FUBINI PROPERTIES FOR FILTER–RELATED σ-IDEALS

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Abstract. Let $\mathcal{M}_{\mathcal{F}}$ be the σ-ideal of meager sets in the Ellentuck topology related to a filter $\mathcal{F}$ on the set of natural numbers. We prove that if $\mathcal{G}$ is a filter with the Baire property and $\mathcal{F}$ is an analytic, non–countably generated p-filter, then the pair $\langle \mathcal{M}_{\mathcal{G}}, \mathcal{M}_{\mathcal{F}} \rangle$ does not have Fubini Property. In fact, there is a Borel set on the plane such that its vertical sections are in $\mathcal{M}_{\mathcal{F}}$ and the complements of its horizontal sections are in $\mathcal{M}_{\mathcal{G}}$. This partially answers a question of Recław [9].

1. Introduction

A *σ-ideal* on an uncountable Polish space X is a family $I \subseteq \mathcal{P}(X)$ which is closed under taking subsets and countable unions.

We always assume that I is *proper*, i.e., $X \notin I$, contains all singletons and has a basis consisting of Borel sets. The latter means that every set from I is covered by a Borel set from I.

Given σ-ideals I and J on Polish spaces X and Y, respectively, we say that the pair $\langle I, J \rangle$ satisfies the *Fubini Property* (FP) if for every Borel subset B of the space $X \times Y$, if all its vertical sections $B_x = \{y : \langle x, y \rangle \in B\}$ are in J, then its horizontal sections $B^y = \{x : \langle x, y \rangle \in B\}$ are in I, for every y outside a set from J. In view of the classical Fubini and Kuratowski–Ulam theorems (see [7, Theorem 8.41]), the pairs $\langle \mathcal{N}_\mu, \mathcal{N}_\nu \rangle$ and $\langle \mathcal{M}(X), \mathcal{M}(Y) \rangle$ satisfy FP. Here $\mathcal{N}_\mu$ is the collection of all subsets of X, having outer measure zero with respect to a Borel σ-finite continuous measure μ on X and $\mathcal{M}(X)$ is the family of all meager subsets of X (we tacitly assume that X has no isolated points, so that the σ-ideal $\mathcal{M}(X)$ contains all singletons).

We say that σ-ideals I and J on spaces X and Y, respectively, are *Borel isomorphic* ($I \equiv_B J$) if there exists a Borel isomorphism $f : X \to Y$ between X and Y such that for $A \in B(X)$, the family of all Borel subsets of X, $A \in I \iff f[A] \in J$. Note that from the point of view of FP, Borel isomorphic ideals are identical. In particular,

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2000 *Mathematics Subject Classification.* Primary 54H05, 54E52, 03E15; Secondary 03E05.

Key words and phrases. σ-ideal, Fubini Property, filter on ω, Ellentuck topology, meager sets.

The author was supported by KBN grant 2 P03A 002 19.

there is essentially one, up to a Borel isomorphism, ccc σ-ideal I (i.e., ccc means that there is no uncountable family of disjoint Borel sets outside I) on a Polish space X such that the complete Boolean algebra $B(X)/(I\cap B(X))$ is atomless and contains a countable dense subset (see e.g. [5], 25.4). It will be referred to as *the category σ-ideal* and denote simply by $\mathcal{M}$. Likewise, there is essentially one σ-ideal of the form $\mathcal{N}_\mu$ for a σ-finite continuous Borel measure μ on an uncountable Polish space and we will denote it simply by $\mathcal{N}$ (see [7], 17.41). Thus the Fubini and Kuratowski–Ulam theorems show essentially just two cases of FP: $\langle\mathcal{N},\mathcal{N}\rangle$ and $\langle\mathcal{M},\mathcal{M}\rangle$.

In [10] I. Recław and the present author gave a number of other examples including $\langle\mathcal{M}, J\rangle$ for every J generated by any family of closed subsets of Y and $\langle\mathcal{N}, J\rangle$ whenever J is generated by some particular collections of closed sets (singletons, compact sets, closed measure zero sets). However, only one new example of a pair of *ccc* σ-ideals satisfying FP was found (depending on the consistency of the existence of a measurable cardinal) and it seemed that the Fubini Property among ccc σ-ideals is a very rare phenomenon.

The search for new ccc examples motivated Recław [9] to study σ-ideals related to filters on the set of natural numbers $\omega=\{0,1,2,\ldots\}$ (we use a standard convention of identifying ω with its cardinality; thus $|\omega|=\omega$).

A *filter on* ω is a family of subsets of ω closed under taking finite intersections and supersets of its elements. We always assume that a filter is proper, i.e., not equal $\mathcal{P}(\omega)$ and contains all cofinite sets.

For a filter $\mathcal{F}$ on ω let $\tau_{\mathcal{F}}$ be the topology on the set $[\omega]^\omega$ generated by the collection $O_{\mathcal{F}}$ of basic open sets of the form $[s,A]=\{X\in[\omega]^\omega:\ s\subseteq X\subseteq s\cup A\}$, where $s\in[\omega]^{<\omega}$ and $A\in\mathcal{F}$. Here $[\omega]^\omega$ ($[\omega]^{<\omega}$, resp.) is the collection of all infinite (finite, resp.) subsets of ω. If $s\in[\omega]^{<\omega}$, then we simply write $[s]$ instead of $[s,\omega]$. Thus $[s]=\{X\in[\omega]^\omega:\ s\subseteq X\}$. Similarly, if $s\in\{0,1\}^{<\omega}$, the set of all finite binary sequences, then we let $[s]=\{f\in\{0,1\}^\omega:\ s\subseteq f\}$.

It turns out (see [9]) that the collection $\mathcal{M}_{\mathcal{F}}$ of $\tau_{\mathcal{F}}$–meager sets is a ccc σ-ideal on the Polish space $[\omega]^\omega$ (with the standard Cantor-set topology, obtained by identifying subsets of ω with their characteristic functions).

A filter $\mathcal{F}$ on ω is *countably generated* if it has a *countable basis*, i.e., a family of members $\{B_n : n\in\omega\}$ of $\mathcal{F}$ such that for every $A\in\mathcal{F}$ there is $n\in\omega$ with $B_n\subseteq A$.

Recław [9] proved, in particular, that under Continuum Hypothesis there is a non–countably generated filter $\mathcal{F}$ such that the pair $\langle\mathcal{M},\mathcal{M}_{\mathcal{F}}\rangle$ has FP.

This example of a pair of ccc σ-ideals satisfying FP is essentially different from the standard $\langle \mathcal{M}, \mathcal{M} \rangle$ one, in view of the following observation.

Proposition 1.1. *The following are equivalent for a filter $\mathcal{F}$ on ω:*

(1) $\mathcal{F}$ is countably generated,
(2) $\mathcal{M}_\mathcal{F} \equiv_B \mathcal{M}$.

Proof. To prove the direction (i) $\Rightarrow$ (ii) it suffices to note that the quotient Boolean algebra $B(X)/(\mathcal{M}_\mathcal{F} \cap B(X))$ is complete, atomless and contains a countable dense subset.

To prove the converse, assume that $\mathcal{M}_\mathcal{F} \equiv_B \mathcal{M}$. Then the algebra $B(X)/(\mathcal{M}_\mathcal{F} \cap B(X))$ has a countable dense subset, so there are basic $\tau_\mathcal{F}$–open sets $[t_n, A_n] \in O_\mathcal{F}$, $n \in \omega$, such that for every $A \in \mathcal{F}$ there is $n \in \omega$ with

$$[t_n, A_n] \setminus [\emptyset, A] \in \mathcal{M}_\mathcal{F}.$$

But the set $[t_n, A_n] \setminus [\emptyset, A]$ is open in the topology $\tau_\mathcal{F}$ and the topological space $([\omega]^\omega, \tau_\mathcal{F})$ is a Baire space (see [9]), so we simply have

$$[t_n, A_n] \subseteq [\emptyset, A].$$

It follows that the collection $\{A_n \setminus N : n \in \omega,\ N \in [\omega]^{<\omega}\}$ is a base of the filter $\mathcal{F}$.

□

On the other hand, Recław [9] proved that if $\mathcal{G}$ is a filter with the Baire property (with respect to the standard topology on $[\omega]^\omega$), then for a large class of filters $\mathcal{F}$ the pair $\langle \mathcal{M}_\mathcal{G}, \mathcal{M}_\mathcal{F} \rangle$ violates FP in the strongest possible sense. Namely, there exists a *0-1 counterexample to FP*, i.e., a (standard) Borel subset of $[\omega]^\omega \times [\omega]^\omega$ such that its vertical sections are in $\mathcal{M}_\mathcal{F}$ and the complements of its horizontal sections are in $\mathcal{M}_\mathcal{G}$. Recław asked (see [9, Problem 2]), whether there exist (standard) Borel filters $\mathcal{F}$, $\mathcal{G}$, at least one not countably generated, such that $\langle \mathcal{M}_\mathcal{G}, \mathcal{M}_\mathcal{F} \rangle$ has FP. Our main result (see Theorem 3.5) gives the following partial answer: if at least one of Borel filters $\mathcal{F}$, $\mathcal{G}$ is a non-countably generated p-filter, then there is a 0-1 counterexample to FP for $\langle \mathcal{M}_\mathcal{G}, \mathcal{M}_\mathcal{F} \rangle$.

2. Preliminaries

All topological and descriptive set-theoretic notions (like the Baire property, Borel or analytic sets) concerning subsets of the space $[\omega]^\omega$ will refer, unless explicitly stated otherwise, to the standard topology, inherited from the Cantor set $\{0, 1\}^\omega$ by identifying subsets of ω with their characteristic functions.

An *ideal on* ω is a family of subsets of ω closed under taking finite unions and subsets of its elements. We always assume that an ideal is proper and contains all finite sets.

For an ideal $\mathcal{I}$ (a filter $\mathcal{F}$, resp.) on ω, $\mathcal{I}^*$ ($\mathcal{F}^*$, resp.) denotes the *dual filter* (*the dual ideal*, resp.), i.e., the filter (the ideal, resp.) on ω consisting of the complements of elements of $\mathcal{I}$ ($\mathcal{F}$, resp.).

We can clearly talk about filters and ideals on any other countably infinite set, by identifying this set with ω via a fixed bijection.

We shall use the following combinatorial characterization of filters with the Baire property, due to Talagrand [12] (see also [1, Theorem 4.1.2]).

The quantifier $\forall^\infty k$ ($\exists^\infty k$, resp.) means: for all but finitely many (for infinitely many, resp.) $k \in \omega$.

Proposition 2.1. *The following are equivalent for a filter $\mathcal{F}$ on ω:*

(1) $\mathcal{F}$ has the Baire property,
(2) There is a sequence $n_0 < n_1 < \dots$ of natural numbers such that

$$(\forall X \in \mathcal{F})(\forall^\infty k)\ X \cap [n_k, n_{k+1}) \neq \emptyset.$$

A particularly important and intensively studied family of filters with the Baire property are analytic filters and among them, analytic p-filters. Recall that a filter $\mathcal{F}$ on ω is a *p-filter* if for every sequence $A_n \in \mathcal{F}$, $n \in \omega$, there is an $A \in \mathcal{F}$ such that $A \setminus A_n$ is finite for all n. There is a vast literature concerning analytic p-filters (or rather their dual ideals, i.e., analytic *p-ideals*) – see Todorcevic [13], Solecki [11] and Farah [2].

Recall that a map $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure on* ω if $\phi(\emptyset) = 0$, $\phi(A) \leq \phi(B)$ whenever $A \subseteq B$ and $\phi(A \cup B) \leq \phi(A) + \phi(B)$ for any A, $B \subseteq \omega$. A submeasure ϕ on ω is *lower semicontinuous* if for all $A \subseteq \omega$ we have $\phi(A) = \lim_{n\to\infty} \phi(A \cap n)$. A lower semicontinuous submeasure ϕ induces *the exhaustive ideal of* ϕ, $\mathrm{Exh}(\phi)$, defined as follows

$$\mathrm{Exh}(\phi) = \{A \subseteq \omega : \lim_{n\to\infty} \phi(A \setminus n) = 0\}.$$

It is not difficult to prove that $\mathrm{Exh}(\phi)$ is a Borel, in fact $F_{\sigma\delta}$, p-ideal – see [2, Lemma 1.2.2].

We shall use the following characterization of analytic p-filters due to Solecki [11].

Theorem 2.2. *The following are equivalent for a filter $\mathcal{F}$ on ω:*

(1) $\mathcal{F}$ is an analytic p-filter,
(2) $\mathcal{F} = \mathcal{I}^*$ where $\mathcal{I} = Exh(\phi)$ for some lower semicontinuous submeasure ϕ on ω.

Examples of other analytic filters are given in 3.3.

We will consider the following notion of reducing one σ-ideal on a Polish to another.

If I_1 and I_2 are two σ-ideals on Polish spaces X_1 and X_2, respectively, we write $I_2 \leq_B I_1$ if there exists a Borel function $\varphi : X_1 \to X_2$ such that $C \in I_2$ if and only if $\varphi^{-1}[C] \in I_1$ for every $C \in \mathbf{B}(X_2)$. So, $I_2 \leq I_1$ means that $I_2 \cap \mathbf{B}(X_2)$ is below $I_1 \cap \mathbf{B}(X_1)$ with respect to the Rudin-Keisler order and the function witnessing is Borel.

The relevance of the preordering $\leq_B$ to FP is revealed by the following fact.

Proposition 2.3. *Let I_1, I_2 be σ-ideals on Polish space X_1 X_2, respectively, and assume that $I_2 \leq_B I_1$. Then for an arbitrary σ-ideal J on a Polish space Y*

(1) *if the pair $\langle I_1, J\rangle$ satisfies FP, then so does the pair $\langle I_2, J\rangle$,*
(2) *if there is a 0-1 counterexample to FP for $\langle I_2, J\rangle$, then there is also one for $\langle I_1, J\rangle$.*

Proof. Let $\varphi : X_1 \to X_2$ be a Borel function such that $C \in I_2$ if and only if $\varphi^{-1}[C] \in I_1$ for every $C \in \mathbf{B}(X_2)$.

Assume that a set $B \in \mathbf{B}(X_2 \times Y)$ is a counterexample (a 0-1 counterexample, resp.) to FP for $\langle I_2, J\rangle$, i.e., $B_x \in J$ for each $x \in X_2$ but $\{y \in Y : B^y \notin I_2\} \in J$ $\quad (X \setminus B^y \in I_2$ for each $y \in Y$, resp.).

Let $A = (\varphi, \mathrm{id}_Y)^{-1}[B]$; then $A \in \mathbf{B}(X_1 \times Y)$ and $A_x = B_{\varphi(x)}$, $A^y = \varphi^{-1}[B^y]$ for every $x \in X_1$, $y \in Y$. It follows that A is a counterexample (a 0-1 counterexample, resp.) to FP for $\langle I_1, J\rangle$.

□

The proof of the following fact is essentially just a refinement of Shelah's argument showing that a forcing satisfying certain conditions adds a Cohen real (see [6, Lemma 8.1]).

Proposition 2.4. *If $\mathcal{G}$ is a filter with the Baire property, then $\mathcal{M} \leq_B \mathcal{M}_{\mathcal{G}}$.*

Proof. Using 2.1, fix a sequence $n_1 < n_2 < \ldots$ of natural numbers such that $n_1 = 0$ and

$$(\forall A \in \mathcal{G})(\forall^\infty k) A \cap [n_k, n_{k+1}) \neq \emptyset. \tag{$*$}$$

Let $X_1 = [\omega]^\omega$ and $X_2 = \{x \in 2^\omega : x(0) = 1\}$. For $n \in \omega$ let $\rho(n)$ be the binary expansion of the unique k such that $n_k \leq n < n_{k+1}$.

Define a function $\varphi : X_1 \to X_2$ by:

$$\varphi(A) = \rho(e_A(0))^\frown \rho(e_A(1))^\frown \ldots,$$

where e_A is the increasing enumeration of A.

We claim that the function φ is Borel (actually: continuous) and for every $C \in \mathbf{B}(X_2)$, $C \in \mathcal{M}(X_2)$ if and only if $\varphi^{-1}[C] \in \mathcal{M}_{\mathcal{G}}$.

To see that φ is continuous note that for any $n \in \omega$ and $A \in [\omega]^\omega$, already first $n+1$ elements of A decide whether $A \in \varphi^{-1}[\{x \in X_2 : x(n) = 1\}]$.

Next let G be open dense in X_2. Then $\varphi^{-1}[G]$ is dense in $\tau_{\mathcal{G}}$. To prove this, take an arbitrary $[s, A] \in O_{\mathcal{G}}$. By the density of G, there is

$t \in \{0,1\}^{<\omega}$ such that

$$[\rho(e_s(0))^\frown \ldots ^\frown \rho(e_s(|s|-1))^\frown t] \subseteq G.$$

By $(*)$, there is $k \in A$, $k > e_s(|s|-1)$ such that $\rho(k)$ extends the sequence t. Then

$$[s^\frown k, A \setminus \{0,\ldots,k\}] \subseteq \varphi^{-1}[G].$$

From what we have just proved, it follows that $\varphi^{-1}[C] \in \mathcal{M}_{\mathcal{G}}$, whenever $C \in \mathcal{M}(X_2)$.

Finally, let $E \in \mathbf{B}(X_2) \setminus \mathcal{M}(X_2)$. Then $E = U \triangle C$ for some open $U \neq \emptyset$ and $C \in \mathcal{M}(X_2)$. Hence $\varphi^{-1}[E]= \varphi^{-1}[U] \triangle \varphi^{-1}[C]$ and we will conclude that $\varphi^{-1}[E] \notin \mathcal{M}_{\mathcal{G}}$ as soon as we realize that $\varphi^{-1}[U] \neq \emptyset$. But there is $r \in [\omega]^{<\omega}$ such that $[r] \subseteq U$ and $r(0) = 1$, so $A \in \varphi^{-1}[U]$ whenever the binary expansion of the least element of A equals r.

□

3. Analytic filters

We say that a filter $\mathcal{F}$ on ω has *property (S)*, if there is a partition $\{A_n : n \in \omega\}$ of ω such that:

(S$_1$) for every $m \in \omega$, $\bigcup_{n>m} A_n \notin \mathcal{F}^*$,

(S$_2$) if $S \subseteq \omega$ is an arbitrary selector of $\{A_n : n \in \omega\}$, i.e., if $|S \cap A_n| = 1$ for each $n \in \omega$, then $S \in \mathcal{F}^*$.

Certainly not all filters have property (S).

Proposition 3.1. *If a filter $\mathcal{F}$ has property (S), then it is not countably generated.*

Proof. Let $\{A_n : n \in \omega\}$ be a partition of ω witnessing property (S) of the given filter $\mathcal{F}$.

Suppose, towards a contradiction, that $\{B_n : n \in \omega\} \subseteq \mathcal{F}$ is a countable basis of $\mathcal{F}$.

By induction construct sequences $\langle n_m \rangle_{m\in\omega}$ and $\langle k_m \rangle_{m\in\omega}$ of natural numbers such that the following conditions are satisfied for each $m \in \omega$:

$$n_m < n_{m+1},$$
$$k_m \in A_{n_m} \cap B_m$$

(at each step just pick $k_{m+1} \in \left(\bigcup_{n>n_m} A_n\right) \cap B_{m+1}$).

Finally, let $S = \{k_m : m \in \omega\}$. Then $|S \cap A_n| \leq 1$ for each $n \in \omega$ but $S \notin \mathcal{F}^*$, since $S \cap B_n \neq \emptyset$ for every $n \in \omega$ – a contradiction.

□

On the other hand, for a wide collection of analytic filters the implication above can be reversed.

Theorem 3.2. *Every non-countably generated analytic p-filter has property (S).*

Proof. Let $\mathcal{F}$ be a non-countably generated analytic p-filter. By Solecki's theorem 2.2, $\mathcal{F}^* = \mathcal{I} = \mathrm{Exh}(\phi)$ for some lower semicontinuous submeasure ϕ on ω.

For each $n \in \omega$ define

$$B_n = \left\{ k \in \omega : \phi(\{k\}) \leq \frac{1}{2^n} \right\}.$$

The key observation is the following

Claim.

$$B_n \notin \mathcal{I} \text{ for every } n \in \omega.$$

Proof of Claim. Suppose otherwise and fix an n with $B_n \in \mathcal{I}$. Then a contradiction will be reached as soon as we see that

$$\mathcal{I} = \{X \subseteq \omega : |X \setminus B_n| < \omega\}, \tag{$*$}$$

since this implies that the filter $\mathcal{F}$ is countably generated.

To prove equality $(*)$, take an arbitrary $X \subseteq \omega$.

If $|X \setminus B_n| < \omega$, then $X = (X \cap B_n) \cup (X \setminus B_n) \in \mathcal{I}$. This shows the inclusion "$\supseteq$".

To prove the converse, assume that $|X \setminus B_n| = \omega$. Then for every $k \in X \setminus B_n$, $\phi(\{k\}) > \frac{1}{2^n}$ and $(X \setminus B_n) \setminus m \neq \emptyset$ for each $m \in \omega$, so

$$\phi(X \setminus m) \geq \phi((X \setminus B_n) \setminus m) > \frac{1}{2^n}.$$

It follows that it is not true that $\lim_{m\to\infty} \phi(X \setminus m) = 0$, hence $X \notin \mathcal{I}$.

This completes the proof of Claim.

Clearly $B_{n+1} \subseteq B_n$ for each $n \in \omega$ and we can moreover assume, without loss of generality, that this inclusion is always proper. The point is that the inclusion is indeed proper for infinitely many n's. For otherwise, there is an n_0 with $B_n = B_{n_0}$ for each $n \geq n_0$. But then for every $k \in B_{n_0}$, $\phi(\{k\}) \leq \frac{1}{2^n}$ for each $n \geq n_0$, which implies $\phi(\{k\}) = 0$. Consequently, $\phi(B_{n_0}) = 0$ which implies $B_{n_0} \in \mathcal{I}$, contradicting the claim above.

Now let $A_n = B_n \setminus B_{n+1}$ for $n > 0$ and $A_0 = \omega \setminus \bigcup_{n>0} A_n$. We will finish the proof by showing that the partition $\{A_n : n \in \omega\}$ witnesses property (S) of $\mathcal{F}$.

So let $m > 0$. Then

$$\bigcup_{n \geq m} A_n = B_n \setminus \bigcap_{m \in \omega} B_n.$$

By Claim, $B_n \notin \mathcal{I}$. On the other hand, $\bigcap_{m\in\omega} B_n \in \mathcal{I}$, since $\phi(\{k\}) = 0$ for each $k \in \bigcap_{m\in\omega} B_n$. It follows that $\bigcup_{n\geq m} A_n \notin \mathcal{I}$.

Finally, let S be an arbitrary selector of $\{A_n : n \in \omega\}$. Then $S = \{k_n : n \in \omega\}$, where $k_n \in A_n \subseteq B_n$ for each $n > 0$. It follows that

$$\phi(S \setminus m) = \lim_{n\to\infty} \phi((S \cap n) \setminus m) \leq \lim_{n\to\infty} \sum_{m \leq k_i < n} \phi(\{k_i\}) \leq \sum_{k_i \geq m} \frac{1}{2^{k_i}},$$

hence $\lim_{m\to\infty} \phi(S \setminus m) = 0$, so $S \in \mathcal{I}$.

□

Examples of filters with property (S) include the filters dual to some other analytic ideals studied in the literature.

Proposition 3.3. *The dual filter of any of the following ideals has property (S):*

(1) the ordinal ideal $\mathcal{I}_\alpha$ on a (countable) ordinal α (of the form ω^β for some ordinal $\beta > 1$), i.e., the family of all subsets of α of strictly smaller ordinal type (see [3]*),*
(2) $NWD(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A}$ is nowhere dense$\}$ (see [4]*),*
(3) $NULL(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A}$ is of Lebesgue measure zero$\}$, where the closure $\overline{A}$ is taken in $\mathbb{R}$ (see [4]*).*

Proof. For each ideal $\mathcal{I}$ on the respective countable set X we find a strictly decreasing sequence $(X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \ldots)$ of subsets of X with the following properties:

(1) $X_n \notin \mathcal{I}$ for each $n \in \omega$,
(2) $\bigcap_{n\in\omega} X_n = \emptyset$,
(3) if $S \subseteq X_0$ is such that $|S \cap (X_n \setminus X_{n+1})| = 1$ for each $n \in \omega$, then $S \in \mathcal{I}$.

Then the partition of X consisting of sets $X \setminus X_0,\ X_0 \setminus X_1,\ X_1 \setminus X_2,\ X_2 \setminus X_3, \ldots$ witnesses property (S) of the filter $\mathcal{I}^*$.

So first let $X = \alpha$ and $\mathcal{I} = \mathcal{I}_\alpha$. Then define $X_n = \alpha \setminus \alpha_n$, where $\langle \alpha_n \rangle_{n\in\omega}$ is a strictly increasing sequence of ordinals smaller than α with $\sup_n \alpha_n = \alpha$.

Now let $X = \mathbb{Q} \cap [0,1]$ and $\mathcal{I}$ is either $NWD(\mathbb{Q})$ or $NULL(\mathbb{Q})$. In this case let $X_n = \mathbb{Q} \cap (0, \frac{1}{n+1}]$.

□

Recław [9, Theorem 8] proved that if $\mathcal{G}$ is a filter with the Baire property and a filter $\mathcal{F}$ contains the filter $\mathcal{F}_0$ on $\omega \times \omega$ generated by all sets of the form $\{(n,m) \in \omega \times \omega : n \geq k\}$ where $k \in \omega$ and $\{(n,m) \in \omega \times \omega : m \geq f(n)\}$ where $f : \omega \to \omega$ (i.e., in the notation of [2, §1.2], the filter dual to the ideal generated by $(Fin \times \emptyset) \cup (\emptyset \times Fin)$), then there is a 0-1 counterexample to FP for $\langle \mathcal{M}_\mathcal{G}, \mathcal{M}_\mathcal{F} \rangle$.

Since the filter $\mathcal{F}_0$ has property (S), the sets $A_n = \{n\} \times \omega$, $n \in \omega$, being a witnessing partition, the following theorem generalizes Recław's result.

Theorem 3.4. *If $\mathcal{G}$ is a filter with the Baire property and $\mathcal{F}$ is a filter with property (S), then there is a 0-1 counterexample to FP for $\langle \mathcal{M}_\mathcal{G}, \mathcal{M}_\mathcal{F}\rangle$.*

Proof. Fix a partition $\{A_n :\ n \in \omega\}$ of ω which witnesses property (S) of the filter $\mathcal{F}$. Identify ω via a fixed bijection with the set $\bigcup_{n\in\omega}(\{n\} \times A_n)$ so that A_n is transformed onto $\{n\} \times A_n$ for every $n \in \omega$.

By 2.3 and 2.4, it suffices to prove that there is a 0-1 counterexample to FP for $\langle \mathcal{M}, \mathcal{M}_\mathcal{F}\rangle$. We shall find it in the space $\prod_{n\in\omega} A_n \times [\bigcup_{n\in\omega}(\{n\} \times A_n)]^\omega$ using the formula:

$$C = \{\langle f, Y\rangle :\ \forall^\infty n \in \omega\ f(n) \notin Y_n\},$$

where $Y_n = \{m \in A_n :\ \langle n, m\rangle \in Y\}$ denotes the vertical section determined by n of an infinite set $Y \subseteq \bigcup_{n\in\omega}(\{n\} \times A_n)$.

More precisely, we shall prove that the set C, which is clearly Borel, has the following properties:

(1) For every $f \in \prod_{n\in\omega} A_n$, the complement of the vertical section
$$C_f = \{Y :\ \forall^\infty n \in \omega\ f(n) \notin Y_n\}$$
determined by f is in $\mathcal{M}_\mathcal{F}$.

(2) For every $Y \in [\bigcup_{n\in\omega}(\{n\} \times A_n)]^\omega$ such that $\exists^\infty n \in \omega\ Y_n \neq \emptyset$, the horizontal section
$$C^Y = \{f :\ \forall^\infty n \in \omega\ f(n) \notin Y_n\}$$
determined by Y is meager in the space $\prod_{n\in\omega} A_n$ (with the product topology of countably many discrete spaces A_n, $n \in \omega$).

(3) The complement of the set $\{Y :\ \exists^\infty n \in \omega\ Y_n \neq \emptyset\}$ is in $\mathcal{M}_\mathcal{F}$.

The rest of the argument is a refinement of Recław's proof of [9, Theorem 8].

To prove (1), let $f \in \prod_{n\in\omega} A_n$ and take an arbitrary $[s, B] \in O_\mathcal{F}$. Then, by property (S_2) of $\mathcal{F}$, $f \in \mathcal{F}^*$, so $B \setminus f \in \mathcal{F}$. But $[s, B\setminus f] \subseteq C_f$ which shows that the complement of the set C_f is nowhere dense in the topology $\tau_\mathcal{F}$.

In order to show (2), fix $Y \in [\bigcup_{n\in\omega}(\{n\} \times A_n)]^\omega$ with the property $\exists^\infty n \in \omega\ Y_n \neq \emptyset$. Then

$$C^Y = \bigcup_{k\in\omega} \bigcap_{n\geq k} \{f :\ f(n) \notin Y_n\},$$

so it suffices to prove that for every k, the set $\bigcap_{n\geq k}\{f :\ f(n) \notin Y_n\}$, which is clearly closed, is nowhere dense. So fix k and let s be a finite sequence such that $s(i) \in A_i$ for any $i \in \mathrm{dom}(s)$. Find $n \geq k$, $n \notin \mathrm{dom}(s)$, with $Y_n \neq \emptyset$ and extend s to the sequence t such that $\mathrm{dom}(t) = \mathrm{dom}(s) \cup \{n\}$ and $t(n) \in Y_n$. Then

$$\{f \in \prod_{n\in\omega} A_n :\ t \subseteq f\} \subseteq \prod_{n\in\omega} A_n \setminus \bigcap_{n\geq k}\{f :\ f(n) \notin Y_n\}.$$

To see that (3) is true, note that

$$\{Y:\ \forall^\infty\ Y_n=\emptyset\}=\bigcup_{k\in\omega}\bigcap_{n\geq k}\{Y:\ Y_n=\emptyset\},$$

so it suffices to prove that for every k, the set $\bigcap_{n\geq k}\{Y:\ Y_n=\emptyset\}$ is nowhere dense in the topology $\tau_{\mathcal{F}}$ (it is clearly closed even in the standard topology).

So fix arbitrary k and $[s,B]\in O_{\mathcal{F}}$. Then, by property (S_1) of $\mathcal{F}$, there is $n\geq k$ such that $(B\setminus s)\cap(\{n\}\times A_n)\neq\emptyset$ or, equivalently, $(B\setminus s)_n\neq\emptyset$. Take $m\in(B\setminus s)_n$ and let $t=s\cup\{\langle n,m\rangle\}$. Clearly,

$$[t,B]\subseteq\bigcup_{n\geq k}\{Y:\ Y_n\neq\emptyset\},$$

which completes the proof.

□

The following two corollaries give a partial answer to Recław's problem [9, Problem 2] if there exist Borel filters $\mathcal{F}$ and $\mathcal{G}$, at least one not countably generated, such that $\langle\mathcal{M}_{\mathcal{G}},\mathcal{M}_{\mathcal{F}}\rangle$ has FP.

Theorem 3.5. *If $\mathcal{G}$ is a filter with the Baire property and $\mathcal{F}$ is a non-countably generated analytic p-filter, then there is a 0-1 counterexample to FP for $\langle\mathcal{M}_{\mathcal{G}},\mathcal{M}_{\mathcal{F}}\rangle$.*

Proof. This immediately follows from 3.4 and 3.2. □

Theorem 3.6. *If $\mathcal{G}$ is a filter with the Baire property and $\mathcal{F}$ is the dual filter to any of the following analytic ideals:*

- *the ordinal ideal $\mathcal{I}_{\omega^\beta}$ for any $\beta>1$,*
- *$NWD(\mathbb{Q})$,*
- *$NULL(\mathbb{Q})$.*

then there is a 0-1 counterexample to FP for $\langle\mathcal{M}_{\mathcal{G}},\mathcal{M}_{\mathcal{F}}\rangle$.

Proof. This immediately follows from 3.4 and 3.3. □

In view of the results above one is tempted to conjecture that every non-countably generated analytic ideal on ω has property (S). This, however, is not true. It may be proved that *the branching ideal $\mathcal{I}_b$ on the full binary tree $\{0,1\}^{<\omega}$, i.e., the family of all subsets A of $\{0,1\}^{<\omega}$ such that A is contained in a finite number of branches of the tree (see* [8]*), is a Borel, in fact F_σ, ideal without property (S).*

4. Ultrafilters

Recall that a filter $\mathcal{F}$ on ω is a *q-filter* if every partition of ω into finite sets A_n, $n\in\omega$, has a selector in $\mathcal{F}$, i.e., there is an $S\in\mathcal{F}$ such that $|S\cap A_n|=1$ for all n.

It is well known and easy to infer from Proposition 2.1 that *if $\mathcal{F}$ is a q-filter, then it does not have the Baire property.* Thus the following result is in a sense orthogonal to Theorem 3.5.

Theorem 4.1. *If $\mathcal{G}$ is a filter with the Baire property and $\mathcal{F}$ is a q-filter, then there is a 0-1 counterexample to FP for $\langle \mathcal{M}_{\mathcal{G}}, \mathcal{M}_{\mathcal{F}} \rangle$.*

Proof. As in the proof of 3.4, it suffices to prove that there is a 0-1 counterexample to FP for $\langle \mathcal{M}, \mathcal{M}_{\mathcal{F}} \rangle$.

We define it in the space $\omega^{\uparrow\omega} \times [\omega]^\omega$ by the formula:

$$C = \{\langle f, Y \rangle : \ \exists n \in \omega \ |Y \cap [f(n), f(n+1))| > 1\},$$

where $\omega^{\uparrow\omega}$ denotes the space of all strictly increasing functions $f : \omega \to \omega$.

The set C is clearly Borel, in fact open.

To see that all its vertical sections are in $\mathcal{M}_{\mathcal{F}}$, fix arbitrary strictly increasing function $f : \omega \to \omega$ and $[s, B] \in O_{\mathcal{F}}$. Since $\mathcal{F}$ is a q-filter, there is $S \in \mathcal{F}$ such that $|S \cap [f(n), f(n+1))| = 1$ for every $n \in \omega$.

But then

$$[s, B \cap S] \subseteq \{Y \in [\omega]^\omega : \ \forall n \in \omega \ |Y \cap [f(n), f(n+1))| \leq 1\} = [\omega]^\omega \setminus C_f$$

which shows that the set C_f is nowhere dense in the topology $\tau_{\mathcal{F}}$.

To prove that all horizontal sections of C are comeager, take an arbitrary $Y \in [\omega]^\omega$. Since the set

$$C^Y = \{f \in \omega^{\uparrow\omega} : \ \exists n \in \omega \ |Y \cap [f(n), f(n+1))| > 1\},$$

is open, we complete the proof by showing that it is dense.

So let $s = (s(0), \ldots, s(n-1))$ be a finite strictly increasing sequence of natural numbers. Find $k > s(n-1)$ such that $|Y \cap [s(n-1), k)| > 1$. Then $f \in C^Y$, whenever $f \in \omega^{\uparrow\omega}$ extends the sequence $s^\frown k$.

$\square$

A q-filter which is an ultrafilter is called a *Q-point.*

Observe that *if an ultrafilter is not a Q-point, then it has property (S).* (As a matter of fact, *an ultrafilter has property (S) if and only if it is not Ramsey*, so *it is consistent with ZFC that all ultrafilters on ω have property (S)* – see [1, §4.5]).

Now, bringing together Theorem 3.4 and Theorem 4.1 we obtain the following corollary.

Theorem 4.2. *If $\mathcal{G}$ is a filter with the Baire property and $\mathcal{F}$ is an ultrafilter, then there is a 0-1 counterexample to FP for $\langle \mathcal{M}_{\mathcal{G}}, \mathcal{M}_{\mathcal{F}} \rangle$.*

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