Extending isometrically invariant measures on $\mathbf{R}^n$ – a solution to Ciesielski's query

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ABSTRACT. We prove that if $m\colon \mathcal{M} \to [0,+\infty]$ is an isometrically invariant σ–finite countably additive measure on $\mathbb{R}^n$, then there exists a countably additive isometrically invariant extension $m'\colon \mathcal{M}' \to [0,+\infty]$ of m such that the canonical embedding $e\colon \mathcal{M}/m \to \mathcal{M}'/m'$ of measure algebras defined by $e([A]_m) = [A]_{m'}$ is not surjective.

This answers a question of Ciesielski [2].

1991 Mathematics Subject Classification.
Primary 28C10; Secondary 03E05.

Key words and phrases: invariant σ–finite measures, isometries of $\mathbb{R}^n$, extensions of measures.

Supported by KBN grant 2 P03A 047 09.

Introduction

Given a group G of isometries of $\mathbb{R}^n$ and a G–invariant σ–finite σ–additive measure $m\colon \mathcal{M} \to [0, +\infty]$ on $\mathbb{R}^n$, does there exist a proper G–invariant extension of m i.e., a G–invariant σ–additive measure $m'\colon \mathcal{M}' \to [0, +\infty]$ such that $\mathcal{M} \subseteq \mathcal{M}'$, $\mathcal{M}' \neq \mathcal{M}$ and $m'|\mathcal{M} = m$?

This question was first posed by Sierpiński (for the case when $G = \mathrm{Isom}(\mathbb{R}^n)$ is the group of all isometries of $\mathbb{R}^n$ and m is an isometrically invariant extension of the Lebesgue measure) and then studied by several authors. (See [1] for historical details.)

Sierpiński's problem was finally solved by Ciesielski and Pelc [3] and subsequent investigations by Ciesielski [1], Krawczyk and Zakrzewski [7, 10]) showed, in particular, that for any G, a proper G–invariant extension of m exists in either of the following cases:

- m is a G–invariant extension of the Lebesgue measure and $\mathcal{M} \neq \mathcal{P}(\mathbb{R}^n)$ (see [7], Theorem 2.10);
- m does not vanish on the set of all points with uncountable G–orbits (see [10], Theorem 2.2).

If we move from measure spaces to the respective measure algebras, then we see that extending the measure m to m' does not necessarily mean extending the associated measure algebra. More precisely, it may happen that

$$\forall Y \in \mathcal{M}' \ \exists X \in \mathcal{M} \ m'(Y \triangle X) = 0.$$

This means that the canonical embedding $e\colon \mathcal{M}/m \to \mathcal{M}'/m'$ of measure algebras defined by $e([A]_m) = [A]_{m'}$ is an isomorphism.

After realizing that the methods by which Sierpiński's problem was solved lead exactly to this type of extensions, Ciesielski [2] posed the problem, whether one could always extend an isometrically invariant measure so that the measure algebra gets extended too. It was known that a technique of Hulanicki [5] easily gives the positive answer under the additional asumption that there is no real–valued measurable cardinal less than or equal to the cardinality of the continuum and it later turned out that in the case when $n = 1$ the positive answer follows from a construction due to Kharazišvili [4].

The aim of this note is to give a strong affirmative answer to Ciesielski's question. It is proved that for any G, if m has a proper G–invariant extension, then it also has one as required.

The method

All measures considered here are assumed to be σ–additive, extended real-valued, vanishing on singletons and σ–finite.

Suppose that G is a subgroup of the group $\mathrm{Isom}(\mathbb{R}^n)$ of all isometries of $\mathbb{R}^n$ and that $m\colon \mathcal{M} \to [0, +\infty]$ is a G–invariant measure on $\mathbb{R}^n$.

Call a set $A \subseteq \mathbb{R}^n$ *almost invariant* if $m(A \triangle gA) = 0$ for every $g \in G$.

The following two ways of obtaining a G–invariant extension $m'\colon \mathcal{M}' \to [0, +\infty]$ of m are well known:

(I) find a G–invariant σ–ideal $\mathcal{I} \not\subseteq \mathcal{M}$ of subsets of $\mathbb{R}^n$ consisting of sets of m inner measure zero, then define:
$\mathcal{M}' = \{(M \cup N_1) \setminus N_2 \colon M \in \mathcal{M} \text{ and } N_1,\ N_2 \in \mathcal{I}\}$,
$m'((M \cup N_1) \setminus N_2) = m(M)$ (see [9]);
(II) find an almost invariant set $A \notin \mathcal{M}$, then define:
$\mathcal{M}' = \{(M_1 \cap A) \cup (M_2 \setminus A) \colon M_1,\ M_2 \in \mathcal{M}\}$,

$$m'((M_1 \cap A) \cup (M_2 \setminus A)) = \frac{1}{2} \cdot \big[\ m^*(M_1 \cap A) + m_*(M_2 \setminus A) + m_*(M_1 \cap A) + m^*(M_2 \setminus A)\ \big],$$

where m_* and m^* stand for the inner and outer measure for m, respectively. (See [8].)

Call m' a *weak extension* of m if $\forall Y \in \mathcal{M}'\ \exists X \in \mathcal{M}\ m'(Y \triangle X) = 0$. Otherwise call it *strong.*

Clearly, the first construction produces a weak extension, whereas the second procedure gives a strong one.

Our idea is to perform both of these methods in two steps: the first (weak) one will supply enough almost invariant sets so that the second (strong) one may be performed. More precisely, we shall use the following lemma.

Lemma 1. *Let m be a G-invariant measure on $\mathbb{R}^n$ and suppose that $\{Q_\alpha \colon \alpha < \omega_1\}$ is a partition of $\mathbb{R}^n$ such that:*

(i) the minimal G–invariant σ–ideal $\mathcal{I}$ of subsets of $\mathbb{R}^n$ containing the family $\{Q_\alpha \colon \alpha < \omega_1\}$ consists of sets of m inner measure zero,

(ii) $\forall g \in G\ \exists \beta < \omega_1\ \forall \alpha > \beta\ gQ_\alpha = Q_\alpha$.

Then m has a strong G–invariant extension.

Proof. First extend m to m' such that $m'(N) = 0$ for every $N \in \mathcal{I}$. Then use a classical argument of Hulanicki [5]. Namely, for every $T \subseteq \omega_1$, the set $A_T = \bigcup_{\alpha \in T} Q_\alpha$ is m'–almost–invariant, since if $g \in G$ and $gQ_\alpha = Q_\alpha$ for all $\alpha > \beta$, then $A_T \triangle gA_T \subseteq \bigcup_{\alpha \le \beta} Q_\alpha \cup g[\bigcup_{\alpha \le \beta} Q_\alpha] \in \mathcal{I}$. It follows that $A_T \notin \operatorname{dom}(m')$ for a certain $T \subseteq \omega_1$, since otherwise it is easy to define a measure on $\mathcal{P}(\omega_1)$, which is impossible. (See e.g. [6], Lemma 27.7.) □

The following fact is a convenient tool for checking condition (i) of the above lemma.

Lemma 2 (Kharazišvili, see [10], Lemma 1.5). *Let m be a G-invariant measure on $\mathbb{R}^n$ and let $\mathcal{I}$ be the minimal G–invariant σ–ideal containing a family $\mathcal{Q}$ of subsets of $\mathbb{R}^n$. If for every sequence $\langle Q_k \colon k < \omega\rangle$ of elements from $\mathcal{Q}$ and arbitrary sequence $\langle g_k \colon k < \omega\rangle$ of elements of G there is a sequence $\langle f_i \colon i < \omega\rangle$ of elements of G such that*

$$m\left(\bigcap_{i<\omega} f_i \left[\bigcup_{k<\omega} g_k Q_k\right]\right) = 0,$$

then $\mathcal{I}$ consists of sets of m inner measure zero. □

The results

The following theorem is a direct answer to Ciesielski's question.

Theorem 3. *Every isometrically invariant measure on $\mathbb{R}^n$ has a strong isometrically invariant extension.*

Proof. Represent $\mathbb{R}$ as the union of a strictly increasing sequence $\langle L_\alpha \colon \alpha < \omega_1\rangle$ of its subfields. For each $\alpha < \omega_1$ let $X_\alpha = L_\alpha^n$, $Q_\alpha = X_{\alpha+1} \setminus X_\alpha$ and let G_α consist of all isometries of the form $gx = Ax + b$, where A is a matrix with all entries in L_α and $b \in L_\alpha^n$. It is enough to check that $\{Q_\alpha \colon \alpha < \omega_1\}$ is a partition of $\mathbb{R}^n$ which satisfies the hypotheses of Lemma 1.

To prove (i), we will use Lemma 2. If $g_k \in \mathrm{Isom}(\mathbb{R}^n)$ and $\alpha_k < \omega_1$ for each $k < \omega$, then there is $\alpha < \omega_1$ such that $\bigcup_{k<\omega} g_k Q_{\alpha_k} \subseteq X_\alpha$. But $(x + X_\alpha) \cap X_\alpha = \emptyset$, whenever $x \notin X_\alpha$.

To prove (ii), note that if $g \in G_\beta$, then $gQ_\alpha = Q_\alpha$ for all $\alpha > \beta$. □

With the help of ideas from [1] and [10] the above result may be generalized as follows. Recall that a measure is complete, if it measures every subset of an arbitrary set of measure zero. (Obviously, it assigns value zero to such a set.) Note that if m is a measure on $\mathbb{R}^n$, invariant with respect to a given group G of isometries, then its measure completion $\bar{m}$, i.e., the minimal extension of m to a complete measure, is G–invariant too. It is also easy to see that if there does not exist a proper G–invariant extension of $\bar{m}$, then m does not have any strong G–invariant extensions. It turns out that the converse is also true, as the next theorem shows. In its proof, we will use the following two lemmas.

Lemma 4. *Let $m \colon \mathcal{M} \longrightarrow [0,\infty]$ be a G-invariant measure on $\mathbb{R}^n$ and let $Y \in \mathcal{M}$ be a G-invariant subset of $\mathbb{R}^n$ of positive measure. Consider a G-invariant measure $m_Y \colon \mathcal{M} \longrightarrow [0,\infty]$ defined by $m_Y(A) = m(A \cap Y)$ for every $A \in \mathcal{M}$. If m_Y has a strong G–invariant extension, then so does m.*

Proof. If $m'_Y \colon \mathcal{M}'_Y \longrightarrow [0,\infty]$ is a strong G–invariant extension of m_Y, then we may define a strong G–invariant extension m' of m on the σ-algebra $\mathcal{M}' = \{A \in \mathcal{M}'_Y \colon A \cap L_G \in \mathcal{M}\}$ by $m'(A) = m'_1(A \cap Y) + m(A \setminus Y)$ for $A \in \mathcal{M}'$. □

The next lemma may be obtained by an easy refinement of the proof of Ciesielski's Lemma 3.2 from [1].

Lemma 5. *Let $m \colon \mathcal{M} \longrightarrow [0,\infty]$ be a G-invariant measure on $\mathbb{R}^n$ and let H be an uncountable subset of G. Then $\mathbb{R}$ is the union of a strictly increasing sequence $\langle L_\alpha \colon 0 < \alpha < \omega_1\rangle$ of its subfields such that if $L_0 = \emptyset$ and $Q_\alpha = L_{\alpha+1}^n \setminus L_\alpha^n$ for every $\alpha < \omega_1$, then $\{Q_\alpha \colon \alpha < \omega_1\}$ is a partition of $\mathbb{R}^n$ which satisfies the following condition:*

- *if $g_k \in G$ and $\alpha_k < \omega_1$ for each $k < \omega$, then there are distinct $h_1, h_2 \in H$ such that*
$$h_1^{-1}\left[\bigcup_{k<\omega} g_k Q_{\alpha_k}\right] \cap h_2^{-1}\left[\bigcup_{k<\omega} g_k Q_{\alpha_k}\right] \subseteq \{x \in \mathbb{R}^n : h_1(x) = h_2(x)\}.$$

□

Theorem 6. *Let G is a group of isometries of $\mathbb{R}^n$ and $m\colon \mathcal{M} \longrightarrow [0,\infty]$ be a G–invariant measure on $\mathbf{R}^n$. If the completion $\bar{m}$ of m has a proper G–invariant extension, then m has a strong G–invariant extension.*

In particular, this happens if the set of all points with uncountable G–orbits has a positive outer measure.

Proof. This is a refinement of the proof of Theorem 2.2 from [10].

Since every strong extension of the completion $\bar{m}$ of m strongly extends m, we may assume that the measure m itself is complete.

Let $L_G = \{x \in \mathbb{R}^n : Gx$ is at most countable$\}$ and consider the following three cases.

Case 1. $\mathbb{R}^n \setminus L_G \notin \mathcal{M}$. Then $L_G \notin \mathcal{M}$ and, by method II, we are done, since L_G is G–invariant.

Case 2. $\mathbb{R}^n \setminus L_G \in \mathcal{M}$ and $m(\mathbb{R}^n \setminus L_G) = 0$.

<u>Note</u> that the set $H = \{g|L_G \colon g \in G\}$ is at most countable. This can be proved by the following easy argument. (Compare [10, Lemma 2.1].)

Take a finite subset B of L_G spaning the affine space generated by L_G. Then the function $g|L_G \mapsto g|B$ is one-to-one and, since Gb is at most countable for each $b \in B$, there are at most countably many functions $f\colon B \mapsto \bigcup_{b\in B} Gb$.

Now, if there is a set $A \subseteq \mathbb{R}^n$, such that $HA \notin \mathcal{M}$, then we are done by method II, since the set HA is G–invariant.

So assume that $HA \in \mathcal{M}$ for every $A \subseteq \mathbb{R}^n$ and let m' be a proper G–invariant extension of m. Note that for any $A \subseteq \mathbb{R}^n$, the following implications are true, where the first holds, since $H \subset G$ is countable, and the last one follows from the completeness of m.
$$m'(A) = 0 \implies m'(HA) = 0 \implies m(HA) = 0 \implies m(A) = 0.$$
This implies that every set $A \in \mathcal{M}' \setminus \mathcal{M}$ witnesses that the measure m' is a strong extension of m. □

Case 3. $\mathbb{R}^n \setminus L_G \in \mathcal{M}$ and $m(\mathbb{R}^n \setminus L_G) > 0$. Then, by Lemma 4, we assume without loss of generality that $m(L_G) = 0$.

Now, let $p \leq n$ be the smallest dimension of an affine subspace of $\mathbb{R}^n$ of positive outer m measure. Notice, that it may also be assumed that

(A): there exists an affine subspace L_0 of dimension p having positive outer m measure and such that the set $\{g[L_0] \colon g \in G\}$ is at most countable.

First note that to prove (A) it is enough to show that

(B): any measure satisfying the negation of (A) can be weakly extended to a measure m' with the property, that every affine subspace of $\mathbb{R}^n$ of dimension $\leq p$ has m' measure zero.

Indeed, if (B) holds, then we can extend m at most $n-p$ times to obtain a weak extension of m satisfying (A).

To see (B) suppose that (A) is false and consider the G–invariant σ-ideal $\mathcal{I}$ generated by the family $\mathcal{Q}$ of all affine subspaces L of dimension p. By method I it is enough to show that $\mathcal{I}$ consists of sets of m inner measure zero. This will be proved by using Lemma 2.

So, let $\langle L_k \colon k<\omega\rangle$ and $\langle g_k \colon k<\omega\rangle$ be arbitrary sequences of elements from $\mathcal{Q}$ and G, respectively. For $n,m\in\omega$ put $L_{n,m}=g_n[L_m]$ and let $K=\bigcup_{n,m\in\omega} L_{n,m}$. For every $n,m\in\omega$ fix $f_{n,m}\in G$ such that $f_{n,m}^{-1}[L_{n,m}]\neq L_{i,j}$ for all $i,j\in\omega$. We have:

$$K\cap\bigcap_{n,m\in\omega} f_{n,m}[K]\subseteq\bigcup_{n,m,i,j\in\omega}(L_{n,m}\cap f_{n,m}[L_{i,j}]).$$

But $L_{n,m}\cap f_{n,m}[L_{i,j}]\neq L_{n,m}$. So, the set $L_{n,m}\cap f_{n,m}[L_{i,j}]$ is either empty or is an affine subspace of dimension less than p and $m(K\cap\bigcap_{n,m\in\omega} f_{n,m}[K])=0$. This finishes the proof of (B) and (A).

Next, define $Y=\bigcup_{g\in G} gL_0$. If $Y\notin\mathcal{M}$, then we are done by method II, since Y is G–invariant. So, assume that $Y\in\mathcal{M}$. Then $m(Y)>0$. By Lemma 4 we can also assume that $m(\mathbb{R}^n\setminus Y)=0$.

<u>Recall</u> that $m(L_G)=0$ and that, by (A), the set $\{g[L_0]\colon g\in G\}$ is at most countable. Since $m(Y)>0$, this implies that $L_0\setminus L_G\neq\emptyset$ i.e. there are points in L_0 with uncountable G–orbits. Consequently, the set $\{g|L_0\colon g\in G\}$ is uncountable.

So fix an uncountable subset H of G such that $h_1|L_0\neq h_2|L_0$ for any distinct $h_1,\ h_2\in H$ and let $\{Q_\alpha\colon\alpha<\omega_1\}$ be a partition from Lemma 5.

It suffices to prove that $\{Q_\alpha\colon\alpha<\omega_1\}$ satisfies conditions of Lemma 1.

To check condition (i), use Lemma 2. So let $\langle Q_k\colon k<\omega\rangle$ and $\langle g_k\colon k<\omega\rangle$ be arbitrary sequences of elements from $\mathcal{Q}$ and G, respectively. Let $K=\bigcup_{k<\omega} g_kQ_k$. By the property of $\langle Q_k\colon k<\omega\rangle$ stated in Lemma 6, there are distinct $h_1,h_2\in H$ such that

$$h_1^{-1}[K]\ \cap h_2^{-1}[K]\ \cap L_0\subseteq\{x\in L_0\colon h_1(x)=h_2(x)\}.$$

But $h_1|L_0\neq h_2|L_0$, so the set $\{x\in L_0\colon h_1(x)=h_2(x)\}$ is either empty or is an affine subspace of dimension less than p. Accordingly,

$$m(h_1^{-1}[K]\ \cap h_2^{-1}[K]\ \cap L_0)=0.$$

Let $\langle L_n\colon n\in\lambda\rangle$, $\lambda\leq\omega$, be a one-to-one enumeration of $\{g[L_0]\colon g\in G\}$. For each $n\in\lambda$ let $L_n=f_n[L_0]$; it follows that

$$m((f_n\circ h_1^{-1})[K]\ \cap(f_n\circ h_2^{-1})[K]\ \cap L_n)=0.$$

Consequently,

$$
\begin{aligned}
m^*\Bigg(\bigcap_{\substack{k\in\lambda\\ i\in\{1,2\}}}(f_k\circ h_i^{-1})[K]\Bigg) &= m^*\Bigg(\bigcap_{\substack{k\in\lambda\\ i\in\{1,2\}}}(f_k\circ h_i^{-1})[K]\cap\bigcup_{n\in\lambda}L_n\Bigg)\\
&\le \sum_{n\in\lambda} m^*\Bigg(\bigcap_{\substack{k\in\lambda\\ i\in\{1,2\}}}(f_k\circ h_i^{-1})[K]\cap L_n\Bigg)\\
&\le \sum_{n\in\lambda} m^*((f_n\circ h_1^{-1})[K]\ \cap(f_n\circ h_2^{-1})[K]\ \cap L_n)=0.
\end{aligned}
$$

Condition (ii) may be established exactly as in the proof of Theorem 3 and we are done.

Finally, note that the last assertion of the theorem follows from considerations from Case 1 and Case 3. □

The following corollary explains the situation for extensions of the Lebesgue measure.

Theorem 7. *Let G be a group of isometries of $\mathbb{R}^n$ and $m\colon \mathcal{M}\longrightarrow[0,+\infty]$ be a G–invariant complete measure which extends the Lebesgue measure on $\mathbf{R}^n$. If $\mathcal{M}\neq\mathcal{P}(\mathbf{R}^n)$, then m has a strong G–invariant extension.*

Proof. This immediately follows from Theorem 6 and the fact that under the above assumptions there always exists a proper G–invariant extension of m. (See [7], Theorem 2.10.) □

Acknowledgement. The author wishes to thank Krzysztof Ciesielski for inspiring correspondence and helpful remarks.

References

[1] K .Ciesielski, *Algebraically invariant extensions of σ–finite measures on Eucledean spaces*, Trans. Amer. Math. Soc. 318 (1990), 261–273.

[2] K. Ciesielski, *Query 5*, Real Analysis Exchange 16(1) (1990/91), 374.

[3] K. Ciesielski, A. Pelc, *Extensions of invariant measures on Euclidean Spaces*, Fund. Math. 125 (1985), 1–10.

[4] A.B. Kharazišvili, *Some applications of the Hamel bases* (in Russian), Bull. Acad. Sci. of the Georgian SSR, 85 (1977), 17–20.

[5] A. Hulanicki, *Invariant extensions of the Lebesgue measure*, Fund. Math. 51 (1962), 111–115.

[6] T. Jech, *Set theory*, Academic Press, 1978.

[7] A. Krawczyk, P. Zakrzewski, *Extensions of measures invariant under countable groups of transformations*, Trans. Amer. Math. Soc. 326 (1991), 211–226.

[8] J. Łoś, E. Marczewski, *Extensions of measure*, Fund. Math. 36 (1949), 267–276.

[9] E. Szpirlajn, *Sur l'extension de la mesure lebesguienne*, Fund. Math. 25 (1935), 551–558.

[10] P. Zakrzewski, *Extensions of isometrically invariant measures on Euclidean spaces*, Proc. Amer. Math. Soc. 110 (1990), 325–331.

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