The uniqueness of Haar measure and set theory

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ABSTRACT. Let G be a group of homeomorphisms of a non-discrete, locally compact, σ-compact topological space X and suppose that a Haar measure on X exists: a regular Borel measure μ , positive on nonempty open sets, finite on compact sets and invariant under the homeomorphisms from G.

Under some mild assumptions on G and X we prove that the measure completion of μ is the unique, up to a constant factor, non-zero, σ-finite, G-invariant measure defined on its domain iff μ is ergodic and the G-orbits of all points of X are uncountable. In particular, this is true if either G is a locally compact, σ-compact topological group acting continuously on X or the space X is uniform and non-separable, and G consists of uniformly equicontinuous unimorphisms of X.

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Introduction.

This paper is a contribution to the theory of uniqueness of invariant measures. In a variety of analytic-geometric situations, there are given a locally compact, σ-compact topological space X and a group G of its homeomorphisms for which a Haar measure exists: a regular Borel measure μ, positive on nonempty open sets, finite on compact sets and invariant under the homeomorphisms from G.

There are two features of the approach presented in this paper which distinguish it from most of the published work on the subject.

The first is that we are dealing with the *measure completion* $\bar{\mu}$ of μ, rather than with μ itself, looking for conditions which guarantee the uniqueness of $\bar{\mu}$ among *all* non-zero, σ*-finite,* invariant measures defined on its domain. The motivation behind this is that in many cases it is just $\bar{\mu}$, not μ that we are really interested in (the typical example is Lebesgue measure on $\mathbf{R}^n$) and that the indicated uniqueness property is strong.

The second is the extensive use of set-theoretic arguments. A classical result of von Neumann tells us that the uniqueness of $\bar{\mu}$ described above is equivalent to the fact that $\bar{\mu}$ is ergodic and every σ-finite invariant measure on the domain of $\bar{\mu}$ is absolutely continuous with respect to $\bar{\mu}$. Now the set-theoretic context of the problem is revealed by the following trivial observation: if there is no set Y with a non-zero, diffused, σ-finite measure defined on $\mathcal{P}(Y)$, the power set of Y, then every diffused, σ-finite measure defined on the domain of $\bar{\mu}$ is absolutely continuous with respect to $\bar{\mu}$.

But the hypothesis of the above implication, commonly referred to as the non-existence of a real-valued measurable cardinal, most probably has the status of an additional set-theoretic axiom which can neither be proved nor disproved within the usual set theory ZFC (see [3]). Hence, if we assume that real-valued measurable cardinals do *not* exist, then the measure $\bar{\mu}$ always has the uniqueness property in question, provided that it is diffused and ergodic. But what if they *do* exist? Fortunately, the above implication in our context admits the following refinement: if there is no non-zero, σ-finite, *invariant* measure defined on $\mathcal{P}(X)$, then every σ-finite invariant measure on the domain of $\bar{\mu}$ is absolutely continuous with respect to $\bar{\mu}$. This implication, together with von Neumann's result quoted above, provides the basic tool for our uniqueness arguments. They are carried out entirely within ZFC but, nevertheless, require measures defined on $\mathcal{P}(X)$ to be handled with mixed measure- and set-theoretic methods.

The main result of this work shows that under various assumptions on X and G, the uncountability of all G-orbits is a necessary and sufficient condition for $\bar{\mu}$ to be unique in the sense described above, provided that it is ergodic. In particular, this is true if either G is a locally compact, σ-compact topological group acting continuously on X or the space X is uniform and non-separable, and G consists of uniformly equicontinuous unimorphisms of X.

The paper is organized as follows. Sections 1 and 2 contain, respectively, measure- and set-theoretic preliminaries. In Section 3 we discuss in detail the problem of existence of a G-invariant, non-zero, σ-finite measure defined on $\mathcal{P}(X)$, where G is an arbitrary group of transformations of a set X. The main uniqueness results are proved in Section 4.

There is an extensive literature on the subject of uniqueness of Haar measures. Chapter XI of P. R. Halmos [4], Chapter VII of I. E. Segal and R. A. Kunze [10], Chapter III of L. Nachbin [6] and §55 of K. R. Parthasarathy [8] are most relevant for our treatment of Haar measure in general. The idea of using set-theoretic considerations in proving the uniqueness property dealt with here is due to Harazišvili [5] who established it for the cases when G is either a group of isometries of $\mathbf{R}^n$ or a subgroup of a topological group X.

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1. Measure-theoretic preliminaries.

This section contains measure-theoretic notation, terminology and a couple of classical results to be used below. The reader is referred to [4], [3], [10], [6] and [8] for more details.

By a *measure on a set* X we mean a countably additive, nonzero function ν defined on a σ-algebra $\mathcal{A}$ of subsets of X and assuming values in $[0, +\infty]$. The triple $(X, \mathcal{A}, \nu)$ is then a *measure space.*

A measure ν on X is *σ-finite* if X is a countable union of sets of finite measure; it is *diffused* if $\nu(\{x\}) = 0$ for every $x \in X$ and *complete* if $\mathcal{N}_\nu \subseteq \mathcal{A}$, where $\mathcal{N}_\nu = \{A \subseteq X : \nu^*(A) = 0\}$, $\nu^*(A) = \inf\{\nu(B) : A \subseteq B \in \mathcal{A}\}$, for $A \subseteq X$.

$(X, \mathcal{A}, \nu)$ is *atomless* if every set $A \in \mathcal{A} \setminus \mathcal{N}_\nu$ has disjoint subsets $C, D \in \mathcal{A} \setminus \mathcal{N}_\nu$ and then ν is *atomless.*

$\mathrm{non}(\mathcal{N}_\nu)$ is the least cardinality of a subset of X not in $\mathcal{N}_\nu$.

$\mathrm{add}(\mathcal{N}_\nu)$ is the least cardinality of a subset of $\mathcal{N}_\nu$ whose union is not in $\mathcal{N}_\nu$.

We say that a group G is a *group of transformations* of a set X if G *acts on* X in the sense that there is a map $\langle g, x\rangle \to gx$ of $G \times X$ into X such that:

(i) for each $g \in G, x \to gx$ is a permutation of X,

(ii) for all $x \in X$ and $g_1, g_2 \in G$, $g_1(g_2x) = (g_1 \cdot g_2)x$.

We then write:

$gA = \{gx : x \in A\}$ where $A \subseteq X$, $g \in G$,

$Gx = \{gx : g \in G\}$ (the *G-orbit* of a point $x \in X$),

$G_x = \{g \in G : gx = x\}$ (the *stabilizer* of a point $x \in X$),

$\mathrm{fix}(g) = \{x \in X : gx = x\}$ where $g \in G$.

$[H]_\nu = \{g \in G : \{x \in X : gx \notin Hx\} \in \mathcal{N}_\nu\}$ where ν is a measure on X and H a subgroup of G.

Given a measure space $(X,\ \mathcal{A},\ \nu)$ and a group G of transformations of X we say that ν is *G-quasi-invariant* (resp.: *invariant*) if $\nu(A) = 0$ implies $\nu(gA) = 0$ (resp.: $\nu(gA) = \nu(A)$) for every $A \in \mathcal{A},\ g \in G$.

Let X, $\mathcal{A}$, ν and G be as above and assume that the measure ν is G-quasi-invariant.
A set $A \in \mathcal{A}$ is *ν-almost-invariant* if $\nu(A \setminus gA) = 0$ for every $g \in G$.
ν is *ergodic* if $\nu(A) = 0$ or $\nu(X \setminus A) = 0$ for any ν-almost-invariant set $A \in \mathcal{A}$.

The following basic result is due to von Neumann [7].

Proposition 1.1. Suppose that G acts on X and $\nu : \mathcal{A} \to [0, \infty]$ is a σ-finite invariant ergodic measure on X. Then if ν' is a σ-finite invariant measure on $\mathcal{A}$, absolutely continuous with respect to ν, then $\nu' = C \cdot \nu$ for a certain constant $C > 0$.

All topological spaces considered in this paper are non-discrete and Hausdorff.

Let X be a topological space.
ν is a *Borel measure on X* if it is defined on $\mathcal{BOR}(X)$, the σ-algebra of *Borel sets* in X i.e. the σ-algebra generated by all open subsets of X.
ν is *regular* if for every $A \in \mathcal{A}$:

$$\begin{aligned} \nu(A) &= \sup\{\nu(D) : D \subseteq A,\ D \text{ compact}\} \\ &= \inf\{\nu(U) : A \subseteq U,\ U \text{ open}\}. \end{aligned}$$

A σ–finite measure space $(X,\ \mathcal{A},\ \nu)$ is a *Radon measure space* if ν is complete, each open subset of X lies in $\mathcal{A}$, every point of X belongs to some open set of finite measure and for every $A \in \mathcal{A},\ \nu(A) = \sup\{\nu(D) : D \subseteq A,\ D \text{ compact}\}$.

Let X be a locally compact, σ-compact topological space and G a *group of homeomorphisms* of X i.e. a group of transformations of X such that for every $g \in G$, the permutation $x \to gx$ is a homeomorphism of X.
ν is a *Haar measure* on X if it is a G-invariant, regular, Borel measure on X, finite on compact sets and positive on nonempty open sets.

The above definition covers two most important cases for which the theory of a Haar measure is developed:
(1) G is a *uniformly equicontinuous group* of *unimorphisms* of a *uniform, uniformly locally compact*, σ-compact space X (see [10, Chapter VII]);
(2) G admits a locally compact, σ-compact topological group topology which makes its action on X continuous as a map from the product space $G \times X$ into X (see [6, Chapter III]).

Let X be a locally compact, σ-compact topological space, G a group of homeomorphisms of X and μ a Haar measure on X. Let $\bar{\mu}$ be the *measure completion* of μ i.e., the unique extension of μ to a complete measure defined on the σ-algebra $\overline{\mathcal{BOR}}(X)$, generated by $\mathcal{BOR}(X) \cup \mathcal{N}_\mu$. Then, clearly, $(X, \overline{\mathcal{BOR}}(X), \bar{\mu})$ is a σ-finite Radon measure space and $\bar{\mu}$ is G-invariant.

The following is a consequence of Proposition 1.1.

Proposition 1.2. Suppose that X is a locally compact, σ-compact topological space, G is a group of homeomorphisms of X and μ is a Haar measure on X. Then $\bar{\mu}$ is a unique, up to a constant factor, σ-finite, invariant measure on $\overline{\mathcal{BOR}}(X)$ if and only if μ is ergodic and any σ-finite invariant measure on $\overline{\mathcal{BOR}}(X)$ is absolutely continuous with respect to $\bar{\mu}$.

2. Set-theoretic preliminaries.

Our set-theoretic notation and terminology are standard. An ordinal is the set of its predecessors and a cardinal is an initial ordinal. In particular, $\omega = \{0, 1, \ldots\}$ is both the set of natural numbers and its cardinality, and ω_1 is the first uncountable cardinal. If A is a set, then $\mathcal{P}(A)$ is its power set and $|A|$ its cardinality.

Most of the set-theoretic content of this paper depends on the notion of a real-valued measurable cardinal and the impact of its existence on measure theory. The basic text for this material is Fremlin [3].

Let κ be a cardinal.

κ is *real-valued measurable* (resp.: *atomlessly measurable*) if there exists a σ-finite, diffused (resp.: atomless) measure m defined on $\mathcal{P}(\kappa)$ such that $\mathrm{add}(\mathcal{N}_m) = \kappa$.

We shall often use a basic result of Ulam, that if m is a σ-finite, diffused (resp.: atomless) measure defined on $\mathcal{P}(X)$, then $\mathrm{add}(\mathcal{N}_m)$ is a real-valued measurable (resp.: atomlessly measurable) cardinal and in either case, the cardinal $\mathrm{add}(\mathcal{N}_m)$ is regular and weakly inaccessible (see [3,1D]).

Suppose that $(X,\, \mathcal{A},\, \nu)$ and $(Y,\, \Sigma,\, \lambda)$ are σ-finite measure spaces and let $P \subseteq X \times Y$.

We write $P_x = \{y \in Y : \langle x, y\rangle \in P\}$ and $P^y = \{x \in X : \langle x, y\rangle \in P\}$ where $x \in X,\ y \in Y$.

We say that P *satisfies the Weak Fubini Property (*WFP) *for* $\nu \times \lambda$ if the following implication is true:

If for every $x \in X,\ P_x \in \mathcal{N}_\lambda$, then there exists $y \in Y$ such that $X \setminus P^y \notin \mathcal{N}_\nu$.

Note that by the Fubini theorem, all elements of the product σ-algebra $\mathcal{A} \otimes \Sigma$ satisfy the WFP for $\nu \times \lambda$. The next three lemmas provide more information.

Lemma 2.1. Suppose that $(X,\, \mathcal{A},\, \nu)$ and $(Y,\, \Sigma,\, \lambda)$ are σ-finite Radon measure spaces. Then all Borel subsets of $X \times Y$ satisfy WFP for $\nu \times \lambda$.

Proof. This follows from the fact that Borel sets in $X \times Y$ belong to the domain of the Radon product measure of ν and λ for which a version of the Fubini theorem is true (see [2, 1.11, 1.13]). $\square$

Lemma 2.2. Suppose that $(X, \mathcal{A}, \nu)$ and (Y, Σ, λ) are σ-finite measure spaces. If $\mathrm{non}(\mathcal{N}_\lambda) < \mathrm{add}(\mathcal{N}_\nu)$, then all subsets of $X \times Y$ satisfy WFP for $\nu \times \lambda$.

Proof. Take an arbitrary set $D \subseteq X \times Y$ and suppose, towards a contradiction, that for every $x \in X$, $D_x \in \mathcal{N}_\lambda$ but for every $y \in Y, X \setminus D^y \in \mathcal{N}_\nu$.

The following trick is well-known:

Take a set $B \subseteq Y$ such that $\lambda^*(B) > 0$ and $|B| = \mathrm{non}(\mathcal{N}_\lambda)$. Let $A = \bigcap_{y \in B} D^y$. Since $|B| < \mathrm{add}(\mathcal{N}_\nu)$, $A \neq \emptyset$.

Take an arbitrary $x \in A$.

Since $\lambda^*(D_x) = 0, B \setminus D_x \neq \emptyset$. But this contradicts the fact that by the definition of A, $B \subseteq \bigcap_{x \in A} D_x$. $\square$

Lemma 2.3. Suppose that $(X, \mathcal{P}(X), m)$ is a σ-finite, atomless measure space and (Y, Σ, λ) is a σ-finite Radon measure space of Maharam type less than $\mathrm{add}(\mathcal{N}_m)$.

Then all subsets of $X \times Y$ satisfy WFP for $m \times \lambda$.

Proof. The point is that if the Maharam type of a σ-finite Radon measure space (Y, Σ, λ) is less than an atomlessly measurable cardinal, then $\mathrm{non}(\mathcal{N}_\lambda) = \omega_1$ (see [3, 6G]).

Hence $\mathrm{non}(\mathcal{N}_\lambda) < \mathrm{add}(\mathcal{N}_m)$ and everything follows now from Lemma 2.2. $\square$

For more on the possibility that all subsets of the product of two well-behaved measure spaces satisfy the WFP the reader is referred to [14].

3. The existence of σ-finite invariant measures on $\mathcal{P}(X)$.

As pointed out in the Introduction, there is a tight connection between our uniqueness problem and that of existence of invariant measures which measure *all* subsets of the given space.

In this section we assume that G is a group of arbitrary transformations of an abstract set X and give necessary and sufficient conditions for the existence of a σ-finite, invariant measure defined on $\mathcal{P}(X)$.

We shall use the following notation:

$O_\kappa(H) = \{x \in X : |Hx| = \kappa\}$ where H is a subgroup of G and κ is a cardinal.

$S_\rho(G) = \{O_\kappa(H) : |H| = \rho, \kappa > \omega\}$ where ρ is a cardinal.

$I_G(m) = \{A \subseteq X : m(gA) = 0 \text{ for every } g \in G\}$ where m is a measure defined on $\mathcal{P}(X)$.

Theorem 3.1. Suppose that $(X, \mathcal{P}(X), m)$ is a probability space and that G is a group of transformations of X. Then the following are equivalent:

1) there exists a σ-finite invariant measure m' defined on $\mathcal{P}(X)$ such that $m \ll m'$;
2) there is no uncountable collection of pairwise disjoint subsets of X outside $I_G(m)$;
3) $S_\rho(G) \subseteq \mathcal{N}_m$ whenever $\omega < \rho < \mathrm{add}(\mathcal{N}_m)$;
4) $S_{\omega_1}(G) \subseteq \mathcal{N}_m$;
5) there exists a countable subgroup H of G such that $G = [H]_m$;
6) there exists a σ-finite invariant measure m' defined on $\mathcal{P}(X)$ such that $\mathcal{N}_{m'} = I_G(m)$;
7) there exists a σ-finite quasi-invariant measure m' defined on $\mathcal{P}(X)$ such that $\mathcal{N}_{m'} = I_G(m)$.

Proof. This comes from the author's previous publications [11], [12] and [13] and all the needed ideas may be found there. In the presented outline, the proof of crucial implication 4) $\to$ 5) is due to D.H. Fremlin.

The implications 1) $\to$ 2), 3) $\to$ 4), 6) $\to$ 7), 6) $\to$ 1) and 7) $\to$ 2) are obvious. So it suffices to prove the implications 2) $\to$ 3), 4) $\to$ 5) and 5) $\to$ 6).

2) $\to$ 3): It is enough to prove that $S_\rho(G) \subseteq I_G(m)$.

So suppose otherwise and let H be a subgroup of G such that $|H| = \rho < \mathrm{add}(\mathcal{N}_m)$ but $m(O_\kappa(H)) > 0$ for a certain cardinal $\kappa > \omega$.

Partition $O_\kappa(H)$ into pairwise disjoint selectors $S_\alpha,\ \alpha < \kappa$, of the collection of all H-orbits of cardinality κ. Since $\kappa > \omega$, one of them, say S_0, lies in $I_G(m)$.

Then $O_\kappa(H) = \bigcup_{\alpha<\kappa} \bigcup_{h \in H} (S_\alpha \cap h^{-1}S_0)$. Since $I_G(m)$ is invariant and closed under taking unions of less than $\mathrm{add}(\mathcal{N}_m)$ elements, this implies that $O_\kappa(H) \in I_G(m)$ contradicting the choice of H.

4) $\to$ 5): Suppose that for every countable subgroup H of G, $G \neq [H]_m$.

Using this assumption define by induction a sequence $\langle h_\alpha : \alpha < \omega_1 \rangle$ of elements of G such that $m(E_\alpha) > 0$ for each $\alpha < \omega_1$, where

$$E_\alpha = \big\{ x \in X : h_\alpha x \notin \{h_\beta x : \beta < \alpha\} \big\}.$$

Let H be subgroup of G generated by the set $\{h_\alpha : \alpha < \omega_1\}$.

Note that $\big\{ x \in X : |\{\alpha < \omega_1 : x \in E_\alpha\}| > \omega \big\} \subseteq O_{\omega_1}(H)$.

But by [1, 1E], $m\big(\big\{ x \in X : |\{\alpha < \omega_1 : x \in E_\alpha\}| > \omega \big\}\big) > 0$, so $m(O_{\omega_1}(H)) > 0$ contradicting the assumption that $S_{\omega_1}(G) \subseteq \mathcal{N}_m$.

5) $\to$ 6): Let $H = \big\{ h_n : n < \omega \big\}$ and fix a selector S of the collection of all H-orbits.

It is easy to check that the following definition works:

$$m'(A) = \sum_{k \le \omega} k \cdot \Big(\sum_{n<\omega} \frac{1}{2^{n+1}} \cdot m(h_n \big\{ x \in S : |A \cap Hx| = k \big\} \Big) \qquad \text{for } A \subseteq X$$

where $\omega \cdot 0 = 0$ and $\omega \cdot t = \infty$, if $t \neq 0$. $\square$

As a corollary we obtain the basic structural theorem for σ-finite, invariant measures defined on $\mathcal{P}(X)$.

Theorem 3.2. If m is a σ-finite quasi-invariant measure defined on $\mathcal{P}(X)$, then there exists a countable subgroup H of G such that $G = [H]_m$. $\square$

4. The uniqueness results.

In this section we assume that X is a locally compact, σ-compact topological space, G is a group of homeomorphisms of X and μ is a Haar measure on X.

We shall need one more piece of notation:
$P(H) = \big\{\langle x, g\rangle \in X \times G : gx \in Hx\big\}$ where H is a subgroup of G.

The following result reveals the key observations for our approach to the uniqueness problem.

Theorem 4.1.

(i) If there is no σ-finite, invariant measure on $\mathcal{P}(X)$, then every σ-finite invariant measure on $\overline{\mathcal{BOR}}(X)$ is absolutely continuous with respect to $\bar{\mu}$.

(ii) If there is a σ-finite, invariant measure on $\mathcal{P}(X)$, then there exists a probability Radon measure ν on X such that $G = [H]_\nu$ for a certain countable subgroup H of G.

Proof.

(i) Take an arbitrary σ-finite invariant measure ν on $\overline{\mathcal{BOR}}(X)$ and suppose, towards a contradiction, that there exists a set $Z \in \overline{\mathcal{BOR}}(X)$ such that $\bar{\mu}(Z) = 0$ but $\nu(Z) > 0$.

Using an exhaustion argument find elements $g_n \in G, n \in \mathbf{N}$, such that the set $Y = \bigcup_{n\in\mathbf{N}} g_n Z$ is ν-almost invariant.

Define a measure m on $\mathcal{P}(X)$ by:

$$m(A) = \nu(A \cap Y) \qquad \text{for } A \subseteq X.$$

By the choice of Y, m is a σ-finite and invariant measure on $\mathcal{P}(X)$, contrary to our assumption that such a measure does not exist.

(ii) By Theorem 3.2, there exists a countable subgroup H of G such that $G = [H]_m$. Replace m by an equivalent probability measure m' on $\mathcal{P}(X)$.

Then the restriction of m' to the σ-algebra $\mathcal{B}(X)$ of Baire sets in X i.e. the σ-algebra generated by all compact G_δ subsets of X, is a probability Baire measure on X, so it can be extended to a Radon measure ν (see [4, 54.D]).

We are going to show that $G = [H]_\nu$.

So fix an arbitrary $g \in G$.

Let $P = P(H)$ and note that the set $X \setminus P^g = \bigcap_{h\in H}\{x \in X : gx \neq hx\}$ is a G_δ subset of X.

Claim. If A is a G_δ subset of X and $\nu(A) > 0$, then there exists a Baire set $B \subseteq A$ with $\nu(B) > 0$.

Proof of Claim. Let $A = \bigcap_{n\in\mathbf{N}} U_n$, where U_n are open. By the regularity of ν, there exists a compact set $C \subseteq A$ such that $\nu(C) > 0$. Now for every $n \in \mathbf{N}$ there is a compact G_δ set C_n such that $C \subseteq C_n \subseteq U_n$ and it suffices to set $B = \bigcap_{n\in\mathbf{N}} C_n$ to get what was required.

Now suppose that $\nu(X \setminus P^g) > 0$. By what has just been proved, there is $B \in \mathcal{B}(\mathrm{X})$ such that $B \subseteq X \setminus P^g$ and $\nu(B) > 0$. But $m'(X \setminus P^g) = 0$, so $m'(B) = 0$ contradicting the fact that $m'|\mathcal{B}(X) = \nu|\mathcal{B}(X)$.

Thus $G = [H]_\nu$. $\square$

In view of Theorems 3.2 and 4.1, the problem emerges of how to determine, given a σ-finite measure ν on X and a countable subgroup H of G, that $G \neq [H]_\nu$.

Lemma 4.2. Suppose that ν is a σ-finite measure on X and H is a countable subgroup of G. Then:

(i) If there exists a σ-finite measure λ on G such that for every $x \in X$, $P(H)_x \in \mathcal{N}_\lambda$ and $P(H)$ satisfies the WFP for $\nu \times \lambda$, then $G \neq [H]_\nu$.

(ii) If the group G admits a locally compact, σ-compact topological group topology, λ is a Haar measure on G, $G_x \in \overline{\mathcal{BOR}}(G)$ for every $x \in X$, all G-orbits are uncountable and $P(H)$ satisfies the WFP for $\nu \times \lambda$, then $G \neq [H]_\nu$.

Proof. Set $P = P(H)$.

(i) Note that $P^g = \{x \in X : gx \in Hx\}$. Hence, by WFP, $\{x \in X : gx \notin Hx\} \notin \mathcal{N}_\nu$ for a certain $g \in G$, which shows that $G \neq [H]_\nu$.

(ii) This will follow from (i) as soon as we prove that $P_x \in \mathcal{N}_\lambda$ for every $x \in X$.

Note that $P_x = \bigcup_{h\in H} h \cdot G_x$.

Since Gx is uncountable, so is the index of the subgroup G_x in the group G. Hence by the σ-finiteness of the measure $\bar{\lambda}$, $\bar{\lambda}(G_x) = 0$.

Consequently, $\bar{\lambda}(P_x) = 0$ for every $x \in \mathrm{X}$. $\square$

We shall need one more auxiliary result.

Lemma 4.3. Suppose that m is a σ-finite, invariant measure on $\mathcal{P}(X)$ and H is a countable subgroup of G such that $G = [H]_m$. If U is an arbitrary open subset of X such that $\bigcup_{g\in G} gU = X$, then $m(X \setminus \bigcup_{h\in H} hU) = 0$; in particular, $m(U) > 0$.

Proof. Let $A = X \setminus \bigcup_{h\in H} hU$ and suppose, towards a contradiction, that $m(A) > 0$. Due to the σ–compactness of X, there is a compact set K such that $m(A \cap K) > 0$. Since $K \subseteq \bigcup_{g\in G} gU$, there is a single $g \in G$ such that $m(A \cap K \cap gU) > 0$. But $A \cap K \cap gU \subseteq \{x \in X : \ g^{-1}(x) \notin Hx\}$ – a contradiction. $\square$

Now we are ready to prove the main theorem; case 6) was first proved by Penconek with a different argument, cases 3) and 4) generalize the results from [9].

Theorem 4.4. Suppose that X is a locally compact, σ-compact topological space, G is a group of homeomorphisms of X and μ is a Haar measure on X.

(i) If $\bar{\mu}$ is a unique, up to a constant factor, G–invariant σ-finite measure on $\overline{\mathcal{BOR}}(X)$, then all orbits are uncountable.

(ii) If μ is ergodic and all orbits are uncountable, then $\bar{\mu}$ is a unique, up to a constant factor, G–invariant σ-finite measure on $\overline{\mathcal{BOR}}(X)$, whenever either of the following conditions holds:

1) For every $g \in G$, there is no real-valued measurable cardinal $\leq |\mathrm{fix}(g)|$;

2) There is no real-valued measurable cardinal $\leq \sup\{|G_x| : x \in X\}$;

3) G admits a locally compact, σ-compact topological group topology such that there is no atomlessly-measurable cardinal $\leq$ the topological weight of G and $G_x \in \overline{\mathcal{BOR}}(G)$ for every $x \in X$;

4) G admits a locally compact, σ-compact topological group topology making the action continuous;

5) The space X is uniform and non-separable, and G consists of uniformly equicontinuous unimorphisms of X;

6) The space X is non-separable, all orbits are dense in X and there is no real-valued measurable cardinal $\leq$ the topological weight of X.

Moreover, in cases 5) and 6) the uncountability of orbits follows automatically from the remaining assumptions.

Proof. (i) Suppose that there exists a countable orbit $\mathcal{O}$. Then the function:

$$\nu(A) = |A \cap \mathcal{O}| \qquad \text{for } A \subseteq \overline{\mathcal{BOR}}(X)$$

is a σ-finite invariant measure on $\overline{\mathcal{BOR}}(X)$. By the uniqueness of $\bar{\mu}$, there exists a constant $c > 0$ such that $\bar{\mu}(A) = c|A \cap \mathcal{O}|$ for every $A \in \overline{\mathcal{BOR}}(X)$. But now, since μ is inner regular for compact sets and assumes positive values on non-empty open sets, this easily implies that every open set is compact. This contradicts the assumption that the space X is non-discrete.

(ii) By Proposition 1.2 and Theorem 4.1.(i), in all cases it suffices to show that there is no σ-finite, invariant measure on $\mathcal{P}(X)$.

So suppose otherwise and let m be a σ-finite, invariant measure on $\mathcal{P}(X)$.

Note that since m is σ-finite, the uncountability of orbits implies that it is diffused. Hence $\mathrm{add}(\mathcal{N}_m)$ is a real-valued measurable cardinal.

Let H be a countable subgroup of G such that $G = [H]_m$, whose existence is guaranteed by Theorem 3.2, and let $P = P(H)$.

Case 1) Take an arbitrary $g \in G$. Note that $P^g = \bigcup_{h\in H} \text{fix}(h^{-1}g)$. But for every $h \in H$, $|\text{fix}(h^{-1}g)| < \text{add}(\mathcal{N}_m)$, hence $m(P^g) = 0$. It follows that $G \neq [H]_m$ contradicting the choice of H.

Case 2) Let $\rho = \sup\{|G_x| : x \in X\}$.

Since the cardinal $\text{add}(\mathcal{N}_m)$ is weakly inaccessible, $\rho^+ < \text{add}(\mathcal{N}_m)$.

If $|G| = \rho$, then $X = \bigcup_{\omega<\kappa\leq\rho} O_\kappa(G)$ so, by Theorem 3.1.3), $m(X) = 0$ — a contradiction.

If $|G| > \rho$, let λ be a measure on G defined by:

$$m(A) = 0 \text{ if } |A| \leq \rho \text{ and } m(A) = 1 \text{ if } |X \setminus A| \leq \rho.$$

Since $\text{non}(\mathcal{N}_\lambda) = \rho^+ < \text{add}(\mathcal{N}_m)$, Lemma 2.2 tells us that P satisfies WFP for $m \times \lambda$.

Moreover, for every $x \in X$, $P_x = \bigcup_{h\in H} hG_x \in \mathcal{N}_\lambda$. So, by Lemma 4.2.(i), $G \neq [H]_m$ – a contradiction.

Case 3) First prove the following

Claim. The measure space $(X, \mathcal{P}(X), m)$ is atomless.

Proof of Claim. Suppose otherwise and fix a set $A \subseteq X$ such that $m(A) > 0$ and for any $B \subseteq X$ either $m(A \cap B) = 0$ or $m(A \setminus B) = 0$. Then, since $G = [H]_m$, for every $g \in G$ there is $h_g \in H$ with $m(\{x \in A : gx \neq h_g x\}) = 0$. Similarly, since the space X is σ-compact, there is a compact set $K \subseteq X$ with $m(A \setminus K) = 0$.

For each $g \in G$ set $D_g = \{x \in K : g(x) = h_g(x)\}$; note that $m(A \setminus D_g) = 0$. It follows that the collection $\{D_g : g \in G\}$ has the finite intersection property and since its members are closed subsets of a compact set, $\bigcap_{g\in G} D_g \neq \emptyset$. But if $x \in \bigcap_{g\in G} D_g$, then $Gx \subseteq Hx$, contradicting the uncountability of Gx.

It follows that $\text{add}(\mathcal{N}_m)$ is an atomlessly-measurable cardinal.

The measure space $(G, \overline{\mathcal{BOR}}(G), \bar{\lambda})$ is Radon and, by the hypotheses, its Maharam type (= its topological weight) is $< \text{add}(\mathcal{N}_m)$. So, by Lemma 2.3, P satisfies WFP for $m \times \bar{\lambda}$.

By Lemma 4.2.(ii), $G \neq [H]_m$ – a contradiction again.

Case 4) By Theorem 4.1.(ii), there is a probability Radon measure ν on X such that $G = [H]_\nu$.

But on the other hand, the set P is Borel in the space $X \times G$, so by Lemma 2.1, it satisfies the WFP for $\nu \times \bar{\lambda}$, λ being a Haar measure on G.

Hence, by Lemma 4.2, $G \neq [H]_\nu$ – a contradiction.

Case 5) Recall that the topology of X is defined by a collection of *separating pseudometrics*, each of which is invariant under the group G (see [10, Corollary 7.3.1]). Using this it is easy to see that if G' is a subgroup of G, then all G'-orbits are dense in X iff $X = cl(\bigcup_{g\in G'} gU)$ for every open $U \neq \emptyset$, where $cl(A)$ is the closure of a set A in the space X.

A contradiction with the nonseparability of X will be reached by proving that every H-orbit is dense in X.

So let U be an arbitrary non–empty open subset of X First note that since μ is ergodic and assumes positive values on non–empty open sets, $X = cl(\bigcup_{g\in G} gU)$. By the preceeding remarks, all G–orbits are dense in X i.e. $X = \bigcup_{g\in G} gV$ for every open $V \neq \emptyset$.

Then, by Lemma 4.3, $m(X \setminus \bigcup_{h\in H} hU) = 0$. It follows that $m(X \setminus cl(\bigcup_{h\in H} hU)) = 0$ which in turn gives $X = cl(\bigcup_{h\in H} hU)$, since by Lemma 4.3, the density of all G-orbits implies that m assumes positive values on non–empty open sets.

Case 6) Fix an open base $\{U_\alpha : \alpha < \rho\}$ for the topology of X of the minimal cardinality ρ. Since all G-orbits are dense in X, for each $\alpha < \rho$, $X = \bigcup_{g\in G} gU_\alpha$, so by Lemma 4.3, $m(X \setminus \bigcup_{h\in H} hU_\alpha) = 0$.

Since $\rho < \mathrm{add}(\mathcal{N}_m)$, it follows that $\bigcap_{\alpha<\rho} \bigcup_{h\in H} hU_\alpha \neq \emptyset$. But if $x \in \bigcap_{\alpha<\rho} \bigcup_{h\in H} hU_\alpha$, then $cl(Hx) = X$, contradicting the non–separability of the space X. $\square$

In view of the proof above, it is tempting to conjecture that the uncountability of all orbits of an arbitrary group of homeomorphisms of a locally compact, σ-compact space X always implies the nonexistence of a σ-finite, invariant measure on $\mathcal{P}(X)$. This, however, is not the case, as has recently been found out by Penconek.

In Penconek's example G is a group of homeomorphisms of $\mathbf{R}^2$ under which the Lebesgue measure l_2 is invariant and ergodic, all G–orbits are uncountable and dense in $\mathbf{R}^2$, and, if the cardinality of the continuum is at least real–valued measurable, there exists a σ-finite, G–invariant measure on $\mathcal{P}(\mathbf{R}^2)$ whose restriction to the σ–algebra $\mathcal{L}_2$ of the Lebesgue measurable subsets of the plane is σ–finite and orthogonal to l_2. In particular, l_2 is a Haar measure with respect to the group G but is not unique in the family of all σ–finite, G–invariant measures defined on $\mathcal{L}_2$ – an "exotic" measure exists.

It turns out, however, that we may eliminate such "exotic" examples by restricting our attention to *locally finite* measures which for every point in the space X assign a positive finite value to one of its open neighbourhoods. This is based on the following observation.

Lemma 4.5. Suppose that all G–orbits are dense in X and m is a σ-finite, invariant measure on $\mathcal{P}(X)$. If U is an arbitrary non–empty open subset of X, then $m(U) = \infty$.

Proof. This follows immediately from Lemma 3.6 of [15], which states that under the given assumptions there exists a countable partition $\langle A_n : n \in \mathbf{N}\rangle$ of U and a sequence $\langle g_n : n \in \mathbf{N}\rangle$ of elements of G such that $\langle g_n A_n : n \in \mathbf{N}\rangle$ forms a partition of X. $\square$

Now we can state our final uniqueness result.

Theorem 4.6. Suppose that X is a locally compact, σ–compact topological space, G is a group of homeomorphisms of X and μ is a Haar measure on X.

If μ is ergodic and all orbits are dense in X, then $\bar{\mu}$ is a unique, up to a constant factor, invariant locally finite measure on $\overline{\mathcal{BOR}}(X)$.

Proof. By Proposition 1.1, it suffices to take an arbitrary locally finite invariant measure ν on $\overline{\mathcal{BOR}}(X)$ and show that it is absolutely continuous with respect to $\bar{\mu}$.

So suppose otherwise and follow the proof of Theorem 4.1.(i) to obtain a σ–finite invariant measure m defined on $\mathcal{P}(X)$, such that for every $A \in \overline{\mathcal{BOR}}(X)$, $m(A) \leq \nu(A)$. But this implies, by Lemma 4.5, that $\nu(U) = \infty$ for every non–empty open subset U of X, contradicting the local finiteness of the measure ν. □

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