SOME SET-THEORETIC ASPECTS OF MEASURE THEORY

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Abstract. This article presents a survey of results concerning some set-theoretic problems arising from the attempts to find a generalization of the notions of length, area and volume, so that as many subsets of the space as possible could be "measured".

1.Introduction

The historical roots of measure theory go back to antiquity, since it has originated from problems concerning three fundamental notions: the length of an interval, the area of a region and the volume of a solid. The development of mathematics made it necessary to generalize these notions, so that more complicated subsets of a given *space* (the real line $\mathbb{R}$, the plane $\mathbb{R}^2$, the space $\mathbb{R}^3$ or, in general, the n-dimensional Euclidean space $\mathbb{R}^n$) could be "measured". Such a generalization, let us call it a *measure* should, of course, satisfy some reasonable conditions:

- the measure of simple sets, such as intervals (squares, cubes, respectively) must be equal to their length (area, volume, respectively); in particular the points of the space must have measure zero,
- the measure should be an *additive* function, i.e., the measure of the union of two (or finitely many) disjoint sets must be equal to the sum of their measures,
- the measure of a set should not depend on its position in the space, i.e., the measure should assign equal values to congruent sets.

A partial solution to the problem of finding a "good" notion of a measure was so-called *Jordan measure*, constructed in 1890 by Peano and Jordan and closely connected to the Riemann integral. The idea behind the Jordan measure is to approximate a given set from within and from outside by finitely many elementary figures (see [15], Appendix B). It turns out that the domain of Jordan measure consists of such bounded subsets A of $\mathbb{R}^n$ whose boundary $\delta(A)$ (i.e., the difference between the closure and the interior of A) is small in the following sense: for every $\varepsilon > 0$ there is a pairwise interior-disjoint collection of finitely many cubes K_i, $i < m$ such that $\delta(A)$ is covered by the union of K_i's and $\sum\limits_{i<m} \mathrm{volume}(K_i) < \varepsilon$. Though the collection, just described, of *Jordan measurable* sets seems large and is indeed large enough from the point of view of the theory of Riemann integral, it is easy to see that it does not even contain all open subsets of $\mathbb{R}^n$.

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But in 1902 Lebesgue invented a different notion of a measure, widely accepted today as the ultimate solution to the problem of finding a satisfactory way to "measure" complicated sets. In contrast to the Jordan measure, the approximation figures may now also consist of countably many elementary areas and the most important new feature of the *Lebesgue measure* is its *countable additivity*, i.e., the measure of the union of countably many pairwise disjoint sets must be equal to the sum of their measures. This is precisely what is needed to develop the theory of the Lebesgue integral.

It turns out that the n-dimensional Lebesgue measure m is the unique object satisfying a short list of reasonable conditions. Strictly speaking, m is the unique non-negative function defined on the collection $\mathcal{L}$ of the so-called *Lebesgue measurable* subsets of $\mathbb{R}^n$ such that:

1. $\mathcal{L}$ is a σ-*algebra* of subsets of $\mathbb{R}^n$, i.e., it is a collection of subsets of $\mathbb{R}^n$ closed under taking complements and countable unions:
   if $A \in \mathcal{L}$, then $\mathbb{R}^n \setminus A \in \mathcal{L}$,
   if $A_i \in \mathcal{L}$ for $i = 0, 1, \dots$, then $\bigcup\limits_{i \in \mathbb{N}} A_i \in \mathcal{L}$.
2. $\mathcal{L}$ contains all open subsets of $\mathbb{R}^n$, hence, by 1., also all closed, F_σ, G_δ etc., sets, i.e., all *Borel* subsets of $\mathbb{R}^n$,
3. $\mathcal{L}$ is closed under taking subsets of null sets, i.e., if $A \in \mathcal{L}$, $m(A) = 0$ and $B \subseteq A$, then $B \in \mathcal{L}$ (and then, of course, $m(B) = 0$),
4. $\mathcal{L}$ is the smallest σ-algebra satisfying the two preceding conditions. In particular, every element of $\mathcal{L}$ differs from a Borel set by a set of measure zero, i.e., $A \in \mathcal{L}$ if and only if there is a Borel set B such that $m(A \triangle B) = 0$, where $\triangle$ denotes the symmetric difference of A and B: $A \triangle B = (A \setminus B) \cup (B \setminus A)$,
5. m is countably additive, i.e., if sets $A_i \in \mathcal{L}$ for $i = 0, 1, \dots$ are pairwise disjoint, then $m(\bigcup\limits_{i \in \mathbb{N}} A_i) = \sum\limits_{i=0}^{\infty} m(A_i)$,
6. $m(I) = 1$, where I is the unit cube in $\mathbb{R}^n$,
7. m is isometrically invariant, i.e., if $A \in \mathcal{L}$ and g is an isometry of $\mathbb{R}^n$, then $g[A] \in \mathcal{L}$ and $m(g[A]) = m(A)$, where $g[A]$ denotes the image of A under g.

It is well-known that Lebesgue measure is a proper extension of the Jordan measure: Lebesgue and Jordan measures assign the same value to every Jordan-measurable set and $\mathcal{L}$ contains many new subsets of $\mathbb{R}^n$. In fact, as we have already pointed out, the Lebesgue measure measures just enough sets from the point of view of the integral calculus. But we may still ask, how good Lebesgue measure really is:

- Does it measure *all* subsets of $\mathbb{R}^n$ or perhaps some *non-measurable* sets still exist?
- If not all subsets of $\mathbb{R}^n$ are measurable, then which sets are measurable and can we extend Lebesgue measure to capture still more sets?
- How about extending Lebesgue measure ultimately, i.e., to a measure defined on the power set $\mathcal{P}(\mathbb{R}^n)$ of the space?

Since these questions involve the unrestricted notion of an *arbitrary subset* of the space, Set Theory enters the scene.

2. The existence of non-measurable sets

The commonly accepted axioms of set theory, known as ZFC (Zermelo-Fraenkel set theory ZF with the Axiom of Choice AC), imply that Lebesgue measure does not measure all sets: there are subsets of $\mathbb{R}^n$ which are non-measurable. A standard example of a set not in $\mathcal{L}$, often presented at more advanced courses of the integral calculus, was given by Vitali in 1905 and is therefore known as the Vitali set.

Let us recall the Vitali's construction (for $n = 1$, which we can do with no loss of generality).

First consider the family $\mathcal{P}$ consisting of all translates of the set of rationals, i.e., sets of form $x + \mathbb{Q} = \{x + q : \ q \in \mathbb{Q}\}$, where $\mathbb{Q}$ denotes the set of all rational numbers and $x \in \mathbb{R}$. It is not difficult to check that $\mathcal{P}$ is a partition of $\mathbb{R}$: it consists of pairwise-disjoint non-empty sets which cover $\mathbb{R}$. Indeed, if $(x_1 + \mathbb{Q}) \cap (x_2 + \mathbb{Q}) \neq \emptyset$ and $x \in \big((x_1 + \mathbb{Q}) \cap (x_2 + \mathbb{Q})\big)$, then $x_1 - x_2 = (x_1 - x) + (x - x_2) \in \mathbb{Q}$, so $x_1 + \mathbb{Q} = x_2 + \mathbb{Q}$.

Now form a *selector* S of $\mathcal{P}$ by picking up exactly one element from each member of $\mathcal{P}$. What you get is precisely a Vitali set. Since every set of the form $x + \mathbb{Q}$ is dense in $\mathbb{R}$, for technical reasons we may additionally assume that $S \subseteq [0, 1]$.

To convince yourself that S is nonmeasurable, argue as follows.

Suppose otherwise, i.e., assume that $S \in \mathcal{L}$; then it makes sense to consider $m(S)$ and there are two cases:

Case 1. $m(S) = 0$. Then the invariance of m under all translations yields $m(q + S) = 0$ for every $q \in \mathbb{Q}$. But it is easy to see that $\mathbb{R} = \bigcup_{q \in \mathbb{Q}} (q + S)$.

Indeed, for a given $x \in \mathbb{R}$ take the unique $s \in S \cap (x + \mathbb{Q})$; then $x = q + s$ for a certain $q \in \mathbb{Q}$.

Finally, since the set of rationals is countable, the countable additivity of m implies that

$$m(\mathbb{R}) = m\Big(\bigcup_{q \in \mathbb{Q}} (q + S)\Big) = \sum_{q \in \mathbb{Q}} m(q + S)) = 0,$$

which is absurd.

Case 2. $m(S) > 0$. In this case consider the collection $\{q + S : \ q \in \mathbb{Q} \cap [0, 1]\}$ consisting of the translates of S by the rationals from the interval $[0, 1]$.

Then, on one hand, since $\bigcup_{q \in \mathbb{Q} \cap [0,1]} (q + S) \subseteq [0, 2]$, we have

$$(*) \qquad m\Big(\bigcup_{q \in \mathbb{Q} \cap [0,1]} (q + S)\Big) \leq 2.$$

But on the other hand, an important property of S is that the translates of S by different rationals are disjoint:

$$q_1 \neq q_2 \text{ implies that } (q_1 + S) \cap (q_2 + S) = \emptyset.$$

Indeed, suppose that $(q_1 + S) \cap (q_2 + S) \neq \emptyset$, i.e., for some $s_1, s_2 \in S$, $q_1 + s_1 = q_2 + s_2$. Then $s_1,\ s_2 \in s_1 + \mathbb{Q}$, so by the choice of S, $s_1 = s_2$, which immediately gives $q_1 = q_2$.

Hence it suffices to take $k > \frac{2}{m(S)}$ different rationals $q_1, \ldots q_k$ from $[0,1]$ to get

$$(**) \qquad m(\bigcup_{i=1}^{k}(q_i + S)) = \sum_{i=1}^{k} m(q_i + S) = k \cdot m(S) > 2,$$

which contradicts inequality $(*)$.

A careful reader must have noticed the non-constructive nature of the proof above. We neither described S as the collection of objects satisfying a certain property nor gave a general rule of picking elements of S out of sets forming the partition $\mathcal{P}$. We just used the fact, that since all these sets are non-empty, a selector "obviously" exists. But to justify this reasoning we need precisely the Axiom of Choice (AC). One of the many equivalent formulations of AC states exactly this: for any family of non-empty, pairwise disjoint sets there exists a *selector*, i.e., a set which has exactly one common point with each member of the family under consideration.

So we can see that the Vitali's construction uses AC in an apparently unavoidable way. But what about other examples of non-measurable sets? Is it at all possible to construct such a set effectively, without the use of AC?

It turns out that the answer is "NO"! In 1964 Solovay proved that in the ZF set theory one cannot prove the existence of non-measurable sets (see [15], Chapter 13 or [8], Section 42 or [10], Section 11). This means that we can consider a consistent mathematical universe in which all subsets of $\mathbb{R}^n$ are Lebesgue measurable. But there two prices to be paid : a mathematical and a metamathematical one.

The mathematical price is that this requires throwing away the Axiom of Choice and without AC mathematics looks very strange. Indeed, many parts of modern mathematics make an essential use of AC . Zorn's lemma, "every set can be well ordered", Tychonoff's product theorem for compact spaces, "every vector space has a basis", existence and uniqueness of algebraic closure of a field, the Hanh-Banach extension theorem – this is just a sample of numerous important theorems whose proofs appeal to AC (see [7] for a thorough discussion of this point).

The metamathematical price is also worth mentioning, though its detailed analysis is beyond the scope of this article.

A very important part of modern set theory is devoted to consistency proofs. To say that a certain statement A (such as, for example, the claim:

"all subsets of $\mathbb{R}^n$ are Lebesgue measurable") is consistent with set theory (typically either ZF or ZFC) means that the negation of A cannot be derived from the other axioms or, equivalently, that adding A to them does not lead to a contradiction (we believe, of course, that no contradiction can be derived from ZF itself; in 1938 Gödel proved that AC is consistent with ZF – see [8], Section 13). A method to prove this is to create a model, i.e., a universe in which all the axioms and the statement A are true. But a construction of such a model takes place within set theory and sometimes requires additional set theoretical assumptions to be carried out. It turns out that these assumptions are often formulated as so-called large cardinal axioms, which claim the existence of sets that are large in some sense.

In our case, Solovay's construction of a model for ZF + "all subsets of $\mathbb{R}^n$ are Lebesgue measurable" uses the assumption, that there exists an uncountable inaccessible cardinal, i.e., an uncountable cardinal κ which is not a successor cardinal, nor a sum of fewer than κ cardinals smaller than κ (see [8], p.28). In 1979 Shelah proved that this is unavoidable: the existence of an inaccessible cardinal is necessary to construct a model in which all sets of reals are Lebesgue measurable.

Let LM denote the assertion that all subsets of $\mathbb{R}^n$ are Lebesgue measurable (the dimension n does not really matter) and let IC abbreviate the statement that there exists an uncountable inaccessible cardinal. Thus the results of Solovay and Shelah may be summarized as follows: the theory ZF+LM is equiconsistent with the theory ZF+IC. Since the latter is known to be essentially stronger than ZF (no model for ZF+IC can be constructed by means of ZF alone), ZF set theory together with the assertion that there are no Lebesgue non-measurable sets is, in a sense, slightly more vulnerable to contradiction than set theory without it. We say that the consistency strength of LM, measured on the large cardinal scale, is exactly that of IC.

3. Which sets are Lebesgue measurable?

From now on let us stick to "ordinary" ZFC set theory which includes AC as one of its axioms. Then, since not all subsets of $\mathbb{R}^n$ are Lebesgue measurable, the question arises, which sets actually are in $\mathcal{L}$. As we know from the preceding section, all constructions of non-measurable sets require "non-effective" methods that make an essential use of AC. So, since we cannot "define" a non-measurable set, we may hope that all "definable" sets are Lebesgue measurable.

The part of set theory which deals with "definable sets" in spaces like $\mathbb{R}^n$ is called the descriptive set theory. In this theory sets are classified according to the complexity of their definitions.

At the lowest stage there are *Borel* sets, i.e., sets from the smallest σ-algebra of subsets of the space, containing all open sets. Thus Borel sets are obtained from open sets by the successive use of the operations of complementation and countable union.

One of the basic properties of Lebesgue measure, listed in Section 1, is that all Borel subsets of $\mathbb{R}^n$ are Lebesgue measurable. In fact, the restriction of m to Borel sets together with the property that $\mathcal{L}$ is closed under taking subsets of measure zero sets, completely determines $\mathcal{L}$: a set $A \subseteq \mathbb{R}^n$ belongs to $\mathcal{L}$ if and only if there exists a Borel set B such that the symmetric difference $A \triangle B$ may be covered by a Borel set of measure zero.

Beyond the Borel sets there are sets obtained from them by the successive use of the operations of continuous image and complementation. Their study began from a crucial discovery of Suslin who, in 1916, noticed, that Lebesgue made in one of his works the mistaken claim that the projection of a Borel subset of the plane is also a Borel set. Suslin's counterexample to Lebesgue's assertion lead to the study of *analytic* sets, which are precisely the continuous images (or, equivalently, projections) of Borel sets and form the lowest level of the hierarchy of so-called *projective sets.* The projective hierarchy consists of analytic or **A** sets, their complements, known as *co-analytic* or **CA** sets, **PCA** sets (the continuous images or, equivalently, projections of co-analytic sets), **CPCA** sets (complements of **PCA**) and so on.

Are projective sets measurable? It turns out that without additional axioms we can decide it only for the first level of the projective hierarchy.

Namely, in 1917, Lusin proved that all analytic, hence also all co-analytic, sets are Lebesgue measurable (see [11], Chapter III).

In 1938 Gödel built a model, called the constructible universe, in which all the ZFC axioms are true but there exists a non-measurable **PCA** set (see [8], Section 41). On the other hand, Martin and Solovay showed, in 1970, the consistency of ZFC plus the proposition that every **PCA** set is Lebesgue measurable; the latter follows, in fact, from the so-called Martin's Axiom plus the negation of the Continuum Hypothesis.

The two theorems just quoted, show that the assertion that every **PCA** set is Lebesgue measurable is independent from ZFC, i.e., the addition of either this statement or its negation yields a noncontradictory system. It is remarkable that proofs of these consistency results do not use large cardinal hypotheses. But, strangely enough, to prove the consistency of ZFC+"all **CPCA** sets are Lebesgue measurable", we must go beyond ZFC. Namely, a modification of Solovay's construction of a model for ZF+LM, carried out with the help of an inaccessible cardinal (see Section 2), gives a model for ZFC+"all projective sets are Lebesgue measurable". Shelah proved, in turn, that the use of IC is unavoidable even if we only want to establish the consistency of ZFC +"all **CPCA** sets are Lebesgue measurable".

Summing up: we may believe that all projective sets are measurable, but the consistency strength of this assertion, measured on the large cardinal scale, is again exactly that of IC.

4. Countably additive extensions of the Lebesgue measure

Since, as pointed out in the preceding sections, the Lebesgue measure leaves many, even simply-defined sets, out of its domain, the question arises if we can *extend* it, so that some new sets become measurable.

This means extending $\mathcal{L}$, the domain of m, to a σ-algebra $\mathcal{M}$, $\mathcal{L} \subsetneq \mathcal{M} \subseteq \mathcal{P}(\mathbb{R}^n)$, of subsets of $\mathbb{R}^n$ and defining a countably additive function $\mu : \mathcal{M} \to [0, \infty]$ such that $\mu(A) = m(A)$ for every $A \in \mathcal{L}$. We can either simply require that $\mathcal{L}$ is properly contained in $\mathcal{M}$ or demand that $\mathcal{M}$ captures a given collection (finite, countable, etc.) of new sets. It would of course be optimal to capture all non-measurable sets, i.e., to extend m to $\mathcal{M} = \mathcal{P}(\mathbb{R}^n)$.

Moreover, we would like the measure μ to be as isometrically invariant as possible.

More precisely, given a subgroup G of the group D_n of all isometries of R^n, a measure μ is called *G-invariant*, if for every isometry $g \in G$, $A \in \mathcal{M}$ implies $g[A] \in \mathcal{M}$ and $\mu(g[A]) = \mu(A)$.

The Lebesgue measure m is itself D_n-invariant and when we try to extend it, we always want an extension to be G-invariant for G as large as possible.

It turns out, however, that even if we completely forget about the invariance requirements, there are still problems, we have to face.

An old technique of Łoś and Marczewski [13] guarantees that we can always extend the Lebesgue measure to a given finite collection of new sets. But already the question, if the same is true for countable collections, cannot be decided on the basis of ZFC.

On one hand, in 1929, Banach and Kuratowski [1] showed that the Continuum Hypothesis (which states that every infinite subset of $\mathbb{R}$ is either countable or has the cardinality of the continuum; CH is well-known to be independent of ZFC) implies the existence of a collection $\{A_i : i \in \mathbb{N}\}$ of subsets of $\mathbb{R}^n$ which cannot be included in the domain of any countably additive measure that extends m.

On the other hand, in 1984, Carlson [2] showed that it is consistent with ZFC to assume that for every countable collection $\{A_i : i \in \mathbb{N}\}$ of subsets of $\mathbb{R}^n$ there is a countably additive extension $\mu : \mathcal{M} \to [0, \infty]$ of m with $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{M}$.

Thus the assertion that Lebesgue measure can always be extended to countably many new sets is independent of ZFC.

So, finally, how about assuming the existence of an extension of m to a countably additive measure defined on $\mathcal{P}(\mathbb{R}^n)$, i.e., to a *universal* measure on $\mathbb{R}^n$?

This also does not seem to lead to a contradiction with ZFC but the metamathematical price to be paid for it is much higher than that for claiming (in the absence of AC, of course) that the Lebesgue measure is itself universal (see Section 2). Namely, by results of Ulam and Solovay, the consistency strength of ZFC+"Lebesgue measure can be extended to a countably additive universal measure", measured on the large cardinal scale, is precisely that of MC, the assertion claiming the existence of the so-called measurable cardinal (see [9], Chapter 23). Such a cardinal is a very very large cardinal indeed. In particular, it is inaccessible and there are plenty of inaccessible cardinals below it. Nevertheless, since it is generally felt that MC is consistent with ZFC, the existence of a universal countably additive extension of the Lebesgue measure is a conceivable hypothesis, whose consequences have been intensively studied for the last three decades (see [6] for a comprehensive survey of contemporary knowledge about this and related subjects).

But how much invariance can we expect from various extensions of the Lebesgue measure, provided that they exist? Well, there are some good and some bad news.

The good news is that according to a result of Marczewski [14], proved in 1935, we can always properly extend m to a D_n-invariant countably additive measure.

Moreover, there is no maximal D_n-invariant extension of m: answering a question posed by Sierpiński in 1936, Ciesielski and Pelc [3] proved, in 1982, that every D_n-invariant countably additive additive extension of the Lebesgue measure has a proper D_n-invariant extension (see [5] or [4] for an account of the long history of this problem and [16] for its generalization).

The bad news is that if we want to extend m by particular given sets, then we cannot expect much invariance from an extension.

First of all, from the proof that the Vitali set, considered in Section 1, is not in $\mathcal{L}$, also shows that it cannot be included in the domain of any D_n-invariant countably additive extension of m. In particular, no universal countably additive extension of m is D_n-invariant (remember that we are now working in ZFC, so this does not contradict the consistency of ZF + "the Lebesgue measure is universal").

In fact, by a result of Krawczyk and Zakrzewski [12], if a universal countably additive extension of m exists, then for a group G of isometries of $\mathbb{R}^n$, the following are equivalent:

1. there is an extension of the Lebesgue measure to a G-invariant countably additive universal measure on $\mathbb{R}^n$,
2. G is discrete, i.e., the G-orbit $\{g(x) : g \in G\}$ of any point $x \in \mathbb{R}^n$ consists of isolated points.

For example, we can imagine an extension of the Lebesgue measure to a universal measure invariant with respect to all translations by the integers but this definitely is a very weak property.

Likewise, Zakrzewski [17] proved that if m can be extended to an arbitrary countable family of subsets of $\mathbb{R}^n$ (this is consistent, by the result of Carlson, quoted above), then for a group G of isometries of $\mathbb{R}^n$, the following are equivalent:

1. for every countable collection $\{A_i : i \in \mathbb{N}\}$ of subsets of $\mathbb{R}^n$ there is a countably additive G-invariant extension $\mu : \mathcal{M} \to [0, \infty]$ of m with $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{M}$,

2. G is discrete.

Finally, to emphasize the fact that the non-discreteness of a group G implies that certain countably additive extensions of m, even by countably many new sets, cannot be G-invariant, let us quote another result from Zakrzewski [17]:

If G is a non-discrete subgroup of D_n, then for every $\varepsilon > 0$ there is an open set $A \subseteq \mathbb{R}^n$ such that $m(A) = \varepsilon$ and there is a partition of A into countably many pieces $\{A_i : i \in \mathbb{N}\}$ which can be rearranged by some isometries $\{g_i : i \in \mathbb{N}\}$ from G to form a partition $\{g_i[A_i] : i \in \mathbb{N}\}$ of $\mathbb{R}^n$.

Now, if there was a countably additive G-invariant extension $\mu : \mathcal{M} \to [0, \infty]$ of m with $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{M}$, then we would have:

$$\begin{aligned}\varepsilon = m(A) = m(\bigcup_{i\in\mathbb{N}} A_i) = \mu(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i=0}^{\infty} \mu(A_i) \\ = \sum_{i=0}^{\infty} \mu\big(g_i[A_i]\big) = \mu(\bigcup_{i\in\mathbb{N}} g_i[A_i]) = \mu(\mathbb{R}^n) = m(\mathbb{R}^n) = \infty,\end{aligned}$$

which is a clearly absurd.

5. Finitely additive extensions of the Lebesgue measure

In the preceding section we considered only countably additive extensions of m. Although countable additivity is a crucial property of a measure from the point of view of the integral theory, measures which are only *finitely* additive also play a very important role in various parts of mathematics (see [15]).

Recall that $\mu : \mathcal{M} \to [0, \infty]$ is finitely additive, if whenever finitely many sets $A_i \in \mathcal{M}$, $i < k$, are pairwise disjoint, $\mu(\bigcup_{i<k} A_i) = \sum_{i<k} \mu(A_i)$; in this case the domain $\mathcal{M}$ of μ is an *algebra* of subsets of $\mathbb{R}^n$, i.e., it is closed under taking complements and *finite* unions.

Our hope, that relaxing the requirement of countable additivity might perhaps change the answers to questions concerning invariant extensions of the Lebesgue measure, is based on at least two facts.

Firstly, the proof that the Vitali set is non-measurable with respect to any D_n-invariant extension μ of m uses the countable additivity of μ in an apparently unavoidable way.

Secondly, using Zorn's Lemma it is not difficult to show the existence of an extension of m to a *universal* finitely additive measure, i.e., to a finitely additive measure defined on $\mathcal{P}(\mathbb{R}^n)$ (whatsoever, no invariance is required).

In view of the latter, let us immediately attack the most urgent question:

- Does there exist an extension of the Lebesgue measure to a universal finitely additive D_n-invariant measure?

Or, more generally:

- For which groups G of isometries of $\mathbb{R}^n$ does there exist a universal, finitely additive G-invariant extension of m?

The answer to the first question is very surprising, since it depends on the dimension n of the space.

On one hand, for $n = 1$ or 2 the answer is "YES": Banach, in 1923, proved that there is an isometry invariant finitely additive universal extension of the Lebesgue measure on the line and on the plane (see [15], Chapter 10).

On the other hand, for $n \geq 3$, the answer is "NO", due to a spectacular result of Hausdorff, Banach and Tarski, known as the Banach-Tarski Paradox: Given any two bounded sets $A,\ B \subseteq \mathbb{R}^n$, each having nonempty interior, there is a partition of A into finitely many pieces $\{A_i : i < k\}$ which can be rearranged by some isometries $\{g_i : i < k\}$ to form a partition $\{g_i[A_i] : i < k\}$ of B (see [15] for a comprehensive survey of this and related results).

Now, if A and B are, say, two balls in $\mathbb{R}^n$, $n \geq 3$, with different radii, then $m(A) \neq m(B)$ and there is even no finitely additive D_n-invariant extension $\mu : \mathcal{M} \to [0,\infty]$ of m with $\{A_i : i < k\} \subseteq \mathcal{M}$. Indeed, otherwise we would have:

$$m(A) = m(\bigcup_{i<k} A_i) = \mu(\bigcup_{i<k} A_i) = \sum_{i<k} \mu(A_i)$$
$$= \sum_{i<k} \mu\big(g_i[A_i]\big) = \mu(\bigcup_{i<k} g_i[A_i]) = \mu(B) = m(B)$$

– a contradiction.

The Banach-Tarski Paradox is widely believed to be the most surprising result of theoretical mathematics. Of course, pieces of Banach-Tarski's "paradoxical decompositions" are chosen with the help of the Axiom of Choice – in a model of ZF+LM (see Section 1) such decompositions do not exist.

But what makes the cases $n = 1,\ 2$ so special? Note that it is the first time in this article, when something really depends on the dimension of the space under consideration!

It turns out that the difference lies in the algebraic structures of the corresponding isometry groups D_n: the groups D_1 and D_2 are solvable, whereas if $n \geq 3$, the group D_n has a free subgroup of rank 2. The latter means that there exist two isometries $g,\ h$ of $\mathbb{R}^n$ such that the equality

$$g^{m_1} \circ h^{l_1} \circ \ldots g^{m_k} \circ h^{l_k} = \mathrm{id}_{\mathbb{R}^n}$$

does not hold for any non-zero integers $m_i,\ l_i$. This property guarantees enough freedom in moving points around to get the desired partitions.

The following answer to the second of two questions, formulated above, clarifies the situation (for a proof, due to Mycielski, see [15], Theorem 11.20).

For a group G of isometries of $\mathbb{R}^n$, the following are equivalent:

1. there is an extension of the Lebesgue measure to a G-invariant finitely additive universal measure on $\mathbb{R}^n$,
2. G has no free subgroup of rank 2.

It is well-known that a solvable group does not contain a free subgroup of rank 2. It follows, in particular, that in any dimension there always exists a universal finitely additive extension of m, invariant under all translations.

In order to see, what we gained by dropping the requirement of countable additivity, the reader should compare the last result with the assertion from Section 4, that a universal *countably* additive extension of m might be invariant only under discrete groups of isometries (provided that it exists at all!).

6. Concluding remarks

The sample of results, collected in this article, indicates that our intuition of a "measure" as a generalization of length, area and volume assigning values to *all* subsets of the space in the isometry invariant way, is incompatible with the abstract idea of an *arbitrary* set. By Banach's result (see Section 5), we could speak of a partial success on the line and of the plane but only at the cost of losing countable additivity . And by Solovay's work (see Section 2), we could believe that the Lebesgue measure itself does the job but the price for it would be the refutation of the Axiom of Choice. Modern mathematics cannot afford that.

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