A NOTE ON WEAK DISTRIBUTIVITY AND CONTINUOUS RESTRICTIONS OF BOREL FUNCTIONS

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Abstract. The aim of this note is to present a simple proof of the following theorem closely connected to earlier results by Recław [2] and Zapletal [3]: if I is a σ-ideal in the σ-algebra $\mathbf{B}(\mathbb{R})$ of the Borel subsets of the reals then the quotient Boolean algebra $\mathbf{B}(\mathbb{R})/I$ is weakly distributive if and only if:

- every Borel set $B \notin I$ contains a compact set $C \notin I$,
- for every Borel set $B \notin I$ and a Borel function $f : B \to \mathbb{R}$ there is a set $C \subseteq B$, $C \notin I$ such that $f \restriction C$ is continuous.

Let X be an uncountable Polish, i.e., separable, completely metrizable, topological space. By $\mathbf{B}(X)$ we denote the σ-algebra of all Borel subsets of X. A *σ-ideal* I on X is a family $I \subseteq \mathbf{B}(X)$ which is closed under taking Borel subsets and countable unions; we always assume that I is *proper*, i.e., $X \notin I$ and contains all singletons. Then the quotient Boolean algebra $\mathbb{B}_I = \mathbf{B}(X)/I$ is σ-complete and atomless. Recall that a σ-complete Boolean algebra $\mathbb{B}$ is called *weakly distributive* (more exactly (ω, ω, ω)-weakly distributive), if for every sequence $\langle P_n : n \in \omega \rangle$ of countable maximal antichains in $\mathbb{B}$, there exists a maximal antichain Q in $\mathbb{B}$ such that each element $q \in Q$ meets only finitely many elements of each P_n. It is easy to see that the weak distributivity of $\mathbb{B}_I$ in particular implies that for every set $B \in I^+ = \mathbf{B}(X) \setminus I$ and a sequence $\langle \mathcal{P}_n : n \in \omega \rangle$ of countable coverings of B by Borel sets, there is a set $C \subseteq B$ such that $C \in I^+$ and C is contained in the union of finitely many sets from $\mathcal{P}_n$ for each $n \in \omega$.

The following theorem was proved by Zapletal [3, Lemma 2.2.3] under the additional assumption that forcing with $\mathbb{B}_I$ is proper. Property (2) below was earlier proved in a more general setting by Recław [2, Corollary 2] under the stronger condition of $(\omega, \cdot, \omega)$-distributivity (which is just like (ω, ω, ω)-distributivity but with no restrictions on the cardinality of antichains P_n imposed) of the algebra $\mathbb{B}_I$. Our simple proof of the assertion that conditions (1) and (2) follow from (ω, ω, ω)-distributivity of $\mathbb{B}_I$ is based on fundamental facts about Borel sets in

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Polish spaces ([1, Theorem 13.1]) and resembles the standard argument showing that a finite Borel measure on a Polish space is inner regular with respect to compact sets (see [1, Theorem 17.11]). For the sake of completeness we also sketch a folklore-like proof of the opposite implication.

Theorem 1. *If I is a σ-ideal on a Polish space (X,τ) then the Boolean algebra $\mathbb{B}_I$ is weakly distributive if and only if:*

(1) every Borel set $B \in I^+$ contains a compact set $C \in I^+$,
(2) for every Borel set $B \in I^+$ and a Borel function $f : B \to Y$ into a second countable space Y there is a set $C \subseteq B$, $C \in I^+$ such that $f \restriction C$ is continuous.

Proof. First assume that the algebra $\mathbb{B}_I$ is weakly distributive.

In order to prove (1) let $B \in I^+$. First find a Polish topology $\tau_B \supseteq \tau$ on X such that $\mathbf{B}(X,\tau_B) = \mathbf{B}(X,\tau)$ and B is clopen in τ_B. Then B with the subspace topology inherited from (X,τ_B) is a Polish space and all topological notions concerning B will refer to this space. Fix a compatible complete metric for B. For each n pick a covering $\mathcal{P}_n = \{B_i^n : i \in \omega\}$ of B by closed balls with $\mathrm{diam}(B_i^n) \le 2^{-n}$. Since the algebra $\mathbb{B}_I$ is weakly distributive, there is a set in I^+ which for every n is contained in the union of finitely sets from $\mathcal{P}_n$; let C be its closure. Then C is closed and totally bounded, and thus compact. Note that since $\tau \subseteq \tau_B$, C is compact in the space (X,τ) as well. This proves (1).

To prove (2) let again $B \in I^+$ and a let $f : B \to Y$ be a Borel function into a second countable space Y. This time find a Polish topology $\tau_f \supseteq \tau$ on X such that $\mathbf{B}(X,\tau_f) = \mathbf{B}(X,\tau)$, B is closed in τ_f and moreover f is continuous as a function defined on B with the subspace topology inherited from (X,τ_f). By (1), there is a compact (in τ_f) set $C \subseteq B$, $C \in I^+$. Then the function $f \restriction C$ is continuous, since the topology of C as a subspace of (X,τ) is the same as its topology as a subspace of (X,τ_f).

For the opposite direction, for each n let $P_n = \{b_i^n : i \in \omega\}$ be a maximal antichain in $\mathbb{B}_I$. Pick Borel representatives B_i^n of b_i^n such that sets $\{B_i^n : i \in \omega\}$ are pairwise disjoint for each $n \in \omega$.

Take an arbitrary $B \in I^+$; it suffices to find a set $C \subseteq B$, $C \in I^+$ such that C has non-empty intersection with finitely many sets $\{B_i^n : i \in \omega\}$ for each $n \in \omega$.

So let $Y = \bigcap_{n\in\omega} \bigcup_{i\in\omega} B_i^n$; clearly, $Y \in \mathbf{B}(X)$ and $X \setminus Y \in I$. Let $B' = B \cap Y \in \mathcal{I}^+$ and define the function $f : B' \to \mathcal{N}$ ($\mathcal{N} = \omega^\omega$ is the Baire space of irrationals) by putting $f(x)(n) = i$ if and only if $x \in B_i^n$. The function f is Borel so (1) and (2) imply that there is a compact set $C \subset B'$, $C \notin I$ such that $f \restriction C$ is continuous. Then $f[C]$ is a compact subset of $\mathcal{N}$ hence it is bounded by a function $g : \omega \to \omega$. But then $i > g(n)$ implies that $C \cap B_i^n = \emptyset$, so C has the desired property. $\square$

Actually, Zapletal [3, Lemma 2.2.3] considers the completion $\overline{\mathbb{B}_I}$ of the Boolean algebra $\mathbb{B}_I$ rather than $\mathbb{B}_I$ itself. However, under the assumption that forcing with $\mathbb{B}_I$ is proper, $\overline{\mathbb{B}_I}$ is weakly distributive if and only if $\mathbb{B}_I$ is weakly distributive (the relevant property, implied by the properness of $\mathbb{B}_I$, is that of its $(\omega, \cdot, \omega_1)$-distributivity i.e., for every sequence $\langle P_n : n \in \omega \rangle$ of maximal antichains in $\mathbb{B}_I$, there exists a maximal antichain Q in $\mathbb{B}_I$ such that each element $q \in Q$ meets at most countably many elements of each P_n). Hence Zapletal's result follows from Theorem 1 .

Actually, we do not know any example of a weakly distributive algebra of the form $\mathbb{B}_I$ whose completion is not weakly distributive.

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References

[1] A.S. Kechris, *Classical descriptive set theory*, Graduate Texts in Math. 156, Springer-Verlag (1995).

[2] I. Recław, *Restrictions to continuous functions and Boolean algebras*, Proc. Amer. Math. Soc. 118 (1993), 791–796.

[3] J. Zapletal, *Descriptive Set Theory and Definable Forcing*, Memoirs Amer. Math. Soc., AMS, Providence, 2004.

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