_r|u_0 - u_n|_{1,2} \cdot |\omega_0 - \omega_n|_2 + \mathrm{Const}_1\, |u_0 - u_n|_{1,2}|\omega_0|_{1,2}.$$

From (3.7.30) we have

$$|\omega_0 - \omega_n|_2 \le \left(\frac{\mathrm{Const}_1}{4\nu_r}|\omega_0|_{1,2} + \frac{1}{2}\right)|u_0 - u_n|_{1,2}. \tag{3.7.31}$$

Let us assume that $|f|_2$ and $|z|_2$ are small enough so that there holds (see (3.7.27))

$$|u_i|_{1,2} \le \frac{\nu_{\min}}{2\,\mathrm{Const}_1} \text{ and } |\omega_i|_{1,2} \le \frac{\nu_{\min}}{2\,\mathrm{Const}_1}, \qquad i = 0, 1, 2, \ldots$$

Then, substituting (3.7.31) into (3.7.29) we derive

$$(\nu_0 + \nu_r)|u_0 - u_n|_{1,2} \le \frac{\nu}{2}|u_0 - u_n|_{1,2} + 2\nu_r \cdot \left(\frac{\mathrm{Const}_1}{4\nu_r}\frac{\nu_{\min}}{2\,\mathrm{Const}_1} + \frac{1}{2}\right)|u_0 - u_n|_{1,2} + |\nu_0 - \nu_n|\frac{\nu}{2\,\mathrm{Const}_1},$$

whence

$$|u_0 - u_n|_{1,2} \le \mathrm{Const}\,|\nu_0 - \nu_n|, \tag{3.7.32}$$

with Const independent of $n$. Using this result in (3.7.30), by Young's inequality we have

$$c_1|\omega_0 - \omega_n|_{1,2}^2 \le \mathrm{Const}\,|u_0 - u_n|_{1,2}^2 + \frac{c_1}{2}|\omega_0 - \omega_n|_{1,2}^2 + \mathrm{Const}\,|\omega_0|_{1,2}|u_0 - u_n|_{1,2},$$

whence

$$|\omega_0 - \omega_n|_{1,2}^2 \le \mathrm{Const}\big(|\nu_0 - \nu_n|^2 + |\nu_0 - \nu_n|\big). \tag{3.7.33}$$

Thus, if $\nu_n \to \nu_0$, then both $u_n$ and $\omega_n$ converge in $V$ to $u_0$, $\omega_0$, respectively. Similarly, using the LBB condition, we prove that $p_n \to p_0$ in $W$, so that the mapping $\lambda \mapsto \mathcal{U}(\lambda)$ is continuous from $\Lambda$ into $V \times W \times V$.

Thus, we have proved the continuity. It remains to prove the nonsingularity of the branch under consideration. Precisely, we shall show that for $\nu$ sufficiently large and $|f|_2$, $|z|_2$ small enough, $D_{\mathcal{U}}G(\lambda, \mathcal{U}(\lambda))$ is an isomorphism from $V \times W \times V$ onto $V \times W \times V$ for all $\lambda \in \Lambda$.

By Lemma 3.6.1, $T$ defined by (3.6.17) is an isomorphism of $H^{-1} \times H^{-1} \to V \times W \times V$. Furthermore, from Lemma 3.6.3 we conclude that $D_{\mathcal{U}}G(\mathcal{U})$ maps into $L^{3/2} \times L^{3/2}$, which is compactly imbedded into $H^{-1} \times H^{-1}$. Hence, $D_{\mathcal{U}}G(\mathcal{U})$ is compact from $V \times W \times V \to H^{-1} \times H^{-1}$.

Therefore $D_{\mathcal{U}}F = I + T \cdot D_{\mathcal{U}}G$ is a compact perturbation of the identity, whence it is a Fredholm operator of index zero, see [**92**]. Then it follows that $D_{\mathcal{U}}F$ is an isomorphism, if only it is surjective. It is sufficient to check whether

$$\mathcal{V} + T \cdot D_{\mathcal{U}}G(\mathcal{U}) \cdot \mathcal{V} = 0 \text{ implies } \mathcal{V} = 0. \tag{3.7.34}$$

Slightly modifying the above proof of the branch continuity result, we conclude that for $\nu_{\min}$ large enough and $|f|_2$, $|z|_2$ sufficiently small, (3.7.34) holds indeed. To avoid repeating virtually the same estimates, we omit the details. □

## 3.8. Newton's method for the discrete problem

Problem 3.4.2, obtained from finite element discretization of Problem 3.4.1, corresponds to equation (3.6.19), which is posed in a finite dimensional space. In this section we present an analysis of the Newton's method for solving this equation for fixed $\lambda \in \Lambda$, where $\Lambda$ is as in Theorem 3.6.6.

We shall consider an iterative method for solving (3.4.9), derived from the Newton's iteration for (3.6.19). Starting from an initial guess $\mathcal{U}_h^0 = (u_h^0, \lambda p_h^0, \omega_h^0)$ we define the iteration step $\mathcal{U}_h^n \mapsto \mathcal{U}_h^{n+1}$ as

$$\mathcal{U}_h^{n+1} = \mathcal{U}_h^n - [D_{\mathcal{U}}F_h(\lambda, \mathcal{U}_h^n)]^{-1} \cdot F_h(\lambda, \mathcal{U}_h^n), \qquad n = 0, 1, \ldots$$

which, in our case, is equivalent to solving some variational problem.

Newton's algorithm. Given $(u_h^n, p_h^n, \omega_h^n) \in V_h \times W_h \times V_h$, find $(u_h^{n+1}, p_h^{n+1}, \omega_h^{n+1}) \in V_h \times W_h \times V_h$, which satisfies

$$\begin{cases} (\nu + \nu_r)(\nabla u_h^{n+1}, \nabla v) + d(u_h^n,\, u_h^{n+1},\, v) + d(u_h^{n+1},\, u_h^n,\, v) - (p_h, \operatorname{div} v) \\ \qquad = 2\nu_r(\operatorname{curl} \omega_h^{n+1}, v) + d(u_h^n,\, u_h^n,\, v) + (f, v), \\ (\operatorname{div} u_h^{n+1}, q) = 0, \\ c_1(\nabla \omega_h^{n+1}, \nabla \xi) + d(u_h^n,\, \omega_h^{n+1},\, \xi) + d(u_h^{n+1},\, \omega_h^n,\, \xi) + c_2(\operatorname{div} \omega_h^{n+1}, \operatorname{div} \xi) \\ \qquad + 4\nu_r(\omega_h^{n+1}, \xi) = 2\nu_r(\operatorname{curl} u_h^{n+1}, \xi) + d(u_h^n,\, \omega_h^n,\, \xi) + (g, \xi) \end{cases} \tag{3.8.35}$$

for all $(v, q, \xi) \in V_h \times W_h \times V_h$.

The following theorem guarantees local quadratic convergence of algorithm (3.8.35) to the solution of the discrete system. The rate of convergence depends on the parameter $\lambda = (\nu + \nu_r)^{-1}$, but is independent of the mesh parameter $h$.

Theorem 3.8.1. *Assume that $\mathcal{U}_h(\lambda) = (u_h(\lambda), \lambda p_h(\lambda), \omega_h(\lambda))$, for $\lambda \in \Lambda$, is as in Theorem 3.6.6 with $h_0$ sufficiently small. For fixed $\lambda$ in $\Lambda$ there exists $r > 0$ such that for all $h \leq h_0$ the Newton method defines a unique sequence $\{\mathcal{U}_h^n\}_{n \in N}$ , which is convergent to $\mathcal{U}_h$ and is all contained in a ball $B(\mathcal{U}_h, r)$, provided the initial guess $\mathcal{U}_h^0$ lies in $B(\mathcal{U}_h, r)$. Moreover, the convergence rate is quadratic,*

$$||\mathcal{U}_h^{n+1} - \mathcal{U}_h||_{V \times W \times V} \leq \text{Const}\, ||\mathcal{U}_h^n - \mathcal{U}_h||^2_{V \times W \times V},$$

*with* Const *independent of* $h$.

PROOF. It suffices to check the assumptions of the classical theorem on the convergence of Newton's method in finite dimensional Banach space in the case when the existence of solution is *a priori* guaranteed, see e.g. [**6**, Chapter VII, Paragraph 2]. Since $\mathcal{U}_h(\lambda)$ lies on a branch of nonsingular solutions, $D_{\mathcal{U}}F_h(\lambda, \mathcal{U}_h(\lambda))$ is an isomorphism on $V_h \times W_h \times V_h$. Moreover,

$$||[D_{\mathcal{U}}F_h(\lambda, \mathcal{U}_h(\lambda))]^{-1}||_{L(V\times W\times V)} \leq \mathrm{Const}_\lambda \tag{3.8.36}$$

where $\mathrm{Const}_\lambda$ is independent of $h$ (see (3.32) in [**46**, Chapter IV, Paragraph 3]). Thus (3.8.36) yields

$$||[D_{\mathcal{U}}F_h(\lambda, \mathcal{U})]^{-1}||_{L(V\times W\times V)} \leq 2\,\mathrm{Const}_\lambda,$$

provided $\mathcal{U} \in V_h \times W_h \times V_h$ is such that

$$||D_{\mathcal{U}}F_h(\lambda, \mathcal{U}) - D_{\mathcal{U}}F_h(\lambda, \mathcal{U}_h(\lambda))||_{L(V\times W\times V)} < \frac{1}{2\,\mathrm{Const}_\lambda}. \tag{3.8.37}$$

Inequality (3.8.37) holds for $||\mathcal{U} - \mathcal{U}_h(\lambda)||_{V\times W\times V} \leq r'$ with $r' > 0$ sufficiently small. Hence the theorem follows from the fact that the second derivative $D^2_{\mathcal{U}\mathcal{U}}F_h(\lambda, \mathcal{U})$ is independent of $\mathcal{U}$ and is bounded independently of $h$. $\square$

CHAPTER 4

# Block preconditioners for nonsymmetric saddle point problems

In the following sections, we discuss a class of methods for solution of the algebraic nonsymmetric saddle point problems arising from mixed finite element discretization of partial differential equations, in particular, the Navier–Stokes equation. We analyse block diagonal and block triangular preconditioners. In both cases, we prove the convergence of the iterative method is independent of the mesh parameter $h$. Some of these results appeared in [**60**] and [**61**].

## 4.1. Introduction

In many applications one has to solve a discrete system of linear equations with block matrix

$$\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix} \equiv \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{4.1.1}$$

where the matrix block $A$ is positive definite, yet not necessarily symmetric, and $C$ is nonnegative. Our motivation for considering this problem comes from the linearized discrete mixed finite element Navier-Stokes equations, where, except some very simple cases, $A$ is nonsymmetric, while usually it is reasonable to assume that it is positive definite [**38**]. $C$ can be zero or positive definite when a stabilized method is applied. Another possible and important source of problems of that kind may be a mixed (stabilized) finite element approximation of indefinite, nonsymmetric second order elliptic equations.

In both mentioned above examples, system (4.1.1) is ill conditioned with respect to the mesh parameter $h$, and our aim is to design and/or analyse preconditioners for this system, for which uniformly good performance (in the whole range of $h$) of an iterative method used to solve (4.1.1) can be achieved.

The case of symmetric blocks $A$ and $C$ has been considered by many authors before, see for example [**35**], [**16**], [**17**], [**19**], [**83**], [**85**] or [**52**], [**53**], [**54**]. These preconditioners have been verified in thorough computational tests (see references listed above and also [**41**]). However, there is little literature for the nonsymmetric case; we overview some works later in this section.

In this chapter we are going to extend to the nonsymmetric case the recent results of Klawonn [**52**], Bramble and Pasciak [**17**], and D'yakonov [**35**] developed for the symmetric saddle point problem. We present new mathematical analysis of algorithms, confirming numerical indications from papers by Elman and Silvester [**38**] and Elman, Silvester and Wathen [**39**]. Our goal is to analyse two strategies of preconditioning (4.1.1) using block diagonal or block triangular matrices.

We concentrate on two basic approaches to preconditioning (4.1.1). The first one is to symmetrize the problem (see, e.g. [**35**], [**17**], [**15**]) with the aid of a block preconditioner built up from preconditioners for symmetric parts of diagonal blocks of (4.1.1). We prove that the resulting system is symmetric with respect to some discrete inner product, and its spectral condition number is independent of the dimension of (4.1.1). Hence, we can apply the conjugate gradient method and conclude that the number of iterations required to reduce the residual by a given factor is bounded independently of the dimension of the problem. The second approach relies on applying the GMRES iterative method to system (4.1.1) preconditioned by some block triangular matrix. This approach was considered recently in [**52**], though only for the symmetric case. We prove that, under minor assumptions, block triangular preconditioner guarantees the GMRES to converge independently of $h$, even in the nonsymmetric case.

For example, in the case of linearized Navier–Stokes equations, the block preconditioner can be built from preconditioners for (symmetric and positive definite) the discrete Laplacian and mass matrices. That gives an application programmer a great opportunity to reuse, in an efficient way, existing very powerful methods (or software) like the domain decomposition or the multigrid method for these simpler problems. Application of domain decomposition preconditioners, like the additive Schwarz method (see Section 4.2.3), results in a well parallelizable algorithm. We believe that this may be a more robust approach than to use custom domain decomposition preconditioners developed especially for the whole saddle point problem.

The key idea which makes the analysis possible, is the choice of an auxiliary inner product in which the crucial parameters are measured. From that point of view, our approach is well understood and is based on choosing this custom inner product dependent on the preconditioner being used.

For the symmetric case it has been recently proved by Klawonn [**52**] that both block diagonal or block triangular preconditioners lead to systems with the condition number independent of $h$. However, this analysis relies strongly on assumed symmetry of the relevant components. Elman and coworkers [**38**] were probably first to analyse how block preconditioners do work in the case of the Oseen equations

$$-\nu\Delta u + (k \cdot \nabla)u + \nabla p = f, \tag{4.1.2}$$

$$\operatorname{div} u = 0, \tag{4.1.3}$$

which serve as a model for the linearized Navier-Stokes equations[1] and lead to a nonsymmetric system. The results of numerical experiments from [**39**] and [**40**] indicated the robustness of block preconditioners in such case, however, there was no complete theory explaining observed behaviour. Elman also proposed another kind of a block preconditioner, so called BFBt preconditioner, designed for nonsymmetric problems [**40**], providing experimental results and partial analysis of the preconditioner. Independently, Klawonn and Starke [**56**] have provided a field of values analysis of block preconditioners for the Oseen problem. Their analysis concerns the *right* preconditioner and uses kind of $H^{-1} \times L^2$ norm for the analysis.

Most of the results in the mentioned above works are given for *exact* solver for the nonsymmetric block. The inexact solver seems to introduce severe difficulties to the analysis, which, in turn, result (at best) in quite complex assumptions.

In contrast, our analysis is all valid for inexact and symmetric block solvers and shows that a sufficient preconditioner can be based just on the symmetric part of the diagonal blocks of $\mathcal{M}$. We perform our analysis in a discrete $H^1 \times L^2$-like norm (which is more natural for the problem), using custom inner products derived from the preconditioner being used.

The plan of remaining sections is as follows. After recalling in Section 4.2 some basic facts related to iterative methods and preconditioning for large systems of linear equations, in Section 4.3 we introduce the framework in which we shall analyse our algorithms. Some useful lemmas are proved in Section 4.4. Section 4.5 is devoted to the analysis of preconditioning methods which lead to symmetric positive definite systems. In the following section, Section 4.6, we analyse a block triangular preconditioner for the GMRES method. We conclude this chapter with Section 4.7, which contains the results of numerical experiments.

## 4.2. Iterative methods and preconditioners

After linearization of a discrete nonlinear partial differential equation (or a system of partial differential equations), one usually ends up with a linear system of algebraic

[1]The convective term $k$ in the Oseen equation is divergence free, which is not necessarily true for the linearized discrete Navier-Stokes system.

equations:

$$Ax = b, \tag{4.2.4}$$

where $A$ is a sparse matrix of very large dimension. In such case, it is not reasonable (or it may be impossible at all) to use direct solvers, such as the Gaussian elimination or the Cholesky–Banachiewicz algorithm. The key difficulties for direct methods are their large memory requirements and high computational cost for very large problems. Moreover, the $LU$ factorization of a sparse matrix often leads to *full* matrices. Iterative methods give an opportunity to overcome the limitations of direct methods.

The performance of most of the iterative methods depends, however, on spectral properties of the matrix. For the conjugate gradient method, which is the method of choice for symmetric positive definite matrices, the convergence rate generally depends on the spectral condition number. Similarly, for methods like GMRES, which are suitable for nonsymmetric or indefinite problems, their convergence also depends on spectral properties of the matrix, but in a more complicated way.

Since matrices derived from the finite element or finite difference approximation of partial differential equations are ill conditioned with respect to the mesh size, some *preconditioning* is necessary to accelerate the convergence of the iterative method being used. The idea of preconditioning is quite simple[2]: one transforms the original system (4.2.4) into

$$A_0^{-1}Ax = A_0^{-1}b, \tag{4.2.5}$$

where $A_0^{-1}$ is a nonsingular matrix with the following properties:

- $A_0^{-1}$ is "easy" to apply, precisely, the system with $A_0$ is cheaper to solve than with matrix $A$;
- the convergence of an iterative method for the system with transformed matrix $A_0^{-1}A$ is faster than for the original problem (4.2.4).

Below we briefly overview popular iterative methods. We focus on matrices in $R^n$, however all these results hold for general linear operators in finite dimensional Hilbert spaces.

**4.2.1. The preconditioned conjugate gradient method.** The preconditioned conjugate gradient (PCG) algorithm, developed in the 1950s as a direct method, nowadays is the iterative method of choice for solving large, sparse problems (4.2.5) with $A$ and $A_0$ being symmetric, positive definite matrices. This well known algorithm is

[2]Here we only present the idea of *left* preconditioning. It is also possible to use right preconditioners, see e.g. [**47**], [**49**].

described in most of the textbooks, see for example [**47**], [**49**], so we only present an overview of its most important features.

This method involves the matrix $A$ and the preconditioner $A_0^{-1}$ only through a matrix multiplication routine, which is important for efficient use of the matrix sparsity and implementation of many good preconditioners. PCG also requires the storage for only four working vectors, which results in little memory consumption.

Assume that $A$ and $A_0$ are symmetric and positive definite. Let us denote by $||\cdot||_A$ the energy norm $||x||_A = \sqrt{x^T Ax} = (Ax, x)_{R^n}^{1/2}$. The following theorem estimates the convergence rate of the PCG method[3]:

THEOREM 4.2.1 (Daniel, see [**30**]). *Denote by* $\kappa$ *the condition number of* $A_0^{-1}A$, *that is,* $\kappa = ||A_0^{-1}A||_2 \cdot ||(A_0^{-1}A)^{-1}||_2$. *Then*

$$||e^n||_A \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^n \cdot ||e^0||_A,$$

*where* $e^k$ *denotes the error on the* $k$*-th step of the iteration.*

Since $\kappa$ may be expressed as the quotient of the maximum and the minimum eigenvalue of $A_0^{-1}A$, $\kappa = \lambda_{\max}(A_0^{-1}A)/\lambda_{\min}(A_0^{-1}A)$, we can easily see that the convergence rate of the conjugate gradient algorithm indeed depends on spectral properties of $A_0^{-1}A$. If we want the PCG to converge fast, we have to choose $A_0^{-1}$ which gives small value of $\kappa$, which means that $A_0^{-1}$ has to squeeze the spectrum of $A$ into a small positive interval distant from zero. Precisely, the desired property is that $A_0$ is uniformly spectrally equivalent to $A$, i.e.

$$\exists m, M > 0 \quad \forall x \in R^n \qquad m \cdot (A_0 x, x)_{R^n} \leq (Ax, x)_{R^n} \leq M \cdot (A_0 x, x)_{R^n},$$

where $m, M$ are independent of $h$.

It is well known that matrices derived from the finite element approximation of partial differential equations are ill conditioned with respect to the mesh size. For example, the finite element discretization of a second order elliptic partial differential equation leads to $\kappa = O(h^{-2})$. Thus, preconditioning is necessary in such case.

Domain decomposition methods lead to good preconditioners with additional nice property useful in the supercomputing environment: high level of parallelism and good scalability. We discuss them later in this section.

[3]If the eigenvalues of $A_0^{-1}A$ cluster in a small number of intervals, the PCG method may perform much better than it is indicated by its condition number, see [**49**].

**4.2.2. The GMRES method.** The generalized minimum residual method (GMRES) was proposed in the 1980s by Saad and Schultz [**84**] and since then has been discussed in many textbooks on the iterative methods, see for example [**47**], [**49**]. This method may be applied to solve nonsymmetric or indefinite systems.

Unlike the conjugate gradient method, GMRES storage requirements grow linearly with the number of iterations, so in practice a restarted version is usually used. This means that after a prescribed number of iterations, the method is started again with previously obtained approximation as the new initial guess. It has been noticed that good preconditioning is vital for GMRES – otherwise the process may not converge at all.

Suppose that we apply the GMRES method to solve (4.2.5). Let $\langle \cdot, \cdot \rangle$ denote an inner product, generated by some symmetric positive definite matrix $B$, i.e. $\langle x, y \rangle = y^T B x$ for all $x, y$. A possible choice is to take $B = A_0$, if only $A_0$ is a symmetric, positive definite matrix. The following theorem estimates the convergence rate of the GMRES:

THEOREM 4.2.2 (Eisenstat, Elman and Schultz, see [**36**]). *Define*

$$\underline{c} = \inf_{x \neq 0} \frac{\left\langle A_0^{-1} A x,\, x \right\rangle}{\langle x,\, x \rangle}, \qquad \bar{c} = \sup_{x \neq 0} \frac{\left\langle A_0^{-1} A x,\, A_0^{-1} A x \right\rangle}{\langle x,\, x \rangle}.$$

*If* $\underline{c} > 0$ *then the GMRES algorithm is convergent and the preconditioned residual on the k-th step,* $r_k = A_0^{-1}(A x_k - b)$, *satisfies*

$$||r_k||_B \leq \left(1 - \frac{\underline{c}^2}{\bar{c}}\right)^{\frac{k}{2}} \cdot ||r_0||_B.$$

The factor in the above estimate depends again on spectral properties of $A_0^{-1}A$ matrix (see also [**89**]), for example, $\underline{c}$ is simply the smallest eigenvalue of the symmetric part of $A_0^{-1}A$. Note however, that the condition $\underline{c} > 0$ is not necessary for the GMRES to converge.

**4.2.3. Schwarz methods.** The Schwarz framework[4], as developed by Lions [**67**], [**66**] and further by Dryja and Widlund [**34**], [**33**], makes it possible to analyse and construct efficient preconditioners for symmetric positive definite problems. These preconditioners are usually referred to as "domain decomposition" methods and are

[4]The origins of the method can be traced back to 1870, when H.A.Schwarz applied an alternating domain decomposition method in order to prove the existence for an elliptic problem in a region consisting of two overlapping domains of simple shape. After 210 years P.-L.Lions renewed the interest to this method as a numerical scheme, see [**87**].

well suited for solving the problem on parallel computers. In the same framework one can analyse other efficient methods, such as the multigrid.

Schwarz algorithms for nonsymmetric or indefinite problems have been considered by Cai and Widlund, see [**25**] and [**22**]. It is also possible to apply domain decomposition to nonlinear problems, see [**32**], [**24**], [**74**].

In addition, one can develop decomposition algorithms for saddle point problems, [**76**], [**77**], [**17**], [**27**], [**55**]. However, a general theory for domain decomposition methods for saddle point problems (analogous to that for the elliptic case) has not been developed yet.

In the reminder of this section we provide only a very brief introduction to basic Schwarz methods for elliptic problems, according to the overview article [**28**]. We refer the reader to a monograph [**87**] for up-to-date details and discussion of various domain decomposition methods and the list of references.

Let $\Omega$ denote a polygon or a polyhedron with triangulation $\mathcal{T}_h$, where $h$ is the mesh size. $\mathcal{T}_h$ is called a fine mesh. Let us consider a linear system (4.2.4) with a symmetric positive definite matrix $A$ derived from a finite element discretization of some selfadjoint, elliptic partial differential operator $\mathcal{L}$ on the fine mesh $\mathcal{T}_h$. As it was mentioned in preceding sections, an iterative solver is limited by ill conditioning of $A$. Domain decomposition algorithms provide a way of construction of good preconditioners for $A$.

In order to introduce the *overlapping Schwarz method*, at first we subdivide $\Omega$ into $p$ non-overlapping (that is, with mutually disjoint interiors) subdomains $\Omega_1, \dots, \Omega_p$,

$$\bar{\Omega} = \bigcup_{i=1}^{p} \bar{\Omega}_i.$$

We assume that individual $\Omega_i$s are sums of triangulation elements $\tau_h$ and they themselves form a triangulation of $\Omega$ with mesh size $H$. Triangulation $\mathcal{T}_H$ is called a coarse mesh. Each $\Omega_i$ is then extended by a stripe of width $\beta H$ and form a bigger subdomain $\hat{\Omega}_i$, which again is supposed to be a sum of elements of fine triangulation $\mathcal{T}_h$ (Figure 4.2.3). If we denote by $A_i$ the discretization of $\mathcal{L}$ restricted to subdomain $\hat{\Omega}_i$, we can introduce the *overlapping additive Schwarz* preconditioner as

$$A_0^{-1} = \sum_{i=1}^{p} R_i^T A_i^{-1} R_i, \tag{4.2.6}$$

where $R_i$ is a restriction mapping, which restricts a discrete function on $\Omega_h$ to the $i$th subdomain $\hat{\Omega}_{ih}$. Accordingly, $R_i^T$ is an extension mapping.

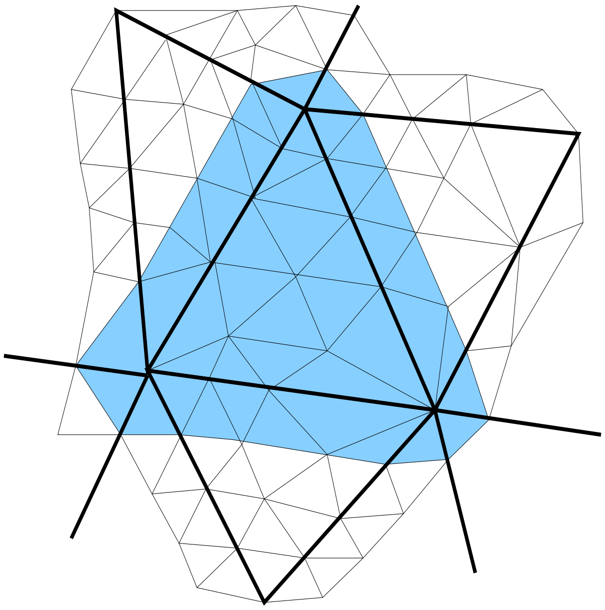

FIGURE 4.2.1. Decomposition of $\Omega$ into overlapping subdomains. Thin triangles constitute fine triangulation $\mathcal{T}_h$, while thick triangles form coarse triangulation $\mathcal{T}_H$, consisting of subdomains $\Omega_i$, $i = 1, \dots, p$. If we denote the central subdomain by $\Omega_i$, then the shadowed area indicates $\hat{\Omega}_i$.

Let us observe that indeed, the additive Schwarz preconditioner defined by (4.2.6) can easily be applied in parallel, since each term of the sum can be computed independently on different processors.

THEOREM 4.2.3 (Dryja and Widlund, see [**34**]). *For $A_0^{-1}$ as in (4.2.6),*

$$\operatorname{cond}(A_0^{-1}A) \leq \text{Const}\, \frac{1}{H^2}\left(1 + \frac{1}{\beta}\right)$$

The addition of a coarse grid solver, which computes the solution of the discrete differential problem on the coarse mesh $\mathcal{T}_H$ formed from subdomains $\Omega_i$, removes the dependence of the condition number on $H$ and, in consequence, on the number of subdomains:

THEOREM 4.2.4 (Dryja, Widlund and Smith, see [**34**], [**33**]). *Consider*

$$A_0^{-1} = R_H^T A_H^{-1} R_H + \sum_{i=1}^{p} R_i^T A_i^{-1} R_i,$$

*where $A_H, R_H$ denote the coarse grid discretization of $\mathcal{L}$ and coarse grid restriction operator, respectively. Then*

$$\text{cond}(A_0^{-1}A) \leq \text{Const}\left(1 + \frac{1}{\beta}\right)$$

*for sufficient amount of overlap.*

The condition number can still be enhanced, if one uses *multiplicative Schwarz* method. The price one pays for better convergence properties is the loss of parallelism[5] in the algorithm, since the multiplicative Schwarz method is quite sequential: after computing $k$th iterate $u^k$, one computes $u^{k+1}$ by solving a sequence of local problems:

$$u^{k+\frac{1}{p+1}} := u^k + R_H^T A_H^{-1} R_H (f - Au^k),$$
$$u^{k+\frac{i+1}{p+1}} := u^{k+\frac{i}{p+1}} + R_i^T A_i^{-1} R_i (f - Au^{k+\frac{i}{p+1}}), \qquad i = 1, \ldots, p.$$

THEOREM 4.2.5 (Bramble, Pasciak, Wang and Xu, see [**14**]). *The error $||u - u^k||_A$ on the kth step of the above multiplicative Schwarz algorithm satisfies*

$$||u - u^k||_A \leq \rho^k ||u - u_0||_A,$$

*with $\rho < 1$ independent of $h$ and $H$, but dependent only on $\beta$ and parameters of the differential problem.*

Finally, let us mention other important domain decomposition algorithms.

Two level Schwarz algorithms (that is, with coarse space) have their natural extension to *multilevel methods*.

So called *substructuring* methods rely on reducing the decomposed problem into the (preconditioned) iterative solve of the interface problem, and then solving in parallel local problems with boundary values known from the previous step. Notice that, in a way, our block preconditioners are inspired by better conditioning of the Schur complement of a symmetric positive definite matrix.

Certain *multigrid methods* can be analysed as space decomposition methods. They are among the fastest methods for elliptic partial differential equations or Stokes problem in velocity–pressure formulation, cf. comparisons in [**37**]. Their potential drawback is that they are difficult for efficient parallel implementation.

Recently, so called *mortar* methods, see [**9**], gained a lot of attention. They fit into the class of structural methods and they relax the interdomain continuity requirements

[5] One can still preserve some amount of parallelism by using a multicolouring technique. Another possibility is to use *hybrid* methods, which combine additive and multiplicative approaches, see e.g. [**23**].

in order to give the user more flexibility in choosing the discretization and solution method on each subdomain.

## 4.3. General assumptions

We consider the saddle point problem (4.1.1) with matrix $\mathcal{M}$, which arises from mixed finite element discretization of partial differential equations, so our framework should reflect the dependence of problem (4.1.1) on the mesh parameter $h$. For theoretical convenience, we shall carry our analysis in terms of linear operators rather than in terms of matrices.

Let $\bar{V}, \bar{W}$ be Hilbert spaces with scalar products denoted by $((\cdot,\cdot))$ and $(\cdot,\cdot)$, respectively. The norms in these spaces, induced by the inner products, will be denoted by $||\cdot||$ and $|\cdot|$.

Consider a family of finite dimensional subspaces indexed by parameter $h \in (0,1)$:

$$V_h \subset \bar{V}, \qquad W_h \subset \bar{W}.$$

If $V_h, W_h$ come from the finite element approximation, the dimension of these subspaces is increasing for $h$ decreasing. Let us introduce three continuous bilinear forms: $a : \bar{V} \times \bar{V} \to R$, $b : \bar{V} \times \bar{W} \to R$, $c : \bar{W} \times \bar{W} \to R$, and assume that $a(\cdot,\,\cdot)$ is $\bar{V}$–elliptic, i.e.

$$\exists \alpha > 0 \quad \forall v \in \bar{V} \qquad a(v,\,v) \geq \alpha ||v||^2 \tag{4.3.7}$$

and that $c(\cdot,\,\cdot)$ satisfies

$$\exists \gamma \geq 0 \quad \forall p \in \bar{W} \qquad c(p,\,p) \geq \gamma |p|^2 \tag{4.3.8}$$

(we allow $\gamma = 0$). $a(\cdot,\cdot)$ may be not symmetric. In what follows we shall consider a family of finite dimensional problems:

PROBLEM 4.3.1. Given $f \in \bar{V}'$ and $g \in \bar{W}'$, find $(u_h, p_h) \in V_h \times W_h$ such that

$$\begin{cases} a(u_h,\,v) + b(v,\,p_h) = \langle\langle f, v\rangle\rangle & \forall v \in V_h, \\ b(u_h,\,w) - c(p_h,\,w) = \langle g, w\rangle & \forall w \in W_h. \end{cases} \tag{4.3.9}$$

Here $\langle\langle\cdot,\cdot\rangle\rangle$ and $\langle\cdot,\cdot\rangle$ denote the dual pairings between $\bar{V}$ and $\bar{V}'$, $\bar{W}$ and $\bar{W}'$, respectively.

Throughout this chapter we shall assume that $V_h$ and $W_h$ satisfy the uniform LBB condition

$$\exists \beta > 0 \quad \forall h \in (0,1) \quad \forall p \in W_h \qquad \beta |p| \leq \sup_{v \in V_h, v \neq 0} \frac{b(v,\,p)}{||v||}. \tag{4.3.10}$$

REMARK 4.3.1. Under above assumptions, it follows from Theorem 3.2.1 and Theorem 2.3.2 that Problem 4.3.1 is uniquely solvable in $V_h \times W_h$ for all $h \in (0,1)$ and the following stability estimate holds:

$$||u_h|| + |p_h| \le \text{Const}(||f||_{\bar{V}'} + ||g||_{\bar{W}'}),$$

with Const independent of $h$.

From now on we shall denote for simplicity the space $V_h$ (resp. $W_h$) by $V$ (resp. $W$). The generic constant "Const", which appears later in this chapter is independent of the parameter $h$.

Let us formulate Problem 4.3.1 in an operator form

PROBLEM 4.3.2. Find $(u,p) \in V \times W$ such that

$$\begin{cases} Au + B^*p = F & \text{in } V, \\ Bu - Cp = G & \text{in } W. \end{cases} \tag{4.3.11}$$

Here, the operators in (4.3.11) are as follows:

$$\begin{aligned} &A : V \to V, \quad ((Au, v)) = a(u,\, v) \quad \forall u, v \in V, \\ &B : V \to W, \quad (Bu, p) = b(u,\, p) \quad \forall u \in V, p \in W, \\ &C : W \to W, \quad (Cp, q) = c(p,\, q) \quad \forall p, q \in W, \end{aligned}$$

while the right hand side is defined by

$$\begin{aligned} &F \in V, \quad ((F, v)) = \langle\langle f, v\rangle\rangle \quad \forall v \in V, \\ &G \in W, \quad (G, w) = \langle g, w\rangle \quad \forall w \in W. \end{aligned}$$

$B^*$ denotes the formal adjoint operator to $B$, i.e. $(Bu,p) = ((u, B^*p))$ for all $u \in V$, $p \in W$.

In what follows, we analyse some preconditioning strategies for Problem 4.3.2, using operator

$$\mathcal{M}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & J_0 \end{pmatrix} \qquad \text{– block diagonal,}$$

or

$$\mathcal{M}_0 = \begin{pmatrix} A_0 & 0 \\ B & -J_0 \end{pmatrix} \qquad \text{– block triangular.}$$

We shall always assume that the finite dimensional operators $A_0 : V \to V$ and $J_0 : W \to W$ satisfy

- $A_0^{-1}$ is easy to apply,

- $((A_0u, v)) = ((A_0v, u))$ for all $u, v \in V$,
- there exist constants $0 < a_0 \le a_1$, independent of $h$, such that

$$a_0((u, u)) \le ((A_0u, u)) \le a_1((u, u)) \qquad \forall u \in V, \tag{4.3.12}$$

and, similarly (see [**52**]),

- $J_0^{-1}$ is easy to apply,
- $(J_0p, q) = (J_0q, p)$ for all $p, q \in W$,
- there exist constants $0 < b_0 \le b_1$, independent of $h$, such that

$$b_0(p, p) \le (J_0p, p) \le b_1(p, p) \qquad \forall p \in W. \tag{4.3.13}$$

In other words, we shall always assume that $A_0$ and $J_0$ define good preconditioners, cf. Section 4.2. For example, in the case of linearized Navier-Stokes equations, $A_0$ may be a preconditioner for the discrete Laplace operator, while $J_0$ may be the pressure mass matrix operator preconditioner. These preconditioners may be constructed using very efficient domain decomposition or multigrid techniques, as mentioned in Section 4.2.3.

The product space $V \times W$ is equipped with a natural scalar product $\langle\cdot,\cdot\rangle$,

$$\left\langle \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right\rangle = ((u, v)) + (p, q),$$

however, we are going to analyse the preconditioned problem using custom inner product $[\cdot,\cdot]$, which is dependent on the preconditioner being used:

$$\left[ \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right] = ((A_0u, v)) + (J_0p, q) = \left\langle \begin{pmatrix} A_0 & 0 \\ 0 & J_0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right\rangle. \tag{4.3.14}$$

Observe that the norms generated by both products are equivalent one to another, with constants independent of $h$.

## 4.4. Operator bounds

Using the variational setting (Problem 4.3.1) we prove the following estimates on operators involved in Problem 4.3.2.

LEMMA 4.4.1. *The norms of operators $A$, $B$, $C$, $A_0$, $J_0$ and their adjoints in adequate spaces are bounded independently of $h$,*

$$\begin{aligned} &||A||_{V\to V}, \quad ||B||_{V\to W}, \quad ||C||_{W\to W}, \\ &||A^*||_{V\to V}, \quad ||B^*||_{W\to V}, \quad ||C^*||_{W\to W}, \quad ||A_0||_{V\to V}, \quad ||J_0||_{W\to W} \le \text{Const}\,. \end{aligned} \tag{4.4.15}$$

PROOF. For any linear operator $L : X \to Y$ where $X, Y$ are Hilbert spaces, there holds

$$||Lx||_Y \leq \sup_{y \in Y, y \neq 0} \frac{(Lx, y)}{||y||_Y}.$$

In particular, since we assumed that $a(\cdot, \cdot)$ is bounded on $\bar{V} \times \bar{V}$, see (4.3.7),

$$||Au|| \leq \sup_{v \in V} \frac{((Au, v))}{||v||} = \sup_{v \in V} \frac{a(u, v)}{||v||} \leq \text{Const } ||u||.$$

Similarly,

$$|Bu| \leq \sup_{p \in W} \frac{(Bu, p)}{|p|} = \sup_{p \in W} \frac{b(u, p)}{|p|} \leq \text{Const } ||u||,$$

and

$$|Cq| \leq \sup_{p \in W} \frac{c(q, p)}{|p|} \leq \text{Const } |q|.$$

The proof for the adjoint operators is analogous. Using (4.3.12) and (4.3.13) we prove, using the symmetry and positive definiteness of $A_0$,

$$||A_0 u||^2 = ((A_0 u, A_0 u)) = ((A_0 A_0^{1/2} u, A_0^{1/2} u)) \leq a_1((A_0^{1/2} u, A_0^{1/2} u) = a_1((A_0 u, u)) \leq a_1^2 ||u||^2,$$

so that $||A_0||_{V \to V} \leq a_1$, and similarly, $||J_0||_{W \to W} \leq b_1$. $\square$

LEMMA 4.4.2. *The norm of the inverse of $A_0$ or $J_0$ is bounded independently of $h$, precisely,*

$$||A_0^{-1}||_{V \to V} \leq \frac{1}{a_0}, \quad ||J_0^{-1}||_{W \to W} \leq \frac{1}{b_0}.$$

PROOF. We carry the proof only for $A_0$ (for $J_0$ it is analogous). Let us fix arbitrary $F \in V$. By (4.3.13), for $v \in V$ such that $A_0 v = F$ there holds

$$a_0 ||v||^2 \leq ((A_0 v, v)) = ((F, v)) \leq ||F|| ||v||.$$

Since $v = A_0^{-1} F$, we obtain

$$a_0 ||A_0^{-1} F|| \leq ||F||.$$

$\square$

LEMMA 4.4.3. *For all $u \in V$,*

$$((A_0^{-1} u, u)) = \sup_{v \in V} \frac{((v, u))^2}{((A_0 v, v))}.$$

PROOF. Notice that $A_0^{-1}$ is, like $A_0$, a symmetric positive definite operator. Let us fix arbitrary $u \in V$. Obviously,

$$\sup_{v\in V} \frac{((v,u))^2}{((A_0 v, v))} \geq \frac{((A_0^{-1}u,u))^2}{((A_0 A_0^{-1}u, A_0^{-1}u))} = ((A_0^{-1}u,u)).$$

On the other hand, the symmetry of $A_0$ together with Schwarz inequality with respect to $((A_0\cdot,\cdot))$ yield for any $v \in V$

$$((u,v))^2 = ((A_0A_0^{-1}u,v))^2 = ((A_0^{-1}u, A_0 v)) \leq ((A_0v,v)) \cdot ((A_0A_0^{-1}u, A_0^{-1}u)),$$

so that

$$\frac{((u,v))^2}{((A_0v,v))} \leq ((A_0^{-1}u,u))$$

for all $v \in V$. □

LEMMA 4.4.4. *For any $F \in V$,*

$$\frac{1}{a_1}||F||^2 \leq ((A_0^{-1}F,F)) \leq \frac{1}{a_0}||F||^2.$$

PROOF. From Lemma 4.4.3 we get the lower bound:

$$((A_0^{-1}F,F)) = \sup_{u\in V}\frac{((F,u))^2}{((A_0u,u))} \geq \frac{((F,F))^2}{((A_0F,F))} \geq \frac{1}{a_1}||F||^2.$$

On the other hand, Lemma 4.4.2 yields

$$((A_0^{-1}F,F)) \leq ||A_0^{-1}||_{V\to V}||F||^2 \leq \frac{1}{a_0}||F||^2,$$

which proves the upper bound. □

LEMMA 4.4.5.

$$||\mathcal{M}||_{V\times W\to V\times W}, \quad ||\mathcal{M}^*||_{V\times W \to V\times W} \leq \text{Const}\,.$$

PROOF. The lemma follows from Lemma 4.4.1:

$$\begin{aligned}
||\mathcal{M}\begin{pmatrix} u \\ p\end{pmatrix}||^2_{V\times W\to V\times W} &= ||\begin{pmatrix} Au + B^*p \\ Bu - Cp\end{pmatrix}||^2_{V\times W\to V\times W} \\
&= ||Au+B^*p||^2 + |Bu-Cp|^2 \leq (||Au|| + ||B^*p||)^2 + (|Bu| + |Cp|)^2 \\
&\leq 2\cdot\left(\left(||A||^2_{V\to V} + ||B||^2_{V\to W}\right)||u||^2 + \left(||B^*||^2_{W\to V} + ||C||^2_{W\to W}\right)|p|^2\right) \\
&\leq \text{Const}\left(||u||^2 + |p|^2\right).
\end{aligned}$$

The norm of $\mathcal{M}^*$ is estimated in a similar way. □

LEMMA 4.4.6. *The solution $(u,p) \in V \times W$ of Problem 4.3.2 satisfies the inequality*

$$||u|| + |p| \leq \mathrm{Const}(||F|| + |G|). \tag{4.4.16}$$

PROOF. Problem 4.3.2 is equivalent to Problem 4.3.1, so estimate (4.3.1) holds. By definition of $F$ and $G$ it follows from Riesz theorem that $||F|| = ||f||_{V'}$ and $|G| = ||g||_{W'}$. Note that Const in (4.4.16) does not depend on $h$. □

## 4.5. Preconditioners leading to symmetric, positive definite problems

In this section we are going to extend the results obtained previously for symmetric saddle point problems in [**35**], [**17**], [**15**]. By symmetrizing the problem we achieve two goals. The first is that the transformed system allows for using the conjugate gradients (PCG) method, which is less memory consuming than the GMRES method. Secondly, the transformed system has its spectral condition number bounded uniformly with respect to $h$, so the number of iterations required to reduce the error by a given factor is independent of $h$. Moreover, no scaling is needed for $A_0$, in contrast to results for preconditioner for the GMRES in the next section. Finally, the analysis remains valid for $A$ only $V^0$-elliptic, see Remarks 4.5.1 and 4.5.2.

**4.5.1. Block diagonal preconditioner.** Using the block diagonal preconditioner

$$\mathcal{M}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & J_0 \end{pmatrix},$$

we transform system (4.3.11) into the following one:

PROBLEM 4.5.1. Find $(u,p) \in V \times W$ such that

$$\mathcal{M}_0^{-1}\mathcal{M}^*\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix} = \mathcal{M}_0^{-1}\mathcal{M}^*\mathcal{M}_0^{-1}\begin{pmatrix} F \\ G \end{pmatrix}. \tag{4.5.17}$$

The operator $\mathcal{P} = \mathcal{M}_0^{-1}\mathcal{M}^*\mathcal{M}_0^{-1}\mathcal{M}$ is selfadjoint with respect to the inner product $[\cdot,\cdot]$, since we have

$$\mathcal{P} = \begin{pmatrix} A_0^{-1} & 0 \\ 0 & J_0^{-1} \end{pmatrix}\begin{pmatrix} A^*A_0^{-1}A + B^*J_0^{-1}B & A^*A_0^{-1}B^* - B^*J_0^{-1}C \\ BA_0^{-1}A - C^*J_0^{-1}B & BA_0^{-1}B^* + C^*J_0^{-1}C \end{pmatrix}.$$

The following theorem guarantees $\mathcal{P}$ is well conditioned:

THEOREM 4.5.1. *There exist positive constants $m_1, m_2$, independent of $h$, such that*

$$m_1\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] \le \left[\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] \le m_2\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right]. \tag{4.5.18}$$

PROOF. Observe that

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] = ((A_0^{-1}(Au + B^*p), Au + B^*p)) + (J_0^{-1}(Bu - Cp), Bu - Cp). \tag{4.5.19}$$

Defining $\tilde{f} = Au + B^*p$ and $\tilde{g} = Bu - Cp$, we obviously have that $(u, p)$ satisfies

$$\begin{cases} Au + B^*p = \tilde{f} \\ Bu - Cp = \tilde{g}, \end{cases} \tag{4.5.20}$$

so from Lemma 4.4.4 and Lemma 4.4.6 we obtain

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] \ge \text{Const}\left(||u||^2 + |p|^2\right).$$

This, with Lemma 4.4.4 yields the lower bound in (4.5.18). In order to prove the upper bound, we return to (4.5.19) and use Lemma 4.4.5.

$$\begin{aligned}\left[\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] &= \left\langle \mathcal{M}^*\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \\ &\le ||\mathcal{M}^*|| \cdot ||\mathcal{M}_0^{-1}|| \cdot ||\mathcal{M}|| \cdot \left\langle\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \\ &\le \text{Const}\left(||u||^2 + |p|^2\right) \le \text{Const}\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right],\end{aligned}$$

since from Lemma 4.4.1 and Lemma 4.4.2 it follows that $||\mathcal{M}_0^{-1}||_{V\times W\to V\times W} \le \text{Const}$, with constant independent of $h$. $\square$

REMARK 4.5.1. It is worth noticing that the $V$-ellipticity assumption in Theorem 4.5.1 on $A$ may be weakened. Indeed, all we need is that the saddle point problem (4.5.20) is uniquely solvable and the stability estimate holds. According to Theorem 2.3.2, $V^0$-ellipticity of $A$ plus the LBB condition is sufficient.

When using the PCG method for solving Problem 4.5.1, one has to compute inner products $[\mathcal{P}x, y]$. Observe that $[\mathcal{P}x, y] = \langle\mathcal{M}^*\mathcal{M}_0^{-1}\mathcal{M}x, y\rangle$ so one has to solve *only one* system with $\mathcal{M}_0$ per inner product. The block diagonal preconditioner is also discussed in Chapter 5 in the context of micropolar equations.

**4.5.2. Block triangular preconditioner.** It is also possible to construct a symmetric preconditioner for the operator $\mathcal{M}$, which is based on a block lower triangular matrix

$$\mathcal{M}_0 = \begin{pmatrix} A_0 & 0 \\ B & J_0 \end{pmatrix}. \tag{4.5.21}$$

We transform the system (4.3.11) into an equivalent one,

$$\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix} = \mathcal{F}, \tag{4.5.22}$$

where

$$\mathcal{P} = \mathcal{M}^*\mathcal{L}_0^*\mathcal{K}_0^{-1}\mathcal{L}_0\mathcal{M}, \qquad \mathcal{F} = \mathcal{M}^*\mathcal{L}_0^*\mathcal{K}_0^{-1}\mathcal{L}_0\begin{pmatrix} f \\ g \end{pmatrix}, \tag{4.5.23}$$

and

$$\mathcal{L}_0^*\mathcal{K}_0^{-1}\mathcal{L}_0 \equiv \begin{pmatrix} I & A_0^{-1}B^* \\ 0 & -I \end{pmatrix}\begin{pmatrix} A_0^{-1} & 0 \\ 0 & J_0^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ BA_0^{-1} & -I \end{pmatrix}. \tag{4.5.24}$$

Observe that $\mathcal{K}_0^{-1}\mathcal{L}_0$ is nothing else but $\mathcal{M}_0^{-1}$ and that $\mathcal{P}$ is symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle$.

THEOREM 4.5.2. *There exist constants $0 < c_0 \le c_1$ independent of parameter $h$, such that*

$$c_0\left\langle\begin{pmatrix} u \\ p \end{pmatrix},\begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \le \left\langle\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix},\begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \le c_1\left\langle\begin{pmatrix} u \\ p \end{pmatrix},\begin{pmatrix} u \\ p \end{pmatrix}\right\rangle$$

*for all $(u,p) \in V \times W$.*

PROOF. First, we shall give the lower bound. As in the previous section, the upper bound will be easy to derive from the boundedness assumptions (see Lemma 4.4.6).

For $(u,p) \in V \times W$ we have

$$\left\langle\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix},\begin{pmatrix} u \\ p \end{pmatrix}\right\rangle = \left\langle\mathcal{K}_0^{-1}\mathcal{L}_0\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix},\mathcal{L}_0\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \tag{4.5.25}$$

and

$$\mathcal{L}_0\mathcal{M} = \begin{pmatrix} A & B^* \\ B(A_0^{-1}A - I) & BA_0^{-1}B^* + C \end{pmatrix}.$$

Therefore,

$$\left\langle\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix},\begin{pmatrix} u \\ p \end{pmatrix}\right\rangle = |||Au + B^*p|||^2_{A_0^{-1}} + |||BA_0^{-1}(A - A_0)u + (BA_0^{-1}B^* + C)p|||^2_{J_0^{-1}}, \tag{4.5.26}$$

where $|||u|||^2_{A_0^{-1}} = ((A_0^{-1}u, u))$ for $u \in V$, and similarly, $|||p|||^2_{J_0^{-1}} = (J_0^{-1}p, p)$ for $p \in W$. Defining $\tilde{f} = Au + B^*p$ and $\tilde{g} = BA_0^{-1}(A - A_0)u + (BA_0^{-1}B^* + C)p$ we obviously have that the pair $(u, p)$ satisfies the system

$$\begin{cases} Au + B^*p = \tilde{f} \\ Cp - Bu = \tilde{g} - BA_0^{-1}(Au + B^*p) = \tilde{g} - BA_0^{-1}\tilde{f}. \end{cases} \tag{4.5.27}$$

From the estimates for the saddle point problem (see Section 4.4) it follows that the solution $(u, p)$ of (4.5.27) satisfies

$$\begin{aligned} ||u|| + |p| \leq \mathrm{Const}(|||\tilde{f}|||_{A_0^{-1}} + |||BA_0^{-1}\tilde{f} - \tilde{g}|||_{J_0^{-1}}) \leq \\ \mathrm{Const}(||B||, ||A_0^{-1}||) \cdot (|||\tilde{f}|||_{A_0^{-1}} + |||\tilde{g}|||_{J_0^{-1}}). \end{aligned} \tag{4.5.28}$$

This estimate yields the lower bound on the spectrum of $\mathcal{P}$. To get the upper bound we return to (4.5.26) and use the estimates from the previous subsection. □

REMARK 4.5.2. It is worth noticing that the $V$-ellipticity assumption in Theorem 4.5.2 on $A$ may be weakened. Indeed, all we need is that the saddle point problem (4.5.27) is uniquely solvable and the stability estimate holds. According to Theorem 2.3.2, $V^0$-ellipticity of $A$ plus the LBB condition is sufficient.

Observe that the preconditioner discussed here is more costly to evaluate than the diagonal one: it requires three applications of $A_0^{-1}$ per inner product, compared to only one $A_0^{-1}$ for diagonal preconditioner analysed in the previous subsection.

## 4.6. Preconditioner for the GMRES method

It is well known that after symmetrizing the system, its condition number increases. This fact is also indicated by numerical experiments (see Section 4.7). Therefore, rather than symmetrizing the system, we may use efficiently preconditioned GMRES method for our problem. If the preconditioner works so good that we obtain satisfactory approximation after few iterations, then the GMRES memory consumption is less painful, and we can benefit from its better convergence properties.

Let us recall from Section 4.2, that the convergence rate of the GMRES method for a system

$$\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} = \mathcal{M}_0^{-1}\begin{pmatrix} f \\ g \end{pmatrix} \tag{4.6.29}$$

may be estimated by means of two parameters,

(4.6.30)
$$\underline{c} = \inf_{(u,p)\neq 0} \frac{\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right]}{\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right]}, \quad \bar{c} = \sup_{(u,p)\neq 0} \frac{\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}\right]}{\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right]}.$$

We remind that $[\cdot,\cdot]$ is defined by (4.3.14) and it generates the norm in which the residual for GMRES is measured.

**4.6.1. Block triangular preconditioner.** In this section we are concerned with block triangular preconditioner,

$$\mathcal{M}_0 = \begin{pmatrix} A_0 & 0 \\ B & -J_0 \end{pmatrix},$$

which was analysed by Klawonn in [**52**] only for the symmetric case[6]. An energy norm approach makes it easier to analyse the preconditioner even in the nonsymmetric case.

THEOREM 4.6.1. *If $A_0$ is scaled so that the symmetric part[7] $A_{symm}$ of $A$ and $A^*A_0^{-1}A$ satisfy*

$$(((A_{symm} - \frac{1}{2}A^*A_0^{-1}A)u, u)) \geq \delta \cdot ((A_0 u, u)) \qquad \forall u \in V, \tag{4.6.31}$$

*for some positive constant $\delta$ independent of $h$, then the convergence rate of the GMRES for (4.6.29) is independent of $h$.*

PROOF. Our aim will be to estimate the quantities $\underline{c}, \bar{c}$ in (4.6.30) independently of $h$. We have

$$\begin{pmatrix} A_0 & 0 \\ 0 & J_0 \end{pmatrix} \cdot \mathcal{M}_0^{-1} = \begin{pmatrix} I & 0 \\ BA_0^{-1} & -I \end{pmatrix},$$

so that

$$\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] = ((Au,\, u)) + ((A_0^{-1}Au,\, B^*p)) + ((A_0^{-1}B^*p,\, B^*p)) + (Cp,\, p).$$

[6]Block triangular based preconditioner was also implicitly proposed and analysed by Bramble and Pasciak in [**16**]. This fact was first noticed in [**52**].

[7]By definition, $A_{symm} = \dfrac{A + A^*}{2}$.

From (4.3.8) we estimate

$$\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] \geq \frac{1}{2}|||Au + B^*p|||^2_{A_0^{-1}} + (((A_{symm} - \frac{1}{2}A^*A_0^{-1}A)u,\, u)) + \frac{1}{2}|||B^*p|||^2_{A_0^{-1}},$$

with energy norm introduced by symmetric and positive definite operator $A_0^{-1}$,

$$|||v|||^2_{A_0^{-1}} = ((A_0^{-1}v,\, v)).$$

Obviously, $|||Au + B^*p|||^2_{A_0^{-1}} \geq 0$ and

$$|||B^*p|||^2_{A_0^{-1}} = (BA_0^{-1}B^*p,\, p) \geq \mathrm{Const}(J_0p,\, p),$$

the latter from (4.3.13) and the LBB condition, see [**16**].

Therefore, if only $A_0$ is scaled properly so that the symmetric part $A_{symm}$ of operator $A$ satisfies (4.6.31) with $\delta$ independent of $h$, then we can conclude that

$$\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] \geq \mathrm{Const}\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right].$$

In order to estimate $\bar{c}$, observe that, using the estimates from Section 4.4,

$$\begin{aligned}\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}\right] &= \left\langle\begin{pmatrix} A_0 & \\ & J_0 \end{pmatrix}\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \\ &= \left\langle\mathcal{M}^*\mathcal{M}_0^{-1}\begin{pmatrix} A_0 & \\ & J_0 \end{pmatrix}\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \\ &= ||\mathcal{M}^*|| \cdot ||\mathcal{M}_0^{-1}|| \cdot ||\begin{pmatrix} A_0 & \\ & J_0 \end{pmatrix}|| \cdot ||\mathcal{M}_0^{-1}|| \cdot ||\mathcal{M}|| \cdot \left\langle\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right\rangle.\end{aligned}$$

In view of Lemma 4.4.5 and Lemma 4.4.1 we conclude that

$$\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}\right] \leq \mathrm{Const}_1\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right].$$

Therefore, $\bar{c} \leq \mathrm{Const}_1$, and this completes the analysis of the block triangular preconditioner. The convergence rate of GMRES for the preconditioned system is independent of the parameter $h$. □

REMARK 4.6.1. Let us notice that the scaling requirement is similar to that of Bramble and Pasciak [**16**] for the triangular preconditioner in the symmetric case.

Similarly, in our case it can be proved that $A_0$ must be scaled in order to preserve positive definiteness of the operator $\mathcal{M}_0^{-1}\mathcal{M}$. Indeed, we have

$$\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] = \frac{1}{2}|||Au + B^*p|||^2_{A_0^{-1}} + ((Au,u)) + \frac{1}{2}|||B^*p|||^2_{A_0^{-1}} - \frac{1}{2}|||Au|||^2_{A_0^{-1}} + (Cp,p). \tag{4.6.32}$$

Assume that $A_0$ is rescaled by a parameter $\epsilon^{-1} > 0$, so we can take new $A_0^{-1} := \epsilon \cdot A_0^{-1}$. We are going to prove that we can choose $\epsilon$ in such a way that the operator $\mathcal{M}_0^{-1}\mathcal{M}$ is no longer positive definite. Replacing $A_0^{-1}$ in (4.6.32) by $\epsilon \cdot A_0^{-1}$ we get

$$\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right]_\epsilon = \frac{\epsilon}{2}\left(|||Au + B^*p|||^2_{A_0^{-1}} + |||B^*p|||^2_{A_0^{-1}} - |||Au|||^2_{A_0^{-1}}\right) + ((Au,u)) + (Cp,p). \tag{4.6.33}$$

Suppose that $0 \neq p \in W$. Now, taking $u = u_H + u_0$, where $u_H$ satisfies

$$Au_H + B^*p = 0,$$

we have $u_H \neq 0$ (by the inf-sup condition (4.3.10) and the ellipticity assumption (4.3.7)) and

$$|||Au + B^*p|||^2_{A_0^{-1}} = |||Au_0|||^2_{A_0^{-1}}, \qquad |||B^*p|||^2_{A_0^{-1}} = |||Au_H|||^2_{A_0^{-1}},$$

whence

$$\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right]_\epsilon = \frac{\epsilon}{2}(|||Au_0|||^2_{A_0^{-1}} + |||Au_H|||^2_{A_0^{-1}} - |||Au|||^2_{A_0^{-1}}) + ((Au,u)) + (Cp,p).$$

Since $u_0$ is arbitrary, we can take for example $u_0 = \pm u_H$. Then

$$\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right]_\epsilon = ((Au,u)) + (Cp,p) \mp \epsilon||Au_H||^2.$$

One can easily see that taking $\epsilon$ large enough, $\left[\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right]_\epsilon$ can be made negative or positive with large absolute value.

The question, whether our triangular preconditioner works well without scaling, remains still open.

REMARK 4.6.2. Observe that for $A = A^* = A_0$ ("exact" preconditioning block), (4.6.31) holds with $\delta = 1/2$ without any scaling. In general case, one may scale $A_0^{-1}$

by a factor $\epsilon = \lambda/\mu$, where $\lambda$ is the smallest eigenvalue of the following generalized eigenproblem,

$$A_{symm} u = \lambda A_0 u, \tag{4.6.34}$$

while $\mu$ is the largest eigenvalue of another generalized eigenproblem,

$$A^* A_0^{-1} A u = \mu A_0 u. \tag{4.6.35}$$

Then (4.6.31) holds with $\delta = \dfrac{\lambda^2}{2\mu}$. Since both $\lambda$ and $\mu$ are bounded from above and below independently of $h$, so are $\epsilon$ and $\delta$. Indeed, if $\lambda$ is an eigenvalue of (4.6.34), then

$$\alpha((u,u)) \le ((A_{symm} u, u)) = \lambda((A_0 u, u)) \le \lambda a_1((u,u)),$$

so $\lambda \ge \dfrac{\alpha}{a_1}$. On the other hand,

$$\lambda a_0 ||u||^2 \le \lambda((A_0 u, u)) = ((A_{symm} u, u)) = ((Au, u)) \le ||A|| \cdot ||u||^2,$$

whence $\lambda \le \dfrac{||A||}{a_0}$. Similarly, if $\mu$ is an eigenvalue of (4.6.35), then

$$\mu a_0 ||u||^2 \le \mu((A_0 u, u)) = ((A^* A_0^{-1} A u, u)) \le ||A^*|| \cdot ||A_0^{-1}|| \cdot ||A|| \cdot ||u||^2,$$

so $\mu \le \dfrac{1}{a_0} \cdot ||A^*|| \cdot ||A_0^{-1}|| \cdot ||A||$. Finally, we have

$$\mu a_1 ||u||^2 \ge \mu((A_0 u, u)) = ((A^* A_0^{-1} A u, u)) = ((A_0^{-1} A u, A u)) \ge \frac{1}{a_1} ||Au||^2.$$

Since, from (4.3.7),

$$\alpha \le \frac{((Au, u))}{((u,u))} \le \frac{||Au|| \cdot ||u||}{((u,u))} = \frac{||Au||}{||u||},$$

it follows that $\mu \ge \dfrac{\alpha^2}{a_1^2}$.

It is also possible to choose the scaling factor as $\epsilon = \tilde{\lambda}/\tilde{\mu}$, where $\tilde{\lambda}$ is the smallest eigenvalue of $A_{symm}$ and $\tilde{\mu}$ is the largest eigenvalue of $A^* A_0^{-1} A$. Then we have

$$((Au, u)) - \frac{\epsilon}{2}((A^* A_0^{-1} A u, u)) \ge \frac{\tilde{\lambda}}{2}((u,u)) \ge \frac{\alpha}{2a_0} \cdot \frac{\tilde{\lambda}}{\tilde{\mu}} \cdot ((\frac{A_0}{\epsilon} u, u)).$$

These eigenvalues may be efficiently estimated by means of the inverse and power iteration, respectively, since both matrices are symmetric. Note however, that in the context of finite element approximation of e.g. Oseen equations, estimating those eigenvalues still requires a solution of a generalized eigenvalue problem of the form $Ax = \lambda Bx$.

Yet another way to obtain a scaling factor $\epsilon$ for which $\delta$ is still independent of $h$ would be to choose any smaller value, e.g.

$$\epsilon = \frac{\alpha}{||A^*||_{V\to V} \cdot ||A_0^{-1}||_{V\to V} \cdot ||A||_{V\to V}},$$

since the smallest eigenvalue of the symmetric part of $A$ is not smaller than $\alpha$, while the greatest eigenvalue of $A^* A_0^{-1} A$ can be estimated from above by $||A^*||_{V\to V} \cdot ||A_0^{-1}||_{V\to V} \cdot ||A||_{V\to V}$. Of course, the latter choice gives us worse constants in the estimate of $\underline{c}$ and has little practical value.

## 4.7. Numerical experiments

We implemented certain preconditioners considered in this chapter and examined their influence on the iterative solution of a nonsymmetric saddle point problem. The test problem was to solve the Oseen equations in a rectangle $\Omega = (-1,1) \times (-1,1)$

$$\begin{cases} -\nu\Delta u + (k \cdot \nabla)u + \nabla p = f, \\ \operatorname{div} u = 0, \end{cases}$$

in $\Omega$, with homogeneous Dirichlet boundary conditions on $u$. The function $k(\cdot,\cdot)$ was defined as in [**38**], [**56**], as a simple vortex,

$$k(x,y) = \begin{pmatrix} 2y(1-x^2) \\ -2x(1-y^2) \end{pmatrix},$$

see Figure 4.7.2 for the plot of the vector field $k$ and Figure 4.7.3 for an example solution of a lid driven cavity problem for the Oseen equations.

For the right hand side vector in our experiments, we always took a random vector with elements uniformly distributed between $(-1,1)$.

We discretized the equation using popular $Q_2$–$Q_1$ Taylor–Hood rectangular finite elements, see Section 3.2. The mesh was uniform in both directions, with $nx$ inner pressure nodes along the $x$-axis and $ny$ inner pressure nodes in $y$-direction, so we had approximately twice as much inner nodes in each direction for each component of the discrete velocity. In all experiments presented below, $nx = ny$. To keep the pressure in the space $L_0^2$, following [**48**], we added to the system a discretized condition $\int p = 0$ using the Lagrange multipliers.

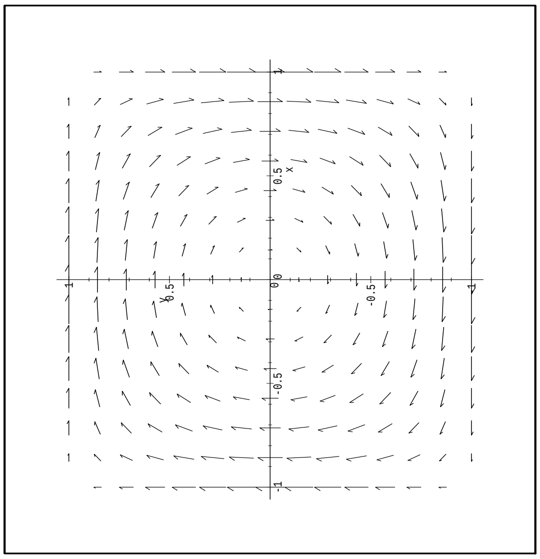

FIGURE 4.7.2. Plot of $k(x, y)$.

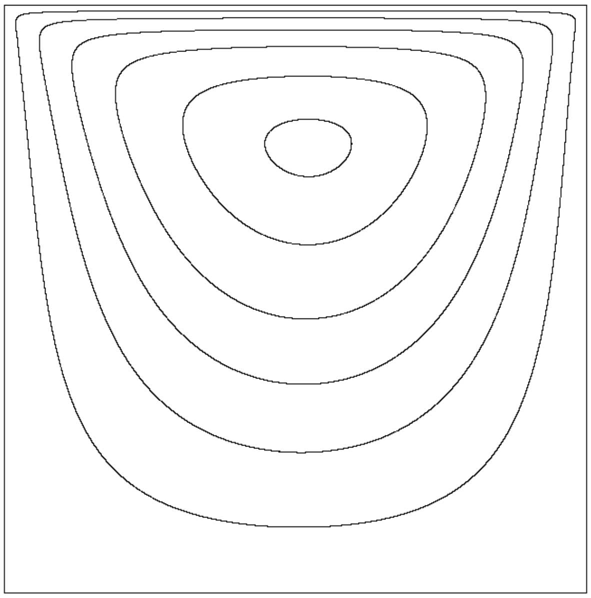

FIGURE 4.7.3. Streamlines for a lid driven cavity problem for the Oseen equation.

All algorithms were implemented using PETSc library [**88**] subroutines for iterative solvers and additive Schwarz preconditioners. We performed our tests in three different solver/preconditioner configurations:

1. GMRES/diag – the GMRES method with block diagonal preconditioner,
2. GMRES/triang – the GMRES method with block triangular preconditioner, see Section 4.6.1,
3. CG/symm – the symmetrized conjugate gradient method with block diagonal preconditioner, see Section 4.5.1.

The stopping criterion was to reduce the initial residual by a factor of $10^6$. A divergence of the iterative process was stated when the number of iterations exceeded 150. We used a restarted version of GMRES, with restart after every 30 iterations. We did not modify the GMRES code provided by PETSc to make use of the auxiliary inner product considered in Section 4.6.1. In all cases, the viscosity parameter $\nu$ was set to one.

First, we conducted experiments under the best available circumstances, that is, we used as the preconditioners: $A_0$ = Laplacian, $J_0$ = mass matrix. Notice that, in contrast to [**40**] and [**56**], we restricted ourselves only to *symmetric* preconditioners, without significant performance degradation as compared to results from the mentioned above papers. Nevertheless, we refer to the above choice of preconditioners as the "exact" preconditioners, since in this case we exactly solve the symmetric part of relevant operators. Our results are listed in Table 4.7.1, and Figure 4.7.4, while convergence histories are presented in Figure 4.7.5.

TABLE 4.7.1. Iteration counts for different solvers and preconditioning strategies: results for $A_0$ = Laplacian, $J_0$ = Mass matrix.

| $nx$ | DOF[8] | GMRES/diag | GMRES/triang | CG/symm |
|---|---|---|---|---|
| 1 | 59 | 21 | 16 | 18 |
| 5 | 387 | 37 | 21 | 42 |
| 10 | 1202 | 38 | 21 | 45 |
| 20 | 4182 | 39 | 21 | 46 |
| 40 | 15542 | 40 | 21 | 49 |
| 80 | 59862 | 39 | 21 | 49 |

Next, we tested the same methods, using the additive Schwarz method implemented in PETSc, for the preconditioners $A_0$, $J_0$. In all tests the preconditioners were based on the standard black-box decomposition into 6 rectangular subdomains

[8]DOF = the total number of degrees of freedom.

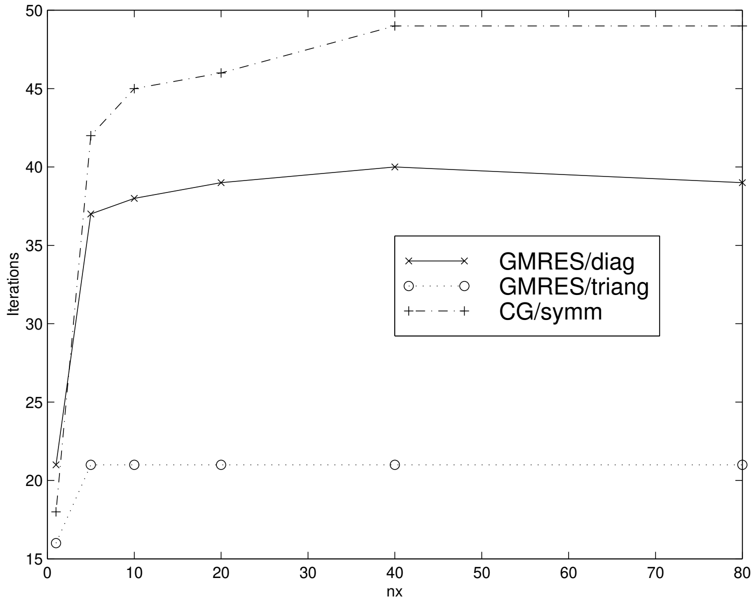


FIGURE 4.7.4. The growth of iterations as a function of problem size for different solvers and preconditioning strategies: the results for $A_0 =$ Laplacian, $J_0 =$ Mass matrix.

with small overlap (2 nodes) provided by PETSc command line options. No coarse space was used in our tests. The results are presented in Table 4.7.2 and Figure 4.7.6.

TABLE 4.7.2. Iteration counts for different solvers and preconditioning strategies: the results for additive Schwarz preconditioners.

| $nx$ | DOF | GMRES/diag | GMRES/triang | CG/symm |
|---|---|---|---|---|
| 1 | 59 | 21 | 18 | 17 |
| 5 | 387 | 71 | 58 | 77 |
| 10 | 1202 | 63 | 47 | 73 |
| 20 | 4182 | 128 | 76 | 127 |
| 40 | 15542 | 127 | 77 | 113 |
| 80 | 59862 | 183 | 108 | 130 |

The experiments confirm theoretical results, showing that the number of iterations reflects the quality of the preconditioning blocks being used. This may be seen very clearly from the experiments with “exact” solvers (Figure 4.7.4) and implicitly from those for the inexact solvers (Figure 4.7.6). When using “exact” (but still only symmetric) preconditioning blocks, the number of iterations is virtually independent

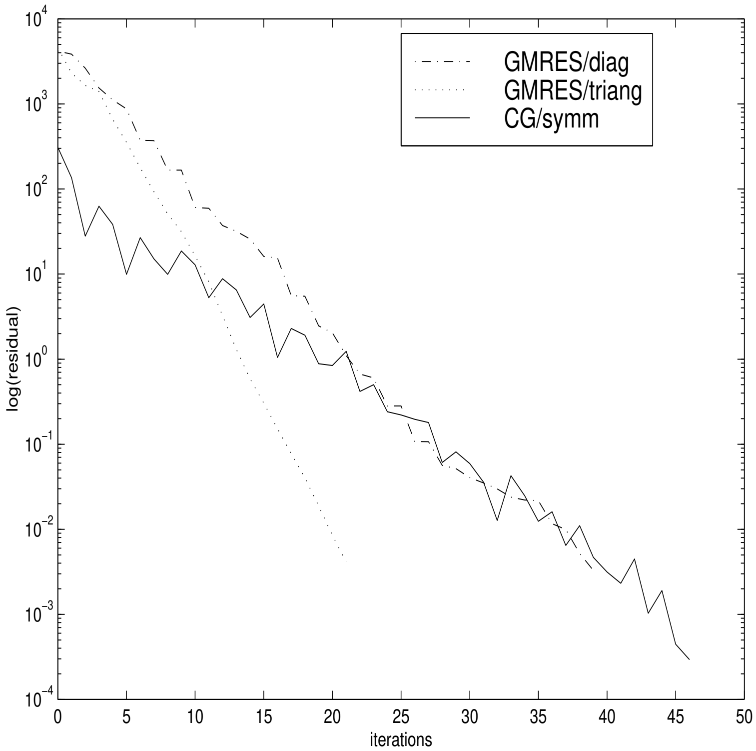


FIGURE 4.7.5. Convergence histories for different solvers, $nx = 20$.

of the mesh size. As one could expect, the triangular preconditioner is most effective. It is more than twice as fast as diagonal preconditioners. The CG/symm combination is slightly slower than GMRES/diag.

For additive Schwarz preconditioning blocks, the number of iterations was of course higher. According to the theory, see Section 4.2.3, the condition number of preconditioned blocks depends linearly on the overlap ratio. In our experiments, we kept the overlap constant and equal to $2h$, so the overlap ratio decreased linearly with the growth of $nx$. Therefore, for our inexact solvers we should have observed the number of iterations increasing as the square root of $nx$, and this is approximately what we actually see, cf. Figure 4.7.6. The superiority of GMRES/triang was not so apparent when additive Schwarz method preconditioners were used: the number of iterations required by symmetrized CG was competitive to that of GMRES/triang. The judgement in this case is even more complicated, when we recall that CG requires significantly less working space than GMRES.

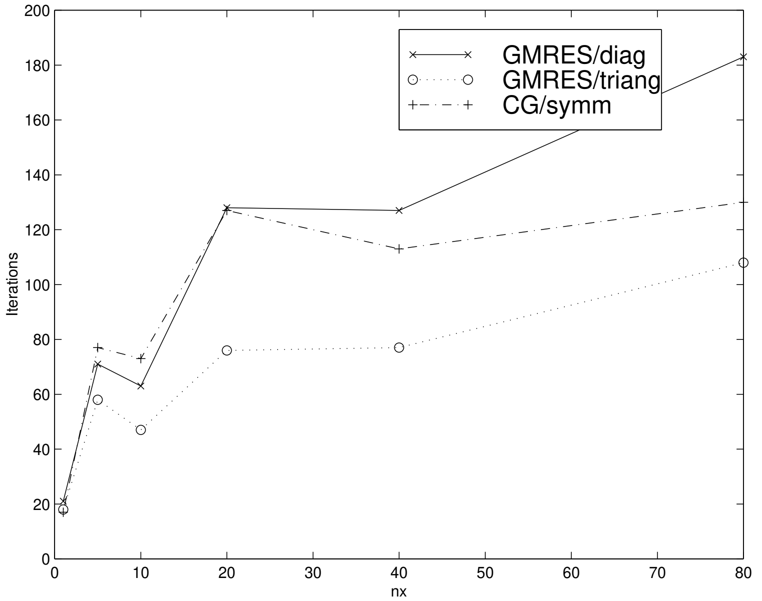


FIGURE 4.7.6. Iterations as the function of problem size for different solvers and preconditioning strategies: the results for additive Schwarz preconditioners.

Let us point out, that without a preconditioner any solver diverged even for a problem as small as $nx = ny = 10$.

We also experimented with the constraint on pressure $p$ other than $\int p = 0$, namely, we used instead a one point Dirichlet condition, $p(x_0) = 0$, at some pressure mesh node $x_0$. Interestingly, the number of iterations was significantly higher than in the previous case. We show them in Table 4.7.3.

TABLE 4.7.3. Iteration counts for Dirichlet condition on pressure. See Table 4.7.1 for comparison.

| $nx$ | DOF | GMRES/diag | GMRES/triang | CG/symm |
|---|---|---|---|---|
| 5 | 387 | 89 | 29 | 74 |
| 10 | 1202 | 99 | 31 | 88 |
| 20 | 4182 | 209 | 33 | 100 |
| 40 | 15542 | – | 36 | 112 |

Figure 4.7.7 presents convergence histories for Dirichlet pressure condition for problem size $nx = ny = 20$. Besides the larger number of iterations, we notice another strange behaviour of the CG/symm solver. At the beginning, the residual

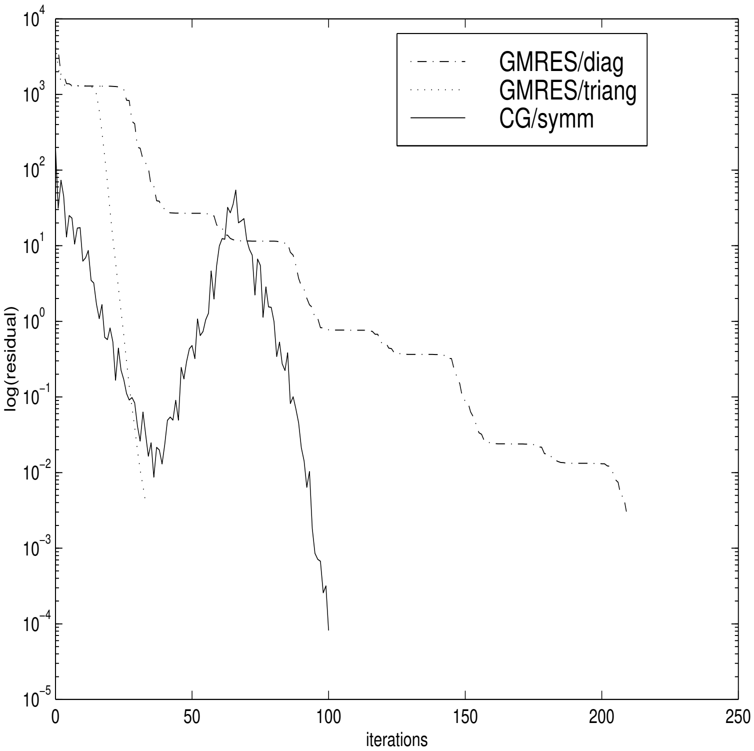


FIGURE 4.7.7. Convergence histories for different solvers and Dirichlet pressure constraint. Note the increase of the residual for CG/symm.

decreases, but then suddenly the residual starts to grow. After a number of iterations, the tendency reverses again, and the residual finally decreases smoothly. More close observation of this phenomenon suggested that the increase of the residual occurs when the solver starts "adding the right constant" to the pressure. Therefore we tried a heuristic modification of the CG symmetrized solver, where we *released* the constraints on $p$, accepting therefore *nonuniqueness* of the solution. Of course, without constraint on $p$ the whole iterative process may be in general divergent. Quite surprisingly, the method worked, and the experiments showed that the divergence happened after the residual was reduced to a very low level. The iterations scores were better, however still greater than those for $\int p = 0$ constraint. We do not have a precise explanation for such a behaviour.

CHAPTER 5

# Preconditioners for micropolar equations

We are going to propose and analyse effective preconditioners for micropolar equations, basing upon the general theory developed in Chapter 4. We shall consider a block diagonal and a hybrid diagonal–triangular preconditioner. In both cases we prove that the condition number of the transformed system is independent of the mesh size.

## 5.1. Introduction and general assumptions

In this chapter, we propose and analyse a preconditioning method for one step of Newton's method described in Section 3.8. Performing one step of Newton's algorithm,

$$(u_h^n, p_h^n, \omega_h^n) \mapsto (u_h^{n+1}, p_h^{n+1}, \omega_h^{n+1}),$$

requires solving a variational problem (3.8.35). In fact, we are actually dealing with a family of such problems, indexed by the mesh parameter $h$. In what follows, we are concerned with preconditioning these problems with respect to $h$, so that the resulting (preconditioned) problem is given by a linear symmetric positive definite operator whose condition number is independent of $h$. Let us denote for short $(u_h^{n+1}, p_h^{n+1}, \omega_h^{n+1})$ by $(u, p, \omega)$. Now, define linear operators $A, B, C, T_A, T_B$ by variational identities for all $u, v \in V_h$ and $w \in W_h$:

$$\begin{aligned}
&A : V_h \to V_h, && ((Au, v)) = (\nu + \nu_r)(\nabla u, \nabla v) + d(u_h^n,\, u,\, v) + d(u,\, u_h^n,\, v),\\
&B : V_h \to W_h, && (Bu, w) = -(\operatorname{div} u, w), \quad B^* : W_h \to V_h, ((B^* w, u)) = -(\operatorname{div} u, w),\\
&C : V_h \to V_h, && ((Cu, v)) = c_1(\nabla \omega, \nabla v) + d(u_h^n,\, \omega,\, v) + c_2(\operatorname{div} \omega, \operatorname{div} v) + 4\nu_r(\omega, v),\\
&T_A : V_h \to V_h, && ((T_A u, v)) = -2\nu_r(\operatorname{curl} u, v),\\
&T_C : V_h \to V_h, && ((T_C u, v)) = -2\nu_r(\operatorname{curl} u, v) + d(u,\, \omega_h^n,\, v).
\end{aligned}$$

Then we can express (3.8.35) in an operator form:

PROBLEM 5.1.1. For $\mathcal{F} = (\phi, \psi, \varphi) \in V_h \times W_h \times V_h$, find $(u, p, \omega) \in V_h \times W_h \times V_h$ such that

$$\mathcal{M}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \equiv \begin{pmatrix} A & B^* & T_A \\ B & 0 & 0 \\ T_C & 0 & C \end{pmatrix}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix} = \mathcal{F}. \tag{5.1.1}$$

If (5.1.1) is used to perform the Newton iteration step of (3.8.35), then the right hand side of the problem is $((\phi, v)) \equiv d(u_h^n,\, u_h^n,\, v) + (f, v)$, $(\psi, w) \equiv 0$, $((\varphi, \xi)) \equiv d(u_h^n,\, \omega_h^n,\, \xi) + (g, \xi)$.

The following lemma is a consequence of Theorem 3.6.6.

LEMMA 5.1.1. *Suppose that the assumptions of Theorem 3.6.6 hold with sufficiently small $h_0$. In addition, let us assume that $(u_h^n, p_h^n, \omega_h^n)$ is close enough to the solution of Problem 3.4.2. Then for any $\mathcal{F} = (\phi, \psi, \varphi) \in V_h \times W_h \times V_h$ there exists a unique solution $(u, p, \omega)$ of Problem 5.1.1 and*

$$||u||_1 + |p|_2 + ||\omega||_1 \leq \mathrm{Const}(||\phi||_1 + |\psi|_2 + ||\varphi||_1)$$

*with* Const *independent of $h$.*

PROOF. The lemma follows from the fact that under assumptions of Theorem 3.8.1, the derivative of $F_h$ is an isomorphism, whose inverse is bounded independently of $h$. □

Let $A_0 : V_h \to V_h$ be a good preconditioner for the discrete Laplace operator, so that

(i) $A_0^{-1}$ is easy to apply,
(ii) $((A_0 u, v)) = ((A_0 v, u))$ for all $u, v \in V_h$,
(iii) there exist constants $\alpha_1, \alpha_2 > 0$, independent of $h$, such that

$$\alpha_1((u, u)) \leq ((A_0 u, u)) \leq \alpha_2((u, u)) \tag{5.1.2}$$

for all $u \in V_h$.

Likewise, we introduce a good preconditioner for the "mass matrix" operator, i.e.

(iv) $J_0^{-1}$ is easy to apply,
(v) $(J_0 p, q) = (J_0 q, p)$ for all $p, q \in W_h$,
(vi) there exist constants $\beta_1, \beta_2 > 0$, independent of $h$, such that

$$\beta_1(Jp, p) \leq (J_0 p, p) \leq \beta_2(Jp, p) \tag{5.1.3}$$

for all $p \in W_h$.

A good choice for $A_0$ and $J_0$ may be to use a variant of an additive Schwarz method, see Section 4.2.3.

## 5.2. Block diagonal preconditioner

Introducing the block diagonal operator matrix

$$\mathcal{M}_0 = \begin{pmatrix} A_0 & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & A_0 \end{pmatrix}, \tag{5.2.4}$$

we finally define a preconditioned version of problem (3.8.35).

PROBLEM 5.2.1. Find $(u, p, \omega) \in V_h \times W_h \times V_h$ such that

$$\mathcal{M}_0^{-1}\mathcal{M}^*\mathcal{M}_0^{-1}\mathcal{M} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} = \mathcal{M}_0^{-1}\mathcal{M}^*\mathcal{M}_0^{-1}\mathcal{F}. \tag{5.2.5}$$

Observe that if one uses a preconditioned conjugate gradient method with the above preconditioner, then each step of PCG method requires solution of three smaller, independent problems with relatively small computational cost.

LEMMA 5.2.1. *The operator $\mathcal{P} = \mathcal{M}_0^{-1}\mathcal{M}^*\mathcal{M}_0^{-1}\mathcal{M}$ is self-adjoint with respect to the auxiliary scalar product $[\cdot,\cdot]$ defined by*

$$\left[ \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} v \\ q \\ z \end{pmatrix} \right] \equiv ((A_0 u, v)) + (J_0 p, q) + ((A_0 \omega, z)). \tag{5.2.6}$$

PROOF. Indeed, $\mathcal{P}$ may be written as

(5.2.7)

$$\mathcal{P} = \begin{pmatrix} A_0^{-1} & 0 & 0 \\ 0 & J_0^{-1} & 0 \\ 0 & 0 & A_0^{-1} \end{pmatrix} \times$$

$$\times \begin{pmatrix} A^*A_0^{-1}A + B^*J_0^{-1}B + T_C^*A_0^{-1}T_C & A^*A_0^{-1}B^* & A^*A_0^{-1}T_A + T_C^*A_0^{-1}C \\ BA_0^{-1}A & BA_0^{-1}B^* & BA_0^{-1}T_A \\ T_A^*A_0^{-1}A + C^*A_0^{-1}T_C & T_A^*A_0^{-1}B^* & T_A^*A_0^{-1}T_A + C^*A_0^{-1}C \end{pmatrix}.$$

$\square$

The main result of this section is an estimate of the condition number of operator $\mathcal{P}$ in the norm induced by $[\cdot,\cdot]$.

THEOREM 5.2.2. *Let $\mathcal{P}$ be as in Lemma 5.2.1. Suppose Lemma 5.1.1 holds and $A_0, J_0$ satisfy assumptions (i) – (vi) of this chapter. Then there exist positive constants $m_1, m_2$, independent of $h$, such that*

$$m_1 \left[ \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] \le \left[ \mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] \le m_2 \left[ \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right]$$

*for any $(u, p, \omega) \in V_h \times W_h \times V_h$.*

PROOF. By (5.2.6) and (5.2.7), we have

$$\begin{aligned} \left[ \mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] &= ((Au, A_0^{-1}Au)) + (Bu, J_0^{-1}Bu) + ((T_C u, A_0^{-1} T_C u)) \\ &\quad + 2((Au, A_0^{-1}B^* p)) + ((B^* p, A_0^{-1} B^* p)) \\ &\quad + 2((Au, A_0^{-1} T_A \omega)) + 2((T_C u, A_0^{-1} C\omega)) \\ &\quad + 2((B^* p, A_0^{-1} T_A \omega)) + ((T_A \omega, A_0^{-1} T_A \omega)) \\ &\quad + ((C\omega, A_0^{-1} C\omega)), \end{aligned}$$

whence

(5.2.8)

$$\begin{aligned} \left[ \mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] &= ((Au + B^* p + T_A \omega, A_0^{-1}(Au + B^* p + T_A \omega))) + (Bu, J_0^{-1} Bu) \\ &\quad + ((T_C u + C\omega, A_0^{-1}(T_C u + C\omega))). \end{aligned}$$

Therefore,

$$\left[ \mathcal{P} \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \right] \ge \mathrm{Const}\big( ||Au + B^* p + T_A \omega||_1^2 + |Bu|_2^2 + ||T_C u + C\omega||_1^2 \big).$$

Now, Lemma 5.1.1 implies that if $(u, p, \omega)$ solves

$$\begin{cases} Au + B^* p + T_A \omega = \tilde{\phi} & \text{in } V_h, \\ \qquad\qquad\quad Bu = \tilde{\rho} & \text{in } W_h, \\ \qquad\quad C\omega + T_C u = \tilde{\psi} & \text{in } V_h, \end{cases}$$

then

$$||u||_1 + |p|_2 + ||\omega||_1 \leq \text{Const}(||\tilde{\phi}||_1 + |\tilde{\rho}|_2 + ||\tilde{\psi}||_1) \tag{5.2.9}$$

with Const independent of $h$. Therefore, taking (5.2.9), (5.1.2) and (5.1.3) into account, we obtain the lower bound

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \geq \text{Const}\left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right].$$

Similarly we can establish the upper bound. Indeed, (5.2.8) yields

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \leq \text{Const}\big(||Au + B^*p + T_A\omega||_1^2 + |Bu|_2^2 + ||T_C u + C\omega||_1^2\big)$$
$$\leq \text{Const}(||Au||_1^2 + ||B^*p||_1^2 + ||T_A\omega||_1^2 + |Bu|_2^2 + ||T_C u||_1^2 + ||C\omega||_1^2).$$

Estimating each term in the sum similarly as in Section 4.4, we arrive at

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \leq \text{Const}\left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right],$$

which completes the proof. □

The system from (5.2.5) can be solved by the conjugate gradient method since its matrix is symmetric and positive definite with respect to the auxiliary inner product introduced in (5.2.6). By Theorem 5.2.2, the number of iterations required for reducing the residual by a given factor is independent of $h$. As it has been pointed out in [**16**], computing the inner product $[\mathcal{P}u, v]$ during the PCG step requires only one solution of a system with the operator $\mathcal{M}_0$.

## 5.3. Hybrid preconditioner

Mixing the diagonal preconditioner for the micropolar system with the triangular one for the Navier–Stokes system, we construct a hybrid, block diagonal–triangular[1] preconditioner

$$\mathcal{M}_0 = \begin{pmatrix} A_0 & 0 & 0 \\ B & -J_0 & 0 \\ 0 & 0 & A_0 \end{pmatrix}. \tag{5.3.10}$$

[1] Of course, formally, this preconditioner is of triangular form.

In order to treat this preconditioner in the framework of Chapter 4 we reorder the equations and the unknowns in such a way that the last equation is the discrete continuity condition, and the vector of unknowns is $(u, \omega, p)$. Then the system (5.1.1) takes the following matrix form:

$$\mathcal{M}\begin{pmatrix} u \\ \omega \\ p \end{pmatrix} \equiv \left(\begin{array}{cc|c} A & T_A & B^* \\ T_C & C & 0 \\ \hline B & 0 & 0 \end{array}\right)\begin{pmatrix} u \\ \omega \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \\ 0 \end{pmatrix}. \tag{5.3.11}$$

If the submatrix

$$\tilde{A} = \begin{pmatrix} A & T_A \\ T_C & C \end{pmatrix} \tag{5.3.12}$$

is positive definite, then the assumptions of Theorem 4.6.1 and Section 4.3 are satisfied for block problem with matrix

$$\begin{pmatrix} \tilde{A} & \tilde{B}^* \\ \tilde{B} & 0 \end{pmatrix}, \tag{5.3.13}$$

where $\tilde{B} = (B, 0)$.

The only assumption which needs a check is the LBB condition (4.3.10):

$$\sup_{(u,\omega)\in V_h\times V_h} \frac{(\tilde{B}\begin{pmatrix} u \\ \omega \end{pmatrix}, p)}{(||u||_1^2 + ||\omega||_1^2)^{\frac{1}{2}}} = \sup_{(u,\omega)\in V_h\times V_h} \frac{(Bu,\, p)}{(||u||_1^2 + ||\omega||_1^2)^{\frac{1}{2}}} = \sup_{u\in V_h} \frac{(Bu,\, p)}{||u||_1} \geq \beta|p|_2,$$

due to the LBB condition for $V_h$ and $W_h$, see Section 3.2, with $\beta$ as in (4.3.10), independent of $h$.

In this way we proved the following theorem:

THEOREM 5.3.1. *Suppose that*

$$\tilde{A} = \begin{pmatrix} A & T_A \\ T_C & C \end{pmatrix}$$

*is $V_h \times V_h$–elliptic. Then for $\mathcal{M}$, $\mathcal{M}_0$ as in (5.1.1), (5.3.10), with $A_0$ properly scaled, the parameters $\underline{c}$ and $\bar{c}$ defined by (4.6.30) are bounded independently of $h$.*

Finally, we prove that the $V_h \times V_h$ ellipticity assumption on $\tilde{A}$ is non-void, at least for diffusion dominated problems.

LEMMA 5.3.2. *Assume that Theorem 3.6.6 is valid and $(\nu + \nu_r)^{-1} \in \Lambda$, where $\Lambda$ is as in Theorem 3.6.6. Then matrix $\tilde{A}$ defined by (5.3.12) is $V_h \times V_h$- elliptic for $||u^n||_1, ||\omega^n||_1$ sufficiently small with respect to $(\nu + \nu_r)$ and $c_1$.*

PROOF. We have,

$$\left\langle \begin{pmatrix} A & T_A \\ T_C & C \end{pmatrix} \begin{pmatrix} u \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ \omega \end{pmatrix} \right\rangle = ((Au,\, u)) + ((T_A\omega,\, u)) + ((T_C u,\, \omega)) + ((C\omega,\, \omega))$$
$$\begin{aligned} &= (\nu+\nu_r)((u,u)) + d(u,\, u^n,\, u) + d(u^n,\, u,\, u) \\ &\quad + c_1((\omega,\omega)) + c_2(\operatorname{div}\omega, \operatorname{div}\omega) + 4\nu_r(\omega,\omega) \\ &\quad - 2\nu_r(\operatorname{curl}\omega, u) - 2\nu_r(\operatorname{curl} u, \omega) + d(u,\, \omega^n,\, \omega) + d(u^n,\, \omega,\, \omega). \end{aligned} \tag{5.3.14}$$

Now, observe that

$$\begin{aligned} 2\nu_r(\operatorname{curl}\omega, u) + 2\nu_r(\operatorname{curl} u, \omega) = 4\nu_r(\operatorname{curl} u, \omega) &\le 4\nu_r\left(\frac{1}{4}|\operatorname{curl} u|_2^2 + |\omega|_2^2\right) \\ &\le \nu_r||u||_1^2 + 4\nu_r|\omega|_2^2. \end{aligned}$$

It remains to prove a bound on the convective terms. By continuity of $d(\cdot,\,\cdot,\,\cdot)$, we have

$$\begin{aligned} d(u,\, u^n,\, u) + d(u^n,\, u,\, u) + d(u,\, \omega^n,\, \omega) + d(u^n,\, \omega,\, \omega) &\le \mathrm{Const}_1\, ||u||_1^2(||u^n||_{1,2} + ||\omega^n||_{1,2}) + \\ &+ \mathrm{Const}_2\, ||\omega||_1^2(||u^n||_{1,2} + ||\omega^n||_{1,2}). \end{aligned}$$

So if only for some $\gamma > 0$

$$||u^n||_{1,2} + ||\omega^n||_{1,2} < \gamma < \min\{\frac{\nu}{\mathrm{Const}_1}, \frac{c_1}{\mathrm{Const}_2}\},$$

then we easily derive the lower bound in (5.3.14),

$$\left\langle \begin{pmatrix} A & T_A \\ T_C & C \end{pmatrix} \begin{pmatrix} u \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ \omega \end{pmatrix} \right\rangle \ge \mathrm{Const}\cdot(||u||_1^2 + ||\omega||_1^2).$$

$\square$

REMARK 5.3.1. For $\nu$ large enough, $h$, $|f|_2$ and $|z|_2$ sufficiently small, and $(u^0, \omega^0)$ sufficiently close to the solution of Problem 3.4.2, all assumptions of Lemma 5.3.2 are satisfied.

Let us comment that in view of the above theorem, the theory of block triangular preconditioners needs further development to treat the case of non-elliptic diagonal blocks. Notice, however, that we do not need additional assumptions in the case of symmetric micropolar equations as discussed in [**7**] (i.e. with convective terms dropped) – since obviously in that case $\tilde{A}$ is elliptic and symmetric. In such a case both preconditioners: diagonal and hybrid, work well possibly after suitable scaling of the hybrid one.

## 5.4. Numerical experiments

Here, we collect and discuss the results of tests of above preconditioners applied to a *symmetric* and linear micropolar system in 2-D on a unit square $\Omega = (0,1)\times(0,1)$,

$$\begin{cases} -\epsilon^2 \dfrac{\partial^2}{\partial x^2}u_1 - \dfrac{\partial^2}{\partial y^2}u_1 + \dfrac{\partial}{\partial x}p = 2N^2 \dfrac{\partial}{\partial y}\omega & \text{in } \Omega, \\ -\epsilon^4 \dfrac{\partial^2}{\partial x^2}u_2 - \epsilon^2 \dfrac{\partial^2}{\partial y^2}u_2 + \dfrac{\partial}{\partial y}p = -2N^2\epsilon^2 \dfrac{\partial}{\partial x}\omega & \text{in } \Omega, \\ \dfrac{\partial}{\partial x}u_1 + \dfrac{\partial}{\partial y}u_2 = 0 & \text{in } \Omega, \\ -\dfrac{R_L}{R_C}\dfrac{\partial^2}{\partial x^2}\omega - \dfrac{R_L}{R_C\epsilon^2}\dfrac{\partial^2}{\partial y^2}\omega + 4N^2\omega & \\ \qquad = -2N^2\left(\dfrac{\partial}{\partial y}u_1 - \epsilon^2\dfrac{\partial}{\partial x}u_2\right) & \text{in } \Omega, \\ u_1 = u_1^0, \quad u_2 = u_2^0, \quad \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

with parameters $R_L = 1.41$, $R_C = 1.0$, $N = 0.71$, $\epsilon = 0.1$, see [**7**] for details related to this problem. We set inflow/outflow polynomial Dirichlet boundary conditions on $u_1, u_2$ and homogeneous Dirichlet conditions on $\omega$.

We discretized the system using Taylor-Hood rectangular $Q_2$–$Q_1$ elements (for details on experimental settings, see Chapter 4.7). We used exact solvers as the preconditioning blocks. Our main goal was to compare different solver/preconditioner strategies for micropolar equations. In addition to CG/symm, GMRES/diag (using block diagonal preconditioner (5.2.4)), and to GMRES/triang (based on hybrid preconditioner (5.3.10)) strategies discussed in Chapter 4.7, we included for comparison

4. CR/diag – the conjugate residual method accompanied by block diagonal preconditioner (5.2.4), as discussed by Klawonn in [**52**].

The results are presented in Table 5.4.1 and Figure 5.4.1. Since the hybrid preconditioner acts more alike to the diagonal one, the number of iterations for GMRES/hybrid and CG/symm is similar. The CR/diag solver is significantly less effective, but still comparable with the above strategies. The GMRES/diag performs surprisingly poor.

TABLE 5.4.1. Iteration counts for various micropolar problem sizes: results for $A_0$ = Laplacian, $J_0$ = Mass matrix.

| $nx$ | DOF | CR/diag | GMRES/diag | GMRES/triang | CG/symm |
|---|---|---|---|---|---|
| 1 | 84 | 21 | 13 | 8 | 14 |
| 5 | 556 | 53 | 53 | 21 | 38 |
| 10 | 1731 | 69 | 90 | 28 | 46 |
| 20 | 6031 | 77 | 90 | 31 | 49 |
| 40 | 22431 | 79 | 99 | 43 | 50 |
| 80 | 86432 | 81 | 113 | 46 | 50 |

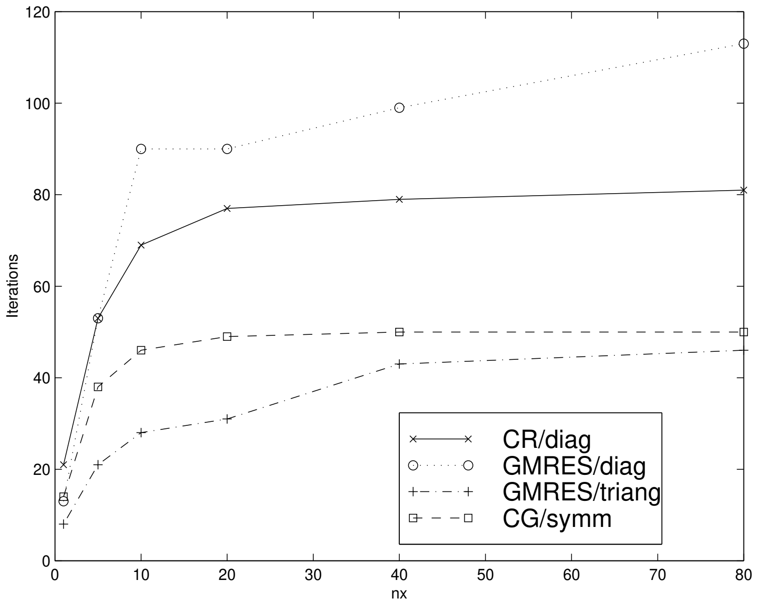


FIGURE 5.4.1. Dependence of number of iterations on the size of micropolar problem, for different solvers and preconditioning strategies: results for $A_0$ = Laplacian, $J_0$ = Mass matrix.

As with the Oseen equations, we conclude that the number of iterations required to decrease initial residual by a given factor is bounded independently from the mesh size (see Figure (5.4.1)). Therefore, the methods proposed and analysed in Chapter 4 can be successfully extended and practically applied to this more complex system.

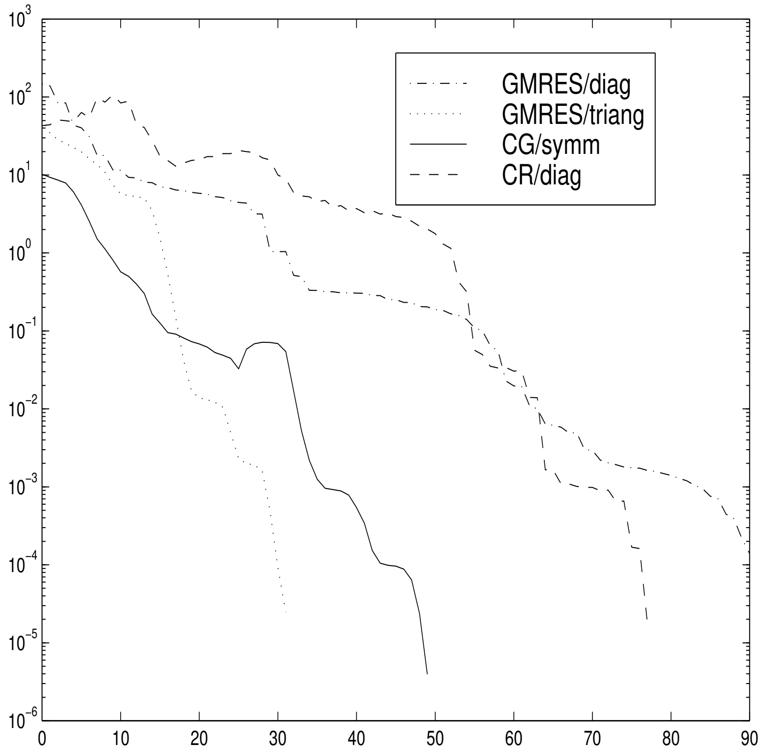


FIGURE 5.4.2. Convergence histories for different solvers.

APPENDIX A

# Function spaces

Differential problems which we analyse in this work, are posed in a weak form in certain Sobolev spaces. For completeness, in this section we list the notation and basic properties of function spaces we use in the thesis. More details can be found in [**1**], [**62**].

Throughout the thesis we use common symbols to denote some standard differential operators, such as the gradient "$\nabla$", the Laplacian "$\Delta$", the divergence "div" or the rotation "curl". The reader may find their complete definitions in almost any textbook on the calculus. Our notation is compatible with [**46**].

Let $\Omega$ denote a bounded open subset in $R^N$, with smooth enough boundary $\partial\Omega$. We denote:

$L^q(\Omega)$ – the space of real functions defined on $\Omega$ with $q$th ($1 \leq q < \infty$) power absolutely integrable with respect to Lebesgue measure in $R^N$, and the norm

$$|\phi|_q = \left(\int_\Omega |\phi|^q\right)^{\frac{1}{q}} \quad \text{for } q < \infty$$

and its obvious modification for $q = \infty$;

$L^2_0(\Omega)$ – the subspace of functions in $L^2(\Omega)$ with zero mean value,

$$L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}.$$

$C_0^\infty(\Omega)$ – the space of smooth functions with compact support in $\Omega$;

$W^{m,q}(\Omega)$ – Sobolev space ($m = 1, 2, 3, \dots : 1 \leq q \leq \infty$) of functions from $L^q(\Omega)$, which generalized derivatives up to the order $m$ are in $L^q(\Omega)$, with norm $\|\cdot\|_{m,q}$ defined as

$$||u||_{m,q} = \left(\sum_{|\alpha|\leq m} |D^\alpha u|_q\right)^{1/q};$$

the seminorm in these spaces will be denoted by $|\cdot|_{m,q}$,

$$|u|_{m,q} = \left(\sum_{|\alpha|=m} |D^\alpha u|_q\right)^{1/q};$$

$W_0^{1,q}(\Omega)$ – the closure of $C_0^\infty(\Omega)$, in the norm $||\cdot||_{1,q}$;

$W^{-1,p}(\Omega)$, $1 \le p < \infty$ – the dual space to $W_0^{1,q}(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$, with the norm

$$||f||_{-1,q} = \sup_{u\in W_0^{1,q}(\Omega), u\neq 0} \frac{\langle f,u\rangle}{||u||_{1,q}},$$

where $\langle\cdot,\cdot\rangle$ denotes a dual pairing between $W^{-1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$;

$V^0(\Omega)$ – the closure of all divergence-free functions from $[C_0^\infty(\Omega)]^N = C_0^\infty(\Omega)\times \ldots \times C_0^\infty(\Omega)$ ($N$-fold Cartesian product) in $||\cdot||_{1,2}$;

$W^{1-\frac{1}{r},r}(\partial\Omega)$ – the space of traces on $\partial\Omega$ of functions from $W^{1,r}(\Omega)$, with usual norm denoted by $[\cdot]_{1-\frac{1}{r},r}$;

$C^0(\bar\Omega)$ – the space of continuous functions on $\bar\Omega$, with the norm $|\cdot|_\infty$;

$C^{0,\alpha}(\bar\Omega)$ – the space of Hölder continuous functions on $\bar\Omega$, with norm denoted by $|\cdot|_{0,\alpha,\Omega}$, $(0<\alpha\le 1)$.

We shall use the symbol $H^m(\Omega)$ to denote the space $W^{m,2}(\Omega)$ and, analogously, we shall write $H_0^1(\Omega)$ while referring to $W_0^{1,2}(\Omega)$.

Where there is no risk of confusion, we write $W^{k,q}$, $H^k$, $H_0^1$, $L_0^2$ instead of $W^{k,q}(\Omega)$, $H^k(\Omega)$, $H_0^1(\Omega)$, $L_0^2(\Omega)$ and use the same symbols for threefold products of such spaces. We equip these product spaces with corresponding product norms.

In particular, we denote the inner product in the space $[L^2(\Omega)]^N$ ($N=1$ or 3) by

$$(u,v) := \sum_{i=1}^N \int_\Omega u_i\, v_i\, dx, \qquad u,v\in [L^2(\Omega)]^N.$$

For the inner product in $[W_0^{1,2}(\Omega)]^N$, $N=1$ or 3, we use the notation

$$((u,v)) := \sum_{j=1}^N (\nabla u_j, \nabla v_j) = \sum_{j=1}^N\sum_{i=1}^N \int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}, \qquad u,v\in [W_0^{1,2}(\Omega)]^N.$$

We shall need the following Poincaré–Friedrichs inequality:

THEOREM A.0.1. *If $\Omega$ is connected and bounded, then there exists* Const $> 0$ *such that*

$$|u|_2 \leq \text{Const}\, |\nabla u|_2, \qquad \forall u \in W_0^{1,2}(\Omega).$$

From this fact it follows that in $W_0^{1,2}(\Omega)$, the seminorm $|\cdot|_{1,2}$ defines a norm which is equivalent to $||\cdot||_{1,2}$. As this norm frequently appears in this thesis, we use a special symbol, $||\cdot||_1$, for this norm:

$$||u||_1 = |u|_{1,2} = |\nabla u|_2 \qquad \forall u \in W_0^{1,2}(\Omega).$$

The following imbedding theorem constitutes relationships between different Sobolev spaces and others:

THEOREM A.0.2. *Let $\Omega$ be an open, bounded subset of $R^N$ with Lipschitz continuous boundary. For real $1 \leq p < \infty$ and integers $n \leq m$ the following imbeddings hold:*

$$W^{m,p}(\Omega) \hookrightarrow \begin{cases} W^{n,q}(\Omega), & \text{if } \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m-n}{N} > 0, \\ C^n(\bar{\Omega}), & \text{if } \dfrac{1}{p} < \dfrac{m-n}{N}. \end{cases}$$

*In addition, the imbedding $W^{m,p}(\Omega) \hookrightarrow W^{n,\tilde{q}}(\Omega)$ is compact for all $\tilde{q}$ which satisfy*

$$1 \leq \tilde{q} < \frac{Np}{N-(m-n)p}, \qquad \text{whenever } N > (m-n)p,$$

*or*

$$1 \leq \tilde{q} < \infty, \qquad \text{if } N = (m-n)p.$$

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