ON A CONSTRUCTION OF UNIVERSALLY SMALL SETS

PIOTR ZAKRZEWSKI

ABSTRACT. We present a new construction of an uncountable subset of the reals which belongs to every σ-ideal I on $\mathbb{R}$ with the property that there is no uncountable family of disjoint Borel sets outside I.

1. INTRODUCTION

A *σ-ideal* on the reals or, more generally, an uncountable Polish (i.e. separable, completely metrizable) topological space X is a family $I \subseteq \mathcal{P}(X)$ which is closed under taking subsets and countable unions. We always assume that I is *proper*, i.e., $X \notin I$, contains all singletons and has a basis consisting of Borel sets. The latter means that every set from I is covered by a Borel set from I. A σ-ideal I is ccc if there is no uncountable family of disjoint Borel sets outside I.

Most important ccc σ-ideals are:

- a *measure σ-ideal* consisting of all subsets of X of (outer) measure zero with respect to a Borel measure μ on X, a countably additive, continuous (i.e., points have measure zero), finite measure defined on the σ-algebra $\mathbf{B}(X)$ of Borel subsets of X,
- a *category σ-ideal* consisting of all meager subsets of X in a Polish topology τ on X such that (X, τ) has no isolated points and $\mathbf{B}(X) = \mathbf{B}(X, \tau)$, the σ-algebra of Borel subsets of X in the topology τ.

A subset A of X is *universally null* (*universally meager*, resp.) if it belongs to every measure (category, resp.) σ-ideal (see [12]). Several methods of constructing uncountable universally null and universally meager sets are known (see [7]). Some of them lead to sets with an apparently stronger property.

A subset A of X will be called *universally small* if it belongs to every ccc σ-ideal on X. Equivalently, there is no ccc σ-ideal $\mathcal{I}$ in the σ-algebra $\mathbf{B}(A)$ of (relative) Borel subsets of A (by a σ-ideal in a σ-algebra $\mathcal{A}$ of subsets of a set A we mean a proper subfamily of $\mathcal{A}$ containing all

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singletons which is closed under taking subsets in $\mathcal{A}$ and countable unions; it is ccc if there is no uncountable family of disjoint sets from $\mathcal{A} \setminus \mathcal{I}$).

From this definition it immediately follows that *if* $\mathbf{B}(A) = \mathcal{P}(A)$, *the power set of* A, *and the cardinality of* A *is less than the first quasi-measurable cardinal (see* [2]*), then* A *is universally small.* In particular, *every* Q*-set of cardinality* ω_1 *is universally small* (A is a Q*-set* iff every subset of A is a relative G_δ).

Known ZFC constructions of uncountable universally small sets use either coanalytic sets (a selector of the constituents – see [7, Theorem 5.3]) or Ulam matrices (see [8]) or special Aronszajn trees of perfect sets (see [5] and [3]).

In this note we present another construction which seems somewhat simpler than the ones mentioned above. It uses the following Fubini–type property of ccc σ-ideals on X – see [11].

Proposition 1.1. *Let* I *be an arbitrary ccc* σ*-ideal on a Polish space* X*. Then every Borel subset* B *of the plane* $X \times X$ *with all vertical sections* $B_x = \{y : \langle x, y\rangle \in B\}$ *countable has horizontal sections* $B^y = \{x : \langle x, y\rangle \in B\}$ *in* I*, for every* y *outside a countable set.*

2. An example of a universally small set

Our construction of a universally small set is based on the following result.

Theorem 2.1. *Let* X *be an uncountable Polish space. Suppose that* A *is a subset of* X *such that there is a set* $Z \in \mathbf{B}(A \times X)$ *with the following properties:*

(1) $\forall x \in A$ *the vertical section* Z_x *is countable,*
(2) the set $\{y \in X : |A \setminus Z^y| \leq \omega\}$ *is uncountable.*

Then A *is universally small.*

Proof. Suppose otherwise and let I be a ccc σ-ideal on X such that $A \notin I$. Find a set $B \in \mathbf{B}(X \times X)$ with $B \cap (A \times X) = Z$.

The set $D = \{x \in X : |B_x| \leq \omega\}$ is co-analytic (see [6, Theorem 29.19]), so it belongs to the σ-algebra generated by $\mathbf{B}(X) \cup I$, the σ-ideal I being ccc (see [6, Theorem 29.13]). In particular, since I has a basis consisting of Borel sets, there are Borel sets C, $E \subseteq X$ with $C \subseteq D \subseteq E$ and $E \setminus C \in I$. Let $B' = B \cap (E \times X)$. Then we still have $B' \cap (A \times X) = Z$ but $|(B')_x| \leq \omega$ for all $x \in X$ outside a set from I, namely for every $x \in C \cup (X \setminus E)$. Hence, Proposition 1.1 implies that the set $\{y \in X : (B')^y \notin I\}$ is countable. Note, however, that since $A \notin I$, if $|A \setminus Z^y| \leq \omega$, then $(B')^y \notin I$. Thus, the set $\{y \in X : |A \setminus Z^y| \leq \omega\}$ is countable – a contradiction. □

In order to see how to construct sets which satisfy the hypotheses of Theorem 2.1, consider the following example.

Example 2.2. Let Y be an arbitrary subset of $\mathbb{R}$ of cardinality ω_1 and suppose that C is a subset of $Y \times Y$ such that $\forall x \in Y\ |C_x| \leq \omega$ and $|Y \setminus C^x| \leq \omega$.

Since $\forall x \in Y\ |C_x| \leq \omega$, there is a countably generated σ-algebra $\mathcal{S}$ of subsets of Y such that $C \in \mathcal{S} \otimes \mathbf{B}(\mathbb{R})$, the σ-algebra generated by the family $\{S \times B : S \in \mathcal{S},\ B \in \mathbf{B}(\mathbb{R})\}$. (see [1]).

Enlarging $\mathcal{S}$, if necessary, we can assume that it separates points. Hence there is a bijection $\phi : Y \to A$ onto a set $A \subseteq \mathbb{R}$ such that $S \in \mathcal{S}$ iff $\phi[S] \in \mathbf{B}(A)$ for every $S \subseteq Y$ (see [6, Proposition 12.2]).

It is now easy to see that the set $A = \phi[Y]$ satisfies the hypotheses of Theorem 2.1 with $Z = (\phi, id_X)[C]$ being a witnessing set.

The universally small set constructed in the example above has cardinality ω_1. Note that with the help of Theorem 2.1 we cannot achieve more. For assume that a set $A \subseteq X$ satisfies the hypotheses of Theorem 2.1 with Z a witnessing set and let $Y = \{y \in X : \ |A \setminus Z^y| \leq \omega\}$. Then the set Z has the following properties:

- $Z \subseteq A \times Y$, where the sets A and Y are uncountable,
- $\forall x \in A\ |C_x| \leq \omega$,
- $\forall y \in Y\ |A \setminus C^y| \leq \omega$

But it is well-known and easy to see that this implies $|A| = |Y| = \omega_1$.

3. Ulam matrices revisited

Recall that an *Ulam ω_1-matrix* in a set A is a collection $\{A_{n,\alpha} : (n, \alpha) \in \omega \times \omega_1\}$ of subsets of A satisfying the following conditions:

(1) $A_{n,\alpha} \cap A_{n,\beta} = \emptyset$ for each $n \in \omega$ and distinct $\alpha, \beta \in \omega_1$,
(2) $|A \setminus \bigcup_{n\in\omega} A_{n,\alpha}| \leq \omega$ for each $\alpha \in \omega_1$.

The standard construction of an Ulam ω_1-matrix in ω_1 is to choose for each $\xi \in \omega_1,\ \xi > 0$, a surjection $g_\xi : \omega \to \xi$ from ω onto ξ and let $A_{n,\alpha} = \{\xi \in \omega_1 : g_\xi(n) = \alpha\}$ (see e.g. [4, Lemma 27.6]).

The possibility of using Ulam matrices in constructions of universally small sets relies on the following well-known fact (see e.g. [8, Lemma 2.6] and [4, Lemma 27.7]).

Proposition 3.1. *If $\mathcal{A}$ is a σ-algebra of subsets of a set A containing all singletons and $\{A_{n,\alpha} : (n, \alpha) \in \omega \times \omega_1\}$ is an Ulam ω_1-matrix in A such that $A_{n,\alpha} \in \mathcal{A}$ for every $n \in \omega,\ \alpha \in \omega_1$, then there is no ccc σ-ideal in $\mathcal{A}$.*

Then, given an Ulam ω_1-matrix $\{A_{n,\alpha} : (n, \alpha) \in \omega \times \omega_1\}$ in ω_1, it is possible to find a countably generated σ-algebra $\mathcal{A}$ of subsets of ω_1 which contains all singletons and all elements of the matrix. Finally, by means of a suitable bijection, the Borel space $\langle \omega_1, \mathcal{A} \rangle$ may be identified with the Borel space $\langle A, \mathbf{B}(A)\rangle$ of relative Borel subsets of a certain subset A of the reals of cardinality ω_1. Then, by Proposition 3.1, the set A is universally small, since the σ-algebra $\mathbf{B}(A)$ contains an Ulam ω_1-matrix in A (for the details see [8]).

In order to see, how the method of constructing universally small sets presented in Section 2 is related to Ulam matrices, we shall now give another proof of Theorem 2.1 based on Proposition 3.1.

Another proof of Theorem 2.1. Let X be an uncountable Polish space and suppose that sets A and Z satisfy the hypotheses of Theorem 2.1. Let $Y = \{y \in X : \ |A \setminus Z^y| \leq \omega\}$; by the remarks at the end of Section 2, $|A| = |Y| = \omega_1$.

Suppose that I is an arbitrary ccc σ-ideal on X such that $A \notin I$. We will achieve a contradiction by showing that there exists an Ulam ω_1-matrix in A consisting of elements of the σ-algebra $\mathcal{A}$ generated by $\mathbf{B}(A) \cup (I \cap \mathcal{P}(A))$ (clearly, the σ-ideal $\mathcal{I}$ is ccc in $\mathcal{A}$).

As before, find a set $B \in \mathbf{B}(X \times X)$ such that $B \cap (A \times X) = Z$ and $|B_x| \leq \omega$ for all $x \in X$ outside a set from I. Then it easily follows from the Lusin-Novikov theorem (see [6], 18.10) and the fact that every set from I is covered by a Borel set from I that B can be written as $\bigcup_{n\in\omega}(g_n \cap B)$, where each g_n is (the graph of) a function $g_n : X \to X$ which is measurable with respect to the σ-algebra generated by $\mathbf{B}(X) \cup I$.

For each $n \in \omega$ let $f_n = g_n|A$ be the restriction of g_n to A. Clearly, $Z = \bigcup_{n\in\omega} f_n$ and every function f_n is measurable with respect to the σ-algebra $\mathcal{A}$. In particular, $f_n^{-1}(y) \in \mathcal{A}$ for every $n \in \omega, \ y \in Y$ and it is easy to check that the collection $\{f_n^{-1}(y) : n \in \omega, \ y \in Y\}$ is an Ulam ω_1-matrix in A.

$\square$

It is perhaps of an independent interest to notice that the last step of the argument above gives an apparently new method of constructing an Ulam ω_1-matrix in ω_1. Just take a subset C of $\omega_1 \times \omega_1$ such that

$$\forall\alpha \in \omega_1 \ |C_\alpha| \leq \omega \text{ and } |\omega_1 \setminus C^\alpha| \leq \omega \tag{$*$}$$

and write $C = \bigcup_{n\in\omega} f_n$ for some functions $f_n : \omega_1 \to \omega_1$. Then the collection $\{f_n^{-1}(\alpha) : n \in \omega, \ \alpha \in \omega_1\}$ is an Ulam ω_1-matrix in ω_1.

Conversely, if $\{A_{n,\alpha} : n \in \omega, \ \alpha \in \omega_1\}$ is an Ulam ω_1-matrix in ω_1, then the set

$$C = \{\langle\beta, \alpha\rangle \in \omega_1 \times \omega_1 : \ \exists n \in \omega \ \beta \in A_{n,\alpha}\}$$

has property $(*)$.

4. Remarks on other constructions of universally null (meager) sets

It would be interesting to find out if other known constructions of universally null (meager, resp.) sets give universally small sets. Recently Recław [10] proved that *if* $\mathcal{R} \subset X \times X$ *is a co-analytic relation in a Polish space* X*, then any set* $A \subset X$ *well ordered by* $\mathcal{R}$ *is universally null and universally meager* (for a Borel relation $\mathcal{R}$ this was earlier proved by Plewik [9]). The proof for universally null (meager, resp.) is based on the Fubini (Kuratowski-Ulam, resp.) theorem.

It turns out that a straightforward refinement of Recław's proof gives a generalization of the above to the class of ccc σ-ideals on X satisfying the following properties:

- (*regularity*) For every set $B \in \mathbf{B}(X \times X)$ the set $\{x \in X : B_x \in I\}$ belongs to the σ-algebra generated by $\mathbf{B}(X) \cup I$,
- (*the Fubini Property*) for every Borel set $B \in \mathbf{B}(X \times X)$, if all its vertical sections B_x are in I, then its horizontal sections B^y are in I, for every y outside a set from I.

Theorem 4.1 (essentially Recław [10])**.** *If* $\mathcal{R} \subset X \times X$ *is a co-analytic relation in a Polish space* X*, then any set* $A \subset X$ *well ordered by* $\mathcal{R}$ *belongs to every ccc* σ*-ideal* I *on* X *which is regular and has the Fubini Property.*

Unfortunately, the only known examples of σ-ideals satisfying the hypotheses of Theorem 4.1 are measure and category σ-ideals.

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Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland
E-mail address: piotrzak@mimuw.edu.pl