Flipping and Forking

LICS 2025

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University of Warsaw

26th June 2025

Tree-like decompositions

Everybody loves tree-like decompositions of graphs!

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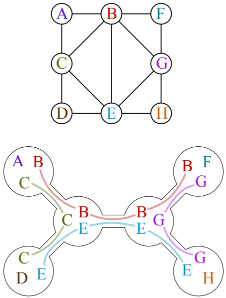

Treewidth

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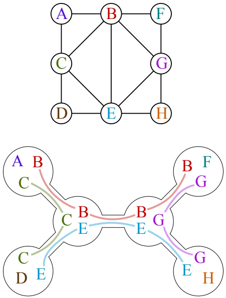

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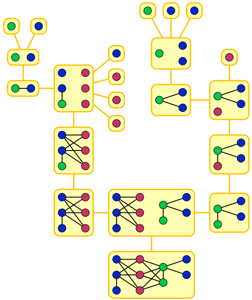

Clique-width

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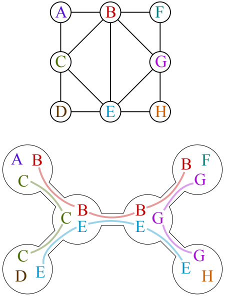

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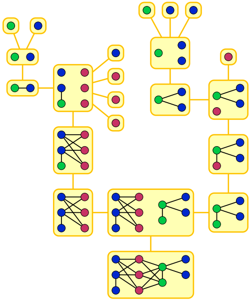

Clique-width

- Useful for dynamic programming on graphs
- Enable efficient model checking algorithms
- Often yield fixed-parameter tractable (FPT) algorithms

Figures adapted from Wikipedia (David Eppstein)

Monadically stable classes of graphs

Definition

A class of graphs is *monadically stable* if it does not transduce the class of all half-graphs.

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A half-graph of order n is a bipartite graph (U, V, E) with $U = \{a_1, \ldots, a_n\}, V\{b_1, \ldots, b_n\}$ and edges $E = \{(a_i, b_j) \mid i \leqslant j\}$.

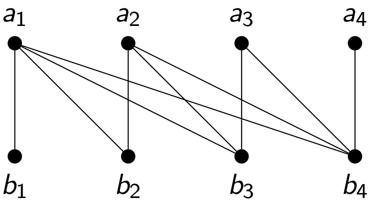

Figure: A half-graph of order $n = 4$.

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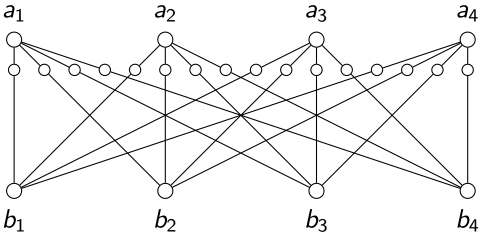

Figure: 1-subdivision of $K_{4,4}$.

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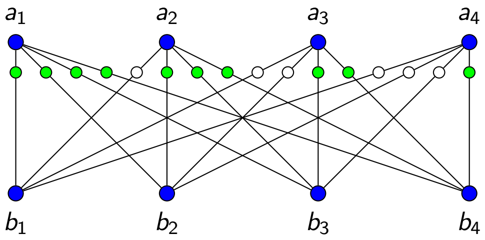

Figure: Colored 1-subdivision of $K_{4,4}$.

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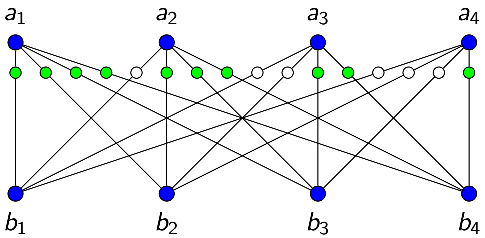

Now let's apply

$$\varphi(x, y) \equiv \mathrm{Blue}(x) \wedge \mathrm{Blue}(y) \wedge \exists z \, (\mathrm{Green}(z) \wedge E(x, z) \wedge E(y, z)) \, .$$

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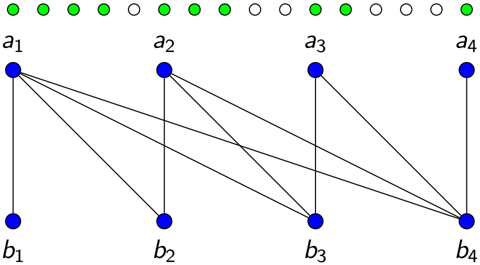

Figure: The result of applying φ to the colored graph.

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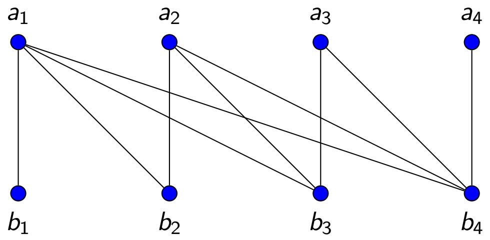

Figure: We take an induced subgraph.

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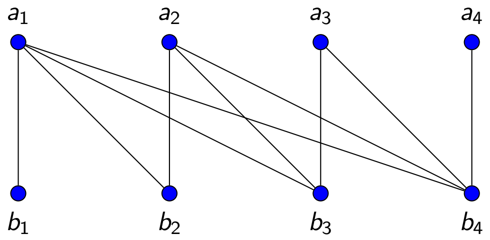
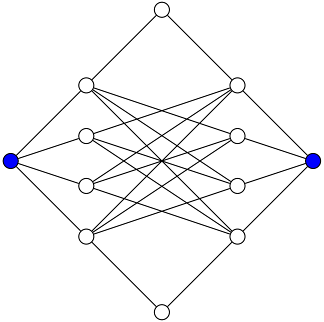


Fact

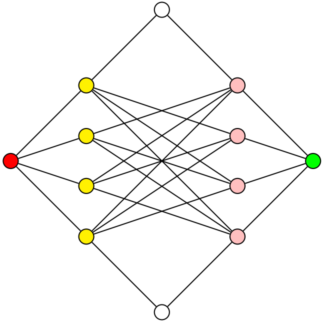
The class of all 1-subdivided bicliques is not monadically stable.

(Definable) flips



We start with a graph and a subset S of its vertices (in blue).

(Definable) flips



We partition the vertices of $V(G)$ by their neighborhoods on S.
Let's flip the edges between these pairs of colors: (○, ○), (○, ○), (○, ○).

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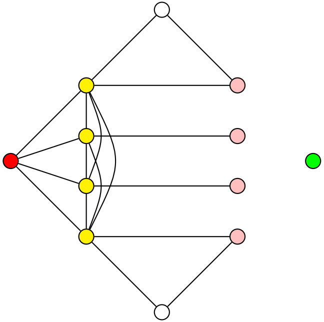
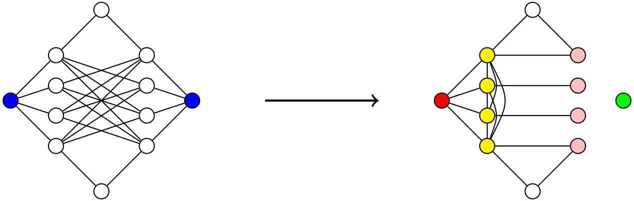


The result of the flip.

(Definable) flips



Definition

Fix a graph G and a set $S \subseteq V(G)$. An S-flip of G is obtained as follows:
1. Partition $V(G)$ by their neighborhoods in S;
2. For each pair or parts (P_1, P_2) (possibly $P_1 = P_2$) either keep the edges between P_1 and P_2 or complement them.

Flipper Game

The radius-r batch-ℓ Flipper game is played on a graph G_1. In round i

1. Flipper chooses a set S_i with $|S_i| \leqslant \ell$ and an S_i-flip H_i of G_i.
2. Connector chooses G_{i+1} as an induced subgraph given by radius-r ball in H_i.

Flipper wins once G_i has size 1.

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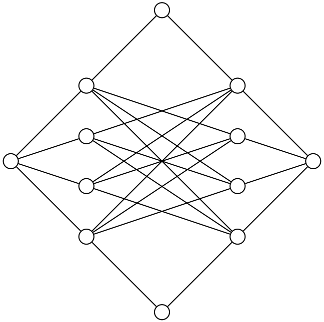

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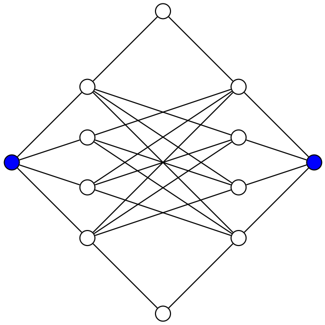

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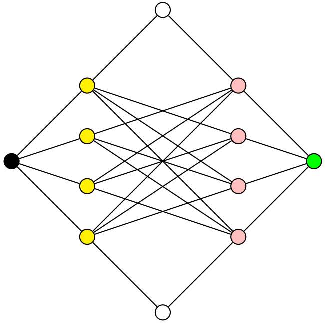

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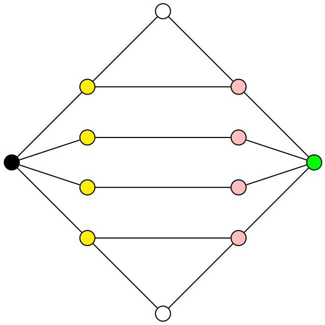

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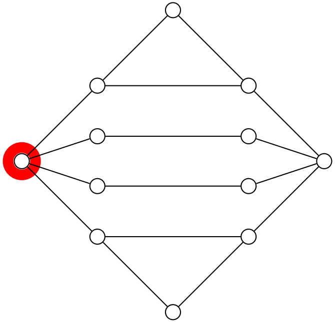

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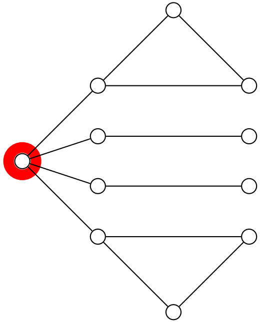

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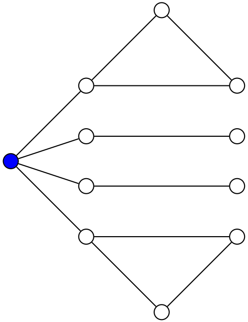

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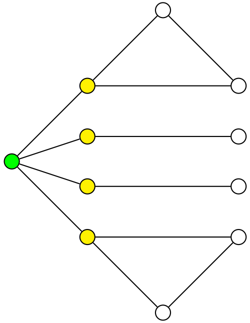

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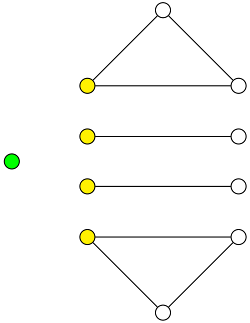

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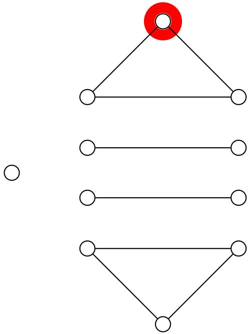

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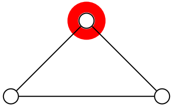

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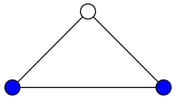

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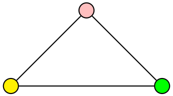

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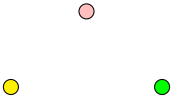

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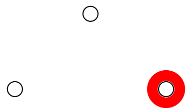

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Theorem [Gajarský, Mählmann, McCarty, Ohlmann, Pilipczuk, P., Siebertz, Sokołowski, Toruńczyk, '23]

The following are equivalent for a class of graphs $\mathcal{C}$:

- $\mathcal{C}$ is monadically stable,
- $\forall r\ \exists k\ \exists \ell$ such that for every graph $G \in \mathcal{C}$ Flipper wins the radius-r batch-ℓ Flipper game on G in at most k rounds.

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This gives tree-like decompositions for monadically stable classes of graphs.

Monadically stable classes of relational structures

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Our contribution:

- We give a definition of (definable) flips for relational structures.

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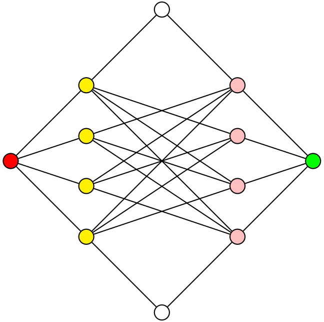
Our contribution:

- We give a definition of (definable) flips for relational structures.
- We characterize monadically stable classes of relational structures combinatorially, in terms of flipper game.
- We characterize forking independence – a fundamental notion in model theory – in terms of flips.

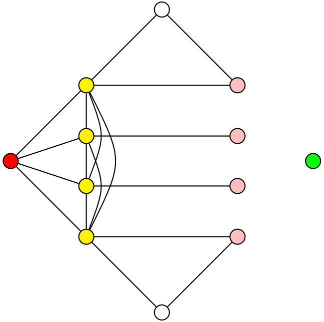
What are flips?



Observation

The edges of an S-flip can be defined by a quantifier free formula $\varphi(x, y)$ with parameters from S.

What are flips?



Flips are reversible – the edges of the original graph can be defined in a flip by a quantifier free formula $\psi(x, y)$ with parameters from S.

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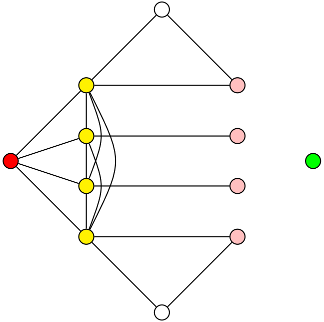

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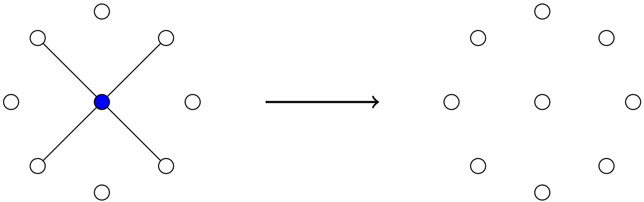

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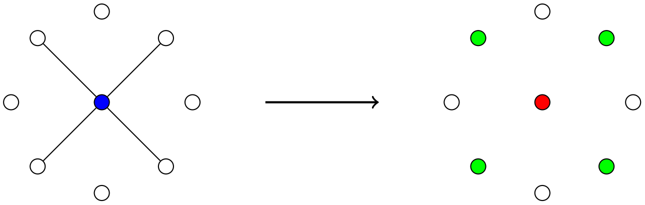

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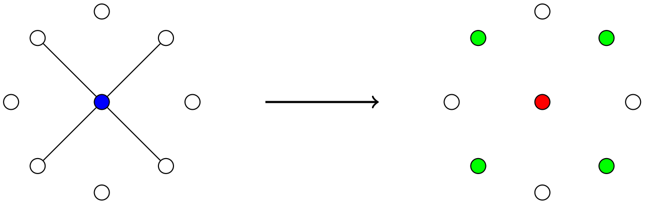

Flips are reversible – the edges of the original graph can be defined in a flip by a quantifier free formula $\psi(x, y)$ with parameters from S.
We need the access to the neighborhood classes in the original graph. Luckily, they are also definable by a quantifier free formula!

Flips of relational structures

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We say that $\mathbf{N}$ is an S-flip of $\mathbf{M}$ if all the formulas mentioned above use only parameters from S.

Example

Fix a rooted tree T. Take $\mathbf{M} = (V(T), \{(u, v, w) \mid u = \mathrm{lca}(v, w)\})$.

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$$\mathbf{M} \models R(x, y, z) \iff \mathbf{N} \models R_0(x, y, z) \vee (x = s \wedge R_1(y, z)) \vee (y = s \wedge R_2(x, z)) \vee (z = s \wedge R_2(x, y)).$$

Flipper game for relational structures

Theorem [P., Toruńczyk]

The following are equivalent for a class of relational structures $\mathcal{C}$:

- $\mathcal{C}$ is monadically stable,
- $\forall r \; \exists k$ such that for every structure $G \in \mathcal{C}$ Flipper wins the radius-r Flipper game on G in at most k rounds.

Flipping and forking

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It generalizes e.g. linear independence in vector spaces and algebraic independence in fields.

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Definition [GMMOPPSST'23, PT'25]

We say that *a is distance-r flip independent from b over C* (denoted $a \perp^r_C b$) if there is a C-flip $\mathbf{N}$ of $\mathbf{M}$ such that the distance between a and b in the Gaifman graph of $\mathbf{N}$ is $\geqslant r$.

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Theorem [P., Toruńczyk]

For every monadically stable structure $\mathbf{M}$, its elementary extension $\mathbf{N}$, and two distinct elements $a, b \in \mathbf{N}$

$$a \underset{\mathbf{M}}{\perp\!\!\!\smile} b \iff \forall r \quad a \overset{r}{\underset{\mathbf{M}}{\perp\!\!\!\smile}} b.$$

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Thank you!