Extending invariant measures on topological groups

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ABSTRACT. Let G be an uncountable Hausdorff topological group.

We prove that if G is either Polish and not locally compact or compact and not zero–dimensional, or the countable product of finite groups with uniformly bounded cardinalities, then every invariant σ–finite measure on G has a proper invariant extension.

We do this by expressing G as the union of a „short" chain of its proper subgroups.

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Introduction.

Let G be an uncountable group.

The aim of this note is to address the question, whether every invariant σ–finite measure on G has a proper invariant extension. Harazišvili [4] and, independently, Erdős and Mauldin [2] proved that if $m : \mathcal{A} \to [0, +\infty]$ is such a measure, then $\mathcal{A} \neq \mathcal{P}(G)$ i.e., the measure m is not *universal* on G. One would like to know if there is, moreover, a *proper invariant extension* of m i.e., a measure $m' : \mathcal{A}' \to [0, +\infty]$ such that $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{A}' \neq \mathcal{A}$ and $m'|\mathcal{A} = m$.

All measures considered here are assumed to be σ–additive, extended real-valued, diffused and σ–finite.

We will say that a group G satisfies the *Measure Extension Property*, MEP, if every invariant measure on G has a proper invariant extension.

By an old result of Hulanicki [6], if there is no real-valued measurable cardinal $\leq |G|$, then G satisfies MEP. Pelc [9] proved that all abelian groups satisfy MEP and conjectured that in fact *all* groups do. Though it is now known that some other algebraically defined classes of groups satisfy MEP, the question, whether Pelc's conjecture is true, remains a major open problem.

The aim of this note is to show that some topologically defined classes of groups satisfy MEP.

Results

Throughout the paper G is assummed to be a Hausdorff topological group.

Our results are as follows.

Theorem 1. *If G is Polish and not locally compact, then G satisfies* MEP.

Theorem 2. *If G is compact and not zero-dimensional, then G satisfies* MEP.

Theorem 3. *If G is the product of a sequence $\langle G_n : n < \omega \rangle$ of finite groups with the property that there is a finite constant N such that $1 < |G_n| < N$ for each $n < \omega$, then G satisfies* MEP.

In the rest of this note we outline proofs of these results.

Suppose that m is an invariant measure on G. Call a set $A \subseteq G$ *almost invariant* if $m(A \triangle gA) = 0$ for every $g \in G$.

We shall need two measure-theoretic lemmas.

Lemma 4 (Hulanicki, [6]). *If there exists a non-measurable almost invariant set, then m has a proper invariant extension.*

Lemma 5 (Harazišvili, [5]). *If there is a set Z such that $m^*(Z) > 0$ and for every sequence $\langle g_k : k < \omega \rangle$ of elements of G there is a sequence $\langle f_i : i < \omega \rangle$ of elements of G such that*

$$\bigcap_{i<\omega} f_i \bigcup_{k<\omega} g_k Z = \emptyset,$$

then m has a proper invariant extension.

The next fact contains the key observation.

Lemma 6. *Suppose that H is a subgroup of G and the index $[G : H]$ of H in G is uncountable. If $m^*(H) > 0$, then m has a proper invariant extension.*

Proof. It suffices to show that the set $Z = H$ has the property from Lemma 5.

So take an arbitrary sequence $\langle g_k : k < \omega \rangle$ of elements of G. Since the index of H in G is uncountable, it is possible to choose for each $i \in \omega$ a $f_i \in G$ such that

$$(*) \qquad f_i \notin \bigcup_{k<\omega} g_i H g_k^{-1}.$$

It suffices now to check that

$$(\bigcap_{i<\omega} f_i \bigcup_{k<\omega} g_k H) \cap \bigcup_{k<\omega} g_k H = \emptyset.$$

So suppose that there is $g \in (\bigcap_{i<\omega} f_i \bigcup_{k<\omega} g_k H) \cap \bigcup_{k<\omega} g_k H$. Fix $i, k \in \omega$ such that $g \in f_i g_k H \cap g_i H$. But then $f_i \in g_i H g_k^{-1}$, contradicting $(*)$. □

Let us say that an infinite, regular cardinal λ is in $\mathrm{Cof}(G)$, the *cofinality spectrum of G*, if

$$G = \bigcup_{\alpha<\lambda} G_\alpha$$

for a certain strictly increasing sequence $\langle G_\alpha : \alpha < \lambda \rangle$ of subgroups of G with $[G : G_\alpha] > \omega$ for each $\alpha < \lambda$.

Note that if G is the union of a chain of length λ of proper subgroups and $\mathrm{cf}(\lambda) > \omega$, then $\mathrm{cf}(\lambda) \in \mathrm{Cof}(G)$.

Recall that a universal measure on a set X is called *uniform* if all sets of cardinalities less than $|X|$ have measure zero.

The proof of the following Main Lemma is based on ideas taken from [6].

Main Lemma. *If there exists $\lambda \in \mathrm{Cof}(G)$ such that there is no universal, uniform measure on λ, then G satisfies* MEP.

Proof. Let m be an invariant measure on G. We will show that m has a proper invariant extension.

Take $\lambda \in \mathrm{Cof}(G)$ and let $\langle G_\alpha : \alpha < \lambda \rangle$ be the corresponding sequence of subgroups.

If there is an $\alpha < \lambda$ such that $m^*(G_\alpha) > 0$, then we are done by Lemma 6.

So assume otherwise and for each $\alpha < \lambda$ let $Q_\alpha = G_{\alpha+1} \setminus G_\alpha$; by what we have just assumed, $m(Q_\alpha) = 0$. It then follows that for every $T \subseteq \lambda$, the set $A_T = \bigcup_{\alpha \in T} Q_\alpha$ is almost-invariant. Finally, for some $T \subseteq \lambda$, the set A_T is non-measurable, since otherwise it is easy to define a uniform universal measure on λ. By Lemma 4, this completes the proof. □

Let d denote the smallest size of a dominating family in ω^ω.

Lemma 7 (Fremlin, [3]**).** *There is no universal uniform measure on cf(d).*

Now Theorems 1, 2 and 3 will follow from Main Lemma as soon as the final three lemmas are established.

Lemma 8. *If G is Polish and not locally compact, then cf(d) $\in \mathrm{Cof}(G)$.*

Proof. Since G is Polish and not σ–compact, the minimal size of a collection of compact sets which covers G equals d.

Let $\{C_\alpha : \alpha < d\}$ be such a collection. In order to see that $\mathrm{cf}(d) \in \mathrm{Cof}(G)$ consider the sequence $\langle G_\alpha : \alpha < d\rangle$, where G_α is the subgroup of G generated by $\bigcup_{\xi<\alpha} C_\xi$. □

Lemma 9. *If G is compact and not zero-dimensional, then $\omega_1 \in \mathrm{Cof}(G)$.*

Proof. The representation theory of compact groups tells us that in this case there is a (continuous) group homomorphism which maps G onto an uncountable closed subgroup G' of the group $U(n, \mathbb{C})$ of all complex $n \times n$ unitary matrices, for some $n > 0$.

It clearly suffices to prove that $\omega_1 \in \mathrm{Cof}(G')$.

The idea (which has origins in Ciesielski's paper [1]) is to represent $\mathbb{C}$ as the union of a strictly increasing sequence $\langle L_\alpha : \alpha < \omega_1\rangle$ of its subfields and then to define G_α as the subgroup of G' consisting of all matrices from G' with entries restricted to L_α. Some care must be taken, however, to secure that $G_\alpha \neq G'$. So let $\mathcal{B}$ be a maximal algebraically independent over $\mathbb{Q}$ subset of the set of all entries of the elements of G'. Represent $\mathcal{B}$ as the union of a strictly increasing sequence $\langle \mathcal{B}_\alpha : \alpha < \omega_1\rangle$ of its subsets and define L_α as the algebraic closure of the field generated by $\mathcal{B}_\alpha \cup \mathbb{Q}$. Finally, consider the sequence $\langle G_\alpha : \alpha \in \omega_1\rangle$. □

Lemma 10. *If G is the product of a sequence $\langle G_n : n < \omega\rangle$ of finite groups with the property that there is a finite constant N such that $1 < |G_n| < N$ for each $n < \omega$, then $\omega_1 \in \mathrm{Cof}(G)$.*

Proof. By a result of van Douwen, there is a strictly increasing sequence $\langle F_\alpha : \alpha < \omega_1\rangle$ of filters in $\mathcal{P}(\omega)$ such that $U = \bigcup_{\alpha<\omega_1} F_\alpha$ is a free ultrafilter in $\mathcal{P}(\omega)$. Let:

$$H = \{g \in G : \{n < \omega : g_n = e_n\} \in U\},$$

where e_n is the neutral element of G_n.

Then H is a normal subgroup of G and $[G : H] < N$.

Fix a selector S of the collection of all H-cosets.

For each $\alpha < \omega_1$ let:

$$H_\alpha = \{g \in G: \ \{n < \omega: \ g_n = e_n\} \in F_\alpha\}$$

and define G_α as the subgroup of G generated by $H_\alpha \cup S$.

Note that $G_\alpha \neq G$, since $[G : H_\alpha] > \omega$ and the subgroup of G generated by S is at most countable. □

Final comments.

The question, which cardinal numbers are in $\mathrm{Cof}(G)$ is interesting in its own right.

Sharp and Thomas [10] proved that if $Sym(\omega)$ is the group of all permutations of ω, then MA implies that $\mathrm{Cof}(Sym(\omega)) = \{2^\omega\}$ (compare this with Lemma 8; in fact, in [10] another proof is given that $\mathrm{cf}(d) \in \mathrm{Cof}(Sym(\omega))$). On the other hand Macpherson and Neumann [8] showed that $\omega \notin \mathrm{Cof}(Sym(\omega)$.

If G is compact and not zero–dimensional (see Lemma 9), then a refinement of our argument establishes that $\omega \in \mathrm{Cof}(G)$. In particular, this is true for all compact Lie groups.

If G is the countable product of finite groups with uniformly bounded cardinalities (see Lemma 10), then it may happen that $\omega \notin \mathrm{Cof}(G)$ (see [7]). On the other hand, it is easy to see that $\omega \in \mathrm{Cof}((\mathbb{Z}_2)^\omega)$.

Note finally, that if G is the countable product of a sequence of at most countable, nontrivial groups, then if all of them are either infinite or have uniformly bounded cardinalities, then G satisfies MEP. It is open what happens in the intermediate case .

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