MEASURES IN LOCALLY COMPACT GROUPS ARE CARRIED BY MEAGER SETS

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ABSTRACT. We show that for a σ-finite diffused Borel measure in a nondiscrete locally bounded topological group there is a meager set whose complement is of measure zero.

1. Introduction

A finite Borel measure in a topological space is called *residual* if it vanishes on nowhere dense sets.

S. Lotz ([14, Lemma 6.11]) proved that a nondiscrete compact zero–dimensional topological group does not admit a nontrivial finite residual measure (and it follows from a result of T. Armstrong ([1, Proposition 5]) that the assumption that the space is zero–dimensional can be dropped). We prove that the same conclusion holds for any nondiscrete locally bounded topological group.

Section 2 contains preliminary material and basic definitions and facts about residual measures. In Section 3 we prove with no help of additional set theoretic assumptions that in a nondiscrete locally compact topological group G there are no residual measures, regardless of the character of G. The problem whether the same is true for subspaces of such a group is discussed in Section 5 and decided under the assumption $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$. The particular case of the Haar measure (that does not have to be σ-finite) is considered in Section 4.

For more information on residual measures we refer the reader to [6, 7, 8] and [15]. Concerning the question which topological properties forbid the existence of residual measures, we refer to [15], [16] and [4].

2. Residual and schismatic measures

All topological spaces we work with are assumed to be Hausdorff, so the term "space" refers to a Hausdorff topological space. A *Borel measure* μ in a space X is a σ-additive mapping that assigns to each Borel subset of X a value from the interval $[0, \infty]$.

A measure μ in X is *trivial* if $\mu X = 0$, *finite* if $\mu X < \infty$, and *σ-finite* if X admits a countable cover by measurable sets of finite measure. It is *diffused* if it vanishes on singletons.

All sets that differ from a Borel set by a negligible set, i.e. by a set contained in a Borel set of measure zero, are considered measurable. A

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measure μ is *outer regular* if each Borel set is contained in a G_δ–set of the same measure.

A measure μ in X is *fully supported* if the union of all open negligible sets is negligible, and *strictly positive* if $\mu(U) > 0$ for every nonempty open set $U \subseteq X$. Obviously, a strictly positive measure is fully supported.

A finite measure μ in X is τ*-additive* if $\mu(\bigcup \mathcal{U}) = \sup\{\mu(U) : U \in \mathcal{U}\}$ holds for any upwards directed family $\mathcal{U}$ of open sets. Every finite τ-additive measure is fully supported.

The symbol ω denotes the first infinite cardinal = the set of nonnegative integers and ω_1 denotes the first uncountable cardinal. The cardinality of the continuum is denoted by $\mathfrak{c}$. The cardinality of a set A is denoted by $|A|$.

We shall also need the notion of a real–valued measurable cardinal. Recall that a cardinal κ is *real–valued measurable* if there is a diffused, finite, nontrivial κ-additive measure defined on the power set of κ (see [9]). The first real–valued measurable cardinal is denoted by $\mathfrak{r}$ (provided that it exists). Then $|X| < \mathfrak{r}$ is an abbreviation of the statement that there is no real–valued measurable cardinal $\leq |X|$.

Recall that a subset A of a space is *meager* if it is a countable union of nowhere dense sets, and *comeager* if it is the complement of a meager set.

Definition 2.1. Let μ be a σ-finite Borel measure in a space X. The measure μ is called

(i) *residual* if $\mu A = 0$ whenever $A \subseteq X$ is meager,
(ii) *schismatic* if there is a meager set $A \subseteq X$ such that $\mu(X \setminus A) = 0$.

For an exhaustive review of residual measures see [6, 7, 8], where residual measures are called *hyperdiffuse*. The notion of a schismatic measure, dual to that of a residual measure, was introduced in [15]. For any σ-finite measure μ in a space X there are unique measures μ_s and μ_r in X such that $\mu = \mu_s + \mu_r$, μ_s is schismatic and μ_r is residual ([15, Proposition 1.2]).

Definition 2.2. A space X is called *schismatic* if every σ-finite diffused Borel measure in X is schismatic. Equivalently, if there are no nontrivial finite diffused residual Borel measures in X.

The reason why we take into account diffused measures only is that otherwise any space with an isolated point would not be schismatic (a Dirac measure is residual if and only if it is concentrated on an isolated point).

It is obvious that a discrete space D is schismatic if and only if $|D| < \mathfrak{r}$. Therefore if X is a space and I is the set of its isolated points, then X is schismatic if and only if $|I| < \mathfrak{r}$, the subspace $X \setminus \overline{I}$ is schismatic and has no isolated points. So the only interesting case is actually that of X without isolated points. The details are worked out in [15].

The following proposition summarizes some basic properties of the schismatic spaces. Assertion (i) comes from [6, Proposition 2.19], see also [15, Corollary 3.7]; statement (iii) is [15, Proposition 2.8(ii)].

Proposition 2.3. (i) *Every metrizable space without isolated points is schismatic.*
(ii) *Every dense subspace of a schismatic space is schismatic.*
(iii) *The preimage of a schismatic space without isolated points under an open continuous mapping is schismatic.*

3. Locally bounded groups are schismatic

A topological group is either discrete or not. The discrete case is not very interesting: As explained above, if G is discrete, then it is schismatic if and only if $|G|< \mathfrak{r}$. The goal of this section is to prove that a nondiscrete locally bounded topological group is schismatic.

Let us recall the notion of a locally bounded group and related concepts. We adhere to the multiplicative notation. A subset A of a topological group G is called *bounded* if for each neighborhood V of the unity there is a finite set $F \subseteq G$ such that $A \subseteq FV$. The group G is called *locally bounded* if some neighborhood of the unity is bounded.

A topological group G has two natural uniform structures called the right and left uniformities. If G is locally bounded, then the uniform completions of these uniformities coincide (even if G is not abelian). We refer to this completion as the *Weil completion* and denote it by $G^\star$. It is known (see e.g. [3, 1.13 and 1.14]) that if G is locally bounded, then $G^\star$ is a locally compact group and G is a dense subgroup of $G^\star$.

We shall need the following two lemmas. Lemma 3.1 can be found e.g. in [11, Theorem 5.14] – the "therefore" part follows from the fact that if H is a clopen subgroup of a locally compact group G and $\varphi : G/H \to G$ is an arbitrary function choosing exactly one point from each left coset of H in G, then the function $(h,c) \mapsto \varphi(c)h$ is a homeomorphism between $H \times G/H$ and G, so G is topologically $H \times G/H$. Lemma 3.2 is a slightly improved version of a theorem of Kakutani and Kodaira [12] and can be found e.g. in [3, Theorem 3.7].

Lemma 3.1. *Let G be a locally compact group. Then G contains a σ-compact clopen subgroup $H \subseteq G$. Therefore there is a space D such that G is homeomorphic to $H \times D$.*

Lemma 3.2. *Let G be a locally compact σ-compact group and $\langle V_n : n \in \omega\rangle$ a sequence of neighborhoods of the unity. Then there is a normal compact subgroup $K \lhd G$ such that $K \subseteq \bigcap_{n\in\omega} V_n$ and the quotient group G/K is metrizable.*

Now we are ready to state the main result of this section.

Theorem 3.3. *Each nondiscrete, locally bounded group is schismatic.*

Proof. Let G be a nondiscrete locally bounded group. Since G is dense in its Weil completion $G^\star$, according to Proposition 2.3(ii) we may assume, without loss of generality, that G itself is locally compact. Let H and D be the sets from Lemma 3.1. Since G is topologically $H \times D$, H is the image of G under an open continuous mapping (the projection). Hence by Proposition 2.3(i) and (iii), it suffices to find an open continuous mapping of H onto a metrizable space Y without isolated points.

Denote by $\mathbf{e}$ the unity of H and by λ a (left) Haar measure on H. Since H is not discrete, $\lambda(\{\mathbf{e}\}) = 0$. Since λ is outer regular (see [3, 1.15]), there is a sequence $\langle V_n : n \in \omega\rangle$ of open neighborhoods of $\mathbf{e}$ in H such that $\lambda\big(\bigcap_{n\in\omega} V_n\big) = 0$. Lemma 3.2 yields a normal compact subgroup $K \lhd H$ such that $\lambda(K) = 0$ and H/K is metrizable. As Haar measure λ is strictly positive, the set K is nowhere dense in H.

Let $\pi : H \to H/K$ be the quotient homomorphism and let $Y = H/K$. Clearly, π is an open continuous mapping and to complete the proof we show that Y has no isolated points.

Indeed, if a point $\pi(h) \in Y$ were isolated, then $\pi^{-1}(\pi(h))$ would be open in H. But $\pi^{-1}(\pi(h)) = hK$ and since K is nowhere dense, so is hK – a contradiction.

□

4. Haar measure

We now investigate the particular case of Haar measure. If G is nondiscrete, locally compact and σ-compact, Haar measure on G is σ-finite. By the preceding theorem, there is a meager set $M \subseteq G$ such that $G \setminus M$ is of Haar measure zero. If G is nondiscrete, locally compact, but not σ-compact, Haar measure on G is not σ-finite. It nevertheless makes sense to ask if some meager set is of positive or full Haar measure.

Proposition 4.1. *Let G be a nondiscrete, locally compact, but not σ-compact topological group and λ its Haar measure. Then each uncountable closed discrete subset of G is of non–σ-finite measure λ. Therefore*

(i) *there is a nowhere dense set $N \subseteq G$ such that $\lambda(N) = \infty$*

(ii) *there is no meager set $M \subseteq G$ such that $\lambda(G \setminus M) = 0$. As a matter of fact, each comeager subset of G is of non–σ-finite Haar measure.*

Proof. Let A be an uncountable closed discrete set in G. We prove that A is of non–σ-finite Haar measure. Identify G with $H \times D$ of Lemma 3.1. As H is σ-compact, $A \cap H$ is at most countable. It follows that the set $I = \{d \in D : (H \times \{d\}) \cap A \neq \emptyset\}$ is uncountable. Since λ is outer regular, if A had σ-finite measure, there would exist an open set $U \supseteq A$ of σ-finite measure and as λ is strictly positive, $\lambda(U \cap (H \times \{d\})) > 0$ for each $d \in I$. Since $\{U \cap (H \times \{d\}) : d \in I\}$ is disjoint and uncountable, we have the desired contradiction.

(i) follows from the obvious facts that first, G being nondiscrete, a discrete closed subset of G must be nowhere dense, and second, that G being non–σ-compact there is an uncountable closed discrete set in G.

To show (ii) it suffices to prove that each comeager set $C \subseteq G$ contains an uncountable closed discrete set. But since G is locally compact, no nonempty open subset of G is meager. Hence C, being comeager, meets the set $H \times \{d\}$, for each $d \in D$. Taking one point of C in each copy of H we obtain an uncountable closed discrete subset of C. □

Remark 4.2. A brief inspection of the proof reveals that the only properties of the Haar measure we needed were outer regularity and strict positivity. Therefore Proposition 4.1 holds for any Borel measure λ in G that is outer regular and strictly positive.

5. Subspaces of locally bounded groups

Here we discuss the question if subspaces of a locally bounded group are schismatic.

If G is a first countable topological group, then G is metrizable (see [3, 1.8]). Therefore Proposition 2.3(i) implies that each subspace of G without isolated points is schismatic. When the local character of G is uncountable, the situation is more complicated. From now on the local character of a topological space X is denoted by $\chi(X)$, and the density of X is denoted by $d(X)$.

We shall make use of the cardinal $\mathrm{non}(\mathcal{N})$ which is defined by

$$\mathrm{non}(\mathcal{N}) = \min\{|A| : A \subseteq \mathbb{R} \text{ is not Lebesgue measurable}\}.$$

Clearly, $\omega < \mathrm{non}(\mathcal{N}) \leq \mathfrak{c}$. We refer to [2, 2.1 and 7.1] for further details.

We are ready to state the main result of this section.

Theorem 5.1. *Let G be a locally bounded group. If $\chi(G) < non(\mathcal{N})$, then each subspace of G without isolated points is schismatic.*

The proof requires some lemmas. Lemma 5.2 is taken from [9, 6G]; it relates $\mathrm{non}(\mathcal{N})$ and the possible value of the first real–valued measurable cardinal, $\mathfrak{r}$. Lemma 5.3 is [16, Lemma 3.2(ii)] and Lemma 5.4 is a modified version of [16, Lemma 3.4(i)].

Lemma 5.2. *$non(\mathcal{N}) > \omega_1 \implies \mathfrak{c} < \mathfrak{r}$.*

Lemma 5.3. *Assume that $\mathfrak{c} < \mathfrak{r}$. Let X be a locally compact space without isolated points such that $\chi(X) < \mathfrak{r}$. If X is not schismatic, then there is a fully supported nontrivial residual measure in X.*

Lemma 5.4. *Let X be a regular Hausdorff space and μ a fully supported finite residual Borel measure in X. Let $E \subseteq X$. If $d(E) < non(\mathcal{N})$, then $\mu(E) = 0$.*

Proof. Denote by $\mu^\sharp$ the outer measure. Assume that $\mu^\sharp(E) > 0$. Let $D \subseteq E$ be a dense set such that $|D| < \mathrm{non}(\mathcal{N})$. By [16, Proposition 2.6(ii)] it follows that $\mu^\sharp(D) = \mu(\overline{D}) = \mu(\overline{E}) > 0$. Define a Borel measure μ_D on D by $\mu_D(B) = \mu^\sharp(B)$ (the *restriction* of μ). Then μ_X is finite and diffused. By [16, Fact 2.2(i)], the measure μ is τ-additive and by [10, Lemma 11.4], μ_D is τ-additive as well. Therefore it does not have an atom. But [10, Theorem 9.2] asserts that if $|D| < \mathrm{non}(\mathcal{N})$, then any finite Borel measure in D has an atom, a contradiction. $\square$

Proof of Theorem 5.1. First of all, as obviously $\chi(G^*) = \chi(G)$, we may assume that G is locally compact. If $\chi(G) = \omega$, then G is metrizable and by Proposition 2.3(i) we are done. So assume that $\chi(G) \geqslant \omega_1$. As, by the assumption, $\chi(G) < \mathrm{non}(\mathcal{N})$, we have $\mathrm{non}(\mathcal{N}) > \omega_1$. Therefore Lemma 5.2 yields $\mathfrak{c} < \mathfrak{r}$. Aiming at a contradiction, assume that $Y \subseteq G$ is a space without isolated points that is not schismatic. According to Proposition 2.3(ii) we may assume that Y is closed and hence locally compact. As $\chi(Y) \leqslant \chi(G) < \mathrm{non}(\mathcal{N}) \leqslant \mathfrak{c} < \mathfrak{r}$, Lemma 5.3 yields a fully supported nontrivial residual measure μ in Y.

Identify G with $H \times D$ of Lemma 3.1. As H is open in G and μ is fully supported, there is $d \in D$ such that $\mu\big(Y \cap (H \times \{d\})\big) > 0$. But on the other hand, since H is a σ-compact group, we have $d(H) \leqslant \chi(H)$, as easily follows from [3, 3.5]. Therefore $d\big(Y \cap (H \times \{d\})\big) \leqslant d(H) \leqslant \chi(G) < \mathrm{non}(\mathcal{N})$ and Lemma 5.4 yields $\mu\big(Y \cap (H \times \{d\})\big) = 0$, a contradiction. $\square$

If the character of G is too large, the conclusion of Theorem 5.1 fails. The following is a partial converse to Theorem 5.1.

Theorem 5.5. *If G is a locally compact group such that $\chi(G) \geq \mathfrak{c}$, then G has a compact subspace $K \subseteq G$ that admits a finite, strictly positive, diffused residual measure. In particular, K is not schismatic.*

Proof. For a space X denote by $\pi\chi(x, X)$ the π–character of X at a point $x \in X$, and by $\pi\chi(X)$ the π–character of X. If $X = G$ is a topological group, then obviously $\pi\chi(x, G) = \pi\chi(G)$ for each $x \in G$. By [3, 3.6], $\pi\chi(G) = \chi(G)$ for any topological group. The theorem now easily follows from [16, Proposition 3.6] that asserts that if X is a locally compact space such that $\pi\chi(x, X) \geq \mathfrak{c}$ for each $x \in X$, then X has a compact subspace K that admits a finite, strictly positive, diffused residual measure. □

The combination of theorems 5.1 and 5.5 yields the following corollary.

Corollary 5.6 (Assume $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$). *Let G be a locally compact group. Then each subspace of G without isolated points is schismatic if and only if $\chi(G) < \mathfrak{c}$.*

Remark 5.7. One of the referees conjectured that in Lemma 5.4 and Theorem 5.1, $\mathrm{non}(\mathcal{N})$ is best possible. For 5.4 it is indeed so: It is enough to take for X the real line provided with the density topology, for μ the Lebesgue measure and for E a nonmeasurable set of the reals of size $\mathrm{non}(\mathcal{N})$. Concerning Theorem 5.1, the conjecture remains unresolved.

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