Measures on algebraic-topological structures

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Introduction

This article is entirely devoted to *countably additive measures invariant under groups of transformations*. Our general framework is a G*-space*, i.e., a pair (X, G), where G is a group acting on a set X. If, moreover, $\mathcal{A}$ is a σ-algebra of subsets of X, we say that $\mathcal{A}$ is G*-invariant* if $gA \in \mathcal{A}$ whenever $g \in G$ and $A \in \mathcal{A}$. Then we also say that G *acts by measurable transformations* on the measurable space $(X, \mathcal{A})$. We say that a (σ-additive, non-zero) measure $\mu : \mathcal{A} \to [0, \infty]$ is G*-invariant* if $\mu(gA) = \mu(A)$ for every $g \in G$ and $A \in \mathcal{A}$, i.e., if for each $g \in G$ the transformation $(g, \cdot) : X \to X$ is *measure-preserving*.

Invariant measures are present in many parts of mathematics including harmonic analysis, ergodic theory and topological dynamics.

A typical example of such a measure is the Lebesgue measure: one of its key properties is that it assigns equal values to congruent sets, so that the measure of a set does not depend on its position in the space.

The Lebesgue measure is a particular example of a Haar measure on a locally compact topological group G: the only, up to a constant factor, regular Borel measure, positive on nonempty open sets, finite on compact sets and invariant under the left shifts from G. Haar measure plays an important role in abstract harmonic analysis and group representation theory.

In ergodic theory one focuses on the case where a single measure-preserving transformation T of a probability measure space $(X, \mathcal{A}, \mu)$ is given, so that X becomes a $\mathbb{Z}$-space via the map $(k, x) \mapsto T^k x$ for $k \in \mathbb{Z}$ and $x \in X$. Invariant measures for more general group actions are also of considerable interest.

But in this article we leave aside these aspects of the theory of invariant measures that are specific for harmonic analysis or ergodic theory. Instead, given a group G acting by measurable transformations on a measurable space $(X, \mathcal{A})$, we focus on the fundamental problems concerning the interplay between algebraic, combinatorial and topological properties of the action and the existence and properties of G-invariant measures defined on $\mathcal{A}$. In particular, we are interested in the way in which invariance of a measure influences the size and the structure of the underlying σ-algebra, and the existence and properties of nonmeasurable sets.

General necessary and sufficient conditions for the existence of finite and σ-finite invariant measures on arbitrary G-spaces are discussed in Section 1. We omit these conditions obtained for the special case of $\mathbb{Z}$-actions which do not have meaningful extensions to more general classes of actions. Such conditions properly belong to classical ergodic theory and are discussed in many standard textbooks on the subject (see e.g. Friedman(1970)). We concentrate on results related to a characterization that was obtained by Hopf (1932) for the case of $\mathbb{Z}$-spaces and was later generalized by Kawada (1944), Hajian and Ito (1969) and others to arbitrary G-spaces. It is formulated in terms of the notion of *countable equidecomposability*, first investigated by Banach and Tarski (1924). We admit after Hopf that such results are not fit for immediate applications but nevertheless they throw a light on the intrinsic nature of G-spaces admitting invariant measures.

Section 2 is devoted to invariant measures defined for *all* subsets of a given set X. Results are parallel to the work of Tarski (1938) on the existence of *finitely* additive invariant measures. Conditions imposed on a G-space X, which refute the existence of a G-invariant σ-finite measure defined on $\mathcal{P}(X)$, at the same time guarantee that for an arbitrary G-invariant σ-finite measure $\mu : \mathcal{A} \to [0, \infty]$ on X, we have $\mathcal{A} \neq \mathcal{P}(X)$, i.e., nonmeasurable sets exist. These investigations are related to the well-known *general measure problem*, whether there exists a σ-additive σ-finite measure defined on the power set of X and vanishing on all singleton sets. Thus the answers, as is well-known, depend on set-theoretic assumptions.

In Section 3 we deal with extensions of invariant measures, the matter closely connected to the invariant version of the general measure problem. Given a G-invariant measure $\mu : \mathcal{A} \to [0, \infty]$ on a G-space X we look for another G-invariant measure $\bar{\mu} : \mathcal{A}' \to [0, \infty]$ such that $\mathcal{A} \subsetneq \mathcal{A}'$ and $\bar{\mu}|\mathcal{A} = \mu$. This reduces to finding a suitable non-μ-measurable set that can be added to the domain of the measure μ.

In Section 4 topology enters into the picture, as we discuss the existence of Borel measures on Polish (i.e. separable, completely metrizable) groups. In particular, standard facts about Haar measure in locally compact groups are summarized. For the sake of conceptual coherence and making the interplay between measure and topology more transparent, we restrict ourselves to the case of second countable groups.

The discussion of invariant properties of Borel measures on Polish spaces is continued in Section 5, where continuous or, more generally, Borel actions of Polish groups are considered. Some parts of this section belong to *descriptive dynamics*, a branch of descriptive set theory that has been recently attracting increasing interest. Our exposition is based on the widely distributed set of lecture notes by A.S. Kechris (1994).

In the last section we come back to extensions of invariant measure, this time concentrating on isometrically invariant measures in Euclidean spaces $\mathbb{R}^n$. We present a survey of results connected with Sierpiński's problem, formulated in Szpilrajn (1935), concerning the existence of a maximal isometrically invariant extension of Lebesgue measure.

1 Invariant measures on arbitrary G-spaces

We say that a group $(G, \cdot, e)$ *acts* on a set X, if there is a map $(g, x) \mapsto gx$ of $G \times X$ into X, called an *action* of G on X, such that for every $x \in X$:

(i) $ex = x$;

(ii) for all $g_1, g_2 \in G,\ g_1(g_2x) = (g_1 \cdot g_2)x$.

We then say that (X, G), or just X, is a *G-space*.

It follows that for each $g \in G$ the function $x \mapsto gx$ is a permutation of X. If G itself consists of permutations of X, then we tacitly assume that the action is given by $gx = g(x)$ and say that G is a *group of permutations* of X.

Given a G-space X a *(G-)orbit* is the set $Gx = \{gx : \ g \in G\}$. A subset $A \subseteq X$ is *(G-)invariant* if it is the union of orbits of its elements or, equivalently, it is closed under the action of G. We say that the action is *transitive*, if X is the only orbit. The action is *free*, if $gx \neq x$ for any $g \neq e$ and $x \in X$.

If G is a subgroup of a group $(X, \cdot)$, then G acts on X by *left shifts*, i.e., $gx = g \cdot x$ for $g \in G,\ x \in X$. Of course, this action is free and it is transitive if and only if $G = X$; G-orbits coincide with the left-cosets of G in X.

If $\mathcal{A}$ is a σ-algebra of subsets of a set X, then we say that $(X, \mathcal{A})$ is a *measurable (*or *Borel) space* . The elements of $\mathcal{A}$ are called *measurable* sets.

By a *measure on X* we mean a σ-additive (unless explicitly stated otherwise), non-zero measure defined on a σ-algebra $\mathcal{A}$ of subsets of X. We *always assume that $\{x\} \in \mathcal{A}$ for any $x \in X$.* A measure $m : \mathcal{A} \to [0, \infty]$ on X is *diffused* if it vanishes on all singleton sets, i.e., $m(\{x\}) = 0$ for all $x \in X$.

Given sets $A, B \in \mathcal{A}$ we write $A \subseteq_m B$, if $m(A \setminus B) = 0$; if $A \subseteq_m B$ and $B \subseteq_m A$, then we write $A =_m B$.

Suppose a group G acts by measurable transformations on a measure space $(X, \mathcal{A}, m)$.

The action of G is *m-free* if $m^*(\{x \in X : gx = x\}) = 0$ for any $g \in G \setminus \{e\}$, where m^* stands for the *outer measure* of m.

A subset $A \subseteq X$ is *(G-)almost invariant* (with respect to m) if $m(gA \triangle A) = 0$ for every $g \in G$, where $\triangle$ denotes the symmetric difference. It is easy to check that the collection

$$\mathcal{A}^* = \{A \in \mathcal{A} : A \text{ is almost invariant}\}$$

is a σ-algebra of subsets of X contained in $\mathcal{A}$.

The measure m is *(G-)quasi-invariant* if $m(A) = 0$ implies $m(gA) = 0$, and *(G-)invariant* if $m(A) = m(gA)$, for every $A \in \mathcal{A}$ and $g \in G$. It is *(G)-ergodic*, if $m(A) = 0$ or $m(X \setminus A) = 0$ for any $A \in \mathcal{A}^*$. Ergodicity of m is equivalent to the fact that the associated measure algebra $\mathcal{A}^*/m$ is the two-element algebra.

Note that if the measure m is quasi-invariant, then the σ-algebra $\mathcal{A}^*$ is invariant. If m is, moreover, σ-finite, then *every $A \in \mathcal{A}$ is contained in a minimal almost invariant set $A^* \in \mathcal{A}^*$*, where minimal means that $A^* \subseteq_m B$, whenever $A \subseteq B$ and $B \in \mathcal{A}^*$ (see e.g. D. Ramachandran and M. Misiurewicz

(1982)). If the group G is countable, then simply $A^* = \bigcup_{g\in G} gA$; in general $A^* = \bigcup_{n\in\mathbb{N}} g_n A$ for certain $g_n \in G$, $n \in \mathbb{N} = \{0, 1, \ldots\}$.

The problem of existence of invariant measures on an arbitrary G-space have been approached from two different but closely related points of view.

(A) In classical ergodic theory the problem was formulated as follows (see P.R. Halmos (1956), p.81).

Given a group G acting by measurable transformations on a σ-finite measure space $(X, \mathcal{A}, m)$, find necessary and sufficient conditions for the existence of a G-invariant, probability or σ-finite measure μ defined on $\mathcal{A}$ and such that $m \ll \mu$, i.e., m is absolutely continuous with respect to μ. It is usually assumed that the measure m is quasi-invariant (in this case elements of G are sometimes called *nonsingular transformations* of the measure space $(X, \mathcal{A}, m)$). It is then required that $\mu \sim m$, i.e., the measures μ and m have the same null sets.

(B) Influenced by Tarski's work on finitely additive invariant measures (see S. Wagon (1986) or the article in this Handbook on "Paradoxes in measure theory" by Mikl"os Laczkovich) one searches for "purely combinatorial" conditions for the existence of an arbitrary probability or σ-finite G-invariant measure, when a group G acts by measurable transformations on a measurable space $(X, \mathcal{A})$, with no measure defined on the σ-algebra $\mathcal{A}$ given in advance.

The two approaches lead to closely related results. The reason is that if all G-orbits are uncountable, then any σ-finite invariant measure defined on the σ-algebra $\mathcal{A}$ has to be diffused but even in relatively simple cases, when the σ-algebra is, say, countably generated, it may happen that it carries no σ-finite, diffused measure at all, even without the additional requirement of G-invariance. So, natural necessary conditions for the existence of an invariant measure have the form: "$\mathcal{A}$ carries a measure m satisfying some additional properties".

In order to formulate the desired conditions, we need to recall the notion of *equivalence by countable decompositions*, introduced by Banach and Tarski (1924). Our terminology is consistent with that from Wagon's monograph (1986).

Given a group G acting by measurable transformations on a measurable space $(X, \mathcal{A})$ we say that sets $A, B \in \mathcal{A}$ are *countably (G-)equidecomposable (in $\mathcal{A}$)*, in symbols $A \sim_\infty B$ in $\mathcal{A}$, if there is a partition of A into countably many sets $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, and elements $g_n \in G$ such that the sets $g_n A_n$ form a partition of B.

A set $A \in \mathcal{A}$ is called *countably (G-)paradoxical (in $\mathcal{A}$)*, if A contains disjoint subsets $A_1, A_2 \in \mathcal{A}$ with $A \sim_\infty A_1 \sim_\infty A_2$ in $\mathcal{A}$.

A set $A \in \mathcal{A}$ is called *G-negligible (in $\mathcal{A}$)* if X contains pairwise disjoint subsets $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, with each $A_n \sim_\infty A$ in $\mathcal{A}$. Let N_G consist of all G-negligible elements of $\mathcal{A}$. It can be proved (see e.g. R. Chuaqui (1977)) that the family N_G is closed under countable unions and taking subsets in $\mathcal{A}$, i.e., it constitutes a σ*-deal in* $\mathcal{A}$. In fact, it is not difficult to prove that *the σ-ideal N_G is generated by countably paradoxical sets in* $\mathcal{A}$, i.e., it is the smallest σ-ideal in $\mathcal{A}$ containing all measurable subsets of X that are countably paradoxical in $\mathcal{A}$.

Additionally suppose that m is a measure defined on $\mathcal{A}$. A set $Z \in \mathcal{A}$ is called *(G-)bounded* (with respect to m), if $Z \sim_\infty A$ in $\mathcal{A}$ implies $m(Z \setminus A) = 0$ for every $A \in \mathcal{A}$, $A \subseteq Z$, i.e., Z is not countably G-equidecomposable in $\mathcal{A}$ with any measure-theoretically proper measurable subset of itself.

It is not difficult to prove that Z *is bounded if and only if the measure* m *vanishes on all measurable subsets* E *of* Z *with the property that* Z *contains pairwise disjoint sets* $E_n \in \mathcal{A}$, $n \in \mathbb{N}$, *with each* $E_n \sim_\infty E$ *in* $\mathcal{A}$ (see e.g. Y. Kawada (1944)).

In particular, we have the following.

Proposition 1.1. *Suppose a group* G *acts by measurable transformations on a measure space* $(X, \mathcal{A}, m)$. *The following are equivalent:*

(i) *The set* X *is* G*-bounded with respect to* m*;*

(ii) *The measure* m *vanishes on all* G*-negligible sets from* $\mathcal{A}$*;*

(iii) *The measure* m *vanishes on all measurable sets that are countably* G*-paradoxical in* $\mathcal{A}$.

The following fundamental theorem on the existence of invariant probability measures was first proved by E. Hopf (1932) for the case of $\mathbb{Z}$-actions and then generalized by Y. Kawada (1944) to arbitrary group actions.

Theorem 1.2. *Suppose a group* G *acts by measurable transformations on a* σ*-finite measure space* $(X, \mathcal{A}, m)$ *and assume that the measure* m *is quasi-invariant. Then the following are equivalent:*

(i) *There exists an invariant probability measure* μ *on* $\mathcal{A}$ *such that* $\mu \sim m$.

(ii) *The set* X *is bounded with respect to* m.

Obviously, any invariant probability measure on $\mathcal{A}$ vanishes on sets that are countably paradoxical in $\mathcal{A}$, so the existence of such a measure μ with $\mu \sim m$ immediately implies that X is bounded with respect to m. The nontrivial part of the Hopf-Kawada theorem asserts that the converse is true.

Given sets $A, B \in \mathcal{A}$ we write $A \ll B$ if there is $C \in \mathcal{A}$ with $C \sim_\infty A$ in $\mathcal{A}$, $C \subseteq_m B$ and for every $Q \in \mathcal{A}^*$, if $Q \subseteq_m B^*$ and $m(Q) > 0$, then $m((B \setminus C) \cap Q) > 0$, i.e., $C \cap Q$ is a measure-theoretically proper subset of $B \cap Q$.

Given sets $A, B, C \in \mathcal{A}$ and $n \in \mathbb{N}$ we write

$$A = nB + C$$

if $A =_m A_1 \cup \cdots \cup A_n \cup C$ where $A_1, \ldots, A_n, C$ are pairwise disjoint sets in $\mathcal{A}$ with each $A_i \sim_\infty B$ in $\mathcal{A}$.

The essence of proof (which uses, as Kawada admits, some ideas taken from J. von Neumann's 1936-37 classical lecture notes on "Continuous geometry" – see von Neumann (1998)) are the following two lemmas.

Lemma 1.3. *For every sets $A, B \in \mathcal{A}$ of positive measure there is a partition (up to a measure zero set)*

$$A^* =_m \bigcup_{n \in \mathbb{N}} Q_n$$

of A^* into pairwise disjoint sets $Q_n \in \mathcal{A}^*$, $n \in \mathbb{N}$, such that for each $n \in \mathbb{N}$ $m(Q_n) > 0$ and

$$A \cap Q_n = n(B \cap Q_n) + R_n, \quad \text{where } R_n \ll B \cap Q_n.$$

Moreover, the partition is unique, up to a measure zero set.

We say that a set $B \in \mathcal{A}$ is a *G-atom* if $m(B) > 0$ and B is minimal with respect to $\ll$ in the sense that $A \ll B$ implies $m(A) = 0$, for any $A \in \mathcal{A}$. A set $Q \in \mathcal{A}^*$ is called *discrete* if $Q =_m B^*$ for a certain G-atom B.

Lemma 1.4. *There are disjoint sets $D, C \in \mathcal{A}^*$ such that $X =_m D \cup C$ and the following conditions are satisfied*

(i) If $m(D) > 0$, then D is discrete;

(ii) If $m(C) > 0$, then there is a decreasing sequence of sets $C_n \in \mathcal{A}$ with $C_0 = C$, $C_n^* =_m C$ and $C_n = 2C_{n+1}$ for every $n \in \mathbb{N}$.

Moreover, such sets C and D are unique, up to measure zero sets.

Taking the lemmas above for granted, the proof is completed as follows.

Assume, without loss a generality, that $m(X) = 1$.

Let D and C be sets from Lemma 1.4. We shall define measures μ_d, μ_c on measurable subsets of D and C, respectively (if $m(D) = 0$ or $m(C) = 0$, then we let $\mu_d \equiv 0$ or $\mu_c \equiv 0$, respectively).

So assume that $m(D) > 0$ and let $D = B^*$ for a certain G-atom B. Take $A \in \mathcal{A}$, $A \subseteq_m D$. By Lemma 1.3 and the fact that B is a G-atom, there exists a partition of D into sets $Q_n \in \mathcal{A}^*$ such that $m(Q_0) = 0$ and

$$Q_n = n(B \cap Q_n) \quad \text{for each } n > 0,$$

Fix $n > 0$ and additionally assume that $A \subseteq_m Q_n$. Then, using again Lemma 1.3, we have

$$A \cap P_k = k(B \cap P_k) \quad \text{for } k = 1, \ldots, n,$$

where $A^* =_m \bigcup_{k=1}^{n} P_k$ is the appropriate partition of A^* into sets $P_k \in \mathcal{A}^*$.

Set

$$\mu_d^{(n)}(A) = \sum_{k=1}^{n} \frac{k}{n} m(P_k).$$

For an arbitrary $A \in \mathcal{A}$ with $A \subseteq_m D$ define

$$\mu_d(A) = \sum_{n=1}^{\infty} \mu_d^{(n)}(A \cap Q_n).$$

Now assume that $m(C) > 0$ and take $A \in \mathcal{A}$, $A \subseteq_m C$. Fix $n \in \mathbb{N}$. Note that $C = 2^n C_n$.

Using Lemma 1.3 we find a partition of A^* into sets $Q_k^{(n)} \in \mathcal{A}^*$, $k = 0, \dots, 2^n$, such that

$$A \cap Q_k^{(n)} = n\left(C_k \cap Q_k^{(n)}\right) + R_k^{(n)} \quad \text{where } R_k^{(n)} \ll Q_k^{(n)}.$$

Then let

$$f_n(A) = \sum_{k=1}^{2^n} \frac{k}{2^n} m(Q_k^{(n)}),$$

and define

$$\mu_c(A) = \lim_{n\to\infty} f_n(A).$$

It can be checked that this definition is correct.

Finally, for an arbitrary $A \in \mathcal{A}$ define

$$\mu(A) = \mu_d(A \cap D) + \mu_c(A \cap C).$$

For details see Kawada (1944).

Kawada (1944) noticed that this proof reveals the following uniqueness property of invariant measures.

Theorem 1.5. *Suppose a group G acts by measurable transformations on a measurable space $(X, \mathcal{A})$. Then any invariant probability measure μ on $\mathcal{A}$ is determined by its values on $\mathcal{A}^*$, the σ-algebra of measurable sets, almost invariant with respect to μ. Moreover, if m is a probability quasi-invariant measure on $\mathcal{A}$ such that X is bounded with respect to m, then there exists a unique invariant probability measure μ on $\mathcal{A}$ such that $\mu \sim m$ and $\mu(A) = m(A)$ for any $A \in \mathcal{A}^*$. In particular, if m is ergodic, then there exists a unique invariant probability measure μ on $\mathcal{A}$ with $\mu \sim m$.*

Shorter proofs of the Hopf-Kawada theorem were later found by Ramashandran and Misiurewicz (1982), and Ramachandran (1989). The simplification was achieved by using the *semigroup of (countable equidecomposability) types* (see Section 5 of Laczkovich's article in this Handbook with the obvious changes). Ramachandran's proof was based on a theorem of Tarski (see Tarski (1938, Theorem 1.55) concerning measures on semigroups. In fact, this theorem yields the existence of a *finitely* additive invariant measure equivalent to a given quasi invariant measure m. The latter, however, suffices, due to the following result of A.P. Caldér"on (1955) (see also N.A. Friedman (1970, Theorem 3.13)).

Theorem 1.6. *Suppose a group G acts by measurable transformations on a σ-finite measure space $(X, \mathcal{A}, m)$ and assume that the measure m is quasi-invariant. If P is a finite, finitely additive, invariant measure defined on $\mathcal{A}$ with $P \sim m$, then the function μ defined by*

$$\mu(A) = \inf \left\{ \sum_{n\in\mathbb{N}} P(A_n) : A_n \in \mathcal{A},\ A \subseteq \bigcup_{n\in\mathbb{N}} A_n \right\}, \quad \textit{for } A \in \mathcal{A},$$

is a finite, σ-additive, invariant measure on $\mathcal{A}$ with $\mu \sim m$.

To formulate another necessary and sufficient condition for the existence of an invariant probability measure we need the notion of a weakly wandering set.

Suppose a group G acts by measurable transformations on a measurable space $(X, \mathcal{A})$. A set $A \in \mathcal{A}$ is called *weakly wandering* (under G) if there exist elements $g_n \in G, n \in \mathbb{N}$, such that $g_n A \cap g_m A = \emptyset$ for $n \neq m$. Clearly, every weakly wandering set is G-negligible.

The following strengthening of the Hopf-Kawada theorem was first proved by A. Hajian and S. Kakutani (1964) for $\mathbb{Z}$-actions and then generalized by A. Hajian and Y. Ito (1969) to arbitrary actions (partial generalizations were obtained by several authors – see the references in Hajian and Ito (1969)).

Theorem 1.7. *Suppose a group G acts by measurable transformations on a σ-finite measure space $(X, \mathcal{A}, m)$ and assume that m is quasi-invariant. Then the following are equivalent.*

(i) *There exists an invariant probability measure μ on $\mathcal{A}$ such that $\mu \sim m$;*

(ii) *The measure m vanishes on all weakly wandering sets in $\mathcal{A}$.*

Under the hypotheses of Theorem 1.7 condition (ii) above is equivalent to each of the following three assertions:

(iii) $m(A) > 0$ implies $\inf_{g\in G} m(gA) > 0$, for every $A \in \mathcal{A}$ – see Hajian and Kakutani (1964);

(iv) For every $\varepsilon > 0$ there is $\delta > 0$ such that $m(A) < \delta$ implies $m(gA) < \varepsilon$, for every $A \in \mathcal{A}$ – see K. Sakai (1978);

(v) For every $\varepsilon > 0$ there is $\delta > 0$ such that $m(A) < \delta$ and $A \sim_\infty B$ imply $m(B) < \varepsilon$, for every $A,\ B \in \mathcal{A}$ – see K. Sakai (1978).

Hajian and Ito used functional analytic methods in their proof, treating the existence of an invariant measure as the problem of finding a fixed point under certain group of linear operators on an appropriate Banach space.

More precisely, let us assume that the measure m is a probability measure and consider the following action of the group G on the Hilbert space $L^2(X, \mathcal{A}, m)$:

$$U_g(f) = f(g^{-1}x)[\omega_g(x)]^{1/2}, \quad \text{for } g \in G,\ f \in L^2(X, \mathcal{A}, m),$$

where ω_g is the Radon-Nikodym derivative with respect to m of the measure $mg^{-1}(A) = m(g^{-1}A)$ for $A \in \mathcal{A}$. Any common fixed point, i.e., a function $f_0 \in L^2(X, \mathcal{A}, m)$ with $U_g(f_0) = f_0$ for all $g \in G$, gives a desired invariant probability measure $\mu \sim m$ by the formula

$$\mu(A) = \int_A f_0^2(x)dm(x), \quad \text{for } A \in \mathcal{A}.$$

Yet another proof of the Hopf-Kawada theorem was given by R. Chuaqui (1977). Chuaqui (quoted in Wagon (1986, Question 9.13)) conjectured that the the existence of a σ-finite measure m on $\mathcal{A}$ which vanishes on all G-negligible sets is sufficient for the existence of a G-invariant probability measure on $\mathcal{A}$, even without the additional requirement that m is quasi-invariant. By a result of Halmos (1947), this was known to be true for $\mathbb{Z}$-actions.

Chuaqui's conjecture was confirmed in Zakrzewski (1993a). More exactly, we have the following result which also generalizes the Hajian-Ito theorem.

Theorem 1.8. *Suppose a group G acts by measurable transformations on a σ-finite measure space $(X, \mathcal{A}, m)$. Then the following are equivalent.*

(i) *There exists an invariant probability measure μ on $\mathcal{A}$ such that $m \ll \mu$;*

(ii) *The measure m vanishes on all G-negligible sets in $\mathcal{A}$;*

(iii) *The measure m vanishes on all weakly wandering sets in $\mathcal{A}$.*

Let us now turn to the conditions for the existence of invariant σ-finite measures.

If a group G acts by measurable transformations on a measure space $(X, \mathcal{A}, m)$, a set $Z \in \mathcal{A}$ is called *$(G$-$)\sigma$-bounded* (with respect to m), if it is a countable union of bounded sets.

In his paper (1944), quoted above, Kawada proved the following.

Theorem 1.9. *Suppose a group G acts by measurable transformations on a σ-finite measure space $(X, \mathcal{A}, m)$ and assume that m is quasi-invariant. Then the following are equivalent.*

(i) *There exists an invariant σ-finite measure μ on $\mathcal{A}$ such that $\mu \sim m$;*

(ii) *The set X is σ-bounded with respect to m;*

(iii) *There is a bounded set $A \in \mathcal{A}$ with $A^* =_m X$.*

This result, for the special case of $\mathbb{Z}$-actions, was later rediscovered by Halmos (1947) who also noticed that in this case the assumption that the measure m is G-quasi-invariant may be omitted if we want the resulting invariant measure only to dominate m in the sense of absolute continuity.

The following theorem from Zakrzewski (1993b) generalizes the results of Kawada and Halmos to arbitrary groups of measurable transformations of any σ-finite measure space $(X, \mathcal{A}, m)$. The new element that arises here is the necessity of adding to the condition that X is σ-bounded another property, which guarantees the existence of a quasi-invariant measure dominating m in the sense of absolute continuity (and is, therefore, superfluous if m itself is assumed to be quasi-invariant).

Let I_G consist of all $A \in \mathcal{A}$ with $m(gA) = 0$ for every $g \in G$. It is clear that I_G is a σ-ideal in $\mathcal{A}$.

We say that a σ-ideal I in $\mathcal{A}$ satisfies the *countable chain condition in $\mathcal{A}$*, if there is no uncountable collection of pairwise disjoint sets in $\mathcal{A} \setminus I$. Clearly, m being σ-finite, the σ-ideal of m-measure zero sets satisfies the countable chain condition in $\mathcal{A}$.

Theorem 1.10. *Suppose a group G acts by measurable transformations on a σ-finite measure space $(X, \mathcal{A}, m)$. Then the following are equivalent.*

(i) *There exists an invariant σ-finite measure μ on $\mathcal{A}$ such that $m \ll \mu$;*

(ii) *The set X is σ-bounded with respect to m and the σ-ideal I_G satisfies the countable chain condition in $\mathcal{A}$.*

Moreover, the following are equivalent:

(i) *The σ-ideal I_G satisfies the countable chain condition in $\mathcal{A}$;*

(ii) *There exists a quasi-invariant, σ-finite measure μ on $\mathcal{A}$ such that $I_G = \{A \in \mathcal{A} : \mu(A) = 0\}$;*

(iii) *There exists a quasi-invariant, σ-finite measure μ on $\mathcal{A}$ such that $m \ll \mu$.*

In fact, the proof of the nontrivial part of Theorem 1.8, asserting the existence of an invariant probability measure μ on $\mathcal{A}$, $m \ll \mu$, boils down to showing that *if m vanishes on all weakly wandering sets in $\mathcal{A}$, then the σ-ideal I_G satisfies the countable chain condition in $\mathcal{A}$.*

In the special case when the group G is countable, the σ-ideal I_G always satisfies the countable chain condition in $\mathcal{A}$. But in fact in this case even an *invariant* σ-finite measure always exists, the measure defined by $\mu(A) = |A \cap Gx|$ for $A \in \mathcal{A}$, i.e., the counting measure concentrated on the orbit Gx of an arbitrary point $x \in X$, being a trivial example.

However, if we look for *diffused* invariant measures, then it follows that *a quasi-invariant σ-finite diffused measure defined on $\mathcal{A}$ exists, provided that the σ-algebra $\mathcal{A}$ carries a σ-finite diffused measure at all.* As is well known, the latter may not be true even if $\mathcal{A}$ is countably generated. On the other hand, even in the case of a $\mathbb{Z}$-action, the existence of a quasi-invariant σ-finite diffused measure does not imply that such an *invariant* measure exists. An example is given in Zakrzewski (1993b). It is based on the result of D.S. Ornstein (1960) that *there exists a $\mathbb{Z}$-action by measurable transformations of the Lebesgue measure space $([0,1],\mathcal{L},\lambda)$ such that the Lebesgue measure λ on the σ-algebra $\mathcal{L}$ of Lebesgue measurable subsets of $[0,1]$ is quasi-invariant and ergodic but there is no invariant, σ-finite measure μ on $\mathcal{L}$ such that $\mu \sim \lambda$.*

Other conditions for the existence of an invariant σ-finite measure, equivalent with a given quasi-invariant one, were obtained in K. Arnold (1968) (see also Zakrzewski (1993b, Theorem 3.3)), J.M. Rosenblatt (1974) (it concerns the case when the group G is finitely generated) and J.M. Rosenblatt (1980) (it concerns the existence of an invariant measure which is *strongly equivalent* with the given quasi-invariant measure in the sense that the measures have the same sets of measure zero and the same sets of infinite measure).

Theorems 1.2 and 1.9 are in fact special cases of a more general theorem from Kawada (1944) on the existence of a strictly positive measure on a Boolean algebra $\mathbb{A}$, invariant with respect to a certain group of automorphisms of $\mathbb{A}$ (see Kawada (1944, Satz 4)). Of course, this is closely related to the problem of finding an invariant measure μ with $\mu \sim m$, if a measure m, quasi-invariant under an action by measurable transformations on a measurable space $(X,\mathcal{A})$ of a group G is given. Indeed, the action of G on X canonically induces the action of G on the associated measure algebra $\mathbb{A} = \mathcal{A}/m$. This Boolean-algebraic point of view is also present in Vladimirov (1965) and Chuaqui (1977). Let us note that the search for invariant measures on Boolean algebras is connected to the study of groups of automorphisms and measure-preserving automorphisms of measure algebras (see Fremlin (2000) for an account of the subject and further references).

A remarkable feature of existence proofs from Kawada (1944) is that they also establish certain uniqueness properties of invariant measures. Let us mention the following generalization of Theorem 1.5: *Suppose a group G acts by measurable transformations on a measurable space $(X,\mathcal{A})$. Let μ be an invariant σ-finite measure on $\mathcal{A}$ and suppose that $A \in \mathcal{A}$ is a bounded set with $A^* =_\mu X$. Then μ is determined by its values on sets of the form $A \cap Q$, where $Q \in \mathcal{A}^*$.* In particular, we have the following corollary established earlier in von Neumann (1940, Lemma 3.3.4).

Proposition 1.11. *Suppose a group G acts by measurable transformations on a measurable space $(X,\mathcal{A})$. If μ is an invariant σ-finite ergodic measure on $\mathcal{A}$, then μ is unique, up to a constant factor, among invariant σ-finite measures ν on $\mathcal{A}$ with $\nu \ll \mu$.*

An account of results connected to the Hopf-Kawada theorem on one hand and a classification of G-spaces with a quasi-invariant measure, studied by H.A. Dye (1959), (1963) and W. Krieger (1969a), (1969b), on the other, may be found in Dang-Ngoc-Nghiem (1973).

2 Nonmeasurable sets for invariant measures

We say that a measure m is *universal on a set* X, if it is defined on the σ-algebra $\mathcal{P}(X)$ of all subsets of X, i.e., if m-nonmeasurable sets do not exist.

A particular case of the problem discussed in the preceding section concerns the existence of invariant universal measures on G-spaces. As is well-known, given a set X, there might be fundamental set-theoretic obstacles preventing the existence of any σ-finite diffused universal measure on X (see e.g. D.H. Fremlin's article (1993) for a thorough discussion of this point). On the other hand, in many cases, the reason why a particular measure cannot be defined on $\mathcal{P}(X)$ is precisely its G-invariance, regardless of any new set-theoretic assumptions which might possibly be added to the usual Zermelo-Fraenkel set theory with the axiom of choice (ZFC).

Let us call a set X *large*, if there exists a σ-finite or, equivalently, probability diffused universal measure on X (in the commonly used terminology this means that there exists a *real-valued-measurable* cardinal $\kappa \leq |X|$ – see e.g. Fremlin (1993, 1C-D)); otherwise call X *small*.

The prototype of arguments employing invariance properties of measures to show the existence of nonmeasurable sets within ZFC is the Vitali construction.

By a *selector* or a *transversal* of a collection $\mathcal{C}$ of nonempty, pairwise disjoint subsets of a set X, we mean a subset $Y \subseteq X$ which intersects every element of $\mathcal{C}$ in exactly one point.

The classical *Vitali subset of* $\mathbb{R}$, constructed by G. Vitali (1905), is a selector of the family of all cosets of the rationals $\mathbb{Q}$, treated as a subgroup of the additive group $\mathbb{R}$. It is easy to note that *a Vitali subset of* $\mathbb{R}$ *is nonmeasurable with respect to any translation-invariant measure on* $\mathbb{R}$ *which assumes a finite non-zero value on* $[0,1]$. In particular, *no translation-invariant probability measure on* $\mathbb{R}$ *is universal.*

In fact, Vitali's argument works equally well for, say, left action of a countable subgroup of an arbitrary uncountable group G. Let H be a subgroup of G and call a *Vitali set of* H a selector of the collection of all left-cosets of H in G. Now, if H is countably infinite and S is a Vitali set for H, then S is nonmeasurable with respect to any finite invariant measure on G. This follows directly from the fact that

$$G = \bigcup_{h \in H} hS,$$

i.e., G can be partitioned into countably many translates of S. The latter shows also that G is countably paradoxical (in arbitrary pieces). (It is in fact not difficult to prove that *in the general case of an arbitrary G-space X, a subset E of X is countably paradoxical (in arbitrary pieces) if and only if there exists a countable subgroup H of G such that the intersection with E of the H-orbit of every point of E is infinite* – see e.g. Zakrzewski (1991a, Lemma 2.2)).

Recall that according to a theorem of Tarski (1938), given an action of a group G on a set X, a *finitely* additive probability invariant measure on X exists if and only if X is not *paradoxical*, i.e., X does not contain disjoint subsets X_1, X_2, each *finitely* equidecomposable with X – see Laczkovich's article in this Handbook for more details.

The exact analog of Tarski's theorem for *countably* additive measures is not true in general, since there may be no σ-additive probability universal measure on X vanishing on all countably paradoxical sets for purely set-theoretic reasons like, say, X being small, which have nothing to do with the obvious obstacle that X is countably paradoxical. So the following special case of Theorem 1.8, proved in Zakrzewski (1991a), seems to be the optimal analog of Tarski's theorem in the countably additive case for arbitrary G-spaces (see, however, Theorem 5.1).

Theorem 2.1. *For an arbitrary G-space (X, G) the following are equivalent:*

(i) *There exists a universal invariant probability measure on X;*

(ii) *There exists a universal probability measure on X vanishing on all countably paradoxical subsets of X.*

Let us now turn to σ-finite measures.

A.B. Kharazishvili (1975) and, independently, P. Erdös and R.D. Mauldin (1976) clarified the situation for the case when a group acts on itself by (say) left shifts (see also C. Ryll-Nardzewski and R. Telgarski (1978) for a generalization of this result).

Theorem 2.2. *Let G be an uncountable group acting on itself by left shifts. Then no σ-finite invariant measure on G is universal.*

Proof. Take a subgroup H of G of cardinality ω_1 and let $S \subseteq G$ be a Vitali set of H. Suppose that μ is a σ-finite invariant universal measure on X. Then we cannot have $\mu(S) > 0$, since this implies that the collection $\{hS : h \in H\}$ consists of uncountably many sets of positive measure. But $\mu(S) = 0$ is also impossible, since then $G = \bigcup_{h \in H} hS$ has measure zero (the union of any family of ω_1-many null sets with respect to a universal σ-finite diffused measure is a null set – see Fremlin (1993, 1D)). $\square$

The above argument proves also the nonexistence on G of a universal σ-finite measure, quasi-invariant with respect to its uncountable subgroup H. On the other hand, *for an arbitrary countable group H acting on a large set X, a universal σ-finite invariant diffused measure always exists.* (*Proof:* Assume that m is a universal probability diffused measure on X. Fix a selector S of the collection of all H-orbits such that $m(S) > 0$. Define the measure μ by:

$$\mu(A) = \sum_{k \in \mathbb{N} \cup \{\infty\}} k \cdot m\left(\{x \in S : |A \cap Hx| = k\}\right) \quad \text{for } A \subseteq X, \tag{$*$}$$

where $\infty \cdot 0 = 0$ and $\infty \cdot t = \infty$, if $t \neq 0$.)

The case of an arbitrary G-space X which admits a universal probability measure m is covered by Theorem 1.10. However, it turns out that now the situation is simpler: *the existence of a quasi-invariant σ-finite universal measure on X implies the existence of an equivalent invariant one.* This is a consequence of the following structural theorem for universal σ-finite quasi-invariant measures established in Zakrzewski (1991a) (for a shorter proof, due to D.H. Fremlin, see Zakrzewski (1997) and Fremlin (1993, 1J)).

Theorem 2.3. *Suppose G is a group of permutations of a set X. If m is a universal quasi-invariant σ-finite measure on X, then there exists a countable subgroup H of G such that*

$$m(\{x \in X : \; gx \notin Hx\}) = 0 \quad \textit{for every} \quad g \in G.$$

Now, under the notation of Theorem 2.3, if S is a selector of the collection of all H-orbits, then formula $(*)$ above defines a G-invariant universal σ-finite measure μ with $\mu \sim m$.

Combining this with Theorem 1.10 we have the following (for more details see Zakrzewski (1991a) and (1997)).

Theorem 2.4. *For an arbitrary G-space (X, G) the following are equivalent:*

(i) *There exists a universal invariant σ-finite measure on X;*

(ii) *There exists a universal quasi-invariant σ-finite measure on X;*

(iii) *There exists a universal σ-finite measure m on X such that there is no uncountable collection of pairwise disjoint sets outside the σ-ideal $I_G = \{A \subseteq X : m(gA) = 0 \text{ for every } g \in G\}$.*

In particular, Theorem 2.3 implies that *if a group G acting on a set X contains an uncountable subgroup G' whose action on X is free, then no universal σ-finite quasi-invariant measure on X exists.* This was earlier proved by Kharazishvili (1983) (see also Kharazishvili (1996)) in the following stronger form: *under the above assumptions, if $\mu : \mathcal{A} \to [0, \infty]$ is a σ-finite quasi-invariant measure on X, then every set $Y \in \mathcal{A}$ with $\mu(Y) > 0$ contains a nonmeasurable* *set.* But in fact, Theorem 2.4 easily implies that, in general, *for an arbitrary G-space X, if there is no universal quasi-invariant σ-finite measure on X, then any σ-finite quasi-invariant measure on X has the property that every measurable subset of X of positive measure contains a nonmeasurable set.* (*Proof:* If $\mu : \mathcal{A} \to [0, \infty]$ is a σ-finite quasi-invariant measure on X and $Y \in \mathcal{A}$ is such that $\mathcal{P}(Y) \subseteq \mathcal{A}$, then the universal measure m on X defined by $m(A) = \mu(A \cap Y)$, for $A \subseteq X$, satisfies condition (iii) above.)

More examples of natural G-spaces admitting no universal σ-finite invariant measures can be found in M. Penconek and Zakrzewski (1994), and Zakrzewski (1997).

While the results above concern the possibility of simultaneously making all subsets of a given G-space X measurable with respect to an invariant measure, there still remains the more modest problem, whether *a particular* subset of X can be "made measurable" by an invariant measure. This question, aimed at getting a deeper insight in the properties of nonmeasurable sets, is motivated by the following facts:

- *every Vitali subset of* $\mathbb{R}$ *is nonmeasurable with respect to any translation-invariant extension of the Lebesgue measure;*
- *there exists a Vitali subset of* $\mathbb{R}$ *which is measurable with respect to a certain σ-finite diffused translation-invariant measure defined on a σ-algebra containing all Lebesgue measurable subsets of* $\mathbb{R}$ (this is due to Kharazishvili (1983), (1991), (1996)).

Along these lines the following theorem of Kharazishvili (1980) generalizes Theorem 2.2 in the case of abelian groups.

Theorem 2.5. *Let G be an uncountable abelian group acting on itself by left shifts. Then there exists a subset A of G nonmeasurable with respect to any σ-finite invariant measure on G.*

It seems open whether the same holds for arbitrary groups. Let us call a subset A of an arbitrary G-space X *absolutely nonmeasurable* if it does not belong to the domain of any σ-finite invariant measure on X. Kharazishvili (1983), (1991), (1996) gave more examples of absolutely nonmeasurable sets in certain natural G-spaces but the problem, posed in Kharazishvili (1996), to find necessary and sufficient conditions for the existence of an absolutely nonmeasurable subset of X, is still open.

We may also look at sets nonmeasurable with respect to extensions of a given invariant measure. Answering a question of A. Pelc (1986), S. Solecki (1993a) obtained the following general result, particular instances of which were earlier proved by Kharazishvili (1978) (the case of a translation invariant extension of the Lebesgue measure on $\mathbb{R}^n$ and Pelc (1986) (the case of a left invariant extension of a (left) Haar measure on a locally compact, σ-compact topological group; see also E. Hewitt and K.A. Ross (1963, 16.13)).

Theorem 2.6. *Suppose G is an uncountable group acting on a set X and μ is a σ-finite invariant measure on X. If the action of G is μ-free, then each set of positive measure contains a subset nonmeasurable with respect to any invariant extension of μ.*

The above holds, in particular, for every uncountable group G acting on itself by left shifts; this gives another strengthening of Theorem 2.2.

Let us emphasize a remarkable role played by Vitali sets in the constructions of sets nonmeasurable with respect to invariant measures. If we define a *Vitali set of a subgroup H* of a group G acting on a set X as a selector of the collection of all H-orbits, then it is certainly true to say that playing with Vitali sets is the basic technique leading to the results above.

Measurability properties of Vitali sets with respect to invariant measures were investigated by Solecki (1993b). He showed that under the hypotheses of Theorem 2.6 one can always find a countable subgroup of G whose Vitali sets behave just like the original Vitali sets of the rationals on the real line. More precisely, *there exists a countable subgroup H of G such that each Vitali set of H is nonmeasurable with respect to any invariant extension of μ.* On the other hand, under certain assumptions a subgroup of full cardinality with properties like that cannot be constructed: *If the cardinality of the group G has uncountable cofinality, G acts freely on a set X and μ is a σ-finite invariant ergodic measure on X, then there exists an invariant extension $\bar{\mu}$ of μ such that each subgroup H of G with $|H| = |G|$ has a $\bar{\mu}$-measurable Vitali set.* For the special case of additive subgroups of $\mathbb{R}$ the latter was strengthened by A. Nowik who proved in (1996/97) that *there exists a translation invariant extension of the Lebesgue measure on $\mathbb{R}$ which measures at least one Vitali set of each uncountable subgroup of $\mathbb{R}$.* Recently, Solecki (2001) showed the same in the context of translation invariant extensions of the Haar measure on any locally compact, second countable abelian group.

Vitali sets of subgroups of $\mathbb{R}$ were also studied by J. Cichoń, A.B. Kharazishvili and B. Węglorz in (1993). They proved, in particular, that *if H is an uncountable analytic (i.e. a continuous image of a Borel set) subgroup of $\mathbb{R}$ then there exist a Vitali set of H which is Lebesgue measurable and all Vitali sets of H are measurable with respect to a certain isometrically invariant extension of the Lebesgue measure on $\mathbb{R}$.*

More information on various set-theoretic aspects of the theory of invariant measures can be found in Kharazishvili (1998).

We complete this section with a result which shows that the assumption that invariant measures under consideration are σ-finite is crucial for all results concerning the existence of nonmeasurable sets.

We say that a measure m defined on a σ-algebra $\mathcal{A}$ of subsets of a set X is *semi-finite* or *semiregular* if every set of positive measure contains a set of positive finite measure.

Zakrzewski (1987) and earlier in the abelian case Pelc (1982), solving a problem posed by V. Kannan and S.R. Raju (1980), proved the following.

Theorem 2.7. *If G is an arbitrary large group, then there exists a universal semi-finite invariant measure on G.*

3 Extensions of invariant measures

We have seen in Section 2 that given a σ-finite invariant measure $\mu\colon \mathcal{A} \to [0,+\infty]$ on a G-space X satisfying some additional conditions, there are nonmeasurable sets which cannot be "made measurable" with respect to any invariant extension of μ. A complementary question is whether there is a nonmeasurable set which *can* be "made measurable" by an invariant extension of μ. More precisely, whether a *proper invariant extension* of μ, i.e., an invariant measure $\bar{\mu}$ defined on an invariant σ-algebra $\mathcal{A}' \supsetneq \mathcal{A}$ and such that $\bar{\mu}|\mathcal{A} = \mu$, exists at all. This problem will be referred to as the *invariant measure extension problem.* A particularly important instance of it, concerning isometrically invariant extensions of the Lebesgue measure, whose investigations were initiated by E. Szpilrajn (Marczewski) in (1935), will be discussed in Section 6.

The obvious necessary condition for the existence of a proper extension of a given invariant σ-finite measure $\mu : \mathcal{A} \to [0,\infty]$ on a G-space X is that it is nonuniversal, i.e., $\mathcal{A} \neq \mathcal{P}(X)$. Moreover, without loss of generality we will always assume that the measure μ is diffused. (For let $C = \{x \in X : \mu(\{x\}) > 0\}$; C is a countable subset of X and $\mu = \mu_1 + \mu_2$ where $\mu_1,\ \mu_2 : \mathcal{A} \to [0,+\infty]$ are defined by $\mu_1(A) = \mu(A \cap C)$ and $\mu_2(A) = \mu(A \setminus C)$. If $\mu(C) > 0$, then μ_1 is a universal measure on X and it follows that the measure μ has an invariant extension if and only if so does the measure μ_2, which is clearly diffused.)

In general, there are the following basic ways of obtaining an invariant extension $\bar{\mu}\colon \mathcal{A}' \to [0,+\infty]$ of μ:

(I) Find a set $B \notin \mathcal{A}$ such that for any sequence $\langle g_n : n \in \mathbb{N}\rangle$ of elements of G, the union $\bigcup_{n\in\mathbb{N}} g_n B$ has inner measure zero. Let $\mathcal{I}$ consist of all subsets of the unions of this form.

Then define:

$$\mathcal{A}' = \{(A \triangle N \colon A \in \mathcal{A} \text{ and } N \in \mathcal{I}\},$$
$$\bar{\mu}\,((A \triangle N) = \mu(A)$$

(see E. Szpilrajn (Marczewski) (1935));

(II) Find an almost invariant set $Z \notin \mathcal{A}$, then define:

$$\mathcal{A}' = \{(A_1 \cap Z) \cup (A_2 \setminus Z) \colon A_1,\ A_2 \in \mathcal{A}\},$$

$$\bar{\mu}\,((A_1 \cap Z) \cup (A_2 \setminus Z)) = \frac{1}{2}\cdot\Big[\ \mu^*(A_1 \cap Z) + \mu_*(A_2 \setminus Z) + \mu_*(A_1 \cap Z) + \mu^*(A_2 \setminus Z)\ \Big],$$

where μ_* and μ^* stand for the inner and outer measure for μ, respectively– see J. Łoś and E. Marczewski (1949).

Recall that a set X is *small*, if there is no σ-finite diffused universal measure on X, i.e., there is no *real-valued-measurable* cardinal $\leq |X|$; otherwise X is *large*.

A. Hulanicki (1962) used method II to prove the following general result.

Theorem 3.1. *Let G be a group of permutations of a set X such that $|G| \leq |X|$ and X is small. If μ is an invariant σ-finite diffused measure which is uniform, i.e., every subset of X of cardinality smaller than $|X|$ has measure zero, then μ has a proper invariant extension.*

Pelc (1986) applied method I to solve the invariant measure extension problem for the case of an abelian group acting on itself as follows.

Theorem 3.2. *Let G be an uncountable abelian group acting on itself by left shifts. Then every σ-finite invariant measure on G has a proper invariant extension.*

Pelc conjectured that the same is true for any uncountable group G. A refinement of Hulanicki's argument that lead to Theorem 3.1 above shows that it is indeed the case when G is small. So one cannot falsify Pelc's conjecture on the basis of ZFC alone, the nonexistence of large sets being relatively consistent with ZFC (see e.g. Fremlin (1993, 1Je). It is also known that the conjecture is true for some other algebraically or topologically defined classes of groups (see Zakrzewski (1996)) but the general question is still open.

Thus the invariant measure extension problem for *uncountable* groups of permutations is not solved even for the case of the left-shift action of a group on itself. On the other hand, subsequent investigations started in Pelc (1986) and then continued in A. Krawczyk and Zakrzewski (1991), brought a satisfactory solution for *countable* permutation groups.

So let $\mu : \mathcal{A} \to [0, +\infty]$ be a σ-finite diffused measure on X, invariant with respect to a countable group G of permutations of X.

A Hulanicki-style argument, based on method II above, shows that *if X is small, then μ always has a proper invariant extension.*

On the other hand, if X is large, then μ can be universal (see the remarks following Theorem 2.2). So assume that μ is defined on a proper σ-algebra $\mathcal{A}$ of subsets of X. We may additionally assume that μ is complete and every invariant subset of X is measurable, since otherwise we can immediately extend μ using method I or II, respectively.

The following result from Krawczyk and Zakrzewski (1991) reveals the structure of such measures. We say that a partition P of a subset Y of X is a *G-partition* if $A \in P$ implies $gA \in P$ for every $g \in G$. We call P a *minimal*

G-partition of Y if there is no G-partition $P' \neq P$ of Y with the property that every element of P is the union of finitely many elements of P'. It turns out that the structure of μ can be completely described in terms of G-partitions of G-orbits.

Theorem 3.3. *Assume that G is a countable group of permutations of a set X and $\mu : \mathcal{A} \to [0, \infty]$ is an invariant σ-finite complete diffused measure defined on X such that every invariant subset $A \subseteq X$ is in the σ-algebra $\mathcal{A}$. Let S be a selector of G-orbits.*

Then there exists a function $s \mapsto P_s$, determined uniquely up to a measure zero set, which assigns to each $s \in S$ a countable G-partition P_s of the orbit Gs, with the following property:

(i) *The σ-algebra $\mathcal{A}$ consists of all sets $A \subseteq X$ such that for μ-almost every $s \in S$, the set $A \cap Gs$ is the union of some pieces of the partition P_s.*

Moreover:

(ii) *For any $A \in \mathcal{A}$*

$$\mu(A) = \sum_{k \in \mathbb{N} \cup \{\infty\}} k \cdot \mu \left(\mathcal{T} \cap \bigcup \left\{ Gs : \ s \in S \quad \text{and} \quad A \cap Gs \in P_s^{(k)} \right\} \right),$$

where $\mathcal{T}$ is a selector of the family $\{P_s : \ s \in S\}$ and $P_s^{(k)}$ consists of sets which are unions of k elements of the partition P_s ($\infty \cdot 0 = 0$ and $\infty \cdot t = \infty$ if $t \neq 0$).

(iii) *The measure μ does not have a proper invariant extension if and only if μ-almost every partition P_s is minimal.*

Now the point is that properties of G-partitions of the orbit Gx of a point $x \in X$ are reflected by algebraic properties of G and the *stabilizer* of x, i.e., the group $G_x = \{g \in G : gx = g\}$. This is due to the fact that *every G-partition P of Gx has the form $P = \{gHx : \ g \in G\}$, where H is a subgroup of G containing G_x. Moreover, such a P is minimal if and only if every proper subgroup of H containing G_x has infinite index in H.* (see e.g. M.I. Kargapolov and Ju.I. Merzljakov (1979)).

In view of this, we have the following corollary.

Theorem 3.4. *Suppose G is a countable group of permutations of a set X.*

(i) *If every nontrivial subgroup H of G has a proper subgroup of finite index in H, then every σ-finite nonuniversal invariant measure μ on X such that the action of G is μ-free has a proper invariant extension.*

(ii) *If there exists a nontrivial subgroup H of G, every proper subgroup of which has infinite index in H, X is large and the action of G on X is free, then there exists a σ-finite nonuniversal invariant measure on X without proper invariant extension.*

The additive group $\mathbb{Q}$ of the rationals is an example of a group every proper subgroup of which has an infinite index. It follows that *if $\mathbb{Q}$ acts freely on a large set X, then there exists a nonuniversal invariant σ-finite (*or even *a probability) measure on X without proper invariant extension.* Further applications to measures on Euclidean spaces $\mathbb{R}^n$, invariant with respect to groups of isometries of $\mathbb{R}^n$, will be discussed in Section 6.

4 Invariant measures on Polish groups

In the previous sections we considered discrete G-spaces and discussed properties of measures which were related only to the algebraic structure of underlying spaces. Now we are going to let topology enter the scene. In order to make the interplay between measure and topology more transparent we decided to stay within the realm of *standard* Borel spaces.

Let $\mathbf{B}(Y)$ or $\mathbf{B}(\tau)$ denote the σ-algebra of Borel subsets of a topological space (Y, τ), i.e., the σ-algebra of subsets of Y generated by the open sets of Y. A measurable space $(X, \mathcal{A})$ is a *standard Borel space* if either X is countable and $\mathcal{A} = \mathcal{P}(X)$ or $(X, \mathcal{A})$ is *isomorphic* to the Borel space $(\mathbb{R}, \mathbf{B}(\mathbb{R}))$, i.e., if there is a bijection f between X and $\mathbb{R}$ such that $A \in \mathcal{A}$ if and only if $f[A] \in \mathbf{B}(\mathbb{R})$, for $A \subseteq X$. Equivalently, $\mathcal{A} = \mathbf{B}(\tau)$ for a certain topology τ on X which is *Polish*, i.e., admits a compatible separable complete metric (see A.S. Kechris (1995, 12.5)).

By a *Borel measure* on a standard Borel space $(X, \mathcal{A})$ we mean a measure defined on $\mathcal{A}$. A Borel measure on a Polish space X is a measure defined on the σ-algebra $\mathbf{B}(X)$. Note that if m is a σ-finite Borel measure on a standard Borel space $(X, \mathcal{A})$, then m is diffused if and only if it is *nonatomic*, i.e., any set of positive measure splits into two sets, each having positive measure.

A *Polish group* is a topological group whose topology is Polish (a survey of some basic facts from the theory of Polish groups may be found in the book of H. Becker and A.S. Kechris (1996)). Examples of Polish groups include (for more examples see Becker and Kechris (1996, 1.3)):

(i) Hausdorff second countable locally compact groups: countable groups (with the discrete topology), Lie groups, the Cantor group $\mathbb{Z}_2^{\mathbb{N}}$;

(ii) $(X, +)$, where X is a separable Banach space;

(iii) The group of homeomorphisms $H(X)$ of a compact metric space (X, d), where the group operation is composition, with the topology it inherits as a G_δ

subset of $C(X,X)$, the Polish space of all continuous functions $f : X \to X$ with the sup metric $d_u(f,g) = \sup_{x\in X} d(f(x), g(x))$, for $f, g \in C(X,X)$;

(iv) The group of isometries $\mathrm{Iso}(X,d)$ or just $\mathrm{Iso}(X)$ of a complete separable metric space (X,d) with the topology of pointwise convergence. It can be proved that if the space (X,d) is compact, then $\mathrm{Iso}(X,d)$ is a compact subgroup of $H(X)$. By a result of J. van Dantzig and B.L. van der Waerden (1928), if the space (X,d) is locally compact and connected, then the group $\mathrm{Iso}(X,d)$ is locally compact (for recent extensions of the latter result to more general classes of spaces see J.D. Clemens, S. Gao and A.S. Kechris (2000)).

The problem of existence of a σ-finite invariant Borel measure on a Polish *locally compact* group was solved by A. Haar (1933), as is well known. The uniqueness part of the following result is due to von Neumann (1936).

Theorem 4.1. *Let G be a Polish locally compact group acting on itself by left shifts. Then there exists a unique, up to a multiplicative positive constant, σ-finite invariant Borel measure on G.*

This measure is called the (left) *Haar measure* on G and we will denote it by μ_G. Similarly, there is a right Haar measure $\mu_G^{'}$ defined by:

$$\mu_G^{'}(B) = \mu_G(B^{-1}) \quad \text{for} \quad B \in \mathbf{B}(G).$$

In order to realize the essential difference of approach compared with the existence arguments from Section 1, let us briefly sketch a more or less standard proof of existence of Haar measure (the "covering ratios" proof – for details see K.R. Parthasarathy (1977, Proposition 54.2) or J. Mycielski (1974)).

Proof. Fix a compatible metric d in G and let B_ε denote the open neighborhood of the identity of radius ε. Without loss of generality we assume that $\overline{B_1}$ is compact. For any compact set $K \subseteq G$ and any $\varepsilon \leq 1$ let

$$E(K,\varepsilon) = \min\{|\mathcal{C}| : \ \mathcal{C} \text{ is a finite covering of } K \text{ with left shifts of } B_\varepsilon\}.$$

and then put

$$\lambda(K) = \lim_{n \to \mathcal{F}} \frac{E(K, 1/n)}{E(\overline{B_1}, 1/n)},$$

where $\mathcal{F}$ is a nonprincipal ultrafilter of subsets of $\mathbb{N}$.

Finally, for every open set $U \subseteq G$ let

$$\mu(U) = \sup\{\lambda(K) : \ K \subseteq U, \ K \text{ compact}\}$$

and for any $B \in \mathbf{B}(G)$ define

$$\mu(B) = \inf\{\mu(U) : \ B \subseteq U, \ U \text{ open}\}.$$

□

From this proof it may be seen that Haar measure is finite on compact sets. In particular, we have

Proposition 4.2. *If G be a Polish locally compact group, then the following are equivalent:*

(i) *Haar measure on G is finite.*

(ii) *The group G is compact.*

For a compact group G it is customary to assume that $\mu_G(G) = 1$.

The uniqueness of Haar measure does not imply that the left and right Haar measures on G coincide (even up to a multiplicative constant), since a (left) *Haar measure is in general not right-invariant.* On the other hand, left *Haar measure is always right-quasi-invariant*, i.e., quasi-invariant with respect to right shifts. This follows from the following more general fact, which is also the first step towards von Neumann's proof of uniqueness of Haar measure.

Proposition 4.3. *Let m_l and m_r be arbitrary σ-finite measures on a Polish locally compact group G. If m_l is left quasi-invariant and m_r is right quasi-invariant, then $m_l \sim m_r$. Consequently, $m_l \sim m_r \sim \mu_G$.*

Proof. Take an arbitrary set $B \in \mathbf{B}(G)$. By Fubini's theorem and the appropriate quasi-invariance of the measures under consideration we have

$$\begin{aligned} m_l(B) = 0 \quad &\Leftrightarrow \quad m_l(g^{-1}B) = 0 \text{ for } m_l\text{-almost all } g \in G \\ &\Leftrightarrow \quad m_r(Bh^{-1}) = 0 \text{ for } m_r\text{-almost all } h \in G \\ &\Leftrightarrow \quad m_r(B) = 0. \end{aligned}$$

□

It is known that any Polish group admits a left-invariant compatible metric (the Birkhoff-Kakutanitheorem – for a proof see e.g. S.K. Berberian (1974, p. 28). In view of a result of S. Banach (1937), *if d is a left-invariant compatible metric on a Polish locally compact group G, then (left) Haar measure on G is invariant under all isometries of the metric space* (G, d) (for various generalizations of this result see I.E. Segal (1949), I.E. Segal and R.A. Kunze (1978, Corollary 7.5.1) and C. Bandt (1983)). If (left) Haar measure on G is right-invariant, G is called *unimodular.* Since right shifts are isometries of the space (G, d) if and only if the metric d is right-invariant, it follows that *every Polish locally compact topological group admitting a compatible (two-sided) invariant metric is unimodular.* Examples of such groups include abelian groups and compact groups.

The next result reveals another important property of Haar measure and is the second step towards the proof of its uniqueness.

Proposition 4.4. *A Haar measure on a locally compact Polish group is ergodic.*

Proof. Let $A \in \mathbf{B}(G)$ be μ_G-almost-invariant. It means that $\chi_A = \chi_{g^{-1}A}$, μ_G-a.e., where χ_A is the characteristic function of A in G. By Fubini's theorem there is $h_0 \in G$ such that $\chi_A(h_0) = \chi_{g^{-1}A}(h_0)$ for μ_G-a.e. $g \in G$. Equivalently, either $gh_0 \in A$ for μ_G-a.e. $g \in G$ or $gh_0 \notin A$ for μ_G-a.e. $g \in G$. By the (right) quasi-invariance of μ_G it follows that $\mu_G(G \setminus A) = 0$ or $\mu_G(A) = 0$, respectively. $\square$

Now the uniqueness of Haar measure follows at once from Proposition 1.11.

A consequence of uniqueness is that unimodular groups may also be characterized as follows: *G is unimodular if and only if Haar measure on G is inversion-invariant, i.e., invariant under the transformation $g \mapsto g^{-1}$.* (*Proof:* If μ_G is right-invariant, then μ_G' is left-invariant. Then, by uniqueness, there is a positive constant c such that $\mu_G(B^{-1}) = \mu_G'(B) = c \cdot \mu_G(B)$ for any $B \in \mathbf{B}(G)$. Now it suffices to take as B any symmetric set, i.e., such that $B = B^{-1}$, of finite positive measure (for example the closure of a symmetric neighborhood of the identity of a sufficiently small diameter) to conclude that $c = 1$. The other implication is easy).

Another consequence of uniqueness is the existence of a continuous homomorphism Δ of G into the multiplicative group of positive real numbers satisfying the following property

$$\mu_G(Ag) = \Delta(g)\mu_G(A) \quad \text{for all } g \in G,\ A \in \mathbf{B}(G).$$

The function Δ is called *the modular function of the group G*. Clearly, *G is unimodular if and only if $\Delta(g) = 1$ for all $g \in G$.* From this it becomes obvious why compact groups are unimodular. (*Proof:* If G is compact, then $1 = \mu_G(G) = \mu_G(Gg) = \Delta(g)\mu_G(G) = \Delta(g)$ for any $g \in G$). It is also easy to see that the function Δ is the Radon-Nikodym derivative $\frac{d\mu_G}{d\mu_G'}$.

It should be underlined that the basics of Haar measure theory, surveyed above, were appropriately generalized by A. Weil (1940) to *arbitrary locally compact groups*. Also, Weil's proof of uniqueness is entirely different from that of von Neumann (other uniqueness arguments were given by H. Cartan (1940) and D.A. Raikov (1942)). A complete account of Haar measure in general locally compact groups may be found in many standard textbooks (for example see E. Hewitt and K.A. Ross (1963), L. Nachbin (1965) and P.R. Halmos (1974)).

Now let us turn to Polish groups which are not locally compact. S. Ulam (quoted by J.C. Oxtoby in (1946)) proved that *if G is a Polish, non-locally-compact group and m is any left quasi-invariant Borel measure assuming at least one positive finite value, then every open subset of G contains uncountably many pairwise disjoint left shifts of a compact set of positive measure.* Combining this with the theorem on existence of Haar measure we immediately obtain a characterization of locally compact groups in terms of invariant measures.

Theorem 4.5. *Assume that G is a Polish group. Then the following are equivalent:*

(i) *G admits a (left) quasi-invariant Borel σ-finite measure;*

(ii) *G is locally compact.*

This characterization can be considerably strengthened. A *standard Borel group* is a group G together with a σ-algebra $\mathcal{A}$ such that $(G, \mathcal{A})$ is a standard Borel space and the mapping $(x, y) \mapsto xy^{-1}$ is a Borel function from $G \times G$ into G. If G is a standard Borel group, then there is, of course, a Polish topology τ such that $\mathcal{A} = \mathbf{B}(\tau)$ but it is not necessarily true that there is one for which (G, τ) is a topological group. On the other hand, if such a topology exists, then it is unique (see e.g. Kechris (1995, 12.24)).

A. Weil (1940) (a weaker result for a broader class of groups) and G.W. Mackey (1957) proved that *if G is a standard Borel group and there exists a quasi-invariant Borel σ-finite measure on G, then there is a topology τ giving its Borel structure and making G a Polish locally compact group.* The idea of a proof is to use a quasi-invariant Borel probability measure μ on G to embed G into the unitary group $U(H)$ of the Hilbert space $H = L^2(G, \mathbf{B}(G), \mu)$ (see e.g. Parthasarathy (1977, §56)).

The Mackey-Weil theorem shows that Polish locally compact groups can be characterized in the following way.

Theorem 4.6. *Assume that G is a standard Borel group. Then the following are equivalent:*

(i) *G admits a quasi-invariant Borel σ-finite measure.*

(ii) *G carries a Polish topology which gives its Borel structure and makes it a locally compact group.*

Though, as we have just seen, in a non-locally-compact Polish group G, for any σ-finite Borel measure on G, the σ-ideal in $\mathbf{B}(G)$ of measure zero subsets of G is neither left nor right invariant, it turns out that every Polish group G carries a certain natural (two-sided) invariant σ-ideal in $\mathbf{B}(G)$. The following concept was introduced by J.P.R. Christensen (1972) for the abelian case and then extended by F. Topsøe and J. Hoffmann-Jørgensen (1980), and J. Mycielski to all Polish groups.

We say that a Borel (or, more generally, a *universally measurable*, i.e., measurable with respect to the completion of every σ-finite Borel measure on G) subset A of G is *Haar null*, if there exists a Borel probability measure m on G such that $m(gAh) = 0$ for all $g, h \in G$.

It is not difficult to prove that *in Polish locally compact groups Haar null Borel sets coincide with the Borel sets of Haar measure zero.* In an arbitrary

Polish group G, *Haar null Borel sets form a σ-ideal in* $\mathbf{B}(G)$ *which is (two-sided) invariant*, i.e., closed under left and right shifts (the closure under countable unions requires proof – see Christensen (1972)). Another similarity of Haar null sets with the Haar measure zero sets, at least for abelian groups, is exhibited by the following result of Christensen (1972) (for the case of a (not necessarily abelian) locally compact group this is the well known *Steinhaus property* which is due to H. Steinhaus (1920) in the case of $G = \mathbb{R}$ and to Weil (1940) in general). *If G is an abelian Polish group and A is a Borel (or universally measurable) set which is not Haar null, then the set AA^{-1} contains an open neighborhood of the identity of G.* It seems to be open, whether the same is true for a broader class of Polish groups.

Motivated by another property of the Haar measure zero sets in locally compact groups Christensen asked, whether every collection of pairwise disjoint, universally measurable sets which are not Haar null is countable. R. Dougherty (1994) obtained negative answer for many particular non-locally compact groups. Finally, Solecki (1996) proved that *if G is a Polish non-locally compact group admitting a (two-sided) invariant metric, then there exists an uncountable collection of closed, pairwise disjoint sets which are not Haar null.* This leads to another characterization of the local compactness of a Polish group (admitting an invariant metric) in measure theoretic terms.

Theorem 4.7. *Assume that G is a Polish group, admitting an invariant metric. Then the following are equivalent:*

(i) *The σ-ideal of Haar null Borel subsets of G satisfies the countable chain condition in* $\mathbf{B}(G)$*;*

(ii) *G is locally compact.*

It again seems to be open, whether the same is true for a broader class of Polish groups.

Let us close this section by pointing out an important role played in the results above by the assumption that invariant measures under consideration are σ-finite. Namely, Oxtoby (1946) proved the following.

Theorem 4.8. *Every Polish group carries a Borel semi-finite invariant measure.*

5 Invariant measures on Polish G-spaces

A G-space X is called a *standard Borel G-space* if G is a standard Borel group, X is a standard Borel space and the action of G on X is a Borel function from

$G \times X$ into X. If G is a Polish group, X is a Polish space and the action is continuous, then X is called a *Polish G-space.* By a fundamental theorem of Becker and Kechris (1996, Theorem 5.2.1) *if G is a Polish group, then any standard Borel G-space $(X, \mathcal{A})$ is Borel isomorphic to a Polish G-space, i.e., $\mathcal{A} = \mathbf{B}(\tau)$ for a certain Polish topology τ on X which makes the action of G on X a continuous function from $G \times X$ into X.*

Polish G-spaces have long been studied in several areas of mathematics including ergodic theory, operator algebras and group representation theory. Of particular interest are actions of Polish locally compact groups which cover the case of the group of integers acting on X via the map $(k, x) \mapsto T^k x$, where T is a homeomorphism of X, *flaws*, i.e., $\mathbb{R}$-actions and actions of Lie groups. Other examples of Polish G-spaces include:

(i) The action of a Polish group G on itself by (say) left shifts;

(ii) The canonical action of a Polish group G on the *quotient space G/H* of all left-cosets of a closed subgroup H of G (it is a Polish space with the quotient topology consisting of all $U \subseteq G/H$ such that $\pi^{-1}[U]$ is open in G, where $\pi(x) = xH$ is the canonical projection) given by $(g, xH) \mapsto (gx)H$, for $g, x \in G$;

(iii) The action of the group of homeomorphisms $G = H(X)$ on a compact metrizable space X given by $(g, x) \mapsto g(x)$ for $g \in H(X)$ and $x \in X$;

(iv) The action of the group of isometries $G = \mathrm{Iso}(X)$ on a Polish space X given by $(g, x) \mapsto g(x)$ for $g \in \mathrm{Iso}(X)$ and $x \in X$.

An excellent account of the topic (and more examples) is given in Becker and Kechris (1996).

The case when G is locally compact is the most important one as far as invariant measures are concerned. There is, however, a general result valid for Borel actions of arbitrary Polish groups, which in this case strengthens the existence theorems from Section 1. Recall that Tarski's theorem asserts that the existence of a *finitely additive* probability universal invariant measure on a G-space X follows already from the assumption that the set X is not paradoxical. Becker and Kechris (1996, Theorem 4.2.1) and earlier M.G. Nadkarni (1990) in the case G is countable, proved the following exact analog of Tarski's theorem for *countably additive* measures on Polish G-spaces.

Theorem 5.1. *Let G be a Polish group and X a standard Borel G-space. Then the following are equivalent:*

(i) *There exists an invariant Borel probability measure on X;*

(ii) *The space X is not countably paradoxical in Borel pieces.*

Note that by the result of Becker and Kechris, quoted at the beginning of this section, we may assume that X is actually a Polish G-space. Then the crux of the proof of the nontrivial implication ((ii) $\Rightarrow$ (i)) lies in the case G is countable.

For assume that we have already proved the theorem for the countable case. Let G_0 be a countable dense subgroup of G and assume that X is not countably G-paradoxical in $\mathbf{B}(X)$. Then X is not countably G_0-paradoxical and by the countable case there exists a G_0-invariant probability Borel measure μ on X. It turns out that μ is also G-invariant, due to the following general fact which may be established by a straightforward continuity argument. *In a Polish G-space any Borel probability measure invariant under a countable dense subgroup of G is G-invariant* (it should be noted that the latter is *not* true for σ-finite measures).

The countable case was proved by Nadkarni for $\mathbb{Z}$-actions but, as noted by Becker and Kechris, the argument works as well for any countable group with some easy modifications in the proof. The following outline is extracted from Becker and Kechris (1996, Section 4).

Let us assume that G is a countable group of homeomorphisms of a Polish space X and that $X \notin N_G$, the σ-ideal in $\mathbf{B}(X)$ consisting of all G-negligible Borel subsets of X, i.e., sets $A \in \mathbf{B}(X)$ such that X contains pairwise disjoint subsets $A_n \in \mathbf{B}(X)$, $n \in \mathbb{N}$, with each $A_n \sim_\infty A$ in $\mathbf{B}(X)$. This is equivalent to the assertion that X is not countably paradoxical in $\mathbf{B}(X)$. We assume, moreover, that all orbits are infinite, since otherwise the result holds trivially.

Denote by $\mathbf{B}^i(X)$ the σ-algebra of invariant Borel sets and recall that since the group G is countable, $A^* = \bigcup_{g \in G} gA$ is the minimal invariant set containing a Borel set $A \subseteq X$.

The idea of the proof is similar to that of the proof of the Hopf-Kawada theorem (Theorem 1.2), outlined in Section 1.

Given sets $A, B \in \mathbf{B}(X)$ we write $A \prec B$ if there is $C \in \mathbf{B}(X)$ with $C \sim_\infty A$ in $\mathbf{B}(X)$, $C \subseteq B$ and for every $x \in B$, $C \cap Gx$ is a proper subset of $B \cap Gx$.

Given sets $A, B, C \in \mathbf{B}(X)$ and $n \in \mathbb{N}$ we write

$$A = nB \oplus C$$

if $A = A_1 \cup \cdots \cup A_n \cup C$ where $A_1, \ldots, A_n, C$ are pairwise disjoint Borel sets with each $A_i \sim_\infty B$ in $\mathbf{B}$(X).

The following two lemmas are analogs of lemmas 1.3 and 1.4.

Lemma 5.2. *For every Borel sets $A, B \subseteq X$ there is a partition (up to a set from N_G)*

$$A^* = \bigcup_{n \in \mathbb{N}} Q_n$$

of A^* into pairwise disjoint Borel sets $Q_n \in \mathbf{B}^i(X)$, $n \in \mathbb{N}$, such that for each $n \in \mathbb{N}$

$$A \cap Q_n = n(B \cap Q_n) \oplus R_n, \quad \text{where } R_n \prec B \cap Q_n.$$

Moreover, the partition is unique, up to a set from N_G.

For every Borel sets $A, B \subseteq X$ and $x \in X$, let

$$[A/B](x) = \begin{cases} n, \text{ if } x \in Q_n, \\ 0, \text{ otherwise,} \end{cases}$$

in the notation of the preceding lemma.

Lemma 5.3. *There is a decreasing sequence of Borel sets C_n with $C_0 = X$, $C_n^* = X$ and $[C_n/C_{n+1}](x) \geq 2$ modulo N_G, for every $n \in \mathbb{N}$.*

Taking the lemmas above for granted, the proof is carried out as follows.

Fix a sequence (C_n) from Lemma 5.3 and for any $A \in \mathbf{B}(X)$ and $x \in X$ let

$$m(A, x) = \lim_{n \to \infty} \frac{[A/C_n](x)}{[X/C_n](x)}.$$

It can be proved that this definition is correct for all x outside a set from N_G and the function $m(A, x)$ has the following properties:

(i) $0 \leq m(A, x) \leq 1$ and $m(X, x) = 1$;

(ii) $m(A, x) = 0$ modulo N_G if and only if $A \in N_G$;

(iii) $m(gA, x) = m(A, gx) = m(A, x)$ modulo N_G;

(iv) $m\left(\bigcup_{n \in \mathbb{N}} A_n, x\right) = \sum_{n=0}^{\infty} m(A_n, x)$ modulo N_G, whenever A_n, $n \in \mathbb{N}$, are pairwise disjoint Borel sets;

(v) If Q is an invariant Borel set and $x \in Q$, then

$$m(A, x) = m(A \cap Q, x) \quad \text{modulo } N_G.$$

Now fix an invariant countable open base $\mathcal{U}$ for the topology of X and let

$$\mu(A) = \inf\left\{\sum_{n \in \mathbb{N}} \tau(A_n) : A_n \in \mathcal{U},\ A \subseteq \bigcup_{n \in \mathbb{N}} A_n\right\}, \quad \text{for } A \in \mathbf{B}(X),$$

where $\tau : \mathcal{U} \to [0, \infty]$ is defined by

$$\tau(A) = m(A, x_0),$$

for a suitably chosen $x_0 \in X$, so that the function μ is an invariant probability Borel measure on X (the point is that checking this requires only *countably* many conditions of the form (iii) and (iv) above plus $m(\emptyset, x_0) = 0$, hence the set E of exceptional points is in N_G and, since $X \notin N_G$, we may choose $x_0 \in X \setminus E$).

For details see Nadkarni (1990) and Becker and Kechris (1996).

As an immediate consequence of theorems 4.1, 4.5 and 5.1, and Proposition 4.2 we have the following characterization of compact Polish groups.

Theorem 5.4. *If G is a Polish group, then the following are equivalent:*

(i) *G is not countably paradoxical in Borel pieces.*

(ii) *G is compact.*

Let us now consider actions of Polish locally compact groups. Classically, of particular importance is the case when a locally compact group G acts continuously on a compact metric space X, i.e., when X is a *compact* Polish G-space. N.N. Bogoliouboff and N.M. Kryloff (1937) proved that *every compact Polish $\mathbb{Z}$-space admits an invariant Borel probability measure.* This was later extended by S.V. Fomin (1950) to actions by solvable groups. More generally, we say that a Polish locally compact group G is *topologically amenable* if there exists a *finitely* additive invariant Borel probability measure on G. Thus *every compact group is topologically amenable*, the left Haar measure being a witness of its amenability. Other examples include abelian and solvable groups (see Wagon (1986, p. 162)). (According to a theorem of A.L.T. Paterson (1986) *topologically amenable groups are characterized among Polish locally compact groups as exactly these groups which are not paradoxical in Borel pieces* – this should be compared with Theorem 5.1 above.)

The following general theorem (see e.g. F.P. Greenleaf (1969) or J.F. Plante (1983)) establishes a necessary condition for the existence of invariant Borel probability measures on compact Polish G-spaces.

Theorem 5.5. *Let G be a Polish locally compact topologically amenable group. Then every compact Polish G-space X admits an invariant Borel probability measure.*

A functional analytic proof of Theorem 5.5 is based on the fact that *a locally compact topologically amenable group G satisfies the following version of the Markov-Kakutani fixed point theorem: if G acts continuously on a compact convex subset K of a locally convex topological space E so that all the transformations $x \mapsto gx$, $g \in G$, are affine (i.e. $g(\alpha x + (1-\alpha)y) = \alpha gx + (1-\alpha)gy$ for $x, y \in K$ and $0 \leq \alpha \leq 1$), then they have a common fixed point* (see e.g. Wagon (1986, p. 163)).

For an arbitrary Polish space X, let $P(X)$ be the space of all probability Borel measures on X, equipped with the topology generated by the maps $m \mapsto \int f dm$, where f varies over the set of all bounded continuous real-valued functions on X. Then $P(X)$ is a Polish space. If, moreover, X is compact, then so is the space $P(X)$ and, by the Riesz Representation Theorem, it may be viewed as a convex, compact subspace of the dual space $C(X, \mathbb{R})^*$ equipped with its weak*-topology (see Kechris 1995, Section 17E).

Now the proof runs as follows: let $E = C(X, \mathbb{R})^*$, $K = P(X)$ and consider the action of G on K defined by $g\mu = \mu g^{-1}$, where $\mu g^{-1}(A) = \mu(g^{-1}A)$ for $A \in \mathbf{B}(X)$. Since this action is continuous, the Markov-Kakutani theorem, quoted above, gives a common fixed point $\mu \in P(X)$, which is an invariant probability Borel measure on X. (This should be compared with the proof of Theorem 1.7 of Hajian and Ito, outlined in Section 1.)

Since the group $\mathrm{Iso}(X)$ of all isometries of a compact metric space (X,d) is compact, hence topologically amenable, and X is a compact Polish $\mathrm{Iso}(X)$-space, we have the following corollary.

Proposition 5.6. *If X is a compact metric space, then there exists an isometry-invariant Borel probability measure on X.*

Returning to general considerations, suppose that G is a Polish locally compact group and X is an arbitrary Polish G-space. Denote by INV_G the set of all invariant probability Borel measures on X. Then INV_G is a convex subset of the Polish space $P(X)$. Let EINV_G denote the set of all ergodic measures $\mu \in \mathrm{INV}_G$.

It follows from Theorem 5.1 that N_G *is the collection of all Borel sets $A \subseteq X$ such that $\mu(A) = 0$ for every $\mu \in INV_G$.* (*Proof: Without loss of generality assume that the group G is countable. If $A \notin N_G$, then $A^* \notin N_G$ and by Theorem 5.1 there is a measure $\mu \in INV_G$ with $\mu(A^*) = 1$*).

A thorough study of INV_G was undertaken by R.H. Farrel (1962) and V.S. Varadarajan (1963). Their main results may be summarized as follows.

Theorem 5.7. *Let G be Polish locally compact group and X a Polish G-space.*

(i) *Any measure $\mu \in INV_G$ is determined by its values on $\mathbf{B}^i(X)$. Moreover, if $INV_G \neq \emptyset$, then any probability measure defined on $\mathbf{B}^i(X)$ which vanishes on every set $A \in \mathbf{B}^i(X) \cap N_G$ has the unique extension to a certain measure $\mu \in INV_G$;*

(ii) *If there exists an invariant probability Borel measure on X, then there also exists an ergodic one;*

(iii) *The sets INV_G and $EINV_G$ are Borel in the space $P(X)$;*

(iv) *(A uniform ergodic decomposition theorem) If $INV_G \neq \emptyset$, then there is a Borel mapping β, $\beta : x \mapsto \beta_x$ from X onto $EINV_G$ such that*

- *The mapping β is constant on G-orbits, i.e., $\beta_{gx} = \beta_x$ for $x \in X$, $g \in G$;*
- *For every $\nu \in EINV_G$, if $X_\nu = \{x \in X : \beta_x = \nu\}$, then $\nu(X_\nu) = 1$;*
- *For any $\mu \in INV_G$,*

$$\mu(A) = \int_X \beta_x(A) d\mu(x) \quad \text{for all} \quad A \in \mathbf{B}(X). \qquad (*)$$

Moreover, the decomposition map β is uniquely determined modulo N_G.

Some remarks are in order concerning the above result.

- Assertion (i) is related to Theorem 1.5. The point is that *when G is locally compact, then for any Borel subset A of a Polish G-space X there exists a set $\tilde{A} \in \mathbf{B}^i(X)$ with $\mu(A \triangle \tilde{A}) = 0$ for any $\mu \in INV_G$ such that $A \in \mathbf{B}^*(X)$* (where $\mathbf{B}^*(X)$ is the σ-algebra of all Borel subsets of X which are almost invariant with respect to μ). It follows that the values of any measure $\mu \in \mathrm{INV}_G$ on $\mathbf{B}^*(X)$ are determined by its values on $\mathbf{B}^i(X)$. This also implies that *a measure $\mu \in INV_G$ is ergodic if and only if $\mu(A) = 0$ or 1 for any $A \in \mathbf{B}^i(X)$.*
- In the "moreover" part of (i) the assumption that $\mathrm{INV}_G \neq \emptyset$ may be omitted. Indeed, in view of Theorem 5.1, this already follows from the existence of any probability measure defined on $\mathbf{B}^i(X)$ and vanishing on each $A \in \mathbf{B}^i(X) \cap N_G$ (since this implies $X \notin N_G$).
- Assertion (ii) is related to the fact that *when G is locally compact, then $EINV_G$ is the set of all extreme points of the convex set INV_G* (this is known to be false for arbitrary Polish groups).
- In assertion (iv) we obtain a partition of X into invariant Borel sets X_ν with the property that (by $(*)$) $\nu \in \mathrm{EINV}_G$ is the only invariant Borel probability measure on the standard Borel G-space X_ν, i.e., X_ν is *uniquely ergodic*. Note that this does not imply that X_ν is a single G-orbit. But it is so if, for instance, the group G is compact (see remarks following Theorem 5.8 below); in this case the partition $\{X_\nu : \nu \in \mathrm{EINV}_G\}$ consists of all G-orbits and if $X_\nu = Gx_0$, then the measure ν is defined by:

  $$\nu(A) = \mu_G(\{g \in G : gx_0 \in A\}) \quad \text{for } A \in \mathbf{B}(X),$$

  where μ_G is the Haar measure on G.

Some generalizations of the Farrel-Varadarajan's results to quasi-invariant measures were obtained by Y.I. Kifer and S.A. Pirogov (1972), K. Schmidt (1977), A. Ditzen (1992), and G. Greschonig and K. Schmidt (2000).

So far in this section we have been looking at invariant *probability* measures only. The case of *σ-finite* measures is less clear in general but it is best understood when the group G is locally compact. Given a Polish G-space X let us say that X is *(G-)homogeneous* if the action of G is transitive, i.e., if X coincides with the only G-orbit. Recall that Δ is the modular function of the group G and $G_x = \{g \in G : gx = x\}$ is the stabilizer of a point $x \in X$. Note that a transitive action of G on a Polish space X can be identified in a natural way with the canonical action of G on the quotient space G/G_x; thus all homogeneous G-spaces are essentially of the form G/H, where H is a closed subgroup of G.

The study of invariant Borel measures on homogeneous G-spaces was undertaken by A. Weil (1940). The main results may be summarized as follows.

Theorem 5.8. *Suppose that G is a Polish locally compact group and X is a Polish G-homogeneous G-space. Fix a point $x_0 \in X$ and denote by δ the modular function of the group G_{x_0}.*

(i) *There exists a quasi-invariant probability Borel measure on X and any two such measures are equivalent.*

(ii) *There exists an invariant σ-finite Borel measure on X if and only if $\delta = \Delta|G_{x_0}$, the restriction of Δ to G_{x_0}. Such a measure, if it exists, is unique, up to a constant factor, and ergodic.*

Proof. (a sketch) In order to prove (i), it suffices to take a probability Borel measure $\mu \sim \mu_G$ on G and then define

$$\nu(A) = \mu(\{g \in G : gx_0 \in A\}) \quad \text{for } A \in \mathbf{B}(X).$$

The equivalence of any two quasi-invariant σ-finite measures and the uniqueness of an invariant σ-finite measure can be proved by a Fubini-type argument similar to that used for the case of the left action of G on itself – see propositions 4.3 and 4.4.

So the main part of the proof is to show that under the assumption that $\Delta|G_{x_0} = \delta$, there is a non-negative Borel function $\rho : X \to \mathbb{R}$ such that the measure λ defined by

$$\lambda(A) = \int_A \rho(x) d\nu(x) \quad \text{for } A \in \mathbf{B}(X),$$

is G-invariant. For details see e.g. Parthasarathy (1977, Proposition 55.6). $\square$

In particular, a unique invariant Borel *probability* measure on a Polish homogeneous G-space X exists, if the group G is a compact. On the other hand, X may possess such a measure without G being compact – conditions for its existence were studied by S.P. Wang (1976) and T.S. Wu and J.S. Yang (1986).

If G is a unimodular locally compact Polish group but H is its closed subgroup which is not unimodular, then the quotient G-space G/H admits no invariant σ-finite Borel measure (an example is given in Parthasarathy (1977, Exercise 55.8).

A special case of a homogeneous G-space is a *homogeneous Polish metric space* (X, d), i.e., an $\mathrm{Iso}(X)$-homogeneous Polish $\mathrm{Iso}(X)$-space. Recall that the group $\mathrm{Iso}(X)$ is Polish but not necessarily locally compact, even if X is so. However, Banach (1937) proved that *if X is a homogeneous Polish locally compact metric space, then there exists an isometrically invariant Borel σ-finite measure*

μ *on* X. The existence of Borel measures on metric spaces with various invariance properties was also studied by L.L. Loomis (1945), (1949), J. Mycielski (1974), (1977), C. Bandt (1983), C. Bandt and G. Baraki (1986), and S. Gao and A.S Kechris (2001, Theorem 6.10).

The more general problem of existence of a Haar-like Borel measure on a locally compact topological space X, invariant with respect to a group G of homeomorphisms of X, have also been considered by several authors. The following general result is due to R.C. Steinlage (1975).

Theorem 5.9. *Let* X *be a Polish locally compact space and* G *a group of homeomorphisms of* X *onto itself. Suppose that, for every disjoint pair of compact subsets of* X*, there is a nonempty open set* $U \subseteq X$ *such that no image of* U *under any element of* G *intersects both of the compact sets. Then there exists an invariant* σ*-finite Borel measure* μ *on* X*. In particular, the condition holds if* G *is a group of isometries of a Polish locally compact metric space* (X, d)*. In this case, if at least one* G*-orbit is dense in* X*, then* μ *is positive on open sets, finite on compact sets and is a unique, up to a constant factor, invariant Borel measure on* X *with these properties.*

As a matter of fact, Steinlage proved the theorem in a more general framework of arbitrary locally compact Hausdorff spaces – it strengthened earlier results of Banach (1937), Segal (1949), Segal and Kunze (1978, Theorem 7.2), Y. Mibu (1958) and J. Poncet (1954) (see also J.W. Roberts (1974/75) where a similar result for compact spaces is proved).

Even if a given Polish G-space is not homogeneous, we can apply Theorem 5.8 to the induced action of G on an arbitrary G-orbit (which is an invariant Borel subset of X – see Becker and Kechris (1996, 2.3.3)). In particular, it follows that *if* G *is a Polish locally compact group, then every Polish* G*-space* X *admits a quasi-invariant Borel probability measure.* But in fact the latter can also be proved by another method which gives some new information.

Proposition 5.10. *Let* G *be a Polish locally compact group,* X *a Polish* G*-space and* m *a probability Borel measure on* X*. Then there exists a quasi-invariant probability Borel measure* m' *on* X *with the following properties:*

(i) $m'(A) = m(A)$ *for any* $A \in \mathbf{B}^i(X)$*;*

(ii) $m' \ll \nu$ *for any quasi-invariant probability Borel measure* ν *on* X *with* $m \ll \nu$*;*

(iii) m *is ergodic if and only if* m' *is ergodic.*

Proof. Let μ_G be a (left) Haar measure on G and take a probability Borel measure $\mu \equiv \mu_G$ on G. Put

$$m'(A) = \int_G m(g^{-1}A)d\mu(g) \quad \text{for } A \in \mathbf{B}(X).$$

It is easy to see that this works. □

Let us finally briefly touch the case when G is a countable group and X is a standard Borel G-space. In this case the existence of an invariant (or quasi-invariant) probability (or σ-finite) Borel measure on X is really a property of the *orbit equivalence relation* E_G, whose equivalence classes are the G-orbits.

Let us say that an equivalence relation E on a set X is *countable* if every equivalence class is countable. The equivalence relation E_G is Borel (as a subset of $X \times X$) and countable. Conversely, J. Feldman and C.C. Moore (1977) proved that *if E is a countable Borel equivalence relation on a standard Borel space X, then there is a countable group G and a Borel action of G on X such that $E = E_G$.* Now, given such an E, a σ-finite Borel measure μ on X is called *E-invariant* (*E-quasi-invariant*, respectively) if for any Borel set $A \subseteq X$ and a Borel $1-1$ map $f : A \to X$ with $f(x)Ex$ for all $x \in A$, we have $\mu(f[A]) = \mu(A)$ ($\mu(f[A]) = 0 \Leftrightarrow \mu(A) = 0$, respectively). It is easy to check that this is equivalent to the assertion that μ is G-invariant (G-quasi-invariant, respectively) for any countable group G and a Borel action of G on X with $E = E_G$.

As we have already seen (Theorem 5.1), Nadkarni (1990) proved that the existence of an E-invariant probability Borel measure on X is equivalent to the fact that X is not countably paradoxical in Borel pieces with respect to any Borel action of a countable group inducing E. Let us say that E is *compressible* if there is a Borel map $f : X \to X$ such that for each $x \in X$, $f\big[[x]_E\big]$ is a proper subset of the equivalence class $[x]_E$ of x.

Now, Nadkarni's theorem may be stated in the following way, with no appeal to any action that induces E.

Theorem 5.11. *Suppose E is a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) *There exists an E-invariant Borel probability measure on X;*

(ii) *E is not compressible.*

Hence the existence of an E-invariant Borel probability measure on X tells us something about the structure of E.

On the other hand, a *σ-finite* E-invariant Borel measure on X trivially exists for any countable Borel equivalence relation E and it may even be proved that *there always exists a diffused one, provided that E has uncountably many equivalence classes.* (*Proof*: By a theorem of J. Silver (1980), *since E is a Borel equivalence relation on a standard Borel space X with uncountably many equivalence classes, there is a nonempty perfect set $S \subseteq X$ which meets every equivalence class in at most one point.* Let m be a Borel probability diffused measure on S and suppose $G = \{g_n : n \in \mathbb{N}\}$, $g_0 = \mathrm{id}_X$, is an arbitrary countable group of Borel automorphisms of X with $E_G = E$. Let $S_n = g_n S$ and define an E-invariant σ-finite Borel measure μ on X as follows

$$\mu(A) = \sum_{n\in\mathbb{N}} m\left(S \cap g_n^{-1}\left[A \cap \left(S_n \setminus \bigcup_{k<n} S_k\right)\right]\right) \quad \text{for} \quad A \in \mathbf{B}(X).)$$

However, the existence of an E-quasi-invariant σ-finite diffused Borel measure which is additionally *E-ergodic*, again expresses an important property of the equivalence relation E.

We say that a σ-finite Borel measure ν on X is *E-ergodic* if $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ for any invariant $A \in \mathbf{B}(X)$, where a subset $A \subseteq X$ is called *invariant* if it is the union of all equivalence classes of its elements. This is equivalent to asserting that ν is G-ergodic for any countable group G of Borel automorphisms of X with $E = E_G$. It follows from Proposition 5.10 that *if there exists a diffused E-ergodic probability Borel measure on X, then there also exists an E-quasi-invariant one.*

A Borel equivalence relation E on a standard Borel space X is called *smooth* if there is a standard Borel space Y and a Borel map $f : X \to Y$ with $xEy \Leftrightarrow f(x) = f(y)$ for any $x, y \in X$. It might be proved that *if E is countable, then E is smooth if and only if E has a Borel transversal* (see Kechris (1995, 18.D).

The following theorem is closely related to the so-called *Glimm-Effros dichotomy* – see Becker and Kechris (1996, Section 3.4).

Theorem 5.12. *Suppose E is a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) *There exists an E-quasi-invariant, ergodic, diffused Borel probability measure on X;*

(ii) *E is not smooth.*

It is not difficult to prove that if a countable aperiodic (i.e., with all equivalence classes infinite) Borel equivalence relation E is smooth, then it is compressible. But if E is not compressible, then by theorems 5.11 and 5.7(ii), there even exists an *E-invariant*, ergodic, diffused Borel probability measure on X. An example of a non-smooth compressible E is the *Vitali equivalence relation* on R (i.e., $xEy \iff x - y \in \mathbb{Q}$).

6 Isometrically invariant measures on Euclidean spaces

The study of isometrically invariant measures on the Euclidean space $\mathbb{R}^n$ is particularly appealing, since it is directly connected to the search for a "good" notion of a measure that would generalize the notions of the length of an interval, the area of a region and the volume of a solid. Though the Lebesgue measure $\lambda_n : \mathcal{L}_n \to [0,\infty]$ has been widely accepted as the ultimate solution to the problem of finding a satisfactory way to "measure" complicated subsets of $\mathbb{R}^n$, we have to face the fact that some Lebesgue nonmeasurable sets still exist (at least if we accept the axiom of choice – see Wagon (1986, Chapter 13) for a discussion of this point). One way to cope with this disadvantage is to consider isometrically invariant extensions of λ_n.

Let G be a subgroup of $\mathrm{Iso}(\mathbb{R}^n)$ – the group of all isometries of $\mathbb{R}^n$. Recall from Section 3 that given an invariant σ-finite measure μ defined on a σ-algebra $\mathcal{A}$ of subsets of $\mathbb{R}^n$ there are two basic ways of obtaining an invariant extension $\bar{\mu}\colon \mathcal{A}' \to [0,+\infty]$ of μ.

The first is to find a nonmeasurable set such that the G-invariant σ-ideal $\mathcal{I}$ it generates in $\mathcal{P}(\mathbb{R}^n)$, consists of sets of inner μ-measure zero. The extension $\bar{\mu}$ is then defined on the σ-algebra generated by $\mathcal{A} \cup \mathcal{I}$ so that $\bar{\mu}(N) = 0$ for any $N \in \mathcal{I}$ (see Szpilrajn (1935)).

The second boils down to finding a μ-almost invariant set $A \notin \mathcal{A}$ and then defining $\bar{\mu}$ on the σ-algebra generated by $\mathcal{A}$ and A (see J. Łoś and E. Marczewski (1949)).

The existence of a proper isometrically invariant extension of λ_n was proved by Szpilrajn (Marczewski) in (1935), using the first method. Szpilrajn constructed a generalized *Bernstein set* $Z \subseteq \mathbb{R}^n$ with the property that any countable union of isometric copies of Z, is neither contained in nor disjoint from an arbitrary nonempty perfect subset P of $\mathbb{R}^n$. In the same paper Szpilrajn stated the problem posed by Sierpiński, whether *every* isometrically invariant extension of Lebesgue measure in $\mathbb{R}^n$ has a proper isometrically invariant extension. It is well known that if the continuum is large, then there exists an extension of λ_n to a universal measure, i.e., a measure defined on $\mathcal{P}(\mathbb{R}^n)$ (see e.g. Fremlin (1993, 1D)). However, no such extension can be isometrically invariant. Indeed, the classical Vitali subset of $\mathbb{R}^n$ is nonmeasurable with respect to any translation-invariant extension of λ_n. (In fact, by a theorem of Kharazishvili (1978), *every set of positive Lebesgue measure contains a subset nonmeasurable with respect to any translation invariant extension of* λ_1). Moreover, by Theorem 2.2, *no universal, translation invariant, σ-finite measure on $\mathbb{R}^n$ exists at all.* Thus no isometrically invariant extension of λ_n is universal and Sierpiński's problem amounts to asking if a maximal one exists.

The negative answer, i.e., the theorem that *no maximal isometrically invariant measure on $\mathbb{R}^n$ exists* was first obtained independently by S.S. Phakadze (1958) and A. Hulanicki (1962) under the additional assumption that the continuum is small (this is closely related to Theorem 3.1). Then Kharazishvili (1980) proved it without any additional set-theoretical assumptions for the one dimensional case. Finally, K. Ciesielski and A. Pelc (1985) got the negative answer in any dimension, showing even more than was required.

Theorem 6.1. *Suppose G is any subgroup of Iso($\mathbb{R}^n$) containing all translations. Then every G-invariant σ-finite measure on $\mathbb{R}^n$ has a proper G-invariant extension.*

The Ciesielski and Pelc's proof is also based on the first method of extending invariant measures, described above. More precisely, it is showed that *the space $\mathbb{R}^n$ is a countable union of G-absolutely negligible sets*, where a set $N \subseteq \mathbb{R}^n$ is called *G-absolutely negligible* if the G-invariant σ-ideal, it generates in $\mathcal{P}(\mathbb{R}^n)$, consists of sets of inner measure zero with respect to *every* G-invariant σ-finite measure on $\mathbb{R}^n$. Now, given a σ-finite G-invariant measure $\mu : \mathcal{A} \to [0, \infty]$ on $\mathbb{R}^n$, if $\mathbb{R}^n = \bigcup_{i \in \mathbb{N}} N_i$, where each N_i is G-absolutely negligible, then $N_{i_0} \notin \mathcal{A}$ for at least one $i_0 \in \mathbb{N}$.

The idea of partitioning $\mathbb{R}^n$ into countably many G-absolutely negligible sets is due to Kharazashvili (1983) who used it for the case when G is the group of all translations of $\mathbb{R}^n$. A more detailed account of the long history of Sierpiński's problem can be found in Ciesielski (1990) or (1989).

In view of Theorem 6.1, Ciesielski and Pelc (1985) stated the problem of characterizing those subgroups G of Iso($\mathbb{R}^n$) for which the conclusion of their result remains true.

Using ideas of Hulanicki (1962) they noticed that if the continuum is small, then no restrictions on G are needed.(For the reasons, explained at the beginning of Section 3, we will assume that all measures, we want to extend, are diffused.)

Ciesielski (1990) proved, in particular, that it is enough to require that either the action of G is transitive (this was earlier proved by Kharazishvili under the additional assumption of the Continuum Hypothesis) or G contains uncountably many translations. In fact, in the same paper Ciesielski studied a more general question: given a subgroup G of Iso($\mathbb{R}^n$) and a σ-finite G-invariant diffused measure μ on $\mathbb{R}^n$ does there exist a proper G-invariant extension of μ? He proved that it is so, provided that G contains an uncountable subset H with the property that every two different elements of H coincide exactly on a set of inner measure μ zero. Finally, Zakrzewski (1990) noticed how to apply the key technical tool from Ciesielski's paper to get the following strengthening.

Theorem 6.2. *Suppose G is a subgroup of Iso($\mathbb{R}^n$) and let μ be a σ-finite G-invariant diffused measure on $\mathbb{R}^n$. If μ does not vanish on the set of points with uncountable G-orbits, then μ has a proper G-invariant extension.*

As a corollary we have the following solution to Ciesielski-Pelc's problem.

Theorem 6.3. *Let G be a subgroup of Iso($\mathbb{R}^n$). The following are equivalent.*

(i) *Every G-invariant σ-finite diffused measure on $\mathbb{R}^n$ has a proper G-invariant extension;*

(ii) *Either the continuum is small or exactly one of the following holds:*

- *All G-orbits are uncountable;*
- *There exists a common fixed point of all elements of G and all other G-orbits are uncountable.*

Theorem 6.3 reduces the problem of finding necessary and sufficient conditions for extendability of a given G-invariant σ-finite diffused measure on $\mathbb{R}^n$ to the case in which G is countable. This case was fully examined by Krawczyk and Zakrzewski (1991) as a particular instance of the general problem, whether a given σ-finite measure, invariant with respect to a countable group of permutations of a set X, has a proper invariant extension. The solution to the latter was presented in Section 3 (see theorems 3.3 and 3.4). Combining it with Theorem 6.3 leads, in particular, to the following characterization.

Theorem 6.4. *Let G be a subgroup of $Iso(\mathbb{R}^n)$. The following are equivalent.*

(i) *Every nonuniversal G-invariant σ-finite diffused measure on $\mathbb{R}^n$ has a proper G-invariant extension;*

(ii) *The set of all points $x \in \mathbb{R}^n$ such that the G-orbit Gx is countable and there is a nontrivial minimal G-partition of Gx is small.*

On the other hand, Krawczyk and Zakrzewski (1991) obtained the following theorem, related to the original Sierpiński's problem.

Theorem 6.5. *Suppose G is any subgroup of $Iso(\mathbb{R}^n)$. Then every nonuniversal G-invariant σ-finite extension of the Lebesgue measure in $\mathbb{R}^n$ has a proper G-invariant extension.*

Note that when we consider arbitrary groups of isometries, we have to assume that a G-invariant extension of λ_n, we want to extend, is nonuniversal. Indeed, as has already been mentioned, if the continuum is large, then λ_n can be extended to a universal measure. But as a matter of fact such an extension is rather poorly invariant with respect to isometries, as another result from Krawczyk and Zakrzewski (1991) shows.

Proposition 6.6. *Assume that the continuum is large and let G be a subgroup of $Iso(\mathbb{R}^n)$. Then the following are equivalent.*

(i) *There is an extension of the Lebesgue measure to a G-invariant universal measure on $\mathbb{R}^n$;*

(ii) *The group G is discrete.*

Here "discrete" refers to the topology inherited from $\mathrm{Iso}(\mathbb{R}^n)$ and is equivalent to the fact that each G-orbit consists of isolated points. For example, we can imagine an extension of the Lebesgue measure to a universal measure invariant with respect to all translations by the integers but this definitely is a very weak property.

To emphasize the fact that the non-discreteness of a group G implies that certain extensions of λ_n, even by countably many new sets, cannot be G-invariant, let us quote an observation from Zakrzewski (1991b): *If G is a non-discrete subgroup of Iso($\mathbb{R}^n$), then every open set $A \subseteq \mathbb{R}^n$ is countably G-equidecomposable with $\mathbb{R}^n$ (in arbitrary pieces).*

Until now we discussed the basic question, whether a given invariant measure can be properly extended in the invariant way. But there are also some interesting results concerning particular properties of such extensions. For example, if we move from measure spaces to the respective measure algebras, then we see that extending a measure $\mu\colon \mathcal{A} \to [0, +\infty]$ to $\bar{\mu} : \mathcal{A}' \to [0, +\infty]$ does not necessarily mean extending the associated measure algebra. More precisely, it may happen that

$$\forall Y \in \mathcal{A}' \; \exists X \in \mathcal{A} \quad \bar{\mu}(Y \triangle X) = 0. \tag{$*$}$$

This means that the canonical embedding $e\colon \mathcal{A}/\mu \to \mathcal{A}'/\bar{\mu}$ of measure algebras defined by $e([A]_\mu) = [A]_{\bar{\mu}}$ is an isomorphism. Let us call such an extension *weak*, otherwise call it *strong*.

Clearly, the first of the two methods of extension, recalled at the beginning of this section, produces a weak extension, while the second gives a strong one. It was already realized by Szpilrajn (1935) that *there exists a strong isometrically invariant extension of the Lebesgue measure λ_n on $\mathbb{R}^n$.*

Every $\mathrm{Iso}(\mathbb{R}^n)$-invariant weak extension of λ_n is $\mathrm{Iso}(\mathbb{R}^n)$-ergodic. Kharazishvili (1983) proved that *there exists a strong Iso($\mathbb{R}^n$)-invariant extension of λ_n which is Iso($\mathbb{R}^n$)-ergodic.*

Another property of a weak extension is that it does not change the *character* of the measure space $(\mathbb{R}^n, \mathcal{A}, \mu)$, i.e., the smallest cardinality κ of a collection $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{A}$ such that for any set $A \in \mathcal{A}$ and every $\varepsilon > 0$ there is $\alpha < \kappa$ with $\mu(A \triangle A_\alpha) < \varepsilon$.

For the Lebesgue measure the character is $\aleph_0$ and the question, whether there exists a translation invariant extension of the Lebesgue measure on $\mathbb{R}$ of uncountable character, was already stated by Szpilrajn (1935). The affirmative answer was given by K. Kodaira and S. Kakutani (1950). In fact, Kakutani and Oxtoby (1950) constructed a translation invariant extension of λ_1 of the maximal possible character (see also Hewitt and Ross (1963, 16.3), where a generalization to the case of Haar measure on a compact metric group is proved).

Theorem 6.7. *There exists a translation invariant extension of the Lebesgue measure on $\mathbb{R}$ of the character $2^{2^{\aleph_0}}$.*

Using the same technique Kharazishvili (1978) proved that *there is an Iso*$(\mathbb{R}^n)$*-invariant extension* $\bar{\mu}\colon \mathcal{A}' \to [0, +\infty]$ *of the Lebesgue measure on* $\mathbb{R}^n$ *of the character* $2^{2^{\aleph_0}}$ *such that no restriction of* $\bar{\mu}$ *to an Iso*$(\mathbb{R}^n)$*-almost invariant (with respect to* $\bar{\mu}$*) set* $Y \in \mathcal{A}'$ *is ergodic.*

After realizing that the methods by which Sierpiński's problem was solved lead to weak extensions, Ciesielski (1990/ 91) posed the problem, whether every isometrically invariant σ-finite measure on $\mathbb{R}^n$ has a strong isometrically invariant extension. The affirmative answer is given in Zakrzewski (1995/96), where it is also proved that *if* G *is any subgroup of Iso*$(\mathbb{R}^n)$*, then every complete* σ*-finite* G*-invariant measure on* $\mathbf{R}^n$ *which has a proper* G*-invariant extension, has a strong* G*-invariant extension as well.*

We conclude this section with another result from Ciesielski and Pelc (1985): *If the continuum is large, then there exists a universal semi-finite isometrically invariant measure on* $\mathbb{R}^n$.

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