A NOTE ON UNIFORM CONTINUITY OF MONOTONE FUNCTIONS

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Abstract. We prove that it is consistent with ZFC that for every non-decreasing function $f : [0,1] \to [0,1]$, each subset of $[0,1]$ of cardinality $\mathfrak{c}$ contains a set of cardinality $\mathfrak{c}$ on which f is uniformly continuous. We show that this statement follows from the assumptions that $\mathfrak{d}^* < \mathfrak{c}$ and $\mathfrak{c}$ is regular, where $\mathfrak{d}^* \leq \mathfrak{d}$ is the smallest cardinality κ such that any two disjoint countable dense sets in $2^{\mathbb{N}}$ can be separated by sets each of which is an intersection of at most κ-many open sets in $2^{\mathbb{N}}$. We establish also that $\mathfrak{d}^* = \min\{\mathfrak{u}, \mathfrak{d}\} = \min\{\mathfrak{r}, \mathfrak{d}\}$, thus giving an alternative proof of the latter equality established by J. Aubrey in 2004.

1. Introduction

The main subject of this note is the following theorem.

Theorem 1.1. *The statement*

(*) *For every non-decreasing function* $f : [0,1] \to [0,1]$*, each subset of* $[0,1]$ *of cardinality* $\mathfrak{c}$ *contains a set of cardinality* $\mathfrak{c}$ *on which* f *is uniformly continuous.*

is independent of ZFC.

Sierpiński [10, Théorème 6] proved that CH implies the negation of $(*)$. Sierpiński considered an increasing function $f : [0,1] \to [0,1]$ which is discontinuous precisely at the rationals in $(0,1)$ (a well-known example of such a function is due to Lebesgue and is defined by letting $L(0) = 0$ and $L(x) = \sum_{\{n\in\mathbb{N}:\ q_n<x\}} 2^{-n}$ for $x \in (0,1]$, where $\{q_n : n \in \mathbb{N}\}$ is an injective enumeration of $\mathbb{Q} \cap [0,1)$). He proved that the (continuous) restriction of f to the set $\mathbb{P}$ of irrationals in $[0,1]$ is not uniformly continuous on any Lusin set in $\mathbb{P}$.

The fact that in some models of ZFC statement $(*)$ can be true will be explained in Section 2, where a new cardinal coefficient $\mathfrak{d}^*$ which plays a crucial role in our considerations is introduced. A combinatorial analysis, giving a precise description of $\mathfrak{d}^*$ in terms of some other known

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cardinal characteristics of the continuum, will be presented in Section 3. Some additional comments will be gathered in Section 4.

2. The statement (∗) is consistently true

Definition 2.1. *Let* Fin *and* cFin *be the families of all finite and cofinite subsets of* $\mathbb{N}$*, respectively. Then* $\mathfrak{d}^*$ *is the minimal cardinality of a cover* K *of* $\mathcal{P}(\mathbb{N})$ *by compact subspaces*[1] *such that for each* $\mathcal{K} \in \mathsf{K}$*, either* $\mathcal{K} \cap \mathsf{Fin} = \emptyset$ *or* $\mathcal{K} \cap \mathsf{cFin} = \emptyset$*.*

Since both $\mathcal{P}(\mathbb{N}) \setminus \mathsf{Fin}$ and $\mathcal{P}(\mathbb{N}) \setminus \mathsf{cFin}$ are homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ which can be covered by $\mathfrak{d}$ many compact subspaces, we have $\mathfrak{d}^* \le \mathfrak{d}$ (here $\mathfrak{d}$ denotes, as usual, the *dominating number*, i.e., the smallest cardinality of a dominating family in $\mathbb{N}^{\mathbb{N}}$ corresponding to the ordering of eventual domination $\le^*$, cf. [3]). It follows that it is consistent with ZFC to assume that $\mathfrak{d}^* < \mathfrak{c}$ and $\mathfrak{c}$ is regular, cf. [3], and we shall show that these conditions yield (∗).

Let us start from the following observation.

Lemma 2.2. *Given two disjoint countable sets* D_0, D_1 *in a compact metrizable space* X*, there is a collection* $\mathscr{K}$ *of compact sets in* X *such that* $|\mathscr{K}| \le \mathfrak{d}^*$*,* $\bigcup \mathscr{K} = X$ *and each element of* $\mathscr{K}$ *hits at most one* D_i*.*

Proof. Using the Cantor–Bendixson theorem we write X in the form $Y \cup Z$, $Y \cap Z = \emptyset$, such that Y is a compact space without isolated points and Z is countable. Now, to get $\mathscr{K}$, it suffices to find a relevant cover of Y and then to extend it by the singletons of Z. Instead, we simply assume that X has no isolated points.

There is a continuous surjection $u : \mathcal{P}(\mathbb{N}) \to X$ such that $u^{-1}(d)$ is a singleton, for $d \in D_0 \cup D_1$. Indeed, if we expand each $d \in D = D_0 \cup D_1$ to a copy P_d of the Cantor set in X, then [7, §45, II, Theorem 4] gives us a continuous surjection $u : \mathcal{P}(\mathbb{N}) \to X$ which is injective on $u^{-1}(\bigcup_{d\in D} P_d)$.

Now, the disjoint sets $u^{-1}(D_i)$, $i = 0, 1$, are countable, and by adopting standard arguments concerning countable dense homogeneity of $2^{\mathbb{N}}$ (cf. [8, Theorem 1.6.9]) one can find a homeomorphism $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $u^{-1}(D_0) \subseteq h(\mathsf{Fin})$ and $u^{-1}(D_1) \subseteq h(\mathsf{cFin})$.

Finally, if K is a collection of compact sets in $\mathcal{P}(\mathbb{N})$ satisfying the conditions in the definition of $\mathfrak{d}^*$, with $|\mathsf{K}| = \mathfrak{d}^*$, the collection $\mathscr{K} = \{(u \circ h)(\mathcal{K}) : \mathcal{K} \in \mathsf{K}\}$ satisfies the assertion of the lemma.

□

To complete the proof of the consistency of statement (∗) (hence also the proof of Theorem 1.1) it is enough to prove the following result.

[1] As usual, we identify $\mathcal{P}(\mathbb{N})$ with $2^{\mathbb{N}}$ using characteristic functions.

Proposition 2.3. *If* $\mathfrak{d}^* < \mathfrak{c}$ *and* $\mathfrak{c}$ *is regular, then statement* $(*)$ *is true.*

Proof. Let a function $f : [0,1] \to [0,1]$ be non-decreasing. Let D be the set of discontinuity points of f and let X be the closure in the square $[0,1] \times [0,1]$ of the graph of f restricted to the set $[0,1] \setminus D$. For each $t \in [0,1]$ let $X(t) = X \cap (\{t\} \times [0,1])$.

Since f is non-decreasing, the set D of discontinuity points of f is countable and for each $d \in D$ we have $X(d) = \{(d, d_0), (d, d_1)\}$, where $d_0 < d_1$, while $X(t) = \{(t, f(t))\}$ for $t \in [0,1] \setminus D$.

Let

$$Q_i = \{(d, d_i) : \ d \in D\}, \ i = 0, 1.$$

Since X is compact, Lemma 2.2 provides a covering $\mathscr{K}$ of X by compact sets such that $|\mathscr{K}| \le \mathfrak{d}^*$ and each element of $\mathscr{K}$ intersects at most one of the sets $Q_0, \ Q_1$.

Let $\pi : X \to [0,1]$ be the projection onto the first coordinate. Since each $K \in \mathscr{K}$ contains at most one point from every pair $X(d) = \{(d, d_0), (d, d_1)\}$, $d \in D \cap \pi(K)$, K is the graph of a continuous function $f_K : \pi(K) \to [0,1]$. Since K is compact, f_K is uniformly continuous. Moreover, f and f_K coincide on $\pi(K) \setminus D$, so f is uniformly continuous on $\pi(K) \setminus D$.

We are ready to address statement $(*)$. Let $E \subseteq [0,1]$ have cardinality $\mathfrak{c}$. Since the sets $\pi(K)$, $K \in \mathscr{K}$, cover $[0,1]$ and $|\mathscr{K}| \le \mathfrak{d}^* < \mathfrak{c}$ (cf. Lemma 2.2), the regularity of $\mathfrak{c}$ implies that there is $K \in \mathscr{K}$ with $|\pi(K) \cap E| = \mathfrak{c}$, and since f is uniformly continuous on the set $(\pi(K) \cap E) \setminus D$, we get $(*)$. □

3. $\mathfrak{d}^* = \min\{\mathfrak{u}, \mathfrak{d}\} = \min\{\mathfrak{r}, \mathfrak{d}\}$

The aim of this section it to prove the equalities announced in its title. Let us recall that the *ultrafilter number* $\mathfrak{u}$ is the minimal size of a base of a free ultrafilter in $\mathcal{P}(\mathbb{N})$, and the *reaping number* $\mathfrak{r}$ is the minimal size of a *reaping subfamily* $\mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$, i.e., a family $\mathcal{B}$ of infinite subsets of $\mathbb{N}$ such that for every $X \subseteq \mathbb{N}$ there exists $B \in \mathcal{B}$ such that either $X \supset B$ or $B \cap X = \emptyset$, see [3] for more information on these cardinals.

Clearly, each ultrafilter base is reaping, hence $\mathfrak{r} \le \mathfrak{u}$, and the strict inequality is consistent, cf. [6]. However, by [1, Corollary 6.3] we cannot have $\mathfrak{r} < \mathfrak{u}$ if $\mathfrak{r} < \mathfrak{d}$, which is clearly equivalent to $\min\{\mathfrak{u}, \mathfrak{d}\} = \min\{\mathfrak{r}, \mathfrak{d}\}$. In the course of our proof of Theorem 3.4 below we reestablish the latter equality, this way giving an alternative and more streamlined proof thereof, as well as giving a natural topological interpretation of $\min\{\mathfrak{u}, \mathfrak{d}\}$.

Definition 3.1.

- *A family* $\mathcal{F}$ *of infinite subsets of* $\mathbb{N}$ *is called a* semifilter*, if it is closed under taking almost-supersets of its elements, i.e., if* $F \in \mathcal{F}$ *and* $F \subseteq^* X \subseteq \mathbb{N}$*, then* $X \in \mathcal{F}$.

- *A family $\mathcal{C}$ of infinite subsets of $\mathbb{N}$ is called* centered *if $\cap\mathcal{C}'$ is infinite for any finite subfamily $\mathcal{C}' \subseteq \mathcal{C}$.*

For a semifilter $\mathcal{F}$ we denote by $\mathcal{F}^+$ the family $\{X \subseteq \mathbb{N} : \forall F \in \mathcal{F}\,(F \cap X \neq \emptyset)\}$. It is clear that $\mathcal{F}^+$ is also a semifilter.

For $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ we denote by $\sim\mathcal{X}$ the family $\{\mathbb{N} \setminus X : X \in \mathcal{X}\}$. It is easy to check that $\mathcal{F}^+ = \mathcal{P}(\mathbb{N}) \setminus \sim\mathcal{F}$. Note that if $\mathcal{F}_0 \subseteq \mathcal{F}_1$ are semifilters, then $\mathcal{F}_0^+ \supseteq \mathcal{F}_1^+$.

The following two facts are probably well-known, but we give their proofs for the sake of completeness. The first one is reminiscent of [3, Theorem 2.10].

Lemma 3.2. *Let $\kappa < \mathfrak{d}$ be a cardinal and*

$$\big\{\langle n_i^\alpha : i \in \mathbb{N}\rangle \,:\, \alpha < \kappa\big\}$$

be a family of increasing number sequences. Then there exists an increasing number sequence $\langle h(j) : j \in \mathbb{N}\rangle$ such that for every finite $A \subset \kappa$ the set

$$J_A = \{j \in \mathbb{N} \,:\, \forall \alpha \in A\, \exists i \in \mathbb{N}\, \big([n_i^\alpha, n_{i+1}^\alpha) \subseteq [h(j), h(j+1))\big)\}$$

is infinite.

Proof. It is enough to find h satisfying the statement of the lemma for singletons $A \subseteq \kappa$, i.e., A of the form $\{\alpha\}$ for some $\alpha \in \kappa$, since without loss of generality we may assume that for every finite $A \subseteq \kappa$ there exists $\gamma \in \kappa$ such that

$$\forall j \in \mathbb{N}\, \forall \alpha \in A\, \exists i \in \mathbb{N}\, \big([n_i^\alpha, n_{i+1}^\alpha) \subseteq [n_j^\gamma, n_{j+1}^\gamma)\big).$$

For every $\alpha \in \kappa$ and $i \in \mathbb{N}$ set $f_\alpha(i) = n_{2i}^\alpha$ and find a strictly increasing $h \in \mathbb{N}^\mathbb{N}$ such that $h(0) = 0$ and $h \not\leq^* f_\alpha$ for all $\alpha \in \kappa$. We claim that $\langle h(j) : j \in \mathbb{N}\rangle$ is as required. Indeed, let us fix $i > 0$ such that $h(i) \geq f_\alpha(i) = n_{2i}^\alpha$. Thus,

$$\{n_k^\alpha : k \leq 2i\} \subseteq \bigcup_{l<i} [h(l), h(l+1)),$$

which gives that $\big|\{k < 2i : n_k^\alpha, n_{k+1}^\alpha$ belong to the same interval $[h(l), h(l+1))\}\big| \geq i$, and consequently $J_{\{\alpha\}}$ defined in the formulation must be infinite because i with $h(i) \geq f_\alpha(i) = n_{2i}^\alpha$ may be taken arbitrarily large. □

Corollary 3.3. *Let K be a family of $< \mathfrak{d}$-many compact subspaces of $\mathcal{P}(\mathbb{N}) \setminus \mathsf{Fin}$. Then there exists a monotone surjection $\phi : \mathbb{N} \to \mathbb{N}$ and a centered family $\mathcal{C}$ of size $|\mathcal{C}| \leq |\mathsf{K}|$ such that $\{\phi[X] : X \in \bigcup K\} \subseteq \mathcal{F}$, where $\mathcal{F}$ is the smallest free filter generated by $\mathcal{C}$.*

Proof. For every $\mathcal{K} \in \mathsf{K}$, $\mathcal{K}$ being compact, there is an increasing number sequence $\langle n_i^\mathcal{K} : i \in \mathbb{N}\rangle$ such that $X \cap [n_i^\mathcal{K}, n_{i+1}^\mathcal{K}) \neq \emptyset$ for any $X \in \mathcal{K}$ and $i \in \mathbb{N}$. Since $|\mathsf{K}| < \mathfrak{d}$, by Lemma 3.2 there exists a strictly increasing function $h : \mathbb{N} \to \mathbb{N}$ such that $h(0) = 0$ and for any finite $\mathsf{K}' \subseteq \mathsf{K}$

the set $C(\mathsf{K}')$ of those $j \in \mathbb{N}$ such that for every $\mathcal{K} \in \mathsf{K}'$ there exists $i \in \mathbb{N}$ with

$$[h(j), h(j+1)) \supseteq [n_i^{\mathcal{K}}, n_{i+1}^{\mathcal{K}}), \tag{3.1}$$

is infinite. Note that the family

$$\mathcal{C} = \{C(\mathsf{K}') : \mathsf{K}' \text{ is a finite subset of } \mathsf{K}\}$$

is centered because by the definition of $C(\mathsf{K}')$ we have

$$C(\mathsf{K}'_0 \cup \mathsf{K}'_1 \cup \cdots \cup \mathsf{K}'_n) = \bigcap_{i \le n} C(\mathsf{K}'_i)$$

for any finite family $\{\mathsf{K}'_0, \mathsf{K}'_1, \ldots, \mathsf{K}'_n\}$ of finite subsets of K.

We claim that $\mathcal{C}$ and $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi^{-1}(j) = [h(j), h(j+1))$ for all $j \in \mathbb{N}$, are as required. Indeed, given $X \in \bigcup \mathsf{K}$, let $\mathcal{K} \in \mathsf{K}$ be such that $X \in \mathcal{K}$. Then

$$\phi[X] = \{j \in \mathbb{N} : X \cap [h(j), h(j+1)) \ne \emptyset\} \supseteq$$
$$\supseteq \big\{j \in \mathbb{N} \,:\, \exists i \in \mathbb{N} \left([h(j), h(j+1)) \supseteq [n_i^{\mathcal{K}}, n_{i+1}^{\mathcal{K}})\right)\big\} \supseteq C(\{\mathcal{K}\}) \in \mathcal{C},$$

which completes the proof. □

For every $X \subseteq \mathbb{N}$ we adopt the following notation:

$$\downarrow X = \mathcal{P}(X), \quad \uparrow X = \{Y \subseteq \mathbb{N} : X \subseteq Y\},$$
$$\downarrow^* X = \{Y \subseteq \mathbb{N} : Y \subseteq^* X\}, \quad \uparrow^* X = \{Y \subseteq \mathbb{N} : X \subseteq^* Y\}.$$

For a subset $\mathcal{L} \subseteq \mathcal{P}(\mathbb{N})$ we set:

$$\uparrow \mathcal{L} = \bigcup \{\uparrow X : X \in \mathcal{L}\}, \quad \downarrow \mathcal{L} = \bigcup \{\downarrow X : X \in \mathcal{L}\},$$
$$\uparrow^* \mathcal{L} = \bigcup \{\uparrow^* X : X \in \mathcal{L}\}, \quad \downarrow^* \mathcal{L} = \bigcup \{\downarrow^* X : X \in \mathcal{L}\}.$$

It is easy to see that if $\mathcal{L}$ is compact, then both $\uparrow \mathcal{L}$ and $\downarrow \mathcal{L}$ are compact, and $\uparrow^* \mathcal{L}$ and $\downarrow^* \mathcal{L}$ are σ-compact.

We are now ready to prove the main result of this section.

Theorem 3.4. $\mathfrak{d}^* = \min\{\mathfrak{u}, \mathfrak{d}\} = \min\{\mathfrak{r}, \mathfrak{d}\}$.

Proof. As we have already noticed, since both $\mathcal{P}(\mathbb{N}) \setminus \mathsf{Fin}$ and $\mathcal{P}(\mathbb{N}) \setminus \mathsf{cFin}$ are homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ which can be covered by $\mathfrak{d}$ many compact subspaces, we have $\mathfrak{d}^* \le \mathfrak{d}$.

Let $\mathcal{B}$ be a reaping family of cardinality $\mathfrak{r}$. For every $B \in \mathcal{B}$ set $\mathcal{K}(B)_d = \downarrow(\mathbb{N} \setminus B)$ and $\mathcal{K}(B)_u = \uparrow B$.

It follows that

$$\mathcal{P}(\mathbb{N}) = \bigcup \{\mathcal{K}(B)_d : B \in \mathcal{B}\} \cup \bigcup \{\mathcal{K}(B)_u : B \in \mathcal{B}\}$$

and $\mathcal{K}(B)_d \cap \mathsf{cFin} = \mathcal{K}(B)_u \cap \mathsf{Fin} = \emptyset$ for all $B \in \mathcal{B}$, hence $\mathfrak{d}^* \le \mathfrak{r}$, which together with the first paragraph yields $\mathfrak{d}^* \le \min\{\mathfrak{r}, \mathfrak{d}\}$.

To show that $\mathfrak{d}^* \geq \min\{\mathfrak{u}, \mathfrak{d}\}$, suppose, towards a contradiction, that there is a family $\mathsf{K} = \mathsf{K}_d \cup \mathsf{K}_u$ of compact subspaces of $\mathcal{P}(\mathbb{N})$ of size $\kappa = |\mathsf{K}| < \min\{\mathfrak{u}, \mathfrak{d}\}$ such that $\bigcup \mathsf{K} = \mathcal{P}(\mathbb{N})$ and

$$\mathcal{K}_0 \cap \mathsf{cFin} = \mathcal{K}_1 \cap \mathsf{Fin} = \emptyset \tag{3.2}$$

for any $\mathcal{K}_0 \in \mathsf{K}_d$ and $\mathcal{K}_1 \in \mathsf{K}_u$.

Replacing each $\mathcal{K} \in \mathsf{K}_d$ (resp. $\mathcal{K} \in \mathsf{K}_u$) with a countable family of compact subspaces of $\mathcal{P}(\mathbb{N})$ covering $\downarrow^* \mathcal{K}$ (resp. $\uparrow^* \mathcal{K}$), we can assume that $\mathcal{U}_u = \bigcup \mathsf{K}_u$ as well as $\mathcal{U}_d = \sim \bigcup \mathsf{K}_d$ are semifilters on $\mathbb{N}$. Set $\mathcal{U} = \mathcal{U}_u \cup \mathcal{U}_d$. It follows that

$$\mathcal{U}_d^+ = \mathcal{P}(\mathbb{N}) \setminus \sim\mathcal{U}_d = \mathcal{P}(\mathbb{N}) \setminus \sim(\sim\bigcup \mathsf{K}_d) = \mathcal{P}(\mathbb{N}) \setminus \bigcup \mathsf{K}_d \subseteq \bigcup \mathsf{K}_u = \mathcal{U}_u,$$

and hence

$$\mathcal{U}^+ \subseteq \mathcal{U}_d^+ \subseteq \mathcal{U}_u \subseteq \mathcal{U}. \tag{3.3}$$

By the construction, the semifilter $\mathcal{U}$ can be covered by κ many compact subspaces. Since $\kappa < \mathfrak{d}$, by Corollary 3.3, there exists a free filter $\mathcal{F}$ generated by at most κ many subsets of $\mathbb{N}$ and a monotone surjection $\phi : \mathbb{N} \to \mathbb{N}$ with $\mathcal{F} \supseteq \{\phi[X] : X \in \mathcal{U}\}$. Moreover, the family $\{\phi[X] : X \in \mathcal{U}\}$ is a semifilter and one easily checks that

$$\mathcal{F}^+ \subseteq \{\phi[X] : X \in \mathcal{U}\}^+ = \{\phi[X] : X \in \mathcal{U}^+\} \subseteq \{\phi[X] : X \in \mathcal{U}\} \subseteq \mathcal{F},$$

the second inclusion being a consequence of (3.3). This shows that $\mathcal{F}$ is an ultrafilter generated by at most κ many sets, which is impossible because $\kappa < \mathfrak{u}$.

It follows from the above that

$$\min\{\mathfrak{u}, \mathfrak{d}\} \leq \mathfrak{d}^* \leq \min\{\mathfrak{r}, \mathfrak{d}\},$$

and therefore these two inequalities must actually be equalities because $\mathfrak{r} \leq \mathfrak{u}$. □

4. Comments

4.1. **Non-decreasing functions versus arbitrary functions.** In [9, Poposition 3.5] the following result is obtained:

Let $g : \mathbb{P} \to [0,1]$ be a continuous function such that the closure of the graph of f hits each section $\{q\} \times [0,1]$, $q \in \mathbb{Q} \cap [0,1]$, in an uncountable set. Then $\mathbb{P}$ cannot be covered by less than $\mathfrak{d}$ sets on which g is uniformly continuous. In particular, if $\mathfrak{d} = \mathfrak{c}$, there is a subset E of $\mathbb{P}$ of cardinality $\mathfrak{c}$ such that g is not uniformly continuous on any subset of E of cardinality $\mathfrak{c}$ (cf. the proof of [9, Theorem 3.4]).

In contrast, Proposition 2.3 yields the following.

Remark 4.1. *Let $h : E \to [0,1]$ be a monotone function on a set $E \subseteq [0,1]$ of cardinality $\mathfrak{c}$. If $\mathfrak{d}^* < \mathfrak{c}$ and $\mathfrak{c}$ is regular, then h is uniformly continuous on a set of cardinality $\mathfrak{c}$.*

Indeed, if $h : E \to [0,1]$ is, say, non-decreasing, then we can extend it to $f : [0,1] \to [0,1]$ which is also non-decreasing (by setting $f(y) = \sup h[E \cap [0,y]]$ for all $y \in [0,1]$, where we set $\sup(\emptyset) = 0$) and then we can apply Proposition 2.3.

Remark 4.1 applied to the continuous function g and the set E described in the opening statement of this subsection, leads to the following.

Remark 4.2. *If $\mathfrak{d}^* < \mathfrak{d} = \mathfrak{c}$ and $\mathfrak{c}$ is regular, then there exists a continuous function $f : E \to [0,1]$ on a set $E \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ such that no restriction of f to a subset of E of cardinality $\mathfrak{c}$ is monotone.*

4.2. **Models of ZFC with $\mathfrak{d}^* < \mathfrak{d}$.** In view of the remarks from the preceding subsection and for the sake of completeness, we would like to mention that an example of a model of ZFC with $\mathfrak{d}^* = \mathfrak{u} = \aleph_1 < \aleph_2 = \mathfrak{d}$ (cf. Theorem 3.4) is provided by the Miller model resulting from the $\aleph_2$-length countable support iteration of Miller forcing over a model of GCH, see [2, Model 7.5.2]. Another model which has the advantage that $\mathfrak{d}^* = \mathfrak{u} < \mathfrak{d}$ can be any prescribed uncountable regular cardinals is presented by Blass and Shelah in [4].

4.3. **A generalization of $\mathfrak{d}^*$.** In view of Lemma 2.2 it is natural to generalize $\mathfrak{d}^*$ as follows.

Definition 4.3. *For a compact metrizable space X and countable disjoint sets D_0, D_1 in X let $\mathfrak{d}^*(X, D_0, D_1)$ be the minimal cardinality of a cover K of X by compact subspaces such that for each $\mathcal{K} \in \mathsf{K}$, either $\mathcal{K} \cap D_0 = \emptyset$ or $\mathcal{K} \cap D_1 = \emptyset$.*

In this notation we have $\mathfrak{d}^* = \mathfrak{d}^*(\mathcal{P}(\mathbb{N}), \mathsf{Fin}, \mathsf{cFin})$ and Lemma 2.2 states that $\mathfrak{d}^*(X, D_0, D_1) \leq \mathfrak{d}^*$ for any two disjoint countable sets D_0, D_1 in an arbitrary compact metrizable space X. We complement this lemma with the following observation (whose proof gives also another method for establishing Lemma 2.2).

Proposition 4.4. *Let X be a compact metrizable space with no isolated points and let D_0, D_1 be two disjoint countable dense subspaces of X. Then $\mathfrak{d}^*(X, D_0, D_1) = \mathfrak{d}^*$ (equivalently, $\mathfrak{d}^*$ is the smallest cardinality κ such that D_0, D_1 can be separated by two sets each of which is the intersection of a collection of at most κ-many open sets in X).*

In the proof we shall need the following well-known fact which can be proved by the standard back-and-forth argument.

Lemma 4.5. *Let $\mathbb{Q}$ be the set of the rational numbers and $\mathbb{Q} = Q_0^0 \cup Q_1^0$, $\mathbb{Q} = Q_0^1 \cup Q_1^1$ be two decompositions of $\mathbb{Q}$ into disjoint dense subspaces. Then there exists an automorphism of $\langle \mathbb{Q}, \leq \rangle$ (hence a homeomorphism) $h : \mathbb{Q} \to \mathbb{Q}$ such that $h[Q_0^0] = Q_0^1$ and $h[Q_1^0] = Q_1^1$.*

Proof of Proposition 4.4. As X has no isolated points and D_0, D_1 are dense in X, both $A = \mathsf{Fin} \cup \mathsf{cFin} \subseteq \mathcal{P}(\mathbb{N})$ and $B = D_0 \cup D_1$ are homeomorphic to the rationals $\mathbb{Q}$ being countable metrizable spaces without isolated points. Let $h : A \to B$ be a homeomorphism such that $h[\mathsf{Fin}] = D_0$ and $h[\mathsf{cFin}] = D_1$ (its existence follows from Lemma 4.5) and let $\beta h : \beta A \to \beta B$ be its homeomorphic extension to the Čech-Stone compactification βA of A.

By [5, Theorem 3.5.7] we have $\beta h[\beta A \setminus A] = \beta B \setminus B$ and analogously, the extensions $f_A : \beta A \to \mathcal{P}(\mathbb{N})$ and $f_B : \beta B \to X$ of the identity maps $i_A : A \to \mathcal{P}(\mathbb{N})$ and $i_B : B \to X$ have the property that $f_A[\beta A \setminus A] = \mathcal{P}(\mathbb{N}) \setminus A$ and $f_B[\beta B \setminus B] = X \setminus B$.

It follows that if K is a family of compact subspaces of $\mathcal{P}(\mathbb{N})$ such as in the definition of $\mathfrak{d}^*$, then

$$\{f_B[\beta h[f_A^{-1}[\mathcal{K}]]] : \mathcal{K} \in \mathsf{K}\}$$

is such as in the definition of $\mathfrak{d}^*(X, D_0, D_1)$; and vice versa, if K is a family of compact subspaces of X such as in the definition of $\mathfrak{d}^*(X, D_0, D_1)$, then

$$\{f_A[\beta h^{-1}[f_B^{-1}[\mathcal{K}]]] : \mathcal{K} \in \mathsf{K}\}$$

is such as in the definition of $\mathfrak{d}^*$. Thus, $\mathfrak{d}^*(X, D_0, D_1) = \mathfrak{d}^*$, which completes the proof.

□

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